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Stability Conditions for Linear Time-Varying Model Predictive Control in Autonomous Driving

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Abstract—This paper presents stability conditions when designing a linear time-varying model predictive controller for lateral control of an autonomous vehicle. Stability is proved via Lyapunov techniques by adding a terminal state constraint and a terminal cost. We detail how to compute the terminal state and the terminal cost for the linear time-varying case, and interpret the obtained results in the light of an autonomous driving application. To determine the stability conditions, the concept of multi-model description is used, where the linear time-varying model is separated into a finite number of time-invariant models that depend on a single parameter. The terminal set is the maximum positive invariant set of the multi-model description and the terminal cost is the result of a min-max optimization that determines the *worst* time-invariant model if used as a prediction model. In fact, in the autonomous driving case, we show that the min-max approach is a convex optimization problem. The stability conditions are computed offline, maintain the convexity of the optimization, and do not affect the execution time of the controller. In simulation, we demonstrate the stabilizing effectiveness of the proposed conditions through an illustrative example of path following with a heavy-duty vehicle.

I. INTRODUCTION

Autonomous vehicles will be a substantial part of the transportation sector in a matter of time [1]. Before autonomous vehicles become available to the general public, they will be deployed in closed environments such as gravel pits, mining areas, construction sites and loading terminals where workers are subject to dangerous environments and tedious tasks [2]. By eliminating the human-in-the-loop factor, we expect to eliminate the number of human fatalities and increase productivity and efficiency in those working sites.

One of the most important modules in the complex chain of an autonomous vehicle system is the motion controller. This is a core component as it is responsible for stabilizing and guiding the vehicle along the given reference path. The work presented in this paper has in mind the applicability of the motion controller in autonomous heavy duty-vehicles for mining application. In our work, we focus on the lateral control of such vehicles and we target smooth driving while maintaining high-accuracy path tracking performance [3], [4]. Aggressive steering results in a higher *wear and tear*, shortening the lifetime of those heavy-duty vehicles and, as a consequence, increasing the costs for the operators.

Model Predictive Control (MPC) is a useful approach to the problem of vehicle control. MPC is very powerful due to its ability to predict the vehicle behavior for a given set

of inputs and to handle nonlinear time-varying models and constraints in a systematic way (e.g., [5]–[12]).

This paper provides new contributions to the MPC stability and autonomous driving fields as it presents

- 1) the offline computation of the terminal constraint and cost for linear time-varying model predictive controller (LTV-MPC) stability;
- 2) the proof of LTV-MPC stability using the novel terminal constraint and cost;
- 3) the interpretation of the MPC parameter tuning influence of these conditions in the autonomous driving case;
- 4) the applicability of the proposed method in real-time applications, as it maintains the convexity and does not affect the execution time of the optimization problem.

In this paper, the nonlinear vehicle model is linearized around the reference path, giving rise to a linear time-varying model (LTV). The vehicle is modeled in the space domain and in a road-aligned coordinate frame along the reference path to exclude time and speed from the dynamics equations. One of the most popular strategies for ensuring controller stability (see [13] and references therein) is to use the optimization value function as a Lyapunov function. Moreover, the analysis is very convenient if incorporating both a terminal cost and a terminal constraint set in the optimal control problem. The terminal cost is chosen, such that it is equal to the infinite-horizon value function in a suitable neighbourhood of the origin (i.e., the terminal constraint set). Hence, it is possible to use the known advantages of an infinite-horizon control, such as stability [14]. We use the notion of multi-plant description [15], in which the LTV model is separated in several linear time-invariant models (LTI). We propose a method to compute the terminal state constraint, which is chosen to be the maximum positive invariant set of the multi-model description. The terminal cost is computed solving a convex min-max optimization problem [16] that leads to the determination of the *worst* time-invariant model if used as a prediction model.

We reinforce the idea that the development of the methods presented in this paper have in mind their practical applicability. Therefore, we seek stability conditions that neither affect the execution time nor the optimization convexity of the controller. However, we are aware that offline-computed solutions may be conservative as they need to cover a larger set of scenarios, rather than computing a less conservative condition online.

The stability of nonlinear MPC is investigated in [17] by designing a non-quadratic terminal state penalty. This is, in general, not implementable in practice due to its high computational cost. Simon et al. investigate terminal set scaling for a reference tracking MPC when the desired set-point is close to the boundary of the feasible set [18]. The authors prove recursive feasibility and convergence to the new sets, but the

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computational burden of such method is prohibitive for real-time applications. Another approach is proposed by Falcone et al. in [19], where the stability conditions of an LTV-MPC used for active steering systems are investigated and verified experimentally. An additional convex constraint bounding a quadratic function of the control effort and the predicted states is computed to ensure stability. However, this requires the MPC to be cast as a Sequential Quadratic Program (SQP) that has typically higher computational burden than a QP. Additionally, simplifications are made, such as considering the model time-invariance by linearizing around the current set point, to reduce the computational complexity.

The rest of this paper is organized as follows. In Section II, the problem of reference tracking using a receding-horizon framework is addressed by developing an LTV-MPC controller; the autonomous driving field is introduced in Section III by deriving a nonlinear spatial-based vehicle model in the road-aligned coordinate frame. In the same section, we derive the terminal constraints and cost that are used for proving LTV-MPC stability; in Section IV, the effectiveness of the stability conditions is demonstrated through an illustrative example using a heavy-duty vehicle model. Finally, in Section V, we provide some concluding remarks and outline future work.

