The Performance of Market Risk Models for Value at Risk and Expected Shortfall Backtesting
In the Light of the Fundamental Review of the Trading Book
KATJA DALNE
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Abstract

The global financial crisis that took off in 2007 gave rise to several adjustments of the risk regulation for banks. An extensive adjustment, that is to be implemented in 2019, is the Fundamental Review of the Trading Book (FRTB). It proposes to use Expected Shortfall (ES) as risk measure instead of the currently used Value at Risk (VaR), as well as applying varying liquidity horizons based on the various risk levels of the assets involved. A major difficulty of implementing the FRTB lies within the backtesting of ES. Righi and Ceretta proposes a robust ES backtest based on Monte Carlo simulation. It is flexible since it does not assume any probability distribution and can be performed without waiting for an entire backtesting period. Implementing some commonly used VaR backtests as well as the ES backtest by Righi and Ceretta, yield a perception of which risk models that are the most accurate from both a VaR and an ES backtesting perspective. It can be concluded that a model that is satisfactory from a VaR backtesting perspective does not necessarily remain so from an ES backtesting perspective and vice versa. Overall, the models that are satisfactory from a VaR backtesting perspective turn out to be probably too conservative from an ES backtesting perspective. Considering the confidence levels proposed by the FRTB, from a VaR backtesting perspective, a risk measure model with a normal copula and a hybrid distribution with the generalized Pareto distribution in the tails and the empirical distribution in the center along with GARCH filtration is the most accurate one, as from an ES backtesting perspective a risk measure model with univariate Student’s t distribution with $\nu \approx 7$ together with GARCH filtration is the most accurate one for implementation. Thus, when implementing the FRTB, the bank will need to compromise between obtaining a good VaR model, potentially resulting in conservative ES estimates, and obtaining a less satisfactory VaR model, possibly resulting in more accurate ES estimates.

The thesis was performed at SAS Institute, an American IT company that develops software for risk management among others. Targeted customers are banks and other financial institutions. Investigating the FRTB acts a potential advantage for the company when approaching customers that are to implement the regulation framework in a near future.

Keywords: Risk Management, Financial Time Series, Value at Risk, Expected Shortfall, Monte Carlo Simulation, GARCH modeling, Copulas, Hybrid Distribution, Generalized Pareto Distribution, Extreme Value Theory, Backtesting, Liquidity Horizon, Basel regulation.
Sammanfattning


Examensarbetet genomfördes vid SAS Institute, ett amerikanskt IT-företag som bland annat utvecklar mjukvara för riskhantering. Tänkbara kunder är banker och andra finansinstitut. Denna studie av FRTB innebär en potentiell fördel för företaget vid kontakt med kunder som planerar implementera regelverket inom en snar framtid.

Nyckelord: Riskhantering, finansiella tidsserier, Value at Risk, Expected Shortfall, Monte Carlo-simulering, GARCH-modellering, Copulas, hybrida distributioner, generaliserad Pareto-fördelning, extremvärdesteori, Backtesting, likviditetshorisonter, Basels regelverk.
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Katja Dalne
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Chapter 1

Introduction

1.1 Financial Risk Management

Risk is defined as the potential of gaining or losing something of value. While uncertainty is a potential and unpredictable outcome, risk is a consequence of action taken even though one is aware of the uncertainty it implies.

Everyone knows that sometimes things does not behave as one can expect. If things go wrong it is of particular interest to investigate how wrong they can go. Financial risk management is about identifying, assessing, managing, reporting and limiting these scenarios within the financial industry. It can be done both qualitatively and quantitatively.

By definition, the risky scenarios occur with low probability which is mathematically represented by the tail of an asset return distribution. Far out in the tail one finds the most rare events which also have the greatest impact if they occur, sometimes referred to as black swans [33 p. 37]. Financial risk managers are interested in modeling these extreme events accurately which is not an easy task due to little or no historical observations from these scenarios.

Financial risk consists of different kinds of risks which are market risk, liquidity risk, credit risk and operational risk.

Market risk: the risk that the value of an investment will decrease due to moves in market factors occurring as a result of recessions, political turmoil, changes in interest rates and foreign exchange rates, access to commodities and capital and globally affecting events such as natural disasters and terrorist attacks.

Liquidity risk: the risk that arises from the decrease of marketability of a
A financial instrument that cannot be traded fast enough to entirely avoid or at least reduce a loss [21, p. 3]. The distinction between liquidity risk and other risks such as market risk is that liquidity risk is indirect, meaning it is a consequential risk occurring due to other risks which provoke liquidity problems for financial institutions [32, p. 406].

**Credit risk**: the risk of a lender that a default on a debt, that arises from a borrower failing to make a required payment, occurs. For the lender it results in lost principal and interest and disruption to cash flows.

**Operational risk**: the risk of a loss occurring due to deficient or failed internal processes being people and systems, or from external events. It can also arise from other classes of risk, such as fraud, security, privacy protection, legal risks, physical or environmental risks.

Within the financial industry today, risk management is one of the most rapidly growing functions. This is probably due to several factors such as the increasingly important role banks and other financial institutions play in our society, as well as the opportunities fresh research and new technology implies for quantitative risk management [32, p. xii].

### 1.2 Risk Control and Bank Regulation

Stability within the financial sector is necessary for any economic growth to arise. The modern society relies on the functioning of banking and insurance systems. Several financial crises have occurred in the past due to banks’ speculative activities involving large risks. In order to avoid these kinds of scenarios the Basel Committee on Banking Supervision was founded. Their objective is to increase understanding of risk and improve the quality of banking supervision worldwide. The Basel Accords are generally adopted for tracking, reporting and exposing market, liquidity, credit and operational risk.

In particular, due to an increased exposure towards market risk by the banks during the last quarter of a century, several regulations have been made by the committee to regulate the banks’ capital requirement. Value at risk (VaR) has become by far the most common risk measure when quantifying market risk and was included from 1996 in Basel’s Market Risk Amendment. It required an estimation of VaR at a confidence level of $99\%$ with a 10-day horizon. The capital requirement, $C(T)$, was computed as follows:

$$C(T) = \max \left( \operatorname{VaR}(T - 1), M \sum_{i=1}^{60} \operatorname{VaR}(T - i) \right)$$

(1.1)
where $3 \leq M \leq 4$ was called the regulatory multiplier and depended on the backtesting performance of the model. In words, the capital requirement took the largest value of the latest VaR and the 60-day average of VaRs weighted with the regulatory multiplier. However, the weaknesses of the regulation and VaR as a risk measure in general became more and more evident and remedies were therefore proposed.

About a decade later, in 2009, the Basel Committee introduced, as a part of the Basel 2.5 accord, a requirement of additional capital for those assets that had turned out to be poorly modeled by the banks during the global financial crisis that started in 2007. In particular it required stressed VaR calculations based on stressed period calibration of the market risk model. This new capital requirement yielded the sum of the regular VaR, see equation (1.1), and the stressed VaR.

However, the accord was created in a hurry as a reaction to the global financial crisis and in practice overlapping sometimes occurred which resulted in double counting the risk. This was the reason the Basel committee decided to improve the framework and published a consultative paper on a "Fundamental Review of the Trading Book" in 2012 [12].

1.3 Fundamental Review of the Trading Book

The "Fundamental Review of the Trading Book" (FRTB) was published in order to improve the Basel framework in the light of the global financial crisis that started in 2007. The improvements proposed capture several issues. From a mathematical point of view, the most intriguing propositions involve replacing VaR by a risk measure called Expected Shortfall (ES) as well as to make liquidity horizons depend on the liquidity of the underlying asset instead of being fixed.

Regarding the replacement of VaR by ES, the Basel committee proposes that the confidence level is to be changed from 99\% to 97.5\%. In particular, two daily VaR’s for both confidence level 99\% and 97.5\% are to be calculated and a daily ES for confidence level 97.5\%. Also, the sum of the VaRs will be replaced by a single stressed ES [32, p. 165]. The change of confidence level for ES will provide a broadly similar level of risk capture as the existing 99\textsuperscript{th} percentile VaR threshold for the normal distribution, while providing a number of benefits, including generally more stable model output and often less sensitivity to extreme outlier observations [3 p. 3]. ES is a coherent risk measure and hence from a mathematical point of view more satisfactory than VaR. The difficulty of replacing VaR by ES lies within the backtesting of a model, which is also admitted by the Basel committee. ES backtesting
methods rely on and have been developed based on VaR backtesting methods, even though they tend to become more complex. This is the reason why both VaR backtesting and ES backtesting will be investigated in this thesis.

Within the banking industry, liquidity is defined as the ability to meet obligations when they enter into force without provoking major losses. It is obtained by overseeing cash flows and keeping a balance between short-term assets and short-term liabilities [32, p. 403]. According to the Basel committee, the definition of a liquidity horizon is the following: "the time required to execute transactions that extinguish an exposure to a risk factor, without moving the price of the hedging instruments, in stressed market conditions" [3, p. 14]. Making liquidity horizons depend on the liquidity of the underlying asset means they should be longer than the current 10 days for illiquid trades [32, p. 8]. Further, this strategy eliminates the need for the regulatory multiplier $M$ to have a minimum value of 3. The Basel committee proposes five liquidity categories ranging from 10 days to 250 days (1 trading year), the shortest horizon being in line with today’s 10-day VaR. In particular, risk factor categories with preassigned liquidity horizons are defined and the banks need to map their risk assets to these risk factor categories, see Table 1.1.
Table 1.1: Overview of the different risk factor categories and varying liquidity horizons in days

<table>
<thead>
<tr>
<th>Risk factor category</th>
<th>Horizon in days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest Rate</td>
<td>X</td>
</tr>
<tr>
<td>Interest Rate ATM Vol</td>
<td>X</td>
</tr>
<tr>
<td>Interest Rate (other)</td>
<td>X</td>
</tr>
<tr>
<td>Credit Spread - sovereign (IG)</td>
<td>X</td>
</tr>
<tr>
<td>Credit Spread - sovereign (HY)</td>
<td>X</td>
</tr>
<tr>
<td>Credit Spread - corporate (IG)</td>
<td>X</td>
</tr>
<tr>
<td>Credit Spread - corporate (HY)</td>
<td>X</td>
</tr>
<tr>
<td>Credit Spread-structured(cash&amp; CDS)</td>
<td>X</td>
</tr>
<tr>
<td>Credit Spread (other)</td>
<td>X</td>
</tr>
<tr>
<td>Equity Price (Large cap)</td>
<td>X</td>
</tr>
<tr>
<td>Equity Price (Small cap)</td>
<td>X</td>
</tr>
<tr>
<td>Equity Price (Large cap) ATM Vol</td>
<td>X</td>
</tr>
<tr>
<td>Equity Price (Small cap) ATM Vol</td>
<td>X</td>
</tr>
<tr>
<td>Equity (other)</td>
<td>X</td>
</tr>
<tr>
<td>FX rate</td>
<td>X</td>
</tr>
<tr>
<td>FX ATM volatility</td>
<td>X</td>
</tr>
<tr>
<td>FX (other)</td>
<td>X</td>
</tr>
<tr>
<td>Energy price</td>
<td>X</td>
</tr>
<tr>
<td>Precious metal price</td>
<td>X</td>
</tr>
<tr>
<td>Other commodities price</td>
<td>X</td>
</tr>
<tr>
<td>Energy price ATM Vol</td>
<td>X</td>
</tr>
<tr>
<td>Precious metal price ATM Vol</td>
<td>X</td>
</tr>
<tr>
<td>Other commodities price ATM Vol</td>
<td>X</td>
</tr>
<tr>
<td>Commodity (other)</td>
<td>X</td>
</tr>
</tbody>
</table>

The specific regulatory adjustment equation is defined as follows:

\[ ES = \sqrt{\left( ES(Q_1)\sqrt{\frac{H_1}{T}} \right)^2 + \left( \sum_{j=1}^{5} ES(Q_j)\sqrt{\frac{H_j - H_{j-1}}{T}} \right)^2} \]

where \( T = 10 \) days represents the base horizon and \( Q_j \) for \( j = 1, \ldots, 5 \) represents the five regulatory liquidity horizons being 10, 20, 60, 120 and 250 days. \( H_j \) also corresponds to the liquidity horizon in days and \( H_j - H_{j-1} \) corresponds to the incremental liquidity horizon in days. \( ES(Q_1) \) consists of all the risk factors while \( ES(Q_j) \) is said to be incremental and represents only the subset of risk factors having a liquidity horizon at least as long as \( j \). For instance, if \( j = 3 \), it corresponds to risk factors with a liquidity horizon greater than or equal to 60 days. Then, \( ES \) is obtained from the base horizon
for all risk factors, $ES(Q_1)$, as well as the sum of $ES(Q_j)$ for subsets of risk factors with longer liquidity horizons. Hence, for each successive $ES(Q_j)$ computation one successively leaves out risk factors that do not have liquidity horizon at least $j$ [32, p. 165-166]. The Basel Committee is hoping that such a framework will deliver a more graduated treatment of risks as well as serving to reduce arbitrage opportunities between the banking and trading books.

The FRTB is expected to go live in 2019 [4, p. 4]. Its impact spans further than just changing model methodology. National supervisors are presumed to finish implementation of standards of the revised market risk by January 2019 and to require banks within their country to report according to the new standards by 2020. A couple of UK and other European banks have already started the implementation and are expected to complete most of their changes by the end of 2017 or the first part of 2018 [10].

1.4 SAS Institute

SAS Institute, founded in 1976, is an American IT company that develops and markets a suite of analytics software that supports their customers in accessing, managing, analyzing and reporting data within decision making. SAS stands for "Statistical Analysis System" and the company is the largest privately held software business in the world. Some software within their portfolio is aimed for risk management and targeted customers are banks and other financial institutions. It is of importance for SAS Institute to be informed about the latest regulations in order to meet their customers’ needs. Investigating the FRTB would increase SAS Institute’s understanding of the regulation and act a potential competitive advantage for the company when developing their risk management software and approaching customers that are to implement this particular regulation framework.

1.5 History of Backtesting Market Risk Models

The concept of backtesting, evaluating and validating financial risk models, has been familiar within institutions several years before 1996, when the Basel Committee started to include VaR within their regulation. The methodology of the risk measure had mainly been developed by J.P. Morgan bank since the late 80’s. Early backtests such as the Unconditional Coverage test, initially invented by Kupiec in 1995 [19], had shown weak results in real applications, mainly since it had been developed on artificial portfolios due to lack of real portfolio data. Christoffersen reviewed the test in 1998 and proposed the Conditional Coverage or Independency test [8]. It examines not only if the number of VaR exceedances is in line with how many one
expects to occur, but if they happen independently of each other. The test received criticism due to its low statistical power for real small historical samples, demanding large samples to be available for it to work in practice. Both tests mentioned above are interval based tests. In 2001, Berkowitz proposed a density based test [5]. Also, he criticized the fact that current tests only measure the number of exceedances but not the magnitude of them. The weakness of his density based test on the other hand, was that it relied on information of the shape of the left tail of the portfolio return distribution, information that is often not available in real applications. Further, Christoffersen and Pelletier proposed in 2004 a test called the duration test that remedy the earlier tests and turned out to be robust.

Even though VaR has been very popular within risk management, as banks and other financial entities such as hedge funds publish outcomes of the risk measure on a regular basis, it has turned into a subject of criticism due to the fact it does not qualify as a coherent risk measure. Therefore, Expected Shortfall (ES), introduced by Artzner et. al. in 1999 [1], was developed to fulfill the criteria of being coherent. Comparative papers have been written on the two risk measures, for instance one of Basak and Shapiro in 2001 [2], who found that when large losses arise, risk management with ES leads to smaller losses than when using VaR. So one could ask why the Basel committee has continued to require reporting of VaR and not ES? The principal reason is the difficulty of backtesting ES. Several efforts have been made to develop methods for this purpose. For instance, in 2000, Mc Neil and Frey invented an extreme value approach [23] and a couple of years later in 2004, the functional delta method was invented by Kerkhof and Melenberg [18]. Both approaches rely on asymptotic test statistics which implies the methods become inaccurate when sample sizes are small. In 2008, Wong proposed to use a saddle-point or small sample asymptotic technique [34]. The advantage of the saddle-point technique is that the method is adapted to the given confidence level in the regulation, being a relatively high quantile, implying that exceedances occur very rarely and therefore the method turns out to be a more robust alternative even for small samples. However, its weakness is that it makes an assumption about normal distribution as well as it regards the full distribution conditional standard deviation as a dispersion measure. The factors mentioned above obviously limits the backtesting of ES for the whole sample period. In 2013, Righi and Ceretta remedy these factors by proposing a new method which uses the dispersion of the truncated distribution instead. Also, the method is not limited to the normal distribution, so that the risk manager is free to use whichever distribution that is the most appropriate [28, 29].
1.6 Paper Contribution

The purpose of the thesis is to provide SAS Institute with a comparative study of different market risk management models to compute VaR and ES in the light of the "Fundamental Review of the Trading Book" proposed by the Basel committee outlined above. In particular, the purpose of this essay is to compare market risk measure models under the standard VaR backtest procedures and the ES backtest procedure proposed by Righi and Ceretta. Due to the fact that Basel proposes multiple confidence levels of VaR backtesting as basis for the Expected Shortfall backtesting, this thesis will investigate backtesting of both risk measures involved.

The report is structured in the following way. In Chapter 1, an introduction is given about the topic as well as the value point of the thesis for the audience as well as SAS Institute. In Chapter 2, various essential mathematical concepts related to risk management in general as well as to the topic are presented along with methodology. In Chapter 3, the implementation of the thesis is described step by step. Along with that, results and discussion are presented. Finally in Chapter 4, a summary and conclusions are presented. Also, suggestions for further research are mentioned.
Chapter 2

Preliminary Theory

In this chapter various essential mathematical concepts related to risk management in general as well as to the topic are presented.

2.1 Probability Distributions

2.1.1 Univariate Distributions

Normal Distribution

The Normal distribution, or sometimes referred to as the Gaussian distribution, is the most common continuous probability distribution. It is often used to model real-valued random variables whose distributions are not known.

The probability density of the normal distribution is given as follows:

$$f(x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $\mu$ is the mean or expectation of the distribution and $\sigma$ is the standard deviation.

The simplest version of the normal distribution is the standard normal distribution which occurs for the special case when $\mu = 0$ and $\sigma = 1$. It has the following probability density function:

$$\phi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

The normal distribution is a symmetric and bell-shaped distribution.

Student’s t Distribution

The Student’s t-distribution is, like the normal distribution, symmetric and bell-shaped but with heavier tails. Hence, it is more suited to model a situ-
ation where the values are more spread from its mean.

The density function of the Student’s t-distribution, $f_\nu$, is given as follows:

$$f_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$  \hspace{1cm} (2.1)

where $\Gamma$ is the gamma function and $\nu$ is the degrees-of-freedom parameter.

**Generalized Pareto Distribution**

A distribution that is commonly used to model tails of another distribution is the generalized Pareto distribution (GPD). It plays an essential role in modeling threshold exceedances. The standard cumulative distribution function of the GPD is given as follows:

$$G_{\xi,\beta}(x) = \begin{cases} 1 - (1 + \xi x/\beta)^{-1/\xi}, & \xi \neq 0 \\ 1 - \exp(-x/\beta), & \xi = 0 \end{cases} \hspace{1cm} (2.2)$$

where $\beta > 0$ and $x \geq 0$ when $\xi \geq 0$ and $0 \leq x \leq -\beta/\xi$ when $\xi < 0$.

The parameter $\xi$ is referred to as the shape parameter and $\beta$ is referred to as the scale parameter [24, p. 275]. Distributions whose tails decrease exponentially, such as the normal distribution, yield a shape parameter equal to zero. Distributions whose tails decrease as a polynomial, such as the Student’s t distribution, yield positive shape parameters. Finally, distributions whose tails are finite, such as the beta distribution, yield negative shape parameters.

### 2.1.2 Multivariate Distributions

When studying several assets and hence several risk factors it is of interest to consider multivariate models for their joint distributions. In particular, spherical and elliptical distributions will be considered.

**Spherical Distributions**

A random vector $X$ is spherically distributed if its distribution is spherically symmetric. This is the case if it is invariant under rotations and reflections [24, p. 89]. Letting $O$ be an orthogonal matrix, where $O$ has real entries and $OO^T = I$, $X$ is spherically distributed if

$$OX \overset{d}{=} X \quad \text{for every orthogonal matrix} \ O.$$
An example of a bivariate spherical distribution is the standard normal distribution \( N_d(0, I) \) [15, p. 274-275].

**Elliptical Distributions**

A random vector \( \mathbf{X} \) is elliptically distributed if there exists a vector \( \mu \), a matrix \( \mathbf{A} \) and a spherically distributed vector \( \mathbf{Y} \) such that

\[
\mathbf{X} \overset{d}{=} \mu + \mathbf{AY}
\]

where \( \mathbf{AA}^T = \Sigma \) and \( \Sigma \) is the dispersion matrix of \( \mathbf{Z} \) [24, p. 66].

**Bivariate Normal Distribution**

When considering two assets in particular, the bivariate normal distribution, \( N_d(\mu, \Sigma) \), is of particular interest. Two random variables \( X \) and \( Y \) are said to be bivariate normally distributed with parameters \( \mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2 \) and \( \rho \) if their probability density function is given by:

\[
f_{XY}(x, y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1 - \rho^2}} \exp \left( - \frac{\left( \frac{x-\mu_X}{\sigma_X} \right)^2 + \left( \frac{y-\mu_Y}{\sigma_Y} \right)^2 - 2\rho \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y}}{2(1 - \rho^2)} \right)
\]

where \( \mu_X, \mu_Y \in \mathbb{R}, \sigma_X, \sigma_Y > 0 \) and \( \rho \in (-1, 1) \) are all constants.

**Bivariate Student’s t-distribution**

When considering two assets in particular the bivariate student’s \( t_\nu \)-distribution for different parameters \( \nu \) is of particular interest. Letting \( X_1 \) and \( X_2 \) be two random variables that fulfills the following:

\[
X = \mu + \frac{\nu}{S_\nu} \mathbf{Z}
\]

where \( \mu \in \mathbb{R}, S_\nu \sim \chi^2_\nu \) and \( \mathbf{Z} \sim N(0, \Sigma) \). The variance is only defined for \( \nu > 2 \) and is then \( \frac{\nu}{\nu - 2} \) [15, p. 277-278].

Naturally, this reasoning can be expanded to hold not only in the bivariate case but in the general case where there are more than two assets involved.
2.2 Financial Time Series

2.2.1 Time Series in General

Definition

A time series is a sequence of observations, data points, of a certain quantity or quantities over a continuous time interval. Also, one can say that a time series model for the observed data \( \{ x_t \} \) is a specification of the means and covariances of a sequence of random variables \( \{ X_t \} \) of which \( \{ x_t \} \) is assumed to be a realization.

Let \((X_t)_{t \in \mathbb{Z}}\) be a stochastic process with finite variance, \(\text{Var}(X_t) < \infty\). The mean function of \(\{X_t\}\) is given as follows:

\[
\mu_X(t) \overset{\text{def}}{=} E(X_t), \quad t \in \mathbb{Z}.
\]

The covariance function of \(\{X_t\}\) is given as follows:

\[
\gamma_X(r, s) = \text{Cov}(X_r, X_s), \quad r, s \in \mathbb{Z}.
\]

\[\text{[6, p. 15].}\]

Stationarity

The time series \(\{X_t, t \in \mathbb{Z}\}\) is weakly stationary if the following holds:

1. \(\text{Var}(X_t) < \infty\) for all \(t \in \mathbb{Z}\),
2. \(\mu_X(t) = \mu\) for all \(t \in \mathbb{Z}\),
3. \(\gamma_X(r, s) = \gamma_X(r + t, s + t)\) for all \(r, s, t \in \mathbb{Z}\).

In words, a weakly stationary process is a stochastic process whose joint probability distribution does not change with time.

The time series \(\{X_t, t \in \mathbb{Z}\}\) is strictly stationary if \(\{X_1, \ldots, X_n\}\) and \(\{X_{1+h}, \ldots, X_{n+h}\}\) have the same joint distributions for all integers \(h\) and positive \(n\) \[\text{[6, p. 15].}\]

Autocorrelation

Let \(\{X_t, t \in \mathbb{Z}\}\) represent a weakly stationary time series. The autocovariance function (ACVF) of \(\{X_t\}\) is \n
\[
\gamma(h) = \text{Cov}(X_{t+h}, X_t)
\]
and the autocorrelation function (ACF) is given by

\[ \rho(h) \overset{\text{def}}{=} \frac{\gamma_X(h)}{\gamma_X(0)} \]  

(2.3)

where \( h \) is referred to as the lag [6, p. 16].

The sample ACF can be visualized in the following way with the value of the ACF for each lag, see Figure 2.1. Here the number of lags is 5, \( h = 5 \). The 95% confidence bounds are indicated by blue lines in the figure and if the sample ACF falls outside these confidence bounds it indicates that the data in the time series is not IID, while on the contrary, if sample ACF falls inside the confidence bounds the data can be seen as IID.

![Sample ACF of a financial time series](image)

Figure 2.1: Sample ACF of a financial time series

When regarding a time series model such as an ARCH or GARCH model, the data that is to be modeled occurs by its square in the formula, see equation (2.7) and (2.8). In this case the ACF should be computed for the square of the actual data.

**White noise**

If the stochastic process \((X_t)_{t \in \mathbb{Z}}\) is a sequence of uncorrelated random variables having zero mean and variance \( \sigma^2 \), then \((X_t)_{t \in \mathbb{Z}}\) is stationary with the following covariance function:

\[ \gamma(h) = \begin{cases} \sigma^2, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0 \end{cases} \]
Such a sequence can be referred to as white noise, $(X_t)_{t \in \mathbb{Z}} \sim WN(0, \sigma^2)$ [6, p. 16].

**Trend and Seasonality**

Many time series have a clear trend or sign of seasonality which needs to be taken into consideration when performing time series analysis. Using the classical decomposition model one can decompose the series as follows:

$$X_t = m_t + s_t + Y_t$$

where $m_t$ is a slowly changing function representing the trend component, $s_t$ is a function with period $d$ representing the seasonal component and $Y_t$ is a random noise stationary time series [6, p. 23].

**2.2.2 Properties of Financial Time Series**

For a time series to be considered financial, the observed quantity needs to be of financial nature. For instance stock prices, commodity prices, interest rates, foreign currency rates and indices, taken over time, form financial time series. A time series for a single risk factor can be modeled as a discrete stochastic process $(X_t)_{t \in \mathbb{Z}}$ representing a group of random variables indexed by integers and defined on the probability space $(\Omega, \mathcal{F}, P)$ [17, p. 5].

Financial time series are commonly modeled as being normally distributed and independent over time although these assumptions have been criticized. Assuming normality is debatable since it has been observed that financial time series most often are leptokurtic and have fatter tails than the normal distribution implies as well as being asymmetric which is not in line with the normal distribution being a symmetric distribution [24, p. 49]. Assuming independence over time is a matter of inconsistency due to the fact that financial time series tend to possess clusters of volatility [31, p. 3]. Typically, financial time series consist of peaceful periods followed by more violent periods when data fluctuations are large. These fluctuations are within finance referred to as the volatility. The size of the volatility indicates the risk of the observed asset. Volatility clustering implies that a large financial return is often followed by an absolute large financial return.

Assume that one has a financial time series represented by $\{Y_t, t \in \mathbb{Z}\}$, where the trend- and seasonal components have been extracted. One may therefore assume $\{Y_t, t \in \mathbb{Z}\}$ to be stationary due to the classical decomposition method mentioned in the Section Trend and Seasonality above. For financial time series a stylized fact is that $\{Y_t, t \in \mathbb{Z}\} \sim WN(\mu, \sigma^2)$ since the autocorrelation is almost negligible. However, dependence is often seen
between observation and thus the series is not IID [24, p. 117]. The time series can be modeled as follows:

$$Y_t = \sigma_t Z_t + \mu$$

where $Z_t$ is standard normal white noise, $\sigma_t$ is the volatility as a function of $X_{t-1}, X_{t-2}, \ldots, X_0$ and $\mu$ is the mean value of the distribution [6, p. 353].

### 2.2.3 Descriptive Statistics

#### Kurtosis

The kurtosis measure provides information about the tails of a distribution and is optimally studied together with other similar measures such as the skewness and the interquartile range [13, p. 507]. The kurtosis is defined as the fourth standardized moment given as follows:

$$k[Y] = \frac{\mu}{\sigma^4} = \frac{E[(Y - \mu)^4]}{(E[(Y - \mu)^2])^2}$$  \hspace{1cm} (2.4)

The kurtosis of the normal distribution is 3. A distribution with fatter tails than the normal distribution have kurtosis greater than 3; distributions with less fat tails have kurtosis less than 3 [21].

#### Skewness

Skewness is a measure of the asymmetry of the probability distribution around its mean. Negative skewness indicates that the tail on the left side of the probability density function is longer or fatter than the right side but does not distinguish between these shapes. Contrarywise, positive skewness indicates that the tail on the right side is longer or fatter than the left side. In cases where one tail is long and the other tail is fat, the skewness measure does not provide any information. The skewness is defined to be the third standardized moment given as follows:

$$\gamma_1 = \frac{\mu}{\sigma^3} = \frac{E[(Y - \mu)^3]}{(E[(Y - \mu)^2])^{3/2}}$$  \hspace{1cm} (2.5)

The skewness of the normal distribution (or any perfectly symmetric distribution) is zero [22].

#### Interquartile Range

The interquartile range (IQR) is a measure of statistical dispersion based on dividing data in the distribution into quartiles. It is equal to the difference between the third and the first quartile given as follows:

$$IQR = Q3 - Q1$$  \hspace{1cm} (2.6)
In other terms, the IQR is equivalent to the difference between the 75th and the 25th percentiles. The IQR of the normal distribution is equal to 1.349 [17, p. 17].

### 2.2.4 Capturing Volatility

When volatility is varying over time, which is the case for most financial time series, GARCH models have been proved to be useful. GARCH stands for General AutoRegressive Conditional Heteroscedasticity and is a generalization of an ARCH model. Other models commonly used within time series analysis, such as ARMA models, assume unconditional standard deviation which means the volatility is independent on time and therefore constant. This is the reason why they do not work very well for financial data [30, p. 477].

