Fixed Income Modeling

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Abstract

Besides financial analysis, quantitative tools play a major role in asset management. By managing the aggregation of large amount of historical and prospective data on different asset classes, it can give portfolio allocation solution with respect to risk and regulatory constraints.

Asset class modeling requires three main steps, the first one is to assess the product features (risk premium and risks) by considering historical and prospective data, which in the case of fixed income depends on spread and default levels. The second is choosing the quantitative model, in this study we introduce a new credit model, which unlike equity like models, model default as a main feature of fixed income performance. The final step consists on calibrating the model.

We start in this study with the modeling of bond classes and study its behavior in asset allocation, we than model the capital solution transaction as an example of a fixed income structured product.
Sammanfattning

Modellering av värdepapper med fast avkastning

Förutom finansiell analys, kvantitativa verktyg spelar en viktig roll i kapitalförvaltningen också. Genom att hantera sammanläggning av stora mängder historiska och framtida uppgifter om olika tillgängsklasser kan dessa verktyg ge placeringslösning med avseende på risk och regulatoriska begränsningar.

Tillgängsklass modellering kräver tre huvudsteg: Den första är att utvärdera produktens funktioner (riskpremie och risker) genom att beakta historiska och framtida uppgifter, som i fallet med fast inkomst beror på spridning och normalnivåer. Den andra är att välja den kvantitativa modellen. I denna studie presenterar vi en ny kreditmodell, som till skillnad från aktieliknande modeller, utformar ”standard” som det viktigaste inslaget i Fixed Income prestanda. Det sista steget består i att kalibrera modellen.

Vi börjar denna studie med modellering av obligationsklasser och med att studera dess beteende i tillgängsallokering. Sedan, modellerar vi kapital lösning transaktionen som ett exempel på en fast inkomst strukturerad produkt.
Acknowledgements

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Last but not least I would like to thank my father and mother, my two elder sisters and their small families and my closest friends. You are the best thing I have in my life.
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1 Introduction

To understand the objective of the thesis, we first describe the work environment and the team mission: ¹

1.1 Company and team description

AXA IM

AXA IM is one of the world leaders in asset management (the 14th by assets under management). Funded in 1994, AXA IM is detained by AXA Group, a leader in financial protection. It offers investment solution mainly for AXA Group but also for third party customers. It is located in 23 countries and has 2400 employees.

It is composed of seven departments which are: AXA IM fixed income, AXA Framlington, AXA Rosenberg (Equity Funds), AXA funds of hedge funds, AXA Real Estate, AXA structured finance and AXA multi assets clients solutions (MACS), each of them is specialized in a specific asset class except MACS department, which is transversal to all AXA IM teams.

Financial engineering team

My master thesis took place within the Financial engineering team within MACS. The team main mission is to provide strategic investment solution advisory by considering the investor’s objectives and constraints.

For that it follows an optimization process:

(1) Products modeling, risk premium and risk calibration.

(2) Trajectories simulation for asset class eligible by the client.

(3) Selection of the optimal portfolio with respect to the risk profile of the client.

Figure 1 shows an example of asset allocation study where we study the impact of including a new asset class in a portfolio, the study shows that we can find a portfolio with the same expected return as the initial portfolio but with a lower risk.

¹AXA IM internal sources
Financial engineering team and product modeling

The project concerns the first part of the optimization process (Product modeling, risk premium and risks calibration). To model the product, the FE team interacts with other AXA IM teams:

**Asset managers:** When the product is not standard, product specialists provide their expertise for a proper understanding of the product and therefore a more proper modeling.

**AXA group economists and strategists:** provide macro-economic forecast on asset class performance and risk forecast, it is the common source with other AXA entities which strengthens the commercial speech.

**Quant team:** The team models the asset classes and their correlation and works also with AXA Derivatives. Calibration methods and parameter fixing might differ from asset allocation to derivatives pricing, for example the implicit volatility, which is widely used in derivatives pricing does not suit for asset management purposes where the historical volatility makes more sense to assess the historical risk of the asset class.

**Sophisticated clients:** The performance of the FE relies on the strength of its assumptions. And the FE results, to be accepted by sophisticated clients which are mostly institutional including insurers and pension funds, must rely on strong assumptions.

**Modeling process:**

The product is first well understood, then the team makes assumptions on assets, risk premium and risk forecast from the economical forecast and historical performance, it then validates a quantitative model and finally calibrate it to match the assumption on risk premium and risks (Figure 2).
1.2 Study objective

The goal of the study is to model fixed income asset classes. While the nominal interest rate is already modeled and calibrated, we focus on the excess return of credit products over the nominal rate.

The first section will model credit bond indexes which are standard products, these assets have generally available literature, benchmarks and market data. We therefore have the following process for bonds modeling(Figure 3).

In the second section, we model a structured product. The product, as opposed to standard asset classes, does not have external literature, no historical benchmark and is a very illiquid asset class. The first part will therefore consist of understanding the product feature from data communicated by the asset management team to determine how we will assess the risk premium and risks , and finally model and calibrate the product(Figure 4).
Figure 4: Capital solution transaction modeling process.
2 Main fixed income asset classes modeling

2.1 Asset class description

Fixed income is a type of investing or budgeting style for which real return rates or periodic income is received at regular intervals and at reasonably predictable levels. The products in this section are widely used, we therefore have available literature on the asset classes features and risks.

We start with a short description of fixed income main asset classes and their main advantage and consideration. We then back-test their historical performance and risk by considering each asset class benchmark. We finally give a focus on default as a key notion for the next step of modeling the credit asset classes.

2.1.1 Fixed income asset classes overview

First, bond asset classes are classified by their rating, investment grade bonds are bonds that are judged by the rating agencies as likely enough to meet payment obligations. A bond is considered investment grade or IG if its credit rating is BBB- or higher by Standard & Poor’s and FITCH or Baa3 or higher by Moody’s.

While High Yield bonds are bonds that have a significant speculative characteristics because of the higher risk of default, they are designed by credit rating agencies as having a lower credit rating. A bond is considered high yield if its credit rating is BB+ or lower by Standard & Poor’s and FITCH or Ba1 or lower by Moody’s. Short duration high yield aims to capture high yield income while minimizing volatility, the average maturity is much limited compared to the overall HY core index (An average expected maturity of three years or less for AXA IM FIIS Europe Short Duration High Yield for example).

While classical bonds are highly liquid, leveraged loans are not issued in a public exchange, but rather are private transactions between the corporation and the lender (bank and/or the investor). Leveraged Loan is a commercial loan provided by a group of lenders. It is first structured, arranged, and administered by one or several commercial or investment banks, known as arrangers and then sold to other banks or institutional investors.

For structured credit asset classes, the most used ones within the team are CLOs. A Collateralized Loan Obligation (CLO) is a vehicle that issues rated debt securities and an unrated equity piece. It provides banks and portfolio managers with a mechanism to outsource risk and optimize economic and regulatory capital management. The proceeds from this issuance are used to purchase a portfolio of predominantly senior secured loans. Coupon and principal payments on the liabilities (the CLO notes) are paid using coupon and principal payments on the assets (the loans), with CLO equity being paid from residual cash flows.
2.1.2 Advantages and considerations

Investment Grade bonds have a low rate of default rate and are highly liquid, while they have a low expected return specially in a low rate environment. At the opposite, High Yield asset class has a high expected return, it has a high volatility and low liquidity (but less than Investment Grade as described in the IMF study \(^2\)), it has also a higher default rate and greater draw downs during crisis.

Leveraged Loans give floating coupon rates, which makes it defensive against the rise of interest rate, it also has a higher seniority than bonds, while it has low liquidity and a limited secondary market, it is also generally callable and not all leverages loans are rated but when they are, they have a low rating. As for leveraged loans, CLO pays floating coupon rates while it offers a high choice from several combinations of risk and reward. But CLOs have low liquidity and are generally callable.

2.1.3 Correlation

In Table 1, we give the historical risk (volatility-Drawdown) of the different Fixed Income Asset Classes Excess Return and the historical correlation over 2003-2015 (the historical excess return for Merrill Lynch Index is detailed in Annex C). We notice a low correlation between Leveraged Loans excess return and other bond indexes while the fact that emerging bonds are denominated in LC decreases the correlation with global indexes.

<table>
<thead>
<tr>
<th></th>
<th>IG Glob</th>
<th>HY Glob</th>
<th>Emerg. HC</th>
<th>Emerg. LC</th>
<th>US Lev Loans</th>
<th>EU Lev Loans</th>
</tr>
</thead>
<tbody>
<tr>
<td>IG Global</td>
<td>100%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HY Global</td>
<td>87,67%</td>
<td>100%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Emerg. HC</td>
<td>60,40%</td>
<td>67,81%</td>
<td>100 %</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Emerg. LC</td>
<td>36,75%</td>
<td>45,55%</td>
<td>48,83%</td>
<td>100%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>US Lev Loans</td>
<td>0,26%</td>
<td>-12,55%</td>
<td>-8,54%</td>
<td>-14,85%</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>EUR Lev Loans</td>
<td>-3,43%</td>
<td>-5,12%</td>
<td>0,50%</td>
<td>-13,61%</td>
<td>80,95%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Table 1: Historical correlation for fixed income classes excess return for benchmarks from 2006 to 2015, (Benchmarks: BOFA Merry Lynch Global corporate index, BOFA Merry Lynch High Yield index for HY Global, JPM EMBI GLOBAL COMPOSITE for Emerging Hard Currency, JPM GBI-EM GLOBAL Composite LC for Emerging Hard Currency, CSLLLITOT Index for US Loans, SPBDEL Index for EU Loans).

2.1.4 Performance and Risk back-test

We perform a back-test on benchmarks for asset classes formerly described by considering the historical volatility and maximum draw down (Figure 5). It shows that as expected, the volatility and draw down risk increase for a lower rating. Leverage Loans have a risk profile close to High Yield Bonds, while

\(^2\)IMF Study, “Market LIQUIDITY-Resilient or fleeting”, p.61
Emerging Bonds since 2003 have a risk profile that lies between IG and High yield bonds.

Figure 5: Historical risk profile by credit class in the left, Risk Profile by rating in the right, the circles size is scaled by the product volatility, BOFA Merry Lynch Global corporate index, BOFA Merry Lynch High Yield index for HY Global, JPM EMBI GLOBAL COMPOSITE for Emerging Hard Currency, JPM GBI-EM GLOBAL Composite LC for Emerging Hard Currency, CSLLL-TOT Index for US Loans, SPBDEL Index for EU Loans, Merry Lynch Indexes for sub rating classes.

2.1.5 A Focus on Default

Borrowers may default when they are unable to make the required payment or are unwilling to honor the debt. The recovery rate is the extent to which principal and accrued interest on a debt instrument that is in default can be recovered, expressed as a percentage of the instrument’s face value.

Since the default and recovery rates are one of the main features of the new model, we investigate the historical default rates for different asset class, whether they are fixed or floating rates and their correlation with the economical situation from S&P and Moody’s data.