A. Preliminaries

In this section, we introduce the main concepts that are used throughout the paper. This section follows the structure and notation of [20].

In this paper, we deal with discrete-time nonlinear systems subject to external inputs:

$$z(k+1) = f(z(k), u(k)), \quad (1)$$

where $z(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are the state and input vectors, respectively. Both systems are subject to state and input constraints

$$z(k) \in \mathcal{Z}, u(k) \in \mathcal{U}, \forall k \in \mathbb{N}_+, \quad (2)$$

where the sets $\mathcal{Z} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are polytopes.

When the system (1) is subject to the feedback control law $u(k) = K(k)z(k)$, where $K(k)$ is a time-varying feedback control gain, we obtain the discrete-time autonomous system:

$$z(k+1) = f_a(z(k)) := f(z(k), K(k)). \quad (3)$$

Definition 1 (One-step controllable set): For the system (1), we denote the one-step controllable set to the set $\mathcal{S}$ as

$$\text{Pre}(\mathcal{S}) = \{z \in \mathbb{R}^n : \exists u \in \mathcal{U}, f(z, u) \in \mathcal{S}\}. \quad (4)$$

Definition 2 (N-step controllable set): For a given target set $\mathcal{S} \subseteq \mathcal{Z}$, the N-step controllable set $\mathcal{K}_N(\mathcal{S})$ of the system (1) subject to the constraints (2) is defined recursively as

$$\mathcal{K}_j = \text{Pre}(\mathcal{K}_{j-1}(\mathcal{S})) \cap \mathcal{Z}, \mathcal{K}_0(\mathcal{S}) = \mathcal{S}, j = \{1, \dots, N\}. \quad (5)$$

Definition 3 (Positive invariant set): A set $\mathcal{O} \subseteq \mathcal{Z}$ is said to be a positive invariant set for the system (3) subject to the constraints in (2), if

$$z(0) \in \mathcal{O} \Rightarrow z(k) \in \mathcal{O}, \forall k \in \mathbb{N}_+.$$

Definition 4 (Maximal positive invariant set): The set $\mathcal{O}^\infty \subseteq \mathcal{Z}$ is the maximal positive invariant set of the

autonomous system (3) subject to the constraints in (2) if $\mathcal{O}^\infty$ is invariant and contains all the invariant sets contained in $\mathcal{Z}$.

Theorem 1.1 (Geometric condition for invariance): A set $\mathcal{O} \subseteq \mathcal{Z}$ is a positive invariant set for the autonomous system (3) subject to the constraints in (2) if and only if

$$\mathcal{O} \subseteq \text{Pre}(\mathcal{O}). \quad (6)$$

A proof is found in [20].

II. LINEAR TIME-VARYING MODEL PREDICTIVE CONTROL

Consider the problem of regulating the discrete-time nonlinear time-varying system (1) with constraints (2) to track a given time-varying reference. Consider the reference state $z_r(k) \in \mathcal{Z}$ and the input $u_r(k) \in \mathcal{U}$. Let $Z_r(k) = \{z_r(k), z_r(k+1), \dots\}$ be the reference state path obtained by applying an input sequence $U_r(k) = \{u_r(k), u_r(k+1), \dots\}$ for $k \geq 0$ to the system (1). Then, the linearization of (1) around $Z_r(k)$ and $U_r(k)$ yields the system

$$\tilde{z}(k+1) = A(\xi(k))\tilde{z}(k) + B(\xi(k))\tilde{u}(k), \quad (7)$$

where $\tilde{z}(k) = z(k) - z_r(k)$ and $\tilde{u}(k) = u(k) - u_r(k)$ and

$$A(\xi(k)) = \left. \frac{\partial f(z, u)}{\partial x} \right|_{\substack{z=z_r(k), \\ u=u_r(k)}}, B(\xi(k)) = \left. \frac{\partial f(z, u)}{\partial u} \right|_{\substack{z=z_r(k), \\ u=u_r(k)}}, \quad (8)$$

where $\xi(k)$ is a parameter vector of the form $\xi(k) = [z_r(k), u_r(k)]^\top$, which is known at every step k . Also, $\xi(k) \in \Xi$ for each k , where Ξ is a closed set containing all the possible values of ξ , such that ($\geq$ and $\leq$ are considered element-wise)

$$\Xi = \{\xi \in \mathbb{R}^{n+m} : \xi_{\min} \leq \xi \leq \xi_{\max}\}.$$

The LTV system (7) is a first-order approximation of the nonlinear system (1). It describes the deviations from the state trajectory $z_r(k)$ when a sequence of inputs $u_r(k)$ is applied to the system.

Let $\gamma \in \Gamma$ represent a model described by a specific pair $(A(\xi), B(\xi))$, where the set Γ is a multi-plant description defined as

$$\Gamma = \{(A, B) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{m \times n} : A = A(\xi), B = B(\xi), \xi \in \Xi\}. \quad (9)$$

Note that each $\gamma \in \Gamma$ is time-invariant and depends on a (known) parameter ξ .

