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Interacting fermions and non-equilibrium properties of one-dimensional many-body systems

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Abstract

Recent experimental progress on ultracold atomic gases have opened up the possibility to simulate many-body systems out of equilibrium. We consider such a system described by the Luttinger model, which is a model of interacting fermions in one spatial dimension.

It is well known that the Luttinger model is exactly solvable using bosonization. This also remains true for certain extensions of the model, e.g., where, in addition, the fermions are coupled to phonons. We give a self-contained account of bosonization, together with complete proofs, and show how this can be used to solve the Luttinger model and the above fermion-phonon model rigorously.

The main focus is on non-equilibrium properties of the Luttinger model. We use the exact solution of the Luttinger model, with non-local interactions, to study the evolution starting from a non-uniform initial state with a position-dependent chemical potential. The system is shown to reach a current-carrying final steady state, in which the universal value of the electrical conductance, known from near-to-equilibrium settings, is recovered. We also study the effects of suddenly changing the interactions and show that the final state has memory of the initial state, which is, e.g., manifested by non-equilibrium exponents in its fermion two-point correlation functions.

Sammanfattning

Nya experimentella framsteg för ultrakalla atomgaser har öppnat upp möjligheten att studera mångpartikelsystem i icke-jämvikt. Vi betraktar ett sådant system beskrivet av Luttinger-modellen, vilket är en modell av växelverkande fermioner i en rumsdimension.

Det är välkänt att Luttinger-modellen är exakt lösbar genom bosonisering. Detta är också sant för vissa utvidgningar av modellen, t.ex. där fermionerna dessutom är kopplade till fononer. Vi ger en fristående redogörelse av bosonisering, tillsammans med fullständiga bevis, och visar hur detta kan användas för att rigoröst lösa Luttinger-modellen och den ovanstående fermion-fonon-modellen.

Huvudfokus är på icke-jämviktsegenskaper hos Luttinger-modellen. Vi använder den exakta lösningen av Luttinger-modellen, med icke-lokala växelverkningar, för att studera utvecklingen från ett olikformigt initialtillstånd med en positionsberoende kemisk potential. Systemet visas nå ett strömförande stabilt sluttillstånd, där det universella värdet av den elektriska ledningsförmågan, känt från studier av system nära jämvikt, återfås. Vi studerar också effekterna av att plötsligt ändra växelverkningarna och visar att sluttillståndet har minne av initialtillståndet, vilket t.ex. manifesteras av icke-jämviktsexponenter i dess fermion-två-punkts-korrelationsfunktioner.

Acknowledgements

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My workplace would not be complete without Farrokh, my office roommate, who is always open for discussing any problem, big or small. Also, no conference in mathematical physics would be as fun without Douglas, my fellow snowboard conqueror of the Chilean Andes. I also want to thank the faculty and my fellow PhD students at the department of theoretical physics at KTH. Lastly, I want to acknowledge the AlbaNova gym: almost all my ideas worth having seem to have been conceived within those walls.

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Contents

Part I

Introduction and background

Chapter 1

Introduction

Many-body systems refer to the study of the collective behavior of large numbers of interacting particles. This thesis is concerned with exactly solvable one-dimensional models of interacting fermions and their use for understanding many-body systems out of equilibrium. Such models were important in the historical development of quantum field theory[1] and its use for understanding many-body systems in equilibrium [72]. One aim of this thesis is to explain their relevance for non-equilibrium quantum physics.

1.1 Exactly solvable models

Exactly solvable means, more or less, what it says: these are models amenable to exact analytical understanding, e.g., models for which all eigenvalues and eigenstates can be computed analytically together with all correlation functions. Examples of one-dimensional many-body systems for which this is the case include the XY model, the XXZ model, and the Luttinger model. The first two are exactly solvable by Bethe ansatz (see, e.g., [33]), and the third was first solved correctly in [48] using bosonization; the latter is one of the main tools we will discuss in this thesis. Another broader term for such models are *integrable* models, although even the definition of this term in quantum physics is (at least) not universally agreed upon; see, e.g., [13] for a discussion of this. However, whatever the choice of definition, there is at least some consensus in that the three models listed above should be classified as integrable.

Exactly solvable models are rare, and exact solutions are beyond reach in most cases. The same is true for integrable models. Models which are not integrable are said to be *non-integrable*. One key difference between integrable and non-integrable models is that integrable models have (sometimes infinitely many) additional conservation laws as compared to non-integrable ones [13]. As will become clear, this will be important later when we address non-equilibrium properties of such models.

[1] By *quantum field theory* we mean quantum models with an infinite number of degrees of freedom.

1.1.1 Luttinger model

We will almost exclusively consider the Luttinger model [68, 44, 48]. This model describes a one-dimensional system of interacting fermions on a line of length $L > 0$. In position space the fermions are described by fields $\psi_r^{(\dagger)}(x)$ for $x \in [-L/2, L/2]$ and $r = \pm$ satisfying the canonical anticommutation relations

$$\{\psi_r(x), \psi_{r'}^\dagger(x')\} = \delta_{r,r'}\delta(x-x'), \quad \{\psi_r(x), \psi_{r'}(x')\} = \{\psi_r^\dagger(x), \psi_{r'}^\dagger(x')\} = 0 \tag{1.1}$$

and antiperiodic boundary conditions. The fermions can either be right- or left-moving, which is indicated by the index r, with $r = +$ meaning right-moving and $r = -$ meaning left-moving; see Figure 1.1.

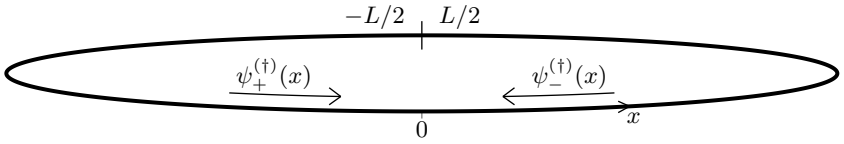


Figure 1.1: Illustration of a one-dimensional system of right- and left-moving fermions with (anti)periodic boundary conditions.

The dynamics is given by the Luttinger Hamiltonian[2] [48, 69]

$$\begin{aligned} H_g = &\sum_{r=\pm} \int_{-L/2}^{L/2} dx\, :\psi_r^\dagger(x)\left(-irv_F\partial_x\right)\psi_r(x): \\ &+ \sum_{r,r'=\pm} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx'\, g_{r,r'}(x-x') :\psi_r^\dagger(x)\psi_r(x)::\psi_{r'}^\dagger(x')\psi_{r'}(x'):, \end{aligned} \tag{1.2}$$

where $:\cdots:$ indicates *Wick* (*normal*) *ordering* (this is made precise in Chapter 2 only for certain operators in momentum space), $v_F > 0$ is the *Fermi velocity*, and the subscript g refers to the interactions

$$g_{r,r'}(x) = \delta_{r,-r'} g_2 V_2(x) + \delta_{r,r'} g_4 V_4(x), \tag{1.3}$$

where g_i are coupling constants and $V_i(x)$ are interaction potentials ($i = 2, 4$). The last term in (1.2) describes fermions interacting through density-density interactions, with the densities $:\psi_r^\dagger(x)\psi_r(x):$ satisfying periodic boundary conditions. The interaction between fermions moving in the same direction is $g_4 V_4(x)$ and that between fermions moving in opposite directions is $g_2 V_2(x)$. These must satisfy certain conditions (see (2.7)), in part, to ensure that the model describes a stable system [48].

The XY and XXZ models mentioned above describe N spins on a one-dimensional lattice with exchange interaction between nearest neighbor spins, where the *lattice spacing* $a_0 > 0$ satisfies $Na_0 = L$. As such, these models are formulated on a discrete space.

[2] The units are such that $\hbar = 1$.

The Luttinger model, on the other hand, is a continuum model since it is formulated on the circle $[-L/2, L/2]$ with (anti)periodic boundary conditions. As will be explained, the XXZ model can be mapped to a system of interacting fermions, while the XX model corresponds to non-interacting fermions. Then, in a certain regime, the Luttinger model can be seen as an approximation of the XXZ model after linearizing the dispersion relations and passing in the *continuum limit* $a_0 \to 0^+$, i.e., the limit where the lattice spacing tends to zero.

1.1.2 Bosonization

As mentioned, the Luttinger model is exactly solvable by *bosonization*, also known as the *boson-fermion correspondence*. This is a well-known tool in condensed matter physics which allows one to describe fermions (particles obeying Fermi-Dirac statistics) in terms of bosons (particles obeying Bose-Einstein statistics), and vice versa; see [19, 21, 22, 23, 32, 60, 61, 64, 65, 69, 71] for a selection of the many reviews and books in the condensed matter physics literature. In particular, we emphasize that bosonization is a collection of precise mathematical results which provide an exact mapping between fermions and bosons; see Section 2.3.

1.1.3 Thermodynamic limit

A reoccurring concept will be that of the *thermodynamic limit* $L \to \infty$. Our construction of the Luttinger model requires L to be finite. However, for expectation values of observables, such as the fermion two-point correlation functions

$$\langle \Psi_g | \psi_r^\dagger(x) \psi_{r'}(x') | \Psi_g \rangle_L \tag{1.4}$$

with respect to the ground state $|\Psi_g\rangle$ of H_g,[3] it is possible to pass in the thermodynamic limit once they are computed for L finite. For instance, the corresponding fermion two-point correlation functions are

$$\langle \Psi_g | \psi_r^\dagger(x) \psi_{r'}(x') | \Psi_g \rangle = \lim_{L\to\infty} \langle \Psi_g | \psi_r^\dagger(x) \psi_{r'}(x') | \Psi_g \rangle_L. \tag{1.5}$$

This extends the model to the real line $\mathbb{R}$, but only on the level of expectation values.