The Figure 6 shows the historical and crisis time default rates taken from the 2014 S&P report. It shows a significant difference between default rates of different High yield sub-rating.
The Figure 7 and 8 \(^3\) show that while default increases during crisis events, recovery rates significantly decreases. Both rates diverge from their mean significantly (+/-2 times their standard deviation) which shows that the hypothesis that the recovery and the default rates are fixed can be improved by considering a stochastic model for default rates.

\(^3\)Frank K. Reilly, David J. Wright, James A. Gentry “Historic changes in the high yield bond market”, University of Illinois, p.72-73
2.2 Excess return modeling

In the former section, we presented the features and historical performance of fixed income asset classes, we now focus on the modeling of the excess return over the nominal interest rate with a new default model.

We first present the theoretical background, we then present the old credit model, we finally study the new CIR Intensity model, we explain why it can offer a better modeling for our fixed income asset classes by modeling defaults and present the closed formula for the credit spread.

2.2.1 Theoretical background

Probability theory

In this section, we first present the definition of a probability space and a filtration:

Definition 2.1. A probability space \((\Omega, F, P)\) is a probability space if:

- \(\Omega\) is a set.
- \(F\) is a \(\sigma\) algebra.
- \(P\) is a function from \(F\) to \([0,1]\) with \(P(\Omega) = 1\) and such that if \(E_1, E_2, \dots \in F\) are disjoint \(P(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} P(E_j)\).

Definition 2.2. A filtration \(\mathcal{F} = \{F_t\}_{t \geq 0}\) on the probability space \((\Omega, F, P)\) is an indexed family of sigma-algebras on \(\Omega\) such that:

1. \(F_t \subseteq F, \forall t \geq 0\).
2. \(s \leq t \Rightarrow F_s \subseteq F_t\).

We now present the definition of stochastic process and Wiener process:

Definition 2.3. Suppose \((\Omega, F, P)\) is a probability space, and that \(I \subseteq R\). Any collection of random variables \(X = X_t, t \in I\) defined on \((\Omega, F, P)\) is called a random process with index set \(I\).
**Definition 2.4.** A stochastic process \( W \) is called a **Wiener process** if the following conditions hold.

1. \( W(0) = 0. \)
2. The process \( W \) has independent increments, i.e if \( r < s \leq t < u \) then \( W(u) - W(t) \) and \( W(s) - W(r) \) are independent stochastic variables.
3. For \( s < t \) the stochastic variable \( W(t) - W(s) \) has the Gaussian distribution \( N[0, \sqrt{t-s}] \).
4. \( W \) has continuous trajectories.

**Bonds and interest rates**

Since we are modeling credit indexes, an important classical credit product is the zero coupon bonds.

**Definition 2.5.** A **zero coupon bond** with maturity date \( T \), also called \( T \)-bond, is a contract which guarantees the holder 1 dollar to be paid on the date \( T \). The price at time \( t \) of a bond with maturity date \( T \) is denoted by \( B(t, T) \).

We now introduce the instantaneous forward rate, the short rate and the money account and risk neutral measure.

**Definition 2.6.** The **instantaneous forward rate with maturity \( T \)**, contracted at \( t \) is defined by

\[
    f(t, T) = -\frac{d\log(B(t, T))}{dt}.
\]

The instantaneous **short rate at time \( t \)** is defined by \( r(t) = f(t, t) \).

The **money account** process is defined by

\[
    dB(t) = r(t)B(t)dt,
\]

\( B(0) = 1 \).

The **risk neutral measure** is the martingale measure which numeraire(\( S_0 \)) is the money account.

**2.2.2 Nominal rate Model**

The notations are the following:

- \( Q \) the risk-neutral probability.
- \( B(t, T) \) the price at \( t \) of a Zero-coupon with maturity \( T \).
- \( r_t \) the instantaneous risk-free rate at \( t \).

The nominal interest rates model is based on the Health-Jarrow-Morton framework.

Let \( W_t^Q = [W_{1,t}^Q, W_{2,t}^Q] \) the dynamic of the Zero-coupon prices is:

\[
    \frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma^B_1(t,T)dW^Q_{1,t} + \sigma^B_2(t,T)dW^Q_{2,t}.
\]
The calibration of the model is not a part of this study and was calibrated formerly within the financial engineering team.

2.2.3 Current credit models

Investment grade

The credit model describes the risky zero coupon diffusion. The risk neutral dynamic of an investment grade zero coupon with maturity $T$ is the following:

$$\frac{d\mathcal{B}(t,T)}{\mathcal{B}(t,T)} = \frac{dB(t,T)}{B(t,T)} + \sigma_{IG} dW_t^{IG} + I_t dN_t.$$ with:

$B(t,T), \mathcal{B}(t,T)$ are respectively the zero coupon risk free and risky prices.

$\sigma_{IG}$ is the additional volatility implied by the credit spread, $W_t$ is a Wiener process, $\sigma_{IG}$ is the volatility of the credit excess return.

$I_t dN_t$ is used to model crisis events, $I_t$ a log normal process, $N_t$ is a poisson process.

A constant risk premium is added to be under the historical measure.

High Yield and Equity

High Yield and equity have currently the same modeling with the Financial Engineering team, they follow a jump diffusion. Under the risk-neutral probability their dynamic is:

$$\frac{dX_t}{X_t} = (r_t - k_t \cdot \lambda - k_t' \cdot \lambda' - k_t'' \cdot \lambda'') + \sigma dW_t + I_t dN_t + J_t dM_t + K_t dD_t.$$ with:

- $N_t, M_t$ and $D_t$ independant Compound Poisson processes.
- $\lambda, \lambda', \lambda''$ are the poisson process intensities.
- $k_t = E[Y_t - 1]$, where $Y_t - 1$ is the random variable percentage change in the stock price if the Poisson event occurs, i.e for $k'(t)$ and $k''(t)$.
- $I_t$ a log normal process, $J_t$ and $K_t$ constant processes.

A constant risk premium is added to be under the historical measure.

Correlation structure

Actually, only High Yield indexes and Equity are correlated since High Yield is modeled as an equity like, while Investment Grade indexes are not correlated (ex US and EU Investment Grades are not correlated) and are also uncorrelated with HY and Equity indexes (Table 2). Nominal interest rate and excess return are not either correlated.
Why a new model?

First for High Yield modeling, We cannot implement buy and hold strategies on bonds, but only bond indexes processes. Furthermore, the actual credit modeling does not model credit default as a main feature of credit products. In addition to that, the old credit model can lead to inconsistent results and the return to maturity can be higher than the initial spread curve for example. Finally, we need to have a better correlation structure between credit asset classes.

The new CIR model aims therefore to offer a better modeling for credit products by modeling default and by taking into consideration spread levels.

2.2.4 New CIR Intensity model

a. Default number modeling

As described before, in the new model we want to model defaults. Defaults are sudden, usually unexpected and cause large, discontinuous price changes. And logically, the probability of default in a short time interval is approximately proportional to the length of the interval.

For CDS pricing for example, we focus on the time of default, \( \tau = \min\{t, N(t) = 1\} \), but since we model a credit index the default event is not the default of all credit component, but a default on a small part of the bond basket, we therefore focus on the number of defaults \( N_t \).

In the following section, we explain the choice of a Poisson process to model defaults.

Poisson process

A counting process \( N(t), t \geq 0 \) is a Poisson process with rate \( \lambda \) if:

(i) \( N(0) = 0 \).

(ii) \( N(t) \) has independent increments.

(iii) \( N(t) - N(s) \sim \text{Poisson}(\lambda(t-s)) \) for \( s < t \).

We recall the Poisson(\( \lambda t \)) distribution:

\[
P(N(t) - N(s) = n) = \frac{(\lambda(t-s))^n}{n!}\exp(-\lambda(t-s)).
\]

Poisson process has 0 as a start time point, is integer-valued, and jump probability over small intervals is proportional to that interval which makes it is suitable
for default modeling.

**Inhomogeneous Poisson process**

Inhomogeneous Poisson process consider a time dependent intensity function \( \lambda(t) \). Let \( \lambda(.) \) be a non-negative intensity function. A Poisson process \( N \) satisfies:

\[
P(N_t - N_s = k) = \frac{(\int_s^t \lambda_u du)^k}{k!} \exp(\int_s^t \lambda_u du).
\]

**Cox process intensity**

We want to model the spread with a stochastic process, we therefore use Cox process that assumes a stochastic intensity. Cox process is now a generalization of the Poisson process, in which the intensity is allowed to be random in such a way that in a particular realization \( \lambda(., \omega) \) of the intensity, the process becomes an inhomogeneous Poisson process.

**b. Risky Zero Coupon pricing with a Cox Intensity process**

**Risky Zero Coupon model definition:**

A zero coupon bond will pay 1 if no default occurs and will pay less depending on the number of defaults.

At maturity \( T \), it gives the following pay-off:

\[
\phi(N_T) = (1 - q)^{N_T - N_t}.
\]

where :

\( N_T - N_t \) is the number of defaults between \( t \) and \( T \).
\( q \) the loss given default.

We model \( N_t \) as a Cox process.

**Risky Zero Coupon pricing:**

The intensity credit models are arbitrage free models that simulate strategies on risky zero coupon bonds. The fundamental pricing formula of a risky zero coupon is :

\[
\overline{B}(t, T) = E[\exp(-\int_t^T r_u du)(1 - q)^{N_T - N_t}|F_t],
\]

\[
\overline{B}(T, T) = (1 - q)^{N_T - N_t}.
\]

where:

- \( q \) : is the loss given default.
• $N$ : is the number of defaults (or credit events).

$N$ is modeled by a Cox process.

We can show that:

$$B(t, T) = E[\exp(-\int_t^T (r_s + q\lambda_s)ds)|F_t].$$

\textbf{Proof.} Since in our case, we consider the case where nominal rates and defaults are independent we have:

$$E[\exp(-\int_t^T r_u du)(1-q)^{N_T-N_t}|\lambda_{t..T}, F_t] = E[\exp(-\int_t^T \lambda_u du)|F_t]E[(1-q)^{N_T-N_t}|\lambda_{t..T}, F_t].$$

$$= E[\exp(-\int_t^T r_u du)|F_t]\sum_{k=0}^{\infty} \frac{(\int_t^T \lambda_u du)^k}{k!}(1-q)^k\exp(-\int_t^T \lambda_u du)$$

$$= E[\exp(-\int_t^T r_u du)|F_t]\sum_{k=0}^{\infty} \frac{(\int_t^T (1-q)\lambda_u du)^k}{k!}\exp(-\int_t^T \lambda_u du)$$

$$= E[\exp(-\int_t^T r_u du)|F_t]\exp(\int_t^T \lambda_u (1-q)du)\exp(-\int_t^T \lambda_u du)$$

$$= E[\exp(-\int_t^T (r_u + q\lambda_u))du|F_t].$$

We therefore have

$$B(t, T) = E[E[\exp(-\int_t^T r_u du)(1-q)^{N_T-N_t}|\lambda_{t..T}, F_t]|F_t] = E[\exp(-\int_t^T (r_u + q\lambda_u))du|F_t].$$

\hfill $\square$

In the following section, we describe the diffusion choice for the cox intensity. We first choose a CIR intensity, and then add a Jump diffusion to model extreme risk scenarios.

