Uncertainty Analysis of the Aerodynamic Coefficients

Filip Söderman*

KTH Royal Institute of Technology,
SE-100 44 Stockholm, Sweden

This thesis treats an error propagation analysis used to estimate the uncertainty of the aerodynamic coefficients. The propagation methods used in this analysis are a Taylor Series Method and a Monte Carlo Method. The Taylor Series Method uses the partial derivatives of each input variable whereas the Monte Carlo Method uses random and repeated samples from the probability density function of each variable. By comparing the results obtained by the different methods, the results can be validated. Coverage intervals with a coverage probability of 95% are calculated along with the percentage contribution each input variable has on the expanded uncertainty. The results showed that the uncertainty of the coefficients varied between 10% and 20% and negligible differences between the methods were observed. More accurate measurements of the dynamic pressure and the position of the center of gravity are needed in order to decrease the uncertainty.

Nomenclature

- \( m \) Aircraft mass
- \( p, q, r \) Angular velocities
- \( u, v, w \) Velocity components
- \( I_{ii} \) Moment of inertia
- \( I_{ij} \) Product of inertia
- \( F_x, F_y, F_z \) External forces
- \( M_x, M_y, M_z \) External moments
- \( n_x, n_y, n_z \) Load factors
- \( F_{xE}, F_{yE} \) Thrust forces
- \( M_{xE}, M_{yE}, M_{zE} \) Moments due to thrust forces
- \( q_0 \) Dynamic pressure
- \( S \) Wing area
- \( b \) Wing span
- \( c \) Mean aerodynamic chord
- \( \epsilon_Y \) Total error in \( Y \)
- \( u_{X_i} \) Standard uncertainty in \( X_i \)
- \( \theta_i \) Sensitivity coefficient
- \( u_c \) Combined standard uncertainty
- \( k \) Coverage factor
- \( U \) Expanded uncertainty
- \( g_Y \) Probability density function
- \( p_p \) Coverage probability
- \( I_p \) Coverage interval

I. Introduction

In dynamic aircraft models several parameters are included that in turn quantify the dependence of aerodynamic forces and moments on the state and control variables. The parameters are in many cases obtained from test flights and wind tunnel tests and to be able to build a valid model it is very important that the accuracy of these parameters are of high order. The concepts of error and error analysis have been a part of metrology for a long time whereas uncertainty is a relatively new quantifiable attribute in metrology [1]. Evaluating the uncertainty is and always has been an important part of a test process to be able to assess its reliability, quality and traceability. Despite using appropriate corrections to known or suspected components of the errors, there still remains doubt about the correctness of a result. By evaluating the uncertainty of a measurement, the degree of goodness can be determined.

With the aerospace industry being a global marketplace it is imperative to have a uniform method for both evaluating and expressing the uncertainty. Results can then be easily compared in all parts of the world. Therefore, the International Organization for Standardization (ISO) addressed the problem and published in 1995, a guide for evaluating the uncertainty. Since then, the Joint Committee for Guides in Metrology (JCGM)\(^1\) has taken over the responsibility and the current version [1] was used in this report. It provides internationally agreed recommendations for evaluation of uncertainty. Additional guides and supplements have been used in this report [2]-[4].

The goal of this thesis is to increase the knowledge concerning the uncertainty of the aerody-

---

*M.Sc. Student Aerospace Engineering, KTH Royal Institute of Technology

\(^1\)An organization composed of several broadly-based international organizations with the task to “Develop and maintain guidance documents addressing the general metrological needs of science and technology.”[3]
namic coefficients. This entails in a better and more accurate evaluation of the results once a flight test is conducted. Evaluation of the data during a flight test is done partly to see the outcome of the test and partly to assess the opportunities and risks during the test. To be able to draw correct conclusions, a deeper knowledge is needed along with the possibility to identify error sources and minimize their effect.

II. Models

The aircraft’s motion can be described by forming the equations of motion. A short derivation of the forces and moments acting on the aircraft will be performed in the following section. The methods used to perform the uncertainty analysis are presented along with a flowchart illustrating the process. A brief outline of the aerodynamic model will also be given.

A. Flight Mechanics

The derivation of the aircraft’s equations of motion are based on Newtonian mechanics considering rigid body motion. Further assumptions made in this report include neglecting time derivatives of mass and inertia, a flat non-accelerating earth and a non-rotating earth. Using these assumptions gives that the force equations can be written as

\[
\begin{align*}
F_x &= m(\dot{u} + qw - rv) \\
F_y &= m(\dot{v} + ru - pw) \\
F_z &= m(\dot{w} + pw - qu)
\end{align*}
\]

where \( m \) is the mass of the aircraft, \( p, q, r \) are angular velocities and \( u, v, w \) are the velocity components. The moment equations are given by