1.2 Non-equilibrium problems

Non-equilibrium physics, in particular for quantum systems, is less developed and generally considered more difficult than the corresponding equilibrium problems. Among the many tools in the literature to address such problems one finds, for instance, the Keldysh formalism, where the main object of study are non-equilibrium Green's functions (see, e.g., [54] for a review), and C^*-dynamical systems (see, e.g., [25, 6]). Recently,

[3] The subscript L in (1.4) indicates that it is computed for L finite.

there has been renewed interest in non-equilibrium quantum physics, spurred by experimental advances on ultracold atomic gases [8]. In such experiments, atoms at very cold temperatures are trapped in optical lattices and used to simulate isolated many-body systems [9]. This has opened up the possibility to experimentally study non-equilibrium properties of isolated quantum systems. On the theoretical side, such systems have been studied out of equilibrium both analytically and numerically, but in particular for interacting systems, such as the XXZ model, there are problems which are still not fully understood [18, 53, 9, 56]. In this thesis we use the exact solution of the Luttinger model to analytically study the evolution in time t of a system of interacting fermions out of equilibrium.

1.2.1 Long time limit

One interesting question is if a system approaches a final steady state in the *long time limit* $t \to \infty$. However, this cannot happen if one considers the full isolated system. The reason is that the time evolution is unitary and thus no information can possibly be lost.[4] However, the same is not true for a subsystem on the interval $[-\ell, \ell]$ with ℓ much smaller than L. Indeed, if boundary effects are not of interest, and in many cases the boundaries are modeled not based on physical considerations but rather mathematical simplicity, then studying such a subsystem would cover the essential physics. This approach corresponds to passing to the thermodynamic limit $L \to \infty$, which removes any boundary effects, and considering local observables, i.e., observables that depend on position, such as the two-point correlation functions in (1.5).

The above explains the importance of the order of the thermodynamic and long time limits, i.e., first $L \to \infty$ and then $t \to \infty$, to obtain information about the final state. Hence, in what follows, the *final state* should be interpreted as the state (of a finite subsystem) obtained after passing to the thermodynamic limit and only then passing in the long time limit.

1.2.2 Domain wall initial state

One way to study non-equilibrium properties of isolated quantum systems is to prepare a system in some initial state and let it evolve in time. One example is a so-called domain wall initial state, in this case a state with a higher density of particles in one half of the system compared to the other. For instance, let $\mu_0 > 0$ be a large chemical potential corresponding to the ground state and consider a chemical potential profile $\mu(x)$ in the form of a smooth domain wall so that $\mu(x) + \mu_0$ interpolates between μ_L on the left side and μ_R on the right; see Figure 1.2. The chemical potential difference is $\mu = \mu_L - \mu_R > 0$, which corresponds to an excess of particles on the left side compared to the right. We assume that μ is much smaller than μ_0.

[4] Assuming the evolution of the system is non-dissipative.

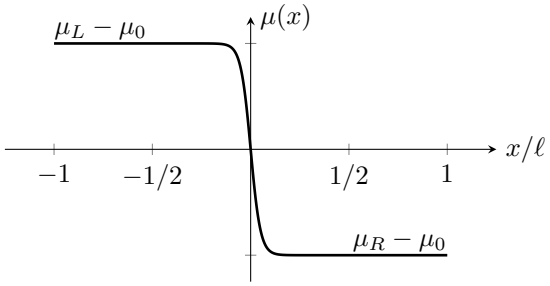


Figure 1.2: Chemical potential profile.

1.2.3 Evolution in a system of non-interacting fermions

Consider a system of non-interacting fermions in a domain wall initial state. This state is denoted $|\Psi_{0,\mu}\rangle$ and is constructed as the ground state of the non-interacting Hamiltonian H_0 coupled to the chemical potential $\mu(x)$ in Figure 1.2 (see Section 4.4 for details). As already mentioned, we will consider expectation values of local observables. Two such observables of interest are the total density

$$R(x,t) = \langle \Psi_{0,\mu} | e^{iH_0 t} \big(:\psi_+^\dagger(x)\psi_+(x): + :\psi_-^\dagger(x)\psi_-(x): \big) e^{-iH_0 t} | \Psi_{0,\mu} \rangle \tag{1.6}$$

and the corresponding current[5]

$$I(x,t) = v_F \langle \Psi_{0,\mu} | e^{iH_0 t} \big(:\psi_+^\dagger(x)\psi_+(x): - :\psi_-^\dagger(x)\psi_-(x): \big) e^{-iH_0 t} | \Psi_{0,\mu} \rangle. \tag{1.7}$$

These are plotted in Figure 1.3 starting from a domain wall initial state. Note that the time evolution is determined by H_0.

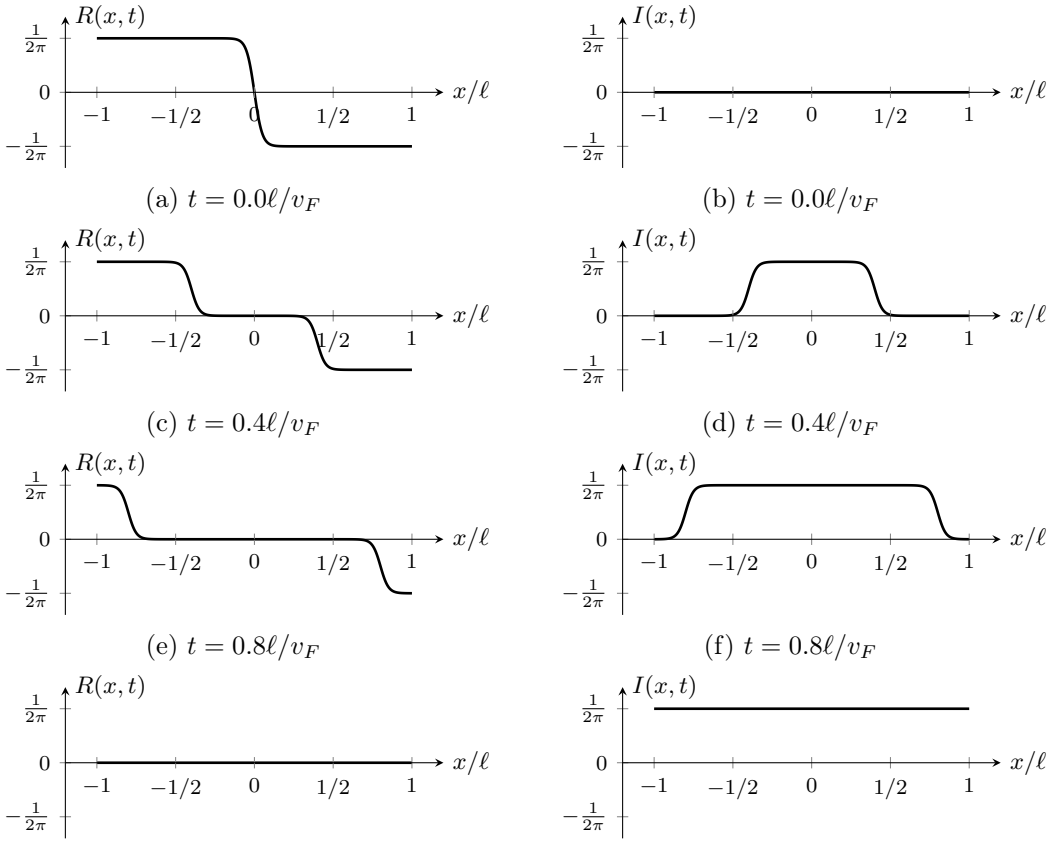
From Figure 1.3 we see that there is a region with vanishing density developing between two fronts moving to the right and left with velocity v_F and $-v_F$, respectively. The current in the same region is non-vanishing and tends to a constant non-zero value I as $t \to \infty$. Hence, starting from a domain wall initial state, the system reaches a current-carrying final state in the long time limit. The electrical conductance, as defined by Landauer [35, 2], in this final state is

$$G = \frac{I}{\mu_L - \mu_R} = \frac{e^2}{2\pi\hbar}, \tag{1.8}$$

where e is the elementary charge.

How the above picture changes when we consider the interacting fermions in the Luttinger model is the topic of Section 4.4. Of particular interest is the question of *universality* of the electrical conductance G in the final state: does it change in the presence of interactions, or does it remain the same, i.e., is it universal?

[5]The total density and the current satisfy the continuity equation $\partial_t R(x,t) + \partial_x I(x,t) = 0$.



(a) $t = 0.0\ell/v_F$ (b) $t = 0.0\ell/v_F$

(c) $t = 0.4\ell/v_F$ (d) $t = 0.4\ell/v_F$

(e) $t = 0.8\ell/v_F$ (f) $t = 0.8\ell/v_F$

(g) $t = 1.2\ell/v_F$ (h) $t = 1.2\ell/v_F$

Figure 1.3: Total density and current for non-interacting fermions starting from a domain wall initial state with $\mu = 1$, $\ell = 200$, and $v_F = 1$.