\textbf{c. Intensity modeling with a CIR Intensity}

\textbf{Model presentation:}

Since we want to keep the intensity $\lambda$ positive, we choose a CIR process under the risk neutral measure $^4$:

$$\begin{align*}
\lambda_t &= y_t + \psi_t, \\
\psi_0 &= \lambda_0 - y_0, \\
dy_t &= k(\mu - y_t)dt + \sqrt{y_t}dW_Q(t)
\end{align*}$$

$^4$John C Cox, Ingersoll, Jonathan E, and Stephen A Ross paper

\textit{A Theory of the Term Structure of Interest Rates Econometrica, 53, 385-407 (1985).}
with:

- \( \psi_t \): a time variant but deterministic function, to match the initial bond prices.
- \( k \): Mean reversion of CIR process.
- \( \mu \): Long term average of CIR process.
- \( v \): Level dependent volatility of CIR process.
- \( y_0 \): Initial value of CIR process.
- \( W_Q(t) \): A standard Brownian motion under the risk-neutral measure \( Q \).

The process does not reach zero if \( 2k\mu > v^2 \).

**Term structure:**

We recall the term structure for a CIR process:

\[
P_{CIR}(t,T,x) = A(t,T) \exp(-B(t,T)x).
\]

where:

\[
\begin{align*}
A(t,T) &= \left[ \frac{2h \exp((k+h)(T-t)/2)}{2h+(k+h)(\exp((T-t)h)-1)} \right]^{\frac{2\mu}{v^2}}, \\
B(t,T) &= \frac{2(\exp((T-t)h)-1)}{2h+(k+h)(\exp((T-t)h)-1)}, \\
h &= \sqrt{k^2 + 2v^2}
\end{align*}
\]

**Proof.** From the short rate dynamics, we have that the CIR model admits an affine term structure (Annex B). We therefore have: \( P_{CIR}(t,T,x) = A(t,T) \exp(-B(t,T)x) \) with:

\[
\begin{align*}
B(t,T) - kB(t,T) - \frac{1}{2}v^2B^2(t,T) &= -1, \\
B(T,T) &= 0
\end{align*}
\]

This equation is a *Riccati equation* with fixed parameters. The second equation for \( A \) is:

\[
\begin{align*}
A_t(t,T) &= k\mu B(t,T), \\
A(T,T) &= 0
\end{align*}
\]

We therefore integrate to get the formula.

**Zero Coupon Spread in the Credit Model:**
We assume that we know the current term structure of risky bonds $S^{mkt}(0, T_i)$ (The initial pure spread zero coupon bond term structure).

Given the term structure of the CIR process we have:

$$S(t, T) = \frac{S^{mkt}(0, T)A(0, t)\exp(-B(0, t)y_0)}{S^{mkt}(0, t)A(0, T)\exp(-B(0, T)y_0)} P_{CIR}(t, T, qy_t).$$

where:

$$\begin{cases} 
P_{CIR}(t, T, x) = A(t, T) \exp(-B(t, T)x), \\
P(t, T) = \left[ \frac{2h \exp((k+h)(T-t)/2)}{2h+(k+h)\exp((T-t)/k)-1} \right] \frac{2x}{2h+(k+h)}, \\
P(t, T) = \frac{2 \exp(h(T-t))}{2h+(k+h)(\exp((T-t)/k)-1)}, \\
h = \sqrt{k^2 + 2q^2} 
\end{cases}$$

Proof. We recall $\lambda_t = y_t + \psi_t$, $\psi_t$ is deterministic and $y_t$ is a CIR process.

We have:

$$S(t, T) = \frac{B(t, T)}{C(t, T)} = E[\exp(-\int_t^T q\lambda_s ds) | F_t] = E[\exp(-\int_t^T qy_s ds) | F_t] g(t, T) = P_{CIR}(t, T, qy_t) g(t, T).$$

where

$$g(t, T) = \exp(-\int_t^T \psi(u) du) = \frac{S^{mkt}(0, T)P_{CIR}(0, t, y_0)}{P_{CIR}(0, T, y_0)S^{mkt}(0, t)} = \frac{S^{mkt}(0, T)A(0, t)\exp(-B(0, t)y_0)}{A(0, T)\exp(-B(0, T)y_0)S^{mkt}(0, t)}.$$

d.Intensity modeling with a CIR Intensity with Jumps

To model crisis events we add jumps to model the simulation of default intensity.
\[
\begin{align*}
\lambda_t &= y_t + \psi_t, \\
\psi_0 &= \lambda_0 - y_0, \\
dy_t &= k(\mu - y_t)dt + \sqrt{y_t}dW_Q(t) + df^{\alpha,\gamma}_t, \\
J^{\alpha,\gamma}_t &= \sum_{i=1}^{M^\alpha_t} Y_i^\gamma \\
J_t &= \sum_{i=1}^{M^\alpha_t} Y_i^\gamma
\end{align*}
\]

\(J_t\) is a Jump process with intensity \(\alpha > 0\) and \(\gamma > 0\) under the risk neutral measure, \(Y_i \sim \exp(1/\gamma)\). The process \(y\) is positive when the parameters verify \(2k\mu > v^2\).

We preserve the attractive feature of positive interest rates implied by the basic CIR dynamics. After adding jumps, we still have closed form formulas to price zero coupon bonds. We denote:

\[
S(t, T) = \frac{B(t, T)}{ZC(t, T)}.
\]

The pure credit bond price can be written as follows:

\[
S(t, T) = \frac{S^{mkt}(0, T)\pi(0, t)}{S^{mkt}(0, 0)\pi(0, T)} \exp(-q\beta(0, T)y_0) \exp(-\beta(t, T)y_t).
\]

with:

\[
\begin{align*}
\pi(t, T) &= A(t, T) \left( \frac{2h exp(\frac{h+k+2\gamma}{2}(T-t))}{2n(h+k+2\gamma)(exp(h(T-t))-1)} \right)^{\frac{-2\gamma\gamma}{q^2-2k\gamma-2\gamma^2}}, \\
\beta(t, T) &= B(t, T)
\end{align*}
\]

Proof. \(^5\)

2.2.5 Modeling credit indexes as rolling bond

In the former section, we presented the credit model that describes the diffusion of a risky zero coupon, however a ZC bond does not have a fixed duration during its lifetime. To model credit indexes, we therefore use rolling bonds to keep a fix duration. A rolling bond is a zero coupon bond of maturity the duration of the index, that get replaced with a fixed frequency with a new Zero Coupon of the same duration:

RollingBond with \(D=5\)years and \(freq = 1\) year,
\(\Rightarrow\) at \(t = 0\) ZC(0, 5),
\(\Rightarrow\) at \(t = 1\), sell ZC(0, 5) and buy ZC(1, 6) (with no cost),
\(\Rightarrow\) at \(t = 2\), sell ZC(1, 6) and buy ZC(2, 7) (with no cost),

We therefore keep a fixed duration when modeling a credit index.

**Correlation Structure**

Currently, the new model includes the correlation between credit asset classes (including between Investment Grade and High Yield) by correlating default intensities. However, there exist no correlation between credit asset classes and equities (Table 3), which is a major disadvantage for the new model, besides the nominal interest rate and excess return are not correlated. A PhD thesis is currently being conducted within the Quant team to have a complete correlation structure.

<table>
<thead>
<tr>
<th></th>
<th>Investment Grade</th>
<th>High Yield</th>
<th>Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment Grade</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>High Yield</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>Equity</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 3: Correlation structure for the new model

**2.3 Risk premium and risk feature computation**

In the former section, we presented the model for credit asset classes. In this section, we will describe the features that we want to calibrate and the computation method we use to assess these features. For that, we first detail available data, then we explain how we set targets for these measures.

**2.3.1 Available data**

**Historical data**

For each asset class (or sub-rating), we have historical data for their benchmark from 1997 to 2016. Table 4 shows the historical data considering Bank Of America Euro BBB corporate index as a benchmark for the BBB asset class. The methodology of computation for Merrill Lynch indexes is detailed in Annex C.

<table>
<thead>
<tr>
<th>Bond Index</th>
<th>Description</th>
<th>Date</th>
<th>Excess Return % 1-month</th>
</tr>
</thead>
<tbody>
<tr>
<td>ER40</td>
<td>Euro Corp BBB</td>
<td>1/31/1997</td>
<td>0.45 %</td>
</tr>
<tr>
<td>ER40</td>
<td>Euro Corp BBB</td>
<td>2/28/1997</td>
<td>0.68 %</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>ER40</td>
<td>Euro Corp BBB</td>
<td>12/31/2015</td>
<td>-0.27 %</td>
</tr>
<tr>
<td>ER40</td>
<td>Euro Corp BBB</td>
<td>1/31/2016</td>
<td>-1.26 %</td>
</tr>
</tbody>
</table>

Table 4: Historical monthly data for excess swap return, BBB as an example, Bank Of America Euro BBB corporate index as a benchmark, Bloomberg.

**Prospective Data**
Since historical data does not assess for future performance, we need prospective assumptions based on economical forecast.

A team of economist and strategist of AXA group provides quarterly assumptions on main asset classes.

As shown in Table 5, prospective data concerns the future level of the 5 year and 7 year spread of the asset class. For default, we have a flat default and recovery rate.

<table>
<thead>
<tr>
<th>EUR</th>
<th>Actual</th>
<th>Q+1</th>
<th>N</th>
<th>N+1</th>
<th>–</th>
<th>N+9</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUR ML Credit 5Y BBB spread over swap</td>
<td>1.62 %</td>
<td>1.56 %</td>
<td>1.45 %</td>
<td>1.30 %</td>
<td>–</td>
<td>1.35 %</td>
</tr>
<tr>
<td>EUR ML Credit 7Y BBB spread over swap</td>
<td>1.83 %</td>
<td>1.73 %</td>
<td>1.55 %</td>
<td>1.60 %</td>
<td>–</td>
<td>1.65 %</td>
</tr>
<tr>
<td>EUR BBB Default Rate</td>
<td>0.10 %</td>
<td>0.10 %</td>
<td>0.10 %</td>
<td>0.10 %</td>
<td>–</td>
<td>0.10 %</td>
</tr>
<tr>
<td>EUR Recovery - Senior or equivalent</td>
<td>34 %</td>
<td>34 %</td>
<td>34 %</td>
<td>34 %</td>
<td>–</td>
<td>34 %</td>
</tr>
</tbody>
</table>

Table 5: Prospective data for excess swap spread, default and recovery rate, BBB as an example, AXA Group Assumptions.