\[
\begin{align*}
M_x &= \dot{p}I_{xx} - \dot{q}I_{xy} - \dot{r}I_{xz} + qr(I_{zz} - I_{yy}) + (r^2 - q^2)I_{yz} \\
&- pqI_{xz} + rpI_{xy} - M_{xE} - N\Delta y_A - C\Delta z_A \\
M_y &= -\dot{p}I_{xy} + \dot{q}I_{yy} - \dot{r}I_{yz} + rp(I_{xx} - I_{zz}) + (p^2 - r^2)I_{xy} \\
&- qrI_{yz} + pqI_{xz} + rH_m - M_{yc} + T\Delta z_A + N\Delta x_A \\
M_z &= -\dot{p}I_{xz} - \dot{q}I_{xy} + \dot{r}I_{yz} + pq(I_{yy} - I_{xx}) + (q^2 - p^2)I_{xz} \\
&- qrI_{xy} - pqI_{yz} + qrI_{xx}
\end{align*}
\]

where \( I_{ii} \) and \( I_{ij} \) are the mass moments of inertia and products of inertia of the aircraft body respectively. Equations (1)-(2) represent the total external forces and moments acting on the aircraft. To obtain the aerodynamic forces and moments used to calculate the aerodynamic coefficients, the contribution from the engine thrust and gravity has to be subtracted. Using the definition of the load factor, the aerodynamic forces \( T, C, \) and \( N \) can be written as

\[
\begin{align*}
T &= mg_n + F_{xE} \\
C &= mg_n + F_{yE} \\
N &= mg_n + F_{zE}
\end{align*}
\]

where \( F_{xE} \) and \( F_{yE} \) are the thrust force components from the engine and \( n_i \) are the load factor components. The aerodynamic moments \( l, m, n \) can be written as

\[
\begin{align*}
l &= \dot{p}I_{xx} - \dot{q}I_{xy} - \dot{r}I_{xz} + qr(I_{zz} - I_{yy}) + (r^2 - q^2)I_{yz} \\
&- pqI_{xz} + rpI_{xy} - M_{xE} - N\Delta y_A - C\Delta z_A \\
m &= -\dot{p}I_{xy} + \dot{q}I_{yy} - \dot{r}I_{yz} + rp(I_{xx} - I_{zz}) + (p^2 - r^2)I_{xy} \\
&- qrI_{yz} + pqI_{xz} + rH_m - M_{yc} + T\Delta z_A + N\Delta x_A \\
n &= -\dot{p}I_{xz} - \dot{q}I_{xy} + \dot{r}I_{yz} + pq(I_{yy} - I_{xx}) + (q^2 - p^2)I_{xz} \\
&- qrI_{xy} - pqI_{yz} + qrI_{xx}
\end{align*}
\]

where \( M_{xE}, M_{yc}, \) and \( M_{zE} \) are the moment components due to the thrust force from the engine, \( H_m \) is the angular momentum due to the spinning rotors and \( \Delta i_A \) are the distances between the aerodynamic reference point and the center of gravity in the S85-system\(^2\). Figure 1 illustrates the aerodynamic forces and moments acting on the aircraft.

\[\text{Figure 1. Aerodynamic forces and moments in the body axis system [6].}\]

The aerodynamic coefficients are then found by normalizing the forces and moments according to

\[
\begin{align*}
C_T &= \frac{T}{q_a S} & C_C &= \frac{C}{q_a S} & C_N &= \frac{N}{q_a S} \\
C_l &= \frac{l}{q_a Sb} & C_m &= \frac{m}{q_a Sc} & C_n &= \frac{n}{q_a Sb}
\end{align*}
\]

where \( q_a, S, b \) and \( c \) are the dynamic pressure, wing area, wing span and mean aerodynamic chord respectively.
B. Uncertainty Analysis

The procedure for the uncertainty analysis in a multivariable case can generally be divided into different steps. The first step in the uncertainty analysis is to develop the data reduction equation (DRE) and the propagation method to use. The DRE is an equation that defines the relation between the quantity of interest and the measured quantities. Examples of DREs are the definitions of the aerodynamic coefficients in Eq. (5). The second step include identifying all possible error sources and estimating the measurement uncertainties. Finally, the last step is to compute and combine the different uncertainty components to provide an uncertainty estimate [2].

The two different methods used to propagate uncertainties through the DREs in this report are a Taylor Series Method (TSM) and a Monte Carlo Method (MCM). These two methods will be described in more detail in the following sections.

1. Taylor Series Method

A general representation of a data reduction equation is given by

\[ Y = f(X_1, X_2, ..., X_j) \]  

where \( Y \) is the measurand, determined from \( j \) quantities \( X_i \) through the relationship specified by \( f \). Each of the input variables \( X_i \) upon which \( Y \) is dependent, may themselves depend on other quantities leading to a complex relationship \( f \). Every input variable have a different standard uncertainty associated with it which then propagates through the DRE and thereby generating an uncertainty in the measurand \( Y \).

