KTH Engineering Sciences

A kernel function approach to exact solutions of Calogero-Moser-Sutherland type models

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Scientific thesis for the degree of Doctor of Philosophy (PhD) in the subject area of Physics.
Stockholm, Sweden 2016

TRITA–FYS 2016:58
ISSN 0280–316X
ISRN KTH/FYS/–16:58–SE
ISBN 978–91–7729–132–9

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Akademisk avhandling som med tillstånd av Kungliga Tekniska högskolan i Stockholm framlägges till offentlig granskning för avläggande av Teknologie Doktorsexamen (TeknD) inom ämnesområdet Fysik 27 oktober i Oskar Kleins auditorium (FR4).

Tryck: Universitetsservice US AB

Abstract

This Doctoral thesis gives an introduction to the concept of kernel functions and their significance in the theory of special functions. Of particular interest is the use of kernel function methods for constructing exact solutions of Schrödinger type equations, in one spatial dimension, with interactions governed by elliptic functions. The method is applicable to a large class of exactly solvable systems of Calogero-Moser-Sutherland type, as well as integrable generalizations thereof. It is known that the Schrödinger operators with elliptic potentials have special limiting cases with exact eigenfunctions given by orthogonal polynomials. These special cases are discussed in greater detail in order to explain the kernel function methods with particular focus on the Jacobi polynomials and Jack polynomials.

Key words: Kernel functions, Calogero-Moser-Sutherland models, Ruijsenaars-van Diejen models, Elliptic functions, Exact solutions, Source Identities, Chalykh-Feigin-Sergeev-Veselov type deformations, non-stationary Heun equation.

Sammanfattning

Denna doktorsavhandling ger en introduktion till konceptet kärnfunktioner och deras roll i den matematiska teorin för speciella funktioner. Av speciellt intresse är andvändningen av kärnfunktionsmetoder i konstruktionen av lösningar till Schrödinger-operatorer i en rumsdiemnsion där interaktions potentialerna är givna i termer av elliptiska funktioner. Dessa metoder är även tillämpliga på en större klass av exakt lösbara system av Calogero-Moser-Sutherland-typ, samt integrabla generaliseringar därav. Schrödinger-operatorer med elliptiska potentialer har kända gränsfall där de exakta lösningarna är ortogonala polynom. Dessa specialfall kommer att behandlas ingående för att ge en mer naturlig förståelse. Vi kommer i främst att studera Jacobi-polynomen och Jack-polynomen.

Nyckelord: Kärnfunktioner, Calogero-Moser-Sutherland-modeller, Ruijsenaars-van Diejen-modeller Elliptiska funktioner, Exakta lösningar, Källidentiteter, CFVS-deformationer, icke-stationära Heun-ekvationen.

Typeset in LaTeX

Preface

This thesis for the degree of Doctor of Philosophy (Ph.D.) in the subject area of Physics is to summarize my research at the Theoretical Physics Department of the Royal Institute of Technology (KTH) in Stockholm, Sweden from 2011 to 2016. The thesis is divided into two parts: the first provides background and complementary results to our research papers. The second consists of four appended scientific papers.

Overview of the thesis

List of papers included in this thesis

I F. Atai, M. Hallnäs, and E. Langmann, *Source Identities and Kernel Functions for Deformed (Quantum) Ruijsenaars Models*, Lett. Math. Phys. (2014) **104**:811.

II F. Atai and E. Langmann, *Deformed Calogero-Sutherland model and fractional Quantum Hall effect*, Preprint: arXiv:1603.06157

III F. Atai and E. Langmann, *Series solutions of the non-stationary Heun equation*, Preprint: arXiv:1609.02525

IV F. Atai, *Integral representation of solution to the non-stationary Lamé equation*, manuscript

The thesis author's contribution to the papers

I I performed the calculations and proved the main results of the Paper, based on an idea by E. Langmann. I also wrote a first draft of the paper and the Paper was finalized in close collaboration.

II The main result of the Paper was proven by me, and I did all the calculations. I wrote a first draft of the paper and the Paper was finalized in close collaboration.

III The results in Sections 2,3, and 5.1 was obtained independently by both authors. I participated in writing the Paper.

Paper not included in this thesis

1. F. Atai, J. Hoppe, M. Hynek, and E. Langmann, *Variational orthogonalization*, Preprint: arXiv:1307.4010

Acknowledgements

Many people have contributed to my development:

First and foremost, I would like to give my most heartfelt thanks to my supervisor Professor Edwin Langmann, for his support, patience, and for introducing me to the exciting fields of special functions and elliptic functions. Words are not enough to express how grateful I am.

My sincerest gratitude to Martin Hallnäs for his insight, comments, and for sharing his vast knowledge of special functions.

I would also like to thank Jack Lidmar and Teresia Månsson for their assistance during my Ph.D. studies.

A big thanks to all my current colleagues in the Theoretical Physics department. Thank you; Karl, Daniel, Patrik, Linnea, Anatoly, John, Göran, Mikael, Stella, Mats, and Stefan for many entertaining conversations around the office. Also to my former colleagues; Johan, Jonas, Hannes, Andreas, Erik, and Richard for creating a hospitable environment. I am also grateful to Olle Edholm for all that he has done, in particular for proofreading this Thesis. I would also like to express my gratitude to my colleagues at the Mathematics department. I am also pleased to have gotten to know Julien and Oskar, whose friendship I cherish. Over the years I have had the privilege of sharing office some truly inspiring people. I would like to thank my former office mates M. Pawellek and J. de Woul. To Per: I will miss our many diverse discussions, and inspiring arguments, over these last few years.

My heartfelt thanks my family for their love, support, and encouragements: My mother Mehri for her never ending support, Rassul, Shahnaz, Hamid, and Annalena for their love. And Bahman for his love of, and thoughts on, physics and mathematics.

To my love: Thank you for your support during this time.

Farrokh Atai
(Stockholm, October 27, 2016)

Contents

Part I

Background and complementary results

Chapter 1

Introduction

The theoretical understanding of our physical surroundings is based on suitable mathematical models that effectively describe and predict phenomena observed in nature. In exceptional cases, the models can be understood completely by exact analytical solutions, and the theory of special functions appears naturally in this context. Other instances require approximate or numerical methods, but even these rely on the exceptional cases for points of reference or benchmarks. The theory of special functions plays a significant role in mathematics and physics, both in topics that can be found in textbooks and in ongoing research, as we will discuss below. Special functions are useful as many of their properties are known in great detail, such as series expansions, integral representations, and generating functions in closed form that are expressed in terms of more elementary functions. Utilizing special functions for our models is advantageous since having suitable representations for solutions allows us to determine many interesting properties, both qualitatively and quantitatively.

Early in the studies of physics or mathematics, special functions are introduced when solving differential equations arising in important problems such as heat transport and wave propagation. In the standard undergraduate courses on electrodynamics, one considers solutions of Laplace and Helmholtz equations in special geometries. These courses provide an introduction to special functions known by the names Laguerre, Bessel, and Legendre, to name a few. In classical mechanics we get differential equations of varying complexity from Newton's second law. The solution of the Kepler problem played an important role in understanding and explaining celestial mechanics. The problems that have been mentioned also have special cases, see *e.g.* Chapter 3 of [1], where the solutions are given in terms of elliptic functions[1] which play an important role in this work, as is discussed below and in Chapter 3. Elliptic functions have been studied extensively by Abel, Jacobi, Weierstrass, Hermite, etc., and have also appeared naturally in many different problems in the modern fields of mathematics and physics, some of which are discussed

[1] Originally, the name was derived from their relation to elliptic integrals, see *e.g.* Section XXII of [2].

below. It suffices, at this point, to describe elliptic functions as natural generalizations of the trigonometric functions, in the same manner as the trigonometric functions generalize the rational functions, and to illustrate with an example that features prominently in our work: The Weierstrass $\wp$-function, defined as

$$\wp(x) \equiv \frac{1}{x^2} + \sum_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(x+2\omega_1 n+2\omega_3 m)^2} - \frac{1}{(2\omega_1 n+2\omega_3 m)^2}, \tag{1.1}$$

is an elliptic function with two distinct periods $(2\omega_1, 2\omega_3)$, whose ratio is not purely real; it generalizes the trigonometric function $1/\sin(\pi x/2\omega_1)^2$ similar as the latter generalizes the rational function $1/x^2$; see (2.2).

The theory of special functions is essentially equivalent to the theory of quantum integrable models, which plays a significant role in the understanding of the microscopic and quantum regime. In order to discuss these matters we must first specify what we mean by a understanding of quantum mechanical models. These models are defined by a Schrödinger operator, or Hamiltonian, $H = T + V$ that represents the energy, meaning that it has the general form with a kinetic term T and a potential term V. Solving the quantum mechanical models is equivalent to finding a complete set of eigenfunctions ψ_n and corresponding eigenvalues E_n, *i.e.*

$$H\psi_n = E_n\psi_n, \tag{1.2}$$

and it is then natural to claim that a model is solved if the eigenfunctions ψ_n, and corresponding eigenvalues E_n, have been determined. We often refer to the solved models as exactly solvable. The problem in (1.2) is, in general, a difficult problem and it is at this point that the theory of special functions enters. A paradigm is the hydrogen atom model which is obtained by a canonical quantization of the Kepler problem and successfully treated by special functions: The model has known eigenvalues and (exact) eigenfunctions in terms of (associated) Laguerre and Legendre polynomials. Another well-known example is the Hermite polynomials which appear as solutions of the simple harmonic oscillator model, defined formally by the Schrödinger operator

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 \quad (x\in\mathbb{R}), \tag{1.3}$$

where $\hbar$ is Planck's constant divided by 2π, $\omega > 0$ the coupling strength, and m the mass of the particle. Special functions, as those mentioned in the above examples, have provided prototypes for numerous other models; one aim is to develop a similar understanding in areas of ongoing research, such as general quantum many-body problems with strongly interacting particles.

In seminal papers [3–5] of Calogero a natural many-variable generalization of (1.3) with strongly interacting, identical particles was solved explicitly. The model can be formally defined by a Schrödinger operator of the form

$$\sum_{j=1}^{N}\Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_j^2} + \frac{m\omega^2}{2}x_j^2\Big) + \sum_{1\le j<k\le N}\frac{\kappa^2}{(x_k-x_j)^2} \tag{1.4}$$

restricted to the region $\{x \in \mathbb{R}^N : x_1 < x_2 < \ldots < x_N\}$, and it is exactly solvable in the sense that it has known eigenvalues in closed form and exact eigenfunctions in terms of a symmetric many-variable generalization of the Hermite polynomials[2] for $\kappa^2 > -\hbar^2/4m$. As we will discuss in Section 5.2, the model exhibits exotic particle statistics when not restricted to the region above [7]. Sutherland [8, 9], inspired by (1.4), considered a system of strongly interacting, identical particles under the influence of a periodic potential, defined by the (formal) Schrödinger operator

$$\sum_{j=1}^{N} -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_j^2} + \sum_{1\le j<k\le N} \frac{\pi^2}{2m\omega_1^2}\frac{g(g-\hbar)}{4\sin(\frac{\pi}{2\omega_1}(x_k-x_j))^2} \tag{1.5}$$

in the region $\{x \in [-\omega_1, \omega_1]^N : -\omega_1 < x_1 < \ldots < x_N \le \omega_1\}$ ($\omega_1 > 0$). The model is often referred to as the *Calogero-Sutherland* (CS) model, or the Sutherland model. It also has explicitly known eigenvalues and eigenfunctions given in terms of particular symmetric polynomials, later identified as the Jack polynomials [10] (see also section 2.2). The theory of symmetric functions have many different applications in both physics and mathematics, such as representation theory [11], algebraic geometry [12], and group theory [13]. (A more extensive discussion can be found in *e.g.* [14] and references therein.) A comprehensive review of these many applications are outside the scope of this thesis but we mention the non-interacting case as an illustrative example: The free fermion[3] case of the CS model (*i.e.* $g = \hbar$ in (1.5)) has explicitly known eigenfunctions in terms of the Schur polynomials, which yield a useful representation of Slater determinants. We would also like to mention that these models have exactly solvable classical limits (*i.e.* taking $\hbar \to 0$ in a suitable manner) known as the Calogero-Moser models [15].

The examples discussed above, and others which are of great interest in our work, fall into a larger family of models that can be defined by formal Schrödinger operators of the general form

$$\sum_{j=1}^{N}\Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_j^2} + U(x_j)\Big) + \sum_{1\le j<k\le N} V(x_k, x_j) \tag{1.6}$$

with particular external potential U and two-body interaction V; see Table 2.1 for examples. The models defined by operators of the form (1.6) are commonly referred to as *Calogero-Moser-Sutherland* (CMS) type models in the literature. The CMS type models have received great interest as they are soluble[4] for a large family of different potentials U and V in (1.6), which also includes potentials that are given in terms of elliptic functions, as we will discuss in Chapter 3. A property of many of the CMS type models, and also the models that are discussed below, is that their

[2] The eigenfunctions are also multiplied by a known ground state wave function that includes a Jastrow type [6] term.

[3] We will not consider spin in this Thesis.

[4] Both quantum integrable and exactly solvable [16] in most cases while only quantum integrable in some [17].

exact eigenfunctions are of the general form $\psi(x) = \psi_0(x)P(z(x))$, with $\psi_0(x)$ the groundstate eigenfunction and $P(z)$ a (symmetric) polynomial that, in many cases, is a natural many-variable generalizations of the classical orthogonal polynomials, such as the Jacobi, Laguerre, Bessel, and Hermite polynomials [18] (see also [16] and references therein). Another remarkable characteristic of the CMS type models is the existence of so-called *kernel functions* which provide a natural tool for constructing eigenfunctions, allowing us to obtain different representations and prove properties of these eigenfunctions, and simple determination of the corresponding eigenvalues (see Chapter 4 for definition and simple examples of kernel functions). An advantageous property of the kernel function method to solve CMS type systems is that it naturally extends to the models with elliptic interactions, as we will explain in Section 4.4. It is known that the models of CMS type can be related to irreducible root systems of classical Lie algebras [17] and Lie super-algebras [19,20], which yields a natural relation between CMS type models and the Laplace operator on symmetric spaces, for special values of the coupling parameters. We will not discuss these relations in detail but will use the nomenclature of root systems, *i.e.* A_{N-1} and BC_N, to distinguish different CMS type models, see also Table 2.1. Different interesting generalizations[5] of the CMS type models have also been constructed, and we now turn to certain generalizations that play a more prominent role in our Papers.

A partially solvable generalization of the CS model (1.5), that is defined by the Schrödinger operator

$$\sum_{j=1}^{N} -\frac{\hbar^2}{2m_j}\frac{\partial^2}{\partial x_j^2} + \sum_{1\leq j<k\leq N} \frac{\pi^2 g(m_j+m_k)(gm_jm_k-\hbar)}{8\omega_1^2 \sin(\frac{\pi}{2\omega_1}(x_j-x_k))^2}, \tag{1.7}$$

was constructed by Sen [21] where the particles have different masses. (Note that the dimension of g here is not the same as previously.) The Sen type generalizations also exists for the models of CMS type [16]. They are all partially solvable in the sense that they have an explicitly known exact groundstate, but a complete set of eigenfunctions have not been found. This results hold true for arbitrary masses and, from a purely mathematical perspective, the particle masses can be arbitrary constants, and we will allow them to also assume negative values. Then the eigenvalue equation for the explicitly known groundstate of (1.7) is, in special cases, equivalent to kernel function identities for pairs of CMS type operators, or deformations thereof; see below. We refer to the groundstate eigenvalue equation for the Sen type operators as *source identities* since it can be used as a source for kernel function identities.

