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Stability and sensitivity of a crossflow-dominated Falkner–Skan–Cooke boundary layer with discrete surface roughness

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With the motivation of determining the critical roughness size, a global stability and sensitivity analysis of a three-dimensional Falkner–Skan–Cooke (FSC) boundary layer with a cylindrical surface roughness is performed. The roughness size is chosen such that breakdown to turbulence is initiated by a global version of traditional secondary instabilities of the crossflow (CF) vortices, instead of an immediate flow tripping at the roughness. The resulting global eigenvalue spectra of the systems are found to be very sensitive to numerical parameters and domain size. This sensitivity to numerical parameters is quantified using the ε -pseudospectrum, and the dependency on the domain is analysed through an impulse response and an energy budget. It is shown that the growth rates increase with domain size, which originates from the inclusion of stronger CF vortices in the baseflow. This is reflected in a change in the rate of advective energy transport by the baseflow. It is concluded that the onset of global instability in a FSC boundary layer as the roughness height is increased does not correspond to an immediate flow tripping behind the roughness, but occurs for lower roughness heights if sufficiently long domains are considered. However, the great sensitivity results in an inability to accurately pinpoint the exact parameter values for the bifurcation, and the large spatial growth of the disturbances in the long domains eventually becomes larger than what can be resolved using finite precision arithmetics.

1. Introduction

Laminar boundary layers over swept aeroplane wings are commonly approximated by the Falkner–Skan–Cooke (FSC) similarity solutions (Falkner & Skan 1931; Cooke 1950). For such flows, the dominating features in low disturbance environments, e.g. free flight, are the spatially developing stationary crossflow (CF) vortices. It is generally established that the stationary CF vortices are

mainly initiated by surface roughness, while free-stream turbulence promotes travelling CF waves (see Saric *et al.* 2003 and the references therein). The CF modes are linked with attachment-line modes (Mack *et al.* 2008) and exhibit convective instability (Wassermann & Kloker 2002), meaning that they act as noise amplifiers. They typically break down to turbulence by secondary instabilities (Högberg & Henningson 1998; Malik *et al.* 1999; Wassermann & Kloker 2002).

In the case of a roughness element inside the boundary layer, the local flow conditions around the roughness can be characterised by a roughness Reynolds number Re_k , which in two-dimensional boundary layers is defined by the roughness height and the local undisturbed streamwise velocity at the roughness height. By increasing Re_k in a two-dimensional boundary layer beyond a certain limit, $Re_{k,t}$, the transition location has been noticed to move upstream to the trailing edge of the roughness (von Doenhoff & Braslow 1961). Due to its importance for the development of laminar wings, the determination of $Re_{k,t}$ has been the subject of several global stability analyses (Loiseau *et al.* 2014; Citro *et al.* 2015; Kurz & Kloker 2016). For roughness elements in a Blasius boundary layer, the above mentioned flow-tripping was found to correspond to the onset of a global flow instability. However, for swept-wing boundary layers, the situation is more complex, as recently demonstrated by Kurz & Kloker (2016). These authors carried out a comprehensive direct numerical simulation (DNS) study accompanied by a global stability analysis and noted a sensitivity of the unstable eigenvalue spectra with respect to numerical parameters and domain size. However, the origin of these issues was not investigated in detail in their work.

Domain dependent spectra for unstable flows are scarcely reported and discussed in the literature. In many cases, the onset of global instability is associated with one pair of complex conjugate eigenvalues that destabilises through a supercritical Hopf bifurcation as some bifurcation parameter is varied (Loiseau *et al.* 2014; Peplinski *et al.* 2015; Citro *et al.* 2015). Such intrinsic oscillations can sometimes be insensitive to numerical details, and be independent of domain size if the overlap between the corresponding direct and adjoint eigenfunctions is included in the domain (Giannetti & Luchini 2007). In contrast, stable eigenvalues along an eigenvalue branch can in many situations be very sensitive with respect to numerical parameters e.g. eigenvalue shifts or domain sizes (Ehrenstein & Gallaire 2005; Alizard & Robinet 2007; Garnaud *et al.* 2013; Cerqueira & Sipp 2014). For such flows, a resolvent or impulse-response analysis may be more appropriate.

In the present work, the global stability of a FSC boundary layer with a cylindrical surface roughness is considered. In particular, the sensitivity of the eigenvalue spectra to numerical parameters and domain dependency is addressed, and an explanation of these are provided.

Case	$[x_{\min}, x_{\max}]$	$[\tilde{x}_{\min}, \tilde{x}_{\max}]$	N ($\mathbb{P}_N$)	elements	grid points
A	$[-20.59, 160]$	$[-20.59, 120]$	8	90,424	$\approx 66 \times 10^6$
B	$[-20.59, 230]$	$[-20.59, 190]$	8	119,574	$\approx 87 \times 10^6$
C	$[-20.59, 300]$	$[-20.59, 260]$	8	146,524	$\approx 107 \times 10^6$
D	$[-20.59, 370]$	$[-20.59, 330]$	$\{8; 10\}$	170,724	$\approx \{124; 227\} \times 10^6$
E	$[-20.59, 440]$	$[-20.59, 400]$	8	195,474	$\approx 142 \times 10^6$

Table 1: Cases considered. $\Omega = [x_{\min}, x_{\max}] \times [0, 20] \times [-12.57, 12.57]$ denotes the computational domain for Cases A–E and $\tilde{\Omega} = [\tilde{x}_{\min}, \tilde{x}_{\max}] \times [0, 20] \times [-12.57, 12.57]$ the corresponding region outside of the support of the sponge function.

