A MULTIPHASE HELE-SHAW FLOW WITH SOLIDIFICATION

KARL JONSSON

Abstract. The one-phase Hele-Shaw flow has a long history and has been extensively studied from several points of view ranging from the fluid dynamical beginnings to complex analysis and integrable systems, see [5].

We prove existence, using the implicit function theorem, of a solution $W_\varepsilon$ in the Bochner space $L^2(0,T;H^1_0(\Omega;\mathbb{R}^m))$ to a non-local in time semi-linear system of coupled PDEs of second order related to obstacle type problems, having the explicit form

$$(0.1) \quad -\Delta u_i^\varepsilon + (1 - \chi_{D_0}) \beta_\varepsilon(u_i^\varepsilon) + \frac{1}{\varepsilon} \sum_{j \neq i} \beta_\varepsilon(u_j^\varepsilon) + \int_0^{\gamma_\varepsilon(t)} \beta_\varepsilon(u_j^\varepsilon(s)) \, ds \beta_\varepsilon(u_i^\varepsilon) = \gamma_\varepsilon(t) \mu^i, \text{ in } (0,T) \times \Omega,$$

with $\Omega \subset \mathbb{R}^n$ a bounded domain, the functions $\beta_\varepsilon(s)$ and $\gamma_\varepsilon(s)$ being regularizations of the Heaviside step function and $s \mapsto (s)^+$ respectively. We show that there is a limit flow $W$ such that $W_\varepsilon \to W$ as $\varepsilon \to 0$ over some subsequence in the space $L^2(0,T;H^1_0(\Omega;\mathbb{R}^m))$.

Contents

1. Introduction 1
2. Notation and assumptions 2
3. The $\varepsilon$-problem 3
4. Fréchet differentiability and solvability of a linear system 5
5. Construction of a solution $W_\varepsilon$ to the $\varepsilon$-problem 10
6. Vanishing-properties of the $\varepsilon$-solutions $W_\varepsilon$ 13
7. Existence of a limit flow $W$ as $\varepsilon \to 0^+$ 13
8. References 14

1. Introduction

Consider a Newtonian fluid constrained in the space between two narrowly separated parallel planes. Within a subregion more fluid is added at a constant rate so that the fluid expands between the plates. The space occupied by the fluid can accurately be described by its projection $D$ onto one of the planes, provided the separation of the planes is small enough. A point on the boundary $\partial D$ of the fluid will move in the outward normal direction $\nu$ with speed proportional to the gradient of the internal pressure $p$ of the fluid. The growing region $D$ has a free boundary $\partial D$ which grows in time. This is the so-called one-phase Hele-Shaw flow.

Date: February 2018.
In this paper we will generalize this situation and consider a multiple-phase Hele-Shaw flow of two or more fluids. The one- and multi-phase Hele-Shaw problem and related problems dealing with quadrature domains has been studied extensively from different approaches, see [4, 3, 1, 5]. In our version of the multi phase flow the phases are separated from the beginning and then evolve according to the one-phase Hele-Shaw flow, at least up until they meet. We wish to model a multi phase flow which fulfills two conditions, namely

- the phases will be \textit{separated} for all times, and
- the phases will fulfill a monotonicity criterion in time, meaning that if one of the phases occupies some part of the ambient space for some time, then it will occupy the same part of space for all future times. The monotonicity will be interpreted as a freezing condition for the phases, so when two phases meet, their joint boundary will be fixed for all future times. Another way to regard this is that the intersection of two phases will \textit{solidify} as they meet. This solidification process will yield a free boundary which grows in time.

One can show that the one phase Hele-Shaw flow can be described as \[ D = D_0 \cup \{ x \in \Omega : u(x) > 0 \} \] where \( u : \Omega \times [0, T) \to \mathbb{R} \) such that \( u(\cdot, t) \in H^1_0(\Omega) \) for some bounded domain \( \Omega \) such that \( \text{spt} \mu \subset D_0 \subset \Omega \subset \mathbb{R}^n \) where \( u \) solves the PDE

\[
-\Delta u + (1 - \chi_{\{u > 0\}}) = t\mu.
\]

For a time \( T > 0, D_0 \subset \Omega \) is the starting configuration for the fluid, \( \chi_A \) is the characteristic function, \( t \) is a time in \([0, T)\) and \( \mu \in H^{-1}(\Omega) \) the measure of finite energy that drives the flow.

To make the multi-phase problem tractable we will essentially consider the following system of \( m \) coupled partial differential equations

\[
-\Delta u^i_\varepsilon(t) + (1 - \chi_{D^i_0})\beta_\varepsilon(u^i_\varepsilon(t)) + \frac{1}{\varepsilon} \sum_{j \neq i} \beta_\varepsilon(u^j_\varepsilon(t)) + \int_0^t \beta_\varepsilon(u^j_\varepsilon(s)) \, ds \beta_\varepsilon(u^i_\varepsilon(t)) = t\mu^i, \quad \text{in } \Omega,
\]

where \( i = 1, \ldots, m, \varepsilon > 0, \) and \( \beta_\varepsilon \) is a regularization of the Heaviside step function depending on the parameter \( \varepsilon \). The idea is to show existence of a solution to this regularized system and then pass in the limit \( \varepsilon \to 0 \) and there obtain a solution to our multi-phase Hele Shaw flow. The coupling terms in the regularized equation are precisely incorporated in order to fulfill the two desired conditions (separation and monotonicity of the phases) that we want to model.

2. Notation and assumptions

We will make use of the following notation, assumptions, and comments in the rest of the article. Note that the list does not always contain full specifications on the type of regularity assumed for the specific entities, instead the list gives an overview of the intended meaning of the symbols.

- \( \Omega \): a bounded domain in \( \mathbb{R}^n \).
- \( D^i_0 \): a collection of subdomains of \( \Omega \) which are pairwise disjoint.
- \( \mu^i \): are elements of \( H^{-1}(\Omega) \) such that \( \text{spt} \mu^i \subset D^i_0 \).
- \( m \): the number of phases in our problem.
\(a(x) \preceq b(x)\): means that there is a universal constant \(C\) such that the inequality \(a(x) \leq Cb(x)\) holds for all designated \(x\).

\(\mathcal{L}(A) = |A|\): \(n\)-dimensional Lebesgue measure of the measurable set \(A \subset \mathbb{R}^n\).

\(\mathcal{H}^s(A)\): \(s\)-dimensional Hausdorff measure, scaled in such a way that \(\mathcal{H}^n = \mathcal{L}\) as measures.