2.3.2 Performance and risk measures

Measures choice:

For the performance measure, we use the 8 years expected excess return since the financial engineering team performs strategic allocation, we therefore need a long term expected return of 8 years which corresponds to an economic cycle. We use prospective data to assess the future expected return of the asset classes, but since received prospective data only include future level spread(Table 5) and not the expected return we use a proxy to assess the risk premium, the methodology is detailed in annex D.

<table>
<thead>
<tr>
<th></th>
<th>EU HY</th>
<th>EU IG</th>
<th>US IG</th>
<th>US HY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annualised expected return</td>
<td>1.19 %</td>
<td>2.21 %</td>
<td>1.77 %</td>
<td>2.73 %</td>
</tr>
</tbody>
</table>

Table 6: Annualized 8 year expected return.

As a risk measure, we use the 1 year total return volatility, since the volatility is one of the most used risk estimator used by investors to assess the asset class risk, we use the total return volatility. We consider that the historical volatility
is a good proxy for the future volatility $\sigma_{yearly} = \sqrt{12}\sigma_{monthly}$ (from historical data).

2.4 Model Calibration

2.4.1 Calibration Goal

In the former section we described the model that we will use for fixed income asset classes and the features that we want to calibrate, in the following section we will describe how we will calibrate the model.

We recall in the following table CIR process with jumps parameters.

- **LossRate** ($q$): Loss given default.
- **$k$**: Mean reversion of CIR process.
- **$\mu$**: Long term average of CIR process.
- **$v$**: Level dependent volatility of CIR process.
- **$y_0$**: Initial value of CIR process.
- **$JF$**: Jump frequency.
- **$JM$**: Jump magnitude.
- **$DRP$**: Default risk premium, percentage of defaults considered.

We note $Parameter = (q, k, \mu, v, y_0, JF, JM, DRP)$.

2.4.2 Objective function for optimization

We use the calibration to make the model match our assumptions on the asset class and therefore we choose the model parameters to minimize:

$$\min Parameter \sum_i \alpha_i |AssetClassQuantitativeFeature_i - ModelCorrespondingValue_i|$$

$\alpha_i$ is used to scale the Asset class quantitative Feature or to prioritize a specific feature.

We consider the features as described in the former subsection:

$$\min Parameter [|TargetE[R(8)] - ModelE[R(8)]| + |Target\sigma_{yearly} - Model\sigma_{yearly}|$$

We therefore have a global non-linear optimization problem and no precise calculation of derivatives is available, we therefore use the Nelson Mead algorithm (Annexe A).

We summarize the algorithm in the following figure:
Nelson Mead Algorithm on the Objective function

\[
\text{Objective function(}\text{Parameter}) = \left| \text{Target}E[R(8)] - \text{Model}E[R(8)] \right| + \left| \text{Target}\sigma_{\text{yearly}} - \text{Model}\sigma_{\text{yearly}} \right|
\]

To compute \( E[R(8)] \), \( \sigma_{\text{yearly}} \) for a given parameter:

<table>
<thead>
<tr>
<th>For one Simulation</th>
<th>After ( N ) simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulate Risk Free and Risky Coupon From :</td>
<td>( N = 10000 )</td>
</tr>
<tr>
<td>(1) the nominal interest rate Model</td>
<td>Compute the Monte Carlo</td>
</tr>
<tr>
<td>(2) Simulate the spread from the CIR Model with jumps</td>
<td>{ ( E[R(8)] ) }, ( \sigma_{\text{yearly}} }</td>
</tr>
</tbody>
</table>

2.4.3 Simulation Tool

We use an internal tool named Apollo, provided by the Quant team within AXA IM, it allows to simulate different asset classes, equity indexes, Zero Coupon bonds, rolling bonds. The tool is organized as an excel add-in while it is encoded in C#. It is used for both product calibration and for strategy tests and asset optimization.

2.4.4 Assets to simulate

We recall the performance and risk measures that we want to compute:

\[
E[R(8)] = E[\ln(\frac{\text{RollingBond}(8)}{\text{RollingBond}(0)})] - \ln(\frac{\text{RollingBond}(8)}{\text{RollingBond}(0)})].
\]

\[
\sigma_{\text{yearly}} = \text{Std}[\ln(\frac{\text{RollingBond}(1)}{\text{RollingBond}(0)})].
\]

We consider 19/04/2016 as a starting date. We therefore simulate a risk-free and a risky rolling bond on time \( T = 0 \) (19/04/2016) and \( T = 1 \) (19/04/2017) and on \( T = 8 \) (19/04/2024).

2.4.5 Model inputs for calibrations

a. Nominal rate parameters

As said before, the nominal rate model (2.2.2) had already been calibrated within the financial engineering team, its parameters and the actual forward curve are both inputs of the model (Table 7,8).

| \( a_n \) | \( 85.9 \% \) |
| \( b_n \) | \( 8.8 \% \) |
| \( \rho_n \) | \( -74 \% \) |
| \( \sigma_n \) | \( 1.6 \% \) |
| \( \eta_n \) | \( 1.2 \% \) |

Table 7: Nominal rate model parameters input.
<table>
<thead>
<tr>
<th>Maturity</th>
<th>Discount rate</th>
<th>Instantaneous Forward</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,044</td>
<td>1,000</td>
<td>-0,0035</td>
</tr>
<tr>
<td>..</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>2</td>
<td>1,003</td>
<td>-0,0014</td>
</tr>
<tr>
<td>3</td>
<td>1,004</td>
<td>-0,0006</td>
</tr>
<tr>
<td>..</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>7</td>
<td>0,983</td>
<td>0,0094</td>
</tr>
<tr>
<td>8</td>
<td>0,971</td>
<td>0,0122</td>
</tr>
</tbody>
</table>

Table 8: Actual nominal spread level bootstrapped from swap prices.

b. Initial Spread

We recall the pricing formula for zero coupon bonds with CIR intensity with jumps:

\[
S(t, T) = \frac{S^{mkt}(0, T)A(0, t)\exp(-B(0, t)\gamma_0)}{S^{mkt}(0, T)A(0, T)\exp(-B(0, t)\gamma_0)} P^{CIR}(t, T, \gamma_t).
\]

where \(S^{mkt}(0, T)\) is the initial pure spread zero coupon bond term structure.

*Why don’t we simply consider the initial spread as the present spread curve for the asset class?*

Initial spread curve can be seen as the best possible excess return in case no default occurs, if we consider the product as a buy and hold to maturity and we only compute the returns and risks to maturity than it is accurate to consider the market initial spread.

Credit Indexes have however frequent re-balancing, it is therefore important to take into consideration the future movements of the spreads.

*Proxy choice*

We note \(EAC7YS(iY)\) and \(EAC8YS(iY)\) the economical assumption first the 7 and 5 years spreads in \(i\) years (we do not have assumptions on other spread levels).

We fix the forward credit spread to the implied forward spread between 5 to 7 years spread level.

\[
\begin{align*}
S(5Y) &= \frac{1}{5} \sum_{i=0}^{5} EAC5YS(iY), \\
S(7Y) &= \frac{1}{7} \sum_{i=0}^{7} EAC7YS(iY), \\
\forall t \leq 5Y S(t) &= S(5Y), \\
\forall t > 5Y S(t) &= (S(5Y) \cdot 5 + Forward \cdot (t - 5))/t,
\end{align*}
\]

with \(Forward = (7 \cdot S(7) - 5 \cdot S(5))/2\)
c. CIR parameters Inputs

Finally, we have the CIR parameter with jumps as inputs, since the goal is to calibrate the model, these parameters are not fixed, but will be changed by the algorithm by using the Nelson Mead algorithm (Table 9).

<table>
<thead>
<tr>
<th>Loss Rate</th>
<th>1 %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>75.98 %</td>
</tr>
<tr>
<td>$\mu$</td>
<td>32.01 %</td>
</tr>
<tr>
<td>$\nu$</td>
<td>57.11 %</td>
</tr>
<tr>
<td>$y_0$</td>
<td>102.10 %</td>
</tr>
<tr>
<td>$JF$</td>
<td>165.14 %</td>
</tr>
<tr>
<td>$JM$</td>
<td>247.25 %</td>
</tr>
<tr>
<td>$DRP$</td>
<td>6.46%</td>
</tr>
</tbody>
</table>

Table 9: IR model with jumps parameters input in the simulation tool.

2.4.6 Model Outputs

Once, the nominal rate and credit parameters are specified, we get $N$ simulation of the asked assets for the specified date (Table 10). We therefore compute the objective function (Table 11) to get the objective function value for each step in the Nelson Mead algorithm.

<table>
<thead>
<tr>
<th>Simulation number</th>
<th>Observation on 19/04/2022</th>
<th>Observation on 19/04/2017</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.787</td>
<td>0.833</td>
</tr>
<tr>
<td>2</td>
<td>0.787</td>
<td>0.782</td>
</tr>
<tr>
<td>3</td>
<td>0.787</td>
<td>0.815</td>
</tr>
<tr>
<td>4</td>
<td>0.787</td>
<td>0.847</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>9998</td>
<td>0.787</td>
<td>0.878</td>
</tr>
<tr>
<td>9999</td>
<td>0.787</td>
<td>0.807</td>
</tr>
<tr>
<td>10000</td>
<td>0.787</td>
<td>0.802</td>
</tr>
</tbody>
</table>

Table 10: Asset class specification in the Appolo tool.

<table>
<thead>
<tr>
<th></th>
<th>Simulated</th>
<th>Target</th>
<th>Difference</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[R(8)]$</td>
<td>1.55 %</td>
<td>1.66 %</td>
<td>0.10 %</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{yearly}$</td>
<td>4.363%</td>
<td>4.12 %</td>
<td>0.24 %</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 11: Objective Function computation for one simulation.

2.4.7 Calibration Results

The following tables present first how the algorithm matches the calibration targets than the calibration parameters. We reach the algorithm for EU and US Investment Grade and High Yield (Table 12,13).
<table>
<thead>
<tr>
<th></th>
<th>EU IG</th>
<th>US IG</th>
<th>US HY</th>
<th>EU HY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Targeted $E[R(8)]$</td>
<td>1.23%</td>
<td>2.05%</td>
<td>3.44%</td>
<td>3.29%</td>
</tr>
<tr>
<td>Calibrated $E[R(8)]$</td>
<td>1.24%</td>
<td>2.06%</td>
<td>3.38%</td>
<td>3.28%</td>
</tr>
<tr>
<td>Targeted $\sigma_{\text{yearly}}$</td>
<td>3.25%</td>
<td>5.28%</td>
<td>8.97%</td>
<td>12.53%</td>
</tr>
<tr>
<td>Calibrated $\sigma_{\text{yearly}}$</td>
<td>3.22%</td>
<td>5.54%</td>
<td>8.95%</td>
<td>13.50%</td>
</tr>
</tbody>
</table>

Table 12: Calibration Results for EU IG, HY and US IG and HY.