1.3 Notation

The units are such that $\hbar = k_B = e = 1$. Table 1.1 contains a list of reserved variables, and sums over these variables range over all allowed values, unless specified otherwise.

Variables	Sets
r, r'	$\{+, -\}$
x, x'	$[-L/2, L/2]$ or $\mathbb{R}$
k, k'	$(2\pi/L)(\mathbb{Z} + 1/2)$ or $\mathbb{R}$
p, p'	$(2\pi/L)\mathbb{Z}$ or $\mathbb{R}$

Table 1.1: Variables and their corresponding sets. The cases where the sets are $\mathbb{R}$ correspond to the thermodynamic limit.

1.4 Organization of the thesis

This thesis is based on two published scientific papers:

- Paper A [37] – In this paper we study a system of interacting fermions described by the Luttinger model coupled to phonons where the interactions are *local*, i.e., delta-like, and we also give a self-contained account of bosonization.
- Paper B [36] – In this paper we study the evolution of the Luttinger model with *non-local interactions*, i.e., interactions with finite range, starting from a domain wall initial state.

The organization of the thesis is as follows. Part I contains introduction and background material. The construction of the Luttinger model and its solution by bosonization are reviewed in Chapter 2, which is the most technical chapter. The fermion-phonon model in Paper A is briefly discussed in Chapter 3. An overview of non-equilibrium properties of exactly solvable models is given in Chapter 4, including a discussion of the Luttinger model out of equilibrium in Paper B. Part II contains the scientific papers.

Chapter 2

Luttinger model

The earliest version of the Luttinger model is due to Tomonaga [68], who obtained it as an approximation of a fermion model with non-linear dispersion relations; see, e.g., [24] and Chapter 4 in [39]. For this reason it is sometimes referred to as the Tomonaga-Luttinger model. It was later rediscovered and formulated as a model of interacting fermions by Luttinger [44], inspired by the work of Thirring [67], with the Tomonaga approximations made a priori, and the first correct solution is due to Mattis and Lieb [48] using bosonization. We recall that the Luttinger model is an example of an exactly solvable model in that all eigenvalues and eigenstates can be computed analytically together with all fermion correlation functions. The model, as formulated below, is in a slightly more general form than that in [48], using notation commonly referred to as *g-ology* (see, e.g., [69]), but the treatment closely follows the solution in [48].

As an important remark, we note that, besides its importance in the historical development of quantum field theory, the Luttinger model is considered a prototype for the equilibrium properties of a larger class of one-dimensional systems referred to as (Tomonaga-)Luttinger liquids [23].

To avoid difficulties with formulating a continuum model in position space,[1] we formulate the Luttinger model in Fourier space (the Fourier transforms are given in Appendix B.3 in Paper A). We thus consider a many-body system of spinless and massless fermions in one spatial dimension with energies rv_Fk, where $v_F > 0$ is the *Fermi velocity*, $k \in (2\pi/L)(\mathbb{Z} + 1/2)$ are momenta, and $r = \pm$ denotes right- ($r = +$) and left- ($r = -$) moving fermions. The energies are measured with respect to a ground state taken as the filled Dirac sea, which corresponds to the large chemical potential $\mu_0 > 0$ in Section 1.2.2, and the momenta are measured relative to the Fermi points $\pm k_F$, where $k_F = \mu_0/v_F$ is the *Fermi momentum*. The corresponding case where the momenta are not measured relative to the Fermi points is illustrated in Figure 2.1, but, as will be clear from Section 2.2, by suitable shifts in the momenta, one can avoid any reference to k_F and μ_0 in the formulation of the Luttinger model.

[1] For instance, field operators have to be interpreted as operator-valued distributions (see, e.g., [12]), and products of field operators in the same point must be defined so to avoid problems with divergencies using methods such as point-splitting (see, e.g., [62, 17]).

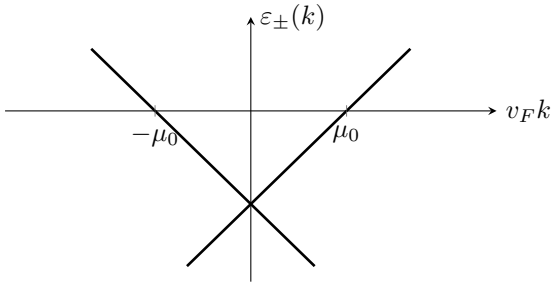


Figure 2.1: Linear dispersion relations $\varepsilon_r(k) = rv_F k - \mu_0$ for right- ($r = +$) and left- ($r = -$) moving fermions. The ground state is taken as the filled Dirac sea, i.e., the state where all fermion states with momenta k below $k_F = \mu_0/v_F$ for $r = +$ and above $-k_F$ for $r = -$ are filled.

2.1 Formulation of the model

Let $\hat{\psi}_r^{(\dagger)}(k)$ for $k \in (2\pi/L)(\mathbb{Z} + 1/2)$ and $r = \pm$ be fermion creation and annihilation operators satisfying[2]

$$\big\{\hat{\psi}_r(k), \hat{\psi}_{r'}^\dagger(k')\big\} = \frac{L}{2\pi}\delta_{r,r'}\delta_{k,k'}, \quad \big\{\hat{\psi}_r(k), \hat{\psi}_{r'}(k')\big\} = \big\{\hat{\psi}_r^\dagger(k), \hat{\psi}_{r'}^\dagger(k')\big\} = 0. \tag{2.1}$$

The non-interacting Hamiltonian in Fourier space is then

$$H_0 = \sum_r \sum_k \frac{2\pi}{L} r v_F k :\hat{\psi}_r^\dagger(k)\hat{\psi}_r(k):, \tag{2.2}$$

where $:\cdots:$ indicates Wick (normal) ordering and can be defined as

$$:A: = A - \langle\Psi_0|A|\Psi_0\rangle \tag{2.3}$$

for operators $A = \hat{\psi}_r^\dagger(k)\hat{\psi}_{r'}(k')$ with $\langle\Psi_0|A|\Psi_0\rangle$ denoting the expectation value of A with respect to the ground state $|\Psi_0\rangle$ of H_0. This ground state is precisely the vacuum defined by

$$\hat{\psi}_r(rk)|\Psi_0\rangle = \hat{\psi}_r^\dagger(-rk)|\Psi_0\rangle = 0 \quad \forall k > 0. \tag{2.4}$$

The fermion Fock space $\mathcal{F}$, which can be thought of as the space of all accessible states in the model, can be constructed from $|\Psi_0\rangle$ using the fermion operators $\hat{\psi}_r^{(\dagger)}(k)$ and is fully determined by (2.1) and (2.4) (see Section II.A in Paper A for details).

The Fourier transforms of the interaction potentials $V_i(x)$ in (1.3) are denoted $\hat{V}_i(p) = \int_{-L/2}^{L/2} dx\, V_i(x)e^{-ipx}$ and are required to satisfy certain conditions (see (2.7)). The Lut-

[2]These are related to the fermion fields $\psi_r^{(\dagger)}(x)$ in (1.2) by the inverse Fourier transform $\psi_r(x) = (2\pi)^{-1/2}\sum_k (2\pi/L)\hat{\psi}_r(k)e^{ikx}$ with $\psi_r^\dagger(x) = \psi_r(x)^\dagger$; see also Section 3 in Paper B.

tinger Hamiltonian in Fourier space can then be written as

$$\begin{aligned} H_g &= H_0 + g_2 H_2 + g_4 H_4, \\ H_2 &= \sum_r \frac{1}{L}\hat{V}_2(0) Q_r Q_{-r} + \sum_r \sum_{p\neq 0} \frac{1}{L}\hat{V}_2(p)\rho_r(p)\rho_{-r}(-p), \\ H_4 &= \sum_r \frac{1}{L}\hat{V}_4(0) Q_r^2 + \sum_r \sum_{p\neq 0} \frac{1}{L}\hat{V}_4(p)\rho_r(p)\rho_r(-p), \end{aligned} \tag{2.5}$$

using fermion *densities* $\rho_r(p)$ and *zero modes* $Q_r = \rho_r(0)$ defined by

$$\rho_r(p) = \sum_k \frac{2\pi}{L} {:}\hat{\psi}_r^\dagger(k)\hat{\psi}_r(k+p){:} \tag{2.6}$$

for $p \in (2\pi/L)\mathbb{Z}$. We recall (cf. Section 1.1.1) that $g_4\hat{V}_4(p)$ is the density-density interaction between fermions moving in the same direction and $g_2\hat{V}_2(p)$ is that between fermions moving in opposite directions. The conditions on the interactions are (see Paper B)

$$\hat{V}_i(p) = \hat{V}_i(-p) \quad \text{for } i = 2, 4, \tag{2.7a}$$

$$\left|g_2\hat{V}_2(p)\right| < \pi v_F + g_4\hat{V}_4(p) \quad \forall p, \tag{2.7b}$$

$$\sum_{p>0} \frac{p\big(g_2\hat{V}_2(p)\big)^2}{\pi v_F\big(\pi v_F + g_4\hat{V}_4(p)\big)} < \infty. \tag{2.7c}$$

The condition in (2.7a) corresponds to the interaction potentials $V_i(x)$ being real. The condition in (2.7b) is a stability condition: the eigenvalues of the model (see (2.17) and (2.29)) would not be real otherwise. The condition in (2.7c) ensures that there exists a unitary operator relating the ground state $|\Psi_0\rangle$ of H_0 to the ground state $|\Psi_g\rangle$ of H_g (see Section IV.E in Paper A for details). As discussed in Section 2.4, the latter is related to questions about the non-trivial limit where the interactions are local.