Assume that a full measurement or estimate of the state $z(t)$ is available at the current time t . Then, we can formulate the following MPC problem,

$$\min_{\tilde{U}_t} \tilde{z}_{t+N|t}^\top Q_f \tilde{z}_{t+N|t} + \sum_{k=t}^{t+N-1} \tilde{z}_{k|t}^\top Q \tilde{z}_{k|t} + \tilde{u}_{k|t}^\top R \tilde{u}_{k|t} \quad (10a)$$

$$\text{subject to } \tilde{z}_{k+1|t} = A(\xi(k|t))\tilde{z}_{k|t} + B(\xi(k|t))\tilde{u}_{k|t}, \quad k = t, \dots, t+N-1, \quad (10b)$$

$$\tilde{z}_{k|t} \in \tilde{\mathcal{Z}}, \quad k = t, \dots, t+N-1, \quad (10c)$$

$$u_{k|t} \in \mathcal{U}, \quad k = t, \dots, t+N-1, \quad (10d)$$

$$\tilde{z}_{t+N|t} \in \tilde{\mathcal{Z}}_f, \quad (10e)$$

$$\tilde{z}_{t|t} = \tilde{z}(t), \quad (10f)$$

where $\tilde{U}_t = \{\tilde{u}_{t|t}, \dots, \tilde{u}_{t+N-1|t}\}$ is the sequence of inputs to be optimized. $\tilde{z}_{t+k|t}$ denotes the difference between the state vector at time $t+k$ predicted at time t and the reference state vector $z_r(t+k)$. The difference between the predicted and the reference state is constrained to be in a convex polytope $\tilde{\mathcal{Z}}$. Here, the notation $z_{t+k|t}$ stands for "the state z at time $t+k$ predicted at time t ". The notation is analogous for $u_{t+k|t}$. Note that $z(t)$ is then the actual state z at time t . The matrices R , Q , and Q_f are positive definite and penalize deviations from the reference input, state, and final state, respectively.

Let $\tilde{U}_t^* = \{\tilde{u}_{t|t}^*, \dots, \tilde{u}_{t+N-1|t}^*\}$ be the optimal solution of (10) at time t . The first element of $\tilde{U}_t^*$ is applied to the system (7) at time t . Let $K : \mathbb{N}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^{m \times n}$ and that satisfies $\tilde{u}_{t|t}^* = K(t)\tilde{z}(t)$ denote the receding horizon control law that associates the optimal input $\tilde{u}_{k|t}^*$ to the current state $\tilde{z}(t)$. The closed loop system then becomes

$$\tilde{z}(k+1) = (A(\xi(k)) + B(\xi(k))K(k))\tilde{z}(k). \quad (11)$$

In the next sampling time, the optimal control problem (10) is solved again using the new state measurements, and the horizon is shifted forward.

In receding-horizon control, the optimization problem may lead us into an infeasible situation (i.e., there does not exist a sequence of control inputs for which the constraints are satisfied). Also, even if the optimization problem is always feasible, the computed optimal control inputs may not lead to an asymptotically stable closed loop system. In general, stability and feasibility are not ensured in (10).

Here, we study how the terminal weight Q_f and the terminal constraint set $\tilde{\mathcal{Z}}_f$ should be chosen such that closed loop stability and feasibility are ensured for all $\gamma \in \Gamma$. Conditions on how the terminal weight Q_f and the terminal constraint set $\tilde{\mathcal{Z}}_f$ should be chosen, such that closed loop stability and feasibility are ensured, have been extensively studied in the past three decades [13]. The main idea is to find a controller F that stabilizes the unconstrained system (7) inside an invariant terminal region $\tilde{\mathcal{Z}}_f$. Then, the closed loop is stable if the cost function (10a) accounts for the infinite-horizon cost. The finite-horizon cost (when the controller is constrained) is added to the infinite-horizon cost (when the controller is unconstrained (i.e., when the system enters $\tilde{\mathcal{Z}}_f$)). Then, for an unconstrained linear system the convergence to the origin is exponential.

III. AUTONOMOUS DRIVING APPLICATION

In this section, we apply the general framework presented in Section II to an autonomous driving application.

A. Spatial-based vehicle model

Fig. 1 depicts the relation between the *global* and the *road-aligned* coordinate frames. The derivation of the spatial-based vehicle model follows [9]. The main idea is to introduce the variable s representing the distance along the reference and model the lateral e_y and heading displacement e_ψ between the vehicle and the road as a function of the space s .

We start by defining the vehicle model in the time-domain. The movement of a car-like nonholonomic vehicle at low speeds (i.e., when the lateral dynamics have negligible influence) is approximately described by its time-domain kinematic equations [21] given by

Fig. 1. A nonlinear dynamic bicycle model including the representation of the road-aligned coordinate system.

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = v \cos(\psi), \\ \dot{y} &= \frac{dy}{dt} = v \sin(\psi), \\ \dot{\psi} &= \frac{d\psi}{dt} = \frac{v}{l} \tan(\delta), \end{aligned} \quad (12)$$

where x and y are the coordinates of the vehicle in the global coordinate system, ψ is the yaw angle, l is the distance between the front and rear axle, v is the longitudinal velocity in the vehicle coordinate system, and δ is the steering angle of the front wheels. The vehicle curvature κ is related with the vehicle steering angle δ by $\kappa = \frac{\tan(\delta)}{l}$.

Accordingly to Fig. 1, we can geometrically derive the relations

$$\begin{aligned} \dot{e}_y &= v \sin(e_\psi), \\ \dot{e}_\psi &= \dot{\psi} - \dot{\psi}_s, \\ \dot{s} &= \frac{\rho_s v \cos(e_\psi)}{\rho_s - e_y}, \end{aligned} \quad (13)$$

where ρ_s is the radius of curvature of the road and ψ_s is the road heading angle.