Using a Taylor Series expansion including only the first-order terms from the expansion and where the partial derivatives are evaluated at the measured values and not the actual true value, gives that the total error in \( Y \) can be written as

\[ \epsilon_Y = \frac{\partial Y}{\partial X_1} \epsilon_{X_1} + \frac{\partial Y}{\partial X_2} \epsilon_{X_2} + \ldots + \frac{\partial Y}{\partial X_j} \epsilon_{X_j} \]  

where \( \epsilon_{X_i} \) represent the total error in the corresponding input variable. The total error for a given input variable \( X_i \) is given by the sum of all contributions encountered during the measurement process and can be written as

\[ \epsilon_{X_i} = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_k \]  

Possible error sources include resolution errors, calibration errors and repeatability. These errors are random variables that follow a special probability distribution which relate the frequency of occurrence of values to the values themselves [2]. The error, \( \epsilon_{X_i} \), is related to the measured quantity \( X_i \) by the simple relation

\[ X_i = X_{i,true} + \epsilon_{X_i} \]  

Figure 2 illustrates a plot of common probability distributions.

For most practical applications, the normal distribution is a valid and relevant distribution to use. Certain criteria have to be fulfilled in order to use the uniform and triangular distributions and therefore have a more limited applicability. The error distribution gives information whether an error is likely or unlikely to occur which in turn makes it possible to estimate the uncertainty. Due to the central limit theorem, the distribution of the measurand can, as given by [1], often be considered normal even though the distribution of each input variable \( X_i \) is not normal.

The variance is in [2] defined as “the mean square dispersion of the distribution about its mean value”. Uncertainty is then defined as the square root of the variance and for a measured value \( X_i \) it can then be written as

\[ u_{X_i} = \sqrt{\text{var}(X_i)} = \sqrt{\text{var}(\epsilon_{X_i})} = u_{\epsilon_{X_i}} \]  

Equation (10) gives that the uncertainty is equal for the measured value and the error. In order to combine the uncertainties from different error sources, the variance addition rule can be applied. This gives that the combined variance for Eq.(7), assuming independent variables, can be written as

\[ \text{var}(\epsilon_Y) = \theta_1^2 u_{X_1}^2 + \theta_2^2 u_{X_2}^2 + \ldots + \theta_j^2 u_{X_j}^2 \]  

where \( \theta_i = \frac{\partial Y}{\partial X_i} \) are called sensitivity coefficients and describe how the output varies with changes in the input and \( u_{X_i} \) is the standard uncertainty of input variable \( X_i \).
The combined standard uncertainty, \( u_c \), is then given by the positive square root of the combined variance
\[
u_c = \sqrt{\sum_{i=1}^{j} \theta_i^2 u_{X_i}^2} \tag{12}
\]

As previously stated, this expression has been derived using the assumption that the input variables \( X_i \) are independent. In the case of dependent variables, the correlation effects would give a contribution to the uncertainty that has to be included.

Despite having \( u_c \) expressing the uncertainty it is often required to define an interval of uncertainty that is expected to encompass the majority of the values that could be attributed to the measurand. This is denoted the expanded uncertainty, \( U \), and is as recommended in [1] obtained at a specific coverage probability (e.g. 95% and 99%) by multiplying the combined standard uncertainty with a coverage factor, \( k \), such that
\[
U = ku_c \tag{13}
\]
Up until this point, no assumption regarding the type of error distribution has been made. This is done when choosing a value on \( k \). As previously stated, the error distribution of the measurand can often be approximated as normal. This allows the use of the t-distribution in order to get an estimation of the coverage factor \( k \). The t-distribution depends on the number of degrees of freedom. In most engineering applications the number of degrees of freedom are large enough to assume a constant value, and a 95% coverage interval will result in a coverage factor \( k = 1.96 \). Using the expanded uncertainty results in a band \( \pm U \) around the measurand that will to 95% (or the coverage probability used) contain the true value of the result.

Assuming a normal error distribution and independent input variables give that the expanded uncertainty can be written as
\[
U = \sqrt{\sum_{i=1}^{j} \theta_i^2 U_{X_i}^2} \tag{14}
\]
where \( U_{X_i} \) is the expanded uncertainty for input variable \( X_i \) with the corresponding sensitivity coefficient \( \theta_i \). The expanded uncertainty for each variable should be expressed using the same coverage factor to yield a result with the correct coverage interval. Equation (14) can then be interpreted to describe the propagation of the overall uncertainty associated with each variable into the overall uncertainty of the final result.

A nondimensionalized form of Eq.(14) is obtained by normalize it with the measurand, \( Y \). Multiplying the result with \( X_i/X_i \), gives that the relative uncertainty can be written as
\[
U = \sqrt{\sum_{i=1}^{j} \left( \frac{X_i}{Y} \theta_i \right)^2 \left( \frac{U_{X_i}}{X_i} \right)^2} \tag{15}
\]
The factor \( U_{X_i}/X_i \) express the relative uncertainty for each variable and the factor \( \frac{X_i}{Y} \theta_i \), called uncertainty magnification factors (UMFs), gives information about the influence each variable has on the uncertainty of the result. Whereas the relative uncertainty for each variable is a number less than one the UMFs can be both larger and smaller than one. An absolute value above one indicates that the influence of the uncertainty in that variable magnifies as it propagates through the DRE whereas an absolute value below one indicates that the influence of the uncertainty decreases as it propagates.