2. Flow configuration and numerical approach

2.1. Baseflow

Here, a FSC boundary layer characterised by a constant free-stream velocity in the spanwise direction, and an accelerating velocity in the streamwise direction is considered,

$$U_{\infty}^*(x^*) = U_0^*(x^*/x_0^*)^m \quad (1)$$

$$W_{\infty}^* = W_0^*. \quad (2)$$

Superscript asterisk denotes dimensional quantities, subscript ∞ the flow outside the boundary layer and subscript zero quantities at the inflow. The acceleration of the flow causes the inviscid streamlines to be curved. In the inviscid region a balance between centrifugal and pressure forces exists, but inside the boundary layer, where the streamwise velocity decreases, a force imbalance arises that generates a CF component perpendicular to the inviscid streamline (Saric *et al.* 2003). This CF component is inflectionally unstable (Gregory *et al.* 1955). The FSC similarity solutions are derived in Appendix Appendix A. See also Falkner & Skan (1931); Cooke (1950) for further details.

All quantities are non-dimensionalised using the displacement thickness δ_0^* and the free-stream streamwise velocity U_0^* . The numerical parameter values used in this study are similar to those used by Högberg & Henningson (1998). The distance from the leading edge to the beginning of the domain is $x_0 = 354.0$, the spanwise free-stream velocity $W_0 = 1.442$, acceleration parameter $m = 0.34207$, and the inflow Reynolds number based on streamwise velocity and displacement thickness is $Re_{\delta_0^*} = 337.9$. A sketch illustrating the setup is given in figure 1a. The computational domain consists of a rectangular box with a spanwise width of 25.14 (chosen to match the wavelength of the most unstable stationary CF mode, Högberg & Henningson 1998), a height of 20.0, and various streamwise lengths according to table 1 (see also figure 2).

(a)

(b)

Figure 1: Overview over the flow case and setup. The sketch in (a) illustrates the coordinate system and the velocity profiles aligned and perpendicular to the inviscid streamline. The surface roughness and the mesh structure is visualised in (b).

Figure 2: Visualisation of the baseflow structures for the various cases listed in table 1 (streamwise component ranging from -0.42 in blue to 1.36 in red is shown). The roughness is centred at the origin.

2.2. Numerical method

The flow is governed by the time-dependent incompressible Navier–Stokes equations subject to constant fluid properties. The governing equations are integrated in time using the high-order DNS-code **Nek5000** (Fischer *et al.* 2008), which is based upon the spectral-element method (SEM). The velocity and pressure fields are discretised using a $\mathbb{P}_N$ – $\mathbb{P}_{N-2}$ formulation, where N th degree Lagrange interpolation polynomials on the Gauss–Lobatto–Legendre quadrature points and $(N - 2)$ th degree Lagrange interpolation polynomials on the Gauss–Legendre quadrature points are chosen as basis functions for the velocity and the pressure, respectively (see Deville *et al.* 2002 for further details).

As inflow conditions, the FSC velocity profiles are prescribed. In the spanwise direction periodic conditions are assigned (only one flow period is simulated), and at the wall an ordinary no-slip condition is used. The outflow boundary conditions read $\partial_x v = 0$, $\partial_x w = 0$ and $Re_{\delta_0}^{-1} \partial_x u - p = -p_a$ (where $p_a = -Re_{\delta_0}^{-1} \partial_x u_{\text{FSC}}$ represent an ambient pressure and subscript FSC denotes FSC profiles). Along the free-stream boundary $u = U_\infty(x)$, $w = W_\infty$ and

$Re_{\delta_0^*}^{-1} \partial_y v - p = -p_b$ (where $p_b = \tilde{p} - Re_{\delta_0^*}^{-1} \partial_y v_{\text{FSC}}$, with $\tilde{p}$ obtained from inserting the FSC velocities into the Bernoulli equation).

In order to obtain a baseflow (below denoted $\mathbf{U}$), selective frequency damping (SFD, Åkervik *et al.* 2006) is used. This applies a volume force that forces the flow towards a temporally low-pass filtered target solution. As the solution approaches the target, the magnitude of the force decreases, thus yielding a steady solution of the unforced Navier–Stokes equations upon convergence. The norm of the difference between the instantaneous and filtered velocity is used as a measure of convergence, and the equations are integrated until this is below 10^{-8} . SFD requires two design parameters to be calibrated – a filter gain χ related to the temporal growth rate of the instability, and a filter width Δ related to its critical frequency. These are chosen to be $\chi = 0.4$ and $\Delta = \pi/8$.

The streamwise extents of the different cases listed in table 1 are illustrated in figure 2. As seen, extending the domain in the streamwise direction implies that stronger CF vortices will be included in the computational domain. Stability analysis of such large domains necessarily requires the use of matrix-free methods. To this end time-stepping techniques as described by e.g. Bagheri *et al.* (2009) are employed.