\(\overline{A}\): closure of the set \(A\).

\(\text{spt } u\): the support of the real measurable function \(u : \Omega \to \mathbb{R}\), that is the set \(\{x \in \Omega : u(x) > 0\}\).

\(\chi_E\): indicator function of the measurable set \(E\).

\(\delta_{ik}\): the Kronecker delta function, \(= 0\) if \(i \neq k\) and \(= 1\) if \(i = k\).

\(u^+, u^-\): for a measurable function \(u : \Omega \to \mathbb{R}\) we define \(u^+(x) := u(x)\) if \(u(x) > 0\) and \(0\) otherwise. We define \(u^-(x) := -u(x)\) if \(u(x) < 0\) and \(0\) otherwise. It follows that \(u^k \geq 0\) and that \(u = u^+ - u^-\).

\(C^{k, \alpha}(\Omega)\): the space of \(k\)-times differentiable functions with derivatives of order \(k\) being Hölder continuous with constant \(\alpha\).

\(\|u\|_{p; \Omega}\): is the \(L^p\)-norm of the function \(u : \Omega \to \mathbb{R}\) defined as \((\int_\Omega |u|^p \, dx)^{1/p}\).

\(C_0^{\infty}(\Omega)\): the set of all infinitely differentiable functions \(\eta : \Omega \to \mathbb{R}\) with \(\text{spt } \eta \subset \Omega\).

\(W^{k,p}(\Omega)\): the Sobolev space of all functions \(f \in L^2(\Omega)\) which have weak derivatives up to order \(k\) which all lie in \(L^p(\Omega)\). The space \(W^{1,2}(\Omega)\) is equipped with the inner product \((f, g) = \int_\Omega fg + \nabla f \cdot \nabla g \, dx\).

\(H^1_0(\Omega)\): the closure of the set \(C_0^{\infty}(\Omega)\) under the norm-topology on \(W^{1,2}(\Omega)\).

\(H^{-1}(\Omega)\): the dual space of the Sobolev space \(H^1_0(\Omega)\).

\(C^k([0, T], B)\): the space of maps \(u : [0, T] \to B\), for some Banach space \(B\), which are \(k\) times continuously differentiable in the \(t\)-variable.

\(L^p((0, T))\): is the Bochner space of measurable functions \(u : [0, T] \to B\) such that \((\int_0^T \|u(t)\|_B^p \, dt)^{1/p}\) is finite.

\(H\): the Heaviside step function \(H : \mathbb{R} \to \mathbb{R}\) defined by \(H(x) = 0\) if \(x < 0\) and \(= 1\) otherwise.

\(\beta_{\varepsilon}\): a sufficiently smooth approximation of the Heaviside function with \(0 \leq \beta_{\varepsilon} \leq 1\), \(\beta_{\varepsilon}(x) = 0\) for \(x \leq -\varepsilon\) and \(\beta_{\varepsilon}(x) = 1\) for \(x \geq \varepsilon\).

\(B_{\varepsilon}\): the primitive function of \(\beta_{\varepsilon}\) defined by \(B_{\varepsilon}(x) = \int_{-\infty}^x \beta_{\varepsilon}(y) \, dy\).

3. The \(\varepsilon\)-Problem

In the upcoming definition we define what a solution to the \(\varepsilon\)-problem means.

**Definition 3.1** (Solution to the \(\varepsilon\)-problem). Given the initial data \(\Omega\), \(D_0^i\), and \(\mu^i\) for \(i = 1, \ldots, m\), and the parameter \(\varepsilon > 0\) then a map

\[
W_{\varepsilon} \in C^1([0, \infty); H^1_0(\Omega; \mathbb{R}^m))
\]

is called a solution to the \(\varepsilon\)-problem if its components \(u_{\varepsilon}^i(t) \equiv W_{\varepsilon}(t)^i\) are non-negative, \(u_{\varepsilon}^i(0) = 0\) a.e. in \(\Omega\), and \(u_{\varepsilon}^i(t) \in H^1_0(\Omega)\) is a weak solution to the system

\[-\Delta u_{\varepsilon}^i(t) + (1 - \chi_{D_0^i})\beta_{\varepsilon}(u_{\varepsilon}^i(t))
+ \frac{1}{\varepsilon} \sum_{j \neq i} \beta_{\varepsilon}(u_{\varepsilon}^j(t)) + \int_0^{\gamma_{\varepsilon}(t)} \beta_{\varepsilon}(u_{\varepsilon}^i(s)) \, ds \beta_{\varepsilon}(u_{\varepsilon}^i(t)) = \gamma_{\varepsilon}(t)\mu^i, \text{ in } \Omega,
\]
for each fixed \( t \geq 0 \) and \( i = 1, \ldots, m \). Explicitly this means, for any \( \phi \in H^1_0(\Omega) \) it holds for any \( i = 1, \ldots, m \) that

\[
\int_\Omega \left[ \nabla u^\varepsilon_i(t) \cdot \nabla \phi + (1 - \chi_{D^\varepsilon_i}) \beta^\varepsilon_i(u^\varepsilon_i(t)) \phi \right] + \frac{1}{\varepsilon} \sum_{j \neq i} \beta^\varepsilon_i(u^\varepsilon_i(t)) + \int_0^{\gamma^\varepsilon_i(t)} \beta^\varepsilon_i(u^\varepsilon_i(s)) \, ds \beta^\varepsilon_i(u^\varepsilon_i(t)) \phi \, dx = \gamma^\varepsilon_i(t) \langle \mu^i, \phi \rangle.
\]

(3.2)

**Remark 3.2.** Heuristically, the intuition for using this type of approximate equation is the following. As \( \varepsilon \searrow 0 \) we will obtain \( u^\varepsilon_i(t) \to u_i^1 \) for some \( u_i^1(t) \) in \( H^1_0(\Omega) \), also having strong convergence in \( L_2(\Omega) \). The term \( (1 - \chi_{D^\varepsilon_i}) \beta^\varepsilon_i(u^\varepsilon_i(t)) \) will converge to \( (1 - \chi_{D^\varepsilon_i}) \chi_{(u^1_i(t))\geq0} \) in the limit which relates to the one-phase Hele-Shaw flow. The last terms on the left hand side of the equation are incorporated in order to assure that the phases in the limit separates at each moment \( t \) (the term \( \frac{1}{\varepsilon} \sum_{j \neq i} \beta^\varepsilon_i(u^\varepsilon_i(t)) \beta^\varepsilon_i(u^\varepsilon_i(t)) \) as well as for (almost) all times before the present time \( t \) (that is the term \( \frac{1}{\varepsilon} \sum_{j \neq i} \int_{\gamma^\varepsilon_i(t)} \beta^\varepsilon_i(u^\varepsilon_i(s)) \, ds \beta^\varepsilon_i(u^\varepsilon_i(t)) \) will guarantee this behavior). The \( \beta^\varepsilon_i \) function will track the support of the functions describing the domains of the flow, for small \( \varepsilon \) essentially being 1 in the support of \( u^\varepsilon_i(t) \) and 0 otherwise. The aim is that we end up with a flow which has separated phases and freezing.