Assuming that $v \neq 0$ and that it is a continuous function, and observing that the spatial derivative can be expressed as a function of the time derivative, namely $\frac{d(\cdot)}{ds} = \frac{d(\cdot)}{dt} \frac{dt}{ds} = \frac{d(\cdot)}{dt} \frac{1}{\dot{s}}$, the spatial-based representation of (12) can then be derived as

$$\begin{aligned} e'_y &= \frac{\dot{e}_y}{\dot{s}} = \frac{\rho_s - e_y}{\rho_s} \tan(e_\psi), \\ e'_\psi &= \frac{\dot{e}_\psi}{\dot{s}} = \frac{(\rho_s - e_y)\kappa}{\rho_s \cos(e_\psi)} - \psi'_s. \end{aligned} \quad (14)$$

We linearize and discretize the nonlinear model (14) (with $\Delta s = vT_s$, where T_s is the sampling period, and v is constant) around a reference trajectory $z_r(k) = [e_{\psi,r}(k), e_{y,r}(k)]^\top = [0, 0]^\top$ for all $k \geq 0$ given by a reference sequence of inputs $U_r(k)$. Thus, we obtain a LTV model of the form (7)

$$\begin{bmatrix} e_y(k+1) \\ e_\psi(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \Delta s \\ -\kappa_r^2(k)\Delta s & 1 \end{bmatrix} \begin{bmatrix} e_y(k) \\ e_\psi(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta s \end{bmatrix} \tilde{\kappa}(k), \quad (15)$$

Fig. 2. Illustration of an autonomous driving situation using an MPC controller. The vehicle motion is predicted using a piecewise linear vehicle model. From the MPC point of view, after the end of the prediction horizon, the path is unknown. However, the path can be assumed to have limited curvature (from $\kappa_{\min}$ to $\kappa_{\max}$). Then, with this information, we can construct a multi-plant description (9) of the vehicle model. From that, it is possible to construct an invariant set to be used as a terminal constraint $\tilde{\mathcal{Z}}_f$ of the MPC controller. By enforcing the last point of the predicted path to lie inside $\tilde{\mathcal{Z}}_f$, stability is proven.

where it is assumed that the reference input curvature $\kappa_r = \kappa_s = \frac{1}{\rho_s}$. The parameter $\xi(k)$ is, in this particular application, the road curvature $\kappa_s(k)$. In fact, Δ_s could also be seen as a parameter, but we opted to set it as constant in our analysis (i.e., the vehicle travels with constant speed).

In summary, we use MPC formulation (10) to regulate a nonholonomic vehicle to the given reference using a linearized and discretized spatial-based kinematic vehicle model (15). The state vector is $\tilde{z}(k) = [e_y(k), e_\psi(k)]^\top$ and the input is $\tilde{u}(k) = \tilde{\kappa}(k) = \kappa(k) - \kappa_r(k)$. Therefore, knowing the reference curvature $\kappa_s(k)$ *a priori* allows the definition of the set Ξ , where $\xi_{\min} = \kappa_{r,\min}$ and $\xi_{\max} = \kappa_{r,\max}$. Fig. 2 illustrates the idea of imposing the last point of the MPC prediction horizon to lie inside the terminal constraint set $\tilde{\mathcal{Z}}_f$. Together with a proper choice of Q_f , closed loop stability and feasibility are ensured for all $\gamma \in \Gamma$.

B. Computation of the terminal constraint $\tilde{\mathcal{Z}}_f$

When the model is LTI, a typical choice for $\tilde{\mathcal{Z}}_f$ is the maximal positive invariant set $\mathcal{O}_{\infty}^{\text{LQR}}$ for the closed loop system (11), where $K(t) = F$, where F is the associated linear quadratic regulator (LQR) gain (i.e., the unconstrained infinite time optimal controller gain).

However, since our model is LTV, there are several maximal positive invariant sets $\mathcal{O}_{\infty}^{\text{LQR}}(\gamma)$ and several different LQR feedback controllers $F(\gamma)$, one for each $\gamma \in \Gamma$. Therefore, we are interested in the maximal positive invariant set $\bar{\mathcal{O}}_{\infty}^{\text{LQR}}$ that is invariant for all $\gamma \in \Gamma$ (i.e., $\bar{\mathcal{O}}_{\infty}^{\text{LQR}} \subseteq \text{Pre}_{\gamma}(\mathcal{O}_{\infty}^{\text{LQR}})$, $\forall \gamma \in \Gamma$). The notation $\text{Pre}_{\gamma}(\cdot)$ denotes the one-step controllable set (see Definition 1) using a specific model γ .

Recall the geometric condition for invariance expressed in Theorem 1.1 and let Γ_d be a discretized version of the multi-plant Γ with a finite number of LTI models. Then, the following recursion

$$\Omega_{k+1} = \bigcap_{\gamma \in \Gamma_d} \text{Pre}_{\gamma}(\Omega_k) \cap \Omega_k, \quad \Omega_0 = \tilde{\mathcal{Z}},$$

Fig. 3. Maximal positive invariant set for each and for all $\gamma \in \Gamma$.

eventually converges to the maximal positive invariant set (see Definition 4) [22], i.e.,

$$\bar{\mathcal{O}}_{\infty}^{\text{LQR}} = \lim_{k \rightarrow \infty} \Omega_k.$$

Note that, in the specific case of the model (15), the different $\mathcal{O}_{\infty}^{\text{LQR}}$ are symmetric with respect to κ_r . It is only necessary to evaluate (the positive or the negative) half of the interval Ξ due to the term $-\kappa_r^2$ in the matrix A of (15). In fact, we only need to compute $\bar{\mathcal{O}}_{\infty}^{\text{LQR}}$ based on the $\mathcal{O}_{\infty}^{\text{LQR}}$ for $\kappa_r = 0$ and $\kappa = \kappa_{r,(\max, \min)}$ as the invariant set at the parameter limits is also invariant to all the parameter values in between.