To obtain information about the percentage contribution each input variable has on the expanded uncertainty, a second nondimensional form of the expanded uncertainty can be formed. Dividing the square of the right hand side of Eq.(14) by \( U^2 \) gives that the uncertainty percentage contribution (UPC) for each input variable \( X_i \) can be defined as
\[
UPC_i = \frac{\theta_i^2 U_{X_i}^2}{U^2} \tag{16}
\]
The UPC of each variable \( X_i \) include both the effect of the UMF and the magnitude of the standard uncertainty of that variable. By comparing the values of the UPCs, information about which variable that gives the largest contribution to the expanded uncertainty is found.

2. Monte Carlo Method

The Monte Carlo method offers an alternative method for calculating the propagation of uncertainties. Whereas TSM uses a first-order Taylor Series expansion to approximate the expanded uncertainty, the MCM gives an numerical approximation of the real distribution function \( G_Y(\eta) \) by propagating of distributions. The core of this method is the use of random and repeated samples of the probability density function (PDF) for each variables \( X_i \), denoted \( g_{X_i}(\xi_i) \). After each sample, the model is evaluate and the distribution function encodes all necessary information about the
measurand $Y$. By using the approximative function, any property of $Y$, such as uncertainty and coverage interval, can be estimated [4].

Figure 3 illustrates the propagation of three input variables PDFs through the model in order to obtain the PDF $(g_Y(\eta))$ for the measurand $Y$.

Figure 3. Propagation of different PDFs through the model $f$.

The effectiveness of the method depends on the number of Monte Carlo iterations, $M$, that are made i.e the number of times the model is evaluated. It can be chosen prior to the run in which case no direct control over the results is obtained. Recommended by [4] is to use a value of $M$ that is large, for example $10^4$ times larger, compared to $1/(1 - p_p)$ with $p_p$ being the coverage probability. A convergence study should also be conducted in order to see if the result has converged towards a steady solution.

In Figure 4 a flowchart illustrates the steps involved when performing an uncertainty analysis using the MCM.

Nominal values for each variable $X_i$

Standard uncertainties for each variable $X_i$

Monte Carlo iterations (1 to $M$ iterations)

Randomly choose an error from the corresponding distribution and add to the nominal value

$X_i(0) + \varepsilon_i(0) \rightarrow X_i(0) + \varepsilon_i(0)$

$\ldots$

$X_n(0) + \varepsilon_n(0) \rightarrow X_n(0) + \varepsilon_n(0)$

Calculate result from data reduction equation

$Y(0) = f(X(0), X(0), \ldots X(0))$

No

$z = M^{-0.5}$

Yes

Calculate combined standard uncertainty, $s_{MC}$

$Y(X, X, \ldots X)$

Calculate a coverage interval, $I_p$

Figure 4. A flowchart illustrating the MCM process.

Nominal values of all variables $X_i$, the number of iterations $M$, standard uncertainties for each variable along with the corresponding error distribution are used as inputs to the MCM. Errors of each variable are then randomly chosen from the corresponding distribution and added to the nominal value. The model is then evaluated using the modified input values and iterated until $M$ iterations have been simulated. The combined standard uncertainty, $u_c$, can then be calculated as the standard deviation, $s_{MCM} = u_c$, using

$$s_{MCM} = \sqrt{\frac{1}{M-1} \sum_{i=1}^{M} (Y_i - \bar{Y})^2} \quad (17)$$

where $\bar{Y}$ is the mean value of $Y$.

The coverage interval for the measurand $Y$ can be determined from the distribution function $G_Y(\eta)$. With $p_p$ being the coverage probability, the endpoints of a $100p_p\%$ coverage interval for $Y$ is given by $G_Y^{-1}(\alpha_p)$ and $G_Y^{-1}(p_p + \alpha_p)$. Where $\alpha_p$, if defined as $\alpha_p = (1 - p_p)/2$, then gives a coverage interval defined by the $(1 - p_p)/2$ and $(1 + p_p)/2$ quantiles which in turn provide a probabilistic symmetric $100p_p\%$ coverage interval [4]. Assuming $p_p = 0.95$, that is a $95\%$ coverage interval, gives that the interval can be expressed as

$$I_p = [G_Y^{-1}(0.025), G_Y^{-1}(0.975)] \quad (18)$$

For the case when the PDF of $Y$ is symmetric about the mean value, the coverage interval will be identical to that interval given by the expanded uncertainty (Eq.(13) and Eq.(14)). In the case of an asymmetric PDF, a different value than that given by $\alpha_p = (1 - p_p)/2$ might be more appropriate to use. Suggested by [4] is then to use the method called the shortest coverage interval to determine the coverage interval. This method has the property that for a single-peak PDF it will contain the most probable value of $Y$. The value of $\alpha_p$ should then be chosen such that

$$\min_{\alpha_p \in (0, 1 - p_p)} g_Y(G_Y^{-1}(\alpha_p + p_p)) - g_Y(G_Y^{-1}(\alpha_p)) \quad (19)$$

If the result of the MCM simulations are distributed symmetrically, the probabilistic symmetric coverage interval, the interval given by the expanded uncertainty in Eq.(13) and Eq.(14) and the shortest coverage interval will all give the same results for a $100p_p\%$ coverage interval.