#### ARCH Model

Letting \( \{Z_t, t \in \mathbb{Z}\} \sim SWN(0,1) \), the stochastic process \( \{Y_t, t \in \mathbb{Z}\} \) represents an ARCH(\( p \)) process if the following holds:

\[
Y_t = \sigma_t Z_t, \quad \text{for all } t \in \mathbb{Z}
\]

where \( \sigma_t \) is determined from the following equation:

\[
\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 \quad \text{for all } t \in \mathbb{Z} \tag{2.7}
\]

where \( \alpha_0 > 0, \alpha_i \geq 0, i = 1, \ldots, p \) and \( Z_t \) is independent of \( \{Y_s, s \leq t\} \) [30, p. 480].

#### GARCH Model

Letting \( \{Z_t, t \in \mathbb{Z}\} \sim SWN(0,1) \), the stochastic process \( \{Y_t, t \in \mathbb{Z}\} \) represents a GARCH(\( p,q \)) process if the following holds:

\[
Y_t = \sigma_t Z_t, \quad \text{for all } t \in \mathbb{Z}
\]

where \( \sigma_t \) is determined from the following equation:

\[
\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \quad \text{for all } t \in \mathbb{Z} \tag{2.8}
\]

where \( \alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0, i = 1, \ldots, p, j = 1, \ldots, q \) and \( Z_t \) is independent of \( \{Y_s, s \leq t\} \) [30, p. 483].
It is now clear that a GARCH process is a generalization of an ARCH process since $\sigma^2$, the squared volatility, can in addition to earlier squared values of the process be dependent of earlier squared volatilities. This is the main reason why GARCH processes are well suited to model volatility clusters. Experiments show that GARCH filtered residuals are almost IID \[31, p. 4\].

A possible drawback of the GARCH model is that it demands a large number of observations to yield accurate parameter estimates.

**GARCH (1,1) Model**

Low order GARCH processes are the most commonly used in practice and GARCH(1,1) in particular since it is considered to be relatively realistic \[30, p. 489\]. The process is defined as follows:

$$Y_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$  \hspace{1cm} (2.9)

**Fitting Data to a GARCH(1,1) Model**

If the sample ACF of a financial time series is small, while the sample ACFs of its absolute values and squares are significantly different from zero, it indicates dependence in the data. A GARCH model is then appropriate to use in order to model the time dependent volatility. The maximum likelihood method can be used to estimate the parameters of the GARCH model. Let $y_1, y_2, \ldots, y_n$ be seen as observations from $Y_1, Y_2, \ldots, Y_n$. To determine the parameters, the likelihood function, essentially being the joint probability density function, is to be maximized with respect to the parameters. The joint probability density function can be written as a product of the conditional density functions as follows:

$$f(Y_1, Y_2, \ldots, Y_T) = f(Y_T|Y_1, Y_2, \ldots, Y_{T-1}) \times f(Y_1, Y_2, \ldots, Y_{T-1}) =$$

$$= f(Y_T|Y_1, Y_2, \ldots, Y_{T-1}) \times f(Y_{T-1}|Y_1, Y_2, \ldots, Y_{T-2}) \times f(Y_1, Y_2, \ldots, Y_{T-2}) =$$

$$= f(Y_T|Y_1, Y_2, \ldots, Y_{T-1}) \times f(Y_{T-1}|Y_1, Y_2, \ldots, Y_{T-2}) \times \cdots \times f(Y_1)$$  \hspace{1cm} (2.10)

When the financial time series of returns is assumed to be conditionally normal the likelihood function for a GARCH(1,1) model yields the following:

$$L(\alpha_0, \alpha_1, \beta_1, \mu|Y_1, Y_2, \ldots, Y_T) =$$

$$\frac{1}{\sqrt{2\pi\sigma_T^2}} e^{-\frac{(Y_T-\mu)^2}{2\sigma_T^2}} \times \frac{1}{\sqrt{2\pi\sigma_{T-1}^2}} e^{-\frac{(Y_{T-1}-\mu)^2}{2\sigma_{T-1}^2}} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(Y_1-\mu)^2}{2\sigma_1^2}}$$  \hspace{1cm} (2.11)
Since \( \ln(L) \) is a monotonically increasing function of \( L \) one can instead regard the following loglikelihood function when maximizing the likelihood:

\[
\ln(L(\alpha_0, \alpha_1, \beta_1, \mu|Y_1, Y_2, ..., Y_T)) =
- \frac{T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \ln(\sigma_t^2) - \frac{1}{2} \sum_{t=1}^{T} \left( \frac{(Y_t - \mu)^2}{\sigma_t^2} \right)
\] (2.12)

As stated above, for a GARCH(1,1) model, the conditional variance depends on the past through the following iterative relationship:

\[
\sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 + \beta_1 \sigma_{t-1}^2
\] (2.13)

This can be substituted into the loglikelihood function so that it only depends on \( Y_t \) and the parameters \( \alpha_0, \alpha_1 \) and \( \beta_1 \) \[6, 24\]. The initial volatility, \( \sigma_0 \), needs to be estimated and is often assumed to be the standard deviation of the data given as follows:

\[
\sigma_0 = \sqrt{\frac{1}{n} \sum_{i=0}^{n} (y_i - \bar{y})^2}
\] (2.14)

Note that when time series are long enough the estimate for the initial volatility is insignificant \[27\] p. 7-8).

**Properties of GARCH Distribution**

The GARCH model tends to explain much of the volatility clustering in a financial time series and GARCH filtered residuals are expected to be approximately IID, \( Z_t \sim \text{IID}(0, 1) \). Then the following properties are obtained:

1. \( E[\sigma_t Z_t] = 0 \)
2. \( E[\sigma_t Z_t, \sigma_{t-k} Z_{t-k}] = 0, \forall k > 0 \)
3. \( E[\sigma_t^2 Z_t^2] = E[\sigma_t^2] \)

which implies an uncorrelated process, although \( E[\sigma_t^2 Z_t^2, \sigma_{t-k}^2 Z_{t-k}^2] \neq 0 \) and \( E[|\sigma_t Z_t|, |\sigma_{t-k} Z_{t-k}|] \neq 0 \) so the process is able to generate excess kurtosis. This is easily seen by applying Hölder’s inequality to the kurtosis of \( \sigma_t Z_t, k(\sigma_t Z_t) \) according to equation (2.4) as follows:

\[
k(\sigma_t Z_t) = \frac{E(\sigma_t Z_t)^4}{(E(\sigma_t Z_t)^2)^2} = k(Z_t) \frac{E\sigma_t^4}{(E\sigma_t^2)^2} \geq k(Z_t)
\]

The unconditional distribution of \( \sigma_t Z_t \), which is equal to the distribution of \( Y_t \), is intuitively a mixture of distributions with small variances and distributions with large variances. We expect the GARCH filtered residuals to be
closer to the normal distribution than the unfiltered ones [32, p. 100].

To test whether the GARCH filtered residuals have kurtosis and skewness matching a normal distribution, a Jarque-Bera test can be performed. The test statistic is defined as follows:

$$JB = \frac{n - j + 1}{6} \left( \hat{\gamma}_1^2 + \frac{1}{4} (\hat{k} - 3)^2 \right)$$

(2.15)

where \(n\) is the number of observations, \(\hat{\gamma}_1\) is the sample skewness, \(\hat{k}\) is the sample kurtosis, and \(j\) is the number of regressors. The null hypothesis is that both skewness and the excess kurtosis is zero. Samples from a normal distribution have an expected skewness of 0 and an expected excess kurtosis of 0, which corresponds to a kurtosis of 3 [20].

**Forecasting Volatility**

To enable computation of future values of financial returns and portfolio values, the future volatilities need to be estimated.

By assuming for the GARCH(1,1) parameters that \(\alpha_1 + \beta_1 < 1\), the \(k\) step forecasted volatility can be computed. Start by considering the unconditional variance, \(\sigma^2\), that can be derived as follows:

\[
\text{Var}(Y_t) = E[Y_t^2] - (E[Y_t])^2 = \\
= E[Y_t^2] = \\
= E[\sigma_t^2 Z_t^2] = \\
= E[\sigma_t^2] = \\
= E[\alpha_0 + \alpha_1 Y_{t-1}^2 + \beta_1 \sigma_{t-1}^2] = \\
= \alpha_0 + \alpha_1 E[Y_{t-1}^2] + \beta_1 \sigma_{t-1}^2 = \\
= \alpha_0 + (\alpha_1 + \beta_1) E[Y_{t-1}^2]
\]

Due to the fact that \(Y_t\) is a stationary process \(\text{Var}(Y_t) = \text{Var}(Y_{t-1}) = E[Y_{t-1}]\) and hence the following holds:

\[
\text{Var}(Y_t) = \alpha_0 + (\alpha_1 + \beta_1) \text{Var}(Y_t)
\]

Therefore, the following holds:

\[
\text{Var}(Y_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}
\]

Deriving the forecast of the next period’s volatility, \(k = 1\), can be done by considering the GARCH(1,1) model, \(\sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 + \beta_1 \sigma_{t-1}^2\) and
substituting \( t \) with \( t + 1 \) as follows:

\[
\hat{\sigma}_{t+1}^2 = \alpha_0 + \alpha_1 E[X^2_t | I_{t-1}] + \beta_1 \sigma_t^2 = \\
= \alpha_0 + \alpha_1 \sigma_t^2 + \beta_1 \sigma_t^2 = \\
= \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2 = \\
= \sigma^2 + (\alpha_1 + \beta_1)(\sigma_t^2 - \sigma^2) = \\
= \frac{\alpha_0}{1 - \alpha_1 - \beta_1} + (\alpha_1 + \beta_1)(\sigma_t^2 - \frac{\alpha_0}{1 - \alpha_1 - \beta_1}) = \\
= (\alpha_1 + \beta_1)(\sigma_t^2 - \frac{\alpha_0}{1 - \alpha_1 - \beta_1}) + \frac{\alpha_0}{1 - \alpha_1 - \beta_1}
\]

By generalizing the formula to apply for an arbitrary \( k \), the \( k \) step forecasted volatility can be computed as follows:

\[
\hat{\sigma}_{t+k}^2 = E[\sigma_{t+k}^2 | \sigma_t^2] = (\alpha_1 + \beta_1)^k \left(\sigma_t^2 - \frac{\alpha_0}{1 - \alpha_1 - \beta_1}\right) + \frac{\alpha_0}{1 - \alpha_1 - \beta_1}
\]

An example of the forecasted volatility has been visualized in Figure 2.2. Here, a time series with 100 data points have been used to make a 30 day forecast.
2.2.5 QQ Plot

QQ plot stands for quantile-quantile plot and is a graphical method to study the distributional properties of empirical data. It plots the sample quantiles of the empirical data versus theoretical quantiles from a chosen distribution. See Figure 2.3 for an example where empirical data is represented by a financial time series and the standard normal distribution is chosen to represent the theoretical distribution.

The shape of the qq plot indicates properties of the empirical data. If the blue line in the graph is linear, it indicates that the empirical data has the same distribution as the theoretical one. If the blue line is S-shaped it indicates that the empirical data has less fat tails than the theoretical distribution, while if it has an inverted S-shape, which is the case in Figure 2.3 it indicates that empirical data has got fatter tails than the theoretical distribution. Hence, in Figure 2.3 the financial time series has got fatter tails than the tails of a standard normal distribution, which is in line with what one can expect since the empirical data is financial [15, p. 236].

Figure 2.3: QQ plot of financial time series against standard normal distribution
2.3 Risk Measures

There exist several ways to measure financial risk. The easiest way is probably to measure the variance of the portfolio returns. However, within financial applications, variance is not a sufficiently good risk measure. As it is defined as the expected square deviation from the mean value, it does not take into account if deviations are positive or negative which is of great importance within financial risk analysis since a positive deviation implies a portfolio profit and a negative deviation implies a portfolio loss. A more suitable risk measure would make difference between positive and negative deviations as well as measuring risk in monetary units. Thus, the risk is translated to the amount of buffer capital needed to be added to a portfolio to assure no unwished outcomes. Before presenting the most commonly used risk measures within finance, meeting the requirements of being suitable, some general theory of risk measures is presented.

According to Hult et. al. [15, p. 161-162], there are several proposed requirements for a risk measure to be considered as good. If \( \rho(X) \) is a function that measures the risk of a stochastic variable \( X \) it is equivalent to the minimum capital needed to be added to the portfolio at time 0 and invested in the reference instrument in order to make the position acceptable. Then the properties of the risk measure can be stated as follows:

Translation invariance (TI):
\[
\rho(X + cR_0) = \rho(X) - c, \quad \forall c \in \mathbb{R}.
\]
By adding the amount \( c \) with risk-free interest rate \( R_0 \) to a portfolio, it reduces the risk by the same amount.

Monotonicity (M):
If \( X_2 \leq X_1 \), then \( \rho(X_1) \leq \rho(X_2) \).
If one knows for sure that portfolio \( X_1 \) will have greater value than portfolio \( X_2 \) in the future, then portfolio \( X_1 \) must be considered less risky.

Convexity (C):
\[
\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \rho(X_1) + (1 - \lambda)\rho(X_2), \quad \text{for any real } \lambda \in [0, 1].
\]
Diversification is rewarded, meaning that spreading one’s investment on several risky positions is better than investing all money in one asset.

Normalization (N):
\[
\rho(0) = 0.
\]
It is acceptable not to invest in any risky assets at all.

Positive homogeneity (PH):
\[
\rho(\lambda X) = \lambda \rho(X), \quad \forall \lambda \geq 0.
\]
Doubling the amount invested in one position, doubles the risk.

**Subadditivity (S):**
\[ \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2). \]

Similarly as to the convexity property, diversification is rewarded, meaning that spreading one’s investment on several risky positions is better than investing all money in one asset.

Different risk measures hold different properties of the ones mentioned above. There are different classes of risk measures. *Coherent measures of risk* satisfies the properties of (TI), (M), (PH) and (S), while *convex measure of risk* satisfies the properties of (TI), (M) and (C) [15, p. 161].

The two most commonly used risk measures considered suitable within finance are *Value at risk* and *Expected shortfall*.

### 2.3.1 Value at Risk

The quantitative risk measure Value at risk (VaR) at level \( p \in (0, 1) \) of a portfolio with value \( X \) at time 1 is given by the following expression:

\[
VaR_p(X) = \min \{ m : P(mR_0 + X < 0) \leq p \}
\]

where \( R_0 \) is the return of a risk-free asset. Translating it into words, the VaR of a position with value \( X \) at time 1 is the smallest amount of money that if added to the position now and invested in the risk-free asset ensures that the probability of a strictly negative value at time 1 is not greater than \( p \) [15, p. 165].

Letting \( X \) be the net gain from the investment, \( X = V_1 - V_0R_0 \), where \( V_0 \) represents the current portfolio value, the discounted loss is computed as follows: \( L = -X/R_0 = V_0 - V_1/R_0 \). In statistical terms \( VaR_p(X) \) is the \((1 - p)\)-quantile of \( L \). If the distribution function of \( L \) is denoted \( F_L \), then

\[
VaR_p(X) = F_L^{-1}(1 - p)
\]


The empirical estimate of VaR can be computed, if given the sample \( L_1, \ldots, L_n \), by the following expression:

\[
\hat{VaR}_p(X) = L_{[np]+1,n}
\]

where \( L_{1,n} \geq \cdots \geq L_{n,n} \) is the ordered sample [15] p. 210].
One of the issues that can arise when dealing with VaR is that there exists possibilities of "hiding risk in the tail". This implies that depending on which level $p$ that has been chosen, VaR does not provide any information about the worst scenario outcomes corresponding to an event whose probability is beyond $p$. Hence, VaR does not take into account what happens far out in the left tail of the distribution of $L$ and therefore the investor has a possibility of hiding risk in that tail [15, p. 175].

Another issue that can arise with VaR is that reducing risk by diversification is not necessarily rewarded. In general an investor would make use of the concept of diversification to reduce the risk since, according to theory, a highly diversified portfolio, investing in many different assets, will have a future value that depends on many independent sources of randomness. Hence, the exposure to risk for one of the assets in particular will become smaller than if all of the initial capital is invested in only one of the assets. Since diversification is not necessarily rewarded the property of subadditivity does not hold and thus VaR is not a coherent risk measure [15, p. 176].

### 2.3.2 Expected Shortfall

Despite the fact that VaR is widely used as a risk measure within the financial industry, it possesses several limitations. According to Einhorn, VaR is like "an airbag that works all the time, except when you have a car accident" [11]. Its major weakness is its ignorance of the far out left tail of the distribution of $X$, beyond level $p$. Potentially, this can lead to careless or in the worst of cases dishonest risk management when missing respectively hiding unlikely but disastrous risks in the left tail. A more accurate risk measure would be the Expected Shortfall (ES), an average of the VaR values below level $p$, defined as follows:

$$ES_p(X) = \frac{1}{p} \int_0^p \text{VaR}_u(X) du$$

ES satisfies the properties of translation invariance, monotonicity, positive homogeneity and sub-additivity. Hence ES is a coherent risk measure.

The empirical estimate of ES is given by the following expression:

$$\hat{ES}_p(X) = \frac{1}{p} \left( \sum_{k=1}^{[np]} \frac{L_{k,n}}{n} + \left( p - \frac{[np]}{n} \right) L_{[np]+1,n} \right)$$

[15, p. 211-212].

ES can also be defined as a conditional mean as follows:

$$ES_p = E(X_t | X_t < \text{VaR}^+_p)$$  \hspace{1cm} (2.16)
which represents the expected value of the day \( t \) loss, conditional on being worse than VaR if \( X_t \) represents the \( t \)-day return [23, p. 3].

### 2.3.3 Risk Distortion Measures and Convexity of Risk Measures

A general way to study risk measures is to regard risk distortion measures that are based on a so-called risk distortion function, allowing a risk measure to have different weights for different parts of the loss distribution.

If one regards a historical realization of a loss distribution given by \( \{L_d\}_{d=1}^D \) with empirical cumulative distribution function \( F_L \), a distortion function \( g(F_L) \) can be defined so that the following holds:

1. \( g \) is non-decreasing and right-continuous
2. \( g(0) = 0 \)
3. \( g(1) = 1 \)

A risk distortion measure is then the expected value of this distorted distribution \( g \). Further, a risk distortion measure is not necessarily coherent. For instance, VaR is a risk distortion measure with the corresponding distortion function:

\[
g = 0 \quad \text{if} \quad t \leq \alpha \quad \text{and} \quad g = 1 \quad \text{if} \quad t > \alpha
\]

where \( \alpha \) is the confidence level and \( t \) is the cumulative probability of \( F_L \).

Also, ES is a risk distortion measure with the following distortion function:

\[
g = 0 \quad \text{if} \quad t \leq \alpha \quad \text{and} \quad g = \frac{t - \alpha}{1 - \alpha} \quad \text{if} \quad t > \alpha
\]

In practice, distortion functions are helpful in measuring the sensitivity of risk and risk contributions to the risk measure itself [32, p. 93-96].

According to what has been mentioned earlier in this chapter where the properties of convex and coherent risk measures have been defined, a coherent risk measure is also a convex risk measure, even though the opposite does not hold [15, p. 162]. The good theoretical properties of ES above VaR becomes evident when studying the distortion function of ES. If \( g \) is convex, then the risk distortion measure is also convex [32, p. 96].

### 2.4 Risk Measure Models

Several methods exist to compute the risk measures. Some of the most common methods will be described below. In 2012 it was estimated that
85% of large banks use historical simulation while 15% use Monte Carlo simulation to compute VaR [25, p. 14].

2.4.1 Delta Method

The delta method is analytical and is the simplest method for computing the risk of a portfolio. It is a linear portfolio approach and relies on the assumption that risk factor returns and the portfolio value are multivariate normally distributed, \( N_d(\mu, \Sigma) \), where \( \mu \) is the mean vector and \( \Sigma \) is the covariance matrix of the distribution [32, p. 24]. The portfolio’s variance \( \sigma_{portfolio}^2 \) can then be computed as follows:

\[
\sigma_{portfolio}^2 = w'\Sigma w
\]  

where \( w \) is the weight of each asset in the portfolio [32, p. 27]

VaR at confidence level \( p \) can then be computed as follows

\[
VaR_p = N^{-1}(p)\sigma_{portfolio} = Z_p\sigma_{portfolio}
\]  

where \( N \) is the cumulative distribution of the normal and \( Z_p = N^{-1}(p) \) is the \( p \)-quantile of the univariate standard normal distribution.

Further, ES can be computed as follows:

\[
ES_p = \sigma_{portfolio}^2 \lambda \left( \frac{VaR_p}{\sigma_{portfolio}} \right) = VaR_p \left( \frac{\phi(Z_p)/(1 - N(Z_p))}{Z_p} \right)
\]  

since \( \lambda(v) = \frac{\phi(v)}{1 - N(v)} \) is the hazard function for the normal distribution [32, p. 28-29].

A strength of the delta approach is that it yields a simple analytical solution when computing VaR. Also, the assumption of normality is straightforward and turns out to be accurate for portfolios consisting of linear instruments. However, the method possesses weaknesses as well. First, when having for instance a portfolio consisting of non-linear instruments, linearization may not always be an accurate approximation of the relationship between the true distribution of the losses and the risk factor returns. Secondly, it has been observed that the assumption of normality is unlikely to hold for a distribution of risk factor returns which is preferably modeled as leptokurtic, heavier-tailed [24, p. 49]. The delta method can quite easily be extended to the multivariate Student’s t case [32, p. 31].

For the multivariate Student’s t distribution VaR can be computed as follows:

\[
VaR_p = t_{\nu}^{-1}(p)\sigma_{portfolio}
\]  

26
where \( t^{-1}_\nu(p) \) is the inverse of the t-distribution function with degrees-of-freedom parameter \( \nu \).

Further, ES can be computed as follows:

\[
ES_p = \sigma_{portfolio} f_\nu(t^{-1}_\nu(p) \frac{\nu + (t^{-1}_\nu(1-p))^2}{\nu - 1})
\]

where \( f_\nu \) is the probability density function of the Student’s t distribution given in equation (2.1) above. Inserting the expression into the density function yields the following expression:

\[
ES_p = \sigma_{portfolio} \frac{\Gamma(\frac{\nu+1}{2})}{1 - p} \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu \pi}} \left( 1 + \frac{(t^{-1}_\nu(p))^2}{\nu} \right)^{-\frac{\nu+1}{2}} \left( \frac{\nu + (t^{-1}_\nu(p))^2}{\nu - 1} \right) \tag{2.21}
\]


2.4.2 Historical Simulation

Historical simulation is a non-parametric method for constructing the cumulative distribution function of asset returns over time using historical observations. If observations consist of asset prices \( S_t \), one can derive the returns as \( R_t = \frac{S_t}{S_{t-1}} \). The empirical cumulative distribution is determined by the bootstrap method which is essentially drawing with replacement.

The strengths of historical simulation are that it is a relatively easy method to implement and is fast to run since it is not computationally intensive. Also, it does not assume any particular distribution of the asset returns since it is non-parametric. This implies that no estimation of volatility or correlation is required - all you need is the time series. However, there are some weaknesses of historical simulation. Firstly, lack of observations may be a problem, in particular when new instruments arrive on the market and there is not enough data available. Also, it assumes by definition that history will repeat itself which is not consistent since there might be some scenarios that have not been captured by history. In particular there is often a lack of data during times of very high volatility since they do not occur that often. Further, the method of historical simulation applies equal weights to all returns of the whole period which is inconsistent with the diminishing predictability of data that are further away from the present. Finally, the historical simulation approach assumes the returns to be IID random variables, which is not always true in reality, especially not when considering financial time series data.

2.4.3 Monte Carlo Simulation

Monte Carlo simulation is a non-parametric method. It is a computational algorithm based on repeated random sampling to obtain an approximate
numerical solution to a quantitative problem. In particular, it simulates the underlying process and then calculates the average result of the process. First, stochastic processes for varying risk factors are specified and correlations and volatilities are estimated from data of risk factor returns. Secondly, return paths are simulated for every risk factor a large number of times. Finally, return relations are collected to a joint distribution of returns, from which risk measures can be computed. In a sense, historical simulation is a special case of a Monte Carlo simulation where risk factor returns are drawn directly from historical data. It is of interest to note that for the Monte Carlo simulation we choose the number of replicates on our own compared to the historical simulation approach which yields that one can determine it to be much greater and hence result in greater accuracy for this method [24, p. 52].

A strength of the Monte Carlo simulation method is that no assumptions of normality are needed. It is an accurate method in the sense that a lot of data is simulated and analysed in order to compute the risk measure. Compared to the delta method, this method is well suited for portfolios with non-linear instruments since there is no assumption about linearized losses. On the other hand, Monte Carlo simulation is computationally intensive and takes more time to run than any of the other methods mentioned. However, the method is expected to become even more popular in the future when the cost of computer hardware decrease and the run times decrease.

2.5 Extreme Value Theory

Extreme value theory (EVT) is the probabilistic theory of extreme scenarios. In other words, large positive and negative outcomes are of interest. In practice, there exists two different methods for extreme value analysis, the Block Maxima method and the Peaks over Threshold method (POT).

2.5.1 Heavy Tails

Empirical studies indicate that daily log returns of financial assets can be modeled by distributions with heavy left tails. The left tail, $F(x)$, is heavy if the following holds:

$$
\lim_{x \to -\infty} \frac{F(x)}{e^{-\lambda x}} = \infty \quad \text{for every } \lambda > 0
$$

where $F$ is the distribution function of log returns. In words, $F$ has a heavy tail if the tail probability $F(x)$ decays slowly as $x$ decreases (when $x \to -\infty$). The normal distribution has a light tail while Student’s t distribution has a heavy tail [15, p. 254].
2.5.2 The Block Maxima Method

Generalized Extreme Value Distribution

The generalized extreme value (GEV) distribution is necessary when studying extreme scenarios. The distribution function of the standard GEV distribution is given as follows:

$$H_\xi(x) = \begin{cases} \exp(-(1 + \xi x)^{-1/\xi}), & \xi \neq 0 \\ \exp(-e^{-x}), & \xi = 0 \end{cases}$$

where $1 + \xi x > 0$. One will obtain a three-parameter family by defining $H_{\xi,\mu,\sigma}(x) \overset{def}{=} H_{\xi}((x - \mu)/\sigma)$ for a real-valued location parameter $\mu$ and a scale parameter $\sigma > 0$. $\xi$ is denoted the shape parameter of the GEV distribution. The distribution is generalized in the sense that the form comprises three different types of distribution subject to the value of $\xi$. One has a

- Fréchet distribution when $\xi > 0$
- Gumbel distribution when $\xi = 0$
- Weibull distribution when $\xi < 0$


Central Limit Theorem

Assuming that $X_1, \ldots, X_n$ are IID random variables with finite variance, the Central Limit Theorem (CLT) says that the normalized sum $(S_n - a_n)/b_n$ converges in distribution when $n \to \infty$. Here $S_n = X_1 + \cdots + X_n$, $a_n = nE(X_1)$ and $b_n = \sqrt{\text{Var}(X_1)}$. Mathematically this is described as follows:

$$\lim_{n \to \infty} P\left( \frac{S_n - a_n}{b_n} \leq x \right) = \Phi(x) \quad x \in \mathbb{R}$$

Block Maxima

One needs to consider a sequence of IID random variables $X_1, \ldots, X_n, n \geq 1$ that represents losses. A block maxima is defined as the normalized maxima $M_n = \text{max}(X_1, \ldots, X_n)$ of IID random variables and a block minima is defined as the normalized minima $M'_n = \text{min}(X_1, \ldots, X_n)$. Further, the block maxima will be used throughout the theory although the derivations concerning the minima are analogous. Now, one needs to assume that the block maxima converges in distribution under a suitable normalization. Recalling the following:

$$P(M_n \leq x) = P(X_1 \leq \ldots, X_n \leq x) = F^n(x),$$
one can conclude, according to the CLT, that there exists real constants $d_n$ and $c_n$, $c_n > 0$, such that

$$\lim_{n \to \infty} P\left(\frac{(M_n - d_n)}{c_n} \leq x\right) = \lim_{n \to \infty} F^n(c_n x + d_n) = H(x)$$

(2.22)

for some non-degenerate distribution function $H(x)$.

If equation (2.22) holds, then $F$ is said to be in the maximum domain of attraction of $H$, $F \in \text{MDA}(H)$ [24, p. 265-266]. In most statistical applications the continuous distributions are in $\text{MDA}(H_\xi)$ for some value of $\xi$ [24, p. 267].

**Fisher-Tippett-Gnedenko Theorem**

If $F \in \text{MDA}(H)$, then $H$ must be a $H_\xi$-distribution, in words a GEV distribution [24, p. 266].

**The Method**

The preliminary step is to derive the block maxima (minima) series from the observations. The most common is to extract the annual maxima (minima). Further, one supposes that these extracted extreme observations are from an unknown distribution $F$ that is assumed to lie within the domain of attraction of an extreme value distribution $H_\xi$ for some $\xi$. After applying time series analysis one can say if the observations are realizations of IID random variables. In that case, by applying the Fisher-Tippett-Gnedenko theorem, the GEV distribution is selected for fitting the data and the true distribution of the $n$-block maximum $M_n$ can be approximated for a large $n$ by the distribution function $H_{\xi, \mu, \sigma}$. One will need to divide daily data into yearly blocks in order to get repeated observations of the $n$-block maximum. The GEV distribution can be fitted using various methods, including maximum likelihood estimation [24, p. 271-272].

The major weakness of the block maxima method is the waste of data since it only uses maximum losses of large blocks for analysis [24, p. 275]. This is the main reason why another method, the Peaks over Threshold method, has become even more used in practice.

**2.5.3 Peaks over Threshold Method**

The Peaks over Threshold (POT) method is being commonly used and is based on threshold exceedance. The model analyzes how many observations that exceed some high level and is suited for practical applications when data of extreme scenarios is limited [24, p. 264].
Generalized Pareto Distribution

The generalized Pareto distribution (GPD), presented in Chapter 2 with distribution function, see equation (2.2), is essential when modeling threshold exceedance. Similarly to the GEV distribution, the GPD can be generalized and possesses the following special cases:

Ordinary Pareto distribution when \( \xi > 0 \)

Exponential distribution when \( \xi = 0 \)

Short-tailed Pareto type II distribution when \( \xi < 0 \)

Also, \( G_{\xi,\beta} \in \text{MDA}(H_\xi) \) for all \( \xi \in \mathbb{R} \), thus the GPD yields a natural model for the excess distribution over a high threshold within EVT [24, p. 276].