The densities $\rho_r(p)$ can be shown to satisfy (see Section II.B in Paper A)

$$\big[\rho_r(p), \rho_{r'}(-p')\big] = r\delta_{r,r'}\frac{Lp}{2\pi}\delta_{p,p'}, \tag{2.8a}$$

$$\rho_+(p)|\Psi_0\rangle = \rho_-(-p)|\Psi_0\rangle = 0 \quad \forall p \geq 0. \tag{2.8b}$$

The zero modes Q_r are Hermitian and are closely related to the *Klein factors* R_r (see Sections II.C and II.D in Paper A for an explicit construction) which are unitary operators satisfying

$$[Q_r, R_{r'}] = r\delta_{r,r'}R_r, \quad R_\pm R_\mp = -R_\mp R_\pm, \tag{2.9a}$$

$$\langle\Psi_0|R_+^{q_+}R_-^{-q_-}|\Psi_0\rangle = \delta_{q_+,0}\delta_{q_-,0} \quad \forall q_+, q_- \in \mathbb{Z}, \tag{2.9b}$$

and where both Q_r and R_r commute with $\rho_r(p)$ for all $p \neq 0$. The commutation relations in (2.8a) indicates that the $\rho_r(p)$ behave as bosons, and together with the Klein factors

these make up the bosonic side of the bosonization mapping between fermions and bosons. This is further explained in Section 2.3.

The physical interpretation of the Q_r are as charge operators and of the R_r as charge-changing operators. The need for introducing the Klein factors is clear if one tries to describe the fermion Fock space $\mathcal{F}$ using only the $\rho_r(p)$: since these operators always preserve the charge of a given state, the Klein factors are necessary to describe states that have non-zero charges.

2.2 Relation to spin chains

Other exactly solvable one-dimensional models of interest, also for non-equilibrium quantum physics as discussed in Section 4.3, are the XY and XXZ models, which we recall are exactly solvable by Bethe ansatz. Below we state these models and give a rudimentary discussion of their relation to the Luttinger model.

Consider a chain of N spins with exchange interaction between nearest neighbors, i.e., a one-dimensional lattice where only spins on neighboring lattice sites interact. Let $J > 0$ be the strength of the exchange interaction and let S_i^x, S_i^y, and S_i^z be spin operators represented by the usual Pauli spin matrices [38]:

$$S_i^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_i^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_i^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.10}$$

We consider only the case of periodic boundary conditions, i.e., we let $S_{N+1}^x = S_1^x$, and similarly for S_i^y and S_i^z.

- XY model:

$$H_{XY} = -J \sum_{i=1}^{N} \left((1+\gamma) S_i^x S_{i+1}^x + (1-\gamma) S_i^y S_{i+1}^y \right), \tag{2.11}$$

  where $-1 < \gamma < 1$ is an anisotropy parameter. Setting $\gamma = 0$ gives the so-called XX model.

- XXZ model:

$$H_{XXZ} = -J \sum_{i=1}^{N} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right), \tag{2.12}$$

  where $\Delta = J_z/J$ denotes the ratio between the strength J_z of the exchange interaction in the z-direction and J in the x- and y-directions. This model is obtained by adding exchange interaction between the spin operators S_i^z to the XX model.

Using a so-called Jordan-Wigner transformation (see, e.g., [38, 43]), the XX model Hamiltonian can be mapped to a Hamiltonian which is quadratic in fermion creation and annihilation operators, and similarly for the XXZ model Hamiltonian. In a suitable regime, these Hamiltonians can be approximated by H_0 and H_g, respectively. We briefly outline this procedure.

The Jordan-Wigner transformation can be expressed as [38]

$$S_j^+ = c_j^\dagger \exp\left(i\pi \sum_{l=1}^{j-1} c_l^\dagger c_l\right), \quad S_j^- = \exp\left(-i\pi \sum_{l=1}^{j-1} c_l^\dagger c_l\right) c_j, \quad S_j^z = c_j^\dagger c_j - \frac{1}{2}, \tag{2.13}$$

where $S_j^\pm = S_j^x \pm i S_j^y$ and $c_j^{(\dagger)}$ are fermion creation and annihilation operators satisfying

$$\left\{c_i, c_j^\dagger\right\} = \delta_{i,j}, \quad \left\{c_i, c_j\right\} = \left\{c_i^\dagger, c_j^\dagger\right\} = 0. \tag{2.14}$$

Using this transformation, and following a Fourier transform, the Hamiltonian for the XXZ model can be written in a form similar to the Luttinger model but where the non-interacting Hamiltonian has the dispersion relation [43]

$$\varepsilon(k) = -J\cos(a_0 k) \tag{2.15}$$

for momenta k, where $a_0 > 0$ is the lattice spacing. We will consider $\varepsilon(k) - \mu$ where μ is a chemical potential introduced to take into account the filled particle states and is fixed by the condition $\varepsilon(k) = \mu$ at the Fermi points $k = rk_F$ for $r = \pm$. (Note that μ here is not the same as in Sections 1.2.2 and 1.2.3.) The Fermi momentum $k_F = \nu\pi/a_0$ is determined by the so-called filling factor ν, i.e., the ratio between the number of filled particle states and the number of accessible states, which means that $\mu = -J\cos(\nu\pi)$. For instance, $\nu = 1/2$ at half filling. It is important to note that, on the lattice, there is an intrinsic *ultraviolet cutoff*[3] on the momenta given by $-\pi/a_0 < k \leq \pi/a_0$ (i.e., the momenta are in the first Brillouin zone).

We take as ground state the filled Dirac sea, i.e., the state where particle states with negative energy, i.e., all k such that $\varepsilon(k) - \mu < 0$, are filled. This is thus determined by the filling factor. The system is assumed to be *close to* but *not at* half filling, i.e., $\nu \neq 1/2$ but close to this value.[4] In addition, we assume that the system is in the low-temperature regime, which has the effect that only processes with small changes in momenta are likely. In this regime, it is motivated to linearize $\varepsilon(k) - \mu$ around the Fermi points $k = rk_F$, since only small changes around these points are of interest. This gives the approximation

$$\varepsilon(k) - \mu \approx rv_F(k - rk_F) = rv_F k - \mu_0 = \varepsilon_r(k) \tag{2.16}$$

for k close to rk_F, where have identified $v_F = Ja_0\sin(\nu\pi)$ and $\mu_0 = v_F k_F = \nu\pi v_F/a_0$ as in Figure 2.1.[5] It is now possible to shift $k \to k + rk_F$ for k close to rk_F, meaning that the momenta are measured relative to the Fermi points, which gives the dispersion relations $\varepsilon_r(k) = rv_F k$ as in the formulation of the Luttinger model in Section 2.1.

As a last step the cutoff is removed so to extend the allowed momenta to all k. This corresponds to the continuum limit $a_0 \to 0^+$, i.e., the limit where the lattice spacing tends

[3] An upper limit on the absolute value of the momenta.

[4] See Footnote 7 on page 16.

[5] There can also be a contribution to μ from the interaction part of the Hamiltonian, i.e., the part containing Δ, but we do not consider this here.

to zero.[6] In this limit, the linearized XXZ model, assuming the system is close to but not at half filling,[7] can be approximated by the Luttinger model with Δ proportional to the coupling constants $g_2 = g_4 = \lambda$ and local interaction potentials $V_2(x) = V_4(x) = \delta(x)$ (see, e.g., [43]). For non-zero Δ, it is necessary to introduce a new implicit cutoff on the momenta in the interactions to ensure that the model is well-defined (cf. Section 2.4); this can be referred to as the Luttinger model being a *partial continuum limit* of the XXZ model. In particular, setting $\Delta = 0$ gives the Hamiltonian H_0 for non-interacting fermions. The Luttinger model formulated in Section 2.1 is a generalization of the above limiting model approximating the XXZ model.

2.3 Exact solution by bosonization

In this section we describe how the Luttinger model can be solved using bosonization. The presentation is similar to [48] and Paper B. A self-contain account of bosonization including complete proofs of the identities below is given in Paper A; see also [16].

The following parameters will be important [69]: the *renormalized Fermi velocity*

$$v_g(p) = v_F \sqrt{\left(1 + g_4 \hat{V}_4(p)/\pi v_F\right)^2 - \left(g_2 \hat{V}_2(p)/\pi v_F\right)^2} \tag{2.17}$$

and the *Luttinger parameter*[8]

$$K_g(p) = \sqrt{\frac{1 + g_4 \hat{V}_4(p)/\pi v_F - g_2 \hat{V}_2(p)/\pi v_F}{1 + g_4 \hat{V}_4(p)/\pi v_F + g_2 \hat{V}_2(p)/\pi v_F}}. \tag{2.18}$$

It follows that the interactions in (2.5) can be written as

$$1 + \frac{g_4 \hat{V}_4(p)}{\pi v_F} = \frac{v_g(p)}{v_F} \frac{K_g(p)^{-1} + K_g(p)}{2}, \quad \frac{g_2 \hat{V}_2(p)}{\pi v_F} = \frac{v_g(p)}{v_F} \frac{K_g(p)^{-1} - K_g(p)}{2}, \tag{2.19}$$

and we also introduce $v_g = v_g(0)$ and $K_g = K_g(0)$.