C. Aerodynamic model

The aerodynamic coefficients from test flights are determined by measuring the load factors, ro-
tational speeds, accelerations and together with the aircraft mass and inertial moments then computed according to Eq.(5). The values obtained from the flight test are then compared with the predictions according to the aerodynamic model. The model gives the predicted coefficients as functions of the measured flight state parameters (such as altitude, Mach number, control surface deflections, aircraft configurations etcetera.). The difference between the two results gives the modelling error which is defined as

\[ \Delta C = C_{\text{flight}} - C_{\text{model}} \] (20)

More information about the aerodynamic model is given in [6].

### III. Results

In this section, the results concerning the uncertainty analysis of the aerodynamic coefficients will be presented. The results obtained by the two methods, TSM and MCM, will be analysed and compared. Input variables that have a large impact on the final uncertainty and magnifies as they propagate through the calculations are identified and listed. For confidentiality reasons the data and results presented have been altered. The overall behaviour of the results illustrated in the figures are, however, still accurate.

#### A. Uncertainties and Coverage intervals

The uncertainty for each of the input variables are all assumed to have a normal distribution and to be independent of each other.

Each manoeuvre is analysed separately and all input variables required to perform the calculations have to be synchronised to the same sampling frequency before the calculations are performed.

The time axis used in the figures is counted in seconds since midnight and the dashed red vertical line indicate the time when the convergence study has been performed. The black lines represent the coverage interval and the aerodynamic model is represented by the magenta coloured line. The manoeuvre presented in this thesis is a combined max stick aft and roll left.

In Table 1, the nominal value, expanded uncertainty and relative uncertainty for the aerodynamic force coefficients are presented when using the TSM.

<table>
<thead>
<tr>
<th>Nominal [-]</th>
<th>Expanded [-]</th>
<th>Relative [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_T )</td>
<td>0.054777</td>
<td>0.003868</td>
</tr>
<tr>
<td>( C_C )</td>
<td>-0.04626</td>
<td>0.001932</td>
</tr>
<tr>
<td>( C_N )</td>
<td>1.021</td>
<td>0.04124</td>
</tr>
</tbody>
</table>

In Table 2, the nominal value, expanded uncertainty and relative uncertainty for the aerodynamic force coefficients are presented when using the MCM.

<table>
<thead>
<tr>
<th>Nominal [-]</th>
<th>Expanded [-]</th>
<th>Relative [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_T )</td>
<td>0.054777</td>
<td>0.003888</td>
</tr>
<tr>
<td>( C_C )</td>
<td>-0.04626</td>
<td>0.001961</td>
</tr>
<tr>
<td>( C_N )</td>
<td>1.021</td>
<td>0.04187</td>
</tr>
</tbody>
</table>

The values presented in Table 1 and Table 2 correspond to the results obtained at the selected time indicated by the dashed red vertical line in the figures. The results show that the difference between the two methods is very small. Due to small standard uncertainties in the input variables along with a linear behaviour of the model, a first order Taylor Series approximation provide a good accuracy which is validated by MCM. Figure 5 illustrates how \( C_T \) changes during the manoeuvre along with the corresponding 95% coverage interval, the aerodynamic model results and the modelling error. The bottom figure illustrates how the expanded and relative uncertainty change.

![Figure 5. Illustration of how \( C_T \) changes during the manoeuvre along with the corresponding uncertainty.](image)

Figure 6 illustrates how \( C_C \) changes during the manoeuvre along with the corresponding 95% coverage interval, the aerodynamic model results and the modelling error. The bottom figure illustrates how the expanded and relative uncertainty change.

![Figure 6. Illustration of how \( C_C \) changes during the manoeuvre along with the corresponding uncertainty.](image)
Figure 6. Illustration of how $C_C$ changes during the manoeuvre along with the corresponding uncertainty.

Figure 7 illustrates how $C_N$ changes during the manoeuvre along with the corresponding 95% coverage interval, the aerodynamic model results and the modelling error. The bottom figure illustrates how the expanded and relative uncertainty change.

The values presented in Table 3 and Table 4 correspond to the results obtained at the selected time indicated by the red vertical line in the figures. Table 3 presents the nominal value, expanded uncertainty and relative uncertainty for the aerodynamic moment coefficients when using the TSM.

Table 3. TSM Moment coefficient uncertainty.

<table>
<thead>
<tr>
<th></th>
<th>Nominal</th>
<th>Expanded</th>
<th>Relative [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_l$</td>
<td>0.003072</td>
<td>0.0001502</td>
<td>8.429</td>
</tr>
<tr>
<td>$C_m$</td>
<td>0.0229</td>
<td>0.003651</td>
<td>20.98</td>
</tr>
<tr>
<td>$C_n$</td>
<td>-0.003784</td>
<td>0.0002234</td>
<td>14.06</td>
</tr>
</tbody>
</table>

Table 4 presents the nominal value, expanded uncertainty and relative uncertainty for the aerodynamic moment coefficients when using the MCM.