<table>
<thead>
<tr>
<th></th>
<th>EU IG</th>
<th>US IG</th>
<th>US HY</th>
<th>EU HY</th>
</tr>
</thead>
<tbody>
<tr>
<td>LR</td>
<td>1.01%</td>
<td>1%</td>
<td>9.32%</td>
<td>1.95%</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>149.85%</td>
<td>69.97%</td>
<td>60.92%</td>
<td>27.59%</td>
</tr>
<tr>
<td>$\mu$</td>
<td>38.86%</td>
<td>69.55%</td>
<td>54.47%</td>
<td>37.55%</td>
</tr>
<tr>
<td>$\nu$</td>
<td>76.19%</td>
<td>63.87%</td>
<td>69.93%</td>
<td>22.99%</td>
</tr>
<tr>
<td>$y_0$</td>
<td>99.24%</td>
<td>101.97%</td>
<td>51.57%</td>
<td>10.10%</td>
</tr>
<tr>
<td>$JF$</td>
<td>68.48%</td>
<td>175.21%</td>
<td>52.94%</td>
<td>188.10%</td>
</tr>
<tr>
<td>$JM$</td>
<td>244.8%</td>
<td>159.97%</td>
<td>26.06%</td>
<td>176.21%</td>
</tr>
<tr>
<td>$DRP$</td>
<td>5.00%</td>
<td>5.74%</td>
<td>34.49%</td>
<td>35.08%</td>
</tr>
</tbody>
</table>

Table 13: Calibration results for EU and US IG, HY.

### 2.4.8 Comparison with old calibration

The Figure 9 shows the new calibrations for credit classes remains homogeneous with other asset classes (Cash-Property-Large Cap Europe). It also shows that risk premium is higher with the new method especially for US where expected Price Return is important. Finally, we notice that credit asset classes have higher sharp ratio than other asset classes which might overweight credit in asset allocation.
Figure 9: Risk-Return Profile with old and New modeling credit classes, Cash, Govies, Property and Large Cap Euro are added for comparison.

2.5 Case Study

In the former sections, we presented the model and how we calibrate the credit asset class, we now test the new model behavior in a case study.

2.5.1 Case Study description

We mix credit asset classes with equity (Large Cap Europe) and govies, Table 14 describes constraints on different asset classes in the simulated portfolios:

<table>
<thead>
<tr>
<th></th>
<th>Minimal Weight</th>
<th>Maximal Weight</th>
<th>Step</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large Cap Europe</td>
<td>0 %</td>
<td>30 %</td>
<td>5 %</td>
</tr>
<tr>
<td>Govies</td>
<td>0 %</td>
<td>40 %</td>
<td>5 %</td>
</tr>
<tr>
<td>EU IG</td>
<td>0 %</td>
<td>40 %</td>
<td>5 %</td>
</tr>
<tr>
<td>US IG</td>
<td>0 %</td>
<td>40 %</td>
<td>5 %</td>
</tr>
<tr>
<td>EU HY</td>
<td>0 %</td>
<td>15 %</td>
<td>5 %</td>
</tr>
<tr>
<td>US HY</td>
<td>0 %</td>
<td>15 %</td>
<td>5 %</td>
</tr>
</tbody>
</table>

Table 14: Constraints on simulated portfolios in the case study.

2.5.2 Simulation inputs

We first have as inputs the nominal interest rate parameters (2.4.5.a) than the initial spread and CIR intensity parameter for every credit asset class (2.4.5.b, 2.4.7), the equity parameters which were calibrated within AXA IM and the correlation matrix from historical data. We finally have the transaction and liquidity costs communicated by the asset management.
2.5.3 Model transition impact on risk-return profile

Figure 10 shows the new model simulation has a narrower volatility range and we notice strong discontinuity in the new model points, this is due to the fact that equity is not correlated with High yield which imply a strong increase in volatility when the weight of equity increases.

![Figure 10: Risk-Return profile with old and new modeling, new model portfolios have a dark blue color.](image)

**Results analysis**

The simulation results depend mainly on the risk-premium target and the correlation structure. To explain the results, we focus on two portfolios (Portfolio 1, Portfolio 2) as specified in Figure 10.

**Portfolio 1:**

The composition of the Portfolio (Table 15) shows that it is only composed of credit indexes while its volatility is higher for the new model. Figure 10 shows that the portfolio expected return is higher, this is due to the fact that the new risk premium targets are higher (Figure 9).

The higher volatility when using the new model is due to the fact that High Yield is modeled as credit indexes in the new model and is correlated with other credit asset classes which increases the portfolio volatility (Table 16).

<table>
<thead>
<tr>
<th></th>
<th>Govies</th>
<th>EUR</th>
<th>US HY</th>
<th>US IG</th>
<th>EU IG</th>
<th>EUR</th>
<th>Old Vol</th>
<th>New Vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio1</td>
<td>35 %</td>
<td>0 %</td>
<td>10 %</td>
<td>15 %</td>
<td>40 %</td>
<td>0 %</td>
<td>2.9 %</td>
<td>3.4 %</td>
</tr>
</tbody>
</table>

Table 15: Portfolio 1 composition.
Correlation between credit indexes and goves

<table>
<thead>
<tr>
<th></th>
<th>EU IG</th>
<th>US IG</th>
<th>US HY</th>
<th>EU HY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Old Model</td>
<td>63.09%</td>
<td>80.15%</td>
<td>-3.43%</td>
<td>-2.43%</td>
</tr>
<tr>
<td>New Model</td>
<td>93.63%</td>
<td>80.22%</td>
<td>34.85%</td>
<td>18.02%</td>
</tr>
</tbody>
</table>

Table 16: Correlation between Govies and other asset classes for the old and new model.

Portfolio 2:

The composition of the Portfolio (Table 17) shows that equity (Large Cap Europe) has a high weight. The lower volatility when using the new model is due to the fact that High Yield indexes are modeled as credit indexes in the new model and are uncorrelated with other equity asset classes which decreases the volatility (Table 18).

<table>
<thead>
<tr>
<th></th>
<th>Govies</th>
<th>EUR HY</th>
<th>US HY</th>
<th>US IG</th>
<th>EU IG</th>
<th>LC EUR</th>
<th>Old Vol %</th>
<th>New Vol %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio2</td>
<td>0 %</td>
<td>15 %</td>
<td>15 %</td>
<td>25 %</td>
<td>5 %</td>
<td>40 %</td>
<td>10.55 %</td>
<td>8.54 %</td>
</tr>
</tbody>
</table>

Table 17: Portfolio 2 composition.

Correlation between credit indexes and LC Europe

<table>
<thead>
<tr>
<th></th>
<th>EU IG</th>
<th>US IG</th>
<th>US HY</th>
<th>EU HY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Old Model</td>
<td>-1.56 %</td>
<td>-0.05%</td>
<td>65.16 %</td>
<td>67.81 %</td>
</tr>
<tr>
<td>New Model</td>
<td>-1.38 %</td>
<td>-1.81 %</td>
<td>-0.29 %</td>
<td>-0.48 %</td>
</tr>
</tbody>
</table>

Table 18: Correlation between LC Europe and other asset classes for the old and new model.

2.5.4 Model transition impact on asset allocation

In this section, we look for the composition of portfolios in the efficient frontier. Figure 11 shows that for EU Investment Grade and for the values of volatility smaller than 8 %, all efficient portfolios have a high Investment Grade weights for the old model. For the new model and since High yield is not correlated with equity, portfolios in the efficient portfolios have a significantly less EU IG weight since it can include more high yield. This does not stand for US Investment Grade, since as shown Figure 9, it has a high sharp ratio.

For high yield and due to high risk premium and decorrelation with equity, both US and EU High yield have high weights in all efficient portfolios for volatility levels between 4% and 9 %, while for the old model, there exist both portfolios with low and high weights for high yield asset classes.
2.6 Study of granularity impact

In this section, we calibrate the subrating of the EU Investment Grade and test if it can provide an improvement in asset allocation.

2.6.1 Stand-Alone result

Calibration Result

We perform the calibrations for EU subratings (Table 19,20). Figure 12 shows first that Investment Grade volatilities stand between AA and A while it has a higher expected return than A for a lower volatility. This stands also for AA that has a higher expected return for a lower volatility compared with AAA, which is not homogeneous with the additional credit risk taken, this can be explained by the fact that the AAA index has a higher duration level and therefore a higher interest rate risk.

<table>
<thead>
<tr>
<th>Targeted $E[R(8)]$</th>
<th>EU IG</th>
<th>EU BBB</th>
<th>EU A</th>
<th>EU AA</th>
<th>EU AAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Targeted $\sigma_{\text{yearly}}$</td>
<td>3.25 %</td>
<td>4.02 %</td>
<td>3.60 %</td>
<td>2.96 %</td>
<td>3.06 %</td>
</tr>
<tr>
<td>Calibrated $E[R(8)]$</td>
<td>1.24 %</td>
<td>1.85 %</td>
<td>1.08 %</td>
<td>0.63 %</td>
<td>0.15 %</td>
</tr>
<tr>
<td>Calibrated $\sigma_{\text{yearly}}$</td>
<td>3.22 %</td>
<td>3.90 %</td>
<td>3.55 %</td>
<td>3 %</td>
<td>3.06 %</td>
</tr>
</tbody>
</table>

Table 19: Calibration results for EU IG and its sub-ratings.
Table 20: Calibration results for EU IG and its sub-ratings.

<table>
<thead>
<tr>
<th>Rating</th>
<th>EU IG</th>
<th>EU BBB</th>
<th>EU A</th>
<th>EU AA</th>
<th>EU AAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>LR</td>
<td>1.01%</td>
<td>1%</td>
<td>7.53%</td>
<td>0.13%</td>
<td>200.00%</td>
</tr>
<tr>
<td>κ</td>
<td>149.85%</td>
<td>151.02%</td>
<td>23.99%</td>
<td>138.62%</td>
<td>29.29%</td>
</tr>
<tr>
<td>μ</td>
<td>38.86%</td>
<td>122.50%</td>
<td>5.95%</td>
<td>35.19%</td>
<td>0.28%</td>
</tr>
<tr>
<td>ν</td>
<td>76.19%</td>
<td>9.76%</td>
<td>10.94%</td>
<td>17.26%</td>
<td>16.18%</td>
</tr>
<tr>
<td>yi</td>
<td>99.24%</td>
<td>69.97%</td>
<td>8.75%</td>
<td>24.98%</td>
<td>16.18%</td>
</tr>
<tr>
<td>JF</td>
<td>68.48%</td>
<td>188.04%</td>
<td>31.60%</td>
<td>193.34%</td>
<td>193.95%</td>
</tr>
<tr>
<td>JM</td>
<td>244.8%</td>
<td>358.72%</td>
<td>12.86%</td>
<td>165.42%</td>
<td>145.09%</td>
</tr>
<tr>
<td>DRP</td>
<td>5.00%</td>
<td>5.00%</td>
<td>5.26%</td>
<td>5.00%</td>
<td>35.14%</td>
</tr>
</tbody>
</table>

Table 21: Core Merry Lynch investment grade index sub-rating composition.