The solution presented here can also be extended to models where, e.g., the fermions are coupled to phonons; this is the topic of Paper A and is reviewed in Chapter 3.

2.3.1 Bosonization

Bosonization, as used here, refers to the following mathematical results:[9]

[6] It is implicitly assumed that J scales as $1/a_0$, so that v_F remains finite. However, since k_F also has the same scaling, this means that $\mu_0 = v_F k_F$ also scales as $1/a_0$, and thus both k_F and μ_0 become infinite in the continuum limit.

[7] The reason for requiring that $\nu \neq 1/2$ when considering the XXZ model is that precisely at half filling other interactions, which we do not want to include, also become important; cf., e.g., [49].

[8] This is a slight abuse of terminology since $K_g(p)$ is a function of momenta for non-local interactions, but it becomes independent of momenta in the limit where the interactions are local.

[9] These identities hold true as quadratic forms on the domain of all finite linear combinations of the eigenstates $|\Psi_{0,\mathbf{m}}\rangle$ of H_0 in (2.27) (see, e.g., Paper A).

(a) *From fermions to bosons*:

$$H_0 = \sum_r \frac{\pi}{L} v_F Q_r^2 + \sum_r \sum_{p>0} \frac{2\pi}{L} v_F \rho_r(-rp)\rho_r(rp) \tag{2.20}$$

together with the non-trivial commutation relations in (2.8a).

(b) *From bosons to fermions*:

$$\psi_r(x;\varepsilon) = L^{-1/2} e^{i\pi r x Q_r/L} R_r^{-r} e^{i\pi r x Q_r/L} \times \exp\left(\sum_{p>0} \frac{\pi}{Lp} e^{-\varepsilon p}\right) \exp\left(r \sum_{p\neq 0} \frac{2\pi}{Lp} \rho_r(p) e^{ipx - \varepsilon|p|/2}\right) \tag{2.21}$$

for $\varepsilon > 0$ with $\psi_r^\dagger(x;\varepsilon) = \psi_r(x;\varepsilon)^\dagger$ define *regularized fields* $\psi_r^{(\dagger)}(x;\varepsilon)$ that converge to the fermion fields $\psi_r^{(\dagger)}(x)$ as $\varepsilon \to 0^+$ in a certain distributional sense specified in Lemma 5.1 in Paper B.

The identity in (2.20) is referred to as *Kronig's identity* and allows one to express the Luttinger Hamiltonian H_g in a form that is quadratic in the densities $\rho_r(p)$, which we recall satisfy bosonic commutation relations. On the other hand, the quantity in (2.21) is a so-called *vertex operator* (see, e.g., [29]) and allows one to express fermion fields using the densities [42, 47, 63]. As mentioned, proofs of these results can be found in Paper A.

2.3.2 Bogoliubov transformation

The bosonized Luttinger Hamiltonian can be diagonalized by a *Bogoliubov transformation* implemented by a unitary operator e^{iS_g} defined by

$$S_g = i \sum_{p\neq 0} \frac{2\pi}{L} \frac{\varphi_g(p)}{p} \rho_+(-p)\rho_-(p), \qquad \tanh 2\varphi_g(p) = -\frac{g_2 \hat{V}_2(p)}{\pi v_F + g_4 \hat{V}_4(p)} \quad \forall p \neq 0, \tag{2.22}$$

where $\varphi_g(-p) = \varphi_g(p) \in \mathbb{R}$ for all $p \neq 0$, so that

$$e^{iS_g} \rho_r(p) e^{-iS_g} = \rho_r(p)\cosh\varphi_g(p) + \rho_{-r}(p)\sinh\varphi_g(p) \quad \forall p \neq 0 \tag{2.23}$$

and

$$e^{iS_g} H_g e^{-iS_g} = H_0 - T_g + D_g + W_g \tag{2.24}$$

with

$$H_0 - T_g = \sum_r \frac{\pi}{L} v_g \left(\frac{K_g^{-1} + K_g}{2} Q_r^2 + \frac{K_g^{-1} - K_g}{2} Q_r Q_{-r}\right), \tag{2.25a}$$

$$D_g = \sum_r \sum_{p>0} \frac{2\pi}{L} v_g(p) \rho_r(-rp)\rho_r(rp), \tag{2.25b}$$

$$W_g = -\sum_{p>0} (v_F - v_g(p))\, p. \tag{2.25c}$$

The operator $H_0 - T_g$ contains the zero modes and D_g is the diagonalized Hamiltonian for the non-zero modes. The c-number W_g is the ground state energy, and the corresponding ground state of H_g is constructed as $|\Psi_g\rangle = e^{-iS_g}|\Psi_0\rangle$. We also note the relation

$$K_g(p) = e^{2\varphi_g(p)}, \tag{2.26}$$

which shows that the Luttinger parameter is directly related to the transformation diagonalizing H_g.

2.3.3 Exact eigenstates and eigenvalues

All eigenvalues and eigenstates of H_g can be constructed as follows. First, the eigenstates of H_0 are

$$|\Psi_{0,\mathbf{m}}\rangle = \prod_r \left(\prod_{p>0} \left(ir\sqrt{2\pi/L|p|} \right)^{m(rp)} \frac{\rho_r(-rp)^{m(rp)}}{\sqrt{m(rp)!}} \right) R_+^{q_+} R_-^{-q_-} |\Psi_0\rangle \tag{2.27}$$

for $\mathbf{m} = \{(m(p))_{p\neq 0}, q_+, q_-\}$ with $m(p) \in \mathbb{N}$ and $q_+, q_- \in \mathbb{Z}$, where at most finitely many of the $m(p)$ are non-zero. Second, the eigenstates of H_g can be expressed as

$$|\Psi_{g,\mathbf{m}}\rangle = e^{-iS_g}|\Psi_{0,\mathbf{m}}\rangle \tag{2.28}$$

with the corresponding eigenvalues

$$\mathcal{E}_{g,\mathbf{m}} = \sum_r \frac{\pi}{L} v_g \left(\frac{K_g^{-1} + K_g}{2} q_r^2 + \frac{K_g^{-1} - K_g}{2} q_r q_{-r} \right) + \sum_{p\neq 0} v_g(p)|p|m(p) + W_g. \tag{2.29}$$

2.3.4 Equilibrium fermion correlation functions

All fermion correlation functions can be computed using the regularized fields in (2.21) and the tools given in Section III in Paper A. For instance, in the thermodynamic limit $L \to \infty$, the only non-zero two-point correlation function is [48]

$$\langle \Psi_g | \psi_r^\dagger(x)\psi_r(x')|\Psi_g\rangle = \frac{i}{2\pi r(x-x') + i0^+} \exp\left(\int_0^\infty dp \frac{\eta_g(p)}{p} (\cos p(x-x') - 1) \right) \tag{2.30}$$

with an interaction-dependent exponent[10] $\eta_g(p) = (K_g(p)^{-1} + K_g(p))/2 - 1$ expressed using the Luttinger parameter in (2.18). The behavior of correlation functions at long distances is modified by such exponents. Take (2.30) as an example: it decays as $O(|x - x'|^{-1-\eta_g})$ for large $|x-x'|$, where $\eta_g = \eta_g(0) = (K_g^{-1} + K_g)/2 - 1$ is a so-called anomalous exponent (see, e.g., Section 4.3 in [59]). For later reference, we note that the correlation function in (2.30) is translation invariant, i.e., it is invariant under shifts $x \to x + y$ and $x' \to x' + y$ for all $y \in \mathbb{R}$ since it only depends on the difference $x - x'$.

[10] This is as slight abuse of terminology as explained in Footnote 8 on page 16.

2.4 Local limit

The *local limit*, i.e., the limit where the interaction range becomes zero, is non-trivial. To see why, denote the interaction range by $a > 0$. To construct a model with local interactions, the parameter a can be seen as a regularization of the model, e.g., in the form of an ultraviolet cutoff on the momenta in the interactions given by $|p| \leq \pi/a$ [72]. As this regularization is removed, i.e., as $a \to 0^+$, divergencies are produced which must be removed by suitable renormalizations; this can be seen as making certain redefinitions of operators and quantities (see, e.g., [3, 52]).

For instance, since the ground state energy W_g diverges as $a \to 0^+$, it is necessary to use an additive renormalization for the Hamiltonian, i.e., we define

$$\tilde{H}_{g;ren} = \lim_{a\to 0^+} \left(e^{iS_g} H_g e^{-iS_g} - W_g \right). \tag{2.31}$$

The fermion fields $\psi_r^{(\dagger)}(x)$, or rather, to be more precise, the regularized fields $\psi_r^{(\dagger)}(x;\varepsilon)$ in (2.21), are more complicated since their correlation functions contain a multiplicative constant $Z_{a,\varepsilon}$ which vanishes as $a \to 0^+$. This requires a multiplicative renormalization, i.e., we define fields proportional to

$$\lim_{a\to 0^+} Z_{a,\varepsilon}^{-1} e^{iS_g} \psi_r^{(\dagger)}(x;\varepsilon) e^{-iS_g}. \tag{2.32}$$

These renormalizations are discussed in detail in Section IV.E in Paper A to which we refer for further discussion. However, as mentioned earlier, the need for renormalizations is related to divergencies produced by the unitary operator e^{iS_g} and questions about when the ground state $|\Psi_g\rangle$ exists in the fermion Fock space $\mathcal{F}$. We also note that the order in which the regularizations are removed, i.e., the order of the limits $\varepsilon \to 0^+$ and $a \to 0^+$, is non-trivial, as will be explained for the fermion-phonon model in Section 3.2.