Table 4. MCM Moment coefficient uncertainty.

<table>
<thead>
<tr>
<th></th>
<th>Nominal</th>
<th>Expanded</th>
<th>Relative [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_l$</td>
<td>0.003072</td>
<td>0.0001515</td>
<td>8.505</td>
</tr>
<tr>
<td>$C_m$</td>
<td>0.0229</td>
<td>0.00368</td>
<td>21.14</td>
</tr>
<tr>
<td>$C_n$</td>
<td>-0.003784</td>
<td>0.0002262</td>
<td>14.24</td>
</tr>
</tbody>
</table>

The results show that the difference between the two methods is very small. Due to small standard uncertainties in the input variables along with a linear behaviour of the model, a first order Taylor Series approximation provide a good accuracy which is validated by MCM. Figure 8 illustrates how $C_l$ changes during the manoeuvre along with the corresponding 95% coverage interval, the aerodynamic model results and the modelling error. The bottom figure illustrates how the expanded and relative uncertainty change.

Figure 9 illustrates how $C_m$ changes during the manoeuvre along with the corresponding uncertainty.

The values presented in Table 3 and Table 4 correspond to the results obtained at the selected time indicated by the red vertical line in the figures. Table 3 presents the nominal value, expanded uncertainty and relative uncertainty for the aerodynamic moment coefficients when using the TSM.

Table 3. TSM Moment coefficient uncertainty.

<table>
<thead>
<tr>
<th></th>
<th>Nominal</th>
<th>Expanded</th>
<th>Relative [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_l$</td>
<td>0.003072</td>
<td>0.0001502</td>
<td>8.429</td>
</tr>
<tr>
<td>$C_m$</td>
<td>0.0229</td>
<td>0.003651</td>
<td>20.98</td>
</tr>
<tr>
<td>$C_n$</td>
<td>-0.003784</td>
<td>0.0002234</td>
<td>14.06</td>
</tr>
</tbody>
</table>

Table 4 presents the nominal value, expanded uncertainty and relative uncertainty for the aerodynamic moment coefficients when using the MCM.

Table 4. MCM Moment coefficient uncertainty.

<table>
<thead>
<tr>
<th></th>
<th>Nominal</th>
<th>Expanded</th>
<th>Relative [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_l$</td>
<td>0.003072</td>
<td>0.0001515</td>
<td>8.505</td>
</tr>
<tr>
<td>$C_m$</td>
<td>0.0229</td>
<td>0.00368</td>
<td>21.14</td>
</tr>
<tr>
<td>$C_n$</td>
<td>-0.003784</td>
<td>0.0002262</td>
<td>14.24</td>
</tr>
</tbody>
</table>
Figure 10 illustrates how $C_n$ changes during the manoeuvre along with the corresponding 95% coverage interval, the aerodynamic model results and the modelling error. The bottom figure illustrates how the expanded and relative uncertainty change.

In Table 5, the coverage interval for the aerodynamic coefficients are presented when using the two methods.

<table>
<thead>
<tr>
<th>Table 5. Coverage intervals.</th>
<th>TSM</th>
<th>MCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_T$</td>
<td>[0.0539, 0.06164]</td>
<td>[0.05404, 0.06178]</td>
</tr>
<tr>
<td>$C_C$</td>
<td>[-0.04819, 0.04433]</td>
<td>[-0.0483, 0.04438]</td>
</tr>
<tr>
<td>$C_N$</td>
<td>[0.9793, 1.062]</td>
<td>[0.9807, 1.064]</td>
</tr>
<tr>
<td>$C_l$</td>
<td>[0.002921, 0.003222]</td>
<td>[0.002924, 0.003227]</td>
</tr>
<tr>
<td>$C_m$</td>
<td>[0.01925, 0.02655]</td>
<td>[0.01952, 0.02689]</td>
</tr>
<tr>
<td>$C_n$</td>
<td>[-0.004007, -0.00356]</td>
<td>[-0.004017, -0.003563]</td>
</tr>
</tbody>
</table>

The values presented in Table 5 correspond to the results obtained at the selected time indicated by the red vertical line in the figures. As the previous results indicated, is the difference between the two methods very small which is also illustrated by the coverage intervals obtained.

To verify that enough iterations are made when using the MCM method a histogram illustrating the error- and frequency distribution along with a convergence study of the expanded uncertainty is made. If the convergence study indicate that the variations are small, the value can be assumed to be a good approximation of the combined standard uncertainty. The execution time of the MCM strongly depends on the number of iterations performed. By starting with a small number of iterations and look at the convergence study, one can decide whether to increase the number of iterations or not. A large difference between the methods indicate that substantial nonlinearities effects may be present and using a first order Taylor Series expansion model is not sufficient.

Figure 11 illustrates a histogram with the distribution of the Monte Carlo simulations of $C_C$. A convergence study of the relative uncertainty is illustrated in the bottom figure.