<table>
<thead>
<tr>
<th>Rating</th>
<th>BBB</th>
<th>A</th>
<th>AA</th>
<th>AAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>43.84%</td>
<td>43.19%</td>
<td>12.46%</td>
<td>0.35%</td>
</tr>
</tbody>
</table>

2.6.2 Replacement of the core index with sub-ratings

Core Index as a portfolio of subrating

We take the Merry Lynch core index as a reference (Table 21). Table 22 shows that the risk-return profile does not change if we replace the core index with its subratings components. Figure 13 shows that the core index is optimal since it belongs to the efficient frontier when we simulate with only sub-rating classes.

Figure 12: Risk-Return profile for EU IG and its subratings.

Figure 13: Risk-Return profile for EU IG and its subratings.

Table 21: Core Merry Lynch investment grade index sub-rating composition.
1 year volatility | 5 years annualized expected return
---|---
Core Index directly simulated | 3.22 % | 1.55 %
Core Index simulated from subratings | 3.29 % | 1.61 %

Table 22: Core Index directly simulated and simulated from its subratings weights.

Figure 13: Simulated portfolios with only EU subrating components, the figure shows that the core index belongs to the efficient frontier.

**Granularity impact in a case study**

We replace the core index in the former case study with its subratings to test if it can provide an improvement in the efficient frontier(Table 23). Figure 14 shows that while simulated portfolios with the core index can be replicated by portfolios simulated with EU IG subratings, the new efficient frontier offers very few improvements, it can be explained by the fact that the core index is already optimal.

<table>
<thead>
<tr>
<th></th>
<th>Minimal Weight</th>
<th>Maximal Weight</th>
<th>Step</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large Cap Europe</td>
<td>0 %</td>
<td>30 %</td>
<td>5 %</td>
</tr>
<tr>
<td>Govies</td>
<td>0 %</td>
<td>40 %</td>
<td>5 %</td>
</tr>
<tr>
<td>EU BBB</td>
<td>0 %</td>
<td>30 %</td>
<td>5 %</td>
</tr>
<tr>
<td>EU A</td>
<td>0 %</td>
<td>30 %</td>
<td>5 %</td>
</tr>
<tr>
<td>EU AA</td>
<td>0 %</td>
<td>10 %</td>
<td>5 %</td>
</tr>
<tr>
<td>EU AAA</td>
<td>0 %</td>
<td>10 %</td>
<td>5 %</td>
</tr>
<tr>
<td>US IG</td>
<td>0 %</td>
<td>40 %</td>
<td>5 %</td>
</tr>
<tr>
<td>EU HY</td>
<td>0 %</td>
<td>15 %</td>
<td>5 %</td>
</tr>
<tr>
<td>US HY</td>
<td>0 %</td>
<td>15 %</td>
<td>5 %</td>
</tr>
</tbody>
</table>

Table 23: Constraints on simulated portfolios in the case study by replacing the core index with its sub-ratings.
Figure 14: Risk-Return profile with the core index in red and sub-ratings in blue for the case study.
3 Modeling of a structured Product using CIR Intensity Model: Capital Solution Transaction

PCS (Partner Capital Solution) is an AXA IM fund composed of transactions called Capital Solution Transactions. The aim of this section is to model this fund and therefore include this product in asset allocation studies. A usual case study is to show how including PCS in an insurance portfolio can improve its risk-return profile.

A Capital Solution Transaction “CST” is a bilateral or syndicated transaction executed with a bank seeking capital relief and free capacity to develop its commercial activity. The aim of this study is to model this product using the CIR model with jumps.

As for Fixed Income products, we will start with a description of the products, we will then explain the modeling and how we assess the risk premium and risk features, we end up with the model calibration and results analysis (Figure 4).

3.1 Product Description

3.1.1 Features

CST provides the originating bank a guarantee amounting to the regulatory capital for a selected loan portfolio. It is constituted of CDS on the first losses of the loan portfolio.

The main investors of the sector are mainly hedge funds with an exception for AXA IM and the Dutch pension fund PGGM. The investment comes mainly as a complementary investment with leveraged loans while the counterpart requires a strong track for the investor which reduces the number of market participants.

Main counterparts are the most sophisticated banks in Europe. While the US market remains relatively limited due to large securitization.

The CST has a fixed income structure and is categorized as an alternative credit. It pays periodic coupons which stand around 10%. This coupon rate is reduced when default rates occurs, it is not however affected by the spread fluctuation on the pool of loans but only by their default.

It has a low attachment point which makes it cover the first losses on the portfolio and therefore release the bank from its regulatory constraints. Small and medium enterprise default risk depend strongly on the economic situation, the attachment point is therefore related to the economic forecast and remains low (around 2%), while well ranked Large Cap experience lower default rates and the attachment point is around 0% (one default over 1500 loans for Large Caps for AXA IM).
3.1.2 Related risks
CST has a low default rate which makes it very sensitive to the default of its underlying loans and therefore it bears a high default risk.

The transaction takes a long time to be closed and there exist a limited number of market participants which makes the CST highly illiquid.

Regulatory constraint makes CSTs callable which may be a potential risk if banks have less regulatory constraints.

3.1.3 Advantages
First, CST has a high coupon (around 10 %) which is attractive, especially in a low rate environment. It also offers an exposure to diversified credit risk exposures held on banks balance sheets. Finally, CST has floating interest rates which makes it defensive against the rise of interest rate.

3.2 Product modeling
We recall the CIR model with jumps:

$$\begin{align*}
\lambda_t &= y_t + \psi_t, \\
\psi_0 &= \lambda_0 - y_0, \\
J_{t}^{\alpha,\gamma} &= \sum_{i=1}^{M_{t}^{\alpha}} Y_{t}^{\gamma}, \\
dy_t &= k(\mu - y_t)dt + \sqrt{y_t}dW_t + dJ_{t}^{\alpha,\gamma}
\end{align*}$$

Since the product is a pure credit product, its performance depends only on the defaults of the underlying pool of assets and not on their performance. It is a buy and hold product and is not rolled like credit indexes, we therefore model the product return as the return of a risky zero coupon with multiple defaults.

$$\overline{B}(t,T) = E[\exp(-\int_{t}^{T} r_u du)(1-q)^{N_{t}^{\alpha}-N_{t}^{\beta}}|F_t].$$

Where $N_t$ is a Cox process with intensity $\lambda$ which is modeled with a CIR model with jumps, by noting $P(t,T)$ the CST price, we therefore have:

$$\ln\left(\frac{P(t,T)}{P(0,T)}\right) = \ln\left(\frac{\overline{B}(t,T)}{\overline{B}(0,T)}\right).$$
3.3 Risk premium and risk features

3.3.1 Available Data

We first have the weight of every transaction in the PCS fund and their respective attachment and detachment point (Table 24). To assess the product risk, we also receive from the asset management expected cash flows for three scenarios, a **mean scenario** that corresponds to the historical average performance of a similar loan basket within AXA IM, a **bull scenario** that corresponds to a good product performance with a small default rate and finally a **bear scenario** that corresponds to the worst historical scenario communicated by the bank (example for BSN 2015: Table 25, 26).

<table>
<thead>
<tr>
<th></th>
<th>Weight</th>
<th>Attachment Point</th>
<th>Detachment Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>BSN 2015</td>
<td>13.47%</td>
<td>0.50%</td>
<td>5.50%</td>
</tr>
<tr>
<td>Craft 2014</td>
<td>19.09%</td>
<td>0 %</td>
<td>5 %</td>
</tr>
<tr>
<td>Elvetia 2</td>
<td>5.74%</td>
<td>0%</td>
<td>5.20%</td>
</tr>
<tr>
<td>Gate 2015</td>
<td>1.19%</td>
<td>0%</td>
<td>10%</td>
</tr>
<tr>
<td>Gaudi 2015</td>
<td>7.71%</td>
<td>2%</td>
<td>8%</td>
</tr>
<tr>
<td>Leicester</td>
<td>4.91%</td>
<td>0%</td>
<td>9.61%</td>
</tr>
<tr>
<td>Makalu</td>
<td>16.16%</td>
<td>0%</td>
<td>4.91%</td>
</tr>
<tr>
<td>Roof 2015</td>
<td>10.42%</td>
<td>1.40%</td>
<td>9.10%</td>
</tr>
<tr>
<td>Start 10</td>
<td>3.94%</td>
<td>1%</td>
<td>7.70%</td>
</tr>
<tr>
<td>Trafn</td>
<td>1.11%</td>
<td>0%</td>
<td>6.50%</td>
</tr>
</tbody>
</table>

Table 24: PCS Fund composition and attachment and detachment point for every transaction, the remaining weight is composed of cash.

<table>
<thead>
<tr>
<th></th>
<th>BULL</th>
<th>Mean</th>
<th>Bear</th>
</tr>
</thead>
<tbody>
<tr>
<td>31/05/2016</td>
<td>-9 708 k</td>
<td>-9 708 k</td>
<td>-9 708 k</td>
</tr>
<tr>
<td>31/07/2016</td>
<td>242 k</td>
<td>242 k</td>
<td>242 k</td>
</tr>
<tr>
<td>....</td>
<td>......</td>
<td>......</td>
<td>......</td>
</tr>
<tr>
<td>31/10/2016</td>
<td>242 k</td>
<td>242 k</td>
<td>242 k</td>
</tr>
<tr>
<td>31/01/2019</td>
<td>242 k</td>
<td>242 k</td>
<td>145 k</td>
</tr>
</tbody>
</table>

Table 25: BSN Expected Cash Flows during Reinvestment Period (all cash flows that come from matured loans or from defaulted loans are reinvested), the period end at 31/01/2019.

<table>
<thead>
<tr>
<th></th>
<th>BULL</th>
<th>Mean</th>
<th>Bear</th>
</tr>
</thead>
<tbody>
<tr>
<td>30/05/2019</td>
<td>1 668 k</td>
<td>5 017 k</td>
<td>2 724 k</td>
</tr>
<tr>
<td>31/07/2019</td>
<td>1 423 k</td>
<td>4 301 k</td>
<td>2 344 k</td>
</tr>
<tr>
<td>....</td>
<td>......</td>
<td>......</td>
<td>......</td>
</tr>
<tr>
<td>30/04/2021</td>
<td>2 792 k</td>
<td>2 560 k</td>
<td>1 399 k</td>
</tr>
</tbody>
</table>

Table 26: BSN Expected Cash Flows during Call Period (all cash flows that come from matured loans or from defaulted loans are not reinvested and given to the investor).
3.3.2 Performance and risk target:

a. Difference with credit index

If compared with former performance and risk targets for credit indexes, the product is very illiquid, volatility can not therefore assess the product risk. We therefore use the $VaR_{1\%}(8Y)$ and $VaR_{1\%}(1Y)$ instead of the volatility.