Briefly, the dynamics of a small perturbation $\mathbf{u}'$ about the fixed point $\mathbf{U}$, is governed by the linearised Navier–Stokes equations

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{u}' = -\nabla p' + \frac{1}{Re_{\delta_0^*}} \nabla^2 \mathbf{u}' + \mathbf{f}' \quad \text{in } \Omega \quad (3a)$$

$$\nabla \cdot \mathbf{u}' = 0 \quad \text{in } \Omega. \quad (3b)$$

This can conveniently be expressed in operator form as $\partial \mathbf{u}' / \partial t = \mathcal{A}_{\mathbf{U}} \mathbf{u}'$, treating pressure as a dependent variable. Upon introducing a global mode ansatz $\mathbf{u}'(\mathbf{x}, t) = \hat{\mathbf{u}}(\mathbf{x}) e^{-i\omega t} + \text{c.c.}$ and discretising (3a)–(3b), one arrives at the generalised eigenvalue problem

$$\mathcal{A}_{\mathbf{U}} \hat{\mathbf{u}} = -i\omega \mathcal{M} \hat{\mathbf{u}}, \quad (4)$$

where $\mathcal{M}$ denotes the mass matrix and $(\hat{\mathbf{u}}, \omega)$ an eigenpair. The eigenvalue problem (4) is solved by integrating the initial condition $\mathbf{u}'(\mathbf{x}, 0)$ in time by a timestepper (Nek5000), below denoted $\mathcal{B}(\Delta t)$. By repeated application of such a timestepper, solution fields separated in time by $\Delta t = l\delta t$ (l denotes the number of timesteps and δt the timestep size) are assembled to span a low-dimensional Krylov space

$$\begin{aligned} \mathcal{K}_m &= \text{span}\{\mathbf{u}'(\mathbf{x}, 0), \mathbf{u}'(\mathbf{x}, \Delta t), \mathbf{u}'(\mathbf{x}, 2\Delta t), \dots, \mathbf{u}'(\mathbf{x}, (m-1)\Delta t)\} \\ &= \text{span}\{\mathbf{u}'(\mathbf{x}, 0), \mathcal{B}(\Delta t)\mathbf{u}'(\mathbf{x}, 0), \mathcal{B}^2(\Delta t)\mathbf{u}'(\mathbf{x}, 0), \dots, \mathcal{B}^{m-1}(\Delta t)\mathbf{u}'(\mathbf{x}, 0)\}. \end{aligned} \quad (5)$$

As a means of determining the leading eigenpairs of (4), the Krylov vectors spanning $\mathcal{K}_m$ are factorised to satisfy an Arnoldi-relation,

$$\mathcal{B}(\Delta t) \mathbf{V}_m = \mathbf{V}_m \mathbf{H}_m + \mathbf{f}_m \mathbf{e}_k^T, \quad (6)$$

Figure 3: Flow visualisation for different k (three spanwise periods shown). (a) $k = 0.9$, $Re_k = 233.96$, (b) $k = 1.3$, $Re_k = 458.01$, (c) $k = 1.7$, $Re_k = 728.63$. Contours of streamwise velocity plotted in light grey and contours of negative λ_2 coloured by streamwise velocity. (a) also shows the coordinate system and the free-stream velocities.

such that $(\hat{\mathbf{u}}, \omega)$ can be approximated from the eigenpairs of the $m \times m$ upper Hessenberg matrix $\mathbf{H}_m$, which represents the projection of $\mathcal{B}(\Delta t)$ onto $\mathcal{K}_m$ spanned by the columns of $\mathbf{V}_m$ ($\mathbf{f}_m \mathbf{e}_k^T$ is the residual of the factorisation).

The implementation (Peplinski *et al.* 2012) is based on the implicitly restarted Arnoldi-method (IRAM) from the ARPACK-library (Lehoucq *et al.* 1997). For these computations periodic boundary conditions are imposed in the spanwise direction, and in all other directions homogeneous Dirichlet conditions are prescribed. A 40 units long sponge region is added at the downstream end of the domain (Peplinski *et al.* 2012). In all the stability computations presented in this work, a tolerance of 10^{-6} has been used for IRAM (see Lehoucq *et al.* (1997) for a discussion on the convergence criterion).

2.3. Surface roughness

A cylindrical surface roughness with diameter $d = 6.0$, visualised in figure 1b, is introduced into the domain with its centre located at the origin (i.e. 20.59 downstream of the inflow). Results from initial non-linear DNSs of roughness elements with heights $k = \{0.9, 1.3, 1.7\}$ corresponding to $Re_k = \{233.96, 458.01, 728.63\}$ (based on the wall-parallel velocity $\sqrt{u_{\text{FSC}}^2 + w_{\text{FSC}}^2}$) in the domain Ω^C , are shown in figure 3. The flow induced by the lowest roughness with $k = 0.9$ (figure 3a) is dominated by large co-rotating CF vortices that emanate from the vortex structure in the near-field roughness wake. After an initial transient phase the flow becomes stationary. Increasing the roughness height to $k = 1.3$ (figure 3b) gives stronger CF vortices that transition to turbulence through a mechanism resembling a secondary instability (cf. Wassermann & Kloker 2002). A probe

inside the boundary layer (marked with a white bullet in figure 3b) measures a band of frequencies around 1.0 (not shown), which can be compared with 0.957 reported by Högberg & Henningson (1998). This suggests that the CF vortices in this case reach sufficient magnitude to be destabilised through secondary instabilities triggered by perturbations introduced by the roughness. Further increase of the roughness height to $k = 1.7$ (figure 3c) gives an immediate breakdown to turbulence behind the roughness reminiscent to those observed in two-dimensional boundary layers (Loiseau *et al.* 2014; Citro *et al.* 2015). Based on this initial DNS study, one can tentatively say that for this setup, the critical roughness height, i.e. the height that first causes flow transition in the domain, lies between 0.9 and 1.3. In light of this, the roughness element with $k = 1.3$ is selected for further analysis.

