Bergman space methods and integral means spectra of univalent functions

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Abstract

We study universal integral means spectra of certain classes of univalent functions defined on subsets of the complex plane. After reformulating the definition of the integral means spectrum of a univalent function in terms of membership in weighted Bergman spaces, we describe the Hilbert space techniques that can be used to estimate universal means spectra from above. Finally, we show that the method of norm expansion used in that context can be applied in a more general setting to reproducing kernel spaces in order to explicitly compute kernel functions.
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Submitted for publication
Introduction

The present thesis for the degree of licentiate of technology consists of an introductory section and three research papers. The first paper, *Spectral notions for conformal mappings: a survey*, has been written jointly with Håkan Hedenmalm and has been submitted for publication. The second paper, *An estimate of the universal means spectrum of conformal mappings*, has previously appeared in the mathematical journal Computational methods and function theory. The third paper, *Norm expansion along a zero variety*, is the result of a collaboration with Håkan Hedenmalm and Serguei Shimorin, and has been submitted for publication.

In this introductory section, we begin by providing the reader with some general background material. We then move on to a discussion of our results and methods and close the introduction by pointing out some remaining problems. References are listed at the end of the introduction.

1 Univalent functions

In this thesis, we consider various classes of holomorphic functions defined on subsets of the extended complex plane, more precisely, the *unit disk*

$$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$$

and the *exterior disk*

$$\mathbb{D}_e = \{ z \in \mathbb{C} \cup \{\infty\} : |z| > 1 \}.$$  

A holomorphic function which is one-to-one is usually called *univalent* or *schlicht*. Univalent functions are examples of *conformal mappings*, that is, mappings with angle-preserving properties.
1.1 The classes $S$ and $\Sigma$

The class $S$ consists of functions $\varphi$ that are univalent in $D$ and, in addition, satisfy the requirements $\varphi(0) = 0$ and $\varphi'(0) = 1$. This normalization means that the Taylor expansion of a function $\varphi \in S$ is of the form

$$\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  \hfill (1.1)

The functions in $S$ can be viewed as conformal mappings of the unit disk onto simply connected domains which contain the origin. For instance, the identity map $\varphi(z) = z$ is in $S$; another example of a function in this class is the \textit{Koebe function}

$$\kappa(z) = \frac{z}{(1-z)^2}.$$  \hfill (1.2)

The Koebe function maps the disk onto the complex plane minus the slit $(-\infty, 1/4]$. As we shall see shortly, the function $\kappa$ plays the part of extremal function in many problem for the class $S$.

There is a close relative of $S$ which is defined in the exterior disk. This is the class $\Sigma$ of functions $\psi$ that are univalent in $D_e$ except for a simple pole at infinity and admit expansions of the form

$$\psi(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}.$$  \hfill (1.3)

The functions in $\Sigma$ map the exterior disk into the extended complex plane while fixing the point at infinity and omitting a compact connected set. If this omitted set has area measure zero we say that $\psi$ is a \textit{full mapping}. For example, the identity map $\psi(z) = z$ belongs to $\Sigma$ but is not a full mapping.

The classes $S$ and $\Sigma$ have been studied extensively since the beginning of the 20th century and we refer the reader to the books [6] and [18] for excellent introductions to the classical theory of univalent functions.

1.2 Pointwise estimates and coefficient problems

As early as 1916, Ludwig Bieberbach discovered that the second Taylor coefficient $a_2$ of any function in $S$ satisfies the inequality $|a_2| \leq 2$ and that equality occurs only for the \textit{Koebe function}

$$\kappa(z) = \frac{z}{(1-z)^2}.$$  \hfill (1.4)
or its rotations \(e^{-i\theta}\kappa(e^{i\theta}z)\). We note that the Koebe function has the expansion
\[
\kappa(z) = \sum_{n=1}^{\infty} nz^n.
\] (1.5)