Chapter 3

Fermions and phonons

There are various possible extensions of the Luttinger model which are still exactly solvable by bosonization. One such extension is to couple the fermions to acoustic phonons, which is the topic of Paper A. This extended model is known since before in the condensed matter physics literature [15, 20, 70].

3.1 Fermion-phonon model

We consider the Luttinger model, formulated as in Chapter 2, with the specific interactions $g_2V_2(x) = \lambda_a(x)/2$ and $g_4V_4(x) = 0$, where $a > 0$ denotes the interaction range and $\lambda_a(x)$ becomes a local interaction, i.e., $\lambda\delta(x)$, in the limit $a \to 0^+$. The fermions are coupled to acoustic phonons by an interaction $g_{P;a}(x)$, which similarly becomes local, i.e., $g_P\delta(x)$, in the limit $a \to 0^+$. In formulating the model the interaction range is kept finite, and thus the model is regularized as discussed in Section 2.4.

The (regularized) fermion-phonon Hamiltonian is

$$\begin{aligned} H = \int_{-L/2}^{L/2} dx \left(\sum_{r=\pm} :\psi_r^\dagger(x) \left(-irv_F\partial_x\right) \psi_r(x): + \frac{1}{2} :\Big(\Pi_P(x)^2 + v_P^2[\partial_x\Phi_P(x)]^2\Big): \right) \\ + \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \Bigg(\lambda_a(x-x') :\psi_+^\dagger(x)\psi_+(x)::\psi_-^\dagger(x')\psi_-(x'): \\ + g_{P;a}(x-x') \sum_{r=\pm} :\psi_r^\dagger(x)\psi_r(x):\partial_x\Phi_P(x') \Bigg), \end{aligned} \tag{3.1}$$

where $v_P > 0$ is the phonon velocity, which must satisfy $v_P < v_F$, and $\Phi_P(x)$ and $\Pi_P(x)$ are Hermitian boson fields satisfying the canonical commutation relations

$$\big[\Phi_P(x), \Pi_P(y)\big] = i\delta(x-y), \quad \big[\Phi_P(x), \Phi_P(y)\big] = \big[\Pi_P(x), \Pi_P(y)\big] = 0 \tag{3.2}$$

and commuting with all fermion fields. Similar to the condition in (2.7b), the interactions $\lambda_a(x)$ and $g_{P;a}(x)$ in the local limit have to satisfy

$$\lambda < 2\pi v_F, \quad 2(g_P/v_P)^2 < 2\pi v_F + \lambda, \tag{3.3}$$

without which the model would describe an unstable system. The condition corresponding to (2.7c) will, however, be violated in the local limit, which again indicates that this limit is non-trivial and requires renormalizations.

3.2 Solution of the model

The solution by bosonization of the fermion-phonon model is discussed in detail in Section IV in Paper A. Here we will only briefly describe the results of that paper.

As for the Luttinger model, the ground state $|\Psi\rangle$ of H is constructed from $|\Psi_0\rangle$ using a unitary operator. Similarly, all eigenvalues and eigenstates are computed analytically together with all fermion correlation functions. However, since the correlation functions are computed using (2.21), the order of the limits $\varepsilon \to 0^+$ and $a \to 0^+$ becomes important. As explained in Section 2.4, after additive renormalization of the Hamiltonian and multiplicative renormalization of the fermion fields, one can write down an effective renormalized fermion-phonon model where the limit $a \to 0^+$ is already taken. Hence, in correlation functions for this model the limit $\varepsilon \to 0^+$ is taken after the local limit. However, in the correlation functions computed for the regularized model in (3.1), the order of the limits is first $\varepsilon \to 0^+$ and then $a \to 0^+$. If the order of the limits cannot be exchanged then these correlation functions define different distributions.

In Paper A we show that the correlation functions of the effective renormalized model agree with the correlation functions of the regularized model in a certain sense as $L \to \infty$ (see Remark 4.9 in Paper A). However, if the limits above can be exchanged, then it would be possible to prove that the correlation functions are identical as distributions, which would give a (more) precise mathematical meaning to how these two models are related.

3.3 Differences in notation with Paper A

Table 3.1 contains a list of important notational differences. Moreover, the *local limit* is referred to as the *continuum limit* in Paper A.

Thesis	Paper A
$\rho_r(p)$	$\hat{J}_r(p)$
g_P	g
$\|\Psi_0\rangle$	Ω
$\|\Psi\rangle$	$\tilde{\Omega}$

Table 3.1: Important notational differences with Paper A.

Chapter 4

Non-equilibrium quantum physics

As discussed in the introduction, recent experimental advances on ultracold atomic gases have opened up the possibility to study isolated quantum systems out of equilibrium. We will restrict ourselves to such systems, but we begin by describing both open and isolated systems.

One important concept will be that of *universality*: a quantity is said to be universal if it does not depend on the precise microscopic details of the model, such as the interactions [59]. Below we will see an example of such a universal result. Moreover, we recall that two models are said to be in the same universality class if at a certain point, typically a so-called critical point, they have the same scaling behavior, such as the same exponents in their correlation functions [11, 59].

4.1 Open and isolated systems

We begin by distinguishing between open and isolated one-dimensional systems.

For an open system it is the couplings to reservoirs that are used to bring the system out of equilibrium. For instance, one can consider a set-up as in Figure 4.1 where the left reservoir have temperature T_L and chemical potential μ_L, and the right reservoir have T_R and μ_R, respectively. One essential question for such set-ups is how to model the couplings: for classical system this has been studied in, e.g., [66, 55], and for quantum systems one uses so-called Lindblad operators [40, 10] studied in, e.g., [28, 27, 5].

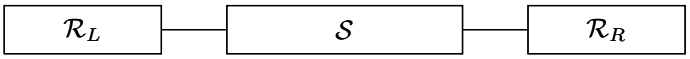


Figure 4.1: One-dimensional system $\mathcal{S}$ coupled to two reservoirs $\mathcal{R}_L$ and $\mathcal{R}_R$ on its left and right side, respectively.

For an isolated system there are no reservoirs and instead it has to be brought out of equilibrium by other means. One way to do so is to consider a quantum quench, i.e., suddenly change a parameter in the Hamiltonian defining the initial state of the system [59]. Examples include, at time $t = 0$, turning off a coupling to an external field, changing

an effective mass (equivalently, changing a gap in the dispersion relations), or changing interactions between particles. The interest in ultracold atomic gases comes from the experimental feasibility of such changes: system properties can be rapidly changed using *Feshbach resonance* (see, e.g., [9] for a review), thereby simulating a quantum quench.

Other possibilities include considering an isolated system on the interval $[-L/2, L/2]$ (ignoring boundary conditions) of length $L > 0$ initially in a partitioned state. This is a state where, e.g., the left side $[-L/2, 0)$ of the system has temperature T_L and chemical potential μ_L, and the right side $(0, L/2]$ has T_R and μ_R, respectively, with some impurity in the origin preventing the two sides from communicating; see, e.g., [7]. This can be modeled as the origin initially ($t < 0$) being totally reflecting, i.e., particles approaching the origin are reflected back to where they came, and at $t = 0$ changing the impurity (i.e., a quantum quench) to being totally transmitting. Examples of questions one usually asks concern how the system evolves with time, whether it reaches a final steady state, and in that case how to describe this final state.

4.2 Integrable and non-integrable systems

An important distinction is that between non-integrable and integrable systems.

Non-integrable systems show ergodic properties in that, evolving from some given initial state, the final state in the long time limit only depends on the total energy and the total number of particles. Such systems are said to exhibit *thermalization*, i.e., they tend to a state in thermal equilibrium and thus can be described by the usual canonical ensemble.

Integrable systems, on the other hand, show non-ergodic properties due to the (sometimes infinitely many) additional conservation laws which constrain the final state. Such systems can still tend to a steady state in the long time limit. In the literature this is referred to as *equilibration*, *relaxation*, or *stabilization*. However, such a state is not in thermal equilibrium, meaning that the systems do not exhibit thermalization. This can, for instance, be manifested by the memory of the initial state due to the additional conservation laws. Instead, it has been proposed that the final state, in general, can be described by a *generalized Gibbs ensemble* [57]. This generalization of the usual (Gibbs) canonical ensemble uses not only the Hamiltonian (corresponding to conservation of energy) but also includes additional conserved quantities. For instance, for the Luttinger model, the corresponding Gibbs measure would not simply be $e^{-\beta H_g}$, where $\beta = 1/T$ is the inverse temperature of the system. Instead, the proposal is to consider $e^{-\beta H_{eff}}$ where H_{eff} is a sum of (sometimes infinitely many) conserved quantities.