We first show that a solution to the \( \varepsilon \)-problem has components which are non-negative.

**Lemma 3.3.** Suppose \( u^\varepsilon_i(t) \) is a solution to the \( \varepsilon \)-problem, then \( u^\varepsilon_i(t) \geq 0 \) in \( \Omega \) for all \( t \in [0, \infty) \) and \( i = 1, \ldots, m \).

**Proof.** By definition \( u^\varepsilon_i(0) = 0 \) in \( \Omega \). Let \( \psi(t) = \min(0, u^\varepsilon_i(t)) \in H^1_0(\Omega) \), then it holds that \( \psi(t) \leq 0, \nabla \psi(t) = \chi_{u^\varepsilon_i(t) < 0} \nabla u^\varepsilon_i(t) \) and \( \nabla \psi(t) \) as well as \( \text{spt} (\beta^\varepsilon \circ u^\varepsilon_i(t)) \cap \text{spt} \psi(t) = \emptyset \). By using \( \psi(t) \) as a test-function in (3.1) yields

\[
\int_\Omega |\nabla \psi(t)|^2 \, dx = \gamma^\varepsilon(t) \langle \mu^i, \psi(t) \rangle \leq 0,
\]

(3.3)

where we use the assumption that \( \mu^i \) and \( \gamma^\varepsilon(t) \) are non-negative for \( t > 0 \). Hence \( \nabla \psi = 0 \) a.e. in \( \Omega \). By connectedness of \( \Omega \) the function \( \psi(t) \) must be constant and \( \psi(t) \in H^1_0(\Omega) \), thereby vanishing on the boundary of \( \Omega \), it follows that \( \psi(t) \) vanishes in \( \Omega \) for all \( t > 0 \), proving the lemma.

In lemma 3.4 a uniform a priori estimate of solutions to the \( \varepsilon \)-problem 3.3 is shown. The main use of the lemma will be in showing that the implicit function theorem can be used iteratively as well as showing that the approximate solutions has a limit as \( \varepsilon \to 0^+ \) along some subsequence.

**Lemma 3.4.** Given a positive number \( T \), if there is a family of maps

\[
W^\varepsilon : [0, T) \to H^1_0(\Omega; \mathbb{R}^m),
\]

for \( \varepsilon > 0 \), with components \( u^\varepsilon_i(t) \), such that \( u^\varepsilon_i(0) = 0 \) and satisfying the system

(7.4)

then for any \( t \in [0, T) \) it holds that

\[
\|u^\varepsilon(t)\|_{H^1_0(\Omega; \mathbb{R}^m)} \lesssim T \|\mu\|_{H^{-1}(\Omega; \mathbb{R}^m)},
\]

(3.4)

independently of \( \varepsilon \).
Proof. By testing equation 3.1 with \( u_i^\varepsilon(t) \in H_0^1(\Omega) \) we obtain

\[
\int_\Omega |\nabla u_i^\varepsilon(t)|^2 \, dx + \int_\Omega (1 - \chi_{D_i^0}) \beta_\varepsilon(u_i^\varepsilon(t)) u_i^\varepsilon(t) \, dx \\
+ \int_\Omega \frac{1}{\varepsilon} \sum_{j \neq i} \left[ \beta_\varepsilon(u_j^\varepsilon(t)) + \int_0^{\gamma_\varepsilon(t)} \beta_\varepsilon(u_j^\varepsilon(s)) \, ds \right] \beta_\varepsilon(u_i^\varepsilon(t)) u_i^\varepsilon(t) \, dx \\
= \gamma_\varepsilon(t) \langle u_i^\varepsilon(t), \mu^i \rangle.
\]

By non-negativity of the terms in the left hand side we obtain, by using the Poincaré inequality, that for some \( C' \) independent of \( \varepsilon \) it holds

\[
\|u_i^\varepsilon(t)\|_{H_0^1(\Omega)}^2 \leq C' T \|\mu^i\|_{H^{-1}(\Omega)} \|u_i^\varepsilon(t)\|_{H_0^1(\Omega)},
\]

for all \( i = 1, \ldots, m \). This gives us a bound for the component \( u_i^\varepsilon(t) \) with \( C_i = C' T \|\mu^i\|_{H^{-1}(\Omega)} \) and setting \( C = \max_{i=1, \ldots, m} C_i \) the lemma follows. \( \square \)

From Lemma 3.4 it follows immediately that the flow \( W_\varepsilon \), considered as an element in the space \( L^p(0, T; H_0^1(\Omega; \mathbb{R}^m)) \), is uniformly bounded with respect to the parameter \( \varepsilon \).

Corollary 3.5. Suppose that \( W_\varepsilon \) is a solution to the \( \varepsilon \)-problem, then \( W_\varepsilon \in L^p(0, T; H_0^1(\Omega; \mathbb{R}^m)) \) for all \( \varepsilon > 0 \) and all \( p \geq 1 \), and

\[
\|W_\varepsilon\|_{L^\infty(0, T; H_0^1(\Omega; \mathbb{R}^m))} \lesssim T \|\mu\|_{H^{-1}(\Omega; \mathbb{R}^m)},
\]

independent of \( \varepsilon \).

4. Fréchet Differentiability and Solvability of a Linear System

We begin this section with two lemmas which are needed in the proof of the fact that the \( \varepsilon \)-problem has a solution. The first lemma shows that a certain map used to define the regularized Hele-Shaw flow between the spaces \( H_0^1(\Omega) \) and \( H^{-1}(\Omega) \) is continuously Fréchet differentiable.