Bieberbach’s result can be used to derive the growth theorem
\[
\frac{|z|}{(1+|z|)^2} \leq |\varphi(z)| \leq \frac{|z|}{(1-|z|)^2}, \quad z \in \mathbb{D},
\] (1.6)
and the distortion theorem or Verzerrungssatz
\[
\frac{1-|z|}{(1-|z|)^3} \leq |\varphi'(z)| \leq \frac{1+|z|}{(1-|z|)^3}, \quad z \in \mathbb{D}
\] (1.7)
for \(\varphi \in \mathcal{S}\). Again, equality in either (1.6) or (1.7) implies that \(\varphi\) is the Koebe function or one of its rotations. Similarly, the argument of the derivative of a function \(\varphi \in \mathcal{S}\) admits the estimate
\[
|\arg \varphi'(z)| \leq 2 \log \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D};
\] (1.8)
this result, however, is not sharp. The sharp rotation theorem is given in [6], p. 35.

In the light of these results, Bieberbach conjectured in 1916 that the coefficients of each function \(\varphi \in \mathcal{S}\) satisfy \(|a_n| \leq n\) for \(n = 2, 3, \ldots\) and that strict inequality holds unless \(\varphi\) is the Koebe function or a rotation of the Koebe function. Using elementary but quite elegant arguments, J.E. Littlewood promptly proved the inequality \(|a_n| < en\) for \(n = 2, 3, \ldots\) in 1925. The Bieberbach conjecture proved to be a real challenge for the mathematical community, and many interesting techniques in complex analysis were developed to obtain various partial results on the Bieberbach conjecture. For instance, in 1923 Karl Löwner (later known as Charles Loewner) derived the famous differential equation that parametrizes single-slit mappings in \(\mathcal{S}\) and used it to prove that \(|a_3| \leq 3\). In recent years, the Löwner differential equation has attracted a considerable amount of attention due to the work of Oded Schramm and others on Stochastic Löwner evolution. The full Bieberbach conjecture was finally proved in 1985 by Louis de Branges (see [5]), to the considerable suprise of researchers in the field of univalent functions.
Let us now turn to univalent functions in the exterior disk. A basic result for the class $\Sigma$ is the \textit{area theorem}. In 1914, Thomas Hakon Grönwall discovered that the Laurent coefficients of a univalent function

$$
\psi(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}
$$

satisfy the inequality

$$
\sum_{n=1}^{\infty} n|b_n|^2 \leq 1,
$$

with equality for functions that are full mappings. This result can be generalized in many ways; one of these generalizations which is due to Prawitz is exploited in the method developed by Hedenmalm and Shimorin in [12]. In particular, the area theorem yields the inequality $|b_1| \leq 1$, which is sharp for the function

$$
\lambda(z) = z - \frac{1}{z}
$$

or rotations of this function. The function $\lambda$ is related to the Koebe function $\kappa$ by means of a square root transformation followed by an inversion:

$$
\lambda(z) = [\kappa(z^{-2})]^{-\frac{1}{2}}.
$$

This fact might lead us to suspect that we should have

$$
|b_n| \leq \frac{2}{n+1}, \quad \text{for} \quad n = 1, 2, \ldots,
$$

and that equality should occur precisely for rotations of

$$
\psi(z) = [\kappa(z^{-n-1})]^{-\frac{1}{n+1}}.
$$

This relation does hold for $n = 2$, but fails for $n \geq 3$ (the correct sharp bound is $|b_3| \leq \frac{1}{2} + e^{-6}$). The conjecture is also false asymptotically, that is, there exist functions such that $b_n \neq O(n^{-1})$ as $n$ tends to infinity. On the other hand, the area theorem guarantees that $b_n = O(n^{-1/2})$ as $n \to \infty$. The coefficient problem for the class $\Sigma$ is still open, although the trivial asymptotic estimates that follow from the area theorem have been improved by James Clunie, Christian Pommerenke, and others. The coefficient problem for $\Sigma$ provides us with one motivation to study the integral means of the derivatives of functions in this class. This connection is discussed in greater detail in the paper [3] of Lennart Carleson and Peter Jones, and in the paper I of this thesis.
1.3 Integral means