3. Global stability and sensitivity analysis

Following the temporal stability framework, and the ansatz function introduced in §2.2, a flow will be termed *globally stable* if the imaginary part of all eigenvalues are negative, and conversely *globally unstable* if at least one eigenvalue has a positive imaginary part. The term *critical* refers to a configuration that is marginally stable, i.e. where the least stable eigenvalue has a vanishing imaginary part (is close to zero).

3.1. Eigenvalue sensitivity

The eigenvalues from the stability analysis of case D are shown in figure 4a. Clearly, all the reported eigenvalues are unstable, which shows that the transition observed in figure 3b actually corresponds to a global flow instability. For this case two polynomial orders for the perturbation (see table 1), as well as two solver tolerances are investigated.

The sensitivity of the spectrum to resolution is investigated by increasing the polynomial order from $N = 8$ to $N = 10$. This amounts to a near doubling in the number of grid points. As seen in figure 4a, the spectra are slightly different but overlapping, which shows that the flow is sensitive to numerical details but sufficiently resolved in space.

The sensitivity of the spectrum to the iterative solver tolerance is much overlooked in this type of studies. The convergence of the velocity and the pressure solver (see Deville *et al.* 2002), is monitored by tracking the norm of the solution residual and the divergence of the velocity, respectively. As shown in figure 4a, changing the solver tolerances from 10^{-10} to 10^{-8} almost doubles the computed growth rates. Sensitivity of the spectra to numerical parameters has been reported for strongly damped eigenmodes by e.g. Cerqueira & Sipp (2014), who concluded that it was impossible to find a unique set of eigenvalues that were intrinsic to their flow. One can view a change in the solver tolerance as effectively imposing a small perturbation on the governing matrix for the flow. As such, the reported eigenvalues should be interpreted as ε -pseudoeigenvalues defined as the eigenvalues of a perturbed matrix, i.e. the

Figure 4: Eigenvalue sensitivity. (a) Spectra for the case D with polynomial order $N = 8$ and tolerance 10^{-10} (o) together with its ε -pseudospectra (logarithmically spaced coloured contours), $N = 10$ and tolerance 10^{-10} (x), $N = 8$ and tolerance 10^{-8} (+). (b) Spectra for case D with sponge length, 30 (◁), 40 (o) and 50 (◇). (c) Spectra for the cases A (*), B (□), C (△), D (o), E (x), growth rates for the case D evaluated from (7) (●).

eigenvalues of $(A + E)$, where A is the governing discretised operator and E a perturbation with $\|E\| \leq \varepsilon$ (Schmid & Henningson 2001). A consequence of this view is for instance that two similar calculations initiated with slightly different random noise never can be expected to give exactly the same eigenvalues. This is because any number σ sampled from a pseudospectral contour line for which the resolvent norm $\|(A - \sigma I)^{-1}\|$ attains a sufficiently large value may be interpreted as an eigenvalue by the solver. Inspection of the pseudospectral

contours suggest that further increases in solver tolerance will not stabilise the eigenvalues in this case, since these curves are both nested and bounded from below. Moreover, determining very sensitive eigenvalues like those plotted in figure 4a to any greater precision is of limited value. The ε -pseudospectrum in figure 4a is computed for the logarithm of the Hessenberg matrix $\mathbf{H}_m$ from the Arnoldi-factorisation (6) (Toh & Trefethen 1996) of case D ($N = 8$, tolerance 10^{-10}) divided by the temporal separation Δt between the Arnoldi-vectors.

In line with the above discussion it is instructive to also consider the sensitivity of the spectrum with respect to the sponge length. This is done by varying the sponge length ± 10 units for case D. As seen in figure 4b, this parameter has a negligible effect on the stability, which suggests that the treatment of the outflow boundary is satisfactory.

3.2. Domain dependency

For the boundary layer studied here, the growth of CF vortices is promoted in the entire domain as visualised in figure 2. Figure 4c shows the eigenvalue spectrum for the cases A–E. In case B, the flow is found to be marginally stable. Notably, the flow destabilises and becomes increasingly unstable as the computational domain is extended downstream. This domain dependency stems from the inclusion of stronger CF vortices, and suggests that the observed global instability is a result of a combination of the flow around the roughness and the stronger local secondary instability associated with the stronger CF vortices. Such a conclusion would be in line with the results of Malik *et al.* (1999), who found that the growth rates of the secondary instability z -modes increased with streamwise location.

3.2.1. Eigenfunctions

For the cases C–E all the obtained eigenfunctions resemble a z -type mode (Högberg & Henningson 1998; Malik *et al.* 1999) as shown in figure 5. The spanwise maximum modulus of all the eigenfunctions is plotted in figure 6. The eigenfunctions consist of two different regions of spatial growth – one located immediately behind the roughness and extending to $x \approx 100$, and another region that coincides with the growth of the CF vortices and extends until the sponge region. Comparison of figure 6 to figure 29 in Kurz & Kloker (2016), shows that their eigenfunctions mainly consist of the first region, whereas the second region, which according to figure 6 becomes dominant for longer domains, in their analysis has been truncated. This could explain why their growth rates did not change significantly with the length of their rather short domains.