Lemma 4.1. Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \), \( M \in L^\infty(\Omega) \) and \( u \in H_0^1(\Omega) \) then for any \( \varepsilon > 0 \) the map

\[
F: \quad H_0^1(\Omega) \quad \longrightarrow \quad H^{-1}(\Omega), \quad u \quad \mapsto \quad M \beta_\varepsilon(u),
\]

explicitly meaning

\[
F(u)[v] = \int_\Omega M(x) \beta_\varepsilon(u(x)) v(x) \, dx,
\]

is continuously Fréchet differentiable with derivative

\[
\frac{\partial F}{\partial u}: \quad H_0^1(\Omega) \quad \longrightarrow \quad \mathcal{B}(H_0^1(\Omega), H^{-1}(\Omega)), \quad u \quad \mapsto \quad M \beta_\varepsilon'(u),
\]

explicitly meaning

\[
\frac{\partial F}{\partial u}(u)[v_1][v_2] = \int_\Omega M(x) \beta_\varepsilon'(u(x)) v_1(x) v_2(x) \, dx.
\]
Proof. To prove this we show that $F$ is Gâteaux differentiable, i.e. we show that the directional derivative $dF(u, h)$ exists for all $u, h$ in $H^1_0(\Omega)$ and equals $M\beta'(u)h$.

By showing that $dF(u, h)$ is linear in $h$ as well as $u \mapsto dF(u, \cdot)$ being continuous, then by \cite{2} Theorem 2.1.13 this implies the desired result.

Let

$$I = F(u + th)[v] - F(u)[v] - M\beta'(u)th[v]$$

where the inequality is independent of $t$.

Claim. It holds that

$$\|I\| \leq \|M\|_{\infty, \Omega}(I_A + I_B),$$

where we want to show that $|I|/|t| \to 0$ as $t \to 0$.

Since $h$ might be unbounded in $\Omega$ we shall decompose the problem in to two domains, one where we have control over the $L^\infty$-norm of $h$ and the rest which will turn out to have small enough Lebesgue measure in order to vanish in a limit procedure. Let $C > 0$ and define $A = \{x \in \Omega : |h(x)| > C\}$ and $B = \{x \in \Omega : |h(x)| \leq C\}$ yielding

$$|I| \leq \|M\|_{\infty, \Omega}(I_A + I_B),$$

for $S = A, B$.

Claim. It holds that

$$I_A \lesssim \frac{|t|}{|C^2|} \|h\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)},$$

where the inequality is independent of $C$ and $t$.

Using the Sobolev imbedding of $H^1_0(\Omega)$ into $L^{2^*}(\Omega)$, where $2^* = \frac{2n}{n-2} = 2 + \frac{4}{n-2}$, and Hölder’s generalized inequality,

$$I_A \leq (|\beta(\cdot)|^2 \beta_2(u) - \beta' \beta_2(u) th) \|v\|_{2^*} \|\beta\|_{2^*} |A| \|h\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)}$$

$$\leq 2|t| \sup_{s \in \mathbb{R}} |\beta'(s)| \|h\|_{2^*} \|v\|_{2^*} \|\beta\|_{2^*} |A| \|h\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)}$$

$$\lesssim \frac{|t|}{C^2} \|h\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)} |A|^\frac{1}{2^*}$$

Furthermore

$$\|h\|_{H^1_0(\Omega)} \geq \|h\|_{2, A} = \left(\int_A |h|^2 \ dx\right)^{1/2} \geq |A|^{\frac{1}{2^*}} C,$$

hence the claim follows.

Claim. It holds that

$$I_B \lesssim \frac{C^2 |t|^2}{|C^2|} \|v\|_{2, \Omega},$$

where the inequality is independent of $C$ and $t$. 
The function $h$ is bounded on $B$ so we have for a.e. $x$
that
\[ |\beta_c(u(x) + th(x)) - \beta_c(u(x)) - \beta'_c(u(x)) th(x)| \leq (th(x))^2 \sup_{s \in \mathbb{R}} |\beta''_c(s)| \lesssim \frac{C^2 t^2}{\varepsilon^2}. \]

Thus
\[ I_B \lesssim \frac{C^2 t^2}{\varepsilon^2} \|v\|_{1,B} \lesssim |\Omega|^{1/2} \frac{C^2 t^2}{\varepsilon^2} \|v\|_{2,\Omega}, \]

which proves the claim.

**Claim.** $|I|/|t| \to 0$ as $t \to 0$.

By the derived estimates, let $C > 0$, then
\[
\frac{|I|}{|t|} \leq \|M\|_{\infty, \Omega} \left( \frac{I_A}{|t|} + \frac{I_B}{|t|} \right)
\leq \frac{1}{\varepsilon C^2} \|h_{1+\frac{q}{4}}^{1/2}(\Omega)\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + \frac{tC^2}{\varepsilon^2} \|v\|_{2,\Omega}.
\]

For given $\nu > 0$ choose $C$ so large that the first term above is less than $\nu/2$, a choice independent of $t$, by subsequently letting $t$ be small we obtain $|I|/|t| < \nu/2 + \nu/2 = \nu$. Hence $|I|/|t| \to 0$ as $t \to 0$.

Hence we have shown the desired result for the Gâteaux derivative, viz.
\[ dF(u, h) = M\beta'_c(u)h. \]

This is clearly linear in $h$ so left to prove is the continuity of the map $u \mapsto dF(u, \cdot)$. Let $u_1, u_2 \in H^1_0(\Omega)$ and set
\[ Z = \mathcal{B}(H^1_0(\Omega), H^{-1}(\Omega)) \text{ and } A = \{ v \in H^1_0(\Omega; \mathbb{R}^2) : \|v\|_{H^1_0(\Omega)} = 1 \text{ for } i = 1, 2 \}. \]

Then
\[
\|dF(u_1, \cdot) - dF(u_2, \cdot)\|_Z = \sup_{\|v\|_{H^1_0(\Omega)} = 1} \|dF(u_1, v_1) - dF(u_2, v_1)\|_{H^{-1}(\Omega)}
= \sup_{v_1, v_2 \in A} \left| \int_{\Omega} M(\beta'_c(u_1) - \beta'_c(u_2))v_1 v_2 \, dx \right|.
\]

Note that the function $\beta'_c(u_1) - \beta'_c(u_2)$ lies in $L^p(\Omega)$ for all $p \geq 1$ since $\beta'_c(u_1) - \beta'_c(u_2) \in L^\infty(\Omega)$ and $\Omega$ is bounded. Continuity of the map follows if we can control the right hand side of the expression with $\|u_1 - u_2\|_{2,\Omega}$.

In the case $n = 2$ it holds that $H^1_0(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $q$ with $1 \leq q < \infty$, so it follows from (4.11) using Hölder’s inequality, that
\[
\|dF(u_1, ..) - dF(u_2, ..)\|_Z \leq \|M\|_{\infty, \Omega} \sup_{v_1, v_2 \in A} \|\beta'_c(u_1) - \beta'_c(u_2)\|_{2,\Omega} \|v_1\|_{4,\Omega} \|v_2\|_{4,\Omega}
\leq \|M\|_{\infty, \Omega} C_3 \|\beta'_c(u_1) - \beta'_c(u_2)\|_{2,\Omega},
\]

for some constant $C_3$ independent of $u_1$ and $u_2$.