A mathematically natural generalization of the CS models was developed by Chalykh, Feigin, Sergeev, and Veselov [19, 20, 22, 23] which incorporates two independent types of strongly interacting, indistinguishable particles and reduce to the standard CMS type models when only one particle type remains. This generalization of the CS model is called the *deformed Calogero-Sutherland* model, and similar

[5] In the sense that the CMS models in (1.6) are obtained as special cases.

generalizations of other systems is referred to as Chalykh-Feigin-Sergeev-Veselov (CFVS) type deformation. The CFVS type deformation of (1.5) formally defined by the differential operator[6]

$$\sum_{j=1}^{N} -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{\tilde{N}} \frac{g\hbar}{2}\frac{\partial^2}{\partial \tilde{x}_j^2} + \sum_{1\leq j<k\leq N} \frac{(\frac{\pi}{\omega_1})^2 g(g-\hbar)}{4\sin(\frac{\pi}{2\omega_1}(x_k-x_j))^2} + \sum_{1\leq j<k\leq \tilde{N}} \frac{(\frac{\pi}{\omega_1})^2 \hbar^2 (g-\hbar)/g}{4\sin(\frac{\pi}{2\omega_1}(\tilde{x}_k-\tilde{x}_j))^2} + \sum_{j=1}^{N}\sum_{k=1}^{\tilde{N}} \frac{(\frac{\pi}{\omega_1})^2 \hbar(\hbar-g)}{4\sin(\frac{\pi}{2\omega_1}(x_j-\tilde{x}_k))^2} \tag{1.8}$$

with $N, \widetilde{N} \in \mathbb{N}_0$ the number of different particles, has exact eigenfunctions given in terms of the so-called *super-Jack polynomials* which are of particular interest in Papers II, as we will elaborate on in Section 5.2. The deformed models are interesting since they have explicitly known eigenvalues and eigenfunctions in closed form, yet a complete understanding of the deformed models is an area of ongoing research. Let us mention one motivating question for our Paper II: The deformed Calogero-Sutherland models does not have a satisfactory interpretation as a standard Quantum Mechanics model since the eigenfunctions are not normalizable (see Paper II for details). It can also be shown that the deformed models reduce to the standard Calogero-Moser models in the classical limit, *i.e.* in this limit there exists no dependence on the second type of particles. A pertinent question is then whether the deformed Calogero-Sutherland model can describe any physical system. In Paper II we propose that the deformed CS model is related to Quantum Field Theory, rather than Quantum Mechanics, and we present an application in the context of the Fractional Quantum Hall Effect (see Paper II).

There exists a one-parameter generalization of the CMS type models known as the *relativistic Calogero-Sutherland models*. The relativistic model originally appeared in the context of the sine-Gordon model [24] and was solved in the classical setting by Ruijsenaars and Schneider [25]. The quantum (trigonometric) Ruijsenaars model can formally be defined by the analytic difference operator[7] [26]

$$\sum_{\substack{j=1\\ \epsilon=\pm}}^{N} \prod_{\substack{k=1\\ k\neq j}}^{N} \Big(\frac{\sin(\frac{\pi}{2\omega_1}(x_k-x_j-\mathrm{i}\epsilon g\beta))}{\sin(\frac{\pi}{2\omega_1}(x_k-x_j))}\Big)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\epsilon\hbar\beta\partial_j} \prod_{\substack{k=1\\ k\neq j}}^{N} \Big(\frac{\sin(\frac{\pi}{2\omega_1}(x_k-x_j+\mathrm{i}\epsilon g\beta))}{\sin(\frac{\pi}{2\omega_1}(x_k-x_j))}\Big)^{\frac{1}{2}}, \tag{1.9}$$

in the region $\{x \in [-\omega_1,\omega_1]^N : -\omega_1 < x_1 < \ldots < x_N \leq \omega_1\}$ and where β is the relativistic deformation parameter, proportional to $1/c$ where c is the speed of light in vacuum. (We set the particle mass in (1.9) to $1/2$ for simplicity.) The operator in (1.9) has known eigenvalues and exact eigenfunctions in terms of the Macdonald polynomials [27]. The Ruijsenaars model in (1.9) has a (quantum) integrable generalization when the trigonometric functions are replaced by elliptic

[6] We set the particle mass to unity for simplicity.

[7] We recall that the operator $-\mathrm{i}\partial_j \equiv -\mathrm{i}\frac{\partial}{\partial x_j}$ is the generator of translations.

ones [26]; see Section 3.2. Is it also known that there exists a CFVS type deformation of the operator in (1.9) [28], and we construct a CFVS type deformation of the elliptic Ruijsenaars model in Paper I. The Ruijsenaars models are known to have integrable generalizations, in the same sense that (1.6) generalizes (1.5), which are known as the van Diejen models [29]. To be more specific: The Ruijsenaars models are related to the A_{N-1} root system, and the van Diejen models generalizes to the other relevant root systems. The models of Ruijsenaars-van Diejen (RvD) type are referred to as relativistic models (see also [30]) due to their reduction to the CMS type models: The analytic difference operator of RvD type reduces to Schrödinger operator of the form (1.6) in the limit $\beta \downarrow 0$ after subtracting the relativistic rest energy. To illustrate this, we consider the reduction of (1.5) which is obtained by multiplying (1.9) with $1/\beta^2$, subtracting the constant N/β^2, and then taking the limit $\beta \downarrow 0$. This yields the Hamiltonian in (1.5).

The CMS and RvD type models have natural generalization where the interactions are given in terms of elliptic functions, rather than rational, hyperbolic, or trigonometric functions, which are amenable to exact treatment. We refer to these types of generalizations as systems with *elliptic potentials*. As we will elaborate further on in Chapter 3, the operators with elliptic potentials can be viewed as the most general in the sense that the other models of CMS, and RvD, type can be obtained by (suitable) limits of the elliptic models. An illustrative example is the elliptic CS model where the $1/\sin^2$ potential in (1.5) is replaced with the Weierstrass $\wp$-function in (1.1), and (1.5) is obtained in the trigonometric limit, *i.e.* $\omega_3 \to +\mathrm{i}\infty$ (see Chapter 3).

A famous example of an operator with an elliptic potential was obtained by Lamé [31, 32] when considering the stationary temperature distribution on an ellipsoid. It was implicitly shown by Lamé that the Laplace equation can be transformed to an eigenvalue equation for a Schrödinger operator with an elliptic potential, known as the *Lamé equation*, when transforming to confocal ellipsoidal coordinates. The aforementioned elliptic CS model can also be viewed as a natural many-variable generalization of the Lamé equation, in the sense that the two variable case of the elliptic CS model reduces to the Lamé equation in the center-of-mass coordinates. We will elaborate more on systems with elliptic potentials in Chapter 3, but would like to conclude this paragraph by mentioning some of the models with elliptic potential which motivated the work in Papers III and IV. A natural generalization of the Lamé equation was introduced by Darboux [33], and independently rediscovered by Treibich and Verdier [34]. This case corresponds to a the Schrödinger operator with a so-called Darboux-Treibich-Verdier (DTV) potential (see Eq. (3.11)) can be reduced to the *Heun differential equation*. (The relation is explained in more details in Section 3.1.) Recent developments have yielded generalizations of the elliptic CMS type models where the period ω_3 (see (1.1)) is treated as an additional variable. These generalized models are often referred to *non-stationary* CMS type models due to their resemblance to a Schrödinger equation with time dependent potential. Constructing solutions of the non-stationary Lamé (see (3.19)) and the non-stationary Heun equations is an area of ongoing research [35–45]. In Paper

III we construct solutions of the non-stationary Heun equation, defined by the eigenvalue equation for the differential operator

$$\frac{\mathrm{i}\hbar^2\pi}{2m\omega_1^2}\kappa\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2m}\sum_{\nu=0}^{3}g_\nu(g_\nu-\hbar)\wp(x+\omega_\nu) \tag{1.10}$$

with $\omega_0\equiv 0$, ω_1 held constant, $\omega_3\equiv\omega_1\tau$, $\omega_2\equiv-\omega_1-\omega_3$, and (arbitrary) model parameters $(A,\{g_\nu\}_{\nu=0}^3)$, as well as the non-stationary Lamé equation[8] by the recursive and perturbative algorithm (to all orders) explained in Sections 5.3. As we will show in Section 3.1, the eigenvalue equation for the operator in (1.10) reduces naturally to the Heun equation in the $\kappa=0$ case.

Plan of the Thesis

In Chapter 2 we consider the limiting cases of the CMS and RvD type models models with elliptic potentials. We also recall standard methods for constructing eigenfunctions.

Chapter 3 discusses the CMS and RvD type models with elliptic potentials.

In Chapter 4 it is demonstrated how the kernel function methods are used for constructing solutions of the CMS and RvD type models.

Chapter 5 gives a brief introduction to the appended scientific papers.

Notation

We start by setting all fundamental constants to unity throughout this thesis, *i.e.* $c=e=\hbar=1$ where c is the speed of light in vacuum, e the elementary charge, and $\hbar$ is Planck's constant divided by 2π. We will (in general) rescale systems with compact support to be on the interval $[-\pi,\pi]$ or tensor products of those; this simply implies that the constant ω_1 (see *e.g.* (1.1) and (1.5)) is set to π, without loss of generality. The general periods are included only when relations between different models are considered. We denote by $\mathbb{N},\mathbb{Z},\mathbb{R}$, and $\mathbb{C}$ the set of positive natural numbers, all integers, real numbers, and complex numbers, respectively. We denote by $\mathbb{N}_0$ the set $\mathbb{N}\bigcup\{0\}$. We also use the notation x for vectors, *i.e.* for V an N dimensional vectors space ($N\in\mathbb{N}$) then the elements in V are denoted by x where $x=(x_1,x_2,\ldots,x_N)$. We define $z(x)=(z(x_1),z(x_2),\ldots)=(z_1,z_2,\ldots)$ for a vector element x and function z. We will use notation $\sum_{j<k}^{N}$ as a short-hand for $\sum_{1\leq j<k\leq N}$ and so forth. We will use superscripts to distinguish the different operators and functions and this notations will carry over different chapters.

[8] The non-stationary Lamé equation is obtained from (1.10) for special values of the parameters, *e.g.* $g_\nu=0$ for $\nu=1,2,3$.

Chapter 2

Exactly solvable systems of CMS and RvD type

A main goal of our research is to construct solutions of CMS and RvD type models *elliptic potentials* (recall that by systems with elliptic potentials we mean models where the interactions are given in terms of elliptic functions; see Chapter 3). The models with elliptic potentials have special limiting cases where the interactions are given in terms of rational, hyperbolic, or trigonometric functions, and these have exact eigenfunctions of the general form $\psi(x) = \psi_0(x)P(z(x))$ with corresponding eigenvalues in closed form (as explained below). In Chapter 4 we explain our kernel function method for constructing the eigenfunctions using such limiting cases as examples. In this Chapter we also recall standard methods to construct these solutions which then are used as benchmarks to compare with results obtained using the kernel function method in Chapter 4.

2.1 Classical orthogonal polynomials

In this Section we consider Schrödinger operators with Pöschl-Teller potentials [46] which are obtained from the Lamé and Heun equations in the trigonometric limit (see Section 3.1). The systems with Pöschl-Teller potentials have exact eigenfunctions given in terms of the classical orthogonal trigonometric polynomials, by which we mean the Chebyshev, Gegenbauer, and Jacobi polynomials, as we will now discuss.

Gegenbauer polynomials

A simple representative for the CMS type models is defined by a Schrödinger operator formally defined as

$$H^{(\mathrm{G})}(x;g) \equiv -\frac{d^2}{dx^2} + g(g-1)\frac{1}{\sin(x)^2} \quad (x \in [-\pi,\pi]), \tag{2.1}$$

which is the simplest (non-trivial) special case of the Pöschl-Teller potential (as explained below). (The coupling parameter is written as $g(g-1)$ for reasons that will become clear below.) The operator in (2.1) is obtained from the Lamé operator (see (3.8)) in the trigonometric limit.

To indicate how such an operator is related to a physical model, we now sketch how it can appear in Quantum Mechanics. Consider two particles confined on the unit circle (recall that the circumference is then 2π) with an interaction that is proportional to the inverse distance squared. Due to translational invariance, we fix the first particles position to 0 and the other particles relative position to x. Using the method of mirror imaging, we then map the system to the real line and the full potential becomes

$$\sum_{m\in\mathbb{Z}} \frac{1}{(x+2\pi m)^2} = \frac{1}{4\sin(\frac{1}{2}x)^2}. \tag{2.2}$$

The (formal) operator in (2.1) have explicitly known, exact eigenfunctions[1] given by

$$\psi_n^{(\mathrm{G})}(x;g) \equiv \sin(x)^g C_n^{(g)}(\cos(x)) \quad (n\in\mathbb{N}_0) \tag{2.3}$$

with $C_n^{(g)}(z)$ the Gegenbauer polynomials[2], and corresponding eigenvalues $(n+g)^2$. We proceed to recall a standard method to derive this results. The operator in (2.1) can be factorized as follows:

$$H^{(\mathrm{G})}(x;g) = (Q^{(\mathrm{G})})^* Q^{(\mathrm{G})} + g^2, \quad Q^{(\mathrm{G})} = -\frac{d}{dx} + g\cot(x), \tag{2.4}$$

with $(Q^{(\mathrm{G})})^*$ the formal adjoint of $Q^{(\mathrm{G})}$, and it follows from (2.4) that the eigenvalues of $H^{(\mathrm{G})}(x;g)$ are positive and bounded from below by g^2. The groundstate eigenfunction can be constructed explicitly by solving the first order differential equation $Q^{(\mathrm{G})}\psi_0 = 0$, and it is a simple exercise to check that the function

$$\psi_0(x;g) = \sin(x)^g \tag{2.5}$$

is the unique solution, up to normalization, of this equation. We now turn our attention to the other eigenfunctions. We make the ansatz that the eigenfunctions are of the form $\psi^{(\mathrm{G})}(x;g) = \psi_0^{(\mathrm{G})}(x;g)P^{(\mathrm{G})}(\cos(x))$, with $\psi_0(x;g)$ in (2.5). We define the differential operator $h^{(\mathrm{G})}(z;g)$ as the similarity transformation of the Schrödinger operator $H^{(\mathrm{G})}(x;g)$ by the groundstate (up to an additive constant) and a change of variables; more specifically, let

$$\begin{aligned}\tilde{h}^{(\mathrm{G})}(x;g) &\equiv \psi_0^{(\mathrm{G})}(x;g)^{-1} H^{(\mathrm{G})}(x;g)\psi_0^{(\mathrm{G})}(x;g) - g^2 \\ &= -\frac{d^2}{dx^2} - g\cot(x)\frac{d}{dx}\end{aligned} \tag{2.6}$$

[1] The index (G) is used in order to distinguish from other cases.

[2] Also known as the ultra-spherical polynomials.

(the second line is obtained by straightforward calculation) and make a variable substitution to $z = \cos(x)$, which transforms (2.6) to the (algebraic) differential operator

$$h^{(\mathrm{G})}(z;g) \equiv (1-z^2)\frac{d^2}{dz^2} - (2g+1)z\frac{d}{dz} \tag{2.7}$$

(it is a simple exercise to check that $h^{(\mathrm{G})}(\cos(x);g) = -\tilde{h}^{(\mathrm{G})}(x;g)$). It follows that the functions $P^{(\mathrm{G})}(z)$ are eigenfunctions of the reduced operator[3] in (2.7), and that $P^{(\mathrm{G})}(z)$ are polynomials if the eigenvalues of (2.7) are $\mathcal{E}^{(\mathrm{G})} = n(n+2g)$, with non-negative integers n. A straightforward inspection shows that the eigenvalue equation for $h^{(\mathrm{G})}$, with eigenvalues $n(n+2g)$, yields the differential equation for the Gegenbauer polynomials $C_n^{(g)}(z)$, *i.e.*

$$\Big((1-z^2)\frac{d^2}{dx_2} - (2g+1)\frac{d}{dx} + n(n+2g)\Big)C_n^{(g)}(z) = 0. \tag{2.8}$$

It follows from standard arguments that the eigenfunctions in (2.3) are orthogonal with respect to the standard $L^2([-\pi,\pi],\mathrm{d}x)$ inner product, *i.e.* for $n \neq m$ we have

$$\langle \psi_n^{(\mathrm{G})} \, | \psi_m^{(\mathrm{G})} \rangle = \int_{-\pi}^{\pi} C_n^{(g)}(\cos(x)) C_m^{(g)}(\cos(x)) \left(\sin(x)^2\right)^g \mathrm{d}x = 0 \quad (n \neq m). \tag{2.9}$$

A simple variable substitution in (2.1) shows that this is equivalent to the orthogonality of the Gegenbauer polynomials, with respect to the $L^2([-1,1],\omega^{(\mathrm{G})}(x;g)\mathrm{d}x)$ inner product, where $\omega^{(\mathrm{G})}(x;g) \equiv (1-x^2)^{g-\frac{1}{2}}$, *i.e.*

$$(C_n^{(g)}, C_m^{(g)}) \equiv \int_{-1}^{1} C_n^{(g)}(x) C_m^{(g)}(x)(1-x^2)^{g-\frac{1}{2}}\mathrm{d}x = 0 \quad (n \neq m). \tag{2.10}$$

In Chapter 4 we use kernel functions for constructing eigenfunctions of CMS type models, and we will use the Gegenbauer polynomials as an illustrative example. To show that the solutions we obtain are indeed the Gegenbauer polynomials one can use the inner product in (2.9), but one can also use the following equivalent condition: Let $P(z)$ be a polynomial of degree n satisfying

$$P(z) = \frac{2^n\Gamma(n+g)}{n!\Gamma(g)}z^n + \mathcal{O}(z^{n-1}). \tag{2.11}$$

If $P(z)$ is an eigenfunction of (2.7) then the polynomial is the Gegenbauer polynomial $C_n^{(g)}$, *i.e.* $P(z) = C_n^{(g)}(z)$, and the eigenvalue is $n(n+2g)$.