Fig. 3 depicts an example of the computation of a $\bar{\mathcal{O}}_{\infty}^{\text{LQR}}$ for the model (15), where $|\kappa_r| \leq 0.18$, $Q = I$, and $R = 1$. The operations with sets are performed using the MPT toolbox for MATLAB [23]. The figure also shows some $\mathcal{O}_{\infty}^{\text{LQR}}$ for different κ_r . The geometric interpretation of the computed set is intuitive. The larger the lateral displacement e_y the larger the displacement in e_ψ as well. Note that, if the vehicle is on the *left* of the path (positive e_y), the heading displacement e_ψ is negative, so that the vehicle always points *inwards* to the path, and vice-versa.

1) *Influence of MPC parameter tuning:* We investigate the influence of weighting matrices Q and R in the computation of $\bar{\mathcal{O}}_{\infty}^{\text{LQR}}$, as they directly influence the LQR control law set $\tilde{\mathcal{F}}$. The MPC tuning is no longer limited to choosing those that yield closed loop stability. In fact, we discuss in Section IV that all the MPC tunings will lead to stability but with different performance. Fig. 4 shows the influence of R and Q on the shape of $\bar{\mathcal{O}}_{\infty}^{\text{LQR}}$. The larger the deviations from the reference input, reflected in R , are penalized, the bigger the set $\bar{\mathcal{O}}_{\infty}^{\text{LQR}}$. If the interpretation is made in the light of using $\tilde{\mathcal{Z}}_f = \bar{\mathcal{O}}_{\infty}^{\text{LQR}}$ as the terminal set in the MPC controller (10), a big terminal set is desired, as the convergence region in which stability can be assured is bigger; however, it is not beneficial to assure stability in cases where the vehicle is too far away from the reference. The higher the cost of being far from the reference (i.e.,

Fig. 4. Maximum positive invariant set for all $\gamma \in \Gamma$. On the left, the input penalization matrix R is varied. On the center, the first element of the state penalization matrix Q , which is related to the lateral displacement e_y , is varied. On the right, the second element of the state penalization matrix Q , which is related to the heading displacement e_ψ , is varied.

Fig. 5. Maximum positive invariant set for all $\gamma \in \Gamma$ for different sampling distances $\Delta_s = vT_s$.

increasing Q_{11}), the smaller the invariant set. Again, in the light of our problem, this is expected. If the lateral deviation from the reference is highly penalized, the controller is too aggressive, resulting in a smaller invariant region and, consequently, in a smaller attraction region, which can lead to problem infeasibility. Finally, the influence of the heading displacement is reflected in a larger e_y region covered and smaller e_ψ region covered. The interpretation lies in the fact that, if heading displacement is expensive, then the invariant set avoids including large e_ψ .

2) *Influence of speed:* We assume constant speed throughout the prediction horizon. This is a valid assumption within the prediction horizon distance, since in autonomous heavy duty-vehicle applications, the driving is not aggressive and the vehicle has (very) slow dynamics. However, the vehicle drives at different speeds, so we investigate how different velocities influence the shape of the maximum positive invariant set of the system (15). The maximum positive invariant set for each (discretized) speed can be computed offline and stored in a look-up table. Then, depending on the current vehicle speed, the terminal state constraint is chosen accordingly. Fig. 5 shows different $\bar{\mathcal{O}}_\infty^{\text{LQR}}$ for different speeds (equivalently the sampling distance Δ_s). It can be seen that the higher the speed the more skewed the shape of the invariant set. As the speed increases, the prediction distance also increases and consequently, the invariant set is typically larger allowing larger e_ψ that are directly correlated with bigger e_y .

C. Computation of the terminal cost Q_f

When the model is LTI (i.e., when for a specific $\xi \in \Xi$), it can be shown that the control law $u(t) = F(\xi)\tilde{z}(t)$ results in an infinite-horizon cost given by

$$J_\infty^*(\tilde{z}(t)) = \tilde{z}(t)^\top P(\xi)\tilde{z}(t) = \sum_{k=t}^{\infty} \tilde{z}_{k|t}^\top Q \tilde{z}_{k|t} + \tilde{u}_{k|t}^\top R \tilde{u}_{k|t}, \quad (16)$$

where $P(\xi)$ is given by the solution of the algebraic Riccati equation for the system (7) for a specific $\xi \in \Xi$ (i.e., for a specific $\gamma \in \Gamma$). Therefore, the terminal cost $\tilde{z}_{t+N|t}^\top Q_f \tilde{z}_{t+N|t}$ is typically chosen as the solution of the algebraic Riccati equation for the system (7) for a given $\xi \in \Xi$.