Assume now that $n > 2$ then using the Sobolev imbedding theorem again
\[ v_i \in L^2(\Omega) \text{ and } \|v_i\|_{2r,\Omega} \leq C_4 \|v_i\|_{H^1_0(\Omega)} \]

\[ v_i \in L^2(\Omega) \quad \text{and} \quad v_i \in L^{2r}(\Omega) \quad \text{for some constant } C_4 \text{ independent of } u_1 \text{ and } u_2. \]
for some $C_4$. Using Hölder's generalized inequality again and (4.11)

\[
|dF(u_1, \cdot) - dF(u_2, \cdot)|_Z \leq |M|_{\infty, \Omega} \sup_A |\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{t/2, \Omega} |u_1|_{2, \Omega} |u_2|_{2, \Omega}
\]

\[
\leq |M|_{\infty, \Omega} C_4^2 |\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{t/2, \Omega}.
\]

**Claim.** $|\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{t/2, \Omega} \leq \|\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)\|_{1, \Omega}^{2/n}$ independently of $u_1$ and $u_2$.

In general, if $w \in L^\infty(\Omega)$ and $\Omega$ is bounded, then for any $p \geq 1$ the interpolation inequality

\[
|w|_{p, \Omega} \leq |w|_{\infty, \Omega}^{1-1/p} |w|_{1, \Omega}^{1/p}
\]

holds. In our specific case, for any $u_1, u_2 \in H^1_0(\Omega)$, the $L^\infty$-norm of $\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)$ will be uniformly bounded with respect to $u_1$ and $u_2$ due to the specific choice of the function $\beta_\varepsilon$, that is

\[
|\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{\infty, \Omega} \leq \frac{C}{\varepsilon} \|u_1 \chi_{\{0 < u_1 < 2\varepsilon\}} - u_2 \chi_{\{0 < u_2 < 2\varepsilon\}}\|_{\infty, \Omega} \leq 4C,
\]

where for some constant $C$.

**Claim.** It holds that

\[
|dF(u_1, \cdot) - dF(u_2, \cdot)|_Z \leq |M|_{\infty, \Omega} \begin{cases} 
|\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{1, \Omega}^{1/2}, & \text{if } n = 2, \\
|\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{1, \Omega}, & \text{if } n > 2,
\end{cases}
\]

independently of $u_1$ and $u_2$.

**Claim.** $|\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{1, \Omega} \leq \varepsilon^{-2} C_2 |u_1 - u_2|_{1, \Omega}$.

For any positive $C_5$, using a similar decomposition of the domain $\Omega$ as before, let $A = \{x \in \Omega: |u_1 - u_2| > C_5\}$ and $B = \{x \in \Omega: |u_1 - u_2| \leq C_5\}$. Since $u_1 - u_2 \in L^2(\Omega)$ we have $|A| \leq KC_5^{-2}$ for some constant $K$ and we get

\[
|\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{1, \Omega} \leq |\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{1, A} + |\beta'_\varepsilon(u_1) - \beta'_\varepsilon(u_2)|_{1, B}
\]

\[
\leq \frac{2C_5}{\varepsilon} |A| + \frac{C_2}{\varepsilon^2} |u_1 - u_2|_{1, B}
\]

(4.16)

\[
\leq \frac{2C_5}{\varepsilon} KC_5^{-2} + \frac{C_2}{\varepsilon^2} |u_1 - u_2|_{1, \Omega}.
\]

Taking the limit as $C_5 \to \infty$ we obtain the desired inequality.

**Claim.** The map $u \mapsto dF(u, \cdot)$ is continuous, more specifically

\[
|dF(u_1, \cdot) - dF(u_2, \cdot)|_Z \leq C \|M\|_{\infty, \Omega} |u_1 - u_2|_{2, \Omega},
\]

for $q = 1/2$ if $n = 2$ and $q = 2/n$ otherwise, where $C$ is independent of $u_1$ and $u_2$.

This follows from the sequence of claims and that $|u_1 - u_2|_{1, \Omega} \leq \|\Omega\|^{1/2} |u_1 - u_2|_{2, \Omega}$.

\[\square\]

**Lemma 4.2.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$, $(0, T)$ an open interval in $\mathbb{R}$, $M \in L^\infty(\Omega)$ and $u \in C(0, T; L^2(\Omega))$ then for any $\varepsilon > 0$ the map

\[
F: (0, T) \to H^{-1}(\Omega),
\]

\[
t \mapsto M \int_0^t \beta_\varepsilon(u(s)) \, ds,
\]
is continuously Fréchet differentiable with derivative

\[
\frac{\partial F}{\partial t} : (0, T) \rightarrow H^{-1}(\Omega),
\]

(4.18)

Proof. Consider for \( \phi \in H^1_0(\Omega) \) with \( \|\phi\|_{H^1_0(\Omega)} = 1 \) and \( h > 0 \),

\[
\left| \int_{\Omega} M \left( \int_0^{t+h} \beta_c(u(s)) \, ds - \int_t^t \beta_c(u(s)) \, ds - h\beta_c(u(t)) \right) \phi \, dx \right|
\]

\[
\leq |M|_{\infty, \Omega} \int_{\Omega} \left| \int_t^{t+h} \beta_c(u(s)) - \beta_c(u(t)) \, ds \right| \phi \, dx
\]

\[
\leq |M|_{\infty, \Omega} \int_{\Omega} \left| \beta_c(u(s)) - \beta_c(u(t)) \right|_{2, \Omega} \phi_2, \Omega
\]

\[
\leq |M|_{\infty, \Omega} h \sup_{s \in [t, t+h]} \left\| \beta_c(u(s)) - \beta_c(u(t)) \right\|_{2, \Omega},
\]

where the right hand side is \( o(h) \) since we assume that \( u \in C(0, T; L^2(\Omega)) \), this proves that the derivative is as stated.