The eigenfunctions in (2.3) can be used in order to form a complete orthogonal basis in the $L^2([0,\pi]),\mathrm{d}x)$, or equivalently, a complete orthogonal basis in the subspace of $L^2([-\pi,\pi])$ satisfying $\psi(-x) = \mathrm{e}^{-\mathrm{i}\pi g}\psi(x)$.

[3]Here and in the following, we refer to the operators of general form $h \equiv \pm\psi_0^{-1}H\psi_0 - E_0$ as the *reduced operators* for the Schrödinger operator H with groundstate ψ_0 and corresponding eigenvalue E_0.

It is clear that the coupling parameter in (2.1) is invariant under $g \to 1-g$ and, for this reason, there are two factorizations of the operator $H^{(\mathrm{G})}$ in (2.1): The second one is like (2.4) with g replaced by $1-g$. This naturally suggests that the functions $\psi_n^{(\mathrm{G})}(x; 1-g)$ are also eigenfunctions of the operator in (2.1). However, it is clear that these eigenfunctions are only L^2 for $1-g > -\frac{1}{2}$. Thus, for $g > \frac{3}{2}$, there is only one unique choice of self-adjoint extension of $H^{(\mathrm{G})}$ in (2.1) allowing for solutions $\psi_n^{(\mathrm{G})}$ in (2.3); for $-\frac{1}{2} < g < \frac{3}{2}$ there are two different self-adjoint extensions of this kind corresponding to the same differential operator. As an illustrative example we consider the case $g = 0$ $(1-g = 1)$ where (2.1) reduces to the second derivative. It is known that the Gegenbauer polynomials reduce to the Chebyshev polynomials of first and second kind, respectively, and using the representation of the Chebyshevs with $z = \cos(x)$ yields the eigenfunctions

$$\frac{n}{2} C_n^{(0)}(\cos(x)) = \cos(nx), \quad \sin(x) C_{n-1}^{(1)}(\cos(x)) = \sin(nx). \tag{2.12}$$

We know from *e.g.* undergraduate courses that (arbitrary) linear combinations of the functions in (2.12) are eigenfunctions of the operator $-\frac{d^2}{dx^2}$, and that we must specify the self-adjoint extension (by imposing *e.g.* Dirichlet b.c.) in order to have a self-adjoint operator. We would also like to mention that the non-normalizable eigenfunctions can often be interesting object of study [37, 38, 47–50].

We can then conclude that the Schrödinger operator in (2.1) is a self-adjoint operator on the domain spanned by the eigenfunctions (2.3) since the eigenfunctions form a complete orthogonal basis with real eigenvalues, the operator is symmetric with respect to the $L^2([-\pi, \pi], \mathrm{d}x)$ inner product, and the operator is bounded from below by g^2.

Jacobi polynomials

There exists a natural one-parameter generalization of the Schrödinger operator in (2.1), formally defined by the Schrödinger operator

$$H^{(\mathrm{J})}(x; g_0, g_1) \equiv -\frac{d^2}{dx^2} + \frac{g_0(g_0-1)}{4\sin(\frac{1}{2}x)^2} + \frac{g_1(g_1-1)}{4\cos(\frac{1}{2}x)^2}, \tag{2.13}$$

with known eigenfunctions

$$\psi_n^{(\mathrm{J})}(x; g_0, g_1) = \sin(\tfrac{1}{2}x)^{g_0} \cos(\tfrac{1}{2}x)^{g_1} P_n^{(g_0-\frac{1}{2}, g_1-\frac{1}{2})}(\cos(x)) \tag{2.14}$$

with $P_n^{(\alpha,\beta)}$ the *Jacobi polynomials*, and corresponding eigenvalues $(n + \frac{1}{2}(g_0+g_1))^2$. The trigonometric identity

$$\frac{1}{\sin(x)^2} = \frac{1}{4\sin(\frac{1}{2}x)^2} + \frac{1}{4\cos(\frac{1}{2}x)^2}$$

clearly shows that (2.13) reduces to (2.1) in special cases. The potential in (2.13) is the general form of the Pöschl-Teller potential [46]. The operator in (2.13) is also

obtained from the Heun equation in (3.11) in the trigonometric limit, as is shown in Chapter 3. Our solutions of the non-stationary Heun equation, in Paper III are constructed such that they reduce to the Jacobi polynomials in the trigonometric limit. As we now discuss, the derivation of (2.14) is very similar to what we explained above, and we therefore can be brief.

The operator in (2.13) can be factorized as

$$H^{(\mathrm{J})}(x;g_0,g_1) = (Q^{(\mathrm{J})})^* Q^{(\mathrm{J})} + \tfrac{1}{4}(g_0+g_1)^2 \tag{2.15}$$

with

$$Q^{(\mathrm{J})} \equiv -\frac{d}{dx} + \tfrac{1}{2}g_0 \cot(\tfrac{1}{2}x) - \tfrac{1}{2}g_1 \tan(\tfrac{1}{2}x) \tag{2.16}$$

and $(Q^{(\mathrm{J})})^*$ the formal adjoint of $Q^{(\mathrm{J})}$. The groundstate of the operator (2.13) can then be solved explicitly by $Q^{(\mathrm{J})}\psi_0^{(\mathrm{J})} = 0$, and is given by

$$\psi_0^{(\mathrm{J})}(x;g_0,g_1) \equiv \sin(\tfrac{1}{2}x)^{g_0} \cos(\tfrac{1}{2}x)^{g_1}, \tag{2.17}$$

with eigenvalue $\frac{1}{4}(g_0+g_1)^2$.

The reduced Schrödinger operator $h^{(\mathrm{J})}(z;g_0,g_1)$ of $H^{(\mathrm{J})}$, given by

$$h^{(\mathrm{J})}(z;g_0,g_1) \equiv -(1-z^2)\frac{d^2}{dz^2} - \big(g_1 - g_0 - (g_0+g_1+1)z\big)\frac{d}{dz}, \tag{2.18}$$

is obtained by a similarity transformation and a change of variable to $z = \cos(x)$ (same as for the Gegenbauer polynomials in (2.6)-(2.7)). The eigenvalues $n(n+g_0+g_1)$, for non-negative integer n, yield the differential equation for the Jacobi polynomials, *i.e.*

$$\left((1-z^2)\frac{d^2}{dz^2} + (g_1 - g_0 - (g_0+g_1+1)z)\frac{d}{dz} - n(n+g_0+g_1)\right) y = 0. \tag{2.19}$$

Thus we can conclude that (2.13) has exact eigenfunctions $\psi_n^{(\mathrm{J})}(x;g_0,g_1)$ $(n \in \mathbb{N}_0)$ given in (2.14). The eigenfunctions in (2.14) are orthogonal with respect to the standard $L^2([-\pi,\pi],\mathrm{d}x)$ inner product, and this implies the orthogonality relation

$$(P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)})_{L^2(\omega^{(\mathrm{J})})} \equiv \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta \mathrm{d}x = 0 \quad (n \neq m), \tag{2.20}$$

$(\alpha = g_0 - \frac{1}{2}, \beta = g_1 - \frac{1}{2})$ for the Jacobi polynomials. The Jacobi polynomials are known to have the expansion

$$P_n^{((g_0-\frac{1}{2},g_1-\frac{1}{2})}(z) = \frac{\Gamma(2n+g_0+g_1)}{2^n n!\Gamma(n+g_0+g_1)} z^n + \mathcal{O}(z^{n-1}) \tag{2.21}$$

which can be used to define a unique solution to (2.19). (Recall that $\Gamma(x)$ is the Euler Gamma function: See also (A.1)).

It is clear by construction that (2.13) reduces to (2.1) for special values of the coupling g_0, g_1 and this extends to the eigenfunctions: It follows from (2.11) and (2.21), that the Jacobi polynomials reduce to the Gegenbauer polynomials when $g_0 = g_1 = g$ as follows,

$$P_n^{(g-\frac{1}{2},g-\frac{1}{2})}(z) = \frac{\Gamma(2g)\,\Gamma\left(n+g+\frac{1}{2}\right)}{\Gamma(n+2g)\,\Gamma\left(g+\frac{1}{2}\right)} C_n^{(g)}(z). \tag{2.22}$$

2.2 Symmetric orthogonal polynomials

The models that have been considered in this Chapter have natural many-variable generalizations that are also exactly solvable. We now turn to examples that features prominently in our Papers, *i.e.* the CMS type model (see (1.5)) and the (quantum) Ruijsenaars models (see (1.9)). These models describe arbitrary number of indistinguishable particles on a circle, and the quantum many-body models we consider can, in many cases, arise from the one-particle models.

The eigenfunctions of the quantum many-variable models are labeled by (positive) integer vectors or partitions, in the same manner that the orthogonal polynomials in Section 2.1 were labeled by non-negative integers n. The physical interpretation of the partitions are, in this work, the corresponding excitation levels for a system of indistinguishable particles since the quantum numbers can be ordered without loss of generality, but there are also different physical interpretations; see Paper II. We proceed to recall some basic definitions of partitions for the convenience of the reader and in order for this Thesis to be self-contained. We will follow the notation and convention in [27] and [13].

Partitions

We define a *partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ as a sequence of non-negative integers λ_j in decreasing order, *i.e.*

$$\lambda_1 \geq \lambda_2 \geq \cdots$$

where only finitely many of the λ_j's are non-zero. The non-zero λ_j of a partition are commonly referred to as *parts* of the partition λ, and the number of parts is its *length*, denoted by $\ell(\lambda)$. The *weight* of this partition, denoted by $|\lambda|$, is the sum over its parts, *i.e.* $|\lambda| \equiv \sum_j \lambda_j$. It is convenient to not make a distinction between partitions that differ only by a sequence of zeros at the end.

Let λ, μ be two partitions and define the *dominance ordering*, denoted by "$\leq$", by

$$\lambda \leq \mu \Leftrightarrow \lambda_1 + \ldots + \lambda_j \leq \mu_1 + \ldots + \mu_j, \quad \forall j. \tag{2.23}$$

The dominance order is only a partial ordering (*i.e.* there exists partition that are not comparable, for example, the partitions $(4,1,0)$ and $(3,2,2)$).

Jack polynomials

We now can turn to the Calogero-Sutherland model and the Jack polynomials. The Calogero-Sutherland model [8,9] is (formally) defined by the Schrödinger operator

$$H_N^{(\mathrm{CS})}(x;g) \equiv \sum_{j=1}^{N} -\frac{\partial^2}{\partial x_j^2} + 2g(g-1)\sum_{j<k}^{N} \frac{1}{4\sin(\frac{1}{2}(x_j-x_k))^2} \tag{2.24}$$

with $x_j \in [-\pi,\pi]$ for all $j = 1,\ldots,N$. It was shown by Sutherland [8,9] that a complete set of eigenfunctions of this operator can be constructed by a method which is remarkably similar to the method we used to construct eigenfunctions of $H^{(\mathrm{G})}$ in (2.1). The Calogero-Sutherland model has exact eigenfunction $\psi_\lambda(x;g,N)$ that are labeled by partitions λ, given by

$$\psi_\lambda(x;g,N) \equiv \psi_0(x;g,N) P_\lambda^{(1/g)}(z), \quad z_j = \mathrm{e}^{\mathrm{i}x_j}, \tag{2.25}$$

with corresponding eigenvalues $\sum_{j=1}^{N}(\lambda_j + \frac{1}{2}g(N+1-2j))^2$, where $P_\lambda^{(1/g)}(z)$ are the *Jack polynomials*. To show this,[4] we proceed with the well-known factorization of the CS Hamiltonian in (2.24): The Schrödinger operator in (2.24) can be factorized as [51]

$$\begin{aligned} H_N^{(\mathrm{CS})}(x;g) &= \sum_{j=1}^{N} (Q_j^{(\mathrm{CS})})^* Q_j^{(\mathrm{CS})} + \frac{1}{12}g^2 N(N^2-1), \\ Q_j^{(\mathrm{CS})} &\equiv -\frac{\partial}{\partial x_j} + g\sum_{j<k}^{N} \cot(\tfrac{1}{2}(x_k - x_j)), \end{aligned} \tag{2.26}$$

with $(Q_j^{(\mathrm{CS})})^*$ the formal adjoint of $Q_j^{(\mathrm{CS})}$. The groundstate is obtained by solving $Q_j^{(\mathrm{CS})}\psi_0 = 0$, for all $j = 1,\ldots,N$. It can be shown by straightforward calculations, and well-known trigonometric identities, that the function

$$\psi_0(x;g,N) \equiv \prod_{j<k}^{N} \sin(\tfrac{1}{2}(x_k - x_j))^g \tag{2.27}$$

solves these equations and thus is the unique groundstate of the operator (2.24).

The Jack symmetric polynomials $P_\lambda^{(1/g)}$ are uniquely defined by the following properties[5]

1. $P_\lambda^{(1/g)} = m_\lambda + \sum_{\nu<\lambda} c_{\lambda,\nu}^{(1/g)} m_\lambda$ with m_λ the monomial symmetric polynomials, defined as
$$m_\lambda(z_1,\ldots,z_N) \equiv \sum_\sigma z_1^{\lambda_{\sigma(1)}} \cdots z_N^{\lambda_{\sigma(N)}} \tag{2.28}$$

[4] The discussion is very similar to the discussion of the Gegenbauer polynomials and will therefore be brief.

[5] Note that the definition in [27] and [52] are for the Jack functions, which are (in a sense) the Jack polynomials in an infinite number of variables.

where the sum is over all distinct permutations of the partition λ, and known coefficients $c_{\lambda,\mu}^{(1/g)}$.

2. $\langle P_\lambda^{(1/g)}, P_\nu^{(1/g)} \rangle'_{1/g} = 0$, for partitions ν, λ such that $\nu \neq \lambda$, where the inner product is defined by

$$\langle P_\lambda, P_\nu \rangle'_{1/g} \equiv \frac{1}{N!} \oint_{|z|=1} \prod_{j=1}^{N} \frac{\mathrm{d}z_j}{2\pi \mathrm{i} z_j} \prod_{j \neq k} (1 - z_j/z_k)^g \ \overline{P_\lambda(z)} \ P_\nu(z). \tag{2.29}$$

(This is a non-trivial result proved in *e.g.* [27] or [52].) One can show that the inner product (2.29) is identical with the standard L^2 inner product of the eigenfunctions of the Sutherland Hamiltonian in (2.25), which is of interest in the physics interpretation of the CS model as a quantum many-body system (we mention this since there are other inner products which are used in the mathematical theory of Jack polynomials [27, 52]). The Jack polynomials can be constructed by many different approaches [8–10, 27, 52], and in Section 4.3 we discuss how the Jack polynomials can be constructed from the kernel function method.

Macdonald polynomials

Here we would also like to mention another many-variable symmetric polynomial, known as the Macdonald polynomials, which depends on an additional parameter.

The Macdonald polynomials are exact eigenfunctions of the algebraic difference operator

$$\mathcal{M}_N(z; p, t) \equiv \sum_{j=1}^{N} \prod_{k \neq j}^{N} \Big(\frac{z_j - t z_k}{z_j - z_k} \Big) \hat{T}_p(z_j) \tag{2.30}$$

with $\hat{T}_q(z_j)$ defined by $\hat{T}_p(z_j) f(z_1, z_2, \ldots z_N) = f(z_1, \ldots z_{j-1}, p z_j, z_{j+1}, \ldots, z_N)$. The Macdonald operator in (2.30) corresponds to the reduced operator for the *relativistic Calogero-Sutherland model*, defined by the analytic difference operator in (1.9), with $p = \mathrm{e}^{-\beta}, t = \mathrm{e}^{-g\beta}$, and $z_j = \mathrm{e}^{\mathrm{i}x_j}$. The relativistic CS model has exact eigenfunctions

$$\psi_\lambda(x; g, \beta) \equiv \psi_0(x; g, \beta) P_\lambda(z(x); p, t) \quad (z_j = e^{\mathrm{i}x_j}), \tag{2.31}$$

with $P_\lambda(z; q, t)$ the Macdonald polynomials and groundstate $\psi_0(x; g\beta, N)$ discovered by Ruijsenaars [53] (it can be found in Paper I, for example).