The Excess distribution

Letting \( X \) be a random variable with distribution function \( F \). Then, the distribution function of the excess distribution over threshold \( u \) yields as follows:

\[
F_u(x) = P(X - u \leq x | X > u) = \frac{F(x + u) - F(u)}{1 - F(u)} \quad \text{for } 0 \leq x < x_F - u
\]

where \( x_F \leq \infty \) is the right endpoint of \( F \).

The Mean Excess

A random variable with finite mean has the following mean excess function:

\[
e(u) = E(X - u | X > u)
\]

[24, p. 276].

Pickands-Balkema-de Haan Theorem

There exists a positive-measurable function \( \beta(u) \) such that

\[
\lim_{u \to x_F} \sup_{0 \leq x < x_F - u} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0
\]

if and only if \( F \in \text{MDA}(H_\xi) \) and \( \xi \in \mathbb{R} \). Thus, when the threshold is raised and \( u \) becomes close to \( x_F \), \( F_u(x) \approx G_{\xi,\beta(u)}(x) \), implying that a distribution for which a normalized maxima converges to a GEV distribution is a distribution for which the excess distribution converges to the GDP [24, p. 277-278].

31
The Method

Suppose one starts with observations of losses, $X_1, \ldots, X_n$, from a loss distribution $F$. Further, one assumes that $F \in \text{MDA}(H_\xi)$. Also, according to the Pickands-Balkema-de Haan theorem, letting $x_F$ be the right endpoint of $F$ one assumes that, for some high threshold $u$, $F_u(x) = G_{\xi,\beta}(x)$ where $0 < x < x_F - u$, $\xi \in \mathbb{R}$ and $\beta > 0$. A random number of the observations, $N_u$, will exceed the threshold level where

$$N_u = \{ X_i > u, 0 \leq i \leq n \}$$

and these are renamed $\hat{X}_1, \ldots, \hat{X}_{N_u}$. For each loss exceedance one needs to compute the size of it, $Z_j = \hat{X}_j - u$. Finally, in order to determine the excess distribution and if the excess data turns out to be IID, one can fit the GPD by, for instance, maximum likelihood estimation to estimate the parameters, and so one has obtained $F_u$ [24, p. 278].

Hybrid Distribution

Based on the POT method described above, a hybrid distribution can be obtained. Given approximately IID random variables being for instance filtered residuals, $Z_{i,t}$, with unknown distribution function $F$ and a high threshold $u_{\text{high}}$, the excesses $Z_{i,t} - u_{\text{high}}$ can be modeled by the GPD with standard cumulative distribution function, see equation (2.2).

The hybrid distribution is set up in the following way. The threshold for the upper tail, $u_{\text{high}}$, is set to some fixed high quantile of the empirical distribution, $u_{\text{high}} = F^{-1}_n(q_{\text{high}})$. Further, it is possible to model the lower tail with a GPD as well. If the lower tail of the empirical distribution only consists of negative values below some negative threshold $u_{\text{low}}$ one may consider the absolute values and the excess of an observation in the lower tail is then the distance from the observation to $u_{\text{low}}$. Hence, $u_{\text{low}}$ is set to be some low quantile of the empirical distribution, $u_{\text{low}} = F^{-1}_n(q_{\text{low}})$.

The GPD can only be used to model the tails of a distribution, not the entire distribution. Therefore some other distribution, for instance the empirical distribution, will be used to model the distribution between the lower and upper tail.

The entire distribution can then be expressed by considering two cases. In the first case one can consider excesses above some threshold $u_{\text{high}}$ of a distribution being modeled with the GPD $G_{\xi,\beta}^{\text{high}}(x - u_{\text{high}})$. For the upper tail, $x \geq u_{\text{high}}$, the cumulative distribution function is given by the following
expression:

\[ P(X \leq x) = P(X \leq u_{\text{high}}) + P(u_{\text{high}} \leq X \leq x) = P(X \leq u_{\text{high}}) + (1 - P(X \leq u_{\text{high}}))G^{\text{high}}_{\xi,\beta}(x - u_{\text{high}}) \]

As mentioned above, \( P(X \leq u_{\text{high}}) \) is estimated by the empirical distribution function as \( P(X \leq u_{\text{high}}) \approx F_{n}(u_{\text{high}}) \). For \( x < u_{\text{high}} \) the remaining distribution is modeled by the empirical distribution \( F_{n}(x) \). The cumulative distribution function for the entire distribution is then modeled as follows:

\[
F(x) = \begin{cases} 
F_{n}(x) & x < u_{\text{high}} \\
F_{n}(u_{\text{high}}) + (1 - F_{n}(u_{\text{high}}))G^{\text{high}}_{\xi,\beta}(x - u_{\text{high}}) & x \geq u_{\text{high}} 
\end{cases}
\]

In the second case one can consider the reversed scenario, excesses below some threshold \( u_{\text{low}} \) of a distribution are being modeled with the GPD \( G^{\text{low}}_{\xi,\beta}(u_{\text{low}} - x) \). Equivalently to the first case, for \( x > u_{\text{low}} \) the remaining distribution is modeled with the empirical distribution \( F_{n}(x) \). The cumulative distribution function for the entire distribution is modeled as follows:

\[
F(x) = \begin{cases} 
F_{n}(u_{\text{low}})(1 - G^{\text{low}}_{\xi,\beta}(u_{\text{low}} - x)) & x \leq u_{\text{low}} \\
F_{n}(x) & x > u_{\text{low}} 
\end{cases}
\]

Combining the two cases yield the following cumulative distribution function for the entire distribution function:

\[
F(x) = \begin{cases} 
F_{n}(u_{\text{low}})(1 - G^{\text{low}}_{\xi,\beta}(u_{\text{low}} - x)) & x \leq u_{\text{low}} \\
F_{n}(x) & u_{\text{low}} < x < u_{\text{high}} \\
F_{n}(u_{\text{high}}) + (1 - F_{n}(u_{\text{high}}))G^{\text{high}}_{\xi,\beta}(x - u_{\text{high}}) & x \geq u_{\text{high}} 
\end{cases}
\]

Note that the shape and scale parameters \( \xi,\beta \) may differ between the two Pareto distribution functions \( G^{\text{low}}_{\xi,\beta} \) and \( G^{\text{high}}_{\xi,\beta} \). Since the empirical data is not necessarily symmetric and the GPD is univariate, for each asset in the reference portfolio, two unique GPDs need to be modeled to obtain unique univariate cumulative distribution functions.

An example of the GPD in comparison with the standard normal distribution, the Student’s t distribution (\( \nu = 3 \)) and the empirical distribution of a financial time series is given for the tail in Figure 2.4.
2.6 Copulas

2.6.1 Background

When considering a multivariate model with several assets it is of particular interest to study the dependence and correlation structures [24, p. 61]. Modeling these structures are among the most important and challenging tasks within multivariate analysis. Copulas are used to illustrate dependence between random variables and are often used within high-dimensional statistical applications. The conventional way to model dependence between random variables is by constructing the joint distribution function [31, p. 6]. When the joint distribution is difficult to derive, copulas become handy. Copulas were discovered relatively recently and the main reason is the developments of market dynamics and financial instruments that have occurred in the recent past. Earlier, the normal distribution was commonly accepted as a reasonable choice of distribution to model financial returns. However, this view has been radically challenged by the reality of today’s market. The presence of non-normality on the market as well as arrival of non-linear instruments has triggered the need for new modeling methods and the use of copulas in particular [7, p. 1-2].
2.6.2 Definition

A copula is a multivariate probability distribution for which the marginal probability distribution of each variable is uniform \([30, p. 175]\). Assume that one is interested in finding a multivariate model for a set of random variables \(X_1, \ldots, X_d\) whose univariate marginal distributions, say \(F_1, \ldots, F_d\) respectively, are known but whose joint distribution is only partially known. One way to determine the multivariate distribution of \(X_1, \ldots, X_d\) is by considering the probability transform and the quantile transform.

The Probability Transform

The probability transform suggests that if \(X\) is a random variable with a continuous distribution function \(F\), then \(F(X)\) is uniformly distributed on the interval \((0,1)\) \([24, p. 186]\).

The Quantile Transform

The quantile transform suggests that if \(U\) is uniformly distributed and if \(G\) is any distribution function, \(G^{-1}(U)\) has distribution function \(G\) \([24, p. 186]\).

Thus, for any random variables \((X_1, \ldots, X_d)\) whose components have continuous univariate marginal distribution functions \(F_1, \ldots, F_d\), \(U_i = F_i(X_i)\) for \(1 \leq i \leq d\) are uniform \((0,1)\)-variables. Then, by definition the following holds:

\[
F(x_1, \ldots, x_d) = P(X_1 \leq x_1, \ldots, X_d \leq x_d) = P(F_1(X_1) \leq F_1(x_1), \ldots, F_d(X_d) \leq F_d(x_d)) = P(U_1 \leq F_1(x_1), \ldots, U_d \leq F_d(x_d))
\]

and the copula may be presented as follows:

\[
C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))
\]

Hence, the copula is the joint distribution function of \((U_1, \ldots, U_d)\) \([31, p. 6]\).

2.6.3 Dependence Measures

The dependence measures to be investigated all yield a scalar measurement for a pair of random variables \((X_1, X_2)\), suited for the bivariate situation, although they can all be extended to the multivariate situation. First, the traditional dependence measure linear correlation is described followed by the more sophisticated and suitable copula-based dependence measures Kendall’s tau and Spearman’s rho.
Linear Correlation

The linear correlation $\rho(X_1, X_2)$ between the random variables $X_1$ and $X_2$ is defined as follows:

$$\rho(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)} \sqrt{\text{Var}(X_2)}}$$

(2.24)

where $-1 \leq \rho(X_1, X_2) \leq 1$. Also, if $X_1$ and $X_2$ are independent, then $\rho(X_1, X_2) = 0$ even though the opposite does not hold [24, p. 201]. The linear correlation, being a linear dependence measure, has the following property for constants $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$:

$$\rho(\alpha_1 X_1 + \beta_1, \alpha_2 X_2 + \beta_2) = \alpha_1 \alpha_2 \rho(X_1, X_2)$$

Therefore, the linear correlation is invariant under strictly increasing linear transformations. However, under strictly increasing non-linear transformations the following holds:

$$\rho(F(X_1), F(X_2)) \neq \rho(X_1, X_2)$$

Hence, in general, linear correlation is not invariant under strictly increasing transformations and therefore not suited as a dependence measure for copula functions as it depends on the marginal distributions [24, 31].

Kendall’s Tau

Rank correlation coefficients only depend on the copula of the bivariate distribution and not on the marginal distributions as the linear correlation does. One of the rank correlation coefficients is called Kendall’s tau [24, p. 206].

Kendall’s tau is defined as follows:

$$\tau(X_1, X_2) = P[(X_1 - X'_1)(X_2 - X'_2) > 0] - P[(X_1 - X'_1)(X_2 - X'_2) < 0]$$

(2.25)

where $(X'_1, X'_2)$ is an independent copy of $(X_1, X_2)$ and which in words is the probability of concordance minus the probability of discordance [15, p. 283]. Kendall’s tau depends only on the unique copula of the risks why it inherits its property of invariance under strictly increasing transformations.

Spearman’s Rho

Another rank correlation coefficient is Spearman’s rho. For the random variables $X_1$ and $X_2$ with marginal distribution functions $F_1$ and $F_2$, Spearman’s rho is defined as follows:

$$\rho_S(X_1, X_2) = \rho(F_1(X_1), F_2(X_2))$$

(2.26)
which in words can be interpreted as the linear correlation of the probability transformed random variables which for continuous random variables is the linear correlation of their unique copula [24, p. 207]. Similarly as Kendall’s tau, Spearman’s rho depends only on the unique copula of the risks why it inherits its property of invariance under strictly increasing transformations.

Coefficients of Tail Dependence

Similar to the rank correlation coefficients, Kendall’s tau and Spearman’s rho, the coefficients of tail dependence are dependence measures depending only on the copula of a pair of random variables \(X_1\) and \(X_2\). Further, these coefficients measures the magnitude of dependence in the tails of a bivariate distribution [24, p. 208]. The coefficient of lower tail dependence is defined as follows:

\[
\lambda_l(X_1, X_2) \overset{df}{=} \lim_{q \to 0} P[X_2 < F^{-1}_2(u)|X_1 < F^{-1}_1(u)]
\]

provided the limit \(\lambda_l(X_1, X_2)\) exists and \(0 \leq \lambda_l(X_1, X_2) \leq 1\). The coefficient of upper tail dependence is defined as follows:

\[
\lambda_u(X_1, X_2) \overset{df}{=} \lim_{q \to 1} P[X_2 > F^{-1}_2(u)|X_1 > F^{-1}_1(u)]
\]

provided the limit \(\lambda_u(X_1, X_2)\) exists and \(0 \leq \lambda_u(X_1, X_2) \leq 1\) [30, p. 185-186]. If \(\lambda_l(X_1, X_2) = 0\) or \(\lambda_u(X_1, X_2) = 0\), \(X_1\) and \(X_2\) are said to be asymptotically independent, otherwise they are said to be asymptotically dependent in the respective tail. In other words, these coefficients computes the probabilities of joint extreme values of the copula [31, p. 7].

2.6.4 Elliptical Copulas

Elliptical copulas is a class of copulas derived from elliptical distributions. Examples of elliptical distributions are the normal distribution and the Student’s t distribution. The corresponding copulas are the Normal copula and the Student’s t copula.

Normal Copula

The normal copula is defined as follows:

\[
C_{R}^{\Phi}(u_1, \ldots, u_d) \overset{df}{=} P(\Phi(X_1) \leq u_1, \ldots, \Phi(X_d) \leq u_d) = \\
\Phi_{R}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))
\]

where \(X\) has multivariate standard normal distribution, \(\Phi_{R}\) is the multivariate distribution function of \(X\). \(\Phi\) denotes the margins of \(\Phi_{R}\) and \(R\) is the linear correlation matrix of the random variables in \(X\) [15, p. 303].
For the bivariate Normal copula there exists relations between the different rank correlation coefficients and the linear correlation as follows:

\[ \tau(X_1, X_2) = \frac{2}{\pi} \arcsin(\rho) \]  
(2.27)

\[ \rho_S(X_1, X_2) = \frac{6}{\pi} \arcsin\left(\frac{1}{2}\rho\right) \]  
(2.28)

Also, the coefficients of tail dependence are of interest. For the bivariate normal copula it can be shown that the following holds:

\[ \lambda_l(X_1, X_2) = \begin{cases} 
0, & \text{if and only if } \mathbf{R} < 1 \\
1, & \text{if } \mathbf{R} = 1 
\end{cases} \]  
(2.29)

and since the normal distribution is from the class of elliptical distributions, thus being radially symmetric, the following holds:

\[ \lambda_u(X_1, X_2) = \begin{cases} 
0, & \text{if and only if } \mathbf{R} < 1 \\
1, & \text{if } \mathbf{R} = 1 
\end{cases} \]  
(2.30)

Hence, the bivariate normal copula is asymptotically independent both in the lower and upper tail if \( \mathbf{R} < 1 \) [7, p. 116].

**Student’s t Copula**

The Student’s t copula is defined as follows:

\[
C_{\nu,R}(u_1, \ldots, u_d) \overset{\text{def}}{=} P(t_{\nu}(X_1) \leq u_1, \ldots, t_{\nu}(X_d) \leq u_d) = t_{\nu,R}^{d}(t_{\nu}^{-1}(u_1), \ldots, t_{\nu}^{-1}(u_d))
\]

where \( \mathbf{X} \) has multivariate Student’s t distribution with \( \nu \) degrees of freedom, mean \( \mu \) for \( \nu > 1 \) and the covariance matrix \( \frac{\nu}{\nu-2}\Sigma \) for \( \nu > 2 \) which is given as follows:

\[
\mathbf{X} \overset{d}{=} \mu + \frac{\sqrt{\nu}}{\sqrt{S}}\mathbf{Z}
\]

where \( \mu \in \mathbb{R}, S \sim \chi_\nu^2 \) and \( \mathbf{Z} \), being multivariate standard normally distributed, is independent of \( S \). Further, \( t_{\nu,R}^{d} \) is the multivariate distribution function of \( \mathbf{X} \), \( t_{\nu} \) denotes the margins of \( t_{\nu,R}^{d} \) and again \( \mathbf{R} \) is the linear correlation matrix of the random variables in \( \mathbf{X} \) [15, p. 303].

For the bivariate Student’s t copula there exists no relation between the rank correlation coefficient Spearman’s rho and the linear correlation as for
the bivariate normal copula although it exists a relation between Kendall’s tau and the linear correlation as follows:

\[ \tau(X_1, X_2) = \frac{2}{\pi} \text{arcsin}(\rho) \]

[31] p. 8]. Also, the coefficients of tail dependence are of interest. For the bivariate Student’s t copula it can be shown that the following holds:

\[
\lambda_l(X_1, X_2) = \begin{cases} 
> 0, & \text{if and only if } R > -1 \\
0, & \text{if } R = 1
\end{cases}
\] (2.31)

and since the Student’s t distribution is from the class of elliptical distributions, thus being radially symmetric, the following holds:

\[
\lambda_u(X_1, X_2) = \begin{cases} 
> 0, & \text{if and only if } R > -1 \\
0, & \text{if } R = 1
\end{cases}
\] (2.32)

Hence, the bivariate Student’s t copula is asymptotically dependent both in the lower and upper tail if \( R > -1 \) [71 p. 118]. In particular the following holds:

\[
\lambda_u(X_1, X_2) = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu + 1)(1-p)}{1 + \rho}}\right)
\] (2.33)


2.7 Backtesting

Backtesting plays an essential role when it comes to validation of a financial risk model. Models are built in an attempt to monitor the reality. Since models often rely on assumptions, for a model to be useful, it needs to be validated. During the financial crisis in 2007 this became even more evident. Model validation includes expert examination, scenario analysis, stress testing and backtesting. Here, backtesting is seen as the most essential part of the validation. Backtesting is used to estimate the predictive performance of a model as if it had been employed during a past period which requires simulating past conditions in a good way. Hence, one limitation of the backtesting method is clearly the need for detailed historical data. Also, it is limited by potential overfitting which easily occurs when the model is complex.

Moving from a VaR model to an ES model implies radical changes in the backtesting methodology. Both backtesting methods for VaR and ES will be discussed as backtesting methods for ES relies on backtesting methods for VaR [32 p. 122]. Backtesting methods are more developed and empirically tested for VaR than for ES [32 p. 166-167]. According to Christoffersen et. al., much more research has been made within the area of risk measure models themselves than on backtesting methods to validate them [9 p. 86].
2.7.1 Backtesting VaR

A reasonable way to test a VaR measure estimation model would be by an exceedance test. Supposing that one-day VaR at confidence level \( p \), where \( p \) is given in per cent, has been computed for a certain time period. By definition, one would then expect that the VaR estimations would be exceeded by the actual outcome in reality for close to 100-\( p \)% of the total number of days. If this is not the case, the estimation model is not reliable.

Let \( I_t \) at time \( t \) be a VaR exceedance indicator given as follows:

\[
I_t = \begin{cases} 
1, & \text{if } \text{VaR}_t \text{ is exceeded} \\
0, & \text{otherwise}
\end{cases}
\]

where \( I_t \) is also referred to as the hit sequence and takes the value 1 when the actual outcome of the loss in reality exceeds the predicted loss (VaR) and 0 otherwise. The expected value of the exceedance rate is the pre-defined level \( p \), \( E[I_t] = p \).

Since 1996, when the Basel committee started to include VaR as a requirement for the banks, this test made up the basis for what they called the "Traffic lights" that determined the multiplier \( 3 \leq M \leq 4 \) according to the number of exceedances falling into one of three colored zones, see equation (1.1).

Due to the nature of VaR, being a quantile measure, the backtesting method is based on statistical evaluation of an interval estimator. A VaR model is considered accurate at its confidence level \( p \) if it satisfies the properties of unconditional coverage and independence [32, p. 123]. If both properties are satisfied the model is said to satisfy the property of conditional coverage. However, a potential drawback of these properties are that they do not account for the magnitude of the exceedance [9, p. 87].

**Unconditional Coverage**

Having data consisting of observed losses in the past \( T \) periods, it is possible to count how many times the computed VaR for that period was exceeded. Too many exceedances indicate that VaR underestimates the risk; while too few exceedances indicate that VaR is overestimating the risk, hence being too conservative. This binary setup yields that the hypothesis test can use the classical Bernoulli trials since it can be seen as a success/failure test [14, p. 11]. Bernoulli trials assumes a binomial distribution for the number of exceedances as follows:

\[
f(x) = \binom{T}{x} p^x (1 - p)^{T-x}
\]
If $T$ is large enough the binomial distribution can be approximated by the normal distribution $N(Tp, Tp(1-p))$ using the normal approximation. If $T$ is not large enough the bootstrap method will be used in order to simulate more data. A likelihood test is formulated in order to find out whether the observed number of exceedances is equal to the expected value of exceedances. The likelihood function of IID random variables, $I_t$, being Bernoulli ($\lambda$)-distributed is given as follows:

$$L(\lambda) = \prod_{t=1}^{T}(1-\lambda)^{1-I_t} \lambda^{I_t}$$

If $T_0$ represents the number of periods where no exceedances have occurred and $T_1$ represents the number of periods where exceedances have occurred the likelihood function can be rewritten as follows:

$$L(\lambda) = (1-\lambda)^{T_0} \lambda^{T_1}$$

$\hat{\lambda}$ is the maximum likelihood estimation of $\lambda$ and is given by the following ratio:

$$\hat{\lambda} = \frac{T_1}{T}$$

The hypothesis can now be formulated as follows:

$$H_0 : \lambda = E[I_t] = p \quad H_1 : \lambda \neq E[I_t] = p$$

Hence, the null hypothesis tests whether on average the coverage rate is consistent with the VaR model. Under the null hypothesis the likelihood-ratio (LR) statistics is given as follows:

$$LR_C = -2\log \frac{L(p)}{L(\lambda)} \quad (2.35)$$

where the p-value of the statistics is given from the $\chi^2_1$-distribution to investigate the statistical significance. The Basel accord regarding backtesting of VaR investigates the unconditional coverage to specify three review zones based on the number of exceedances for supervisory, where the confidence level is set to $p = 99\%$ [32, p. 123-124].

**Independence**

Within financial risk management the coverage property is not sufficient due to the nature of financial returns where volatility clusters often occur and tend to make exceedances clustered as well [9, p. 85]. An overall evaluation may show acceptable results but local evaluation at any given time turns out to be incorrect if only using the coverage property. Since VaR is a conditional measure it possesses serial independence and therefore the independence property is another important property of a VaR model. Several
methods to test the clustering of exceedances exist, for instance the indepen-
dence test and the duration test.

**Independency Test**

The independency property implies that any two exceedances, \( I_t \), must be independent from each other. Assuming the alternative hypothesis, the random variables \( I_t \) being dependent on each other, their sequence can be modeled as a first-order Markov chain with the following transition probabilities given in the matrix as follows:

\[
\Pi = \begin{bmatrix}
1 - \lambda_{01} & \lambda_{01} \\
1 - \lambda_{11} & \lambda_{11}
\end{bmatrix}
\]

where \( \lambda_{xy} = Pr(I_t = y | I_{t-1} = x) \). Hence, \( \lambda_{01} \) is the probability that a no-exceedance is followed by an exceedance and \( \lambda_{11} \) is the probability that an exceedance is followed by an exceedance. Further, the likelihood function can be constructed as follows:

\[
L(\Lambda) = (1 - \lambda_{01})^{T_{00}} \lambda_{01}^{T_{01}} (1 - \lambda_{11})^{T_{10}} \lambda_{11}^{T_{11}}
\]

where \( T_{xy} \) is the number of times a \( x \) event is followed by a \( y \) event and \( \hat{\lambda}_{01} \) and \( \hat{\lambda}_{11} \) are the maximum likelihood estimations given by the following ratios:

\[
\hat{\lambda}_{01} = \frac{T_{01}}{T_{00} + T_{01}}, \quad \hat{\lambda}_{11} = \frac{T_{11}}{T_{10} + T_{11}}
\]

Also, if \( I_t \) is independent, \( \hat{\lambda}_{01} = \hat{\lambda}_{11} = \lambda = \frac{T_{01} + T_{11}}{T} \). Under the null hypothesis the likelihood-ratio (LR) statistics is given as follows:

\[
LR_I = -2 \log \left( \frac{L(\hat{\lambda})}{L(\Lambda)} \right)
\]

where the p-value of the statistics is given from the \( \chi^2 \)-distribution to investigate the statistical significance [32, p. 124-125].

**Duration Test**

There exists another test to study the independency of exceedances which is called the duration test. It is based on modeling the distribution of the durations, being the periods between the exceedences measured in days [9, p. 86]. If VaR has been perfectly modeled, the durations should have a mean of \( \frac{1}{1-p} \), be exponentially distributed and therefore possess the memoryless
property. Under the alternative hypothesis, the exceedances can be modeled by a Weibull-distribution given as follows:

\[
f(x) = a^b b x^{b-1} \exp(-ax)^b
\]

which possesses the property that when \( b < 1 \), the distribution of durations has memory and hence the exceedances are dependent. However, when \( b = 1 \), the distribution reduces to a special case, the exponential distribution, being memoryless. Under the null hypothesis of independence the likelihood function is given as follows:

\[
L(\beta) = \prod_{t=1}^{T_1-1} \beta \exp(-\beta D_t)
\]

where \( D_t = t_t - t_{t-1} \) is the duration of time between two exceedances with \( t_t \) denoting the day of exceedance number \( t \) and \( T_1 \) is the number of periods for which an exceedance occurred. Under the null hypothesis the likelihood-ratio (LR) statistics is given as follows:

\[
LR_D = -2 \log \left( \frac{L(\beta)}{L(\hat{\beta}, 1)} \right)
\]

where the p-value of the statistics is given from the \( \chi^2_1 \)-distribution to investigate the statistical significance [32, p. 125]. Duration-based tests have shown better statistical power than other tests [9, p. 86].

Note that if one cannot assume that the sequence of \( I_t \)'s can be modeled as a first-order Markov chain there exists a more general form of testing independency [9, p. 88-90].

### 2.7.2 Backtesting ES

It is commonly accepted that ES models are more difficult to backtest than VaR models due to the nature of ES as a risk measure not being quantile-based. It has been proposed that the initial part of backtesting an ES model should be a density estimate evaluation, since the ES is a distributional-based risk measure. Different density forecast evaluation methods exist such as the non-parametric Kolmogorov-Smirnov test or the likelihood-based Berkowitz test [32, p. 125-127]. Further efforts have been made to develop methods for backtesting ES in an accurate way. For instance, the extreme value approach by McNeil and Frey in 2000 and the functional delta method by Kerkhof and Melenberg in 2004. A more robust method was invented by Wong in 2008, the saddle-point technique. However, due to its weaknesses, that it makes an assumption about normal distribution as well as the fact that it regards the full distribution conditional standard deviation as a dispersion measure, a more sophisticated and flexible method was proposed by Righi and Ceretta in 2013. The method uses the dispersion of a truncated distribution by
the estimated VaR upper limit, it is not limited to the normal case and it allows one to test if each individual VaR exceedance is significantly different from the ES. Moreover, a Monte Carlo simulation algorithm is presented to determine the significance of the backtest.

The Method

Assuming financial assets follow a GARCH process, recalling the following:

\[ R_t = Y_t + \mu_t, \quad Y_t = \sigma_t Z_t \]  
\[ \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \quad \text{for all } t \in \mathbb{Z} \]

where \( R_t \) represents the log return of an asset, \( \mu_t \) is the mean of an asset, \( \sigma^2 \) is the variance of an asset; \( \alpha_0, \alpha_i \) and \( \beta_j \) are parameters and \( Z_t \) is white noise. Using equation (2.16) as well as (2.38) yields the following expression for calculating the one-day ES at confidence level \( p \):

\[ ES_p = \mu_{t+1} + \sigma_{t+1} E_{t+1}(Z_{t+1}|Z_{t+1} < F^{-1}(p)) \]  
\[ (2.39) \]

where \( F \) denotes the probability distribution function of \( Z_t \). Now, \( ES_p \) is said to be the truncated distribution of \( Z_{t+1} \) with upper limit \( F^{-1}(p) \). A truncated distribution is a conditional distribution resulting from restricting a domain of some other probability distribution. Further, the dispersion around this expected value, denoted the shortfall deviation (SD), will be considered when backtesting ES. Hence, SD is the square root of the truncated variance for some quantile conditional to the probability \( p \) given as follows:

\[ SD_p = \sqrt{\text{Var}(R_{t+1}|R_{t+1} < \text{Var}R_{t+1})} = \sqrt{\sigma_{t+1}^2 \text{Var}(Z_{t+1}|Z_{t+1} < F^{-1}(p))} \]  
\[ (2.40) \]

In previously mentioned ES backtesting methods the whole sample standard deviation is used, while this test only considers the standard deviation for which very large negative returns occur, which is of particular interest when managing risk [28, p. 2-4].

The next step of the method can be performed either analytically or numerically. A numerical approach suggests estimating \( ES_p \) and \( SD_p \) by numerical simulation. In practice, it is realized by simulating a time series of returns \( R_t \) and then compute VaR as the \( p^{th} \) quantile of the simulated returns’ empirical distribution. Further, \( ES_p \) and \( SD_p \) are computed as the expected value and the standard deviation of the values below the computed VaR. A clear advantage of this approach is the simplicity of implementation. However, a disadvantage is the lack of correctly specified dynamics,
for instance, the ignorance of volatility clustering. A remedy would be to consider a fully parametric approach, based on dynamical modeling, which requires analytical derivations of the expectations and variances, ES and SD.

The moment generating function of a truncated distribution with upper limit \( b \) is obtained as follows:

\[
E(X^n) = \frac{1}{F(b)} \int_{-\infty}^{b} x^n f(x) dx
\]  

(2.41)

where \( F \) is the cumulative distribution function and \( f \) is the density function. Applied to the case considered to compute \( ES_p \) and \( SD_p \), \( X \) is \( Z_t \) and \( b \) is \( F^{-1}(p) \). Equation [2.41] is applicable to any probability distribution with finite moments.