To prove continuity of the derivative consider for \( \phi \in H^1_0(\Omega) \) with \( \|\phi\|_{H^1_0(\Omega)} = 1 \)

\[
\left| \int_{\Omega} M(\beta_c(u(s)) - \beta_c(u(t))) \phi \, dx \right| \leq |M|_{\infty, \Omega} \left| \beta_c(u(s)) - \beta_c(u(t)) \right|_{2, \Omega}
\]

which shows the result since \( \left| \beta_c(u(s)) - \beta_c(u(t)) \right|_{2, \Omega} \rightarrow 0 \) as \( s \rightarrow t \). \(\square\)

**Lemma 4.3.** Suppose that \( \lambda^{i,j} \) are non-negative elements of \( L^\infty(\Omega) \) for all \( i, j = 1, \ldots, m \). Then the linear system of equations

(4.19)

\[-\Delta w^i + \sum_j \lambda^{i,j} w^j = \eta^i \text{ in } \Omega\]

for \( w^i \in H^1_0(\Omega) \) and \( i = 1, \ldots, m \) is uniquely solvable for each right hand side \( \eta = (\eta^1, \ldots, \eta^m) \in H^{-1}(\Omega; \mathbb{R}^m) \).

Proof. Consider the bilinear form \( B : H^1_0(\Omega; \mathbb{R}^m) \times H^1_0(\Omega; \mathbb{R}^m) \rightarrow \mathbb{R} \)

defined by

(4.20)

\[ B(w, \varphi) = \sum_{i,j} \int_{\Omega} \nabla w^i \cdot \nabla \varphi^j + \lambda^{i,j} w^i \varphi^j \, dx. \]

then \( B \) is bounded

(4.21)

\[ |B(w, \varphi)| \leq \sum_{i,j} \| \nabla w^i \|_{2, \Omega} \| \nabla \varphi^j \|_{2, \Omega} + \| \lambda^{i,j} \|_{\infty, \Omega} \| w^i \|_{2, \Omega} \| \varphi^j \|_{2, \Omega}
\]

\[ \leq C \| w \|_{H^1_0(\Omega; \mathbb{R}^m)} \| \varphi \|_{H^1_0(\Omega; \mathbb{R}^m)}, \]

for some constant \( C \) independent of \( w \) and \( \varphi \). Furthermore we have the coercivity condition

(4.22)

\[ B(w, w) \geq \| \nabla w^i \|^2_{2, \Omega} \geq C_1 \| w \|^2_{H^1_0(\Omega; \mathbb{R}^m)}, \]

for some \( C_1 > 0 \). Hence for any \( \eta \in H^{-1}(\Omega; \mathbb{R}^m) \) there is by the Lax-Milgram theorem a unique \( w \in H^1_0(\Omega; \mathbb{R}^m) \) such that \( B(w, \varphi) = \langle \varphi, \eta \rangle \) for all \( \varphi \in H^1_0(\Omega; \mathbb{R}^m) \), which exactly proves the lemma. \(\square\)
5. CONSTRUCTION OF A SOLUTION $W_\varepsilon$ TO THE $\varepsilon$-PROBLEM

The upcoming theorem 5.1 ties the results together and shows that the $\varepsilon$-problem has a solution.

**Theorem 5.1.** For any $\varepsilon > 0$ there is a solution $W_\varepsilon \in C^1([0, \infty); H^1_0(\Omega; \mathbb{R}^m))$ to the $\varepsilon$-problem.

**Proof.** The proof relies upon the implicit function theorem and compactness properties of the families of solutions on smaller intervals, principles which are used to build a solution iteratively.

Fix $\varepsilon > 0$ and assume that the flow $W_\varepsilon \in C([0, \eta]; H^1_0(\Omega; \mathbb{R}^m))$ is given for some $\eta \geq \varepsilon$, then let

$$G_\varepsilon : [0, \eta + \varepsilon] \times H^1_0(\Omega; \mathbb{R}^m) \to H^{-1}(\Omega; \mathbb{R}^m)$$

$$(t, u) \mapsto \Delta u^i - (1 - \chi_{D_0^i}) \beta_\varepsilon(u^i) - \frac{1}{\varepsilon} \sum_{j \neq i} [\beta_\varepsilon(u^j) + \oint_{(t)} \beta_\varepsilon(u^j(s)) ds] \beta_\varepsilon(u^i) + \gamma_\varepsilon(t) \mu^i,$$

for all $i = 1, \ldots, m$. It is clear that the map $G_\varepsilon$ is well defined for all $u \in H^1_0(\Omega; \mathbb{R}^m)$ and $0 \leq t \leq \eta + \varepsilon$ since the existing history of the system, $u_\varepsilon(t)$ for $t \leq \eta$, only is incorporated in to the definition of $G_\varepsilon$ up to $t = \eta$.

**Step 1:** Given the initial data for the problem and a positive real number $T$ we fix $\varepsilon > 0$ and define

$$W_\varepsilon(t) \equiv 0 \text{ for all } 0 \leq t \leq \varepsilon,$$

and consider the corresponding function $G_\varepsilon$ defined as above. Thus

$$G_\varepsilon(t, W_\varepsilon(t)) = 0 \text{ for all } t \in [0, \varepsilon],$$

where we use that $0 \leq \gamma_\varepsilon(t) \leq (t - \varepsilon)^+ = 0$ for all $t \in [0, \varepsilon]$. Hence

$$W_\varepsilon \in C^1([0, \varepsilon]; H^1_0(\Omega; \mathbb{R}^m)).$$

**Step 2:** Suppose that for some $\eta > \varepsilon$ we have

$$W_\varepsilon \in C^1([0, \eta]; H^1_0(\Omega; \mathbb{R}^m))$$

and that

$$\|W_\varepsilon(t) - W_\varepsilon(\eta)\|_{2, \Omega} \to 0$$

as $t \to \eta$ where $W_\varepsilon(\eta) \in H^1_0(\Omega; \mathbb{R}^m)$ and such that for corresponding $G_\varepsilon$ it holds that

$$G_\varepsilon(t, W_\varepsilon(t)) = 0$$

for all $t \in [0, \eta]$. 
The partial Fréchet derivatives of $G_\varepsilon$ in the variable $u$ takes the form, using that the Fréchet derivative of $\Delta$ is $\Delta$ and Lemma 4.1,

$$\frac{\partial G^i}{\partial u^j}(t, u)(h) = \Delta h^i - (1 - \chi_{D_j}) \beta'_\varepsilon(u^i)h^i - \frac{1}{\varepsilon} \sum_{j \neq i} [\beta_\varepsilon(u^j) + \int_0^{\sigma_t(t)} \beta_\varepsilon(u^j_s(s)) \, ds] \beta'_\varepsilon(u^i)h^i$$

and

$$\frac{\partial G^i}{\partial u^j}(t, u)(h) = -\frac{1}{\varepsilon} \beta_\varepsilon(u^j)h^j,$$