2.3 Exactly solved systems of CMS type

As we mentioned in the introduction, the CMS type models can be defined by N-variable Schrödinger operator of the general form

$$\sum_{j=1}^{N} -\frac{\partial^2}{\partial x_j^2} + U(x_j) + \sum_{j<k}^{N} V(x_k, x_j) \tag{2.32}$$

with particular potentials $U(x)$ and $V(x,y)$; see Table 2.1 for examples. (We wish to stress that these lists are not complete.)

The models of CMS type share the common property that they have exact solutions of the form

$$\psi_\lambda(x) = \psi_0(x) P_\lambda(z(x)), \tag{2.33}$$

where ψ_0 are explicitly known groundstate eigenfunctions, λ a partition of length less than or equal to the number of particles N, and P_λ polynomials (see *e.g.* [16] and references therein). The CMS type operators all have a Quantum Mechanical interpretation as Hamiltonians.

There are also CMS type operators where U and V in (2.32) are given in terms of elliptic potentials: see Table 2.2. The elliptic CMS type models are known to be integrable, *i.e.* there exists N algebraically independent, and mutually commuting, higher order partial differential operators [17, 19, 29, 56–58]:

$$\sum_{j=1}^{N} (-\mathrm{i})^p \frac{\partial^p}{\partial x_j^p} + \text{ l.o.} \quad (p \in \mathbb{N}), \tag{2.34}$$

where l.o. stands for terms that are lower order in derivatives. Constructing the eigenfunctions of the CMS type models with elliptic potentials is an area of ongoing research [59–62] as many of the standard methods used for non-elliptic systems cannot be generalized to the elliptic case. In Chapter 4 we illustrate how solutions of the CMS type models can be constructed using the method of kernel functions, and that the kernel function methods are also applicable for the CMS models with elliptic potentials. The kernel function method was used in [16] in order to construct the eigenfunctions for the A_{N-1} potential in Table 2.1 by the recursive algorithm described in Section 4.2. We now turn to discuss the CMS systems with elliptic potentials and in particular, the Lamé and Heun equation.

Table 2.1: **Potentials of CMS type models**

Case	$U(x)$	$V(x,y)$
I	$\omega^2 x^2$	$g(g-1)\frac{1}{(x-y)^2}$
I (A_{N-1})	0	$\omega^2(x-y)+g(g-1)\frac{1}{(x-y)^2}$
I	$\omega^2 x^2+g_0(g_0-1)\frac{1}{x^2}$	$g(g-1)\Big(\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\Big)$
II (A_{N-1})	0	$g(g-1)\frac{1}{2\sin(\frac{1}{2}(x-y))^2}$
II (BC_N)	$\frac{g_0(g_0-1)}{2\sin(\frac{1}{2}x)^2}+\frac{g_1(g_1-1)}{2\cos(\frac{1}{2}x)^2}$	$\frac{1}{2}g(g-1)\Big(\frac{1}{\sin(\frac{1}{2}(x-y))^2}+\frac{1}{\sin(\frac{1}{2}(x+y))^2}\Big)$
III (C_N)	$-g_0(g_0-1)\frac{1}{\cosh(x)^2}$	$\frac{1}{2}g(g-1)\Big(\frac{1}{\sinh(\frac{1}{2}(x-y))^2}-\frac{1}{\cosh(\frac{1}{2}(x+y))^2}\Big)$
III	$\omega^2\mathrm{e}^{-2x}+\omega(1+2c)\mathrm{e}^{-x}$	$g(g-1)\frac{1}{2\sinh(\frac{1}{2}(x-y))^2}$

Examples of potentials for CMS type models adapted from [16] and [17]. The case indicates rational (I), trigonometric (II), hyperbolic (III), or elliptic (IV) potentials, as is explained in Chapter 3: See Figure 3.1. The known relations [17,54,55] to root systems of Lie algebras is indicated in the parenthesis.

Table 2.2: **Elliptic potentials of CMS type models**

Case	$U(x)$	$V(x,y)$
IV (A_{N-1})	0	$2g(g-1)\wp(x-y)$
IV (BC_N)	$\sum_{\nu=0}^{3} g_\nu(g_\nu-1)\wp(x+\omega_\nu)$	$2g(g-1)\Big(\wp(x-y)+\wp(x+y)\Big)$

The elliptic CMS potentials associated to the A_{N-1} and BC_N root systems [17,41,54,55,63,64]. The A_{N-1} model is often referred to as the *elliptic Calogero-Sutherland model* and the BC_N model is known as the *Inozemtsev model* [63].

Chapter 3

Systems with elliptic potentials

In this chapter we discuss the systems of CMS and RvD type where the interactions are given in terms of elliptic functions. A famous example of a differential equation with an elliptic potential was derived by Lamé [31, 32] in 1838 when considering the stationary temperature distribution on the surface of an ellipsoid. The Lamé equation in algebraic form (see below) is known to be a special case of the Heun differential equation (this relation is discussed when considering the Heun equation in the form of a Schrödinger operator with the Darboux-Treibich-Verdier potential [33, 34, 65]). The Lamé and Heun equations are discussed in Section 3.1.

As we have mentioned in Chapter 1, the operators with elliptic potentials can be considered as the most general of the CMS type Schrödinger operators, in the sense that the other potentials arise from them in suitable limits: See Figure 3.1. These relations can be obtained from the series representation of the Weierstrass $\wp$-function in (1.1), which imply the following results: For $x \neq 0(\mathrm{mod}(2\omega_1, 2\omega_3))$,

$$\begin{aligned}\lim_{\omega_3\to+\mathrm{i}\infty}\wp(x|\omega_1,\omega_3) &= \sum_{n\in\mathbb{Z}}\frac{1}{(x+2\omega_1 n)^2} - 2\sum_{n\in\mathbb{N}}\frac{1}{(2\omega_1 n)^2}\\ &= \frac{\pi^2}{4\omega_1^2\sin(\frac{1}{2}\frac{\pi}{\omega_1}x)^2} - \frac{\pi^2}{12\omega_1^2} \quad (\Im(\omega_1)=0);\end{aligned} \tag{3.1}$$

$$\lim_{\omega_1\to\infty}\wp(x|\omega_1,\mathrm{i}a) = -\frac{\pi^2}{4a^2\sinh(\frac{\pi}{2a}x)^2} + \frac{\pi^2}{12a^2} \quad (\omega_3=\mathrm{i}a\in\mathrm{i}\mathbb{R}_+). \tag{3.2}$$

It follows from (3.1), and (3.2), that

$$\lim_{\omega_1\to\infty}\lim_{\omega_3\to+\mathrm{i}\infty}\wp(x|\omega_1,\omega_3) = \lim_{\omega_3\to+\mathrm{i}\infty}\lim_{\omega_1\to\infty}\wp(x|\omega_1,\omega_3) = \frac{1}{x^2}. \tag{3.3}$$

The limits discussed above are representative for all elliptic functions: It was proven by Weierstrass that any elliptic function can be expressed as rational functions of $\wp(x)$, with suitable choices of half periods ω_1 and ω_3 (see *e.g.* Chapter XX of [2]); it follows from Weierstrass Theorem that the limiting cases above imply similar relations for all elliptic functions.

Preliminaries

For the convenience of the reader, we will recall some of the notations and properties of the functions that are used in the following sections. (We refer to Appendix A.3 for the definitions, representations, and properties of the elliptic and quasi-elliptic functions.)

We recall that the Weierstrass $\wp$-function is defined as

$$\wp(x) \equiv \frac{1}{x^2} + \sum_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(x+2n\pi+2m\pi\tau)^2} - \frac{1}{(2n\pi+2m\pi\tau)^2}, \tag{3.4}$$

and we denote the Jacobi theta functions by ϑ_ν ($\nu = 1,2,3,4$): See Eqs. (A.19a)-(A.19d). The (non-zero) *half-periods* of the elliptic functions are denoted by[1] ω_1 and ω_3, where we set the half-periods to

$$\omega_1 = \pi,\ \omega_3 = \pi\tau \quad (\Im(\tau) > 0),$$

without loss of generality, and also introduce the shorthand notation $\omega_0 \equiv 0$ and $\omega_2 \equiv -\omega_1 - \omega_3 = -\pi(1+\tau)$ for convenience. The ratio of the half-periods cannot be purely real, *i.e.* $\Im(\tau) \neq 0$; we choose $\Im(\tau) > 0$, without loss of generality. We define the elliptic *nomé* as $q \equiv \exp(\mathrm{i}\pi\tau)$ and the constants e_1, e_2, and e_3 by the value of the Weierstrass $\wp$-function at the half-periods ω_1, ω_2, and ω_3, respectively, *i.e.*

$$e_1 \equiv \wp(\pi), \quad e_2 \equiv \wp(\pi(1+\tau)), \quad e_3 \equiv \wp(\pi\tau). \tag{3.5}$$

[1] The commonly used notation is ω_1 and ω_2 [2].

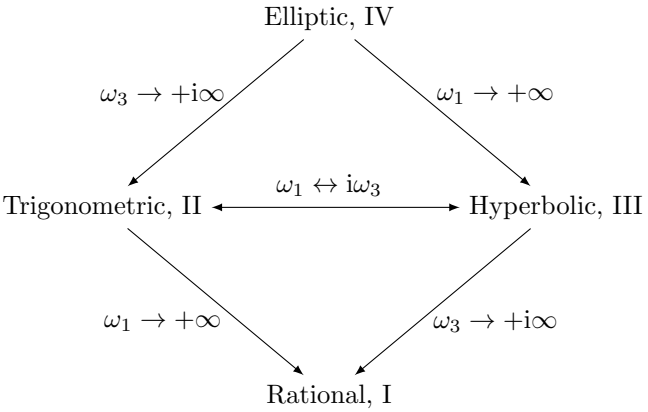

Figure 3.1: The relations of the CMS and RvD models. The diagram is adapted from [30].

3.1 One variable cases

Lamé equation

The Lamé equation can be written as

$$\frac{d^2}{dx^2}\psi = (A\wp(x) + B)\psi \tag{3.6}$$

for some constants A and B. We mention this as the Lamé equation can be written in other forms (see *e.g.* (3.9)), and that we use the form in (3.6) of the Lamé equation as it fits our purpose: We note that Eq. (3.6) reduces to

$$\frac{d^2}{dx^2}\psi = (A\frac{1}{4\sin(\frac{1}{2}x)^2} + (B - \frac{1}{12}A))\psi \tag{3.7}$$

in the trigonometric limit $\Im(\tau) \to +\infty$ (see Eq. (3.1)). (Recall that we set $\omega_1 = \pi$ and $\omega_3 = \pi\tau$.)

It is convenient to write the Lamé equation as an eigenvalue equation for the Schrödinger operator

$$H^{(\mathrm{L})}(x; g) \equiv -\frac{d^2}{dx^2} + g(g-1)\wp(x) \quad (x \in [-\pi, \pi]), \tag{3.8}$$

where the constant A is written as $g(g-1)$ (see also Section 2.1). We note that the elliptic functions are not necessarily real-valued: The Weierstrass $\wp$-function takes real values, for $x \in [-\pi, \pi]$, only when τ is purely imaginary. This clearly shows that, for arbitrary τ ($\Im(\tau) > 0$), the operator $H^{(\mathrm{L})}$ in (3.8), and other Schrödinger operators with elliptic potentials, may not necessarily be symmetric operators.

The eigenfunctions of the operator (3.8), known as the *Lamé functions*, were originally constructed by Lamé [31, 32] for special values of the coupling constant and in more generality by Hermite (see *e.g.* [2]) and Ince [66, 67].

The Lamé equation in algebraic form, obtained by substituting to the variable $z = \wp(x)$, is given by

$$\Big(-\frac{d^2}{dz^2} - \frac{1}{2}(\frac{1}{z-e_1} + \frac{1}{z-e_2} + \frac{1}{z-e_3})\frac{d}{dz} + \frac{(Az+B)}{4(z-e_1)(z-e_2)(z-e_3)}\Big)\psi = 0. \tag{3.9}$$

This is a special case of the Heun differential equation which we now proceed to discuss.

Heun equation

The Heun differential equation is a second order Fuchsian differential equation with four (regular) singular points $(0, 1, t, \infty)$, defined as

$$\frac{d^2y}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-t}\right)\frac{dy}{dx} + \frac{\alpha\beta x - q_{\mathrm{H}}}{x(x-1)(x-t)}y = 0 \tag{3.10}$$

with parameters[2] $q_{\rm H}, \alpha, \beta, \gamma, \delta, \epsilon$ satisfying $\alpha + \beta + 1 = \gamma + \delta + \epsilon$.

We can also write the Heun differential equation as an eigenvalue equation for the Schrödinger operators with the Darboux-Treibich-Verdier potential, given by

$$H^{(\rm DTV)}(x; \{g_\nu\}_{\nu=0}^3) \equiv -\frac{d^2}{dx^2} + \sum_{\nu=0}^{3} g_\nu(g_\nu - 1)\wp(x + \omega_\nu). \tag{3.11}$$

where the parameters $\{g_\nu\}$ and $\alpha, \beta, \gamma, \delta, \epsilon$ can have many different relations, *e.g.*

$$\gamma = g_0 + \tfrac{1}{2},\ \delta = g_1 + \tfrac{1}{2},\ \epsilon = g_2 + \tfrac{1}{2}, \tag{3.12}$$

$$\alpha\beta = \frac{1}{4}(g_0 + g_1 + g_2 + g_3)(g_0 + g_1 + g_2 - g_3 + 1), \tag{3.13}$$

and the auxiliary parameter $q_{\rm H}$ directly proportional to the eigenvalues of (3.11). It is known that the Heun differential equation (3.10) is the eigenvalue equation for the reduced Schrödinger operator for $H^{(\rm DTV)}$ in (3.11). (A detailed derivation of the relation between (3.10) and (3.11) can be found in [61].) The operator in (3.11) is also the single-particle case of the Inozemtsev Hamiltonian, *i.e.* the $\rm BC_1$ case in Table 2.2.

3.2 Many-variable models

Elliptic CS model

There exists a natural many-variable generalization of the Lamé equation, commonly referred to as the *elliptic Calogero-Sutherland* (eCS) model, defined by the Schrödinger operator

$$H_N^{(\rm eCS)}(x; g) \equiv \sum_{j=1}^{N} -\frac{\partial^2}{\partial x_j^2} + 2g(g-1)\sum_{j<k}^{N} \wp(x_j - x_k), \tag{3.14}$$

with $x_j \in [-\pi, \pi]$ for all $j = 1, \ldots, N$. It follows from Eq. (3.1) that the operator (3.14) reduces to the CS operator in (2.24) in the trigonometric limit, *i.e.*

$$\lim_{\Im(\tau)\to\infty} H_N^{(\rm eCS)} = H_N^{(\rm CS)} - \frac{g(g-1)N(N-1)}{24}, \tag{3.15}$$

and the eCS model can be viewed as an elliptic generalization of the CS model. The elliptic Calogero-Sutherland model was originally shown to be quantum integrable in [17] and explicit eigenfunctions have been constructed for integer values of the coupling g [36, 68]. The eigenfunctions of (3.14) have also been considered using standard perturbation theory with expansion in the nomé q, using the Jack

[2] We stress that the auxiliary parameter $q_{\rm H}$ is not the same as the elliptic nomé q.

polynomials [59,69,70]. The eigenfunctions of (3.14) were constructed in [64,71,72] by the perturbative[3] algorithm which we discuss in Section 4.4.

The eigenfunctions of the elliptic CS model is assumed to be of similar form as (2.25): The eigenfunctions, constructed in [64,71,72] of the form

$$\psi_\lambda^{(\mathrm{eCS})}(x;g,N) \equiv \prod_{j<k}^{N} \left(\tfrac{1}{2}q^{-\frac{1}{4}}\vartheta_1(\tfrac{1}{2}(x_k-x_j))\right)^g \mathcal{P}_\lambda(z(x),q^2) \quad (z_j=\mathrm{e}^{\mathrm{i}x_j}, q=\mathrm{e}^{\mathrm{i}\pi\tau}) \tag{3.16}$$

with ϑ_1 the odd Jacobi theta function and $\mathcal{P}_\lambda$ an elliptic generalization of the Jack polynomials, *i.e.* the functions $\mathcal{P}_\lambda$ reduces to the Jack polynomials in the trigonometric limit $q \to 0$. The eigenfunctions in (3.16) reduce to the eigenfunctions (2.25) in the trigonometric limit.