In the papers in this thesis, we study the behavior of univalent functions in the mean and at the boundary. For \( \tau \in \mathbb{C} \) we introduce the integral means of the derivative of a function \( \varphi \in S \) as

\[
M_\tau[\varphi'](r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| |\varphi'(re^{i\theta})|^\tau \right| d\theta, \quad 0 < r < 1.
\] (1.12)

For real values of \( \tau \), the quantity \( M_\tau[\varphi'] \) measures the expansion associated with the map \( \varphi \), and if we consider complex \( \tau \), we also take rotation into account. This can be seen by writing

\[
|(|(\varphi')^\tau| = |\varphi'|^\Re \tau e^{-\Im \tau \arg \varphi'}.
\]

The inequalities (1.7) and (1.8) show that there exists a positive number \( \beta \), depending on \( \tau \), such that

\[
M_\tau[\varphi'](r) = O\left( \frac{1}{(1-r)^\beta} \right) \quad \text{as} \quad r \to 1^-.
\] (1.13)

For \( \varphi \in S \), we define \( \beta_\varphi(\tau) \) as the infimum of all \( \beta \) such that (1.13) holds; this is the integral means spectrum of \( \varphi \). Next, the universal means spectrum for the class \( S \) is defined as

\[
B_S(\tau) = \sup_{\varphi \in S} \beta_\varphi(\tau).
\] (1.14)

An application of the Hölder inequality shows that \( B_S \) is a convex function. For real \( \tau \), we can easily obtain a trivial estimate of \( B_S \):

\[
0 \leq B_S(\tau) \leq \max\{3\tau, -\tau\}, \quad \tau \in \mathbb{R}.
\]

In 1976, J. Feng and T. H. MacGregor (see [8]) found the values of \( B_S(\tau) \) for \( \tau \) in a certain interval:

\[
B_S(\tau) = 3\tau - 1, \quad \text{for} \quad \frac{2}{5} \leq \tau \leq +\infty.
\] (1.15)

Then, in 1994, Lennart Carleson and Nikolai Makarov determined \( B_S \) for negative \( \tau \) smaller then a certain number \( R_{CM} \) (see [4]):

\[
B_S(\tau) = -\tau - 1, \quad \text{for} \quad -\infty \leq \tau \leq R_{CM}.
\] (1.16)
It is known that $R_{CM} \leq -2$, but the exact value of $R_{CM}$ has not yet been determined. The famous Brennan conjecture in conformal mapping is equivalent to saying that $R_{CM} = -2$, or that $B_S(-2) = 1$. We note that the integral means of the Koebe function are

$$\beta_n(\tau) = \begin{cases} 
3\tau - 1 & \text{if } \tau \geq \frac{1}{3}, \\
0 & \text{if } -1 \leq \tau < \frac{1}{3}, \\
-\tau - 1 & \text{if } \tau < -1
\end{cases}$$

and we see that when $t \geq 2/5$ or $t < R_{CM}$, the Koebe function is once again extremal. For small $\tau$, we should expect more complicated mappings to have large integral means (cf. the remarks in [4]).

For real $\tau$ in the interval $R_{CM} < \tau < 2/5$, it remains an open problem to compute the universal means spectrum. There does exist an upper estimate of $B_S(\tau)$ due to James Clunie and Christian Pommerenke which covers all real values of $\tau$:

$$B_S(\tau) \leq \tau - \frac{1}{2} + \left(4\tau^2 - \tau + \frac{1}{4}\right)^{\frac{1}{2}}, \quad \tau \in \mathbb{R},$$

(1.17)

but unfortunately, this estimate is not sharp. Better estimates of $B_S(\tau)$ for small (complex) $\tau$ were recently found by Håkan Hedenmalm and Serguei Shimorin in the papers [12] and [13]:

$$\limsup_{|\tau| \to 0} \frac{B_S(\tau)}{|\tau|^2} \leq 0.3798 \ldots$$

(1.18)

A well-known conjecture due to Philipp Kraetzer asserts that

$$B_S(\tau) = \frac{\tau^2}{4}, \quad \text{for } -2 \leq \tau \leq 2.$$  

The basis for this are the numerical results of [16] and earlier conjectures of Carleson and Jones.