for all $j \neq i$ and all $h \in H^1_0(\Omega; \mathbb{R}^m)$. It is clear that the map

$$\frac{\partial G_\varepsilon}{\partial u}(t, u) : H^1_0(\Omega; \mathbb{R}^m) \rightarrow H^{-1}(\Omega; \mathbb{R}^m)$$

is linear and bounded for fixed $u$. Continuity of this map follows from Lemma 4.1. The partial time derivative is the element

$$\frac{\partial G^i}{\partial t}(t, u) = -\beta_\varepsilon(t - \varepsilon) \frac{1}{\varepsilon} \sum_{j \neq i} \beta_\varepsilon(u^j(\gamma_\varepsilon(t))) \beta'_\varepsilon(u^i) + \beta_\varepsilon(t - \varepsilon) \mu^i$$

in $H^{-1}(\Omega)$ which also is continuous as a map $t \mapsto \frac{\partial G^i}{\partial t}(t, u)$ which follows from Lemma 4.2 since we assume $W_\varepsilon \in C([0, \eta]; L^2(\Omega; \mathbb{R}^m))$.

Hence $G_\varepsilon$ is in $C^1([0, \eta + \varepsilon) \times H^1_0(\Omega; \mathbb{R}^m), \mathbb{R})$. Invertibility of the linear map $\frac{\partial G^i}{\partial u}(t, u)$ at the point $(t, u) = (\eta, u_\varepsilon(\eta))$ follows from Lemma 4.3. We are now ready to use the implicit function theorem.

**Step 3:** By the implicit function theorem there exists an interval $(\eta - \delta, \eta + \delta) \subset (\eta - \varepsilon, \eta + \varepsilon)$, and a $C^1$ map

$$g : (\eta - \delta, \eta + \delta) \rightarrow H^1_0(\Omega; \mathbb{R}^m)$$

such that

$$G_\varepsilon(t, g(t)) = 0 \text{ for all } t \in (\eta - \delta, \eta + \delta).$$

We build the solution to our system by defining

$$W_\varepsilon(t) = g(t) \text{ for all } t \in (\eta, \eta + \delta).$$

Also note that by the uniqueness part of the implicit function theorem that $W_\varepsilon(t) = g(t)$ for all $t \in (\eta - \delta, \eta]$, hence we actually have that $W_\varepsilon \in C^1([0, \eta + \delta); H^1_0(\Omega; \mathbb{R}^m))$.

**Step 4:** Now we have a map $W_\varepsilon \in C([0, \eta + \delta); H^1_0(\Omega; \mathbb{R}^m))$ which solves equation 3.1 for all $t \in [0, \eta + \delta]$. By Lemma 3.4 the element $W_\varepsilon$ in $H^1_0(\Omega; \mathbb{R}^m)$ is uniformly bounded. Since $H^1_0(\Omega; \mathbb{R}^m)$ is a reflexive space, then if we choose a sequence $t = t_k \nearrow \eta + \delta$ then since $\{W_\varepsilon(t_k)\}_k$ is a bounded sequence we can extract a subsequence, again denoted by $\{t_k\}_k$, such that $W_\varepsilon(t_k) \rightharpoonup v$ in $H^1_0(\Omega; \mathbb{R}^m)$ for some $v \in H^1_0(\Omega; \mathbb{R}^m)$. By the compact embedding of $H^1_0(\Omega)$ in $L^2(\Omega)$ we can extract subsequences of $\{t_k\}_k$ such that the convergence of $u_\varepsilon^i(t_k)$ to $u^i$ is strong in $L^2(\Omega)$. 

A MULTIPHASE HELE-SHAW FLOW WITH SOLIDIFICATION 11
The found element \( v \) satisfies \( G_\varepsilon(\eta + \delta, v) = 0 \), that is
\[
\int_\Omega \nabla v^i \cdot \nabla \phi \, dx + \int_\Omega (1 - \chi_D^\varepsilon) \beta_\varepsilon(v^i) \phi \, dx \\
+ \int_\Omega \frac{1}{\varepsilon} \sum_{j \neq i} \beta_\varepsilon(v^j) + \int_0^{\tau_\varepsilon(\eta + \delta)} \beta_\varepsilon(u^j_\varepsilon(s)) \, ds \beta_\varepsilon(v^i) \phi \, dx \\
= \gamma_\varepsilon(\eta + \delta)(\phi, \mu_i),
\]
for all \( i = 1, \ldots, n \) and \( \phi \in H^1_0(\Omega) \). Indeed, the weak convergence \( u_\varepsilon(t_k) \to v \) clearly implies that
\[
\int_\Omega \nabla u^i_\varepsilon(t_k) \cdot \nabla \varphi \, dx \to \int_\Omega \nabla v^i \cdot \nabla \varphi \, dx,
\]
and for any \( M \in L^\infty(\Omega) \) we have
\[
\left| \int_\Omega M(\beta_\varepsilon(u^i_\varepsilon(t_k)) - \beta_\varepsilon(v^i)) \phi \, dx \right| \leq \| M \|_{L^\infty(\Omega)} \int_\Omega |\beta_\varepsilon(u^i_\varepsilon(t_k)) - \beta_\varepsilon(v^i)| |\phi| \, dx \\
\leq \| M \|_{L^\infty(\Omega)} C_\varepsilon \int_\Omega |u^i_\varepsilon(t_k) - v^i| |\phi| \, dx \\
\leq \| M \|_{L^\infty(\Omega)} C_\varepsilon \| u^i_\varepsilon(t_k) - v^i \|_{L^2(\Omega)} \| \phi \|_{L^2(\Omega)} \to 0
\]
as \( k \to \infty \). Using the result above with
\[
(5.9) \quad M_1 = (1 - \chi_{D^\varepsilon_0}) \quad \text{and} \quad M_2 = \frac{1}{\varepsilon} \sum_{j \neq i} \beta_\varepsilon(v^j) + \int_0^{\tau_\varepsilon(t)} \beta_\varepsilon(u^j_\varepsilon(s)) \, ds,
\]
where
\[
(5.10) \quad \| M_1 \|_{L^\infty(\Omega)} \leq 1 \quad \text{and} \quad \| M_2 \|_{L^\infty(\Omega)} \leq (m - 1)(1 + \eta + \delta)/\varepsilon
\]
which both are independent of \( k \), the claim that \( G_\varepsilon(\eta + \delta, v) = 0 \) follows.