As seen in figure 7, the strength of the first upstream region increases with frequency. A close-up on the vortical structures and the eigenfunctions in the vicinity of the roughness (figure 8) reveals that these evolve around the central vortex leg developing from the recirculation region behind the roughness. The relative weakness of this region for the low-frequency z -modes plotted in figure 7a and 8a compared to the corresponding region for high-frequency z -modes, is

Figure 5: Three spanwise periods of the eigenfunction of case D corresponding to $\omega = 1.5327$, at $x = 327$ (view from upstream). The real part of the eigenfunction is plotted, with positive and negative values coloured in red and blue, respectively, and contours of the streamwise baseflow component in black.

Figure 6: Spanwise maximum modulus of the eigenfunctions for the cases B (red), C (cyan), D (blue) and E (black). The eigenfunctions are rescaled with their energy integrated over $\tilde{\Omega}^B$ to simplify comparison.

most likely related to the longer spatial wavelengths associated with the former modes plotted in figure 7b and 8b. It seems that the longer wavelengths of these modes may not be promoted by the thin vortex structures in the near-wake.

Regarding the eigenfunctions in case A and B, the identification of z -modes is less clear, due to the low amplitude of the CF vortices in these cases. Notable is that some low-frequency and stationary modes appear in the eigenvalue spectra for these cases (figure 4c). However, these are all stable and not associated with the transition and global instability of the flow.

3.2.2. Impulse response

In order to analyse the noted domain dependency in greater detail, an impulse-response analysis similar to those performed by Brandt *et al.* (2003); Peplinski *et al.* (2015) is carried out for the cases D and E. As initial perturbation a wavepacket-like disturbance (see Bech *et al.* 1998) is chosen that is oriented in the direction of the inviscid streamline and placed upstream of the roughness. If a flow is convectively unstable, disturbances are advected away from the location of initiation such that the flow in the limit of long times relaxes to its original

Figure 7: The streamwise velocity component of the real part of a sample of eigenfunctions for case D plotted in the xz -plane at the wall-normal position $y = 2$ (view from below). Three periods of the domain are shown with positive and negative values coloured in red and blue, respectively, and contours of the streamwise baseflow component in black. The insets in the upper left corners indicate which mode in the spectrum that is plotted.

Figure 8: Location of the eigenfunctions plotted in figure 7 relative to the vortex legs developing from the roughness near-wake into the far-wake CF vortices. The vortex structures are shown with the λ_2 -criterion (Jeong & Hussain 1995) (green). For visualisation purposes, the contour levels of the eigenfunctions in frame (a) are 10 times smaller than those in frame (b).

state. In contrast, if the flow is absolutely or globally unstable, disturbances will regenerate at their initial location and grow in amplitude while they spread in the upstream and downstream directions. Hence, for a convectively unstable flow the trailing edge velocity of the wavepacket is positive, whereas for an absolutely or globally unstable flow, it is non-positive (Schmid & Henningson 2001).

Figure 9: The energy of the wavepacket integrated over spanwise planes for different times (colour) as a function of group velocity $c_g = x/t$ for the cases D (—) and E (---). The vertical solid lines shows trailing edge velocities c_g^- for $t \leq 200$ and $t \geq 300$.

In figure 9, the energy of the wavepacket is plotted against the group velocity $c_g = x/t$. As shown, the evolution of the impulse is similar for the cases D and E (the lines are nearly indistinguishable), until the wavepacket in the case D reaches the sponge region and gets dampened. By examining the intersection of the curves at the rear and front of the wavepacket at different times, the trailing and the leading edge velocity (c_g^- and c_g^+) of the wavepacket can be estimated. Initially for $t \leq 200$ this intersection gives $c_g^- \approx 0.058$, which suggests that the flow is convectively unstable (i.e. globally stable) and the wavepacket will be advected out of the domain. This is however not the case for larger times. Extrapolating a least-squares fit to the energy curves for $t \geq 300$ and $x \leq 6.8$ yields $c_g^- \approx -0.014$, which implies that the flow is globally unstable. From figure 9 no well-defined leading edge c_g^+ can be determined, which is most likely due to the fact that the flow accelerates and the wavepacket propagates along the curved CF vortices.

The behaviour of the wavepacket is in line with figure 4c and confirms that there is a changeover to global instability as stronger CF vortices are included in the domain. The reason for this is the connection between the roughness near-wake and the far-wake CF depicted by the eigenfunctions in figure 6. As the energy of the wave packet is amplified by the secondary unstable CF vortices in the far-wake, the energy levels in the near-wake also increases. This alters the evolution of the wavepacket, which starts to spread also in the upstream direction. For even longer times, the amplitude in the far-wake becomes so large that the connection between the near- and far-wake becomes impossible to resolve using double precision arithmetics. A related effect can be seen at $t = 400$ for the case E in figure 9 as a flat middle region corresponding to the noise level set by the solver tolerance (10^{-10}). At this point the tail of the wave packet is below this noise level and changes slope, which implies that

Figure 10: Energy budget of the cases C ($\triangle$), D ($\circ$) and E ($\times$) showing $\mathcal{P}$ (red), $\mathcal{E}$ (blue) and $\mathcal{T}$ (magenta) integrated over $\tilde{\Omega}^C$ together with the growth rates (black).

flow physics is erroneously represented. Given this result, further increases in domain size will be questionable.