The second difference is that while we were calibrating with the expected excess return for fixed income indexes, we calibrate with the total expected return.

b. Internal Rate of Return (IRR)

We recall the IRR definition is the rate that make the initial price match the discounted cash flows:

$$\text{Price}_0 = \frac{\text{ScenarioCashflow}_1}{(1+\text{IRR})^1} + \frac{\text{ScenarioCashflow}_2}{(1+\text{IRR})^2} + \ldots + \frac{\text{ScenarioLastCashflow}}{(1+\text{IRR})^n}.$$ 

\[c.\text{Risk premium and risk measures}\]

We use the followings proxies to assess the risk premium and VaR for the transactions and for the fund:

For a single transaction

- $E[R(8)] = IRR$ of the mean scenario.
- $VaR_{5\%}(8Y) = IRR$ of the bear scenario.
- $VaR_{5\%}(1Y)$ is communicated by the asset management, a stress test is applied in the first year and than default rates evolves as for the mean scenario.

For the PCS Fund

We consider the weights of each transaction in the fund (Table 27). We exclude transaction close to their maturity since we are considering long term assumptions and because it gives a risk return profile homogeneous with other asset classes (Figure 15).

- $E[R(8)] = \sum_1^8 w_i E[R_i(8)].$
- $E[VaR] = \sum_1^8 w_i E[VaR_i].$

<table>
<thead>
<tr>
<th>Measure</th>
<th>By considering all transaction</th>
<th>By excluding transactions that are close to their maturity (Elvetia 2, Leicester, Start 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[R(8)]$</td>
<td>5,57 %</td>
<td>5,34 %</td>
</tr>
<tr>
<td>$VaR_{5%}(8Y)$</td>
<td>-2,95 %</td>
<td>-3,90 %</td>
</tr>
<tr>
<td>$VaR_{5%}(1Y)$</td>
<td>-24,5%</td>
<td>-29,12 %</td>
</tr>
</tbody>
</table>

Table 27: Risk Premium and VaR for the PCS fund by considering all transactions and by excluding transaction that are close to their maturity.
3.3.3 Performance and risk measure model computation

Since we model the fund price as a risky Zero Coupon bond, we have:

\[
\begin{align*}
E[R(8)] &= E[\ln(\frac{B(8,8)}{B(0,8)})], \\
VaR^{5\%}(8Y) &= VaR^{5\%}[\ln(\frac{B(8,8)}{B(0,8)})], \\
VaR^{5\%}(1Y) &= VaR^{5\%}[\ln(\frac{B(1,8)}{B(0,8)})]
\end{align*}
\]

We therefore simulate a risky zero coupon with maturity 8 for t=0,1,8.

3.4 Model Calibration

3.4.1 Objective Function

We have the following objective function:

\[
\begin{align*}
\text{Min}_{\text{Parameter}} & \quad \left| \text{Target}E[R(8)] - \text{Model}E[R(8)] \right| + \\
& \quad \left| \text{Target}VaR^{5\%}(8Y) - \text{Model}VaR^{5\%}(8Y) \right| + \frac{1}{10}\left| \text{Target}VaR^{5\%}(1Y) - \text{Model}VaR^{5\%}(1Y) \right|
\end{align*}
\]

where Parameter = (q, k, \mu, v, y_0, JF, JM, DRP).

Remark: We choose the \( \frac{1}{10} \) as the \( VaR^{5\%}(1Y) \) since it has a high scale compared to other measures.

3.4.2 Calibration Algorithm

As for credit indexes we use the Nelson Mead algorithm (Annexe A).

We summarize the algorithm in the following figure:
Nelson Mead Algorithm on the Objective function

\[
\text{Objective function}(\text{Parameter}) = \\
|\text{Target}_E[R(8)] - \text{Model}_E[R(8)]| + \\
|\text{Target}_\text{VaR}_{5\%}(8Y) - \text{Model}_\text{VaR}_{5\%}(8Y)| + \\
\frac{1}{10}|\text{Target}_\text{VaR}_{5\%}(1Y) - \text{Model}_\text{VaR}_{5\%}(1Y)|.
\]

To compute The \(E[R(8)], \text{VaR}_{5\%}(8Y)\) and \(\text{VaR}_{5\%}(1Y)\) For a given parameter:

<table>
<thead>
<tr>
<th>For one Simulation</th>
<th>After N simulation</th>
</tr>
</thead>
</table>
| Simulate Risk Free and Risky Coupon From : 
(1)The nominal interest rate Model. 
(2) Simulate the spread from the CIR Model with jumps. | \(N = 10000\) |

3.4.3 Results

Calibration Results

Table 28 and 29 summarize the calibration results and parameters:

<table>
<thead>
<tr>
<th></th>
<th>Simulated</th>
<th>Target</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E[R(8)])</td>
<td>5.38%</td>
<td>5.34%</td>
<td>0.035%</td>
</tr>
<tr>
<td>(\text{VaR}_{5%}(8Y))</td>
<td>-3.893%</td>
<td>-3.893%</td>
<td>0.004%</td>
</tr>
<tr>
<td>(\text{VaR}_{5%}(1Y))</td>
<td>-29.69%</td>
<td>-29.119%</td>
<td>0.572%</td>
</tr>
</tbody>
</table>

Table 28: Calibration Results.

<table>
<thead>
<tr>
<th></th>
<th>Simulated</th>
<th>(y_0)</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(LR)</td>
<td>19.93%</td>
<td>51.96%</td>
<td></td>
</tr>
<tr>
<td>(\kappa)</td>
<td>58.61%</td>
<td>(JF)</td>
<td>29.14%</td>
</tr>
<tr>
<td>(\mu)</td>
<td>49.53%</td>
<td>(JM)</td>
<td>19.34%</td>
</tr>
<tr>
<td>(\nu)</td>
<td>0.26%</td>
<td>(DRP)</td>
<td>48.17%</td>
</tr>
</tbody>
</table>

Table 29: Calibration Parameters.

Return analysis

1 year return:

The histogram (Figure 16) shows that the 1 year return is centered around 6% and has large positive and negative tails.
8 years return:
For one year return, excess return depend only on $N_T$ realization since $B(T,T) = (1 - q)^{N_T-N_t}$, we therefore have discontinuous value while we have a low probability of getting big losses. (Table 30, Figure 17).

<table>
<thead>
<tr>
<th>8 years default number</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability of each scenario</td>
<td>27.2 %</td>
<td>34.4 %</td>
<td>23.6 %</td>
<td>9.6 %</td>
<td>3.4 %</td>
<td>1.1 %</td>
<td>0.4 %</td>
</tr>
<tr>
<td>8 years annualized return</td>
<td>10 %</td>
<td>6.5 %</td>
<td>3 %</td>
<td>-0.4 %</td>
<td>-3.8 %</td>
<td>-7.3 %</td>
<td>-14.3 %</td>
</tr>
</tbody>
</table>

Table 30: 8 years return scenarios for the calibrated model, 10000 simulations.
Conclusion

The new model offers a better modeling for Fixed Income asset classes by modeling defaults and crisis behavior. However, the study shows that since the model does not allow correlation with equities, it can be misleading to consider it in asset allocation studies, since it can overweight for example High yield and equities which are uncorrelated and therefore under-estimate the real risk of the portfolio.

However, the model can be used for studies with only credit asset classes and for modeling structured product like PCS which are highly uncorrelated with other asset classes. For other studies, a new PhD thesis is currently being conducted to have a complete correlation structure.

Furthermore, the new methodology for computing a proxy for the risk premium includes the price return as an important component, it implies however a stronger sharp ratio which might overweight credit classes and implies less diversified portfolios.

We finally checked that the core index is optimal in its subrating composition since it belongs to the efficient frontier. For PCS, the model succeeds to have the features asked by the asset management with a strong long term expected return with a high short term risk taken from communicated scenarios.
Appendices

A Nelson Mead

Nelder Mead (or downhill simplex) solves global non-linear optimization problem when precise calculation of derivatives is not available. Like other algorithms for this kind of optimization problem, it will not necessarily find the global minimum given the initial values.

The Nelder-Mead algorithm, or downhill simplex algorithm is one of the best known algorithms for multidimensional unconstrained optimization without derivatives. The Nelder-Mead algorithm is designed to solve the classical unconstrained optimization problem of minimizing a given nonlinear function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \). This method uses only function values at some points in \( \mathbb{R}^n \) and does not try to form an approximate gradient at any of these points. The Nelder-Mead method is simplex-based. A simplex \( S \) in \( \mathbb{R}^n \) is defined as the convex hull of \( n+1 \) vertices \( x_0, \ldots, x_n \in \mathbb{R}^n \).

The algorithm begins by constructing an initial simplex \( S \) with \( n+1 \) vertices \( x_0, \ldots, x_n \) around a given input point \( x_0 \in \mathbb{R}^n \). Then we transform the simplex by reflecting, expanding, or contracting the worst point. One iteration of the Nelder-Mead method consists of the following steps.

a.Ordering: determine the \( h, s, l \) of the worst, second worst and the best vertex, respectively, in the current working simplex \( S \) \( f_h = \max_j f_j, \; f_s = \max_{j \neq h} f_j, \; f_l = \min_{j \neq h} f_j \)

b.Centroid: calculate the centroid \( c := \frac{1}{n} \sum_{j \neq h} x_j \)

c.Transformation: compute the new working simplex from the current one. First try to replace only the worst vertex \( x_h \) with a better point by using reflection, expansion or contraction with respect to the best side. If this succeeds, the accepted point becomes the new vertex of the working simplex. If this fails, shrink the simplex towards the best vertex \( x_l \). In this case, \( n \) new vertices are computed.

Simplex transformation in the Nelder-Mead method are controlled by four parameters \( \alpha \) for reflection, \( \beta \) for contraction, \( \mu \) for expansion and \( \psi \) for shrinkage. They should satisfy the following constraints

\[ \alpha > 0, \; 0 < \beta < 1, \; \mu > 1, \; \mu > \alpha, \; 0 < \psi < 1 \]

The standard values, used in most implementations, are
\[ \alpha = 1, \beta = 1/2, \mu = 2, \psi = 1/2 \]

The following algorithm describes the working simplex transformations in step \( c \), and the effects of various transformations are shown in the corresponding figures. The new working simplex is shown in red.

a. Reflect: compute the reflection point \( x_r := c + \alpha (c - x_h) \) and \( f_r := f(x_r) \). If \( f_l \leq f_r < f_s \), accept \( x_r \) and terminate the iteration.

b. Expand: If \( f_r < f_l \), compute the expansion point \( x_e := c + \gamma (x_h - c) \) and \( f_e := f(x_e) \). If \( f_e > f_r \), accept \( x_r \) and terminate the iteration. Otherwise, accept \( x_r \) and terminate the iteration.

c. Contract: If \( f_r \geq f_s \), compute the contraction point \( x_c \) by using the better of the two points \( x_h \) and \( x_r \)

a. Outside: If \( f_s \leq f_r < f_h \), compute \( x_c := c + \beta (x_r - c) \) and \( f_c := f(x_c) \). If \( f_c \leq f_r \), accept \( x_c \) and terminate the iteration. Otherwise, perform a shrink transformation.