Therefore we define
\[
(5.11) \quad W_\varepsilon(\eta + \delta) \equiv v.
\]
In summary, we have an element \( W_\varepsilon \in C^1([0, \eta + \delta); H^1_0(\Omega; \mathbb{R}^n)) \) satisfying\(^3.1\) and where \( W_\varepsilon(\eta + \delta) \in H^1_0(\Omega; \mathbb{R}^n) \) also is defined and satisfying\(^3.1\) as well as
\[
\| W_\varepsilon(t) - W_\varepsilon(\eta + \delta) \|_{2, \Omega} \to 0
\]
as \( t \to \eta + \delta \) where \( W_\varepsilon(\eta + \delta) \in H^1_0(\Omega; \mathbb{R}^n) \).

**Step 5:** The same procedure can be repeated indefinitely, starting at step 2.

**Step 6:** The final step is to show that this construction actually defines \( W_\varepsilon(t) \) for all \( t \geq 0 \). By construction we have a solution \( W_\varepsilon(t) \) for \( t \) in an entire interval \( I \) on the real line. Suppose that this interval is bounded from above and assume that \( \sup I \notin I \), then the same reasoning as in step 4 shows that we can extend the solution up to \( \sup I \) and apply the implicit function theorem to enlarge \( I \), which is a contradiction, hence \( I \supset \mathbb{R}^+ \).  

□
6. Vanishing-Properties of the \( \varepsilon \)-solutions \( W_\varepsilon \)

The following lemma concerns a vanishing property which deals with the interaction of two phases at a specific point in time.

**Lemma 6.1.** For \( \varepsilon > 0 \) let \( W_\varepsilon \) be a solution to the \( \varepsilon \)-problem then for any \( t \in [0, T] \), any given \( \delta > 0 \) and two separate phases \( i \neq j \) it holds that

\[
\left| \left\{ x \in \Omega : u_i^\varepsilon(t) > \delta \right\} \cap \left\{ x \in \Omega : u_j^\varepsilon(t) > \delta \right\} \right| \leq \frac{C}{\delta} \varepsilon
\]

for some constant \( C \) which may depend on the initial data but independent of \( \delta \) and \( \varepsilon \).

**Proof.** Consider for any \( \varepsilon \) the solution \( W_\varepsilon \) to the \( \varepsilon \)-problem. Then \( u_i^\varepsilon(t) \) is a well defined element in \( H_0^1(\Omega) \) for every \( t \in [0, T] \). By considering equation \( 3.5 \) and taking one of the left hand terms and applying absolute values we obtain

\[
\frac{1}{\varepsilon} \int_\Omega \left| \sum_{j \neq i} \beta_\varepsilon(u_j^\varepsilon(t)) \beta_\varepsilon(u_i^\varepsilon(t)) u_i^\varepsilon(t) \right| \, dx \leq \left| \gamma_\varepsilon(t) \| u_i^\varepsilon(t) \|_{H_0^1(\Omega)} \| u_i^\varepsilon(t) \|_{H^{-1}(\Omega)} \right|
\]

\[
\leq C'T^2 \| u_i^\varepsilon \|^2_{H^{-1}(\Omega)} = C,
\]

since all terms in the left hand side of \( 3.5 \) are non-negative and by the uniform bound on the family \( \{ u_i^\varepsilon(t) \}_{\varepsilon > 0} \). Note that \( C' \) is independent of \( \varepsilon \), hence \( C \) is as well.

For some \( i \) and \( j \), such that \( i \neq j \), then given any \( t > 0 \) and \( \delta > 0 \) we consider the set \( A_{ij}^\delta \subset \Omega \) where both \( u_i^\varepsilon(t) \) and \( u_j^\varepsilon(t) \) are greater than \( \delta \), viz.

\[
A_{ij}^\delta \equiv \left\{ x \in \Omega : u_i^\varepsilon(t) > \delta \right\} \cap \left\{ x \in \Omega : u_j^\varepsilon(t) > \delta \right\}.
\]

Then

\[
\frac{1}{\varepsilon} \int_\Omega \sum_{k \neq i} \beta_\varepsilon(u_k^\varepsilon(t))^k \beta_\varepsilon(u_i^\varepsilon(t)) u_i^\varepsilon(t) \, dx \geq \frac{1}{\varepsilon} \int_{A_{ij}^\delta} \beta_\varepsilon(u_i^\varepsilon(t)) \beta_\varepsilon(u_i^\varepsilon(t)) u_i^\varepsilon(t) \, dx
\]

\[
\geq \frac{1}{\varepsilon} \int_{A_{ij}^\delta} \delta \, dx
\]

\[
= \frac{\delta}{\varepsilon} |A_{ij}^\delta|,
\]

where the first inequality follows by positivity of all factors and the second holds for all \( \varepsilon \) small enough. Upon combining \( 6.2 \) and \( 6.4 \) we get the desired result

\[
|A_{ij}^\delta| \leq \frac{C}{\delta} \varepsilon.
\]

\( \Box \)

7. Existence of a limit flow \( W \) as \( \varepsilon \to 0^+ \)

In Theorem 7.1 we will show that there is a limit flow \( W \) in \( L^2(0, T; H_0^1(\Omega; \mathbb{R}^m)) \) such that the solutions \( W_\varepsilon \) will weakly converge \( W \) as \( \varepsilon \to 0^+ \).

**Theorem 7.1.** For any positive number \( T \) there is some sequence \( \varepsilon = \varepsilon_k \to 0^+ \) and an element \( W \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^m)) \) such that the flow \( W_\varepsilon \) converges weakly to \( W \) in \( L^2(0, T; H_0^1(\Omega; \mathbb{R}^m)) \) over \( \varepsilon_k \).
Proof. According to Corollary\,3.5
\[ \|W_\varepsilon\|_{L^2(0,T;H^1_0(\Omega;\mathbb{R}^m))} \leq C \] (7.1)
for some constant \( C \) independent of \( \varepsilon \). Thus for any sequence \( \varepsilon = \varepsilon_k \to 0^+ \) the corresponding solutions \( W_\varepsilon \) in the space \( L^2(0,T;H^1_0(\Omega;\mathbb{R}^m)) \) will be uniformly bounded. Hence there is a subsequence, keeping the name \( \varepsilon_k \), and an element \( W \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^m)) \) such that \( W_\varepsilon \) converges weakly to \( W \) in \( L^2(0,T;H^1_0(\Omega;\mathbb{R}^m)) \) as \( \varepsilon_k \to 0^+ \). \( \Box \)

References


Department of mathematics, KTH, SE-100 44, Stockholm, Sweden
E-mail address: karljo@kth.se