4.3 Spin chains out of equilibrium

Analytically, the XY model has been studied using so-called C^*-dynamical systems. Such a system is defined by the C^*-algebra generated by the spin operators S_i^x and S_i^y for all lattice sites i together with the dynamics corresponding to the Hamiltonian in (2.11) [25]. One important result concerns the long time limit $t \to \infty$, where, starting from a

partitioned initial state with temperatures T_L and T_R to the left and right, respectively, the final state factorizes into right- and left-moving fermions. In this final steady state, the right-moving fermions have temperature T_L, i.e., they come from the left, and the left-moving fermions have T_R, i.e., they come from the right [25, 6, 51, 50]. Similar results have been obtained for models in conformal field theory (see, e.g., [7] and references therein) for cases where there is also a chemical potential difference between the left and right sides.

As explained in Section 2.2, the XX model (and also the XY model) can be mapped to non-interacting fermions by a Jordan-Wigner transformation. Similarly, the XXZ model can be mapped to interacting fermions, which makes it an interesting example for studying systems of interacting fermions out of equilibrium. This has been studied analytically using the Bethe ansatz solution in [41], but it still seems unclear whether this approach allows for a full analytical understanding of the time evolution. In addition to such analytical studies there are several numerical results, such as, e.g., [4, 34, 58], where the evolution of spin chains and the resulting final state in the long time limit are studied. In [58] transport properties in the final state, such as the electrical conductance, are also investigated. We will return to this last topic for the Luttinger model below.

4.4 Luttinger model out of equilibrium

Due to its linear dispersion relations, the Luttinger model can be viewed as a less complicated example of an interacting model as compared to the XXZ model. It is, in a sense, more amenable to a full analytical understanding. Quantum quenches in the Luttinger model of the type where the interactions are turned on or off at $t = 0$ have been studied in [14, 26] for local interactions and in [46] for non-local interactions. In Paper B this is extended to more general quantum quenches of non-local interactions together with turning off the coupling to an external field producing a domain wall initial state.

4.4.1 Luttinger model in an external field

We consider the Luttinger model, formulated as in Chapter 2, with the specific interactions $g_2 = g_4 = \lambda$ and $V_2(x) = V_4(x) = V(x)$, where $V(x)$ is a non-local interaction potential, i.e., the interaction range is finite. The Hamiltonian is denoted by H_λ instead of H_g, the corresponding ground state is denoted by $|\Psi_\lambda\rangle$ instead of $|\Psi_g\rangle$, and similarly for all other quantities depending on the interactions. We recall that the fermion fields $\psi_r^{(\dagger)}(x)$ satisfy antiperiodic boundary conditions, and thus that the densities $:\psi_r^\dagger(x)\psi_r(x):$ satisfy periodic boundary conditions, and similarly for other quantities.

The evolution of the system is studied starting from a domain wall initial state produced by an external field

$$W(x) = -\frac{\tanh(x/\delta) - \tanh((2x+L)/2\delta) - \tanh((2x-L)/2\delta)}{2} \tag{4.1}$$

corresponding to the chemical potential profile $\mu(x) = \mu W(x)$ in Figure 1.2. We recall that $\mu > 0$, which means that the left chemical potential μ_L is larger than the right μ_R.

The last two terms in (4.1) ensures that $W(x)$ satisfies periodic boundary conditions but become irrelevant in the thermodynamic limit; see Figure 4.2. We also note that this field explicitly breaks translation invariance in the initial state, as will be clear from (4.5) and (4.6). The Luttinger model is coupled to $W(x)$ as defined by the Hamiltonian

$$H_{\lambda,\mu} = H_\lambda - \mu \sum_r \int_{-L/2}^{L/2} dx\, W(x) {:}\psi_r^\dagger(x)\psi_r(x){:}, \tag{4.2}$$

and we take the ground state $|\Psi_{\lambda,\mu}\rangle$ of $H_{\lambda,\mu}$ as our initial state. The system is evolved using the Hamiltonian $H_{\lambda'} = H_{\lambda',0}$ with no external field, where, in general, λ' is different from λ. As explained in the introduction, the idea is to study expectation values of local observables with respect to the state $|\Psi_{\lambda,\mu}^{\lambda'}(t)\rangle = e^{-iH_{\lambda'}t}|\Psi_{\lambda,\mu}\rangle$, restricted to the subsystem on the interval $[-\ell,\ell]$ with ℓ much smaller than L; see Figure 4.2.

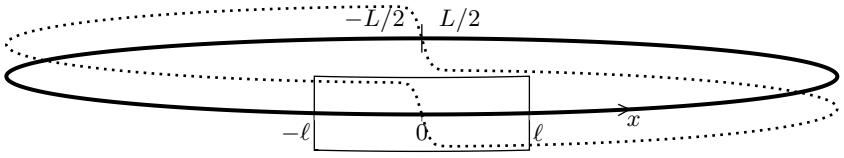


Figure 4.2: Illustration of the subsystem on the interval $[-\ell,\ell]$ and its relation to the system on the circle $[-L/2, L/2]$ with (anti)periodic boundary conditions for ℓ much smaller than L. The external field $W(x)$ in (4.1) is illustrated by the black dotted line.

In Paper B we show that the system reaches a final steady state in the long time limit, in so far expectation values of the total density, the current, and the fermion two-point correlation functions are concerned. Here we will only summarize some of these results.

4.4.2 Total density and current

The total density

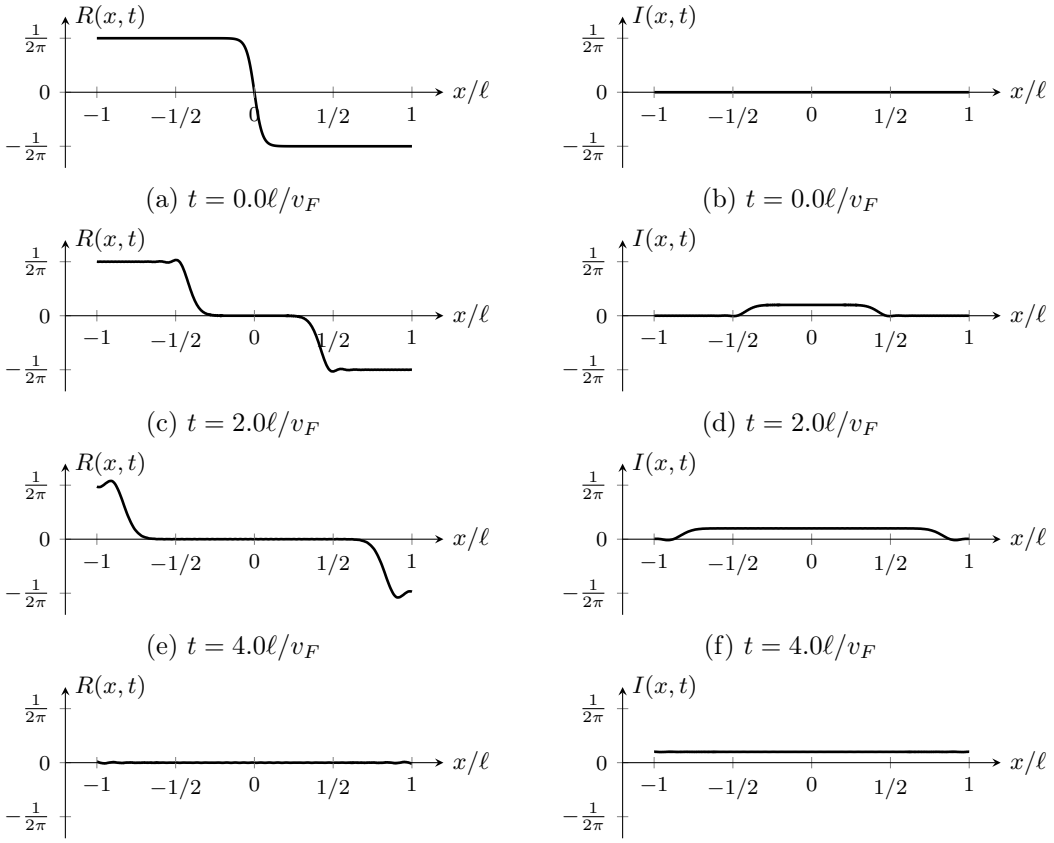
$$R(x,t) = \langle\Psi_{\lambda,\mu}^{\lambda'}(t)|\big({:}\psi_+^\dagger(x)\psi_+(x){:} + {:}\psi_-^\dagger(x)\psi_-(x){:}\big)|\Psi_{\lambda,\mu}^{\lambda'}(t)\rangle \tag{4.3}$$

and the corresponding current

$$I(x,t) = v_F\langle\Psi_{\lambda,\mu}^{\lambda'}(t)|\big({:}\psi_+^\dagger(x)\psi_+(x){:} - {:}\psi_-^\dagger(x)\psi_-(x){:}\big)|\Psi_{\lambda,\mu}^{\lambda'}(t)\rangle \tag{4.4}$$

are plotted in Figure 4.3 using the same interactions as in Figures 3 and 4 in Paper B. Similar to Figure 1.3, we see that there is a region with vanishing density developing between two fronts moving to the right and left. Since the interactions are non-local, the velocities depend on momenta, which means that the evolution is dispersive. This is manifested by the small oscillations traveling ahead of the fronts in Figure 4.3. However, the velocities of the fronts themselves are determined by the group velocity at $p = 0$ and can be shown to be v_g and $-v_g$, respectively.[1] As before, the current in the same region is non-vanishing and tends to a constant non-zero value I as $t \to \infty$.