As can be seen in the convergence study, the result converges towards an acceptable stable solution after about 10000 iterations. The distribution is seen to be normal where the black and green vertical lines correspond to the coverage interval and shortest coverage interval respectively. With a coverage probability, $p_p = 0.95$, this means that there is a 95% probability that the value is within this interval. Similar behaviour of the convergence study and distribution are obtained for the other aerodynamic coefficients.

As can be seen in Figure 11 the difference between the shortest coverage interval and the probabilistic symmetric coverage interval is very small. This is because there is a symmetric distribution about the mean value as illustrated by the histogram. The small difference is due to the randomness in the MCM process which make some interval lengths shorter and some longer. The difference between the TSM and the MCM is in this case negligible and for a faster and more efficient analysis, the TSM should be used. In the case of a larger standard uncertainty in the input variables $u_X$, the distributions become more skewed towards higher values. If there is a larger and more significant difference between the methods, the MCM results should be seen to give the most accurate results.
B. UMFs and UPCs

In Eq.(15), the uncertainty magnification factors are defined as factors giving information about the influence each variable has as it propagates through the DRE.

The UMFs change from manoeuvre to manoeuvre and from sample to sample. Generally the variables giving a UMF larger than one, are the same for different manoeuvres but the values change.

The moment coefficients have increasing propagating uncertainties in the position of the center of gravity. UMFs larger than one for $C_T$ is $FG$, which is the gross thrust, the mass $m$, the load factor $n_x$ and the ram drag $FR$. For $C_l$ the UMFs for the variables $m$, $\dot{p}$, $I_{xx}$ and $n_x$ are also larger than one. Whereas for $C_n$, the UMFs for $m$, $\dot{r}$ and $I_{zz}$ are larger than one.

When analysing the UPCs for each coefficient the contribution each variable has on the expanded uncertainty is clearly given. Figure 12 illustrates a time series plot of how the UPCs for $C_N$ changes during the manoeuvre.

![Figure 12](image_url)

**Figure 12.** An illustration of how the UPCs of $C_N$ changes during the manoeuvre.

As seen in Figure 12, the contribution from the dynamic pressure $q_a$ is dominating with a small contribution from the mass $m$. Similar plots are generated for each coefficient and for $C_T$, the dynamic pressure $q_a$ along with the gross thrust $FG$ and the ram drag $FR$ give the largest contributions. For $C_C$, the contribution from the dynamic pressure $q_a$ is dominating whereas the contribution from the aircraft mass $m$, the load factor $n_y$ constitutes the remaining part. For $C_l$, the contributions from the position of the center of gravity $y_{cg}$, $z_{cg}$ and the dynamic pressure $q_a$ give the largest contributions. For $C_m$, the dominating contribution is given by the $x$-position of the center of gravity $x_{cg}$ and a small contribution from the dynamic pressure $q_a$. For $C_n$, the contributions from the dynamic pressure $q_a$, the $x$-position of the center of gravity $x_{cg}$ and the angular rate $\dot{r}$ give the largest contributions to the expanded uncertainty.

IV. Discussion

The results suggest that the TSM can be used for an accurate, fast and efficient analysis of the uncertainties. No large discrepancies were found between the methods suggesting the first order TSM to be insufficient. The uncertainties change depending on the manoeuvre and coefficient but generally vary between 10% and 20%.

As seen clearly in both Figure 6 and Figure 10, the relative uncertainty might not always give very representative estimation of the uncertainty. This is because, as the nominal value approaches zero, the relative uncertainty will approach infinity. An approach to address this issue by finding and replacing these values with interpolated values has been made. However, some problems still remain and a different solution should be considered.

During steady level flight and during manoeuvres where values close to zero are expected on the aerodynamic coefficients, the expanded uncertainty will give a better estimation of the uncertainty. One can in general say that small values on the coefficients result in relative uncertainty values that should be seen not to give reasonable results. This effect can be seen in Figure 6. As the value of the coefficient increases, the relative uncertainty stabilizes and no spikes appear even though the expanded uncertainty still varies.

This is something that also affects the UPCs. Variables having an absolute standard uncertainty dominates the uncertainty percentage contribution during steady level flight to then decrease as the manoeuvre starts. Therefore it might be more interesting to focus the analysis on the time segment during the manoeuvre when stick inputs are done by the pilot. The relative uncertainty then gives values which are more stable and gives a better estimation of the uncertainty.

Not all variables included in the equations required to calculate the aerodynamic coefficients are measured directly but are instead calculated from other measurements. The angular accelerations $\ddot{p}$, $\ddot{q}$ and $\ddot{r}$ are examples of such variables. They are determined by numerically differentiate the measured angular rates. These signals are, as stated in [7], usually noisy and it is therefore important that this step is done accurately in order not to distort the signals further.
Another example of a variable that is not measured directly is the engine thrust. It is computed using measured signals from the engine. In case of fast transients of the power level setting, the engine model does not simulate the thrust accurately. Therefore, in order to get good estimates it is desired to let the engine variables stabilize before a manoeuvre is initiated [6].