In the case of a truncated standard normal distribution, the analytical form of ES and SD can be derived by first computing the truncated expected value with upper limit \( b \) as follows:

\[
E(X|X < b) = -\frac{\phi(b)}{\Phi(b)}
\]

where \( \phi(b) \) is the density function of the standard normal distribution and \( \Phi(b) \) is the cumulative distribution function of the standard normal distribution \([28, p. 4]\). By substituting this expression with the one in equation (2.39) the following yields:

\[
ES_p = \mu_{t+1} - \sigma_{t+1} \frac{\phi(F^{-1}(p))}{\Phi(F^{-1}(p))}
\]

For a normal distribution \( F^{-1}(p) = \Phi^{-1}(p) \). Hence, the the following holds:

\[
ES_p = \mu_{t+1} - \sigma_{t+1} \frac{\phi(\Phi^{-1}(p))}{p}
\]

Regarding \( SD_p \) one begins with computing the truncated variance with upper limit \( b \) as follows:

\[
Var(X|X < b) = \left( a - b \frac{\phi(b)}{\Phi(b)} - \left( \frac{\phi(b)}{\Phi(b)} \right)^2 \right)
\]

By substituting this expression with the one in equation (2.40), the following holds:

\[
SD_p = \sqrt{\sigma_{t+1}^2 \left( 1 - F^{-1}(p) \frac{\phi(F^{-1}(p))}{\Phi(F^{-1}(p))} - \left( \frac{\phi(F^{-1}(p))}{\Phi(F^{-1}(p))} \right)^2 \right)}
\]

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Again, for a normal distribution, \( F^{-1}(p) = \Phi^{-1}(p) \). Hence, the following holds:

\[
SD_p = \sqrt{\sigma_t^2 \left( 1 - \Phi^{-1}(p) \frac{\phi(\Phi^{-1}(p))}{p} - \left( \frac{\phi(\Phi^{-1}(p))}{p} \right)^2 \right)} \quad (2.42)
\]

In the case of a truncated Student’s t distribution, the analytical form of ES and SD can be derived by first computing the truncated expected value with upper limit \( b \) as follows:

\[
E(X | X < b) = \frac{1}{2\sqrt{\nu \beta(\nu, \frac{1}{2})} t_{\nu}(b)} \left( b^2 h \left( \frac{1+\nu}{2}, 1, 2, -\frac{b^2}{2} \right) \right)
\]

where \( \beta \) is the Beta function and \( h \) is the Gauss hyper geometric function (p. 5). These are given as follows:

\[
\beta(a, b) = \int_0^1 w^{a-1}(1-w)^{b-1} dw
\]

\[
h(a, b, c, x) = \sum_{k=0}^{\infty} \frac{a_k b_k}{c_k} \frac{x^k}{k!}
\]

where \( a_k, b_k \) and \( c_k \) denotes the respective ascending factorials. In the same way as ES and SD were analytically derived for the normal distribution case, the truncated expected value is substituted in equation (2.39) and yields the following:

\[
ES_p = \sigma_t t_{\nu}^2 \frac{1}{2\sqrt{\nu \beta(\nu, \frac{1}{2})} t_{\nu}(F^{-1}(p))} \left( (F^{-1}(p))^2 h \left( \frac{1+\nu}{2}, 1, 2, -\frac{(F^{-1}(p))^2}{2} \right) \right)
\]

For a Student’s t distribution, \( F^{-1}(p) = t_{\nu}^{-1}(p) \). Hence, the following holds:

\[
ES_p = \sigma_t t_{\nu}^2 \frac{1}{2\sqrt{\nu \beta(\nu, \frac{1}{2})} t_{\nu}(F^{-1}(p))} \left( (t_{\nu}^{-1}(p))^2 h \left( \frac{1+\nu}{2}, 1, 2, -(t_{\nu}^{-1}(p))^2 \right) \right)
\]

Regarding \( SD_p \), one begins with computing the truncated variance with upper limit \( b \) for the Student’s t distribution as follows:

\[
Var(X | X < b) = \frac{1}{3\sqrt{\nu \beta(\nu, \frac{1}{2})} t_{\nu}(b)} \left( b^3 h \left( \frac{1+\nu}{2}, 3, 5, -\frac{b^2}{2} \right) \right)
\]

Again, with \( F^{-1}(p) = t_{\nu}^{-1}(p) \) and by substituting this expression with the one in equation (2.40), the following holds:

\[
SD_p = \sqrt{\sigma_t^2 t_{\nu}^2 \frac{1}{3\sqrt{\nu \beta(\nu, \frac{1}{2})} t_{\nu}(F^{-1}(p))} \left( (t_{\nu}^{-1}(p))^3 h \left( \frac{1+\nu}{2}, 3, 5, -(t_{\nu}^{-1}(p))^2 \right) \right)}
\]
An alternative form of the above expression without the Gauss hyper geometric function can be derived to be the following:

\[
SD_p = \left( \frac{\sigma_{t+1}}{\sqrt{\nu \pi}} \right) \Gamma\left(\frac{\nu}{2}\right) \Gamma\left(\frac{\nu - 2}{2}\right) \Gamma\left(\frac{\nu - 2 + 1}{2}\right) \sqrt{\nu - 2 - \nu^2} \sqrt{\nu^2 - 2 - \nu^2} t_{\nu}^{-1}(p) \sqrt{\nu - 2 - \nu}
\]

where \( t_{\nu}^{-1}(p) \) is the distribution function for Student’s t with \( \nu - 2 \) degrees of freedom and \( f_{\nu}(x) \) is given in equation (2.1) [16].

So, ES and SD are the estimates of the expected loss and its dispersion respectively which can be computed for each day a VaR exceedance occur. Further, the backtesting procedure is realized by testing if the day \( k \) exceedance is significantly worse than the expected VaR by regarding the following backtest ratio:

\[
BT_{t+k} = \frac{R_{t+k} - ES_{t+k}^p}{SD_{t+k}^p}
\]

which yields how many dispersion measures apart the actual loss is from its expected value. By applying equation (2.38), (2.39) and (2.40) the following can be derived:

\[
BT_{t+k} = \frac{\mu_{t+k} + \sigma_{t+k} Z_{t+k} - (\mu_{t+k} + \sigma_{t+k} E(Z_{t+k}|Z_{t+k} < F^{-1}(p)))}{\sqrt{\sigma_{t+k}^2 Var(Z_{t+k}|Z_{t+k} < F^{-1}(p))}}
\]

Simplification yields the following expression:

\[
BT_{t+k} = \frac{Z_{t+k} - E(Z_{t+k}|Z_{t+k} < F^{-1}(p))}{\sqrt{Var(Z_{t+k}|Z_{t+k} < F^{-1}(p))}}
\]

The alternative hypothesis for the test is that the actual loss is worse than the expected loss. For instance, Monte Carlo simulation can be used to implement the method. For implementation the following algorithm is to be applied:
1. Simulate $N$ times a sample with $n$ IID random variables $u_{ij}$ from a chosen distribution $F$, where $i = 1, ..., n$ and $j = 1, ..., N$.

2. For each sample $N$, calculate $E[u_{ij}|u_{ij} < p(u_{ij})]$ and $\text{Var}(u_{ij}|u_{ij} < p(u_{ij}))$ where $p(u_{ij})$ denotes the $p^{th}$ quantile of $u_{ij}$.

3. For every $u_{ij} < p(u_{ij})$, form the following simulated backtest ratio:

$$BT_{ij} = \frac{u_{ij} - E[u_{ij}|u_{ij} < p(u_{ij})]}{\sqrt{\text{Var}(u_{ij}|u_{ij} < p(u_{ij}))}}$$

4. Given a stated test significance level $\alpha$, determine the critical value as the median $\alpha$-quantile of all of the $BT_{ij}$ series.

5. Given the actual $BT_{t+k}$, determine the test p-value as the median of $Pr(BT_{ij} < BT_{t+k})$.

The advantages of this method in relation to the ones previously mentioned is that it computes the dispersion of the truncated distribution by the estimated VaR upper limit, instead of the dispersion of the whole probability function. Also, it is not limited to the Gaussian case since other probability distribution functions as well as empirical distributions can be used. Finally, it allows for backtesting of individual VaR exceedances so that the risk manager does not need to wait for an entire period of VaR exceedances to be able to perform the ES backtesting [28, p. 4-6].
Chapter 3

Results and Discussion

In this chapter the implementation of the thesis is described step by step, results are presented and a discussion of the results is held.

MATLAB has been used as computational tool throughout the thesis.

3.1 The Portfolio

In order to implement the outlined issues of interest, a portfolio of various financial assets is needed as reference. The portfolio will contain empirical data in the form of financial asset prices and rates. It is of importance to form a portfolio that represents an as realistic one as possible, where different asset classes such as stock prices, interest rates, exchange rates (FX) and possibly derivatives are represented. In particular, it is of interest to cover as many of the different asset categories outlined in the FRTB according to varying liquidity horizons, as possible, see Table 1.1.

The reference portfolio that will be considered throughout this thesis, consists of stock prices, FX and bond indices. Each financial time series needs to form a sufficiently large sample as well as to represent both low and high volatile periods, which will be the case since data is collected from 2005 and onwards. For the time being, the Basel Committee is proposing that the observation horizon must go back at least to 2005 in order to be able to calibrate the risk metrics to stressed periods [3, p. 18]. Since previous research on the topic has mostly been performed on data from the U.S. market, data from the Swedish market will mainly be used throughout the thesis. Also, since the department of risk at SAS Institute Sweden primarily works with companies within the Nordics, it is of special interest to perform the analysis on a Swedish reference portfolio.

The stocks have been chosen according to varying size of company and busi-
ness area, as well to represent varying volatility levels. The reason for this is to obtain an as diversified investment as possible. To accomplish the criteria of varying size of company, stocks from both Large Cap and Small Cap from the Nordic list have been included in the reference portfolio, where Large Cap represents companies with a market capitalization value over 1 billion EUR and Small Cap represents companies with a market capitalization value less than 150 million EUR. In the case both A and B stocks are available within a company, the ones having highest compounded return have been chosen. For stocks that have been splitted during the actual period, the stock prices have been adjusted. The volatility level has been determined according to three different categories; low volatile, medium volatile and high volatile stocks and are presented in Table 3.1. Here, the volatility is calculated as the standard deviation for the entire sample expressed on yearly basis and in per cent. The table is applicable for the volatility level of stocks and FX.

<table>
<thead>
<tr>
<th>Volatility Level</th>
<th>Volatility on Yearly Basis (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>&lt;25</td>
</tr>
<tr>
<td>Medium</td>
<td>25-40</td>
</tr>
<tr>
<td>High</td>
<td>&gt;40</td>
</tr>
</tbody>
</table>

Table 3.1: Definition of volatility levels

From Table 3.1 it can be concluded that in order to choose which stocks and FX to include in the reference portfolio, the standard deviation needs to be computed for the entire time series of the actual period for available assets and then a set of assets can be chosen so that each volatility level is represented in the reference portfolio.

The interest rates have been represented by choosing different bond indices. A Swedish total bond index (OMRX TBOND), representing an interest rate with low volatility, and an US (S&P) high yield bond (SPUSHY BOND), representing a credit spread with higher volatility. Unfortunately no empirical data of a Swedish high yield bond was available.

All assets chosen to be included in the reference portfolio are displayed in Table 3.2 to deliver an overview of the empirical financial data used in the thesis. In total there are 14 assets making up this reference portfolio.
<table>
<thead>
<tr>
<th>Asset</th>
<th>Type</th>
<th>Cap</th>
<th>Volatility Level</th>
<th>Split</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astra Zeneca</td>
<td>Stock</td>
<td>Large</td>
<td>Low</td>
<td>No</td>
</tr>
<tr>
<td>Fabege</td>
<td>Stock</td>
<td>Large</td>
<td>Medium</td>
<td>Yes</td>
</tr>
<tr>
<td>H&amp;M B</td>
<td>Stock</td>
<td>Large</td>
<td>Low</td>
<td>Yes</td>
</tr>
<tr>
<td>Intrum Justitia</td>
<td>Stock</td>
<td>Large</td>
<td>Medium</td>
<td>No</td>
</tr>
<tr>
<td>Swedish Match</td>
<td>Stock</td>
<td>Large</td>
<td>Low</td>
<td>No</td>
</tr>
<tr>
<td>Telia</td>
<td>Stock</td>
<td>Large</td>
<td>Medium</td>
<td>Yes</td>
</tr>
<tr>
<td>Boliden</td>
<td>Stock</td>
<td>Large</td>
<td>High</td>
<td>Yes</td>
</tr>
<tr>
<td>Lundin Petroleum</td>
<td>Stock</td>
<td>Large</td>
<td>High</td>
<td>No</td>
</tr>
<tr>
<td>Poolia B</td>
<td>Stock</td>
<td>Small</td>
<td>High</td>
<td>No</td>
</tr>
<tr>
<td>USD/SEK</td>
<td>FX</td>
<td></td>
<td>Low</td>
<td></td>
</tr>
<tr>
<td>EUR/SEK</td>
<td>FX</td>
<td></td>
<td>Low</td>
<td></td>
</tr>
<tr>
<td>USD/JPY</td>
<td>FX</td>
<td></td>
<td>Low</td>
<td></td>
</tr>
<tr>
<td>OMRX TBOND</td>
<td>Bond Index</td>
<td></td>
<td>Low</td>
<td></td>
</tr>
<tr>
<td>SPUSHY BOND</td>
<td>Bond Index</td>
<td></td>
<td>High</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2: Overview of the financial assets in the reference portfolio

From Table 3.2 it can be seen that the reference portfolio consists of different kinds of assets where varying size of company and business area, spanning from medical industry to manufacturing, have been achieved for the stocks and varying volatility levels are present for the assets in general. Unfortunately, no low or medium volatile Small Cap stock as well as no medium or high volatile FX was possible to include in the reference portfolio due to lack of empirical data of that kind.

The data has mainly been collected from the public Nasdaq OMX web page [26], although a few time series were provided by one of the largest Swedish banks, Swedbank. Data has been collected over a historical time period spanning from May 18, 2005 to May 3, 2016. The historical time period covers almost eleven consecutive years including the global financial crisis that started in 2007 and continued for a couple of years resulting in an unstable market. Further, data has been collected on a daily basis and in total 2,779 data points for all assets quoted on business days have been collected. For some assets a few data points were missing and in those cases data points were added by linear interpolation.

Depending on what risk level the investor is willing to take, different weights can be assigned for each asset in the reference portfolio. In this thesis, the perspective of an investor allocating equal weights to all assets in the portfolio has been chosen. Thus every asset of the 14 in total represents $\frac{1}{14} \approx 7.14\%$ of the portfolio and the weight coefficient is set to $w = \frac{1}{14}$. 

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Visualising the empirical data for the different assets in the reference portfolio over approximately eleven years result in the following figures where Figure 3.1 represents the stock prices, Figure 3.2 represents the FX rates and Figure 3.3 represents the rates of the bond indices.

Figure 3.1: Prices over time for each stock in the reference portfolio
Figure 3.2: Rates over time for each FX in the reference portfolio
Both Figure 3.1 and Figure 3.2 show the effects of the global financial crisis that took place from 2007 to 2009 by the drop of prices and rates during that time period. Some companies and FX were more affected than others. For instance, regarding Figure 3.1 AstraZeneca was heavily affected while SwedishMatch was less affected. However, all stocks were more or less negatively affected by the financial crisis. Further, regarding Figure 3.2, the USD/SEK rate dropped more during the crisis than the EUR/SEK rate did. Also, the USD/JPY rate dropped, indicating the negative impact of the global financial crisis for FX. Regarding Figure 3.3, one can conclude that the S&P high yield bond (SPUSHY BOND) had its rate decreasing due to the global financial crisis, while the Swedish total bond index (OMRX TBOND) was not affected in particular. Further, the different levels of volatility can clearly be seen in the three figures by studying the behavior of the time series curves, some being more volatile than others, which has been desired when selecting the assets for the reference portfolio.

Prices and rates are commonly denoted $S_{i,t}$ at time $t$ for asset $i$. In financial analysis it is common to consider the logarithmic returns, referred to as log returns, instead of prices, rates or returns, due to the mathematical
advantages it implies. The daily log returns are computed as follows:

$$Y_{i,t} = \log \left( \frac{S_{i,t}}{S_{i,t-1}} \right)$$

Visualising the log returns over time for the different assets in the reference portfolio yields the following, see Figure 3.4.

Figure 3.4: Log returns over time for each asset in the reference portfolio

From Figure 3.4 it is clearly seen that volatility clustering occur more or less for all assets. The volatility clustering indicates dependence in the time series and can be computed by regarding the sample ACF of the squared log returns, see equation (2.3). The reason why the squared log returns are regarded is explained in the Section Autocorrelation in Chapter 2 and are in particular applied here for comparative reasons since GARCH models will be applied in further analysis. Visualizing the sample ACF of the squared
log returns, similarly to Figure 2.1 in Chapter 2 but with \( h = 20 \), yields the following Figure 3.5.

In Figure 3.5 it is seen that for all assets, the sample ACF falls outside the confidence bounds of 95% for the major part of the lags which indicates dependence at that specific lag. It can be concluded that all assets possess sample autocorrelation for at least some lags and hence there is dependence in all financial time series included in the reference portfolio.

Sample ACF of the log returns (non-squared) can be found in Appendix 1 in case of interest in comparing the behavior with the sample ACF of the squared log returns.

Further, the following Table 3.3 reports a range of descriptive statistics on
the log returns for each asset in the reference portfolio according to equation (2.4), (2.5) and (2.6). Mean and standard deviation (Std Dev) are quoted in per cent on a yearly basis according to the following formula: \( \sigma \times \sqrt{T} \times 100 \).

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean</th>
<th>Std Dev</th>
<th>Kurtosis</th>
<th>Skewness</th>
<th>IQR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astra Zeneca</td>
<td>0.217</td>
<td>23.070</td>
<td>9.522</td>
<td>-0.169</td>
<td>0.015</td>
</tr>
<tr>
<td>Faberge</td>
<td>0.359</td>
<td>35.198</td>
<td>9.257</td>
<td>0.271</td>
<td>0.022</td>
</tr>
<tr>
<td>H&amp;M B</td>
<td>0.447</td>
<td>24.805</td>
<td>6.900</td>
<td>0.041</td>
<td>0.016</td>
</tr>
<tr>
<td>Intrum Justitia</td>
<td>0.984</td>
<td>31.540</td>
<td>12.317</td>
<td>0.146</td>
<td>0.019</td>
</tr>
<tr>
<td>Swedish Match</td>
<td>0.609</td>
<td>23.979</td>
<td>7.100</td>
<td>-0.029</td>
<td>0.016</td>
</tr>
<tr>
<td>Telia</td>
<td>0.020</td>
<td>25.381</td>
<td>10.420</td>
<td>-0.485</td>
<td>0.016</td>
</tr>
<tr>
<td>Boliden</td>
<td>0.946</td>
<td>48.692</td>
<td>9.179</td>
<td>0.123</td>
<td>0.028</td>
</tr>
<tr>
<td>Lundin Petroleum</td>
<td>0.566</td>
<td>44.154</td>
<td>13.075</td>
<td>0.165</td>
<td>0.025</td>
</tr>
<tr>
<td>Poolia B</td>
<td>-0.569</td>
<td>43.378</td>
<td>7.687</td>
<td>0.168</td>
<td>0.026</td>
</tr>
<tr>
<td>USD/SEK</td>
<td>0.054</td>
<td>12.686</td>
<td>6.040</td>
<td>-0.092</td>
<td>0.009</td>
</tr>
<tr>
<td>EUR/SEK</td>
<td>0.002</td>
<td>6.969</td>
<td>6.349</td>
<td>0.292</td>
<td>0.005</td>
</tr>
<tr>
<td>USD/JPY</td>
<td>-0.007</td>
<td>10.180</td>
<td>7.149</td>
<td>-0.289</td>
<td>0.007</td>
</tr>
<tr>
<td>OMRX TBOND</td>
<td>0.252</td>
<td>3.962</td>
<td>7.225</td>
<td>-0.153</td>
<td>0.003</td>
</tr>
<tr>
<td>SPUSHY BOND</td>
<td>0.512</td>
<td>4.669</td>
<td>46.025</td>
<td>-1.900</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Table 3.3: Descriptive statistics of the log returns of each asset

From Table 3.3 it can be seen that the mean of the log returns is close to zero for all assets. On the other hand, both standard deviation, kurtosis, skewness and interquartile range (IQR) vary to a certain extent for the different assets. The standard deviation can be interpreted as the volatility which indicates some assets have higher volatility than others. Both kurtosis, skewness and interquartile range measure the fatness and variability of the tails. It can be noted that the kurtosis is higher than 3 for all assets which indicates, recalling equation (2.4), their distributions all have fatter tails than the normal distribution. The skewness is not equal to 0 for any of the assets which indicates, recalling equation (2.5), they are all more or less skewed. Some assets possess very fat tails, for instance the SPUSHY BOND with kurtosis 46 and a skewness of -1.9 indicating it is heavily left skewed. Also, recalling equation (2.6), the IQR is definitely below 1.349, representing the normal distribution, which is in line with the fact that the log returns have fatter tails than the tails of a normal distribution.

### 3.2 Univariate Returns and GARCH Modeling of Data

Since volatility clustering has been observed within the time series for all assets in the reference portfolio, a GARCH filtering of the data is necessary.
A suitable method to use is the GARCH model, see equation (2.8) in Chapter 2. According to Ruppert, a GARCH(1,1) model is particularly useful when modeling the dynamics of conditional volatility, see Subsection GARCH (1,1) Model in Chapter 2 [30, p. 489]. By performing a maximum likelihood estimation, the parameters for the GARCH(1,1) model, $\hat{\alpha}_{i,0}$, $\hat{\alpha}_{i,1}$ and $\hat{\beta}_{i,1}$, are estimated for each asset $i$ according to the equations (2.10), (2.11), (2.12), (2.13) and (2.14) and are presented in Table 3.4.
<table>
<thead>
<tr>
<th>Asset</th>
<th>Parameter estimation</th>
<th>Standard Error</th>
<th>t statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astra Zeneca</td>
<td>$\hat{\alpha}_0=7.51981e-06$</td>
<td>$4.47582e-07$</td>
<td>16.801</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_1=0.0755617$</td>
<td>$0.00543622$</td>
<td>13.8997</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_1=0.889135$</td>
<td>$0.00660566$</td>
<td>134.602</td>
</tr>
<tr>
<td>Fabegge</td>
<td>$\hat{\alpha}_0=2.54872e-06$</td>
<td>$9.31092e-07$</td>
<td>2.73734</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_1=0.0595971$</td>
<td>$0.005008159$</td>
<td>11.9156</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_1=0.9358787$</td>
<td>$0.0054173$</td>
<td>172.757</td>
</tr>
<tr>
<td>H&amp;M B</td>
<td>$\hat{\alpha}_0=7.00324e-06$</td>
<td>$7.13346e-07$</td>
<td>9.81745</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_1=0.0507161$</td>
<td>$0.00842127$</td>
<td>109.166</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_1=0.91932$</td>
<td>$0.00660566$</td>
<td>134.602</td>
</tr>
<tr>
<td>Intrum Justitia</td>
<td>$\hat{\alpha}_0=5.63555e-06$</td>
<td>$1.24187e-06$</td>
<td>4.55244</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_1=0.0612247$</td>
<td>$0.006112051$</td>
<td>10.0138</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_1=0.945018$</td>
<td>$0.00842127$</td>
<td>280.315</td>
</tr>
<tr>
<td>Swedish Match</td>
<td>$\hat{\alpha}_0=8.06261e-06$</td>
<td>$9.38e-07$</td>
<td>8.59554</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_1=0.0582776$</td>
<td>$0.00508857$</td>
<td>11.4527</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_1=0.91932$</td>
<td>$0.00842127$</td>
<td>115.338</td>
</tr>
<tr>
<td>Telia</td>
<td>$\hat{\alpha}_0=5.55355e-06$</td>
<td>$1.24187e-06$</td>
<td>4.55244</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_1=0.0612247$</td>
<td>$0.006112051$</td>
<td>10.0138</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_1=0.945018$</td>
<td>$0.00842127$</td>
<td>280.315</td>
</tr>
<tr>
<td>Boliden</td>
<td>$\hat{\alpha}_0=6.31732e-06$</td>
<td>$3.51128e-07$</td>
<td>9.81745</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_1=0.0507161$</td>
<td>$0.00842127$</td>
<td>109.166</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_1=0.91932$</td>
<td>$0.00842127$</td>
<td>115.338</td>
</tr>
<tr>
<td>Lundin Petroleum</td>
<td>$\hat{\alpha}_0=4.44814e-07$</td>
<td>$2.43243e-07$</td>
<td>1.83547</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_1=0.0475721$</td>
<td>$0.00479547$</td>
<td>9.02077</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_1=0.943803$</td>
<td>$0.00435454$</td>
<td>216.194</td>
</tr>
<tr>
<td>Poolia B</td>
<td>$\hat{\alpha}_0=3.19424e-06$</td>
<td>$3.06294e-07$</td>
<td>10.4287</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_1=0.302246$</td>
<td>$0.0226884$</td>
<td>13.3216</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_1=0.331549$</td>
<td>$0.0256972$</td>
<td>12.9022</td>
</tr>
</tbody>
</table>

Table 3.4: Estimated GARCH parameters for each asset in the reference portfolio
From Table 3.4 it can be concluded that $\hat{\beta}_1 > \hat{\alpha}_1$ for all assets which according to a GARCH(1,1) model, see equation (2.9), indicates that the volatility estimate, $\hat{\sigma}_{i,t}$, is more dependent on the previous volatility estimate, $\hat{\sigma}_{i,t-1}$, than the data itself, indicating that GARCH modeling is a suitable model to apply. Further, it can be concluded that the standard errors of the parameter estimates are small and that the t statistics are satisfactory in general. A t statistic greater than 2 indicates statistical significance at 95% confidence level and is present for almost all parameter estimates for all the assets in the reference portfolio.

As a next step, the daily conditional variances can be recursively computed by applying the GARCH(1,1) model according to equation (2.9) for each asset with the respective specified estimated parameters from Table 3.4 above as follows:

$$\hat{\sigma}_{i,t}^2 = \hat{\alpha}_{i,0} + \hat{\alpha}_{i,1} \hat{Y}_{i,t-1}^2 + \hat{\beta}_{i,1} \hat{\sigma}_{i,t-1}^2$$

These daily conditional variances $\hat{\sigma}_{i,t}^2$ for each asset in the reference portfolio are visualized in the following Figure 3.6.

![Daily Conditional Variances for all Assets](image)

Figure 3.6: Daily conditional variances for each asset in the reference portfolio
From Figure 3.6 it is seen that the stocks Boliden, LundinPetroleum and Poolia are the assets that have the largest daily conditional variances, which is in line with their volatility level displayed in Table 3.2.

GARCH filtered residuals, $Z_{i,t}$, can now be computed as follows:

$$Z_{i,t} = \frac{Y_{i,t}}{\hat{\sigma}_{i,t}}$$

The fitted GARCH models are satisfactory if the squared GARCH filtered residuals turn out to be approximately IID. The sample ACF of these squared GARCH filtered residuals will be regarded according to the explanation in the Section Autocorrelation in Chapter 2 and yields Figure 3.7.

![Figure 3.7: Sample ACF of squared GARCH filtered residuals for each asset in the reference portfolio](image-url)
From Figure 3.7 it can be concluded that with one exception, being the SPUSHY BOND asset, every asset now seems to have approximately IID GARCH filtered residuals since the ACFs fall within the confidence bounds. Hence the GARCH(1,1) model can be seen as a satisfactory model to use.

Sample ACFs of the GARCH filtered residuals (non-squared) can be found in Appendix 2 in case of interest in comparing the behavior with the sample ACFs of the squared GARCH filtered residuals.

To further investigate the success of the GARCH filtration, one can study the descriptive statistics of the GARCH filtered residuals according to Sections 2.2.3 and 2.2.4 in Chapter 2, see Table 3.5:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean</th>
<th>Std Dev</th>
<th>Kurtosis</th>
<th>Skewness</th>
<th>IQR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astra Zeneca</td>
<td>-0.0069</td>
<td>0.9990</td>
<td>7.6077</td>
<td>-0.3543</td>
<td>1.1350</td>
</tr>
<tr>
<td>Faberge</td>
<td>0.0048</td>
<td>0.9998</td>
<td>4.6708</td>
<td>-0.2006</td>
<td>1.1483</td>
</tr>
<tr>
<td>H&amp;M B</td>
<td>-0.0005</td>
<td>0.9991</td>
<td>6.2744</td>
<td>-0.0102</td>
<td>1.1300</td>
</tr>
<tr>
<td>Intrum Justitia</td>
<td>0.0065</td>
<td>0.9977</td>
<td>10.2476</td>
<td>0.7340</td>
<td>1.0449</td>
</tr>
<tr>
<td>Swedish Match</td>
<td>0.0014</td>
<td>0.9979</td>
<td>7.1132</td>
<td>-0.2471</td>
<td>1.1261</td>
</tr>
<tr>
<td>Telia</td>
<td>0.0026</td>
<td>0.9998</td>
<td>10.1695</td>
<td>-0.7208</td>
<td>1.0960</td>
</tr>
<tr>
<td>Boliden</td>
<td>-0.0057</td>
<td>0.9977</td>
<td>7.6986</td>
<td>-0.3639</td>
<td>1.1082</td>
</tr>
<tr>
<td>Lundin Petroleum</td>
<td>-0.0146</td>
<td>0.9995</td>
<td>10.8881</td>
<td>-0.3032</td>
<td>1.0608</td>
</tr>
<tr>
<td>Poolia B</td>
<td>-0.0001</td>
<td>0.9998</td>
<td>8.1765</td>
<td>0.1151</td>
<td>0.9629</td>
</tr>
<tr>
<td>USD/SEK</td>
<td>0.0028</td>
<td>0.9975</td>
<td>3.8195</td>
<td>0.0891</td>
<td>1.2077</td>
</tr>
<tr>
<td>EUR/SEK</td>
<td>0.0044</td>
<td>0.9979</td>
<td>4.2202</td>
<td>0.1626</td>
<td>1.2107</td>
</tr>
<tr>
<td>USD/JPY</td>
<td>0.0037</td>
<td>0.9971</td>
<td>6.7913</td>
<td>-0.1903</td>
<td>1.1163</td>
</tr>
<tr>
<td>OMRX TBOND</td>
<td>-0.0041</td>
<td>0.9893</td>
<td>4.8738</td>
<td>-0.1281</td>
<td>1.2398</td>
</tr>
<tr>
<td>SPUSHY BOND</td>
<td>0.0043</td>
<td>0.8061</td>
<td>7.4146</td>
<td>-0.3767</td>
<td>0.8186</td>
</tr>
</tbody>
</table>

Table 3.5: Descriptive statistics of GARCH filtered residuals of each asset in the reference portfolio

Table 3.5 confirms that the GARCH filtered residuals for each asset now all have means approximately 0 and standard deviations close to 1, which supports the fact that they are now approximately (0,1) distributed standardized residuals. Further, recalling equation (2.4), kurtosis is still more than 3 for all series which implies excess kurtosis. However, the kurtosis has decreased for all series after applying GARCH filtration. This is in line with what one can expect according to the Section Properties of GARCH Distribution in Chapter 2, describing the fact that a GARCH model can capture some of the excess kurtosis seen in unconditional returns but not all non-normality and hence the GARCH filtered residuals have a distribution being closer to normal even though it is still a mixture of distributions. Further,
recalling equation (2.5), the skewness is equal to 0 for none of the assets which indicates that all series are more or less fat tailed and skewed. To test whether the computations of the statistical measures are statistically significant or not, the Jarque-Bera test has been applied according to equation (2.15). The test statistic yields that the null hypothesis of GARCH filtered residuals being normally distributed according to skewness and excess kurtosis is rejected for all assets in the reference portfolio at confidence level 95%. Also, recalling equation (2.6), the IQR is now a bit higher than for the log returns, although it is not equal to 1.349 for any of the assets in the reference portfolio and hence the GARCH filtered residuals are not normally distributed.