3.3. Energy budget

To gain further insight into the quantities most sensitive to the domain size, an analysis of the energy budget is carried out. The computed eigenfunctions are substituted into the linearised evolution equation for the kinetic perturbation energy, upon which an expression for the growth rate ω_i can be derived as

$$\omega_i = \frac{1}{E_{\tilde{\Omega}}} \int_{\tilde{\Omega}} (\mathcal{P} - \mathcal{E} + \mathcal{T} + \Pi + \mathcal{D}) d\mathbf{x}, \quad (7)$$

where $E_{\tilde{\Omega}} = \int_{\tilde{\Omega}} (\hat{u}_j^{r2} + \hat{u}_j^{i2}) d\mathbf{x}$ represents the kinetic energy in $\tilde{\Omega}$ (superscript r and i refer to real and imaginary parts of the eigenfunctions and $j \in \{1, 2, 3\}$). The terms on the right-hand side of (7) correspond to production ($\mathcal{P}$) and dissipation ($\mathcal{E}$) of energy and transport of energy by advection ($\mathcal{T}$), pressure perturbation (Π) and diffusion ($\mathcal{D}$). These terms written in tensor notation

Figure 11: Energy production ($\mathcal{P}$) and dissipation ($\mathcal{E}$) of the eigenmode shown in figure 7b. Frame (a) and (b) shows the spatial location of $\mathcal{P}$ and $\mathcal{E}$, respectively, in relation to the CF vortex at the streamwise position $x = 327$ (view from upstream). Frame (c) shows an iso-contour of $\mathcal{P}$ (red) and $\mathcal{E}$ (blue) in the near-field together with the vortex structures of the baseflow visualised with the λ_2 -criterion (Jeong & Hussain 1995) (green). For visualisation purposes, the contour level of $\mathcal{P}$ is 10 times larger than that of $\mathcal{E}$.

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$$\begin{aligned}\mathcal{P} &= - \left(\hat{u}_l^r \hat{u}_j^r \frac{\partial U_l}{\partial x_j} + \hat{u}_l^i \hat{u}_j^i \frac{\partial U_l}{\partial x_j} \right), \quad \mathcal{E} = \frac{1}{Re_{\delta_0^*}} \left(\frac{\partial \hat{u}_l^r{}^2}{\partial x_j} + \frac{\partial \hat{u}_l^i{}^2}{\partial x_j} \right), \\ \mathcal{T} &= - \left(\hat{u}_l^r \frac{\partial \hat{u}_l^r}{\partial x_j} U_j + \hat{u}_l^i \frac{\partial \hat{u}_l^i}{\partial x_j} U_j \right), \quad \Pi = - \left(\hat{u}_l^r \frac{\partial \hat{p}^r}{\partial x_j} + \hat{u}_l^i \frac{\partial \hat{p}^i}{\partial x_j} \right) \delta_{lj}, \\ \mathcal{D} &= \frac{1}{Re_{\delta_0^*}} \left(\frac{\partial \hat{u}_l^r{}^2}{\partial x_j} + \hat{u}_l^r \frac{\partial^2 \hat{u}_l^r}{\partial x_j \partial x_j} + \frac{\partial \hat{u}_l^i{}^2}{\partial x_j} + \hat{u}_l^i \frac{\partial^2 \hat{u}_l^i}{\partial x_j \partial x_j} \right),\end{aligned}\quad (8)$$

in which $j, l \in \{1, 2, 3\}$. Comparison of the growth rates given by the global stability analysis to those obtained by integrating (7)–(8) for the case D over $\tilde{\Omega}^C$ shows excellent agreement (see figure 4c).

Focusing on the longer domains (the cases C–E) and choosing $\tilde{\Omega}^B$ as domain of integration, the terms mainly contributing to the growth rates are identified as $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{T}$, whereas other terms are orders of magnitude smaller. It is seen in figure 10 that production (red) and dissipation (blue) of energy nearly coincide for all the three cases, and that the entire change in growth rate can be attributed to a decrease in the advection rate of perturbation energy by the baseflow (magenta). As $\mathcal{T}$ decreases, energy will be transported out of the domain at a lower rate, which explains the higher growth rates.

Since the integral value of $\mathcal{P}$ and $\mathcal{E}$ are seen to be roughly independent with respect to the domain size, the spatial structure of these are investigated in more detail. By including the minus sign in the definition of $\mathcal{P}$, its interpretation becomes analogous to that of the corresponding term in a turbulent kinetic

energy budget. Thus, gradients of the baseflow working against the velocity perturbations withdraw energy from the baseflow and transfer it to the perturbation field. As shown in figure 11a, the region of positive energy production aligns closely with the frontal region of the CF vortices, similarly to what was found by Malik *et al.* (1999) considering solely the dominant production term (compare with their figure 10). Plotting $\mathcal{E}$ as is done in figure 11b, shows that the same regions also are responsible for dissipating energy. In the upstream region, figure 11c shows that the production and dissipation of perturbation energy is confined to the central vortex leg along which the eigenmode evolves. This suggests that the onset of global instability does not arise from an instability related to the counter-rotating vortex-pairs surrounding the roughness, but from the baseflow structures associated with the flow over the roughness element and into the recirculation region. In figure 11, $\mathcal{P}$ and $\mathcal{E}$ are shown for the high-frequency z -mode plotted in figure 7b. The corresponding structures for the low-frequency z -mode in figure 7a has a similar shape and location, although the magnitude of these quantities in the near-wake are orders of magnitude lower than for the high-frequency mode.