In the case of LTV systems, we need to determine an upper-bound of the infinite-horizon predicted cost for all $\gamma \in \Gamma$. In other words, we determine the *worst* time-invariant model if used as a prediction model, which would lead to

the largest value of the value function among all models in the multi-model description. Therefore, we need to solve

$$\begin{aligned} \tilde{z}(t)^\top \bar{P} \tilde{z}(t) &\geq \max_{\xi \in \Xi} \{ \\ \min_{\tilde{U}_t} &\sum_{k=t}^{\infty} \tilde{z}_{k|t}^\top Q \tilde{z}_{k|t} + \tilde{u}_{k|t}^\top R \tilde{u}_{k|t} \\ \text{subject to } &\tilde{z}_{k+1|t} = A(\xi(k|t)) \tilde{z}_{k|t} + B(\xi(k|t)) \tilde{u}_{k|t}, \\ &\tilde{z}_{t|t} = \tilde{z}(t) \}. \end{aligned} \quad (17)$$

However, we know that the infinite-horizon cost is defined as (16). Also, the cost of applying the LQR control law to the system is evaluated in an unconstrained situation, (i.e., when the system is in the maximal positive invariant set $\tilde{\mathcal{Z}}_f$). Thus, we can rewrite (17) as

$$\begin{aligned} \tilde{z}(t)^\top \bar{P} \tilde{z}(t) &\geq \max_{\xi \in \Xi} \tilde{z}(t)^\top P(\xi) \tilde{z}(t) \\ \text{subject to } &\tilde{z} \in \tilde{\mathcal{Z}}_f. \end{aligned} \quad (18)$$

Since $\tilde{\mathcal{Z}}_f$ is a convex polytope, the problem (18) can be recast only considering the vertices $\mathcal{V}_z = \{v_{z,1}, v_{z,1}, \dots, v_{z,N_v}\}$ of the polytope, where $v_{z,i} \in \mathbb{R}^2$, $\forall i = \{1, \dots, N_v\}$, defines a vertex coordinates of the polytope $\tilde{\mathcal{Z}}_f$. Hence,

$$\begin{aligned} \tilde{z}(t)^\top \bar{P} \tilde{z}(t) &\geq \max_{\xi \in \Xi} \tilde{z}(t)^\top P(\xi) \tilde{z}(t) \\ \text{subject to } &\tilde{z}_t \in \mathcal{V}_z. \end{aligned} \quad (19)$$

The problem (19) can be recast as an Euclidean distance maximization by performing a change of variables such that $w_t = L(\xi) \tilde{z}_t$, where $L(\xi)$ is a lower triangular matrix obtained by the Cholesky decomposition of the matrix $P(\xi)$. Thus,

$$\begin{aligned} \tilde{z}(t)^\top \bar{P} \tilde{z}(t) &\geq \max_{\xi \in \Xi} w_t^\top w_t \\ \text{subject to } &w_t \in \mathcal{V}_w, \end{aligned} \quad (20)$$

where $\mathcal{V}_w = \{Lv_{z,1}, Lv_{z,2}, \dots, Lv_{z,N_v}\}$.

To find the closed-form expression for $P(\xi)$ for all $\xi \in \Xi$ we solve symbolically (i.e., as a function of ξ) the associated algebraic Riccati equation and chose the only positive definite solution. Then, the problem (20) is reduced to finding ξ that gives the largest distance to all the vertices $\mathcal{V}_w$ at the same time. The problem (20), when solved with respect to a convex set computed as in Section III-B for the system (15) is, in fact, a convex problem within the domain of Ξ , and therefore the maximum is obtained at one of the bounds of Ξ , as shown in Fig. 6. There, the value function is normalized with respect to the minimum cost for each κ_r , which is set to 0 for ease of interpretation. Thus, $\bar{P} = P(\xi^*)$, where ξ^* is one of the bounds of Ξ , gives the largest infinite-horizon predicted cost with respect to all systems $\gamma \in \Gamma$, and associated with it there exists a control law $u(t) = \bar{F} \tilde{z}(t) = F(\xi^*) \tilde{z}(t)$.

D. LTV-MPC Stability and Feasibility

In this section, we prove stability of the proposed LTV-MPC (10) using Lyapunov techniques. Again, we follow the notation and argumentation flow used in [20]. The terminal constraint and terminal cost are determined via the methods explained in Section III.

Fig. 6. Normalized value function of the optimization problem (20), using $\tilde{\mathcal{Z}}_f$ equal to $\bar{\mathcal{O}}_\infty^{\text{LQR}}$ shown in Fig. 3. In this case, $\tilde{\mathcal{Z}}_f$ has six different vertices, but there are only three distance plots due to vertices symmetry with respect to the origin.

Assumption 1: The state penalization matrices Q and Q_f and the input penalization matrix R are symmetric positive definite, i.e., $Q = Q^\top \succ 0$, $Q_f = Q_f^\top \succ 0$, and $R = R^\top \succ 0$.

Assumption 2: The sets $\tilde{\mathcal{Z}}$, $\tilde{\mathcal{Z}}_f$, and $\mathcal{U}$ are closed and contain the origin in their interior.

Assumption 3: $\tilde{\mathcal{Z}}_f$ is control invariant and $\tilde{\mathcal{Z}}_f \subset \tilde{\mathcal{Z}}$.

Assumption 4: There is no model mismatch between the prediction model and the plant model, i.e.,

$$\begin{aligned} A(\xi(k|t)) &= A(\xi(k-n|t+n)) \\ B(\xi(k|t)) &= B(\xi(k-n|t+n)), \end{aligned} \quad (21)$$

for all $t > 0$, $k = t, \dots, t + N - 1$, and $n = 0, \dots, k - 1$.

To prove stability of the MPC controller under the proposed terminal constraints and terminal cost, we need to ensure recursive feasibility of the controller (i.e., the computed control sequence U_{t+1}^* is feasible at time $t+1$ whenever the optimization at time t is feasible).