A QQ plot is a useful tool to investigate the distribution of empirical data, see Section 2.2.5. QQ plots of the GARCH filtered residuals together with different theoretical distributions are presented in Figure 3.8 with the theoretical distribution being the normal distribution and Figure 3.9 with the theoretical distribution being the Student’s t distribution with $\nu = 5$. 
Figure 3.8: QQ plot of GARCH filtered residuals for each asset in the reference portfolio versus the normal distribution
Figure 3.9: QQ plot of GARCH filtered residuals for each asset in the reference portfolio versus the Student’s t distribution with $\nu = 5$

Figure 3.8 again indicates that the GARCH filtered residuals for the different assets in the reference portfolio have heavier tails than the normal distribution due to the inverted S-shape of the blue lines in the graphs, recalling Figure 2.3 in Chapter 2. Further, replacing the theoretical distribution by the Student’s t distribution with $\nu = 5$ results in Figure 3.9. It yields almost linearly shaped blue lines in the graphs for the assets in the reference portfolio which indicates that the GARCH filtered residuals could be approximately Student’s t distributed with $\nu = 5$, although the skewness of the assets earlier discussed indicates this is not possible since the Student’s t distribution is a symmetric distribution.

Now, when it has been shown that GARCH(1,1) modeling is successful in the sense it was able to yield approximately uncorrelated IID standardized
residuals, GARCH parameters were estimated for a rolling 250 day time window for all empirical data. GARCH filtered residuals were computed with the estimated parameters and will be used throughout the rest of the thesis.

3.3 Multivariate Returns and Co-dependence

3.3.1 Correlation Estimations and Visualization (before GARCH)

Bivariate scatter plots of the log returns for all 14 assets in the reference portfolio over the entire time period are presented in Figure 3.10 in order to show the dependence behavior between the assets (here numbered 1-14). The red lines in the graphs are regression lines fitted to each asset pair’s log returns.
Figure 3.10: Bivariate scatter plots of the log returns of the 14 assets in the reference portfolio over the entire time period with corresponding regression lines, before GARCH filtration is performed.
In Figure 3.10, the bivariate scatter plots for each asset pair display how the assets’ log returns are correlated to different extents according to the varying shape of the distributions in the graphs as well as the regression lines. Regarding the regression lines, a positive slope indicates positive correlation between the assets and a negative slope indicates negative correlation between the assets. It can therefore be concluded that the reference portfolio consists of assets with both positive, approximately zero and negative correlation. To get an even more precise picture of the correlation, a correlation matrix can be on the following form according to equation (2.24):

\[
R = \begin{pmatrix}
1 & \rho & \ldots & \ldots & \rho \\
\rho & 1 & \rho & \ldots & \rho \\
\vdots & \rho & \ddots & \rho & \vdots \\
\rho & \ldots & \rho & 1 & \rho \\
\rho & \ldots & \ldots & \rho & 1
\end{pmatrix}
\]  

(3.1)

Correlation estimations of the log returns for all assets in the reference portfolio over the whole time period are computed according to equation (3.1) with each element of the matrix representing the correlation for the corresponding asset pair in Figure 3.10.
From the above correlation estimation matrix, \( \hat{R} \), it is possible to draw conclusions about the precise dependence between the assets in the reference portfolio. A correlation equal to 1 indicates that the assets have maximum (perfect) positive dependence which means their log returns changes equally.

\[
\begin{pmatrix}
1.00 & 0.237 & 0.314 & 0.206 & 0.267 & 0.297 & 0.235 & 0.215 & 0.111 & 0.125 & 0.050 & 0.223 & -0.111 & 0.194 \\
0.237 & 1.000 & 0.437 & 0.381 & 0.255 & 0.351 & 0.427 & 0.369 & 0.174 & -0.195 & -0.210 & 0.259 & -0.249 & 0.294 \\
0.314 & 0.437 & 1.000 & 0.372 & 0.305 & 0.437 & 0.456 & 0.377 & 0.152 & -0.232 & -0.217 & 0.219 & -0.281 & 0.219 \\
0.206 & 0.381 & 0.372 & 1.000 & 0.264 & 0.325 & 0.391 & 0.327 & 0.149 & -0.174 & -0.161 & 0.200 & -0.241 & 0.210 \\
0.267 & 0.255 & 0.305 & 0.264 & 1.000 & 0.289 & 0.219 & 0.227 & 0.099 & -0.056 & -0.083 & 0.168 & -0.117 & 0.119 \\
0.297 & 0.351 & 0.437 & 0.325 & 0.289 & 1.000 & 0.408 & 0.376 & 0.154 & -0.188 & -0.155 & 0.210 & -0.243 & 0.233 \\
0.235 & 0.427 & 0.456 & 0.391 & 0.249 & 0.408 & 1.000 & 0.499 & 0.189 & -0.277 & -0.215 & 0.205 & -0.284 & 0.240 \\
0.215 & 0.369 & 0.377 & 0.327 & 0.227 & 0.376 & 0.499 & 1.000 & 0.175 & -0.286 & -0.214 & 0.197 & -0.272 & 0.235 \\
0.111 & 0.174 & 0.152 & 0.149 & 0.099 & 0.154 & 0.189 & 0.175 & 1.000 & -0.067 & -0.076 & 0.066 & -0.125 & 0.094 \\
0.125 & -0.195 & -0.232 & -0.174 & -0.056 & -0.188 & -0.277 & -0.268 & -0.067 & 1.000 & 0.628 & 0.120 & 0.273 & -0.196 \\
0.050 & -0.210 & -0.217 & -0.161 & -0.083 & -0.155 & -0.215 & -0.214 & -0.076 & 0.628 & 1.000 & -0.117 & 0.248 & -0.205 \\
0.223 & 0.259 & 0.219 & 0.200 & 0.168 & 0.210 & 0.205 & 0.197 & 0.066 & 0.120 & -0.117 & 1.000 & -0.368 & 0.146 \\
-0.111 & -0.249 & -0.281 & -0.241 & -0.117 & -0.243 & -0.284 & -0.272 & -0.125 & 0.273 & 0.248 & -0.368 & 1.000 & -0.105 \\
0.194 & 0.294 & 0.219 & 0.210 & 0.119 & 0.233 & 0.240 & 0.235 & 0.094 & -0.196 & -0.205 & 0.146 & -0.105 & 1.000
\end{pmatrix}
\]
and in the same direction over time during both periods of market downturns and market recovery. If the correlation is equal to -1 the assets have maximum (perfect) negative dependence which indicates that their log returns changes in opposite directions. A correlation equal to zero indicates that there is no relationship between the assets, that they are independent of each other. When constructing a portfolio it is of high interest to obtain an as diversified portfolio as possible with as low correlation between the assets as possible in order to lower the total risk of the investment. In recent times it has become harder to identify financial markets with low correlation due to the effects of the globalization. For instance, a stock market fall in Europe directly affects the stock market in Asia and vice versa. The correlation estimation matrix, $\hat{R}$, confirms the fact that there are both positively and negatively correlated assets in the reference portfolio which indicates it is a potentially good investment.

For further analysis on a more detailed level, a set of representative assets in the reference portfolio have been chosen to consist of AstraZeneca, LundinPetroleum, USD/SEK, OMRX TBOND and SPUSHY BOND. Bivariate scatter plots of the log returns of these representative assets over the entire time period with regression lines are presented in Figure 3.11.
From Figure 3.11 it is possible to conclude by regarding for instance the scatter plot of the two stocks AstraZeneca and LundinPetroleum, that the regression line has a positive slope which indicates they are positively correlated regarding the entire time period. This is in line with what one can expect as they are both present on the same stock market. Further, regarding the scatter plot of the AstraZeneca stock and the OMRX TBOND, one can conclude that they have slightly negative correlation. Also this is in line with what one can expect since they are two assets coming from different asset classes and hence possess no or little negative correlation. To give an even more detailed picture of the assets’ relationship, correlation estimations of the log returns to the corresponding representative assets over the entire time period are computed according to equation (3.1) and are presented.
Equation (3.2) confirms the conclusions drawn above about the positive correlation between the two stocks (AstraZeneca and LundinPetroleum) as well as the negative correlation between the stock and the total bond index (AstraZeneca and OMRX TBOND).

Until now, correlations have been computed for the entire time period of approximately 11 years. Now, computing the correlation estimations of the log returns for the same representative assets as above, on a yearly basis with rolling 250 trading days, yields how correlation estimations change over time, see Figure 3.12.
By studying Figure 3.12 and the dark green line in particular, one can conclude that the two stocks, AstraZeneca and LundinPetroleum, have been positively correlated on a yearly basis during the entire time period except for a couple of months in December 2009 and January 2010. The economical reason why the stocks are positively correlated is that moves in market factors have a higher impact on their returns than the fact that the stocks represent different business areas on the market. Further, it can be concluded by regarding the black line that AstraZeneca and the OMRX TBOND that were said to be negatively correlated, actually have a largely fluctuating correlation over time, meaning it is sometimes negative and sometimes positive. However, it turned out to be negative during the global financial crisis as well as during a couple of years around 2011-2013, indicating that these assets could have acted hedges during these particular periods. In addition, the turquoise line indicates that the total bond index OMRX TBOND could have acted hedge to the second stock as well, LundinPetroleum, due to negative correlation during most of the time period. During stressed periods it
can be seen that the correlation decreases in particular and hence the total bond index is a good choice of hedge in a portfolio where the majority of assets are stocks, particularly during highly volatile periods.

### 3.3.2 Copula Dependence Measures (before GARCH)

According to equation (2.25) and equation (2.27) in particular, Kendall’s tau $\tau$ for the normal and Student’s t copula can be computed from the estimated correlations in equation (3.2) for the log returns of some representative assets in the reference portfolio as above to be the following:

$$
\hat{\tau} = \begin{pmatrix}
1.000 & 0.138 & 0.080 & -0.071 & 0.124 \\
0.138 & 1.000 & -0.185 & -0.176 & 0.151 \\
0.080 & -0.185 & 1.000 & 0.176 & -0.125 \\
-0.071 & -0.176 & 0.176 & 1.000 & -0.067 \\
0.124 & 0.151 & -0.125 & -0.067 & 1.000
\end{pmatrix}
$$

(3.3)

Comparing equation (3.3) with the linear correlation estimates in equation (3.2), one can conclude that the values have slightly changed although they remain either negative or positive respectively. The reason for this is explained in Section 2.6.3 and is due to some drawbacks of the linear correlation as a dependence measure. In particular, the values are now less extreme in the sense they are less positive or less negative respectively and hence moving from linear correlation to Kendall’s tau has implied less dependence between the assets.

Further, according to equation (2.26) and equation (2.28) in particular, Spearman’s rho $\rho_S$ can be computed for the normal copula by using the estimated correlations for the log returns of some representative assets in the reference portfolio as above in equation (3.2) to obtain the following:

$$
\hat{\rho}_S = \begin{pmatrix}
1.000 & 0.205 & 0.120 & -0.106 & 0.124 \\
0.205 & 1.000 & -0.274 & -0.261 & 0.225 \\
0.120 & -0.274 & 1.000 & 0.262 & -0.187 \\
-0.106 & -0.261 & 0.262 & 1.000 & -0.100 \\
0.185 & 0.225 & -0.187 & -0.100 & 1.000
\end{pmatrix}
$$

(3.4)

The same properties hold for Spearman’s rho as for Kendall’s tau in comparison with linear correlation according to Section 2.6.3. Comparing the equations (3.2), (3.3) and (3.4) indicate that the estimates of Spearman’s rho falls somewhere in between the linear correlation estimates and the estimates of Kendall’s tau.

According to the equations (2.29) and (2.30) as well as the fact that the normal copula is asymptotically independent both in the lower and upper
tail if $R < 1$ [7 p. 116], the following holds for the normal copula:

$$\lambda_l = \lambda_u = 0$$

According to equations (2.31) and (2.32) as well as the fact that the Student’s t copula is asymptotically dependent both in the lower and upper tail if $R > -1$ [7 p. 118] and that the Student’s t copula is symmetric since the Student’s t distribution is elliptical, yields that $\lambda_u = \lambda_l$ and using equation (2.33) yields the following result for $p = 0.99$ and $\nu = 3$:

$${\hat{\lambda}}_{u,\nu=3} = {\hat{\lambda}}_{l,\nu=3} = \begin{pmatrix} 0.896 & 0.868 & 0.862 & 0.846 & 0.866 \\ 0.868 & 0.896 & 0.828 & 0.830 & 0.869 \\ 0.862 & 0.828 & 0.896 & 0.871 & 0.838 \\ 0.846 & 0.830 & 0.871 & 0.896 & 0.846 \\ 0.866 & 0.869 & 0.838 & 0.846 & 0.896 \end{pmatrix} \quad (3.5)$$

For $p = 0.99$ and $\nu = 7$ the corresponding result is the following:

$${\hat{\lambda}}_{u,\nu=7} = {\hat{\lambda}}_{l,\nu=7} = \begin{pmatrix} 0.847 & 0.805 & 0.797 & 0.773 & 0.803 \\ 0.805 & 0.847 & 0.748 & 0.750 & 0.806 \\ 0.797 & 0.748 & 0.847 & 0.809 & 0.762 \\ 0.773 & 0.750 & 0.809 & 0.847 & 0.774 \\ 0.803 & 0.806 & 0.762 & 0.774 & 0.847 \end{pmatrix} \quad (3.6)$$

From equation (3.5) and (3.6) it is seen that the assets have tail dependence to a certain extent and that the tail dependence coefficients gets lower when the degrees of freedom, $\nu$, increases which is in line with what one can expect for the dependence of the tail of a Student’s t multivariate distribution.

Kendall’s tau, Spearman’s rho and tail dependence coefficients of the log returns before GARCH filtration for all assets in the reference portfolio are presented in Appendix 6.

### 3.3.3 Unconditional and Conditional Dependence (after GARCH)

According to previous sections, the same dependence measures will be computed and visualized, but now for the GARCH filtered residuals instead, in order to make comparisons when GARCH effects have been accounted for.

Bivariate scatter plots of the GARCH filtered residuals of the set of representative assets in the reference portfolio over the entire time period together with regression lines are presented in Figure 3.11.
Figure 3.13: Bivariate scatter plots with regression lines of some representative assets in the reference portfolio over the entire time period after GARCH filtration has been performed.

Figure 3.13 is to be analysed together with the correlation estimation matrix of the GARCH filtered residuals, similarly to the previous sections. The estimated correlations yield according to equations (2.24) and (3.1) for the same set of representative assets the following:

$$
\hat{R}_{garch} = \begin{pmatrix}
1.000 & 0.176 & 0.135 & -0.103 & 0.144 \\
0.176 & 1.000 & -0.229 & -0.214 & 0.199 \\
0.135 & -0.229 & 1.000 & 0.224 & -0.166 \\
-0.103 & -0.214 & 0.224 & 1.000 & -0.086 \\
0.144 & 0.199 & -0.166 & -0.086 & 1.000 \\
\end{pmatrix}
$$

(3.7)

It can be concluded when comparing $\hat{R}$ and $\hat{R}_{garch}$, see the equations (3.2) and (3.7) together with the Figures 3.11 and 3.13, that when GARCH filtering has been applied the absolute values of the estimated correlations are lower for the majority of asset pairs in the reference portfolio. This is in...
line with the fact that GARCH filtered residuals show little or no depen-
dence, being approximately IID according to the Sections GARCH Model
and Properties of GARCH Distribution in Chapter 2. Even though the
GARCH modeling is performed for univariate time series, the fact that they
are approximately IID implies that market factors are taken into account
and hence in the multivariate case there is less correlation since assets in the
same market are correlated in general.

According to equation (2.25) and equation (2.27) in particular, Kendall’s
tau $\tau$ for the normal and Student’s t copula can be computed from the es-
timated correlations in equation (3.7) for the GARCH filtered residuals for
the same set of representative assets as above to be the following:

$$
\hat{\tau}_{garch} = \begin{pmatrix}
1.000 & 0.113 & 0.086 & -0.066 & 0.092 \\
0.113 & 1.000 & -0.147 & -0.138 & 0.128 \\
0.086 & -0.147 & 1.000 & 0.144 & -0.106 \\
-0.066 & -0.138 & 0.144 & 1.000 & -0.055 \\
0.092 & 0.128 & -0.106 & -0.055 & 1.000
\end{pmatrix} \quad (3.8)
$$

Comparing equation (3.8) with the linear correlation estimates computed
in equation (3.7), one can conclude that the values have slightly changed
although they remain either negative or positive respectively. The reason
for this is explained in Section 2.6.3 and is due to some drawbacks of linear
correlation as a dependence measure. In particular, the values are now less
extreme in the sense they are less positive or less negative respectively and
hence moving from linear correlation to Kendall’s tau has implied less de-
pendence between the assets.

Further, according to equation (2.26) and equation (2.28) in particular,
Spearman’s rho $\rho_S$ can be computed for the normal copula by using the
estimated correlations for the GARCH filtered residuals of some representa-
tive assets in the reference portfolio as above in equation (3.2) to obtain the
following:

$$
\hat{\rho}_{S,garch} = \begin{pmatrix}
1.000 & 0.169 & 0.129 & -0.099 & 0.138 \\
0.169 & 1.000 & -0.219 & -0.205 & 0.190 \\
0.129 & -0.219 & 1.000 & 0.215 & -0.159 \\
-0.099 & -0.205 & 0.215 & 1.000 & -0.082 \\
0.138 & 0.190 & -0.159 & -0.082 & 1.000
\end{pmatrix} \quad (3.9)
$$

The same properties hold for Spearman’s rho as for Kendall’s tau in com-
parison with linear correlation according to Section 2.6.3. Comparing the
equations (3.7), (3.8) and (3.9) indicate that the estimates of Spearman’s
rho falls somewhere in between the linear correlation estimates and the es-
timates of Kendall’s tau.
According to the equations (2.29) and (2.30), as well as the fact that the normal copula is asymptotically independent both in the lower and upper tail if $R < 1$ [7, p. 116], the following holds for the normal copula:

$$\lambda_l = \lambda_u = 0$$

According to equations (2.31) and (2.32), as well as the fact that the Student’s t copula is asymptotically dependent both in the lower and upper tail if $R > -1$ [7, p. 118] and that the Student’s t copula is symmetric since the Student’s t distribution is elliptical, yields that $\lambda_u = \lambda_l$ and using equation (2.33) yields for $p = 0.99$ and $\nu = 3$ the following result:

$$\hat{\lambda}_{u,\nu=3,\text{garch}} = \hat{\lambda}_{l,\nu=3,\text{garch}} = \begin{pmatrix}
0.896 & 0.865 & 0.863 & 0.846 & 0.864 \\
0.865 & 0.896 & 0.834 & 0.836 & 0.867 \\
0.863 & 0.834 & 0.896 & 0.868 & 0.841 \\
0.846 & 0.836 & 0.868 & 0.896 & 0.848 \\
0.864 & 0.867 & 0.841 & 0.848 & 0.896 
\end{pmatrix}$$

(3.10)

For $p = 0.99$ and $\nu = 7$ the corresponding result is obtained as follows:

$$\hat{\lambda}_{u,\nu=7,\text{garch}} = \hat{\lambda}_{l,\nu=7,\text{garch}} = \begin{pmatrix}
0.847 & 0.802 & 0.798 & 0.774 & 0.799 \\
0.802 & 0.847 & 0.757 & 0.759 & 0.804 \\
0.798 & 0.757 & 0.847 & 0.806 & 0.766 \\
0.774 & 0.759 & 0.806 & 0.847 & 0.776 \\
0.799 & 0.804 & 0.766 & 0.776 & 0.847 
\end{pmatrix}$$

(3.11)

From the equations (3.10) and (3.11) it is seen that there is tail dependence and that the tail dependence coefficients gets lower when the degrees of freedom, $\nu$, increases, which is in line with what one can expect for the dependence of the tail of a Student’s t multivariate distribution.

Bivariate scatter plots with regression lines, correlation estimations, Kendall’s tau, Spearman’s rho and tail dependence coefficients of the GARCH filtered residuals for the entire time period for all assets in the reference portfolio are presented in Appendix 7.

### 3.4 Risk Measure Models

Throughout the thesis, one-day calculations have been made for the risk measures. In the Basel regulation a 10-day VaR is required for reporting, although, in practice, it is computed by scaling a one-day VaR calculation with $\sqrt{10}$. Backtesting is always performed on a daily basis and since the aim of this thesis is to examine the backtesting procedures, one-day calculations of the risk measures are performed throughout this thesis.
As it is of interest to examine different risk measure models by determining the behavior of the backtesting procedures, three different model families will be implemented in this thesis.

3.4.1 Model Family 1

Here the unfiltered residuals are regarded so no GARCH filtration is performed. The model parameters, $\mu$ and $\Sigma$, are computed for the unfiltered residuals and equation (2.17) is applied to compute the portfolio variance. The risk measures VaR and ES are then computed analytically for the normal distribution by applying the equations (2.18) and (2.19) and for the Student’s $t$ distribution with $\nu = 3, 7, 15, 20$ by applying the equations (2.20) and (2.21).

3.4.2 Model Family 2

Here the GARCH filtered residuals are regarded so a GARCH filtration is performed. The model family is equivalent to model family 1 except that the diagonal of the covariance matrix, $\Sigma$, is replaced by the forecasted volatilities computed from the estimated parameters of the GARCH model used for the filtration. The decomposition of the covariance matrix is made by a LDL decomposition as follows:

$$\Sigma = LDL^\prime =$$

$$= \begin{pmatrix}
\sigma_{1,t} & 0 & \cdots & 0 \\
0 & \sigma_{2,t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_{14,t}
\end{pmatrix} \begin{pmatrix}
1 & \rho_{1,2} & \cdots & \rho_{1,14} \\
\rho_{2,1} & 1 & \cdots & \rho_{2,14} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{14,1} & \rho_{14,2} & \cdots & 1
\end{pmatrix} = \begin{pmatrix}
\sigma_{1,t} & 0 & \cdots & 0 \\
0 & \sigma_{2,t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_{14,t}
\end{pmatrix}$$

Now, replacing $\sigma_{1,t}, \ldots, \sigma_{14,t}$ in $L$ with its GARCH forecasted volatilities yields the following:

$$\Sigma_{Garch} = LDL^\prime =$$

$$= \begin{pmatrix}
\hat{\sigma}_{1,t} & 0 & \cdots & 0 \\
0 & \hat{\sigma}_{2,t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{\sigma}_{14,t}
\end{pmatrix} \begin{pmatrix}
1 & \rho_{1,2} & \cdots & \rho_{1,14} \\
\rho_{2,1} & 1 & \cdots & \rho_{2,14} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{14,1} & \rho_{14,2} & \cdots & 1
\end{pmatrix} = \begin{pmatrix}
\hat{\sigma}_{1,t} & 0 & \cdots & 0 \\
0 & \hat{\sigma}_{2,t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{\sigma}_{14,t}
\end{pmatrix}$$

The model parameters $\mu$ and $\Sigma_{Garch}$ are hence computed and equation (2.17) with covariance matrix $\Sigma_{Garch}$ instead of $\Sigma$ is applied to compute the portfolio variance. The risk measures VaR and ES are then computed analytically for the normal distribution by applying equation (2.18) and (2.19) and for...
the Student’s t distribution with \( \nu = 3, 7, 15, 20 \) by applying equation (2.20) and (2.21).

### 3.4.3 Model Family 3

Similarly to model family 2, the GARCH filtered residuals are regarded here so a GARCH filtration is performed. Instead of computing the risk measures analytically one will use Monte Carlo simulation for this model family, where random variables are simulated from a copula with the correlation matrix of the GARCH filtered residuals. In order to allow skewness, the tails will be modeled with the generalized Pareto distribution (GPD) according to equation (2.2), resulting in a hybrid distribution for the GARCH filtered residuals according to the Section Hybrid Distribution in Chapter 2.

In this thesis the threshold for the upper tail, \( u_{\text{high}} \), is set to be the 90% quantile (\( q_{\text{high}} = 0.90 \)) of the empirical distribution, \( u_{\text{high}} = F^{-1}_n(0.90) \). The threshold for the lower tail, \( u_{\text{low}} \), is set to be the 10% quantile (\( q_{\text{low}} = 0.10 \)) of the empirical distribution, \( u_{\text{low}} = F^{-1}_n(0.10) \). The GPD can only be used to model the tails of a distribution, not the center. In this thesis, there is so much data available in the center that it is possible to choose either the empirical distribution or fit it to a normal distribution. It was determined that the empirical distribution was to be used for the center of the distribution resulting in a GPD-empirical hybrid distribution. The main reason for modeling the tails with the GPD is due to the non-normality represented by excess kurtosis and skewness of GARCH filtered residuals found in previous analysis, see Section Univariate Returns and GARCH Modeling of Data in Chapter 3. Also, the QQ plots in the same section suggest residuals to be Student’s t distributed with \( \nu = 5 \), however, the skewness is still present and therefore GPD is better suited since it is not a symmetric distribution, allowing for skewness to take place as each tail is modeled separately with unique parameters.

As the GPD is univariate, for each asset in the reference portfolio, two unique GPDs are estimated to obtain unique univariate cumulative distribution functions to set up the hybrid distribution. The GPD parameters are first computed for the entire time series of 2,779 data points with the following result for the parameters, see Table 3.6.
Table 3.6: Estimated GPD parameters for each asset in the reference portfolio over the entire time period

<table>
<thead>
<tr>
<th>Asset</th>
<th>Threshold</th>
<th>$\hat{\xi}$</th>
<th>$\hat{\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astra Zeneca</td>
<td>$u_{\text{high}}$=1.0996</td>
<td>0.0748</td>
<td>0.5684</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.1120</td>
<td>0.2486</td>
<td>0.5306</td>
</tr>
<tr>
<td>Fabege</td>
<td>$u_{\text{high}}$=1.2166</td>
<td>-0.0173</td>
<td>0.5494</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.1662</td>
<td>0.0087</td>
<td>0.6568</td>
</tr>
<tr>
<td>H&amp;M B</td>
<td>$u_{\text{high}}$=1.1506</td>
<td>0.2042</td>
<td>0.4811</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.1550</td>
<td>0.1644</td>
<td>0.5373</td>
</tr>
<tr>
<td>Intrum Justitia</td>
<td>$u_{\text{high}}$=1.1293</td>
<td>0.2659</td>
<td>0.5266</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.0940</td>
<td>0.1268</td>
<td>0.5037</td>
</tr>
<tr>
<td>Swedish Match</td>
<td>$u_{\text{high}}$=1.2077</td>
<td>0.0428</td>
<td>0.5683</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.1320</td>
<td>0.2006</td>
<td>0.4948</td>
</tr>
<tr>
<td>Telia</td>
<td>$u_{\text{high}}$=1.1101</td>
<td>0.0258</td>
<td>0.6016</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.0707</td>
<td>0.3415</td>
<td>0.4664</td>
</tr>
<tr>
<td>Boliden</td>
<td>$u_{\text{high}}$=1.1035</td>
<td>0.0836</td>
<td>0.5758</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.1389</td>
<td>0.1897</td>
<td>0.5426</td>
</tr>
<tr>
<td>Lundin Petroleum</td>
<td>$u_{\text{high}}$=1.1252</td>
<td>0.1179</td>
<td>0.5364</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.1289</td>
<td>0.1994</td>
<td>0.5360</td>
</tr>
<tr>
<td>Poolia B</td>
<td>$u_{\text{high}}$=1.1311</td>
<td>0.1198</td>
<td>0.6352</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.1296</td>
<td>0.1111</td>
<td>0.5841</td>
</tr>
<tr>
<td>USD/SEK</td>
<td>$u_{\text{high}}$=1.2536</td>
<td>-0.0628</td>
<td>0.6088</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.1912</td>
<td>-0.1653</td>
<td>0.6652</td>
</tr>
<tr>
<td>EUR/SEK</td>
<td>$u_{\text{high}}$=1.2071</td>
<td>0.0136</td>
<td>0.6069</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.2072</td>
<td>-0.0220</td>
<td>0.5395</td>
</tr>
<tr>
<td>USD/JPY</td>
<td>$u_{\text{high}}$=1.1504</td>
<td>0.1413</td>
<td>0.4987</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.1524</td>
<td>0.0593</td>
<td>0.6379</td>
</tr>
<tr>
<td>OMRX TBOND</td>
<td>$u_{\text{high}}$=1.1826</td>
<td>0.0509</td>
<td>0.5110</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-1.2102</td>
<td>0.0527</td>
<td>0.5414</td>
</tr>
<tr>
<td>SPUSHY BOND</td>
<td>$u_{\text{high}}$=-0.8417</td>
<td>0.0996</td>
<td>0.4703</td>
</tr>
<tr>
<td></td>
<td>$u_{\text{low}}$=-0.9088</td>
<td>0.0966</td>
<td>0.5732</td>
</tr>
</tbody>
</table>

It is seen from Table 3.6 that the estimated shape parameter $\hat{\xi}$ takes both negative and positive values while the estimated scale parameter $\hat{\beta}$ is always positive, which is in line with the definition in equation (2.2) in Chapter 2. According to the Section Generalized Pareto Distribution in Chapter 2, positive shape parameters indicate distributions whose tails decrease as polynomials, being fatter than the tails of the normal distribution, while negative tail parameters indicate distributions whose tails are finite and hence less fat than the tails of the normal distribution. Regarding Table 3.6 the majority of the shape parameters are positive and one can therefore conclude that the majority of the estimated tails are fatter than the tails of the normal...
distribution.