Elliptic Ruijsenaars model

We would also like to mention the elliptic (quantum) Ruijsenaars model, defined by the analytic difference operators

$$S^{\pm}(x;g\beta) \equiv \sum_{j=1}^{N} \prod_{k\neq j} \left(f_{\mathrm{IV}}^{\mp}(x_k-x_j;g,\beta)^{\frac{1}{2}}\right) \mathrm{e}^{\mp\mathrm{i}\beta\frac{\partial}{\partial x_j}} \prod_{k\neq j} \left(f_{\mathrm{IV}}^{\pm}(x_k-x_j;g,\beta)^{\frac{1}{2}}\right) \tag{3.17}$$

with

$$f_{\mathrm{IV}}^{\pm}(x;g,\beta) \equiv \frac{\vartheta_1(\frac{1}{2}(x\pm\mathrm{i}g\beta))}{\vartheta_1(\frac{1}{2}x)}. \tag{3.18}$$

The operator in (3.17) is known to be quantum integrable [26]. Results in the literature [73] suggest that this operator has eigenfunctions $\mathcal{P}_\lambda(z;q,p,t)$ which provide and elliptic generalization of the Macdonald polynomials mentioned in Section 2.2. The construction of these elliptic Macdonald polynomials is an open area of research, to our knowledge.

3.3 Non-stationary elliptic equations

The work by Olshanetsky and Peremolov [17,54,55] showed that the CMS, and RvD, type models can be related to the irreducible root systems of (classical) Lie algebras, as was discussed in Chapter 1. This gave a natural relation between Schrödinger operators of CMS type and the Laplace operator in symmetric spaces. It was shown by Etingof, Frenkel and Kirillov [35,74,75] that this approach could be extended to affine Kac-Moody algebras. Of particular interest is the operator

$$\frac{2\mathrm{i}}{\pi}\kappa\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial x^2} + g(g-1)\wp(x) \tag{3.19}$$

[3]The algorithm is so-called perturbative due to an expansion in a suitable parameter. We stress that the algorithm is not based on perturbation theory.

with $\pi\tau$ the half-period of the $\wp$-function. The eigenvalue equation for the operator in (3.19) is often referred to as the *non-stationary Lamé equation* due to the resemblance with a time-dependent Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x,t) = H(x,t)\psi(x,t). \tag{3.20}$$

The non-stationary Lamé equation is a one-parameter generalization of Lamé equation in (3.6), with model parameters (κ, g) that are allowed to take arbitrary values.

The representation theoretic work by Etingof, *et. al.*, was generalized in [45] to the Heun case: The *non-stationary Heun equation* is defined as the eigenvalue equation for the differential operator

$$\frac{\mathrm{i}}{\pi}\kappa\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial x^2} + \sum_{\nu=0}^{3} g_\nu(g_\nu - 1)\wp(x+\omega_\nu). \tag{3.21}$$

(Recall that $\omega_0 \equiv 0, \omega_1 = \pi, \omega_3 = \pi\tau$, and $\omega_2 \equiv -\omega_1 - \omega_3$.) The non-stationary Heun equation has also appeared when considering different physical models such as quantum statistical physics for (artificial) spin-ice models [37, 38, 42–44], string theory [36, 76], and exact 4-point correlation functions of the quantum Liouville model [39].

In Papers III and IV, the solutions of the non-stationary Heun, and non-stationary Lamé equation, are constructed using the kernel function methods, which we proceed to discuss in the following Chapter.

Chapter 4

Kernel functions

In this Chapter, we introduce and discuss kernel functions with a particular focus on their role in the theory of special functions related to CMS, and RvD, type models: We show how the kernel function methods can be used to construct exact eigenfunctions of the CMS type models with the different potentials in Tables 2.1 and 2.2. We will mainly consider the CMS, and RvD, type models that were discussed in Chapter 3 and the models which are obtained from the elliptic models in the trigonometric limit; see Chapter 2. The results below illustrates the use of kernel functions as a tool for studying special functions.

Preliminaries

We start by establishing some of the basic terminology that is used in this Thesis and our Papers. A function $K(x,y)$ is called a *kernel function* for a pair of operators $(H(x),\tilde{H}(y))$ if there exists a constant C such that the functional identity

$$(H(x)-\tilde{H}(y)-C)K(x,y)=0 \tag{4.1}$$

holds. We refer to the functional identity in (4.1) as the *kernel function identity*. (We write $H(x)$ to indicate that the operator acts on functions depending on the variable x.) The cases that are of interest in this work will mainly consist of second order differential operators or difference operators that have appeared in Chapters 2 and 3. An illustrative example is the Schrödinger operator $H^{(\mathrm{G})}$ in (2.1) where it is a simple exercise to check the kernel function identity

$$\Big(H^{(\mathrm{G})}(x;g)-H^{(\mathrm{G})}(y;g')\Big)\frac{\sin(x)^g\sin(y)^{g'}}{(\cos(x)-\cos(y))^{g+g'}}=0, \tag{4.2}$$

holds true (this can be verified by straightforward computations using well-known trigonometric identities). Note that the kernel function can always be multiplied by any non-zero constant without changing the kernel function identity in (4.1). It is therefore convenient not to make a distinction between kernel functions that differ by a multiplicative (non-zero) constant.

4.1 Source Identities

A systematic approach to finding (explicit) kernel functions is based on the concept of *source identities*. The kernel functions, and kernel function identities, that are discussed in the proceeding chapters are derived from source identities in Refs. [16, 21, 41, 77, 78].

The source identity for the trigonometric Calogero-Sutherland model is due to Sen [21], who proved that the exact groundstate eigenfunction of the Schrödinger operator in (1.7) is

$$\prod_{j<k}^{\mathcal{N}} \sin(\tfrac{1}{2}(X_j - X_k))^{g m_j m_k} \tag{4.3}$$

(To state the result below, we find it convenient to denote the particle number by $\mathcal{N}$, and particle coordinates by X_j for $j = 1, \ldots, \mathcal{N}$.) The identities that are of particular interest in our research corresponds to the special cases where the mass parameters are restricted to $m_j \in \{1, -1, 1/g, -1/g\}$. To illustrate how the source identity in (1.7), and (4.3), can yield other identities, we consider the special case where

$$(X_j, m_j) = \begin{cases} (x_j, 1), & j = 1, \ldots, N \\ (\tilde{x}_{j-N}, -1/g), & j = N+1, \ldots N + \tilde{N} = \mathcal{N} \end{cases} \tag{4.4}$$

and it follows from a straightforward check that (1.7) reduces to (1.8), and (4.3) becomes the deformed generalization of the CS groundstate. This special case also yields the well-known variant of the groundstate eigenvalue equation for the deformed Calogero-Sutherland model. It is also often more useful to consider the source identities as the proof for special cases can be more involved than the proof of the source identity [16, 41].

4.2 Classical orthogonal polynomials

In order to illustrate the use of the kernel functions, we demonstrate how the well-known eigenfunctions of the Schrödinger operators with the Pöschl-Teller potential (see Section 2.1) can be constructed using kernel functions. In these examples, we reproduce results well-known from other methods (see Chapter 2). In the later sections we will proceed to show how the kernel function methods are can be generalized to models with elliptic potentials. (Recall the Schrödinger operator $H^{(\mathrm{G})}$ in (2.1), $\psi_n^{(\mathrm{G})}$ in (2.3), and the Gegenbauer polynomials $C_n^{(g)}$.)

For g and g' any complex numbers, the function

$$K^{(\mathrm{G})}(x, y; g, g') \equiv \frac{\sin(x)^g \sin(y)^{g'}}{2^{2g}(\sin(\frac{1}{2}(x+y)) \sin(\frac{1}{2}(x-y)))^{g+g'}} \tag{4.5}$$

satisfy the kernel function identity in (4.7), for the pair of operators $(H^{(\mathrm{G})}(x; g), H^{(\mathrm{G})}(y; g'))$ and $C = 0$. We now proceed to show how the kernel function in (4.5) can be used to

construct integral transforms that map a (generalized) eigenfunction of the operator $H^{(\mathrm{G})}(y;g')$ to the eigenfunctions $\psi_n^{(\mathrm{G})}(x;g)$ in (2.3). In particular, we consider the cases where $(g,g') = (g,0),(g,g)$, and $(g,g-m)$, with g arbitrary (m a positive integer), as they are illustrative for our discussion in Section 4.4, and Papers III and IV.

Basic example

We use the case $g' = 0$ as a simple example to illustrate the usefulness of kernel functions for constructing eigenfunctions of $H^{(\mathrm{G})}$: We now show how the function

$$\int_{\mathcal{C}_\epsilon} \frac{\mathrm{d}y}{2\pi} K^{(\mathrm{G})}(x,y;g,0)\mathrm{e}^{-\mathrm{i}(n+g)y}, \tag{4.6}$$

with $\mathcal{C}_\epsilon$ a suitable contour in the complex plane (as explained below), is equal to the eigenfunctions $\psi_n^{(\mathrm{G})}$ in (2.3). Before we proceed to show this relation, we would like to mention that this simple example can be generalized to the elliptic case for special values of the parameters.

Consider the action of the operator $H^{(\mathrm{G})}(x;g)$ on the function in (4.6). It follows from straightforward calculations that

$$\begin{aligned} &H^{(\mathrm{G})}(x;g)\int_{\mathcal{C}_\epsilon} \frac{\mathrm{d}y}{2\pi} K^{(\mathrm{G})}(x,y;g,0)\mathrm{e}^{-\mathrm{i}(n+g)y} \\ &= -\int_{\mathcal{C}_\epsilon} \frac{\mathrm{d}y}{2\pi}\mathrm{e}^{-\mathrm{i}(n+g)y}\frac{d^2}{dy^2}K^{(\mathrm{G})}(x,y;g,0) \\ &= B.T. + (n+g)^2\int_{\mathcal{C}_\epsilon} \frac{\mathrm{d}y}{2\pi} K^{(\mathrm{G})}(x,y;g,0)\mathrm{e}^{-\mathrm{i}(n+g)y} \end{aligned} \tag{4.7}$$

by using the kernel function identity and integrating by parts (twice). The term $B.T.$ comes from the partial integration and is given by

$$B.T. \equiv \int_{\mathcal{C}_\epsilon} \frac{\mathrm{d}y}{2\pi}\frac{\partial}{\partial y}\Big(K^{(\mathrm{G})}(x,y;g,0)\frac{\partial}{\partial y}\mathrm{e}^{-\mathrm{i}(n+g)y} - \mathrm{e}^{-\mathrm{i}(n+g)y}\frac{\partial}{\partial y}K^{(\mathrm{G})}(x,y;g,0)\Big).$$

By choosing a suitable[1] integration path, *e.g.* $\mathcal{C}_\epsilon = [-\pi,\pi] + \mathrm{i}\epsilon$ with $\epsilon > 0$, we get that B.T. $= 0$, and that (4.6) is an eigenfunction of the operator $H^{(\mathrm{G})}(x;g)$. Expanding (4.6), and the variable substitution $t = \exp(\mathrm{i}y)$ yields

$$\sin(x)^g \oint_{|t|<1} \frac{\mathrm{d}t}{2\pi\mathrm{i}t}(1+2\cos(x)t+t^2)^{-g}t^{-n}. \tag{4.8}$$

It follows from Cauchy's (integral) theorem that (4.6) is well-defined, independent of ϵ, and identically zero unless n is a non-negative integer. Expanding the integrand

[1] To be more specific: An integration path such that the boundary contributions vanish and the integral is well-defined.

as a power series in t shows that the integral represents a polynomial, in $\cos(x)$, of order n; more specifically,

$$\oint_{|t|<1} \frac{\mathrm{d}t}{2\pi \mathrm{i}t}(1+2\cos(x)t+t^2)^{-g}t^{-n} = \frac{2^n\Gamma(n+g)}{n!\Gamma(g)}\cos(x)^n + \text{ l.d.}, \tag{4.9}$$

with " l.d. " for the terms of lower degree in $\cos(x)$. It follows from the uniqueness of the eigenfunctions (see Chapter 2) that (4.6) is identical with the eigenfunction $\psi_n^{(\mathrm{G})}(x;g)$ in (2.3),*i.e.*

$$\psi_n^{(\mathrm{G})}(x;g) = \int_{\mathcal{C}_\epsilon} \frac{\mathrm{d}y}{2\pi} K^{(\mathrm{G})}(x,y;g,0)\mathrm{e}^{-\mathrm{i}(n+g)y}. \tag{4.10}$$

Using the kernel function method, we have now re-discovered a well-known expression for the generating functions of the Gegenbauer polynomials (see (A.8)), *i.e.*

$$C_n^{(g)}(z) = \oint_{|t|<1} \frac{\mathrm{d}t}{2\pi \mathrm{i}t} \frac{t^{-n}}{(1-2zt+t^2)^g} \tag{4.11}$$

which is equivalent to Eq. (4.10).

Integral equation

To illustrate another use of kernel functions, we consider the case $g' = g$. The integral transform of the eigenfunction $\psi_n^{(\mathrm{G})}(y;g)$ in (2.3) yield the integral equations

$$\psi_n^{(\mathrm{G})}(x) = \frac{1}{\lambda_n}\int_{\mathcal{C}_\epsilon} \frac{\mathrm{d}y}{2\pi} K^{(\mathrm{G})}(x,y)\psi_n^{(\mathrm{G})}(y), \tag{4.12}$$

for $\lambda_n = 2^{2g}n!\Gamma(2g)/\Gamma(n+2g)$ and, by the uniqueness of the eigenfunctions, yields the relation

$$C_n^{(g)}(z) = \frac{1}{\lambda_n}\oint_{|t|<1} \frac{\mathrm{d}t}{2\pi \mathrm{i}t} \frac{(1-t^2)^{2g}C_n^{(g)}(\frac{1}{2}(t+t^{-1})}{(1-2zt+t^2)^{2g}} \tag{4.13}$$

for the Gegenbauer polynomials. It follows from a straightforward check that the coefficients of the Gegenbauer polynomials (see (A.7)) satisfy this relations and it is possible to show that this integral equation characterizes the polynomials completely. There also exists similar integral equations for other CMS type systems: We would like to mention the work by Whittaker [79, 80], who constructed integral equations for the solutions of the Lamé equation,[2] and the ellipsoidal harmonics with the use of kernel functions.

[2] Whittaker considered the form where the potential is given by $\mathrm{sn}(x)^2$, where sn is the elliptic generalization of the sine-function [2].

Iterative integral representation

The discussion above demonstrated that the construction of eigenfunctions of a CMS type differential operator is equivalent to solving an integral equation using kernel functions. The kernel functions also allows for explicit integral representation of the eigenfunctions via an iterative scheme, that is easily generalized to the many-variable, and elliptic cases. As an illustrate example, we now discuss the iterative integral representation of the Gegenbauer polynomials: Let $\psi_n^{(\mathrm{G})}(x;g')$ in (2.3) be an eigenfunction of (2.1) for coupling g'. The integral transform with the kernel function $K^{(\mathrm{G})}(x,y;g,g')$, *i.e.*

$$\int_{\mathcal{C}_\epsilon} \frac{\mathrm{d}y}{2\pi} K^{(\mathrm{G})}(x,y;g,g')\psi_n^{(\mathrm{G})}(y;g'), \tag{4.14}$$

is well-defined iff $g-g' \in \mathbb{Z}$ and yields a polynomials in $\cos(x)$ of degree $n+g-g'$: The function in (4.14) is given by linear combinations of the functions

$$\sin(x)^g \oint_{|t|<1} \frac{\mathrm{d}t}{2\pi \mathrm{i}t} \frac{t^{\nu+g-g'}(1-t^2)^{2g'}}{(1-2\cos(x)t+t^2)^{g+g'}} \quad (t=\mathrm{e}^{\mathrm{i}y}), \tag{4.15}$$

with $\nu = -n, \dots n-1, n$. Choosing $g' = g-m$, with $g-m>0$, yields the recursive relation

$$C_{n-m}^{(g)}(\cos(x)) = \frac{1}{\kappa_{n,m}} \oint_{|t|<1} \frac{\mathrm{d}t}{2\pi \mathrm{i}t} \frac{t^m(1-t^2)^{2(g-m)}}{(1-2\cos(x)t+t^2)^{2g-m}} C_n^{(g-m)}(\tfrac{1}{2}(t+t^{-1})), \tag{4.16}$$

with $\kappa_{n,m} = (\mathrm{e}^{-\mathrm{i}\pi g}\Gamma(g)\Gamma(n-2m-2g)/n!\Gamma(2g-m)\Gamma(g-m))$. The construction of the eigenfunctions then becomes an iteration of the procedure above for the Gegenbauer polynomials in the integral. The recursive scheme can be summarized as follows: The kernel function in (4.14), with $g' = g-m$, is the kernel of an integral transform that maps eigenfunctions of the $H^{(\mathrm{G})}$, with parameter $g-m$, to eigenfunctions of the $H^{(\mathrm{G})}$ with parameter g. Suppose then that there exists a complete set of eigenfunctions of the operator $H^{(\mathrm{G})}$ with parameter $g=p_0$, for some p_0, then the eigenfunctions of $H^{(\mathrm{G})}$, for parameter $g = Nm+p_0$ with $N \in \mathbb{N}$, can be obtained by the integral transform. The eigenfunctions for $g = Nm+p_0$ will then be given by an N-fold integral. To illustrate this we consider the case where $m=1$, then the eigenfunctions $\psi_n^{(\mathrm{G})}$, for coupling $g=N$, can be expressed as

$$\psi_n^{(\mathrm{G})}(x) = \Big(\prod_{j=0}^{N-1} \int_{\mathcal{C}_{j\epsilon}} \frac{\mathrm{d}y_j}{2\pi}\Big)\Big(\frac{1}{\kappa_{n-j,1}} K^{(\mathrm{G})}(y_j, y_{j+1}; N-j, N-j-1)\Big)\mathrm{e}^{-\mathrm{i}(n+N)y_N} \tag{4.17}$$

where $\mathcal{C}_{j\epsilon} = [-\pi,\pi] + \mathrm{i}j\epsilon$ with $\epsilon > 0$, and we identity $x = y_0$, for brevity. We mentioned that the iterative integral method can also be used in the elliptic cases, and in Paper IV we use the kernel functions in order to construct the solutions of the non-stationary Lamé equation in (3.19); see also Section 4.4. We would like to mention that the integer $m = g-g'$ in the discussion above is arbitrary for the trigonometric model. In the elliptic case, it is fixed by the parameter κ (see Section 3.3).