Theorem 3.1 (LTV-MPC Feasibility): The problem (10) is feasible for all $t \geq 0$ if $Q_f = \bar{P}$, $\tilde{\mathcal{Z}}_f = \bar{\mathcal{O}}_\infty^{\text{LQR}}$, and if $z(0) \in \mathcal{K}_N(\bar{\mathcal{O}}_\infty^{\text{LQR}})$.

Proof: If $z(0) \in \mathcal{K}_N(\bar{\mathcal{O}}_\infty^{\text{LQR}})$, then the system is feasible at $t = 0$ (i.e., there exists a sequence of N inputs that brings the system to the set $\bar{\mathcal{O}}_\infty^{\text{LQR}}$). By definition of $\bar{\mathcal{O}}_\infty^{\text{LQR}}$, the system constraints are instantaneously satisfied at all points in $\bar{\mathcal{O}}_\infty^{\text{LQR}}$, and $\bar{\mathcal{O}}_\infty^{\text{LQR}}$ is invariant under the control law $u(t) = \bar{F} \tilde{z}(t)$. Thus, the system is feasible for all $t \geq 0$. ■

Theorem 3.2 (LTV-MPC stability): Consider the model (7), its constraints (2), and the LTV-MPC controller (10). Let Assumptions 1 - 4 hold. The terminal penalty matrix Q_f is chosen as $\bar{P}$, and the terminal constraint $\tilde{\mathcal{Z}}_f$ is chosen as $\bar{\mathcal{O}}_\infty^{\text{LQR}}$. Then, the state of the closed loop system converges to the origin. Moreover, the origin of the closed loop system is asymptotically stable with domain of attraction $\mathcal{K}_N(\bar{\mathcal{O}}_\infty^{\text{LQR}})$.

Proof: Consider problem (10) at time t . We drop the argument $\xi(t)$ from A and B , and γ from F for the sake of simplicity. Let $z(t) \in \mathcal{K}_N(\bar{\mathcal{O}}_\infty^{\text{LQR}})$ and let $\tilde{U}_t^* = \{\tilde{u}_{t|t}^*, \dots, \tilde{u}_{t+N-1|t}^*\}$ be the optimizer of problem (10) and $\tilde{Z}_t^* = \{\tilde{z}_{t|t}^*, \dots, \tilde{z}_{t+N|t}^*\}$ be the corresponding optimal state trajectory. After the implementation of $\tilde{u}_t^*$ and using Assumption 4, we obtain $\tilde{z}(t+1) = \tilde{z}_{t+1|t} = A\tilde{z}(t) + B\tilde{u}_{t|t}^*$. Let $J_N^*(\tilde{z}(t))$ be the optimal total predicted cost of (10a) when applying the $\tilde{U}_t^*$ to the system state $z(t)$.

Consider now problem (10) for $t+1$. We construct an upper bound on $J_N^*(z(t+1))$. Consider the sequence $\tilde{U}_{t+1} = \{\tilde{u}_{t+1|t}^*, \dots, \tilde{u}_{t+N|t}^*, \bar{F}\tilde{z}_{t+N|t}^*\}$ and the corresponding state trajectory resulting from the initial state $z(t+1)$, $\tilde{Z}_{t+1} = \{\tilde{z}_{t+1|t}^*, \dots, \tilde{z}_{t+N|t}^*, (A+B\bar{F})\tilde{z}_{t+N|t}^*\}$. Let $J_N(\tilde{z}(t+1))$ be the predicted cost (10a) when applying $\tilde{U}_{t+1}$ to $z(t+1)$. We show that $J_N^*(\tilde{z}(t))$ is a Lyapunov function for the closed loop system. Using the results obtained in Section III-C it is possible to derive the following result,

$$\begin{aligned} J_N(\tilde{z}(t+1)) &= J_N^*(\tilde{z}(t)) - (\tilde{z}_{t|t}^\top Q \tilde{z}_{t|t} + \tilde{u}_{t|t}^\top R \tilde{u}_{t|t}^*) \\ &\quad + \tilde{z}_{t+N|t}^\top ((A+B\bar{F})^\top \bar{P} (A+B\bar{F}) \\ &\quad + Q + \bar{F}^\top R \bar{F} - \bar{P}) \tilde{z}_{t+N|t}, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \tilde{z}_{t+N|t}^\top \bar{P} \tilde{z}_{t+N|t} &\geq \tilde{z}_{t+N|t}^\top ((A+B\bar{F})^\top \bar{P} (A+B\bar{F}) \\ &\quad + Q + \bar{F}^\top R \bar{F}) \tilde{z}_{t+N|t} > 0, \end{aligned} \quad (23)$$

and therefore, since $\bar{P}$ gives the largest infinite-horizon predicted cost with respect to all systems $\gamma \in \Gamma$,

$$J_N(\tilde{z}(t+1)) \leq J_N^*(\tilde{z}(t)) - (\tilde{z}_{t|t}^\top Q \tilde{z}_{t|t} + \tilde{u}_{t|t}^\top R \tilde{u}_{t|t}^*), \quad (24)$$

and since $J_N^*(\tilde{z}(t+1)) \leq J_N(\tilde{z}(t+1))$,

$$J_N^*(\tilde{z}(t+1)) - J_N^*(\tilde{z}(t)) \leq -(\tilde{z}_{t|t}^\top Q \tilde{z}_{t|t} + \tilde{u}_{t|t}^\top R \tilde{u}_{t|t}^*), \quad (25)$$

where the right-hand side is clearly negative definite due to the positive definiteness of Q and R .