The empirical CDFs for the GARCH filtered residuals together with estimated GPD tails for each asset in the reference portfolio have been visualized in Appendix 3. It is found that the estimated tail distributions fit the empirical data very well in the tails and one can therefore conclude that the hybrid distribution models the entire distribution much better than if a normal or Student’s t distribution would be used for this purpose.

The GPD parameters were now estimated for a rolling 250 day time window, hence resulting in 2,529 sets of estimated GPD parameters.

Compared to the previous model families, this model family uses simulation to make up the distribution of log returns, based on the hybrid distribution with GPD for the tails and empirical distribution for the center, from which the risk measures VaR and ES can be computed. A copula is used to capture the multivariate dependence structure between the different assets in the reference portfolio. An algorithm has been stated in order to outline the order of computations, first for simulation from a normal copula, then for simulation from a Student’s t copula.

Algorithm for Simulation of Log Return Distribution from Normal Copula:

1. Find the correlation matrix $R$ of the filtered residuals $Z_{i,t}$
2. Find the Cholesky decomposition $A$ of $R$
3. Simulate $n$ independent random variables $z_1, z_2, \ldots, z_n$ from $N(0, 1)$
4. Set $x = Az$
5. Transform $x$ to uniform variables by applying $u_i = \Phi(x_i)$ for $i = 1, \ldots, n$ and where $\Phi$ denotes the univariate standard normal distribution function
6. Transform back to residuals by applying hybrid distribution framework according to equation (2.23). If the uniform variable is higher than or equal to $u_{\text{high}}$ or lower than or equal to $u_{\text{low}}$, the inverse cumulative distribution function is applied, otherwise the empirical filtered residual is chosen
7. Transforming the new residuals to log returns is made by multiplying them with the corresponding forecasted volatility and adding the mean value $\mu$
Algorithm for Simulation of Log Return Distribution from Student’s t Copula:

1. Find the correlation matrix $R$ of the filtered residuals $Z_{i,t}$
2. Find the Cholesky decomposition $A$ of $R$
3. Simulate $n$ independent random variables $z_1, z_2, \ldots, z_n$ from $N(0, 1)$
4. Simulate a random variate $s$ from $\chi^2_\nu$ independent of $z_1, z_2, \ldots, z_n$
5. Set $y = Az$
6. Set $x = \sqrt{(\nu/s)y}$
7. Transform $x$ to uniform variables by applying $u_i = T_\nu(x_i)$ for $i = 1, \ldots, n$ and where $T_\nu$ denotes the univariate Student t distribution function
8. Transform back to residuals by applying hybrid distribution framework according to equation (2.23). If the uniform variable is higher than or equal to $u_{\text{high}}$ or lower than or equal to $u_{\text{low}}$, the inverse cumulative distribution function is applied, otherwise the empirical filtered residual is chosen
9. Transforming the new residuals to log returns is made by multiplying them with the corresponding forecasted volatility and adding the mean value $\mu$

The risk measures are then computed from the simulated log return distribution for the normal case as well as for the Student’s t case with $\nu = 3, 7, 15, 20$. The reason for implementing the Student’s t copula is essentially due to the tail dependence shown in the equations (3.5) and (3.6) and the parameter $\nu$ was set to fixed values instead of being estimated by calibration to the entire reference portfolio since that procedure rarely leads to an accurate estimation.
To be able to determine how many Monte Carlo simulations that are needed for model family 3 in order to result in accurate risk measure estimations, the model’s VaR and ES at a certain confidence level were computed for a specific day for a growing number of Monte Carlo simulations from the normal copula. The day was chosen to be February 12, 2007, but could have been any other day within the sample except one of the 250 first days used for calibration of the model. The procedure was repeated 1,000 times to eliminate statistical uncertainty. Figure 3.14 visualizes the number of Monte Carlo simulations needed for 99% VaR and Figure 3.15 visualizes the corresponding number for 99% ES.

Figure 3.14: Convergence of 99% VaR for increasing number of Monte Carlo simulations (green line) with 95% confidence bounds (red lines)
From the Figures 3.14 and 3.15 one can observe that both 99% VaR and 99% ES fall inside the confidence bounds and that they seem to stabilize around 10,000 Monte Carlo simulations. This is line with the routines of most commercial banks that make 5,000-10,000 Monte Carlo simulations to perform their risk calculations. The number of Monte Carlo simulations used for model family 3 was set to 10,000.

### 3.4.4 An Overview of all Model Families

In order to overview the different model families that are to be investigated throughout this thesis, Table 3.7 presents all model families with their specific properties.
### Model Setup

<table>
<thead>
<tr>
<th>Model Family</th>
<th>GARCH filtration</th>
<th>Analytical/Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model Family 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
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<td>Analytical</td>
</tr>
<tr>
<td>Students t, $\nu = 3$</td>
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<td>Analytical</td>
</tr>
<tr>
<td>Students t, $\nu = 7$</td>
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<td>Analytical</td>
</tr>
<tr>
<td>Students t, $\nu = 15$</td>
<td>No</td>
<td>Analytical</td>
</tr>
<tr>
<td>Students t, $\nu = 20$</td>
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<td>Analytical</td>
</tr>
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<td></td>
</tr>
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<td>Normal</td>
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<td>Analytical</td>
</tr>
<tr>
<td>Students t, $\nu = 3$</td>
<td>Yes</td>
<td>Analytical</td>
</tr>
<tr>
<td>Students t, $\nu = 7$</td>
<td>Yes</td>
<td>Analytical</td>
</tr>
<tr>
<td>Students t, $\nu = 15$</td>
<td>Yes</td>
<td>Analytical</td>
</tr>
<tr>
<td>Students t, $\nu = 20$</td>
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<td>Analytical</td>
</tr>
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<td>Model Family 3 (Copula)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
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</tr>
<tr>
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<td>Simulation</td>
</tr>
<tr>
<td>Students t, $\nu = 7$</td>
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<td>Simulation</td>
</tr>
<tr>
<td>Students t, $\nu = 15$</td>
<td>Yes</td>
<td>Simulation</td>
</tr>
<tr>
<td>Students t, $\nu = 20$</td>
<td>Yes</td>
<td>Simulation</td>
</tr>
</tbody>
</table>

Table 3.7: An overview of the different model families

### 3.5 Backtesting VaR

Starting with the 251st observation in the sample of 2,779 residuals, the models are re-estimated each day using the last 250 day’s residuals, representing one trading year, also named rolling forward, for calibration. Hence, 2,529 VaR forecasts are produced by computing each of the model families’ one-day VaR for the following day. The VaR forecasts over time are then compared to the actual historical outcomes, the log return realizations. If a VaR forecast turns out to be greater than the log returns realization, there is an exceedance according to equation (2.34). When computing VaR at confidence level $p$ per cent one expects $p$ per cent exceedances out of the total number of data points in the historical sample. Several VaR backtests are performed to investigate the reliability of the different model families implemented.

#### 3.5.1 Model Family 1

For $p = 99\%$, visualizing the forecasted VaR (and ES) together with the historical actual log return realizations yields for the normal case Figure 3.16.
From Figure 3.16 it can be concluded that VaR always takes a smaller value than ES, which is in line with the definition of the risk measures, see Section 2.3 in Chapter 2. Further, it can be concluded that the risk measure estimates increase during high volatile periods. When the historical actual log return realization is greater than the risk measure estimate, an exceedance of the risk measure estimate has occurred and is present in the figure at several times, particularly during highly volatile periods.

For the Student’s t case, with four different values of the degrees of freedom parameter $\nu$, the corresponding results can be seen in Figure 3.17.
From Figure 3.17 it can in addition to the conclusions regarding Figure 3.16 be concluded that both VaR and ES are higher for the Student’s t case with decreasing degrees of freedom parameter, $\nu$, than for the normal case, which is in line with what can be expected since the tails of the Student’s t distribution are fatter resulting in higher risk measure estimates. Higher risk measure estimates give rise to less exceedances of the risk measures, which is also seen in the figure.

Regarding a different confidence level, $p = 95\%$, visualizing the forecasted VaR (and ES) together with the historical actual log return realizations yields for the normal case Figure 3.18.
Figure 3.18: VaR and ES together with historical log return realizations for model family 1 (normal case), $p = 95\%$

From Figure 3.18 it can be concluded when comparing it to Figure 3.16 that decreasing the confidence level from $p = 99\%$ to $p = 95\%$ implies a decrease in the risk measure estimates and hence more risk exceedances. This is what is expected since by definition, a lower confidence level will yield more exceedances for the respective risk measure estimates.

For the Student’s $t$ case, with four different values of the degrees of freedom parameter, $\nu$, the corresponding results can be seen in Figure 3.19.
Figure 3.19: VaR and ES together with historical log return realizations for model family 1 (Student’s t case with $\nu=3, 7, 15, 20$), $p = 95\%$

Similarly to the conclusions about the Student’s t case when $p = 99\%$, Figure 3.19 indicates that the risk measures are higher for the Student’s t case with decreasing degrees of freedom parameter, $\nu$, than for the normal case, which is in line with what can be expected since the tails of the Student’s t distribution are fatter resulting in higher risk measure estimates. Decreasing the confidence level implies also for the Student’s t case that more exceedances of the risk measure estimates occur which is by definition the expected result.

Overall, when comparing the Figures 3.16, 3.17, 3.18 and 3.19 it can be concluded that the number of exceedances increases when $p$ decreases. Also, moving from normal distribution to Student’s t distribution implies higher risk measure estimates and hence less exceedances, which is due to the fat tail behavior of the Student’s t distribution in comparison with the normal distribution.
3.5.2 Model Family 2

For $p = 99\%$, visualizing the forecasted VaR (and ES) together with the historical actual log return realizations yields for the normal case Figure 3.20.

![Figure 3.20: VaR and ES together with historical log return realizations for model family 2 (normal case), $p = 99\%$](image)

From Figure 3.20 it can, similarly to model family 1, be concluded that VaR always takes a smaller value than ES, which is in line with the definition of the risk measures, see Section 2.3 in Chapter 2. Further, it can be concluded that the risk measure estimates increase during highly volatile periods and that exceedances of the risk measures mainly occur during these periods.

Comparing Figure 3.20 with the corresponding figure for model family 1, Figure 3.16, it displays the GARCH effects that have been obtained by performing a GARCH filtration and can be seen by the gradual increase of the risk measure estimates during highly volatile periods.

For the Student’s t case, with four different values of the degrees of freedom parameter, $\nu$, the corresponding results can be seen in Figure 3.21.
Figure 3.21: VaR and ES together with historical log return realizations for model family 2 (Student’s t case with $\nu = 3, 7, 15, 20$), $p = 99\%$

From Figure 3.21 it can in addition to the conclusions regarding Figure 3.20 and similarly to analysis regarding model family 1 be concluded that both VaR and ES are higher for the Student’s t case with decreasing degrees of freedom parameter, $\nu$, than for the normal case, which is in line with what can be expected since the tails of the Student’s t distribution are fatter resulting in higher risk measure estimates. Similarly to the normal case represented by Figure 3.20 the GARCH effects are present here and seen by the gradual increase of the risk measure estimates during highly volatile periods.

Regarding a different confidence level, $p = 95\%$, visualizing the forecasted VaR (and ES) together with the historical actual log return realizations yields for the normal case Figure 3.22.
From Figure 3.22 it can be concluded, when comparing it to Figure 3.20, that decreasing the confidence level from $p = 99\%$ to $p = 95\%$ implies a decrease in the risk measure estimates and hence more risk measure exceedances. Similarly to the analysis regarding model family 1, this is in line with the expectations since by definition a lower confidence level will yield more exceedances for the respective risk measure estimates. The GARCH effects are not affected by the confidence level chosen and are therefore present to the same extent as for the previous cases within model family 2.

For the Student’s t case, with four different values of the degrees of freedom parameter, $\nu$, the corresponding results can be seen in Figure 3.23.
Figure 3.23: VaR and ES together with historical log return realizations for model family 2 (Student’s t case with $\nu=3, 7, 15, 20$), $p = 95\%$

Similarly to the conclusions about the Student’s t case when $p = 99\%$, Figure 3.23 indicates that both VaR and ES are higher for the Student’s t case with decreasing degrees of freedom parameter, $\nu$, than for the normal case, which is in line with what can be expected since the tails of the Student’s t distribution are fatter resulting in higher risk measure estimates. Decreasing the confidence level implies also for the Student’s t case that more exceedances occur.

Overall, when comparing the Figures 3.20, 3.21, 3.22 and 3.23 and similarly to the analysis regarding model family 1, the number of exceedances increases when $p$ decreases which is what one can expect. Also, moving from normal distribution to Student’s t distribution implies higher risk measure estimates which is due to the fat tail behavior of the Student’s t distribution. Further, one can note the effect of applying a GARCH filtration when moving from model family 1 to model family 2 as both VaR and ES increase gradually during highly volatile periods which is in line with the expected result of modeling the volatility with a GARCH model.
3.5.3 Model Family 3

For $p = 99\%$, visualizing the forecasted VaR (and ES) together with the historical actual log return realizations yields for simulation from a normal copula Figure 3.24.

![Figure 3.24: VaR and ES together with historical log return realizations for model family 3 (normal copula), $p = 99\%$](image)

From Figure 3.24 it can, similarly to model family 1 and 2, be concluded that VaR always takes smaller values than ES, which is in line with the definition of the risk measures, see Section 2.3 in Chapter 2. Further, it can be concluded that the risk measure estimates increase during highly volatile periods. Comparing Figure 3.24 with the corresponding figures for model family 1, Figure 3.16, and model family 2, Figure 3.20, shows that GARCH effects are equally present for model family 3 as for model family 2 due to the gradual increase of the risk measure estimates during highly volatile periods. Further, the risk measure estimates are higher for model family 3 than for the other model families due to the hybrid distribution with GPD tails. Also, the risk measure estimates are here produced by Monte Carlo simulation which is the reason for the oscillating shape of the curves of the risk estimates in the graphs.

When simulating from a Student’s $t$ copula instead, with four different val-
ues of the degrees of freedom parameter, $\nu$, the corresponding results yield Figure 3.25.

![Figure 3.25: VaR and ES together with historical log return realizations for model family 3 (Student’s t copula with $\nu=3, 7, 15, 20$), $p = 99\%$](image)

From Figure 3.25 it can in addition to the conclusions regarding Figure 3.24 and similarly to analysis regarding model family 1 and 2 be concluded that both VaR and ES are higher when simulating from a Student’s t copula with decreasing degrees of freedom, $\nu$, than from a normal copula, which is in line with what can be expected since the tails of the Student’s t distribution are fatter resulting in higher risk measure estimates. Similarly to the normal case represented by Figure 3.24, the GARCH effects are present here by observing the gradual increase of the risk measure estimates during highly volatile periods. Also, the impact of implementing a simulation method, in comparison with an analytical method, is present in the curves of the risk estimates by the oscillating shape.

Regarding a different confidence level, $p = 95\%$, visualizing the forecasted VaR (and ES) together with the historical actual log return realizations yields
for simulation from a normal copula Figure 3.26.

From Figure 3.26, it can be concluded when comparing it to Figure 3.24, that a decrease of the confidence level from $p = 99\%$ to $p = 95\%$ implies, similarly as it did for model family 1 and 2, a decrease in the risk measure estimates and hence more risk measure exceedances. Similarly to the analysis regarding model family 1 and 2, this is in line with the expectations since by definition a lower confidence level will yield more exceedances for the respective risk measure estimates. The GARCH effects are not affected by the confidence level chosen and are therefore present to the same extent as for the previous copula simulations.

When simulating from a Student’s $t$ copula instead, with four different values of the degrees of freedom parameter, $\nu$, the corresponding results yield Figure 3.27.
Figure 3.27: VaR and ES together with historical log return realizations for model family 3 (Student’s t copula with $\nu=3, 7, 15, 20$), $p = 95\%$

Similarly to the conclusions about the Student’s t case when $p = 99\%$, Figure 3.27 indicates that both VaR and ES are higher when simulating from a Student’s t copula with decreasing degrees of freedom parameter, $\nu$, than from a normal copula, which is in line with what can be expected since the tails of the Student’s t distribution are fatter resulting in higher risk measure estimates. Decreasing the confidence level implies also for the Student’s t copula that more exceedances occur.

Overall, when comparing the Figures 3.24, 3.25, 3.26 and 3.27 and similarly to the analysis regarding model family 1 and 2, the number of exceedances increases when $p$ decreases which is what one can expect. Also, moving from simulation from a normal copula to simulation from a Student’s t copula implies higher risk measure estimates which is due to the fat tail behavior of the Student’s t distribution. Further, one can note the effect of applying GARCH filtration when moving from model family 1 to model family 3 since both VaR and ES increase gradually during highly volatile periods which is in line with the expected result of modeling the volatility with a
GARCH model. Also, regarding model family 3, the risk measure estimates are here produced by Monte Carlo simulation instead of analytical computations, which is the reason for the oscillating shape of the curves of the risk estimates.

### 3.5.4 VaR Exceedances

In this section the number of actual VaR exceedances are computed according to equation (2.34).

<table>
<thead>
<tr>
<th>Actual VaR Exceedances</th>
<th>$p = 99%$</th>
<th>$p = 95%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected # of exceedances</td>
<td>25.29 (1.00%)</td>
<td>126.45 (5.00%)</td>
</tr>
<tr>
<td>Model Family 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
<td>53 (2.10%)</td>
<td>129 (5.10%)</td>
</tr>
<tr>
<td>Students t, $\mu=3$</td>
<td>5 (0.20%)</td>
<td>52 (2.06%)</td>
</tr>
<tr>
<td>Students t, $\mu=7$</td>
<td>18 (0.71%)</td>
<td>87 (3.44%)</td>
</tr>
<tr>
<td>Students t, $\mu=15$</td>
<td>35 (1.38%)</td>
<td>110 (4.35%)</td>
</tr>
<tr>
<td>Students t, $\mu=20$</td>
<td>38 (1.50%)</td>
<td>113 (4.47%)</td>
</tr>
<tr>
<td>Model Family 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
<td>46 (1.82%)</td>
<td>114 (4.51%)</td>
</tr>
<tr>
<td>Students t, $\mu=3$</td>
<td>6 (0.24%)</td>
<td>43 (1.70%)</td>
</tr>
<tr>
<td>Students t, $\mu=7$</td>
<td>14 (0.55%)</td>
<td>81 (3.20%)</td>
</tr>
<tr>
<td>Students t, $\mu=15$</td>
<td>34 (1.34%)</td>
<td>98 (3.88%)</td>
</tr>
<tr>
<td>Students t, $\mu=20$</td>
<td>36 (1.42%)</td>
<td>99 (3.91%)</td>
</tr>
<tr>
<td>Model Family 3 (Copula)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
<td>22 (0.87%)</td>
<td>110 (4.35%)</td>
</tr>
<tr>
<td>Students t, $\mu=3$</td>
<td>17 (0.67%)</td>
<td>59 (2.33%)</td>
</tr>
<tr>
<td>Students t, $\mu=7$</td>
<td>17 (0.67%)</td>
<td>86 (3.40%)</td>
</tr>
<tr>
<td>Students t, $\mu=15$</td>
<td>19 (0.75%)</td>
<td>92 (3.64%)</td>
</tr>
<tr>
<td>Students t, $\mu=20$</td>
<td>24 (0.95%)</td>
<td>96 (3.80%)</td>
</tr>
</tbody>
</table>

Table 3.8: Number of expected vs actual exceedances of VaR for the different models and at different confidence levels $p$

Table 3.8 confirms the analysis outlined in the previous section, that the number of exceedances increase when $p$ decreases. Also, moving from a normal distribution to a Student’s t distribution implies less exceedances which is due to higher risk measure estimates caused by the fat tail behavior of the Student’s t distribution. Further, one can note the effect of applying a GARCH filtration when moving from model family 1 to model family 2 and 3 by fewer exceedances occurring for those model families. Finally, applying the hybrid distribution with GPD tails yields in general more accurate numbers of exceedances (1% and 5% respectively), which is due to the fact that GPD tails with fitted parameters very well models financial data.

In particular, regarding the number of exceedances for $p = 99\%$ in Table
it seems that both model family 1 and 2 together with Student’s t distribution where $7 < \nu < 15$ yields the most satisfying model according to the number of exceedances, being as close as possible to 1%. For $p = 95\%$ it seems that model family 1 with normal distribution is the most satisfying model according to number of exceedances, being as close to 5% as possible.

The VaR exceedances for $p = 99\%$ respectively $p = 95\%$ have been visualized in Appendix 4 and Appendix 5 to give a closer picture of their distribution and magnitude.

**3.5.5 Duration**

Is is of interest to measure the distribution of the exceedances, not only their number. Hence, the number of days between the exceedances, the duration, is measured for the different confidence levels regarded. For $p = 99\%$, see Table 3.9.

<table>
<thead>
<tr>
<th>Duration (in days)</th>
<th>$\leq 1$</th>
<th>$\leq 2$</th>
<th>$\leq 5$</th>
<th>$\leq 10$</th>
<th>$\leq 50$</th>
<th>$\leq 250$</th>
<th>$\leq 500$</th>
</tr>
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<tbody>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td>52</td>
</tr>
<tr>
<td>Students t, $\nu=3$</td>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=7$</td>
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<td>2</td>
<td>2</td>
<td>10</td>
<td>13</td>
<td>16</td>
</tr>
<tr>
<td>Students t, $\nu=15$</td>
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<td>2</td>
<td>5</td>
<td>10</td>
<td>25</td>
<td>31</td>
<td>34</td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
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<td>3</td>
<td>6</td>
<td>11</td>
<td>27</td>
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<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
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<td>9</td>
<td>14</td>
<td>34</td>
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<td>1</td>
<td>1</td>
<td>5</td>
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<td>13</td>
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<td>2</td>
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<td>8</td>
<td>12</td>
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</tr>
<tr>
<td>Students t, $\nu=15$</td>
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<td>2</td>
<td>2</td>
<td>9</td>
<td>15</td>
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<td>2</td>
<td>3</td>
<td>15</td>
<td>19</td>
<td>23</td>
</tr>
</tbody>
</table>

Table 3.9: Duration (days between VaR exceedances) for the different model families, $p = 99\%$.

An as high duration as possible is to prefer as it indicates longer time between occurred VaR exceedances and hence less dependence. From Table 3.9 it is possible to conclude that the durations increase when moving from model family 1 and model family 2 to model family 3. An increasing duration implies that the number of days between VaR exceedances increases which...
implies increased independence between the VaR exceedances provoking less VaR exceedance clustering.

For the corresponding results regarding $p = 95\%$, see Table 3.10.

<table>
<thead>
<tr>
<th>Duration (in days)</th>
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<th>≤5</th>
<th>≤10</th>
<th>≤50</th>
<th>≤250</th>
<th>≤500</th>
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<tbody>
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<td>Model Family 1</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
<td>11</td>
<td>22</td>
<td>40</td>
<td>63</td>
<td>120</td>
<td>127</td>
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<td>10</td>
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<td>74</td>
<td>85</td>
<td>86</td>
</tr>
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<td>Students t, $\nu=15$</td>
<td>10</td>
<td>21</td>
<td>36</td>
<td>54</td>
<td>98</td>
<td>108</td>
<td>109</td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
<td>10</td>
<td>21</td>
<td>36</td>
<td>55</td>
<td>102</td>
<td>111</td>
<td>112</td>
</tr>
<tr>
<td>Model Family 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
<td>9</td>
<td>19</td>
<td>34</td>
<td>57</td>
<td>102</td>
<td>112</td>
<td>113</td>
</tr>
<tr>
<td>Students t, $\nu=3$</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>31</td>
<td>39</td>
<td>42</td>
</tr>
<tr>
<td>Students t, $\nu=7$</td>
<td>8</td>
<td>14</td>
<td>18</td>
<td>35</td>
<td>68</td>
<td>78</td>
<td>80</td>
</tr>
<tr>
<td>Students t, $\nu=15$</td>
<td>9</td>
<td>18</td>
<td>30</td>
<td>47</td>
<td>85</td>
<td>96</td>
<td>97</td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
<td>9</td>
<td>18</td>
<td>30</td>
<td>48</td>
<td>86</td>
<td>97</td>
<td>98</td>
</tr>
<tr>
<td>Model Family 3 (Copula)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
<td>9</td>
<td>20</td>
<td>36</td>
<td>61</td>
<td>101</td>
<td>107</td>
<td>109</td>
</tr>
<tr>
<td>Students t, $\nu=3$</td>
<td>1</td>
<td>6</td>
<td>12</td>
<td>21</td>
<td>48</td>
<td>56</td>
<td>58</td>
</tr>
<tr>
<td>Students t, $\nu=7$</td>
<td>4</td>
<td>11</td>
<td>24</td>
<td>36</td>
<td>77</td>
<td>83</td>
<td>85</td>
</tr>
<tr>
<td>Students t, $\nu=15$</td>
<td>6</td>
<td>11</td>
<td>20</td>
<td>35</td>
<td>83</td>
<td>89</td>
<td>91</td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
<td>5</td>
<td>11</td>
<td>26</td>
<td>45</td>
<td>86</td>
<td>93</td>
<td>95</td>
</tr>
</tbody>
</table>

Table 3.10: Duration (days between VaR exceedances) for the different model families, $p = 95\%$

Table 3.10 does not show any obvious trend for the duration when moving from model family 1 to model family 2 as well as when moving from model family 2 to model family 3. The table will instead be useful to analyze together with the duration tests further down in the analysis.

### 3.5.6 VaR Backtests

#### Unconditional Coverage

The unconditional coverage test is performed to examine if the proportion of VaR exceedances is consistent with the selected confidence level by computing the likelihood ratio according to equation \(2.35\). A p-value less than 0.05 (5\%) in Table 3.11 indicates that the null hypothesis $H_0$, claiming consistency of VaR exceedances, can be rejected and hence the model is inconsistent.
Table 3.11: p-values for the unconditional coverage test for the different model families and at different confidence levels $p$

<table>
<thead>
<tr>
<th>Model Family 1</th>
<th>Normal</th>
<th>Students t, $\nu=3$</th>
<th>Students t, $\nu=7$</th>
<th>Students t, $\nu=15$</th>
<th>Students t, $\nu=20$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0.124</td>
<td>0.067</td>
<td>0.018</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model Family 2</th>
<th>Normal</th>
<th>Students t, $\nu=3$</th>
<th>Students t, $\nu=7$</th>
<th>Students t, $\nu=15$</th>
<th>Students t, $\nu=20$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0.014</td>
<td>0.098</td>
<td>0.044</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model Family 3 (Copula)</th>
<th>Normal</th>
<th>Students t, $\nu=3$</th>
<th>Students t, $\nu=7$</th>
<th>Students t, $\nu=15$</th>
<th>Students t, $\nu=20$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.501</td>
<td>0.078</td>
<td>0.078</td>
<td>0.189</td>
<td>0.795</td>
</tr>
</tbody>
</table>

Table 3.11 confirms the majority of conclusions drawn earlier in this chapter regarding the performance of the different model families and distributions according to number of exceedances. Regarding $p = 99\%$ in particular, it confirms that model family 1 with Student’s t distribution where $\nu = 7, 15$ is satisfactory as p-values are greater than 0.05 indicating consistency of the models. Also, model family 2 with Student’s t distribution where $\nu = 15$ as well as model family 3 with all different distributions are satisfactory due to p-values greater than 0.05. Regarding $p = 95\%$, the conclusion drawn earlier in the chapter according to the number of exceedances still holds as the p-value for model family 1 with normal distribution is greater than 0.05. In addition to this, model family 1 with Student’s t distribution where $\nu = 15, 20$, as well as both model family 2 and 3 with normal distribution, yield greater p-values than 0.05 and can therefore be seen as satisfactory models.

**Conditional Coverage (Independence)**

The conditional coverage test (independency test) is performed to examine the dependence between the exceedances by computing the likelihood ratio according to equation (2.36). A p-value less than 0.05 (5%) in Table 3.12 indicates that the null hypothesis $H_0$, claiming independence between VaR exceedances, can be rejected and hence there is dependence between the VaR exceedances.
Table 3.12: p-values for the conditional coverage (independence) test for the different model families and at different confidence levels $p$.

<table>
<thead>
<tr>
<th>Model Family</th>
<th>$p = 99%$</th>
<th>$p = 95%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>0</td>
<td>0.240</td>
</tr>
<tr>
<td>Students t, $\nu=3$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Students t, $\nu=7$</td>
<td>0.270</td>
<td>0</td>
</tr>
<tr>
<td>Students t, $\nu=15$</td>
<td>0.150</td>
<td>0.027</td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
<td>0.053</td>
<td>0.055</td>
</tr>
<tr>
<td>Normal</td>
<td>0</td>
<td>0.137</td>
</tr>
<tr>
<td>Students t, $\nu=3$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Students t, $\nu=7$</td>
<td>0.044</td>
<td>0</td>
</tr>
<tr>
<td>Students t, $\nu=15$</td>
<td>0.058</td>
<td>0.002</td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
<td>0.036</td>
<td>0.002</td>
</tr>
<tr>
<td>Normal</td>
<td>0.658</td>
<td>0.059</td>
</tr>
<tr>
<td>Students t, $\nu=3$</td>
<td>0.189</td>
<td>0</td>
</tr>
<tr>
<td>Students t, $\nu=7$</td>
<td>0.189</td>
<td>0</td>
</tr>
<tr>
<td>Students t, $\nu=15$</td>
<td>0.365</td>
<td>0.002</td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
<td>0.768</td>
<td>0.012</td>
</tr>
</tbody>
</table>

Table 3.12 confirms the majority of conclusions drawn regarding satisfactory models according to number of exceedances as well as unconditional coverage in Table 3.11. The only changes consist for $p = 99\%$ of model family 1 with Student’s t distribution where $\nu = 20$, which in addition to the other satisfactory models is satisfactory due to a p-value greater than 0.05, indicating independence between VaR exceedances. Regarding $p = 95\%$, the conclusions drawn earlier hold except for model family 1 with Student’s t distribution where $\nu = 15$, now having a smaller p-value than 0.05 and hence not being satisfactory in terms of independence between VaR exceedances.