Standard uncertainties of the variables not measured directly or where any kind of manipulation of the data were made before the calculations, were obtained from Monte Carlo simulations. Several different manoeuvres where analysed and compared to identify possible differences.

By analysing if the values obtained by the aerodynamic model lies within the coverage interval, decisions can be made whether any corrections should be made to the aerodynamic model or not. However, one should keep in mind that there are uncertainties in the variables used as inputs to the aerodynamic model as well. This makes it difficult to say whether the differences are due to model errors or something else. Variables that have a large contribution to the uncertainty in the aerodynamic model might only have a small contribution in the flight mechanic equations and vice versa.

As given by the UPCs, the dynamic pressure was found to be one of the variables that had a large impact on the uncertainty for all coefficients. It is not measured directly but it is instead obtained by a combination of the Mach number, altitude and total temperature in the ambient air. Performing a sensitivity analysis following the one-at-a-time design (OATD) [8], gives information about how the uncertainty in the dynamic pressure can be apportioned to the uncertainty in its input. Changing one input variable at the time and then computing the standard deviation of the output gives that the uncertainty in the Mach number dominates the uncertainty in the dynamic pressure. In order to decrease the uncertainty in the aerodynamic coefficients, a more accurate measurement of the Mach number is needed.

The position of the center of gravity was also found to be one of the variables giving a large contribution to the uncertainty. It is computed by correcting the center of gravity of the dry aircraft by measurements of the fuel content in the tanks. New methods or a deeper analysis to estimate the fuel content might therefore be required to increase the accuracy in the position of the center of gravity.

The engine gross thrust \( FG \) gives one of the largest contribution to the uncertainty in the tangential force coefficient \( C_T \). The current engine model, relies heavily on the measurements and calculations of the exhaust nozzle exit area and the turbine discharge pressure. Therefore any measurement- and calculation errors in these variables will result in a decrease of the accuracy of the gross thrust calculations.

V. Conclusion

Emphasis in this thesis has been placed on increasing the knowledge concerning the uncertainty of the aerodynamic coefficients, how uncertainties propagate through the data reduction equations and how the data reduction influences the expanded uncertainty of the coefficients.

The uncertainty propagation methods used were a Monte Carlo Method and a Taylor Series Method. From the results it can be concluded that the Taylor Series Method works well and produce results that are validated by the Monte Carlo Method. The Taylor Series Method was also found to have a much faster execution time than the Monte Carlo Method since it does not require any iterations for the solutions to converge.

The uncertainties change between the coefficients and from manoeuvre to manoeuvre but vary in most cases between 10% and 20%. In general, larger uncertainties were found in the moment coefficients, especially in the pitch moment coefficient \( C_m \). The variables that dominated the expanded uncertainty were the dynamic pressure, \( q_a \) and the position of the center of gravity, especially the x-position \( x_{cg} \).

By comparing and studying the time histories of the flight test results and the aerodynamic model, any significant discrepancies can be identified. If the model results are outside the coverage interval a more thorough analysis might be required. Decisions whether to update the current aerodynamic model based solely on this analysis are, however, not recommended. There are uncertainties in the variables used in the aerodynamic model which makes it difficult to say whether the discrepancies are due to model errors or something else and a more thorough analysis should be performed.

The accuracy relies upon that good instrumentation measurement techniques and data preparations methods are used. The instrumentation system must be of high fidelity with adequate resolution, sample rates, filtering and few or unknown
time delays between parameters. In order to decrease the uncertainty in the coefficients, the uncertainty in the Mach number used to determine the dynamic pressure, the uncertainty in the fuel measurements giving the position of the center of gravity and the uncertainty associated with the exhaust nozzle exit area and the turbine discharge pressure giving the thrust of the engine should be analysed further.

VI. Future Work

The results and methods presented in this thesis have given opportunities for further development and improvement. The following ideas can be considered for future work.

The assumption that all errors have a normal distribution is not something that have been verified. A deeper analysis should be performed to see if this assumption is correct. One might also look deeper into if any of the variables in the equations can be neglected in order for a simplified analysis without affecting the result significantly.

Another thing to look at would be if longitudinal and lateral manoeuvres should be analysed separately and if some manoeuvres are better suited for an uncertainty analysis than others. Can any conclusions be drawn about the uncertainty in similar manoeuvres? More tests need to be analysed of the same type of manoeuvre before any conclusion regarding any specific similarities can be made.

It might also be interesting to analyse the uncertainties in a real-time evaluation. The current scripts are developed to be used in a post-process stage where there are less restrictions on, for example, execution time.

Development of used software and model updates are ongoing and may require smaller changes to the scripts. As previously mentioned should the method for evaluating the relative uncertainty for values of the coefficients close to zero be revised and calculations of the dynamic pressure, the engine thrust and position of center of gravity should be analysed further for a decrease of the uncertainty.

Acknowledgments

I would like to thank my supervisors at both Saab and KTH for the help and support during this thesis work. It has been a great experience and I have grown both personally and professionally. Lastly I would like to express my gratitude and appreciation to my family and girlfriend for all the love and support throughout my education.

References