Inside: If \( f_r \geq f_h \), compute \( x_c := c + \beta (x_h - c) \) and \( f_c := f(x_c) \). If \( f_c \leq f_h \), accept \( x_c \) and terminate the iteration. Otherwise, perform a shrink transformation.

Shrink: compute \( n \) new vertices \( x_j := x_l + \psi (x_r - x_l) \), for \( j = 0, \ldots, n \), with \( j \neq l \). The iteration stops when the working simplex \( S \) is sufficiently small in some sense.
B Term structure equation and Affine term structure

Reference 6, 7

B.1 The general one-factor diffusion model

Consider a market of interest rate sensitive contracts and assume no arbitrage.
Let the money market account be defined by:
\[
\frac{dB}{B} = r(t)B(t)dt
\]
where \( r \), the short rate, is a diffusion process, defined by the SDE
\[
dr(t) = \mu(t, r(t))B(t)dt + \sigma(t, r(t))dW(t)
\]
under the natural probability measure \( P \), with \( \mu \) and \( \sigma \) sufficiently regular functions and \( W \) a \( P \)-Wiener process.

B.2 Market Price of interest rate risk

Let’s assume the price at time \( t \) of \( T \)-bonds to be a (smooth) function \( F^T \) of time \( t \) and the short rate \( r(t) \), i.e \( p(t, T) = F^T(t, r(t)) \) for every \( t \leq T \). Function \( F^T \) has to satisfy the terminal condition
\[
F^T(T, r) = 1
\]
By Ito formula:
\[
dF^T = \mu^T F^T dt + \sigma^T F^T dW
\]
Where:
\[
\mu^T = \frac{F^T \mu + \frac{1}{2} \sigma^2 F^T}{F^T}, \quad \sigma^T = \frac{\sigma F^T}{F^T}
\]
For two bonds, a \( T \)-bond and an \( s \)-bond, with \( T \neq s \), and consider a relative self-financing strategy \( u(r) = (u^T(t, r), u^s(t, r)) \) of them. If we denote by \( V \) its price, we have that:
\[
\frac{dV}{V} = (u^T \mu^T + u^s \mu^s) dt + (u^T \sigma^T + u^s \sigma^s) dW
\]
We want to choose a relative portfolio in order to eliminate risk, i.e the \( dW \) in (2). To do this, for each \( t \) and for each \( r \) we have to solve the system:
\[
\begin{cases}
u^T(t, r) + u^s(t, r) = 1 \\
u^T(t, r)\sigma^T(t, r) + u^s(t, r)\sigma^s(t, r) = 0
\end{cases}
\]
This system admits one and only one solution if and only if \( \sigma^T(t, r) \neq \sigma^s(t, r) \) and, under this assumption, the solution is

6 Thomas BJORK, Arbitrage Theory in Continous Time book, Chapter 24
7 Claudio Pacati lecture Short Rate Models (2012)
\[
\begin{align*}
\begin{cases}
  u^T(t,r) &= -\frac{\sigma^r(t,r)}{\sigma^T(t,r)}
  \\
  u^s(t,r) &= \frac{\sigma^T(t,r)}{\sigma^s(t,r)}
\end{cases}
\end{align*}
\]

We finally have:

\[
\frac{dV}{V} = \mu^s \sigma^T - \mu^T \sigma^s dt
\]

Since we already gave an asset with (locally) risk-free dynamics, we have:

\[
\frac{\mu^s \sigma^T - \mu^T \sigma^s}{\sigma^s} = r(t)
\]

or, written in an equivalent way:

\[
\frac{\mu^s(t,r(t)) - r(t)}{\sigma^s(t,r(t))} = \frac{\mu^T(t,r(t)) - r(t)}{\sigma^T(t,r(t))}
\]

The result is therefore that:

\[
\lambda(t,r) = \frac{\mu^T(t,r(t)) - r(t)}{\sigma^T(t,r(t))}
\]

\(\lambda\) depends only on \(t\) and \(r\) and not on \(T\), being hence the same for all \(T\)-bonds and can be interpreted as the market price of risk.

### B.3 Term Structure equation

From the previous section (market price of risk & Ito Lemma) we have the following term structure of risk:

\[
F^T_t + (\mu - \lambda \sigma)F^T_t + \frac{1}{2}\sigma^2 F^T_{rr} - rF = 0
\]

The last equation is the term structure under the \(P\)-dynamics, under the martingale dynamics and since \(\mu^{\text{under } Q} = \mu^{\text{under } P} + \lambda \sigma\), we get the following term structure equation

\[
F^T_t + \mu F^T_t + \frac{1}{2}\sigma^2 F^T_{rr} - rF = 0
\]

### B.4 Affine Term Structure

**Definition**: If the term structure has the form:

\[
p(t,T) = F(t,r(t);T),
\]

where \(F\) has the form

\[
F(t,r;T) = \exp(A(t,T) - B(t,T)r),
\]

where \(A\) and \(B\) are deterministic functions, then the model is said to posses an affine term structure (ATS)

From the term structure equation we have

\[
A_t(t,T) - 1 + B_t(t,T)r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0,
\]

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The boundary value $F(T, r; T) = 1$ implies

$$
\begin{align*}
A(T, T) &= 0 \\
B(T, T) &= 0
\end{align*}
$$

**Proposition** Assume that $\mu$ and $\sigma$ are of the form

$$
\begin{align*}
\mu(t, r) &= \alpha(t) r + \beta(t) \\
\sigma(t, r) &= \sqrt{\gamma(t)} r + \psi(t)
\end{align*}
$$

Then the model admits an Affine Term Structure, where $A$ and $B$ satisfy the system

$$
\begin{align*}
B_t(t, T) + \alpha(t)B(t, T) - \frac{1}{2}\gamma(t)B^2(t, T) &= -1 \\
B(T, T) &= 0
\end{align*}
$$

$$
\begin{align*}
A_t(t, T) &= \beta(t)B(t, T) - \frac{1}{2}\psi(t)B^2(t, T) \\
A(T, T) &= 0
\end{align*}
$$

**Proof.** From the equation on $A$ and $B$ we have:

$$
A_t(t, T) - \beta(t)B(t, T) + \frac{1}{2}\psi(t)B^2(t, T) - 1 + B_t(t, T) + \alpha tB(t, T) - \frac{1}{2}\gamma(t)B^2(t, T) = 0
$$

Since the equation holds for all $t, T$ and $r$ we have:

$$
B_t(t, T) + \alpha(t)B(t, T) - \frac{1}{2}\gamma(t)B^2(t, T) = -1
$$

and

$$
A_t(t, T) = \beta(t)B(t, T) - \frac{1}{2}\psi(t)B^2(t, T)
$$
Excess return Merry Lynch Excess return methodology

Excess return is a measure of relative value that neutralizes the interest rate and yield curve risk of a bond, thereby isolating that portion of its performance that is attributed solely to credit and optionality risks. Excess return is equal to a bond’s total return minus the total return of a risk-matched basket of governments or interest rate swaps.

C.1 Excess Return Components:
There are two main components to excess return:
1. the additional interest income that accrues to the security during the period as a result of a higher starting yield relative to duration-matched governments or swaps
2. the effect of any change in spread during the period on the relative price movement of the security versus risk-matched governments or swaps.

C.2 The hedge basket:
The hedge basket is comprised of fair value governments (or swaps) that together are key rate duration-matched to the bond at six key nodes: the 6-month, 2-year, 5-year, 10-year, 20-year and 30-year points on the curve. The hedge basket is also matched to the bond’s currency of denomination. For example, a sterling-denominated corporate bond is compared to U.K. Gilts or sterling interest rate swaps.

C.3 Key Rate Duration
Key rate duration measures a security’s price sensitivity to shifts at ”key” points along the yield curve. Key duration rates are especially useful for securities with embedded options such as call options or prepayment options.

Why it matters?
Key rate duration is a measure of how a security’s value changes when its yield changes by 1% for a certain maturity.

The formula for key rate duration is:

Key Rate Duration = \( \frac{P_- - P_+}{(2 \cdot 0.01 \cdot P_0)} \)

Where:

- \( P_- \) is the security price after a 1% decrease in yield
- \( P_+ \) is the security price after a 1% increase in yield
- \( P_0 \) is the original security price

The sum of these partial duration corresponds to the effective duration.
D Prospective risk premium computation

Notation:

- $\sigma_{\text{yearly}}$: the yearly total return volatility.
- $E[R(T)]$: the expected return for a maturity $T$.
- $8YS(t)$: the 8 year spread from economical assumptions at time $t$ (ex: $8YS(2)$ is the assumption of the 8 year spread level after two years from now).
- $D$: duration of the credit index.
- $R$: Recovery rate.
- $Default$: Default rate over 8 years.

Currently risk premium proxy

Currently, we use the following proxy for the expected return within the financial engineering team:

$$E[IR(8)] = ((1 + \text{Mean}(8YS(0..8)))^8 - 1) \cdot (1 - E[DefaultRate](1 - R)).$$

It is the return generated from coupon payment for bonds that did not default. We take the mean of spread levels to take into consideration future re-balancing of the bond index.

The problem with this methodology is that we neglect the market price change of the credit index while the rebalancing of the index will make it sensitive to spread changes. We therefore include the price return in the new methodology.

New risk premium proxy

We split the risk premium into three components $E[R(8)] = E[IR(8) + PR(8) + LN(8)]$, where:

$$E[IR(8)] = ((1 + \text{Mean}(8YS(0..8)))^8 - 1) \cdot (1 - E[DefaultRate](1 - R))$$

is the income return which is equivalent to the total risk premium in the current methodology.

$$E[PR(T)] = (8YS(8) - 8YS(0)) \cdot D \cdot (1 - E[DefaultRate](1 - R))$$

is the return generated from market price fluctuation for bonds that did not default (can be negative).

$$E[LN(8)] = E[Default](1 - R)$$

represent the Losses generated from defaults by taking into consideration the recovery rate.

EU Investment Grade sub-ratings as an example:

We apply the methodology for each sub-rating of IG(Table 6). We notice that except for AAA, expected price return remains relatively low compared with the expected income return.
We also have that all spread expected variation (from AXA economical forecast) are viewed to get higher in the 8 future years except for BBB which explains the negative price return.

<table>
<thead>
<tr>
<th></th>
<th>Price 8Y Return (Duration to Spread Variation)</th>
<th>Excess 8Y Income Return</th>
<th>8Y Total Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>-0.71 %</td>
<td>1.57 %</td>
<td>0.81%</td>
</tr>
<tr>
<td>AA</td>
<td>-0.98 %</td>
<td>4.55 %</td>
<td>3.41%</td>
</tr>
<tr>
<td>A</td>
<td>-0.95 %</td>
<td>8.94 %</td>
<td>7.73%</td>
</tr>
<tr>
<td>BBB</td>
<td>0.79 %</td>
<td>14.10 %</td>
<td>14.37%</td>
</tr>
</tbody>
</table>

Table 31: Expected excess return decomposition.
References