**Duration**

The duration test is another way to measure the independence and is performed by computing the likelihood ratio according to equation (2.37). A p-value less than 0.05 (5%) in Table 3.13 indicates that the null hypothesis $H_0$, claiming independence between VaR exceedances, can be rejected and hence there is dependence between the VaR exceedances.
Table 3.13: p-values for the duration test for the different model families and at different confidence levels $p$

<table>
<thead>
<tr>
<th>Model Family 1</th>
<th>p-value Duration</th>
<th>$p = 99%$</th>
<th>$p = 95%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>0</td>
<td>0.764</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=3$</td>
<td>0.004</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=7$</td>
<td>0.279</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=15$</td>
<td>0.023</td>
<td>0.181</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
<td>0.010</td>
<td>0.291</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model Family 2</th>
<th>p-value Duration</th>
<th>$p = 99%$</th>
<th>$p = 95%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>0</td>
<td>0.296</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=3$</td>
<td>0.016</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=7$</td>
<td>0.037</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=15$</td>
<td>0.065</td>
<td>0.015</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
<td>0.027</td>
<td>0.019</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model Family 3</th>
<th>p-value Duration</th>
<th>$p = 99%$</th>
<th>$p = 95%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>0.574</td>
<td>0.156</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=3$</td>
<td>0.092</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=7$</td>
<td>0.092</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=15$</td>
<td>0.163</td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>Students t, $\nu=20$</td>
<td>0.918</td>
<td>0.006</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.13 confirms the majority of conclusions drawn earlier in this chapter regarding the performance of the different model families and distributions according to number of exceedances as well as the unconditional coverage test and the conditional coverage (independence) test. The only changes consist for $p = 99\%$ of model family 1 with Student’s t distribution where $\nu = 15, 20$ does not show satisfactory results due to p-values smaller than 0.05.

Summing up, the following models yield satisfactory for all VaR backtests, see Table 3.14:

Table 3.14: Satisfactory models that have passed all VaR backtests for the different levels of $p$
From Table 3.14 it is possible to conclude that only a few models pass all VaR backtests. Here the number of exceedances has not been taken into account and hence some models might not be suitable choices according to that.

As it was earlier mentioned for \( p = 99\% \), when moving from model family 1 to model family 2, the number of exceedances decreases according to Table 3.8. In particular, those exceedances that happen in clusters almost vanishes, which is due to the fact that GARCH filtration has been applied. Note that the model families 1 and 2 together with Student’s t distribution where \( \nu = 7, 15 \) seemed to be having approximately the correct number of exceedances, however, they are shown to happen in clusters by the VaR backtests in this section for most of the values of \( \nu \). Moving from model family 2 to model family 3 implies that the number of exceedances decreases even more. This is due to the fact that tails are modeled with GPD as an addition to the GARCH filtration. Model family 3 seems to have too few exceedances for all distributions, however, it yields favorable p-values for the VaR backtests, indicating that exceedances happen independently of each other. It can be concluded that modeling the volatility by a GARCH model has a positive impact as well as using a simulation method with GPD tail modeling. Out of the satisfactory models in Table 3.14 for \( p = 99\% \), the most suitable choice is therefore a normal copula with GARCH filtration and a hybrid GPD-empirical distribution applied (model family 3 - normal), as it shows satisfactory p-values in all VaR backtests as well as having the most correct number of exceedances (as close to 1% as possible) in Table 3.8.

For \( p = 95\% \), model family 1 with normal distribution seems to be a suitable choice. First, it has approximately the correct number of exceedances (as close to 5% as possible) while all other model families show to few. Further, the p-values for the VaR backtests are satisfying for this model according to Table 3.14. The reason why GARCH modeling does not seem to be needed here is that there are by definition so many exceedances when \( p = 95\% \), that in total it looks like exceedances are independent since the p-values are satisfactory even for model family 1. A quick look in Table 3.10 with the durations for model family 1 show that the exceedances actually are clustered due to low durations, where many exceedances turn out to occur on consecutive days. However, this is a reasonable result since such a relatively low quantile indicates we are not far out in the tail and hence the normal distribution is a good approximation of the distribution. Therefore, one should strive to choose an as simple model as possible for lower quantiles, hence choosing model family 1 with normal distribution out of the satisfactory models for \( p = 95\% \) in Table 3.14.

One could argue that moving from model family 1 to model family 2 does not imply as much improvement as one could expect for a GARCH filtration.
in terms of number of exceedances and p-values for the VaR backtests. This could be due to the fact that 250 days are used for calibration which is not necessarily an optimal choice. It has been chosen to replicate the way banks calibrate their market risk models today.

3.6 Backtesting ES

To be able to perform the backtesting of ES outlined in Section 2.7.2, VaR backtests are necessary. The reason for this is mentioned in the empirical illustration in [28], where it is explained that the ES backtesting heavily relies on an accurate VaR model, which is determined by VaR backtesting. In particular, reliable VaR predictions and hence correct predictions of $ES$, $SD$ and $BT_{t+k}$ are essential components of the ES backtesting. This is the reason why VaR backtests have been performed in the previous section of this thesis. Although it was only some of the VaR models that turned out to be satisfactory, see Table 3.14, the unsatisfactory VaR models will still be examined from an ES backtesting perspective. It is of interest to find out if they may also be unsatisfactory from an ES backtesting perspective.

A significant advantage of the ES backtesting method considered, is that it allows for testing of individual VaR exceedances. Hence, the risk manager does not need to wait for an entire period of VaR exceedances to be able to perform the ES backtesting. In most cases, one single VaR exceedance is sufficient to obtain information about the model performance. This is why this particular ES backtest is very useful, especially during highly volatile periods, which will be illustrated in the following section.

Starting with the 251st observation in the sample of 2,779 returns, the models are re-estimated each day using the last 250 day’s returns, representing one trading year, also named rolling forward, for calibration. Hence, 2,529 ES forecasts are produced by computing each of the model families’ one-day ES for the following day. The ES forecasts over time are then compared to the actual historical outcomes, the log return realizations. If an ES forecast turns out to be greater than the log returns realization, there is an ES exceedance.

As before, a restriction of -3 for each asset’s simulated log returns has been applied for model family 3, the non-analytical model family. The restriction represents a drop of 300% of the asset’s log value from one trading day to another. The reason it has been implemented is both for computational reasons and adaption to reality, recalling the trading curb or circuit breaker, which is a financial regulatory instrument that acts to prevent stock market crashes from occurring.
3.6.1 Righi and Ceretta Method

The backtesting of ES is implemented in the light of the method proposed by Righi and Ceretta outlined in Section 2.7.2. It is applied for the different risk models by calculating an analytical shortfall deviation, $SD_p$, according to the equations (2.42) and (2.43) for model family 1 and 2 respectively, and an empirical $SD_p$ according to equation (2.40) for model family 3. The actual backtest ratio, $BT_{t+k}$, is computed according to equation (2.44) for all model families. Further, a one-tailed test, with the alternative hypothesis that the loss that has occurred is worse than the expected one, is performed by using Monte Carlo simulation to achieve a robust result, see the algorithm in Section 2.7.2. In this thesis, the algorithm is run for 1,000 samples (N) of length 10,000 (n) making up 10,000,000 simulations in total. For model family 3 a bootstrap method, i.e., sampling with replacement, is performed from a large sample of length 50,000. It is worth noting that a start-up period has been implemented for model family 3 and hence the first exceedance is always left out of the results since computations of $SD_p$ are not reliable in that case as the variance of a single observation is 0.

The ES backtest is implemented for the confidence levels $p = 99\%$, $p = 97, 5\%$ and $p = 95\%$, as it is of particular interest to include backtesting for the actual confidence level proposed by Basel in the FRTB, see Section 1.3.

The results for the ES backtest are now to be presented. First, the results regarding significance level $p = 99\%$ and the three different model families are given in the Tables 3.15, 3.16 and 3.17. The tables include the test results for the observations for which the five lowest (if such) p-values, indicating rejection of the null hypothesis at a test significance level of 5%, were obtained. For a summary of the ES backtesting results regarding all observations, see Appendix 8.
\[
p = 99\%
\]

<table>
<thead>
<tr>
<th>Model family 1</th>
<th>Obs.</th>
<th>Date</th>
<th>VaR exceed.</th>
<th>ES exceed.</th>
<th>( BT_{t+k} )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>14</td>
<td>23 May 2006</td>
<td>0.0275</td>
<td>0.0255</td>
<td>4.5800</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>618</td>
<td>13 Oct 2008</td>
<td>0.0359</td>
<td>0.0322</td>
<td>3.0703</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>1013</td>
<td>10 May 2010</td>
<td>0.0197</td>
<td>0.0177</td>
<td>3.1196</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>602</td>
<td>19 Sep 2008</td>
<td>0.0273</td>
<td>0.0241</td>
<td>2.6970</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td>1334</td>
<td>9 Aug 2011</td>
<td>0.0157</td>
<td>0.0136</td>
<td>2.3037</td>
<td>0.04</td>
</tr>
<tr>
<td>Student’s t, ( \nu = 3 )</td>
<td>14</td>
<td>23 May 2006</td>
<td>0.0236</td>
<td>0.0192</td>
<td>2.9245</td>
<td>0.02</td>
</tr>
<tr>
<td>Student’s t, ( \nu = 7 )</td>
<td>14</td>
<td>23 May 2006</td>
<td>0.0259</td>
<td>0.0231</td>
<td>3.8447</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>618</td>
<td>13 Oct 2008</td>
<td>0.0329</td>
<td>0.0276</td>
<td>2.4408</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>1013</td>
<td>10 May 2010</td>
<td>0.0181</td>
<td>0.0152</td>
<td>2.4867</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>602</td>
<td>19 Sep 2008</td>
<td>0.0247</td>
<td>0.0202</td>
<td>2.0936</td>
<td>0.05</td>
</tr>
<tr>
<td>Student’s t, ( \nu = 15 )</td>
<td>14</td>
<td>23 May 2006</td>
<td>0.0263</td>
<td>0.0237</td>
<td>4.0303</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>618</td>
<td>13 Oct 2008</td>
<td>0.0337</td>
<td>0.0289</td>
<td>2.6013</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>1013</td>
<td>10 May 2010</td>
<td>0.0185</td>
<td>0.0159</td>
<td>2.6480</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>602</td>
<td>19 Sep 2008</td>
<td>0.0254</td>
<td>0.0213</td>
<td>2.2480</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 3.15: Results ES backtesting Model Family 1 \( p = 99\% \)

From Table 3.15 it can be concluded which observations for the different cases of model family 1 that rejects the null hypothesis of a loss significantly different from the one expected. It has already been concluded regarding the VaR backtests that model family 1 with normal distribution is not a satisfactory model, see Table 3.14. From an ES backtesting perspective this is confirmed since, for instance observation 14, is more than 4 SD apart from the ES, indicating a significant loss. In total there are 5 observations that reject the null hypothesis at a test significance level of 5%, see Table 3.15. The fact that 5 observations out of 53, for which VaR exceedances occur, representing 9.4% of the observations, reject the null hypothesis, implies that it occurs more often than 5% of the time and so the model is not satisfactory. Regarding a test significance level of 10%, 8 observations yield rejection of the null hypothesis and as 8 out of 53 observations represents more than 10%, again, model family 1 with normal distribution is proven to be unsatisfactory.

For the other cases it is seen that the lower the value of the degrees of freedom parameter \( \nu \) for the Student’s t distribution, the more conservative the model seems to be as less observations exist for which the null hypothesis is rejected in relation to the number of VaR exceedance observations. For \( \nu = 3 \), the model is never rejected, possibly indicating a too conservative model. According to Table 3.14 for model family 1, the Student’s t distribution with \( \nu = 7 \) is considered satisfactory from a VaR backtesting perspective. From an ES backtesting perspective one can see that the
only observation rejecting the null hypothesis, observation 14, is almost 3 SD apart from the ES which represents an extreme loss. However, as only 1 observation out of 18, for which VaR exceedances occur, representing 5.6% of the observations, yields rejection of the null hypothesis, it can be concluded that the result is in line with the chosen test significance level and may therefore be considered satisfactory from an ES backtesting perspective.

Finally, it can be concluded from the dates corresponding to the observations in the table, that the models are particularly rejected during highly volatile periods. In May 2006, the financial markets suffered from the fact that investors worldwide were worried about inflation and rising interest rates. During the autumn of 2008 the global financial crisis peaked which explains the observations for September and October that year. In 2010 the European debt crisis was a fact and in May that year the Greek government announced austerity measures to be put in place which led to increased instability on the financial markets. In August 2011 there was a sharp drop in stock prices in stock exchanges worldwide which was due to fear of contamination of the European sovereign debt crisis to Spain and Italy. Note that these dates, representing observations from highly volatile financial events, will be present in most tables throughout this section.

<table>
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<tr>
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<td>9 Aug 2011</td>
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Table 3.16: Results ES backtesting Model Family 2 $p = 99\%$
From Table 3.16 it can be concluded that the different models within model family 2 are rejected to approximately the same extent as for model family 1. The difference between the model families is that the variance is time dependent here as GARCH modeling has been applied. As a result, the VaR exceedances are less correlated and occur more seldom. Regarding Table 3.14 for model family 2, modeling with the Student’s t distribution with \( \nu = 15 \) had been concluded to be satisfactory from a VaR backtesting perspective. Comparing the Tables 3.15 and 3.16 for that particular distribution yield that for model family 2 rejection of the null hypothesis only happen for 3 observations, representing 8.8\% of the total number of observations for which VaR exceedances occur, instead of 4 observations regarding model family 1, representing 11.4\%. Hence, model family 2 with Student’s t distribution with \( \nu = 15 \) is more in line with the chosen significance level of the backtest than model family 1. However, model family 2 with Student’s t distribution with \( \nu = 7 \) yields 1 observation out of 14 observations for which VaR exceedances occur, representing 7.1\%, and can therefore be considered even more satisfactory from an ES backtesting perspective.

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<td>Student’s t, ( \nu = 20 )</td>
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Table 3.17: Results ES backtesting Model Family 3 \( p = 99\% \)

From Table 3.17 it can be concluded that the models within model family 3 are never rejected except once for the case with Student’s t distribution with \( \nu = 15 \) and observation 618, representing 5.3\% of the total number of observations for which VaR exceedances occur. Therefore this particular distribution is considered satisfactory from an ES backtesting perspective, while the other may be too conservative. According to Table 3.14 for model family 3, all models were considered satisfactory from a VaR backtesting perspective and so this statement is not violated from an ES backtesting perspective for this model family with Student’s t distribution with \( \nu = 15 \). The reason model family 3 is successful, yet maybe too conservative, is due to the GARCH filtration, resulting in less dependence between risk measure estimations as well as the modeling with the GPD. GPD extrapolation of the tails results in relatively high estimates of ES, which has been seen already.
in Section 3.5.3. It can be concluded that this kind of modeling seems to be particularly suitable for financial data, although sometimes too conservative when applying it for risk measures that take into account the full tail information such as ES.

As a next step, the ES backtest results will be presented for $p = 97.5\%$ and the three different model families, see Tables 3.18, 3.19 and 3.20. The tables include the test results for the observations for which the five lowest (if such) p-values, indicating rejection of the null hypothesis at a test significance level of 5\%, were obtained. For a summary of the ES backtesting results regarding all observations, see Appendix 8.

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Table 3.18: Results ES backtesting Model Family 1 $p = 97.5\%$

From Table 3.18, in comparison with the results for confidence level $p = 99\%$, it can be concluded for model family 1 that when the confidence level of the ES now is lowered, a larger amount of VaR and ES exceedances occur and
so more observations reject the null hypothesis, which is by definition what one can expect with a lower confidence level. However, the percentage of observations yielding rejections of the null hypothesis in relation to the total number of observations for which VaR exceedances occur should remain at a level of 5% if models are satisfactory from an ES backtesting perspective. Comparing the different cases for model family 1, approximately the same pattern is found as for $p = 99\%$, where Student’s t distribution with $\nu = 3$ is potentially too conservative and Student’s t distribution with $\nu = 7$ seems to be satisfactory from an ES backtesting perspective as just a few rejections occur. In particular, for the Student’s t distribution with $\nu = 7$, it is seen that 4 observations out of 51 yield rejection of the null hypothesis, representing 7.8% of the total number of observations for which VaR exceedances occur. This can be seen as a rather satisfactory model as it is approximately in line with the significance level chosen for the ES backtest. However, for the Student’s t distribution with $\nu = 15$, only 5 observations out of 70 yield rejection of the null hypothesis, representing 7.1% of the total number of observations for which VaR exceedances occur. Therefore, from an ES backtesting perspective, the latter model is preferable as it is more in line with the significance level chosen for the ES backtest, although the difference is very small.
<table>
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<td>1334</td>
<td>9 Aug 2011</td>
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</table>

| Student’s t, ν = 3 |
|---|---|---|---|---|---|
| Obs. | Date | VaR exceed. | ES exceed. | BT_{t+k} | p-value |
| 14 | 23 May 2006 | 0.0261 | 0.0216 | 3.1115 | 0.018 |
| 618 | 13 Oct 2008 | 0.0363 | 0.0287 | 2.4565 | 0.032 |
| 1013 | 10 May 2010 | 0.0193 | 0.0150 | 2.2798 | 0.036 |

| Student’s t, ν = 7 |
|---|---|---|---|---|---|
| Obs. | Date | VaR exceed. | ES exceed. | BT_{t+k} | p-value |
| 14 | 23 May 2006 | 0.0276 | 0.0244 | 3.8372 | 0.008 |
| 618 | 13 Oct 2008 | 0.0387 | 0.0334 | 3.1222 | 0.016 |
| 1013 | 10 May 2010 | 0.0207 | 0.0177 | 2.9293 | 0.02 |
| 1334 | 9 Aug 2011 | 0.0174 | 0.0144 | 2.4092 | 0.032 |
| 602 | 19 Sep 2008 | 0.0282 | 0.0232 | 2.3606 | 0.036 |

| Student’s t, ν = 15 |
|---|---|---|---|---|---|
| Obs. | Date | VaR exceed. | ES exceed. | BT_{t+k} | p-value |
| 14 | 23 May 2006 | 0.0279 | 0.0249 | 3.9880 | 0.004 |
| 618 | 13 Oct 2008 | 0.0392 | 0.0343 | 3.2602 | 0.012 |
| 1013 | 10 May 2010 | 0.0210 | 0.0182 | 3.0639 | 0.016 |
| 602 | 19 Sep 2008 | 0.0286 | 0.0240 | 2.4850 | 0.028 |
| 1334 | 9 Aug 2011 | 0.0177 | 0.0149 | 2.5344 | 0.028 |

Table 3.19: Results ES backtesting Model Family 2 p = 97.5%

From Table 3.19 it can be concluded that model family 2 yields approximately the same results as for model family 1, see Table 3.18. The only difference is that the null hypothesis is rejected for the Student’s t distribution with ν = 7 once more for model family 1 than 2, indicating only rejections for 6.9% of the total number of observations for which VaR exceedances occur for model family 2 compared to 7.8% for model family 1. Hence model family 2 is more satisfactory from an ES backtesting perspective regarding this particular distribution.
From Table 3.20 it can be concluded that at most 2 observations yield rejection of the null hypothesis. This represents $3.8\%$ of the total number of observations for which VaR exceedances occur, which is below the accepted frequency of rejections of the test, being $5\%$, and hence indicating a probably too conservative model. This is, again, due to the GPD extrapolation of the tails resulting in relatively high ES estimates and hence few rejections of the null hypothesis. The only observations that reject the model for some of the distributions take place during an extremely high volatile period, the peak of the global financial crisis in 2008.

As a next step, the ES backtest results are presented for $p = 95\%$ and the three different model families, see Tables 3.21, 3.22 and 3.23. The tables include the test results for the observations for which the five lowest (if such) $p$-values, indicating rejection of the null hypothesis at a test significance level of $5\%$, were obtained. For a summary of the ES backtesting results regarding all observations, see Appendix 8.

<table>
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Table 3.20: Results ES backtesting Model Family 3 $p = 97.5\%$
### Model family 1

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Table 3.21: Results ES backtesting Model Family 1 \( p = 95\% \)

From Table 3.21 it can be concluded that a larger amount of observations reject the null hypothesis than for higher confidence levels of ES, which is what one can expect as a lower confidence level yields more rejections by definition. Regarding Table 3.14 and the confidence level of ES concerned, modeling with the normal distribution seems to be satisfactory for all model families from a VaR backtesting point of view. Here it is seen that the normal distribution give rise to observations that are more than 5 SD apart from ES which does not yield a satisfactory result. Therefore, it is from an ES backtesting perspective to prefer modeling with Student’s t distribution. In particular, the Student’s t distribution with \( ν = 20 \) yields rejection of the null hypothesis for 5.3% of the observations for which VaR exceedances occur. Since this is in line with the chosen significance level of the ES backtest, this particular distribution is the most satisfying from an ES backtesting perspective.
### Model family 2

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<td>Student’s t, $\nu = 15$</td>
<td>14</td>
<td>23 May 2006</td>
<td>0.0301</td>
<td>0.0270</td>
<td>4.4223</td>
<td>0.004</td>
</tr>
<tr>
<td></td>
<td>618</td>
<td>13 Oct 2008</td>
<td>0.0430</td>
<td>0.0378</td>
<td>3.6778</td>
<td>0.008</td>
</tr>
<tr>
<td></td>
<td>1013</td>
<td>10 May 2010</td>
<td>0.0231</td>
<td>0.0202</td>
<td>3.4770</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>602</td>
<td>19 Sep 2008</td>
<td>0.0321</td>
<td>0.0273</td>
<td>2.8848</td>
<td>0.018</td>
</tr>
<tr>
<td></td>
<td>1334</td>
<td>9 Aug 2011</td>
<td>0.0198</td>
<td>0.0169</td>
<td>2.9354</td>
<td>0.018</td>
</tr>
</tbody>
</table>

Table 3.22: Results ES backtesting Model Family 2 $p = 95\%$

From Table 3.22 the same conclusion can be drawn as for Table 3.21 that the normal distribution give rise to observations that are almost 5 SD apart from ES which does not yield a satisfactory result. On the other hand, the Student’s t distribution with $\nu = 7$ and $\nu = 15$ both yield rejection of the null hypothesis for 6.1% of the observations for which VaR exceedances occur. Since this is approximately in line with the chosen significance level of the ES backtest, these particular distributions are the most satisfying from an ES backtesting perspective.
From Table 3.23 it can be concluded that at most 3 observations yield rejection of the null hypothesis, representing 2.7% of the total number of observations for which VaR exceedances occur. This is below the accepted frequency of rejections being 5% and happens for the normal distribution. This is, again, due to the GPD extrapolation of the tails resulting in few rejections of the null hypothesis, possibly indicating too conservative risk models, where the normal distribution is the less conservative one.

Summing up, the following models yield the most satisfactory per model family for the ES backtests, see Table 3.24:

<table>
<thead>
<tr>
<th>Most satisfactory ES models per model family</th>
<th>p = 99%</th>
<th>p = 97.5%</th>
<th>p = 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model Family 1</td>
<td>Students t, ν=7</td>
<td>Students t, ν=15</td>
<td>Students t, ν=20</td>
</tr>
<tr>
<td>Model Family 2</td>
<td>Students t, ν=7</td>
<td>Students t, ν=7</td>
<td>Students t, ν=7</td>
</tr>
<tr>
<td>Model Family 3</td>
<td>Students t, ν=15</td>
<td>Normal</td>
<td>Students t, ν=15</td>
</tr>
</tbody>
</table>

Table 3.24: The most satisfactory models per model family for different levels of p

From Table 3.24 together with previous reasoning on the results for the

117
ES backtesting, it can be concluded that some of the models that passed through the VaR backtesting, see Table 3.14, will not be satisfactory from an ES backtesting point of view and vice versa. In general, the models that were satisfactory from a VaR backtesting perspective seem to be too conservative from an ES backtesting point of view. In particular, it is of interest to analyze which model is the most suitable to implement for ES with confidence level $p = 97.5\%$, as this has been proposed by the FRTB, see Section 1.3. From Table 3.24 together with previous comparisons between model families it can be concluded that from an ES backtesting perspective model family 2 with Student’s t distribution with $\nu = 7$ yields the most satisfying results.

It is worth remembering that this particular ES backtest method is performed for every single observed VaR exceedance while other ES backtests mentioned in Section 2.7.2 consider the entire sample period of such observations. In the empirical illustration in [28], the ES backtest proposed by McNeil and Frey in 2000 is implemented and yields less extreme p-values than the ones obtained with the ES backtesting method considering every individual observation. This can be due to the fact that most of the exceedances may be small and hide just a few large exceedances in test approaches using the full sample. Therefore, one can argue that the method proposed by Righi and Ceretta could identify unsatisfactory models that have few exceedance outliers better than previously proposed ES backtests. This is due to the very different nature of the test, considering individual observations rather than the entire sample period.

Now, recall the proposition in the FRTB concerning the change of confidence level $p$ from 99% to 97.5% when implementing ES as a risk measure. In particular, it proposes to use ES with $p = 97.5\%$ together with backtests on VaR with $p = 97.5\%$ and $p = 99\%$. According to the previously mentioned results and analysis of the VaR backtesting, the best VaR model for high quantiles is to implement model family 3, a normal copula with GARCH filtration and a hybrid GPD-empirical distribution. It satisfies the requirements of having approximately the expected number of VaR exceedances as well as VaR exceedances being uncorrelated. However, the ES backtests for $p = 97.5\%$ yield model family 2 with univariate Student’s t distribution with $\nu = 7$, an approach with GARCH filtration, to be the most appropriate model to implement, as model family 3 yields possibly too conservative results. Therefore, one can conclude that a good VaR model does not necessarily imply a good ES model, as it seems to yield too conservative from an ES backtesting perspective.
Chapter 4

Summary and Conclusions

In accordance with the paper contribution of this thesis outlined in the beginning of this report, it can be concluded that various market risk measure models have been compared under both VaR and ES backtesting. In particular, standard VaR backtests as well as the ES backtest proposed by Righi and Ceretta have been implemented and analyzed.

Concerning the backtesting of VaR, different risk models are preferable to use depending on the confidence level $p$. For a high quantile, such as $p = 99\%$, the most suitable choice seems to be a normal copula with GARCH filtration and a hybrid GPD-empirical distribution applied (model family 3 - normal), as it shows satisfactory p-values in the VaR backtests as well as having the most correct number of exceedances (as close to 1\% as possible). For a lower quantile, such as $p = 95\%$, the simple model family 1 with no GARCH filtration and normal distribution seems to be the most suitable choice due to approximately correct number of exceedances (as close to 5\% as possible) as well as satisfactory p-values for the VaR backtests. Even though the lack of GARCH effects is present for this model, as exceedances are clustered to some extent, the relatively low quantile indicates that we are not far out in the tail and so the normal distribution is a good approximation of the distribution.

From an ES backtesting perspective, model family 2 with univariate Student’s t distribution with $\nu = 7$ and GARCH filtration yields the most satisfying results overall. This is because the number of rejections of the null hypothesis of losses being significantly different from the ones expected is in line with what one can expect when setting the significance level of the backtest to 5\%. Model family 3, for almost all confidence levels $p$, seems to be too conservative as rejections of the null hypothesis happen much less than 5\% of the time. This is probably due to the extrapolation of the tails when applying GPD, resulting in relatively high ES estimates and thus relatively
low ES exceedances which yield losses being significantly closer to the ones expected, yet too close for the actual significance level of the ES backtest. Hence, one can conclude that the model that yields satisfactory from a VaR backtesting perspective, having approximately the expected number of VaR exceedances and where these take place independently of each other, does not seem to yield a good ES model. Overall, in the light of the confidence levels proposed in the FRTB, models that are satisfactory from a VaR backtesting perspective turn out to be probably too conservative from an ES backtesting perspective. Recalling that an accurate VaR model is necessary for obtaining a good ES model, therefore imposes potential issues when implementing the FRTB. As a result, a trade-off between a sufficiently good VaR model and a sufficiently good ES model will be needed when implementing the FRTB. Risk management divisions at banks, that are to compute ES for $p = 97.5\%$ according to the FRTB, are therefore recommended to compromise whether they want to implement an as good VaR model as possible, accepting the fact that ES estimates might be too conservative, or if they prefer to use a less accurate VaR model and hence resulting in more unstable ES estimates which is potentially a riskier scenario. In either way, they are recommended to implement GARCH filtering of the log returns, as this has shown to be successful for both VaR and ES modeling. Implementation of a risk measure model built on GARCH filtering, a normal copula and a hybrid GPD-empirical distribution would potentially be a good choice if the bank can accept a good VaR model with possibly a more conservative ES profile, and, implementation of a risk measure model built on GARCH filtering and the univariate Student’s $t$ distribution with $\nu \approx 7$ would potentially be a good choice if the bank is concerned about conservative ES estimations that can result from GPD tail extrapolation.

In order for SAS Institute to investigate the FRTB even more, there are some suggestions on further research that could be explored in order to obtain a more accurate picture of the regulation framework. A list of possible improvements is presented here:

1. Perform a more elaborated GARCH filtration, letting $p$ and $q$ take other values than 1

2. Perform a MLE of the parameters $\nu$ for the Student’s $t$ distribution in model family 1 and 2 instead of using the fixed values $\nu = 3, 7, 15, 20$. For model family 3, a more suitable choice would be to implement a grouped $t$ copula which is a generalization of the Student’s $t$ copula. That allows various $\nu$ depending on the marginal data
3. Use more powerful hardware in order to be able to run the Monte Carlo simulations for larger samples used within model family 3 and the ES backtests

4. Investigate in detail other ES backtesting frameworks than the one proposed by Righi and Ceretta

5. Add derivatives and other nonlinear instruments to the reference portfolio to find out if this has an impact on the results

6. Perform analysis on a reference portfolio consisting of assets that fits the varying liquidity horizon categories in an even better way. The Basel Committee recognizes that five buckets is relatively broad even though it is nonetheless an improvement to the current regime, which assumes all assets in a portfolio are equally liquid

7. Perform the entire analysis with varying liquidity horizon applied according to what is proposed in the FRTB. This was initially meant to be included within this thesis, however, it turned out to be out of scope and therefore SAS Institute is highly recommended to perform this analysis
Appendix

Appendix 1

Figure 4.1: Sample ACF of the log returns
Appendix 2

Figure 4.2: Sample ACF of the GARCH filtered residuals
Appendix 3

Visualizing the empirical CDFs for the GARCH filtered residuals together with estimated GPD tails for each asset in the reference portfolio yields the following, see Figure 4.3.