[1]Recall that $v_g = v_g(0)$ where $v_g(p)$ is the renormalized Fermi velocity in (2.17). The group velocity is $v_g^{\rm g}(p) = d(v_g(p)p)/dp$, and $v_g^{\rm g}(0) = v_g$ since the interactions in Fourier space are even (see (2.7a)).



(a) $t = 0.0\ell/v_F$ (b) $t = 0.0\ell/v_F$

(c) $t = 2.0\ell/v_F$ (d) $t = 2.0\ell/v_F$

(e) $t = 4.0\ell/v_F$ (f) $t = 4.0\ell/v_F$

(g) $t = 6.0\ell/v_F$ (h) $t = 6.0\ell/v_F$

Figure 4.3: Total density and current for interacting fermions described by the Luttinger model with non-local interactions starting from a domain wall initial state with $\mu = 1$, $\ell = 200$, and $v_F = 1$. (For details about the interactions see Figures 3 and 4 in Paper B.)

4.4.3 Non-equilibrium fermion correlation functions

In the thermodynamic limit, the only non-zero two-point correlation function is

$$\langle \Psi^{\lambda'}_{\lambda,\mu}(t)|\psi^\dagger_r(x)\psi_r(x')|\Psi^{\lambda'}_{\lambda,\mu}(t)\rangle = e^{-irv_F^{-1}A_r(x,x',t)(x-x')}S_r(x,x',t), \tag{4.5}$$

where $A_r(x,x',t)$ depends on the external field (see Equation (27) in Paper B for details) and

$$\begin{aligned} S_r(x,x',t) = &\frac{i}{2\pi r(x-x')+i0^+} \\ &\times \exp\left(\int_0^\infty dp \frac{\eta_{\lambda,\lambda'}(p)-\gamma_{\lambda,\lambda'}(p)\cos(2pv_{\lambda'}(p)t)}{p}\left(\cos p(x-x')-1\right)\right) \end{aligned} \tag{4.6}$$

with non-equilibrium exponents

$$\begin{aligned} \eta_{\lambda,\lambda'}(p) &= \frac{K_\lambda(p)(K_{\lambda'}(p)^{-2}+1)+K_\lambda(p)^{-1}(K_{\lambda'}(p)^2+1)}{4}-1, \\ \gamma_{\lambda,\lambda'}(p) &= \frac{K_\lambda(p)(K_{\lambda'}(p)^{-2}-1)+K_\lambda(p)^{-1}(K_{\lambda'}(p)^2-1)}{4}. \end{aligned} \tag{4.7}$$

In general,[2] these are different from the corresponding equilibrium exponents $\eta_{\lambda,\lambda}(p) = \eta_\lambda(p)$ and $\gamma_{\lambda,\lambda}(p) = 0$ in (2.30) for $g_2 = g_4 = \lambda$ and $V_2(x) = V_4(x) = V(x)$. In the long time limit, the expression in (4.5) simplifies to (see Theorem 2.3 in Paper B)

$$\begin{aligned} &\lim_{t\to\infty}\langle \Psi^{\lambda'}_{\lambda,\mu}(t)|\psi^\dagger_r(x)\psi_r(x')|\Psi^{\lambda'}_{\lambda,\mu}(t)\rangle \\ &\qquad = \frac{ie^{-irv_F^{-1}(r\mu K_\lambda v_{\lambda'}/2v_\lambda)(x-x')}}{2\pi r(x-x')+i0^+}\exp\left(\int_0^\infty dp\frac{\eta_{\lambda,\lambda'}(p)}{p}\left(\cos p(x-x')-1\right)\right). \end{aligned} \tag{4.8}$$

Note that the two-point correlation function in (4.5) is not translation invariant, but that translation invariance is recovered in the long time limit as seen in (4.8). Moreover, the system retains memory of the initial state, even for infinite times, since it is clear from (4.8) that the final state depends on both λ and λ'.

4.4.4 Final state

In the long time limit, it follows from the above that the system tends to a final steady state in which translation invariance is recovered. Concerning the computed expectation values, this final state can effectively be described as a one-dimensional system coupled to reservoirs on each side as in Figure 4.4. The left reservoir has chemical potential μ_+, corresponding to right-moving particles, and the right reservoir has μ_-, corresponding to left-moving particles. These chemical potentials can be read from the two-point correlation function in (4.8), as explained in Paper B. In general, μ_+ and μ_- depend on the

[2] When $\lambda \neq \lambda' \neq 0$.

interactions and are thus different from μ_L and μ_R, which are only present in the initial state as described by Figure 1.2. Flowing through the system is a current

$$I = G(\mu_+ - \mu_-) = G_{\lambda,\lambda'}(\mu_L - \mu_R) \tag{4.9}$$

with a universal electrical conductance $G = 1/2\pi$, i.e., it does not depend on the microscopic details of the model, in this case the interactions. However, the coefficient $G_{\lambda,\lambda'}$ relating I to $\mu_L - \mu_R$ is *non-universal* since it depends on both λ and λ'.

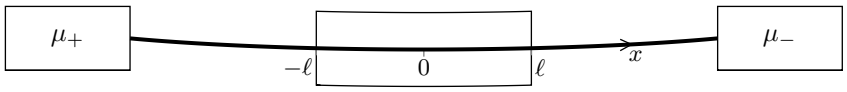


Figure 4.4: Effective description of the final state as a subsystem on the interval $[-\ell, \ell]$ coupled to two reservoirs on its left and right sides with chemical potentials μ_+ and μ_-, respectively.

The universal conductance G was found in, e.g., [2, 1, 45, 31] in near-to-equilibrium settings. In fact, the effective description in Figure 4.4 of the final state is reminiscent of the explanation of the universal conductance in [45]: the explanation given in [45] is that the current only depends on the reservoirs, which are usually modeled as Fermi gases, which are non-interacting, and not on the microscopic details of the system between the reservoirs, which is modeled by the Luttinger model. However, the result in Paper B is obtained in a full non-equilibrium approach and have the important difference that I and $\mu_\pm$ are non-universal and depend on the interactions before (λ) and after (λ') the quench. It is also interesting to note that the non-universal coefficient $G_{\lambda,\lambda'}$ is a generalization of the early proposal of a non-universal value for the conductance in the Luttinger model by Kane and Fisher [30]. This can be seen by setting $\lambda = \lambda'$, in which case $G_{\lambda,\lambda} = K_\lambda/2\pi$, which is precisely the value found in [30].[3]

Moreover, when there is no change of the interactions, i.e., for $\lambda' = \lambda$, we show that the final state, can be modeled as the ground state of the effective Hamiltonian[4]

$$H_{eff} = H_\lambda - (\mu_+ - \mu_0)Q_+ - (\mu_- - \mu_0)Q_-. \tag{4.10}$$

This suggests a generalized Gibbs ensemble where, in addition to the total energy given by H_λ, the additional conserved quantities are the charges $Q_\pm$. This ensemble is given by a Gibbs measure $e^{-\beta H_{eff}}$ using the effective Hamiltonian in (4.10), from which the ground state is obtained in the zero-temperature limit $\beta \to \infty$.

As a last remark, we note that one should be cautious when claiming that certain properties or behaviors are universal. For instance, as mentioned in the beginning of

[3] Recall that $K_\lambda = K_\lambda(0)$ where $K_\lambda(p)$ is the Luttinger parameter in (2.18).

[4] At least concerning the total density, the corresponding current, and the fermion two-point correlation functions in the long time limit.

Chapter 2, the Luttinger model is considered to be a prototype of the Luttinger liquid equivalence class concerning equilibrium properties [23]. However, one should be careful about generalizing this to non-equilibrium properties. For instance, non-integrable models are believed to exhibit thermalization while this is not expected for integrable models. As an example, we saw that the Luttinger model retains memory of the initial state in the long time limit, while the above suggests that the same would not be true for non-integrable models. This difference between non-integrable and integrable systems is an interesting question attracting a lot of current research in non-equilibrium quantum physics.

4.5 Differences in notation with Paper B

Table 4.1 contains a list of important notational differences. Moreover, *generalized Gibbs ensemble* is referred to as *generalized canonical ensemble* in Paper B.

Thesis	Paper A
$\psi_r^\dagger(x)$	$\psi_r^+(x)$
$\psi_r(x)$	$\psi_r^-(x)$
$\hat{\psi}_r^\dagger(k)$	$(2\pi/L)^{-1/2}a_{r,k}^+$
$\hat{\psi}_r(k)$	$(2\pi/L)^{-1/2}a_{r,k}^-$

Table 4.1: Important notational differences with Paper B.

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Part II

Scientific papers

Paper A

Construction by bosonization of a fermion-phonon model
E. Langmann and P. Moosavi
J. Math. Phys. 56, 091902 (2015)

My contribution: independently reworked computations for the bosonization review part, did most of the computations for the fermion-phonon model part, and co-wrote the paper.

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Paper B

Steady states and universal conductance in a quenched Luttinger model
E. Langmann, J. L. Lebowitz, V. Mastropietro, and P. Moosavi
Commun. Math. Phys. (2016)

My contribution: worked out all computations, performed all simulations and created all figures, participated in the discussion of the results, and co-wrote the paper.

Copy of Paper B not attached

Copy of Paper B not attached

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