Recursive algorithm

We now turn to the recursive algorithm for constructing eigenfunctions of the CMS type models. In particular, we consider a special case of the results in [16] and Paper III, given by the Schrödinger operator with the general Pöschl-Teller potential in Section 2.1, *i.e.* $H^{(\mathrm{J})}$ in (2.13). The recursive algorithm can heuristically be summarized as follows: We can construct eigenfunctions of the CMS type models in a domain (as explained below) which is not the desired domain from a physics, and often mathematics, point of view. We refer to these formal solutions as the *singular solutions* the CMS type Schrödinger operators. The integral transform, with the corresponding kernel function, maps the singular solutions to the standard eigenfunctions, as discussed in Chapter 2, for a suitably chosen integration contour. We proceed to illustrate the recursive algorithm for the Pöschl-Teller potential.

Recall the definition of the operator $H^{(\mathrm{J})}$ in (2.13), and the corresponding exact eigenfunctions

$$\psi_n^{(\mathrm{J})}(x) = \sin(\tfrac{1}{2}x)^{g_0}\cos(\tfrac{1}{2}x)^{g_1}P_n^{(g_0-\frac{1}{2},g_1-\frac{1}{2})}(\cos(x)) \tag{4.18}$$

with $P_n^{(\alpha,\beta)}$ the Jacobi polynomials. Let λ, g_0, g_1 be arbitrary complex constants and define the function $K^{(\mathrm{J})}$ as

$$K^{(\mathrm{J})}(x,y;g_0,g_1,\lambda) \equiv \frac{\sin(\frac{1}{2}x)^{g_0}\cos(\frac{1}{2}x)^{g_1}\sin(\frac{1}{2}y)^{\lambda-g_0}\cos(\frac{1}{2}y)^{\lambda-g_1}}{\mathrm{e}^{\mathrm{i}\frac{1}{2}\pi(\lambda-g_0)}2^{g_0+g_1}(\sin(\frac{1}{2}(x+y))\sin(\frac{1}{2}(x-y)))^{\lambda}}. \tag{4.19}$$

Then

$$(H^{(\mathrm{J})}(x;g_0,g_1) - H^{(\mathrm{J})}(y;\lambda-g_0,\lambda-g_1))K^{(\mathrm{J})}(x,y;g_0,g_1,\lambda) = 0. \tag{4.20}$$

We consider the eigenfunctions of the operator $H^{(\mathrm{J})}(y)$ as linear combinations of the functions $f_n = \mathrm{e}^{-\mathrm{i}(n+s)y}$, with n integer and $s \in \mathbb{R}$ arbitrary (for now), in the domain $\Im(y) > 0$. The operator $H^{(\mathrm{J})}(y;\lambda-g_0,\lambda-g_1)$ acts on the functions f_n as follows

$$H^{(\mathrm{J})}f_n = (n+s)^2 f_n - \sum_{\nu\in\mathbb{N}}\gamma_\nu f_{n-\nu}, \tag{4.21}$$

with

$$\gamma_\nu = 4\nu\Big((\lambda-g_0)(\lambda-g_0-1) - (-1)^\nu(\lambda-g_1)(\lambda-g_1-1)\Big). \tag{4.22}$$

This follows from the expansion of the operator $H^{(\mathrm{J})}$ as

$$H^{(\mathrm{J})}(y;\lambda-g_0,\lambda-g_1) = -\frac{d^2}{dy^2} + \sum_{\nu\in\mathbb{N}}\gamma_\nu \mathrm{e}^{\mathrm{i}\nu y} \quad (\Im(y) > 0), \tag{4.23}$$

where we use

$$\frac{1}{\sin(\frac{1}{2}y)^2} = -4\sum_{\nu\in\mathbb{N}}\nu\mathrm{e}^{\pm\mathrm{i}\nu y} \quad (\Im(y) \gtrless 0), \tag{4.24}$$

and shows that the action of the operator has lower triangular structure: $H^{(\mathrm{J})}f_n$ is given as a linear superposition of functions f_m with $m \leq n$. The singular eigenfunctions of the operator are then given by $F_n = f_n + \sum_{m<n} \alpha_n(m) f_m$ where $\alpha_n(m)$ can be computed exactly (see *e.g.* [16]). The integral transform with the kernel function in (4.19) will then transform the singular eigenfunctions to the eigenfunctions $\psi_n^{(\mathrm{J})}$ in (2.14): It follows from simple trigonometric identities that the function in (4.19) can be expressed as

$$K^{(\mathrm{J})}(x,y) = \psi_0^{(\mathrm{J})}(x) \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}(g_0+g_1)y}(1-\mathrm{e}^{\mathrm{i}y})^{\lambda-g_0}(1+\mathrm{e}^{\mathrm{i}y})^{\lambda-g_1}}{(1-2\cos(x)\mathrm{e}^{\mathrm{i}y}+\mathrm{e}^{2\mathrm{i}y})^{\lambda}}. \tag{4.25}$$

The integral transform is well-defined, for integration contour $\mathcal{C}_\epsilon$, if we choose $s = \frac{1}{2}(g_0+g_1)$ (see above), and that the integral transform of the singular solutions are given as linear combinations of the functions

$$\psi_0^{(\mathrm{J})}(x) \int_{-\pi+\mathrm{i}\epsilon}^{\pi+\mathrm{i}\epsilon} \frac{\mathrm{d}y}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}ny}(1-\mathrm{e}^{\mathrm{i}y})^{\lambda-g_0}(1+\mathrm{e}^{\mathrm{i}y})^{\lambda-g_1}}{(1-2\cos(x)\mathrm{e}^{\mathrm{i}y}+\mathrm{e}^{2\mathrm{i}y})^{\lambda}} \tag{4.26}$$

with the same coefficients $\alpha_n(m)$ as above (up to a normalization constant). Cauchy's theorem implies that the functions in (4.26) are well-defined, and of the form $\psi_0^{(\mathrm{J})}(x;g_0,g_1)P(\cos(x))$ with P a polynomial of degree $n \in \mathbb{N}_0$. It follows from the uniqueness of the eigenfunctions that the integral transform of the singular solutions are the eigenfunction $\psi_n^{(\mathrm{J})}$ in (2.14), *i.e.*

$$\begin{aligned}\psi_n^{(\mathrm{J})}(x;g_0,g_1) = \psi_0^{(\mathrm{J})}(x)\Big(&\int_{-\pi+\mathrm{i}\epsilon}^{\pi+\mathrm{i}\epsilon} \frac{\mathrm{d}y}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}ny}(1-\mathrm{e}^{\mathrm{i}y})^{\lambda-g_0}(1+\mathrm{e}^{\mathrm{i}y})^{\lambda-g_1}}{(1-2\cos(x)\mathrm{e}^{\mathrm{i}y}+\mathrm{e}^{2\mathrm{i}y})^{\lambda}} \\ &+ \sum_{m<n} \alpha_n(m) \int_{-\pi+\mathrm{i}\epsilon}^{\pi+\mathrm{i}\epsilon} \frac{\mathrm{d}y}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}my}(1-\mathrm{e}^{\mathrm{i}y})^{\lambda-g_0}(1+\mathrm{e}^{\mathrm{i}y})^{\lambda-g_1}}{(1-2\cos(x)\mathrm{e}^{\mathrm{i}y}+\mathrm{e}^{2\mathrm{i}y})^{\lambda}}\Big).\end{aligned} \tag{4.27}$$

We stress that the arguments above are heuristic and that the recursive algorithm is mathematically rigorous; see Paper III. We would also like to point out that the combination of the recursive algorithm, and the result for the Gegenbauer polynomials above (see (A.8)) can yield the explicit coefficient for expansion of Jacobi polynomials in terms of the Gegenbauer polynomials.

4.3 Symmetric polynomials

The kernel function methods can naturally be generalized to many-variable CMS type models and the corresponding symmetric polynomials. To illustrate this, we discuss how the Jack polynomials, and also Macdonald polynomials, can be constructed using the kernel functions methods above. (Recall the partitions in Section 2.2 and the dominance ordering, denoted by "$\leq$", in (2.23).) The function

$K(x,y) = K(x,y;g,N,M,p)$, given by

$$K^{(\mathrm{CS})}(x,y) = \frac{\mathrm{e}^{\mathrm{i}p(\sum_{j=1}^{N} x_j - \sum_{k=1}^{M} y_k)} \prod_{j<k}^{N} \left(\sin(\tfrac{1}{2}(x_k - x_j))^g\right) \prod_{j<k}^{M} \left(\sin(\tfrac{1}{2}(y_k - y_j))^g\right)}{\prod_{j=1}^{N} \prod_{k=1}^{M} \sin(\tfrac{1}{2}(x_j - y_k))^g} \tag{4.28}$$

with N, M positive integers, x, y independent complex variables, p an arbitrary constant, satisfies the kernel function identity

$$\left(H_N^{(\mathrm{CS})}(x;g) - H_{M\,(\mathrm{CS})}(y;g) - C_{N,M}\right) K^{(\mathrm{CS})}(x,y) = 0 \tag{4.29}$$

for $H_N^{(\mathrm{CS})}$ in (2.24) and $C_{N,M} = (g^2/12)(N-M)((N-M)^2-1) + (N-M)p^2$.

The kernel function in (4.28), for $N = M$ can be used as an integral transform kernel for the recursive algorithm. Here we take the monomial basis, labeled by a partition n of length N, given by

$$f_n = \mathrm{e}^{-\mathrm{i}n_1^+ y_1} \mathrm{e}^{-\mathrm{i}n_2^+ y_2} \cdots \mathrm{e}^{-\mathrm{i}n_N^+ y_N}, \quad n_j^+ = n_j + s_j, \tag{4.30}$$

with $s \in R^N$, in the domain

$$0 > \Im(y_1) > \Im(y_2) > \ldots > \Im(y_N)$$

The action of the CS Hamiltonian is given by

$$H^{(\mathrm{CS})} f_n = \sum_{j=1}^{N} (n_j^+)^2 f_n - 4g(g-1) \sum_{\nu \in \mathbb{N}} \sum_{j<k}^{N} \nu f_{n-\nu(e_j - e_k)} \tag{4.31}$$

with e_j is the standard basis in $\mathbb{Z}$, *i.e.* $(e_j)_l = \delta_{j,l}$, which follows from the many-variable, hypergeometric expansion of the potential, *i.e.*

$$H^{(\mathrm{CS})} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial y_j} - 4g(g-1) \sum_{j<k}^{N} \sum_{\nu \in \mathbb{N}} \nu \mathrm{e}^{\mathrm{i}\nu(y_j - y_k)} \tag{4.32}$$

valid in the domain (see above). It follows by a straightforward check that the operator acts as a lower triangular matrix, *i.e.* $H^{(\mathrm{CS})} f_n$ is given by linear combinations of f_m with $m \leq n$ w.r.t. the dominance order, and can be diagonalized; see [78,81]. The expansion of the kernel function, in this domain, is given by

$$\mathrm{e}^{\mathrm{i}(p+gN)(\sum_{j=1}^{N} x_j - y_j)} \psi_0(x;g,N) \frac{\prod_{j=1}^{N} \mathrm{e}^{\mathrm{i}\frac{1}{2}g(N+1-2j)y_j} \prod_{j<k} (1 - \mathrm{e}^{\mathrm{i}(y_j - y_k)})^g}{\prod_{j,k=1}^{N} (1 - \mathrm{e}^{\mathrm{i}x_j - \mathrm{i}y_k})^g} \tag{4.33}$$

up to an irrelevant constant, which shows that the integral transform is well-defined if $s_j = \frac{1}{2}g(N+1-2j)$, for $j = 1, \ldots, N$, and that the integral transform will produce

a symmetric polynomials in $\mathrm{e}^{\mathrm{i}x_j}$ (multiplied with the groundstate eigenfunction) for $p = -gN$. It follows from the uniqueness that the integral transform of the singular solutions are the eigenfunctions in (2.25).

We would like to mention that the symmetric polynomials can also be constructed by iterative integrals, as explained in Section 4.2 for the Gegenbauer polynomials. This was proven in [82] for the Jack polynomials using the kernel function in (4.28), with $N > M$, which was derived from a CFT consideration of the CS model that is similar to the work in Paper II. The iterative integral representation for the Macdonald polynomials is due to Mimachi and Noumi [83]. Recent developments by Hallnäs and Ruijsenaars in [84–86] for the hyperbolic A_{N-1} case (see Table 2.1) shows that the iterative integral method can also be extended to CMS type models where the eigenfunctions are not given in terms of polynomials.

4.4 Elliptic potentials

The kernel function methods for the CMS models with elliptic potentials are remarkably similar to the discussion in Section 4.2. A notable difference is in the recursive algorithm has to be supplemented by an additional series expansion, which we refer to as the perturbative algorithm. We consider the Lamé equation in order to illustrate the perturbative algorithm. The discussion follows Section 4.2 and will be brief.

(Recall the Jacobi ϑ-functions (see also Appendix A.3) and the Lamé operator $H^{(\mathrm{L})}(x;g)$ in (3.8).) Let ϑ_1 denote the odd Jacobi ϑ-function (see (A.19a)) and g a complex constant. The function

$$\frac{\vartheta_1(x)^g\vartheta_1(y)^g}{\Big(\vartheta_1(\frac{1}{2}(x+y))\vartheta_1(\frac{1}{2}(x-y))\Big)^{2g}} \tag{4.34}$$

satisfies the kernel function identity

$$\Big(H^{(\mathrm{L})}(x;g) - H^{(\mathrm{L})}(y;g)\Big)K(x,y;g) = 0. \tag{4.35}$$

with $H^{(\mathrm{L})}$ in (3.8). The singular eigenfunctions are constructed in terms of plane-waves $\exp(-\mathrm{i}(n+s)y)$, $n \in \mathbb{Z}, s \in \mathbb{R}$ as before. It follows from known identities of the Weierstrass $\wp$-function that the operator $H^{(\mathrm{L})}(y;g)$ can be expanded in a hypergeometric series: Let $0 < \Im(y) < \pi\Im(\tau)$, then

$$H^{(\mathrm{L})}(y;g) = -\frac{d^2}{dy^2} - 4g(g-1)\sum_{\nu\in\mathbb{Z}}\nu\frac{1}{1-q^{2\nu}}\mathrm{e}^{\mathrm{i}\nu y} - 4g(g-1)\frac{\eta_1}{\pi}, \tag{4.36}$$

with η_1/π in (A.25). It follows that the $H^{(\mathrm{L})}$ operator does not act as a lower triangular matrix on the space that is spanned by the plane-waves, and the singular eigenfunctions are then not as straightforward to construct. It was shown in [64] that an additional expansion in the nomé q will yield an algorithm which is close

to the recursive algorithm; see also Paper III. Standard perturbation theory of the elliptic models have usually chosen the nomé q, satisfying $|q| < 1$, as a natural small parameter in the perturbative expansion [59, 61].