Figure 4.3: Empirical CDFs for the GARCH filtered residuals together with estimated GPD tails for each asset in the reference portfolio
Appendix 4

The exceedances for all the different model families with $p = 99\%$ are visualized in Figure 4.3. The first row of graphs represents model family 1, the second row represents model family 2 and the third row represents model family 3. The first column represents the normal case and the rest of the columns represents the Student t case with $\nu=3, 7, 15$ and 20.

Figure 4.4: VaR exceedances for all model families, $p = 99\%$
Appendix 5

The exceedances for all the different model families with $p = 95\%$ are visualized in Figure 4.5.

Figure 4.5: VaR exceedances for all model families, $p = 95\%$
Appendix 6

Kendall’s tau for the log returns of all assets in the reference portfolio:
the reference portfolio: Spearman’s rho for the log returns and the normal copula for all assets in

\[
\begin{pmatrix}
1.000 & 0.152 & 0.203 & 0.132 & 0.172 & 0.192 & 0.151 & 0.138 & 0.071 & 0.080 & 0.032 & 0.143 & -0.071 & 0.124 \\
0.152 & 1.000 & 0.288 & 0.249 & 0.164 & 0.228 & 0.281 & 0.241 & 0.111 & -0.125 & -0.135 & 0.167 & -0.160 & 0.190 \\
0.203 & 0.288 & 1.000 & 0.243 & 0.197 & 0.288 & 0.302 & 0.236 & 0.097 & -0.149 & -0.139 & 0.140 & -0.182 & 0.140 \\
0.132 & 0.249 & 0.243 & 1.000 & 0.170 & 0.211 & 0.256 & 0.212 & 0.095 & -0.111 & -0.103 & 0.128 & -0.155 & 0.135 \\
0.172 & 0.164 & 0.197 & 0.170 & 1.000 & 0.187 & 0.160 & 0.146 & 0.063 & -0.036 & -0.053 & 0.107 & -0.075 & 0.076 \\
0.192 & 0.228 & 0.288 & 0.211 & 0.187 & 1.000 & 0.267 & 0.245 & 0.098 & -0.120 & -0.099 & 0.135 & -0.156 & 0.150 \\
0.151 & 0.281 & 0.302 & 0.256 & 0.160 & 0.267 & 1.000 & 0.333 & 0.121 & -0.178 & -0.138 & 0.131 & -0.183 & 0.154 \\
0.138 & 0.241 & 0.246 & 0.212 & 0.146 & 0.245 & 0.333 & 1.000 & 0.112 & -0.184 & -0.137 & 0.126 & -0.175 & 0.151 \\
0.071 & 0.111 & 0.097 & 0.095 & 0.098 & 0.121 & 0.112 & 1.000 & -0.042 & -0.049 & 0.042 & -0.080 & 0.060 \\
0.080 & -0.125 & -0.149 & -0.111 & -0.036 & -0.120 & -0.178 & -0.184 & -0.042 & 1.000 & 0.432 & 0.076 & 0.176 & -0.125 \\
0.032 & -0.135 & -0.139 & -0.103 & -0.053 & -0.099 & -0.138 & -0.137 & -0.049 & 0.432 & 1.000 & -0.075 & 0.159 & -0.131 \\
0.143 & 0.167 & 0.140 & 0.128 & 0.107 & 0.135 & 0.131 & 0.126 & 0.042 & 0.076 & -0.075 & 1.000 & -0.240 & 0.093 \\
-0.071 & -0.160 & -0.182 & -0.155 & -0.075 & -0.156 & -0.183 & -0.175 & -0.080 & 0.176 & 0.159 & -0.240 & 1.000 & -0.067 \\
0.124 & 0.190 & 0.140 & 0.135 & 0.076 & 0.150 & 0.154 & 0.151 & 0.060 & -0.125 & -0.131 & 0.093 & -0.067 & 1.000 \\
\end{pmatrix}
\]
Tail dependence coefficients for the log returns and the Student’s t copula

\[
\hat{\rho}_s = \begin{pmatrix}
1.000 & 0.227 & 0.301 & 0.197 & 0.256 & 0.285 & 0.225 & 0.205 & 0.106 & 0.120 & 0.048 & 0.214 & -0.106 & 0.185 \\
0.227 & 1.000 & 0.420 & 0.366 & 0.244 & 0.337 & 0.410 & 0.355 & 0.166 & -0.187 & -0.201 & 0.248 & -0.238 & 0.282 \\
0.301 & 0.420 & 1.000 & 0.358 & 0.292 & 0.420 & 0.440 & 0.362 & 0.145 & -0.222 & -0.207 & 0.209 & -0.270 & 0.299 \\
0.197 & 0.366 & 0.358 & 1.000 & 0.253 & 0.312 & 0.376 & 0.314 & 0.142 & -0.166 & -0.154 & 0.192 & -0.230 & 0.201 \\
0.256 & 0.244 & 0.292 & 0.253 & 1.000 & 0.277 & 0.239 & 0.218 & 0.094 & -0.053 & -0.080 & 0.160 & -0.112 & 0.113 \\
0.285 & 0.337 & 0.420 & 0.312 & 0.277 & 1.000 & 0.392 & 0.361 & 0.147 & -0.180 & -0.148 & 0.201 & -0.232 & 0.223 \\
0.225 & 0.410 & 0.440 & 0.376 & 0.239 & 0.392 & 1.000 & 0.482 & 0.181 & -0.265 & -0.206 & 0.196 & -0.272 & 0.230 \\
0.205 & 0.355 & 0.362 & 0.314 & 0.218 & 0.361 & 0.482 & 1.000 & 0.168 & -0.274 & -0.205 & 0.188 & -0.261 & 0.225 \\
0.106 & 0.166 & 0.145 & 0.142 & 0.094 & 0.147 & 0.181 & 0.168 & 1.000 & -0.064 & -0.073 & 0.063 & -0.120 & 0.089 \\
0.120 & -0.187 & -0.222 & -0.166 & -0.053 & -0.180 & -0.265 & -0.274 & -0.064 & 1.000 & 0.609 & 0.114 & 0.262 & -0.187 \\
0.048 & -0.201 & -0.207 & -0.154 & -0.080 & -0.148 & -0.206 & -0.205 & -0.073 & 0.609 & 1.000 & -0.112 & 0.237 & -0.196 \\
0.214 & 0.248 & 0.209 & 0.192 & 0.160 & 0.201 & 0.196 & 0.188 & 0.063 & 0.114 & -0.112 & 1.000 & -0.353 & 0.139 \\
-0.106 & -0.238 & -0.270 & -0.230 & -0.112 & -0.232 & -0.272 & -0.261 & -0.120 & 0.262 & 0.237 & -0.353 & 1.000 & -0.100 \\
0.185 & 0.282 & 0.209 & 0.201 & 0.113 & 0.223 & 0.230 & 0.225 & 0.089 & -0.187 & -0.196 & 0.139 & -0.100 & 1.000
\end{pmatrix}
\]

for all assets in the reference portfolio with \( p = 0.99 \) and \( \nu = 3 \).
The corresponding result for tail dependence for $p = 0.99$ and $\nu = 7$ is the following:

$$
\hat{\lambda}_{u,\nu=3} = \hat{\lambda}_{l,\nu=3} = \\
\begin{bmatrix}
0.896 & 0.869 & 0.873 & 0.867 & 0.870 & 0.872 & 0.869 & 0.868 & 0.862 & 0.862 & 0.858 & 0.868 & 0.846 & 0.866 \\
0.869 & 0.896 & 0.878 & 0.876 & 0.870 & 0.874 & 0.878 & 0.875 & 0.865 & 0.838 & 0.836 & 0.870 & 0.832 & 0.872 \\
0.873 & 0.878 & 0.896 & 0.875 & 0.872 & 0.878 & 0.879 & 0.876 & 0.864 & 0.834 & 0.836 & 0.868 & 0.829 & 0.868 \\
0.867 & 0.876 & 0.875 & 0.896 & 0.870 & 0.873 & 0.876 & 0.873 & 0.864 & 0.840 & 0.841 & 0.867 & 0.833 & 0.867 \\
0.870 & 0.870 & 0.872 & 0.870 & 0.896 & 0.871 & 0.869 & 0.868 & 0.861 & 0.850 & 0.848 & 0.865 & 0.845 & 0.862 \\
0.872 & 0.874 & 0.878 & 0.873 & 0.871 & 0.896 & 0.877 & 0.875 & 0.864 & 0.839 & 0.842 & 0.867 & 0.833 & 0.869 \\
0.869 & 0.878 & 0.879 & 0.876 & 0.869 & 0.877 & 0.896 & 0.881 & 0.866 & 0.829 & 0.836 & 0.867 & 0.828 & 0.869 \\
0.868 & 0.875 & 0.876 & 0.873 & 0.868 & 0.875 & 0.881 & 0.896 & 0.865 & 0.828 & 0.836 & 0.867 & 0.830 & 0.869 \\
0.862 & 0.865 & 0.864 & 0.864 & 0.861 & 0.864 & 0.866 & 0.865 & 0.896 & 0.849 & 0.848 & 0.859 & 0.844 & 0.861 \\
0.862 & 0.838 & 0.834 & 0.840 & 0.850 & 0.839 & 0.829 & 0.828 & 0.849 & 0.896 & 0.885 & 0.862 & 0.871 & 0.838 \\
0.858 & 0.836 & 0.836 & 0.841 & 0.842 & 0.836 & 0.836 & 0.848 & 0.885 & 0.896 & 0.845 & 0.869 & 0.837 \\
0.868 & 0.870 & 0.868 & 0.867 & 0.865 & 0.867 & 0.867 & 0.859 & 0.862 & 0.845 & 0.896 & 0.818 & 0.864 \\
0.846 & 0.832 & 0.829 & 0.833 & 0.845 & 0.833 & 0.828 & 0.830 & 0.844 & 0.871 & 0.869 & 0.818 & 0.896 & 0.846 \\
0.866 & 0.872 & 0.868 & 0.867 & 0.862 & 0.869 & 0.869 & 0.869 & 0.861 & 0.838 & 0.837 & 0.864 & 0.846 & 0.896
\end{bmatrix}
$$
\[
\hat{\lambda}_{\nu = 7} = \hat{\lambda}_{\nu = 7} = \\
\begin{pmatrix}
0.847 & 0.807 & 0.812 & 0.804 & 0.809 & 0.811 & 0.806 & 0.796 & 0.797 & 0.791 & 0.806 & 0.773 & 0.803 \\
0.807 & 0.847 & 0.820 & 0.817 & 0.808 & 0.815 & 0.820 & 0.816 & 0.802 & 0.762 & 0.760 & 0.808 & 0.754 & 0.811 \\
0.812 & 0.820 & 0.847 & 0.816 & 0.812 & 0.820 & 0.821 & 0.816 & 0.800 & 0.756 & 0.759 & 0.805 & 0.748 & 0.805 \\
0.804 & 0.817 & 0.816 & 0.847 & 0.809 & 0.813 & 0.817 & 0.813 & 0.799 & 0.765 & 0.766 & 0.804 & 0.755 & 0.804 \\
0.809 & 0.808 & 0.812 & 0.809 & 0.847 & 0.810 & 0.807 & 0.806 & 0.795 & 0.779 & 0.776 & 0.801 & 0.772 & 0.797 \\
0.811 & 0.815 & 0.820 & 0.813 & 0.810 & 0.847 & 0.818 & 0.816 & 0.800 & 0.763 & 0.767 & 0.804 & 0.755 & 0.806 \\
0.806 & 0.820 & 0.821 & 0.817 & 0.807 & 0.818 & 0.847 & 0.824 & 0.803 & 0.749 & 0.759 & 0.804 & 0.748 & 0.807 \\
0.805 & 0.816 & 0.816 & 0.813 & 0.806 & 0.816 & 0.824 & 0.847 & 0.802 & 0.748 & 0.759 & 0.803 & 0.750 & 0.806 \\
0.796 & 0.802 & 0.800 & 0.799 & 0.795 & 0.800 & 0.803 & 0.802 & 0.847 & 0.778 & 0.777 & 0.792 & 0.771 & 0.795 \\
0.797 & 0.762 & 0.756 & 0.765 & 0.779 & 0.763 & 0.749 & 0.748 & 0.778 & 0.847 & 0.831 & 0.797 & 0.800 & 0.762 \\
0.791 & 0.760 & 0.759 & 0.766 & 0.776 & 0.767 & 0.759 & 0.759 & 0.777 & 0.831 & 0.847 & 0.772 & 0.807 & 0.760 \\
0.806 & 0.808 & 0.805 & 0.804 & 0.801 & 0.804 & 0.804 & 0.803 & 0.792 & 0.797 & 0.772 & 0.847 & 0.733 & 0.799 \\
0.773 & 0.754 & 0.748 & 0.755 & 0.772 & 0.755 & 0.748 & 0.750 & 0.771 & 0.809 & 0.807 & 0.733 & 0.847 & 0.774 \\
0.803 & 0.811 & 0.805 & 0.804 & 0.797 & 0.806 & 0.807 & 0.806 & 0.795 & 0.762 & 0.760 & 0.799 & 0.774 & 0.847 
\end{pmatrix}
\]
Appendix 7

Bivariate scatter plots together with regression lines of the GARCH filtered residuals for all 14 assets in the reference portfolio:
Figure 4.6: Bivariate scatter plots of the assets in the reference portfolio after GARCH filtration is performed
Correlation estimations of the GARCH filtered residuals for all assets in the reference portfolio:

\[ R_{garch} = \begin{pmatrix}
1.000 & 0.228 & 0.306 & 0.196 & 0.246 & 0.279 & 0.237 & 0.176 & 0.101 & 0.135 & 0.056 & 0.203 & -0.103 & 0.144 \\
0.228 & 1.000 & 0.382 & 0.330 & 0.241 & 0.336 & 0.383 & 0.305 & 0.132 & -0.129 & -0.160 & 0.220 & -0.212 & 0.263 \\
0.306 & 0.382 & 1.000 & 0.318 & 0.277 & 0.391 & 0.410 & 0.309 & 0.133 & -0.159 & -0.173 & 0.190 & -0.245 & 0.213 \\
0.196 & 0.330 & 0.318 & 1.000 & 0.234 & 0.288 & 0.336 & 0.250 & 0.137 & -0.091 & -0.094 & 0.167 & -0.184 & 0.200 \\
0.246 & 0.241 & 0.277 & 0.234 & 1.000 & 0.264 & 0.237 & 0.189 & 0.095 & -0.027 & -0.073 & 0.148 & -0.113 & 0.122 \\
0.279 & 0.336 & 0.391 & 0.288 & 0.264 & 1.000 & 0.372 & 0.314 & 0.134 & -0.136 & -0.126 & 0.175 & -0.220 & 0.218 \\
0.237 & 0.383 & 0.410 & 0.336 & 0.237 & 0.372 & 1.000 & 0.426 & 0.163 & -0.213 & -0.187 & 0.175 & -0.255 & 0.230 \\
0.176 & 0.305 & 0.309 & 0.250 & 0.189 & 0.314 & 0.426 & 1.000 & 0.142 & -0.229 & -0.179 & 0.138 & -0.214 & 0.199 \\
0.101 & 0.132 & 0.133 & 0.137 & 0.095 & 0.135 & 0.163 & 0.142 & 1.000 & -0.056 & -0.083 & 0.055 & -0.103 & 0.097 \\
0.135 & -0.129 & -0.159 & -0.091 & -0.027 & -0.136 & -0.213 & -0.229 & -0.056 & 1.000 & 0.583 & 0.187 & 0.224 & -0.166 \\
0.056 & -0.160 & -0.173 & -0.094 & -0.073 & -0.126 & -0.187 & -0.179 & -0.083 & 0.583 & 1.000 & -0.096 & 0.237 & -0.179 \\
0.203 & 0.220 & 0.190 & 0.167 & 0.148 & 0.175 & 0.175 & 0.138 & 0.055 & 0.187 & -0.096 & 1.000 & -0.369 & 0.102 \\
-0.103 & -0.212 & -0.245 & -0.184 & -0.113 & -0.220 & -0.255 & -0.214 & -0.103 & 0.224 & 0.237 & -0.369 & 1.000 & -0.086 \\
0.144 & 0.263 & 0.213 & 0.200 & 0.122 & 0.218 & 0.230 & 0.199 & 0.097 & -0.166 & -0.179 & 0.102 & -0.086 & 1.000
\end{pmatrix} \]
Kendall’s tau for the GARCH filtered residuals and the normal and Student’s t copula for all assets in the reference portfolio:

\[
\hat{\tau}_{\text{garch}} = \begin{pmatrix}
1.000 & 0.147 & 0.198 & 0.125 & 0.158 & 0.180 & 0.152 & 0.113 & 0.065 & 0.086 & 0.036 & 0.130 & -0.066 & 0.092 \\
0.147 & 1.000 & 0.250 & 0.214 & 0.155 & 0.218 & 0.250 & 0.197 & 0.084 & -0.082 & -0.102 & 0.141 & -0.136 & 0.169 \\
0.198 & 0.250 & 1.000 & 0.206 & 0.179 & 0.256 & 0.269 & 0.200 & 0.085 & -0.102 & -0.111 & 0.122 & -0.158 & 0.136 \\
0.125 & 0.214 & 0.206 & 1.000 & 0.150 & 0.186 & 0.218 & 0.161 & 0.087 & -0.058 & -0.060 & 0.107 & -0.118 & 0.128 \\
0.158 & 0.155 & 0.179 & 0.150 & 1.000 & 0.170 & 0.152 & 0.121 & 0.061 & -0.017 & -0.047 & 0.095 & -0.072 & 0.078 \\
0.180 & 0.218 & 0.256 & 0.186 & 0.170 & 1.000 & 0.242 & 0.203 & 0.086 & -0.087 & -0.081 & 0.112 & -0.141 & 0.140 \\
0.152 & 0.250 & 0.269 & 0.218 & 0.152 & 0.242 & 1.000 & 0.280 & 0.104 & -0.137 & -0.120 & 0.112 & -0.164 & 0.148 \\
0.113 & 0.197 & 0.200 & 0.161 & 0.121 & 0.203 & 0.280 & 1.000 & 0.091 & -0.147 & -0.115 & 0.088 & -0.138 & 0.128 \\
0.065 & 0.084 & 0.085 & 0.087 & 0.061 & 0.086 & 0.104 & 0.091 & 1.000 & -0.036 & -0.053 & 0.035 & -0.066 & 0.062 \\
0.086 & -0.082 & -0.102 & -0.058 & -0.017 & -0.087 & -0.137 & -0.147 & -0.036 & 1.000 & 0.396 & 0.120 & 0.144 & -0.106 \\
0.036 & -0.102 & -0.111 & -0.060 & -0.047 & -0.081 & -0.129 & -0.115 & -0.053 & 0.396 & 1.000 & -0.061 & 0.152 & -0.115 \\
0.130 & 0.141 & 0.122 & 0.107 & 0.095 & 0.112 & 0.112 & 0.088 & 0.035 & 0.120 & -0.061 & 1.000 & -0.241 & 0.065 \\
-0.066 & -0.136 & -0.158 & -0.118 & -0.072 & -0.141 & -0.164 & -0.138 & -0.066 & 0.144 & 0.152 & -0.241 & 1.000 & -0.055 \\
0.092 & 0.169 & 0.136 & 0.128 & 0.078 & 0.140 & 0.148 & 0.128 & 0.062 & -0.106 & -0.115 & 0.065 & -0.055 & 1.000
\end{pmatrix}
\]
Spearman's rho for the GARCH filtered residuals and the normal copula for all assets in the reference portfolio:

\[
\begin{bmatrix}
1.000 & 0.219 & 0.293 & 0.187 & 0.236 & 0.268 & 0.227 & 0.169 & 0.097 & 0.129 & 0.053 & 0.194 & -0.099 & 0.138 \\
0.219 & 1.000 & 0.367 & 0.316 & 0.231 & 0.322 & 0.368 & 0.293 & 0.126 & -0.123 & -0.153 & 0.210 & -0.203 & 0.252 \\
0.293 & 0.367 & 1.000 & 0.305 & 0.266 & 0.376 & 0.394 & 0.296 & 0.127 & -0.152 & -0.165 & 0.182 & -0.235 & 0.203 \\
0.187 & 0.316 & 0.305 & 1.000 & 0.224 & 0.276 & 0.323 & 0.239 & 0.131 & -0.086 & -0.090 & 0.159 & -0.176 & 0.192 \\
0.236 & 0.231 & 0.266 & 0.224 & 1.000 & 0.253 & 0.227 & 0.181 & 0.091 & -0.026 & -0.070 & 0.141 & -0.108 & 0.116 \\
0.268 & 0.322 & 0.376 & 0.276 & 0.253 & 1.000 & 0.357 & 0.301 & 0.129 & -0.130 & -0.121 & 0.167 & -0.210 & 0.208 \\
0.227 & 0.368 & 0.394 & 0.323 & 0.227 & 0.357 & 1.000 & 0.410 & 0.156 & -0.204 & -0.179 & 0.167 & -0.244 & 0.220 \\
0.169 & 0.293 & 0.296 & 0.239 & 0.181 & 0.301 & 0.410 & 1.000 & 0.136 & -0.219 & -0.172 & 0.132 & -0.205 & 0.190 \\
0.097 & 0.126 & 0.127 & 0.131 & 0.091 & 0.129 & 0.156 & 0.136 & 1.000 & -0.054 & -0.079 & 0.053 & -0.098 & 0.093 \\
0.129 & -0.123 & -0.152 & -0.086 & -0.026 & -0.130 & -0.204 & -0.219 & -0.054 & 1.000 & 0.565 & 0.179 & 0.215 & -0.159 \\
0.053 & -0.153 & -0.165 & -0.090 & -0.070 & -0.121 & -0.179 & -0.172 & -0.079 & 0.565 & 1.000 & -0.092 & 0.227 & -0.172 \\
0.194 & 0.210 & 0.182 & 0.159 & 0.141 & 0.167 & 0.167 & 0.132 & 0.053 & 0.179 & -0.092 & 1.000 & -0.354 & 0.098 \\
-0.099 & -0.203 & -0.235 & -0.176 & -0.108 & -0.210 & -0.244 & -0.205 & -0.098 & 0.215 & 0.227 & -0.354 & 1.000 & -0.082 \\
0.138 & 0.252 & 0.203 & 0.192 & 0.116 & 0.208 & 0.220 & 0.190 & 0.093 & -0.159 & -0.172 & 0.098 & -0.082 & 1.000
\end{bmatrix}
\]
Tail dependence coefficients for the GARCH filtered residuals and the Student’s t copula for all assets in the reference portfolio with $p = 0.99$ and $\nu = 3$: 
The corresponding result for tail dependence for $p = 0.99$ and $\nu = 7$ is the following:

$$
\hat{\lambda}_{u,\nu=3,\text{garch}} = \hat{\lambda}_{p=3,\text{garch}} = \\
\begin{pmatrix}
0.896 & 0.868 & 0.872 & 0.867 & 0.869 & 0.871 & 0.869 & 0.865 & 0.861 & 0.863 & 0.858 & 0.867 & 0.846 & 0.864 \\
0.868 & 0.896 & 0.876 & 0.873 & 0.869 & 0.874 & 0.876 & 0.872 & 0.863 & 0.844 & 0.841 & 0.868 & 0.836 & 0.870 \\
0.872 & 0.876 & 0.896 & 0.873 & 0.871 & 0.876 & 0.877 & 0.872 & 0.863 & 0.841 & 0.840 & 0.866 & 0.833 & 0.867 \\
0.867 & 0.873 & 0.873 & 0.896 & 0.869 & 0.871 & 0.874 & 0.869 & 0.863 & 0.847 & 0.847 & 0.865 & 0.839 & 0.867 \\
0.869 & 0.869 & 0.871 & 0.869 & 0.896 & 0.870 & 0.869 & 0.866 & 0.861 & 0.852 & 0.849 & 0.864 & 0.845 & 0.862 \\
0.871 & 0.874 & 0.876 & 0.871 & 0.870 & 0.896 & 0.875 & 0.873 & 0.863 & 0.843 & 0.844 & 0.865 & 0.835 & 0.868 \\
0.869 & 0.876 & 0.877 & 0.874 & 0.869 & 0.875 & 0.896 & 0.878 & 0.865 & 0.836 & 0.839 & 0.865 & 0.832 & 0.868 \\
0.865 & 0.872 & 0.872 & 0.869 & 0.866 & 0.873 & 0.878 & 0.896 & 0.863 & 0.834 & 0.839 & 0.863 & 0.836 & 0.867 \\
0.861 & 0.863 & 0.863 & 0.863 & 0.861 & 0.863 & 0.865 & 0.863 & 0.896 & 0.850 & 0.848 & 0.858 & 0.846 & 0.861 \\
0.863 & 0.844 & 0.841 & 0.847 & 0.852 & 0.843 & 0.836 & 0.834 & 0.850 & 0.896 & 0.884 & 0.866 & 0.868 & 0.841 \\
0.858 & 0.841 & 0.840 & 0.847 & 0.849 & 0.844 & 0.839 & 0.830 & 0.848 & 0.884 & 0.896 & 0.847 & 0.869 & 0.839 \\
0.867 & 0.868 & 0.866 & 0.865 & 0.864 & 0.865 & 0.865 & 0.863 & 0.858 & 0.866 & 0.847 & 0.896 & 0.817 & 0.861 \\
0.846 & 0.836 & 0.833 & 0.839 & 0.845 & 0.835 & 0.832 & 0.836 & 0.846 & 0.868 & 0.869 & 0.817 & 0.896 & 0.848 \\
0.864 & 0.870 & 0.867 & 0.867 & 0.862 & 0.868 & 0.868 & 0.867 & 0.861 & 0.841 & 0.839 & 0.861 & 0.848 & 0.896
\end{pmatrix}
\[
\begin{pmatrix}
0.847 & 0.806 & 0.812 & 0.803 & 0.807 & 0.810 & 0.807 & 0.802 & 0.795 & 0.798 & 0.791 & 0.804 & 0.774 & 0.799 \\
0.806 & 0.847 & 0.817 & 0.813 & 0.807 & 0.814 & 0.817 & 0.812 & 0.798 & 0.771 & 0.767 & 0.805 & 0.759 & 0.808 \\
0.812 & 0.817 & 0.847 & 0.812 & 0.810 & 0.817 & 0.819 & 0.812 & 0.798 & 0.767 & 0.765 & 0.803 & 0.754 & 0.805 \\
0.803 & 0.813 & 0.812 & 0.847 & 0.806 & 0.810 & 0.814 & 0.808 & 0.798 & 0.775 & 0.775 & 0.801 & 0.763 & 0.804 \\
0.807 & 0.807 & 0.810 & 0.806 & 0.847 & 0.809 & 0.807 & 0.803 & 0.795 & 0.783 & 0.777 & 0.799 & 0.773 & 0.797 \\
0.810 & 0.814 & 0.817 & 0.810 & 0.809 & 0.847 & 0.816 & 0.812 & 0.798 & 0.770 & 0.771 & 0.802 & 0.758 & 0.805 \\
0.807 & 0.817 & 0.819 & 0.814 & 0.807 & 0.816 & 0.847 & 0.820 & 0.801 & 0.759 & 0.763 & 0.802 & 0.753 & 0.806 \\
0.802 & 0.812 & 0.812 & 0.808 & 0.803 & 0.812 & 0.820 & 0.847 & 0.799 & 0.757 & 0.764 & 0.799 & 0.759 & 0.804 \\
0.795 & 0.798 & 0.798 & 0.795 & 0.795 & 0.798 & 0.801 & 0.799 & 0.847 & 0.779 & 0.776 & 0.791 & 0.747 & 0.795 \\
0.798 & 0.771 & 0.767 & 0.775 & 0.783 & 0.770 & 0.759 & 0.757 & 0.779 & 0.847 & 0.829 & 0.803 & 0.806 & 0.766 \\
0.791 & 0.767 & 0.765 & 0.775 & 0.777 & 0.771 & 0.763 & 0.764 & 0.776 & 0.829 & 0.847 & 0.775 & 0.807 & 0.764 \\
0.804 & 0.805 & 0.803 & 0.801 & 0.799 & 0.802 & 0.802 & 0.799 & 0.791 & 0.803 & 0.775 & 0.847 & 0.732 & 0.795 \\
0.774 & 0.759 & 0.754 & 0.763 & 0.773 & 0.758 & 0.753 & 0.759 & 0.774 & 0.806 & 0.807 & 0.732 & 0.847 & 0.776 \\
0.799 & 0.808 & 0.805 & 0.804 & 0.797 & 0.805 & 0.806 & 0.804 & 0.795 & 0.766 & 0.764 & 0.795 & 0.776 & 0.847
\end{pmatrix}
\]
### Appendix 8

<table>
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<tr>
<th>p = 99%</th>
<th># of VaR exceedances</th>
<th># of ES exceedances</th>
<th>max ES exceedance</th>
<th>Smallest p-value</th>
<th>Median p-value</th>
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<tr>
<td>Normal</td>
<td>St. t, ν = 3</td>
<td>58</td>
<td>30</td>
<td>0.0422</td>
<td>0.03</td>
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<td>18</td>
<td>8</td>
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<td>St. t, ν = 15</td>
<td>35</td>
<td>17</td>
<td>0.0276</td>
<td>0.01</td>
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<tr>
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<td>St. t, ν = 20</td>
<td>38</td>
<td>21</td>
<td>0.0289</td>
<td>0.01</td>
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<tr>
<td>Model family 1</td>
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<td>12</td>
<td>0.0286</td>
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<td>10</td>
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<td>0.0334</td>
<td>0.0040</td>
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<td>67</td>
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### Table 4.1: Summarized results of the ES backtesting with all observations included
Bibliography