4. Conclusion and discussion

It is shown in this paper that a Falkner–Skan–Cooke (FSC) boundary layer with a roughness element large enough to excite crossflow (CF) vortices, but small enough to avoid an immediate flow tripping, is globally unstable if sufficiently strong CF vortices are induced in the domain. The corresponding eigenvalue spectra are rather flat and very sensitive to numerical parameters. Study of the approximate ε -pseudospectrum (Toh & Trefethen 1996) shows that a minute change in the setup, e.g. solver tolerance, can move the eigenvalues from a level of $\varepsilon \sim 10^{-8}$ to $\varepsilon \sim 10^{-5}$. Such a sensitivity implies that it is impossible to determine a unique converged set of eigenvalues for this flow. The growth rates of the eigenfunctions also increase with domain size, which is reflected in the energy budget as a decrease in the advection rate of perturbation energy by the baseflow. An impulse-response analysis shows that there is a changeover from convective to global instability as stronger CF vortices are included in the domain.

A question is whether the growth rates will increase unboundedly with the length of the domain. In case of longer domains, even stronger CF vortices will be included, whose amplitudes eventually will saturate. Given the budget in §3.3, it seems unphysical that the transport term $\mathcal{T}$ will change sign for a sufficiently long domain. Hence, if the mode shape remain unchanged and the results in figure 10 continue to hold, such that $\mathcal{T} \rightarrow 0$ as $\tilde{x}_{\max} \rightarrow \infty$, an upper bound for the growth rates can be estimated to be $\omega_i \approx 0.074$.

It might be argued that the reason for the domain dependency arises since the overlap between the direct and adjoint eigenfunctions stretches beyond the extent of the domain (Giannetti & Luchini 2007). In this flow case, strong CF vortices far downstream of the roughness element are necessary in order to have

a global instability. Hence, one can speculate that such an overlap region is not localised and would be distributed over the entire CF vortices. An investigation of the adjoint spectra is beyond the scope of this work, but preliminary computations indicate that they exhibit similar sensitivity. Therefore it will not be possible to determine a one-to-one correspondence between the direct and adjoint spectra, which would make such an analysis less rigorous.

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Appendix A. FSC boundary layer equations

The similarity solutions of Falkner & Skan (1931) and Cooke (1950), subject to boundary conditions $f = f' = g = 0$ at $\eta = 0$ and $f' \rightarrow 1$, $g \rightarrow 1$ as $\eta \rightarrow \infty$, read

$$f''' + ff'' + \beta_H(1 - f'^2) = 0, \quad (9)$$

$$g'' + fg' = 0. \quad (10)$$

These equations are derived from Prandtl's boundary layer equation (see Schlichting 1979) and are solved with a shooting method using Newton-Raphson and fourth-order Runge-Kutta. Following Schlichting (1979), a stream function (ψ^*) and a dimensionless wall-normal coordinate (η) can be defined as

$$\psi^* = \sqrt{\frac{2}{m+1}} x^* U_\infty^* \nu f(\eta), \quad u^* = \frac{\partial \psi^*}{\partial y^*}, \quad v^* = -\frac{\partial \psi^*}{\partial x^*} \quad (11)$$

$$\eta = y^* \sqrt{\frac{m+1}{2} \frac{U_\infty^*}{\nu x^*}}, \quad (12)$$

from which the streamwise and wall-normal base flow components can be computed. These together with the assumption of constant spanwise velocity gives

$$u_{\text{FSC}} = U_\infty(x) f', \quad (13)$$

$$v_{\text{FSC}} = \frac{1}{2} \sqrt{\frac{2}{m+1} \frac{U_\infty}{x} \frac{1}{Re_{\delta_0^*}}} [(1-m)f'\eta - (1+m)f], \quad (14)$$

$$w_{\text{FSC}} = W_\infty g, \quad (15)$$

where $U_\infty(x) = (1 + x/x_0)^m$ and $W_\infty = W_0^*/U_0^*$ are derived from (1) and (2), and f' , f and g are obtained from equation (9) and (10).