It is known that the kernel functions for CMS type models with elliptic potentials, can only satisfy kernel function identities, such as (4.35) under certain restrictions on the parameters or the particle numbers. These restrictions are often referred to as *balancing conditions*. One can show that the kernel functions satisfy *generalized kernel function identities* if the balancing condition is removed: The function

$$K^{(\mathrm{L})}(x,y,\tau;g,g') \equiv \frac{\vartheta_1'(0)^{g+g'}\vartheta_1(x)^g\vartheta_1(y)^{g'}}{(\vartheta_1(\frac{1}{2}(x-y))\vartheta_1(\frac{1}{2}(x+y)))^{g+g'}} \tag{4.37}$$

with g, g' arbitrary complex constants, $\vartheta_1(x) = \vartheta_1(x|\pi\tau)$, $\vartheta_1(y) = \vartheta_1(y|\pi\tau)$, satisfies the generalized kernel function identity

$$\Big(\frac{2\mathrm{i}}{\pi}(g-g')\frac{\partial}{\partial\tau} + H_{(\mathrm{L})}(x;\tau,g) - H(y;\tau,g') - c_1\Big)K^{(\mathrm{L})}(x,y,\tau;g,g') = 0, \tag{4.38}$$

for $c_1 = 2(g-g')(1-g-g')\eta_1/\pi$ [41]. It follows from that the integral transform maps solutions of the non-stationary Lamé equation with coupling $\kappa = g-g', g'$ to solutions with coupling $\kappa = g-g', g$; as discussed in Paper IV.

We now turn to the Heun equation and some new results. Define g_ν^μ ($\mu,\nu = 0,\ldots,3$) as

$$\begin{pmatrix} g_0^\mu \\ g_1^\mu \\ g_2^\mu \\ g_3^\mu \end{pmatrix} = \begin{pmatrix} \delta_{\mu,0} & \delta_{\mu,1} & \delta_{\mu,2} & \delta_{\mu,3} \\ \delta_{\mu,1} & \delta_{\mu,0} & \delta_{\mu,3} & \delta_{\mu,2} \\ \delta_{\mu,2} & \delta_{\mu,3} & \delta_{\mu,0} & \delta_{\mu,1} \\ \delta_{\mu,3} & \delta_{\mu,2} & \delta_{\mu,1} & \delta_{\mu,0} \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \tag{4.39}$$

with g_ν the model parameters of the Heun equation (see (3.11)). (The Jacobi ϑ-functions given in (A.19a)-(A.19b), and the operator $H^{(\mathrm{DTV})}(x;\{g_\nu^\mu\}_{\nu=0}^3)$ in (3.11).)

Theorem 4.4.1. *Let $\mu \in \{0,1,2,3\}$ be fixed, g_ν^μ as in (4.39) and $\tilde{g}_\nu^\mu = \lambda - g_\nu^\mu$, with $\lambda = \frac{1}{2}\sum_{\nu=0}^3(g_\nu - \frac{1}{4}\kappa)$. The function*

$$K_\mu^{(DTV)}(x,y,\tau;g,\tilde{g}) \equiv \frac{\prod_{\nu=0}^3(\vartheta_{\nu+1}(\frac{1}{2}x)^{g_\nu}\vartheta_{\nu+1}(\frac{1}{2}y)^{\tilde{g}_\nu^\mu})}{\Big(\vartheta_{\mu+1}(\frac{1}{2}(x-y))\vartheta_{\mu+1}(\frac{1}{2}(x+y))\Big)^\lambda} \tag{4.40}$$

satisfies the generalized kernel function identity

$$\Big(\frac{\mathrm{i}}{\pi}\kappa\frac{\partial}{\partial\tau} + H^{(DTV)}(x;\{g_\nu\}_{\nu=0}^3) - H^{(DTV)}(y;\{\tilde{g}_\nu^\mu\}_{\nu=0}^3) - C_{1,1}\Big)K_\mu^{(DTV)}(x,y,\tau) = 0 \tag{4.41}$$

with constant $C_{1,1} = 2\kappa(1-\lambda)(\eta_1/\pi)$ and η_1 in (A.25).

(The $\mu = 0$ case was proven in [41]. The cases where $\mu = 1,2,3$ are new, to our knowledge, and the proof is given in Section B.)

The result in Theorem 4.4.1 for the non-stationary Heun equation yields the following results as special cases.

Corollary 4.4.2. *The function in* (4.40) *for* $\kappa = 0$ (i.e. $\lambda = \frac{1}{2}\sum_{\nu=0}^{3} g_\nu$) *is a kernel function for a pair of Schrödinger operators with the Darboux-Treibich-Verdier potential,* i.e.

$$\Big(H^{(DTV)}(x; g_\nu) - H^{(DTV)}(y; \tilde{g}^\mu_\nu)\Big) K^{(DTV)}_\mu(x, y) = 0, \quad (\mu \in \{0, 1, 2, 3\}) \tag{4.42}$$

for $\{g_\nu\}_{\nu=0}^3$ *complex constants,* $\tilde{g}^\mu_\nu = (\frac{1}{2}\sum_{\nu=0}^{3} g_\nu) - g^\mu_\nu$, *and* $H^{(DTV)}$ *in* (3.11).

The $\mu = 0$ case was obtained in [41] and the $\mu = 3$ case was obtained in [87] as the limit of a kernel function for the elliptic van Diejen model. (The $\mu = 1$, and $\mu = 2$, cases are new, to our knowledge, but follows from the results in [41] and [87] by a simple shift.)

Corollary 4.4.3. *For* $g_\nu = g$ *for all* $\nu = 0, \ldots 3$ *the kernel function in* (4.40) *reduces to*

$$K^{(L)}_\mu(x, y, \tau; g, \kappa) \equiv \vartheta_1(0)^{2g-\kappa} \frac{\vartheta_1(x)^g \vartheta_1(y)^{g-\kappa}}{\Big(\vartheta_{\mu+1}(\frac{1}{2}(x+y))\vartheta_{\mu+1}(\frac{1}{2}(x-y))\Big)^{2g-\kappa}} \tag{4.43}$$

and satisfies the generalized kernel function identity

$$\Big(\frac{2i}{\pi}\kappa\frac{\partial}{\partial\tau} + H^{(L)}(x; g) - H^{(L)}(y; g-\kappa) - c_{(L)}\Big) K^{(L)}_\mu(x, y, \tau) = 0 \tag{4.44}$$

with constant $c_{(L)} = 4\kappa(1 + \kappa - 2g)(\eta_1/\pi)$.

In paper III, we use the kernel function in (4.42), with $\mu = 1$, for the perturbative algorithm for the non-stationary Heun equation, and the kernel function in (4.43) is used for iterative integral representations of the non-stationary Lamé equation in Paper IV.

Chapter 5

Introduction to scientific papers

The purpose of this chapter is to introduce the results that are obtained in the four scientific papers appended to this Thesis.

5.1 Paper I

In Paper I we consider the Ruijsenaars models (*i.e.* A_{N-1} relativistic Calogero-Sutherland models) with different types of interaction given in terms of rational, hyperbolic, trigonometric, and elliptic functions. The Ruijsenaars models are expressed in terms of the function $s(x)$, defined as

$$s(x) \equiv \begin{cases} x & \text{(Rational case (I))} \\ 2\sin(\frac{1}{2}x) & \text{(Trigonometric case (II))} \\ (a/\pi)\sinh(\pi x/a) & \text{(Hyperbolic case (III))} \\ \vartheta_1(\frac{1}{2}x|\mathrm{i}\frac{1}{2}a) & \text{(Elliptic case (IV))} \end{cases} \tag{5.1}$$

with ϑ_1 the Jacobi theta function in (A.19a). This is notation is convenient as it allows for a unified approach (as shown in Paper I).

We construct a restricted Sen type generalization (source identity) of the Ruijsenaars model where the mass parameters only take values in the set

$$\Lambda = \{m_0, -m_0, -1/gm_0, 1/gm_0\}$$

with m_0 some fixed non-zero constant (see also Section 4.1. The Sen type generalization is defined by the (formal) analytic difference operators

$$\mathcal{S}^{\pm}(x; m, g, \beta) \equiv \sum_{j=1}^{N} \frac{s(\mathrm{i}gm_j\beta)}{\mathrm{i}g\beta s'(0)} \mathcal{A}_j^{\mp}(x; m, g, \beta) \mathrm{e}^{\mp \mathrm{i}\frac{\beta}{m_j}\frac{\partial}{\partial x_j}} \mathcal{A}_j^{\pm}(x; m, g, \beta) \tag{5.2}$$

with particular interaction $\mathcal{A}_j^{\pm}$ that can be found in Paper I. In order to illustrate the unified approach, we consider the special case where all the mass parameters are set to unity and the interactions reduce to the interaction in (3.17)-(3.18).

The operator in (5.2) is interpreted as a relativistic generalization of the Sen Hamiltonian in (1.7) as it is obtained by

$$\lim_{\beta\to 0}\big(\beta^{-2}\left(\mathcal{S}^{+}+\mathcal{S}^{-}\right)-\sum_{j=1}^{N}m_j\beta^{-2}\big). \tag{5.3}$$

For the elliptic case, we can only construct an exact groundstate of (5.2) under the restriction of the balancing condition $\sum_{j=1}^{N} m_j = 0$ (see Section 4.4).

The source identity was then used in order to give a unified derivation of previously well-known identities in [26, 28, 88] and the CFVS type deformation of the trigonometric Ruijsenaars model [28]; see Sections 3 and 4 of Paper I . The new results in Paper I is a CFVS type deformations of the Ruijsenaars model when the interactions are given by elliptic functions (Case IV above). The integrability of the elliptic CFSV type deformation of (1.9) is conjectured by the existence of an explicit kernel function.

5.2 Paper II

It is known that the CS model has a collective field theory representation [82, 89–91] that is related to exotic particles known as anyons [92, 93] (explained below) and the Fractional Quantum Hall Effect (FQHE). The Calogero and CS models have naturally appeared in the theory of the FQHE [7, 48, 89, 94, 96, 97] and anyons. Anyons are quasiparticles that exhibit exotic particle statistics, *i.e.*

$$\phi(x)\phi(y)=\mathrm{e}^{\pm \mathrm{i}\alpha\pi}\phi(y)\phi(x)\quad \text{for } x\gtrless y \text{ and } \alpha\in\mathbb{R} \tag{5.4}$$

which are not restricted to only fermions ($\alpha = 1$) or bosons ($\alpha = 0$), and they can appear in lower dimensional systems. The FQHE was exceptional (when discovered) due to the high precision of measurement results which intuitively suggest a theoretical explanation by an exactly solvable systems. The results in Paper II show that the previously known relation between the FQHE and the CS model can naturally be extended to the deformed model. We would like to mention that the collective field theory can also be extended to include the elliptic Calogero-Sutherland model in (3.14) [71, 72].

The collective field theory (CFT) approach is also closely related to the approach in [98] utilizing symmetric functions (SF). It follows from the fact that the CFT and SF approaches are different representations of the (extended) Heisenberg algebra (see *e.g.* [99]). We prove that the collective field theory construction of the CS model, defined by a operator, which we denote by $\mathcal{H}^{\nu,3}$, and which is explicitly known (see Paper II), is in fact the collective field theory description of the deformed Calogero-Sutherland (dCS) model, defined by the differential operator in (1.8): Define the vertex operator $\phi_\nu(x)$ in terms of free bosons that satisfy a $U(1)$ affine Kac-Moody

algebra, *i.e.* $\hat{\rho}_n$ ($n \in \mathbb{Z}$) and $\hat{R}$ satisfying

$$[\hat{\rho}_n, \hat{\rho}_m] = n\delta_{n+m,0}, \quad \hat{\rho}_n^\dagger = \hat{\rho}_{-n}, \quad (n, m \in \mathbb{Z}) \tag{5.5}$$

$$[\hat{\rho}_n, \hat{R}] = \delta_{n,0}\hat{R}, \quad \hat{R}^\dagger = \hat{R}^{-1}. \tag{5.6}$$

We prove that the differential operator defining the deformed CS model in (1.8) (see also Eq. (5) in Paper II), which we denote by $H_{N,\tilde{N}}$, acting on a product of vertex operators is the action of the operator $\mathcal{H}^{\nu,3}$ on the same product of vertex operators: To be precise, for $g = \nu^2$ the following holds

$$\begin{aligned}&\mathcal{H}^{\nu,3}\phi_\nu(x_1)\dots\phi_\nu(x_N)\phi_{-\frac{1}{\nu}}(\tilde{x}_1)\cdots\phi_{-\frac{1}{\nu}}(\tilde{x}_{\tilde{N}})|0\rangle \\ =&(H_{N,\tilde{N}}(x,\tilde{x}) + c)\phi_\nu(x_1)\dots\phi_\nu(x_N)\phi_{-\frac{1}{\nu}}(\tilde{x}_1)\cdots\phi_{-\frac{1}{\nu}}(\tilde{x}_{\tilde{N}})|0\rangle,\end{aligned} \tag{5.7}$$

with c constant and $|0\rangle$ the vacuum state satisfying $\hat{\rho}_n|0\rangle = 0, n \geq 0$, and $\langle 0|R^Q|0\rangle = \delta_{Q,0}$. We would like to mention that these results were anticipated in the physics literature [94].

In Paper II we utilize the relation to the deformed Calogero-Sutherland model, using known mathematical results about the super-Jack polynomials, in order to construct a set of orthogonal eigenstates for the collective field theory: The dCS model is known [23] to have exact eigenfunctions

$$\psi_\lambda(x,\tilde{x};g) = \psi_0(x,\tilde{x};g)SP_\lambda(z,\tilde{z};g), \quad z_j = \mathrm{e}^{\mathrm{i}x_j}, \tilde{z}_j = \mathrm{e}^{\mathrm{i}\tilde{x}_j}, \tag{5.8}$$

labeled by partitions in the so-called fat $(N,\tilde{N})$-hook, *i.e.* $\lambda_{\tilde{N}+1} \leq \tilde{N}$ and $\ell(\lambda)$ arbitrary. (We wish to stress that the function ψ_0 is not a groundstate eigenfunction in the quantum mechanical meaning. We abuse physics terminology and often refer to ψ_0 as the *deformed groundstate*.)

We construct the orthogonal eigenstates as

$$\prod_{j=1}^{N}\int_{\mathcal{C}_\epsilon}\frac{\mathrm{d}x_j}{2\pi}\prod_{k=1}^{\tilde{N}}\int_{\mathcal{C}_{\epsilon'}}\frac{\mathrm{d}\tilde{x}_k}{2\pi}\psi_\lambda(-x,-\tilde{x};\nu^2)\phi_\nu(x_1)\dots\phi_\nu(x_N)\phi_{-\frac{1}{\nu}}(\tilde{x}_1)\cdots\phi_{-\frac{1}{\nu}}(\tilde{x}_{\tilde{N}})|0\rangle, \tag{5.9}$$

which we refer to as *super-Jack states*. The orthogonality of the super-Jack states follows from known identities [19,23,95], such as the integral relations of the super-Jack polynomials

$$\prod_{j=1}^{N}\int_{\mathcal{C}_\epsilon}\frac{\mathrm{d}x_j}{2\pi}\prod_{k=1}^{\tilde{N}}\int_{\mathcal{C}_{\epsilon'}}\frac{\mathrm{d}\tilde{x}_k}{2\pi}\psi_0(x,\tilde{x})^2 SP_\lambda(z,\tilde{z})SP_\mu(z^{-1},\tilde{z}^{-1}) = 0 \quad (\lambda \neq \mu), \tag{5.10}$$

and that the differential operator $H_{N,\tilde{N}}$ in (1.8) is symmetric if we were to consider (5.10) as an inner product. (The integral is well-defined for $\epsilon \neq \epsilon'$ and we use the shorthand $z^{-1} = (z_1^{-1}, \dots)$ etc.)