The paper [12] also contains numerical estimates of $B_S(\tau)$ for real values of $\tau$. In paper II of this thesis we use similar methods and obtain a slight improvement of the estimates of [12]. For instance, we find that $B_S(-1) \leq 0.388$ and $B_S(-2) \leq 1.206$.

We proceed to define the integral means of the derivative of a function $\psi \in \Sigma$ by setting

$$M_r[\psi'](r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |[\psi'(re^{i\theta})]'|d\theta, \quad 1 < r < +\infty.$$  

(1.19)

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There exists a sharp estimate on the logarithm of the derivative of a function \( \psi \in \Sigma \), this is the inequality

\[
|\log \psi'(z)| \leq \log \frac{|z|^2}{|z|^2 - 1}, \quad z \in \mathbb{D}. \tag{1.20}
\]

In this setting, we are interested in the behavior of the integral means as \( r \to 1^+ \). We define the integral means spectrum \( \beta_\psi(\tau) \) of a function \( \psi \in \Sigma \) for complex \( \tau \) as the infimum of all real \( \beta \) such that

\[
M_r[\psi'](r) = O\left( \frac{1}{(1-r)\beta} \right), \quad \text{as} \quad r \to 1^+; \tag{1.21}
\]

in view of the previous inequality, there always exists such a \( \beta \). We then define the universal means spectrum just like before, by setting

\[
B_\Sigma(\tau) = \sup_{\psi \in \Sigma} \beta_\psi(\tau). \tag{1.22}
\]

The trivial estimate for \( B_\Sigma(\tau) \) reads

\[
0 \leq B_\Sigma(\tau) \leq |\tau|, \quad \tau \in \mathbb{R}; \tag{1.23}
\]

and it is known that

\[
B_\Sigma(\tau) = |\tau| - 1, \quad \text{for} \quad \tau \in (-\infty, R_{CM}] \cup [2, +\infty). \tag{1.24}
\]

Here, the constant \( R_{CM} \) is the same as before. Again, the open problem is to determine \( B_\Sigma \) for small values of \( \tau \). Actually, it holds that

\[
B_S(\tau) = \max \{ B_\Sigma(\tau), 3\tau - 1 \}, \quad \tau \in \mathbb{R}; \tag{1.25}
\]

and hence it would be enough to compute \( B_\Sigma \) to determine both universal means spectra. This result is due to Nikolai Makarov (see [17]), and his result has been extended to complex values of \( \tau \) by Ilia Binder in an unpublished manuscript (see [2]).

It is also of interest to study the integral means of bounded univalent functions in the disk. We denote by \( S_b \) the class of functions that are univalent and bounded in \( \mathbb{D} \), and satisfy \( \varphi(0) = 0 \). For the class of bounded functions, it is known that

\[
B_{S_b}(\tau) = \tau - 1, \quad \text{for} \quad \tau \geq 2.
\]
and in [14], Jones and Makarov establish that
\[ BS_b(2 - \tau) = 1 - \tau + O(\tau^2), \quad \text{as} \quad \tau \to 0. \]

One should note that the universal means spectra of functions in \( \Sigma \) and this class of bounded mappings of the unit disk are the same (see the paper [3] or paper I of this thesis).

2 Bergman spaces

In our investigations of universal means spectra of conformal mappings, we rely to a great extent on Hilbert space techniques. More precisely, we find it useful to consider certain classes of Hilbert spaces of holomorphic functions called Bergman spaces.

The function and operator theory of the Bergman spaces is of course interesting in its own right, and many important results have been obtained in recent years. We refer the reader to the books [7] and [11] for treatments of Bergman spaces in the unit disk of \( \mathbb{C} \). The books [16] and [20] include sections on Bergman spaces in higher dimensions.