REFERENCES

- ÅKERVIK, E., BRANDT, L., HENNINGSON, D. S., HØPFFNER, J., MARXEN, O. & SCHLATTER, P. 2006 Steady solutions of the Navier–Stokes equations by selective frequency damping. *Phys. Fluids* **18** (6).
- ALIZARD, F. & ROBINET, J.-C. 2007 Spatially convective global modes in a boundary layer. *Phys. Fluids* **19** (11).
- BAGHERI, S., ÅKERVIK, E., BRANDT, L. & HENNINGSON, D. S. 2009 Matrix-free methods for the stability and control of boundary layers. *AIAA J.* **47** (5), 1057–1068.
- BECH, K. H., HENNINGSON, D. S. & HENKES, R. A. W. M. 1998 Linear and nonlinear development of localized disturbances in zero and adverse pressure gradient boundary-layers. *Phys. Fluids* **10** (6), 1405–1418.
- BRANDT, L., COSSU, C., CHOMAZ, J.-M., HUERRE, P. & HENNINGSON, D. S. 2003 On the convectively unstable nature of optimal streaks in boundary layers. *J. Fluid Mech.* **485**, 221–242.
- CERQUEIRA, S. & SIPP, D. 2014 Eigenvalue sensitivity, singular values and discrete frequency selection mechanism in noise amplifiers: the case of flow induced by radial wall injection. *J. Fluid Mech.* **757**, 770–799.
- CITRO, V., GIANNETTI, F., LUCHINI, P. & AUTERI, F. 2015 Global stability and sensitivity analysis of boundary-layer flows past a hemispherical roughness element. *Phys. Fluids* **27** (8).
- COOKE, J. C. 1950 The boundary layer of a class of infinite yawed cylinders. *Mathematical Proceedings of the Cambridge Philosophical Society* **46** (04), 645–648.
- DEVILLE, O., FISCHER, P. F. & MUND, E. H. 2002 *High-Order Methods for Incompressible Fluid Flow*. Cambridge University Press.
- VON DOENHOFF, A. E. & BRASLOW, A. L. 1961 The effect of distributed surface roughness on laminar flow. In *Boundary layer and Flow Control: its Principles and Application* (ed. G. Lachmann), pp. 657–681. Pergamon Press.
- EHRENSTEIN, U. & GALLAIRE, F. 2005 On two-dimensional temporal modes in spatially evolving open flows: the flat-plate boundary layer. *J. Fluid Mech.* **536**, 209–218.
- FALKNER, V. M. & SKAN, S. W. 1931 Some approximate solutions of the boundary layer equations. *Phil. Mag.* **12** (80), 865–896.
- FISCHER, P. F., LOTTES, J. W. & KERKEMEIER, S. G. 2008 nek5000 Web page. <http://nek5000.mcs.anl.gov>.
- GARNAUD, X., LESSHAFFT, L., SCHMID, P. J. & HUERRE, P. 2013 Modal and transient dynamics of jet flows. *Phys. Fluids* **25** (4).
- GIANNETTI, F. & LUCHINI, P. 2007 Structural sensitivity of the first instability of the cylinder wake. *J. Fluid Mech.* **581**, 167–197.
- GREGORY, N., STUART, J. T. & WALKER, W. S. 1955 On the stability of three-dimensional boundary layers with application to the flow due to a rotating disk. *Philos. T. Roy. Soc. A* **248** (943), 155–199.
- HÖGBERG, M. & HENNINGSON, D. 1998 Secondary instability of cross-flow vortices in Falkner-Skan-Cooke boundary layers. *J. Fluid Mech.* **368**, 339–357.
- JEONG, J. & HUSSAIN, F. 1995 On the identification of a vortex. *J. Fluid Mech.* **285**, 69–94.

- KURZ, H. B. E. & KLOKER, M. J. 2016 Mechanisms of flow tripping by discrete roughness elements in a swept-wing boundary layer. *J. Fluid Mech.* **796**, 158–194.
- LEHOUCQ, R. B., SORENSSEN, D. C. & YANG, C. 1997 *ARPACK User's Guide: Solution of Large Scale Eigenvalue Problems by Implicitly Restarted Arnoldi Methods*. SIAM.
- LOISEAU, J.-C., ROBINET, J.-C., CHERUBINI, S. & LERICHE, E. 2014 Investigation of the roughness-induced transition: global stability analyses and direct numerical simulations. *J. Fluid Mech.* **760**, 175–211.
- MACK, C. J., SCHMID, P. J. & SESTERHENN, J. L. 2008 Global stability of swept flow around a parabolic body: connecting attachment-line and crossflow modes. *J. Fluid Mech.* **611**, 205–214.
- MALIK, M. R., LI, F., CHOUDHARI, M. M. & CHANG, C.-L. 1999 Secondary instability of crossflow vortices and swept-wing boundary-layer transition. *J. Fluid Mech.* **399**, 85–115.
- PEPLINSKI, A., SCHLATTER, P., FISCHER, P. F. & HENNINGSON, D. S. 2012 Stability Tools for the Spectral-Element Code Nek5000; Application to Jet-in-Crossflow. In *ICOSAHOM'12, International Conference on Spectral and High Order Methods*.
- PEPLINSKI, A., SCHLATTER, P. & HENNINGSON, D. S. 2015 Global stability and optimal perturbation for a jet in cross-flow. *Eur. J. Mech. B-Fluid.* **49**, 438–447.
- SARIC, W. S., REED, H. L. & WHITE, E. B. 2003 Stability and transition of three-dimensional boundary layers. *Annu. Rev. Fluid Mech.* **35**, 413–440.
- SCHLICHTING, H. 1979 *Boundary-Layer Theory*, 7th edn. McGraw-Hill, Inc.
- SCHMID, P. J. & HENNINGSON, D. S. 2001 *Stability and Transition in Shear Flows. Applied Mathematical Sciences* v. 142. Springer.
- TOH, K. C. & TREFETHEN, L. N. 1996 Calculation of pseudospectra by the Arnoldi iteration. *SIAM J. Sci. Comp.* **17** (1), 1–15.
- WASSERMANN, P. & KLOKER, M. 2002 Mechanisms and passive control of crossflow-vortex-induced transition in a three-dimensional boundary layer. *J. Fluid Mech.* **456**, 49–84.