5.3 Paper III

Paper III considers the non-stationary Heun equation in (1.10) for arbitrary model parameters $(\kappa, \{g_\nu\}_{\nu=0}^3)$. We show that constructing eigenfunctions of the operator in (1.10) is equivalent to finding solutions of a differential-difference equation: The kernel functions are used in order to construct a particular set of basis functions $\{F_m(x)\}_{m\in\mathbb{Z}}$ (as described in Section 4.2). The solutions of the non-stationary Heun equation is constructed by linear combinations of the basis functions F_m, *i.e.*

$$\psi_n(x) = \sum_{m\in\mathbb{Z}} \alpha_n(m) F_m(x) \tag{5.11}$$

where the coefficients $\alpha_n(m)$ satisfy the differential-difference equation in Section 3 of Paper III. In Section 5 of Paper III we present two algorithms for computing the coefficients $\alpha_n(m)$, expanded in a power series in the nomé q. We note that these series solutions of the non-stationary Heun equation is especially suited when the model parameters take complex values. However, there are difficulties with the perturbative algorithm when the model parameters take integer values.

A remarkable feature of the kernel function method is that the methods developed for the elliptic Calogero-Sutherland model in [64] generalizes to the non-stationary Heun equation, with model details only changing building blocks of the solutions.

5.4 Paper IV

In Paper IV we present a recursive scheme for constructing solutions of the non-stationary Lamé equation, which yields iterative integral representations of these solutions (as explained in Section 4.2). The integral step for the iterative integral representation of the Gegenbauer polynomials are determined by the model parameter κ, *i.e.* $\kappa = g - g'$, in generalization of (4.15). This shows that the iterative integral representations are especially suited for integer values of the model parameters and serves as a complementary approach to the perturbative algorithm in Paper III.

A remarkable property is the existence of Hilbert-Schmidt class integral transform operators that allow us to construct iterative integral representations of the solutions for arbitrary values of the model parameters. (Recall that in Section 4.2 there was a requirement $g - g' \in \mathbb{Z}$.)

The recursive procedure in Paper IV is then used to construct explicit integral representation for special cases of the model parameters.

Appendix A

Special functions

We collect standard definitions and results about special functions: a standard reference is [2], for example.

A.1 Transcendental functions

The Euler Γ function can expressed as

$$\Gamma(x) = \frac{1}{z}\mathrm{e}^{-\gamma z}\prod_{l=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}, \tag{A.1}$$

with $\gamma = \lim_{m\to\infty}\sum_{l=1}^{m}(1/n) - \ln(m) \approx 0.5772157$ the Euler constant. The Euler $\Gamma(x)$ function has simple poles at $x \in -\mathbb{N}_0$ which can be seen from the difference equation

$$\Gamma(x+1) = x\Gamma(x) \tag{A.2}$$

The (raising) Pochhammer symbol $(x)_n$ is defined as

$$(x)_n = x(x+1)\cdots(x+n-1) \quad (n \in \mathbb{N}_0), \tag{A.3}$$

for non-negative integers n. The Pochhammer symbol can be represented as

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{A.4}$$

for all $x, n \in \mathbb{C}$.

A.2 Classical orthogonal polynomials

The Gegenbauer polynomials $C_n^{(g)}$ are the unique solutions of the differential equation

$$\Big((1-x^2)\frac{d^2}{dx^2} - (2g+1)x\frac{d}{dx} + n(n+2g)\Big)C_n^{(g)}(x) = 0 \quad (g > 0, n \in \mathbb{N}_0), \tag{A.5}$$

with expansion

$$C_n^{(g)}(x) = \frac{2^n (g)_n}{n!} x^n + \text{ l.o.} \tag{A.6}$$

where $(g)_n$ is the raising Pochhammer symbol; see (A.3) and (A.4).

The Gegenbauer polynomials have the series representation

$$C_n^{(g)}(x) = \frac{(2g)_n}{n!} \sum_{\nu=0}^{n} \frac{(-n)_\nu (n+2g)_\nu}{(g+\frac{1}{2})_\nu} \Big(\frac{1-x}{2}\Big)^\nu \tag{A.7}$$

The Gegenbauer polynomials have a generating function given by

$$\frac{1}{(1-2x\xi+\xi^2)^g} = \sum_{n\in\mathbb{N}_0} C_n^{(g)}(x)\xi^n \tag{A.8}$$

where the series is absolutely convergent for x in the interior of an ellipse in the complex plain with focal points ± 1 and semi-major axis $\frac{1}{2}(\xi+\xi^{-1})$. For $x \in [-1,1]$ the series is absolutely and uniformly convergent for $|\xi| < 1$.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are the unique solutions of the differential equation

$$\Big((1-x^2)\frac{d^2}{dx^2} + (\beta-\alpha-(\alpha+\beta+2))x\frac{d}{dx} + n(n+\alpha+\beta+1)\Big)P_n^{(\alpha,\beta)}(x) = 0 \quad (n \in \mathbb{N}_0), \tag{A.9}$$

with expansion

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n n!} x^n + \text{ l.d.} \tag{A.10}$$

The Jacobi polynomials have the series representation

$$P_n^{(\alpha,\beta)}(x) = \sum_{l=0}^{n} \frac{(n+\alpha+\beta+1)_l (\alpha+l+1)_{n-l}}{l!(n-l)!} \left(\frac{x-1}{2}\right)^l \tag{A.11}$$

A.3 Elliptic functions

In this appendix we will introduce the elliptic (and quasi-elliptic) functions that we use in this work. We mainly follow the convention in [2].

Definition A.3.1. Let ω_1 and ω_3 be any two non-zero numbers such that ω_3/ω_1 is not purely real. A *double-periodic function* is any function satisfying

$$f(z+2\omega_1) = f(z), \quad f(z+2\omega_3) = f(z)$$

for all z in $\mathbb{C}$ and an *elliptic function* is any double-periodic function that is meromorphic in the finite part of the complex plane.

We refer to the constants ω_1, ω_3 as the *half-periods* of the elliptic functions and define $\omega_2 \equiv -\omega_1 - \omega_3$.

Definition A.3.2. Let ω_1 and ω_3 be as above. A *quasi-elliptic function* is any meromorphic function $f(x)$ that satisfies either

$$f(x+2\omega_j) = g_j(x)f(x) \quad \text{or} \quad f(x+2\omega_j) = f(x) + g_j(x) \quad (j=1,2,3) \tag{A.12}$$

for some analytic functions $g_j(x)$ $(j=1,2,3)$.

Definition A.3.3. The Weierstrass elliptic $\wp$ function is defined as

$$\wp(x|\omega_1,\omega_3) = \frac{1}{x^2} + \sum_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(x+\Omega_{n,m}(\omega_1,\omega_3))^2} - \frac{1}{\Omega_{n,m}(\omega_1,\omega_3)^2} \tag{A.13}$$

with the lattice $\Omega_{n,m}(\omega_1,\omega_3) = 2n\omega_1 + 2m\omega_3$.

It follows from Definition A.3.3 that the Weierstrass $\wp$ function satisfies

$$\wp(cx \mid c\omega_1, c\omega_3) = c^{-2}\wp(x|\omega_1,\omega_3) \tag{A.14}$$

for all non-zero, complex constants c.

The Weierstrass $\wp$-function satisfies the non-linear differential equations

$$(\wp'(x))^2 = 4(\wp(x)-e_1)(\wp(x)-e_2)(\wp(x)-e_3) \tag{A.15}$$

$$\wp''(x) = 2\sum_{j=1}^{3}\prod_{k\neq j}\wp(x) - e_k \tag{A.16}$$

with $e_j \equiv \wp(\omega_j)$ for $j=1,2,3$.

The Weierstrass $\wp$-function is also known to satisfy the half-period shift relations

$$\wp(x+\omega_\nu) = e_\nu + \frac{(e_\nu-e_\mu)(e_\nu-e_\varepsilon)}{\wp(x)-e_\nu} \tag{A.17}$$

for $\nu=1,2,3$ and $\mu,\varepsilon\in\{1,2,3\}$ such that $\mu\neq\nu\neq\varepsilon$ and $\epsilon\neq\mu$, and

$$\sum_{\nu=0}^{3}\wp(x+\omega_\nu|\omega_1,\omega_3) = 4\wp(2x|\omega_1,\omega_3) = \wp(x|\tfrac{1}{2}\omega_1,\tfrac{1}{2}\omega_3). \tag{A.18}$$

We set the half-periods to $(\omega_1,\omega_3) = (\pi,\tau)$. Eq. (A.14) allows simple rescaling to arbitrary periods.

The Jacobi ϑ-functions are defined as

$$\vartheta_1(x) \equiv 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin((2n+1)x), \tag{A.19a}$$

$$\vartheta_2(x) \equiv 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos((2n+1)x), \tag{A.19b}$$

$$\vartheta_3(x) \equiv 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nx), \tag{A.19c}$$

$$\vartheta_4(x) \equiv 1 - 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nx). \tag{A.19d}$$

where $\vartheta_\nu(x) = \vartheta(x,q) = \vartheta_\nu(x|\pi\tau)$ with $q = \exp(\mathrm{i}\pi\tau)$. It follows from the Jacobi triple product identity that the Jacobi ϑ-functions can be expressed as

$$\vartheta_1(x,q) = 2q^{\frac{1}{4}}\sin(x)\prod_{l=1}^{\infty}(1-q^{2l})(1-2q^{2l}\cos(2x)+q^{4l}) \tag{A.20a}$$

$$\vartheta_2(x,q) = 2q^{\frac{1}{4}}\cos(x)\prod_{l=1}^{\infty}(1-q^{2l})(1+2q^{2l}\cos(2x)+q^{4l}) \tag{A.20b}$$

$$\vartheta_3(x,q) = \prod_{l=1}^{\infty}(1-q^{2l})(1+2q^{2l-1}\cos(2x)+q^{4l-2}) \tag{A.20c}$$

$$\vartheta_4(x,q) = \prod_{l=1}^{\infty}(1-q^{2l})(1-2q^{2l-1}\cos(2x)+q^{4l-2}) \tag{A.20d}$$

It follows from the definition that the Jacobi ϑ-functions satisfy the shift relations

$$\begin{aligned}\vartheta_1(x\pm\tfrac{\pi}{2}\tau) &= \pm\mathrm{i}\mathrm{e}^{\mp\mathrm{i}x}q^{-\frac{1}{4}}\vartheta_4(x), \quad \vartheta_2(x\pm\tfrac{\pi}{2}\tau) = \mathrm{e}^{\mp\mathrm{i}x}q^{-\frac{1}{4}}\vartheta_3(x)\\ \vartheta_3(x\pm\tfrac{\pi}{2}\tau) &= \mathrm{e}^{\mp\mathrm{i}x}q^{-\frac{1}{4}}\vartheta_2(x), \quad \vartheta_4(x\pm\tfrac{\pi}{2}\tau) = \pm\mathrm{i}\mathrm{e}^{\mp\mathrm{i}x}q^{-\frac{1}{4}}\vartheta_1(x).\end{aligned} \tag{A.21}$$

The function $\vartheta_\nu(x)$ has simple zeros at the points $x = \omega_\nu/2(\mathrm{mod}(\omega_1,\omega_3))$ with $\omega_1 = \pi, \omega_3 = \pi\tau$.

The Jacobi ϑ_ν functions all satisfy the differential equation

$$(\frac{\pi\mathrm{i}}{4}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial\tau})\vartheta_\nu(x|\pi\tau) = 0 \quad (\nu = 1,2,3,4). \tag{A.22}$$

The Jacobi functions satisfy the double angle formula

$$\vartheta_1(2x) = \frac{2}{\vartheta_1'(x)}\vartheta_1(x)\vartheta_2(x)\vartheta_3(x)\vartheta_4(x) \tag{A.23}$$

The Weierstrass $\wp$-function can be expressed in terms of the Jacobi functions as

$$\wp(x+\omega_\nu) = \frac{1}{4}\Big(\Big(\frac{\vartheta_{\nu+1}'(\frac{1}{2}x)}{\vartheta_{\nu+1}(\frac{1}{2}x)}\Big)^2 - \frac{\vartheta_{\nu+1}''(\frac{1}{2}x)}{\vartheta_{\nu+1}(\frac{1}{2}x)}\Big) - \frac{\eta_1}{\pi} \tag{A.24}$$

with

$$\frac{\eta_1}{\pi} = -\frac{1}{12}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{1}{12} - 2\sum_{l=1}^{\infty}\frac{q^{2l}}{(1-q^{2l})^2} \tag{A.25}$$

Appendix B

Proof of Theorem

In this Section we present the proof of Theorem 4.4.1. We recall that Theorem 4.4.1 concerned the generalized kernel function identity

$$\Big(\frac{i}{\pi}\kappa\frac{\partial}{\partial\tau}+H^{(\mathrm{DTV})}(x;\{g_\nu\}_{\nu=0}^3)-H^{(\mathrm{DTV})}(y;\{\tilde{g}_\nu^\mu\}_{\nu=0}^3)-C_{1,1}\Big)K_\mu^{(\mathrm{DTV})}(x,y,\tau)=0 \tag{B.1}$$

for $K_\mu^{(\mathrm{DTV})}(x,y,\tau)$ in (4.40). As we already mentioned, the $\mu=0$ case was proven in [41] (see also Paper IV). The $\mu=1$ case follows from their results by $x\to x+\pi$ where

$$\begin{aligned}\sum_{\nu=0}^{3}g_\nu(g_\nu-1)\wp(y+\pi+\omega_\nu)=g_0(g_0-1)\wp(y+\pi)+g_1(g_1-1)\wp(y+2\pi)\\+g_2(g_2-1)\wp(y-\pi\tau)+g_3(g_3-1)\wp(y+\pi+\pi\tau).\end{aligned} \tag{B.2}$$

The periodicity of the Weierstrass $\wp$-function shows that (B.2) equal $\sum_{\nu=0}^3 g_\nu^1(g_\nu 1-1)\wp(y+\omega_{nu})$ with g_ν^1 given in (4.39). The $\mu=2$ case follows from the same argument as above from the $\mu=3$ case. The $\mu=3$ case follows from the simple observation: If $K_1^{(\mathrm{DTV})}(x,y,\tau)$ satisfies the generalized kernel function identity in (4.40) then $K_1^{(\mathrm{DTV})}(x,y+\pi\tau,\tau)$ must satisfy

$$\Big(\frac{\mathrm{i}}{\pi}\kappa\frac{\partial}{\partial\tau}+H(x;\{g_\nu\}_{\nu=0}^3)-H(y+\pi\tau;\{\tilde{g}_\nu^\mu\}_{\nu=0}^3)-\mathrm{i}\kappa\frac{\partial}{\partial y}-C_{1,1}\Big)K_\mu(x,y+\pi\tau,\tau)=0. \tag{B.3}$$

It follows from the relations in (A.21) that

$$\begin{aligned}K_1(x,y+\pi\tau,\tau;\{g_\nu\}_{\nu=0}^3,\{\tilde{g}_\nu\}_{\nu=0}^3)=\mathrm{e}^{-\mathrm{i}\frac{1}{2}(2\lambda-\sum_{\nu=0}^3 g_\nu)(y+\frac{\pi}{2}\tau)}\\\times K_4(x,y+\pi\tau;\{g_\nu\}_{\nu=0}^3,\{\tilde{g}_\nu^3\}_{\nu=0}^3)\end{aligned} \tag{B.4}$$

with K_4 in (4.40) and g_ν^3 in (4.39) for $\nu=0,\ldots,3$. A similarity transformation of (B.3) by the exponential function in (B.4) yields the generalized kernel function

identity in (4.42), *i.e.*

$$\mathrm{e}^{-\mathrm{i}\frac{1}{2}\kappa(y+\frac{\pi}{2}\tau)}\Big(\frac{\mathrm{i}}{\pi}\kappa\frac{\partial}{\partial\tau}+\frac{\partial^2}{\partial y^2}-\mathrm{i}\kappa\frac{\partial}{\partial y}\Big)\mathrm{e}^{\mathrm{i}\frac{1}{2}\kappa(y+\frac{\pi}{2}\tau)}=\frac{\mathrm{i}}{\pi}\kappa\frac{\partial}{\partial\tau}+\frac{\partial^2}{\partial y^2} \tag{B.5}$$

(where we used $2\lambda-g_0-g_1-g_2-g_3=-\kappa$).

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Part II

Scientific Papers

Paper I

Source Identities and Kernel Functions for Deformed (Quantum) Ruijsenaars Models

Farrokh Atai, Martin Hallnäs and Edwin Langmann.

Lett. Math. Phys. (2014) **104**:811

Paper II

Deformed Calogero-Sutherland model and fractional Quantum Hall effect

Farrokh Atai and Edwin Langmann

Preprint: arXiv:1603.06157

Paper III

Series solutions of the non-stationary Heun equation

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Paper IV

Integral representation of solution to the non-stationary Lamé equation

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