Equation (25) and the Assumption 1 ensure that $J_N^*(\tilde{z}(t))$ strictly decreases along the state trajectories of the closed loop system (11) for any $\tilde{z} \in \mathcal{K}_N(\bar{\mathcal{O}}_\infty^{\text{LQR}})$, $\tilde{z} \neq 0$. In addition to the fact that $J_N^*(\tilde{z}(t))$ decreases, it is also lower-bounded by zero and since the state trajectories generated by the closed loop system (11) starting from any $\tilde{z}(0) \in \mathcal{K}_N(\bar{\mathcal{O}}_\infty^{\text{LQR}})$ lie in $\mathcal{K}_N(\bar{\mathcal{O}}_\infty^{\text{LQR}})$ for all $t \geq 0$. Equation (25), and Assumptions 2 and 3 are sufficient to ensure that the state of the closed loop system converges to the origin as $t \rightarrow \infty$ if the initial state lies in $\mathcal{K}_N(\bar{\mathcal{O}}_\infty^{\text{LQR}})$.

To show that the origin of the closed loop system (11) is asymptotically stable, we resort again to Lyapunov theory. $J_N^*(\tilde{z}(t))$ is positive definite due to Assumption 1. Furthermore, (25) states that $J_N^*(\tilde{z}(t))$ decreases along the closed loop trajectories. Then, $\tilde{z}_{t|t}^\top Q \tilde{z}_{t|t} + \tilde{u}_{t|t}^\top R \tilde{u}_{t|t}^* \rightarrow 0$ as $t \rightarrow \infty$. ■

Fig. 7. Modified Scania G480 construction truck used as experimental and research platform.

Fig. 8. Simulation of the MPC controller (10) using the vehicle model (15). The goal is to track a straight line with constant speed of 10 m/s, with different MPC parameter tuning and evaluate the effectiveness of using the stability conditions derived in Sections III-B and III-C. The vehicle starts with a lateral displacement of 1 meter from the path.

IV. SIMULATION EXAMPLE

In this section, we demonstrate the effectiveness of using the terminal constraint $\tilde{Z}_f = \bar{\mathcal{O}}_\infty^{\text{LQR}}$ and cost $Q_f = \bar{P}$ through an illustrative example in simulation. In this example, we simulate the behaviour of a 4-axes nonlinear bicycle model with 2 steering axles in the front and 2 traction axles in the back, which is based on a modified Scania G480 construction truck [4] that can be seen in Fig. 7.

The main purpose of the example is to show the effectiveness of the stability conditions derived in this paper. There-

fore, driving performance (e.g., accuracy or smoothness) is not the focus of our analysis, but yet we seek clear evidence of (in)stability. The example consists in driving a straight line with constant speed of 10 m/s. The vehicle starts with a 1 meter lateral displacement from the reference path. We compare the MPC controller (10) with and without terminal constraint and cost. The road-aligned kinematic model (14) is used as a prediction model. The only parameter modified in each simulation is the first element of the state penalization matrix Q_{11} , which accounts for the penalization of the vehicle lateral displacement with respect to the reference path. The other parameters are $Q_{22} = 10$ and $R = 10$, and the prediction horizon is $N = 7$ steps, using a spatial discretization of $\Delta_s = 1$ m. The controller runs at 50 Hz.

Fig. 8 shows the deviation from the path and the steering angle computed during the simulation. It is clear that the performance differences between using or not the stability conditions when $Q_{11} = 1$ are marginal. However, at $Q_{11} = 5$ the differences in performance are more noticeable, and it is clear that while the controller with stability conditions converges much faster to the reference path, the controller without stability conditions takes much longer and oscillates much more. Eventually, by increasing Q_{11} , the controller becomes unstable, and this is demonstrated when $Q_{11} = 10$. At about 6 seconds of simulation, the vehicle doubled the initial lateral displacement from the reference path and the simulation stops. Using the stability conditions, the behaviour is very similar with $Q_{11} = 5$ and the controller appears stable.

In this case, the prediction model is always, in fact, LTI, since the road is straight. However, from the point of view of the MPC, the reference is unknown after the last point of the prediction horizon. The remaining reference path can have any (bounded) curvature, since the derived stability conditions are precisely applied to the last point. Consequently, stability is always guaranteed accordingly to Theorem 3.2.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we investigated the stability conditions (i.e., terminal constraint and cost) when designing an LTV-MPC for reference tracking in an autonomous vehicle. To predict the vehicle motion in the MPC, we derived a nonlinear spatial-based kinematic model on a road-aligned coordinate frame. The determination of the terminal constraint and cost is performed using a multi-model representation to span the possible vehicle models within a predefined domain. We proposed an algorithm to compute the maximal positive invariant set for all the models in the multi-model description and chose it as a terminal constraint. Also, we used a convex min-max optimization to determine the *worst* time-invariant model if used as a prediction model, and, with that, we computed the terminal cost. With this conditions, we proved stability of a reference tracking LTV-MPC through Lyapunov arguments. Finally, a simulation example is presented, where the effectiveness of the stability conditions is clear. When the stability conditions are not used, there are tuning combinations that led to controller instability. On the other hand, when using the stability conditions, the controller was, by design, stable.

As future work, we would like to deploy this controller in a real vehicle to verify the effectiveness of such conditions. Also, it would be very interesting to investigate a way to compute the set of tuning parameters lead to controller

instability without the stability conditions. Also, we should investigate these conditions in the light of robust MPC to mitigate the influence of model uncertainties in the controller performance.

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