2.1 Bergman spaces in \( \mathbb{C} \)

We again consider the unit disk in the complex plane and supply \( \mathbb{D} \) with the probability measures
\[ dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \frac{dx dy}{\pi}, \quad z = x + iy; \]
where we take \(-1 < \alpha < +\infty\). The weighted Lebesgue space \( L^2_\alpha(\mathbb{D}) \) consists of measurable functions \( f \) in \( \mathbb{D} \) with
\[ \|f\|_\alpha^2 = \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < +\infty; \quad (2.1) \]
and the inner product in the space \( L^2_\alpha(\mathbb{D}) \) is given by the expression
\[ \langle f, g \rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z). \quad (2.2) \]

The weighted Bergman space \( A^2_\alpha(\mathbb{D}) \) is the subspace of \( L^2_\alpha(\mathbb{D}) \) consisting of holomorphic functions. Being a closed subspace of \( L^2_\alpha(\mathbb{D}) \), the Bergman space is a Hilbert space.
In fact, the space \( A^2_{\alpha}(\mathbb{D}) \) is a reproducing kernel space. One can show that the point evaluation functional
\[
\Phi_w : A^2_{\alpha}(\mathbb{D}) \ni f \mapsto f(w)
\]
is bounded at each point \( w \in \mathbb{D} \). Hence, by the Riesz representation theorem, there exists a function \( k_w \in A^2_{\alpha}(\mathbb{D}) \) with the reproducing property
\[
f(w) = (f, k_w)_{\alpha}, \quad w \in \mathbb{D}.
\]
(2.3)
We write \( k(z; w) = k_w(z) \) and consider \( k \) as a function on \( \mathbb{D} \times \mathbb{D} \); it is holomorphic in the first argument and anti-holomorphic in the second. An explicit computation involving an orthonormal basis for \( A^2_{\alpha}(\mathbb{D}) \) yields an expression for the kernel function:
\[
k(z; w) = \frac{1}{(1 - wz)^{2+\alpha}}.
\]
Let us assume that \( f \in A^2_{\alpha}(\mathbb{D}) \) has Taylor expansion \( f = \sum_{n=0}^{\infty} a_n z^n \).
There is an analogue of the Parseval formula for the Bergman space \( A^2_{\alpha}(\mathbb{D}) \) which reads
\[
\|f\|^2_{\alpha} = \sum_{n=0}^{\infty} \frac{n!}{(\alpha + 2)_n} |a_n|^2.
\]
Here, we have used the Pochhammer notation
\[
(x)_n = x(x+1)(x+2) \cdots (x+n-1).
\]
This expression of the norm in terms of Taylor coefficient can be used to derive a relation between the norm of a Bergman space functions and its derivatives (see [12] for details):
\[
0 \leq C(\alpha, n)\|f\|^2_{\alpha} - \|f^{(n)}\|^2_{\alpha+2n} = O(\|f\|^2_{\alpha+\theta}), \quad 0 < \theta \leq 1.
\]
(2.4)
This result is one of main tools we rely on when employing Bergman space methods in the study of integral means spectra of conformal mappings.

Returning for a moment to univalent functions, we find that an equivalent definition of the integral means spectrum of a mapping \( \varphi \in S \) in terms of Bergman spaces is to take
\[
\beta_{\varphi}(\tau) = \inf\{\alpha + 1 : (\varphi')^{\tau/2} \in A^2_{\alpha}(\mathbb{D})\}.
\]
(2.5)
This means that if we know that
\[ \| (\varphi')^{r/2} \|_\alpha^2 < +\infty \]
holds for every \( \varphi \in S \), then \( B_S(\tau) \leq \alpha + 1 \). Similarly, the universal means spectrum for the class \( \Sigma \) can be described in terms of weighted Bergman spaces in the exterior disk.

The advantage of these reformulations is that we can now use Hilbert space methods, such as orthogonal projections, to study conformal mappings. These ideas were developed in the paper [12] of Hedenmalm and Shimorin and constitute the foundation for the work in this thesis.

2.2 Bergman spaces in \( \mathbb{C}^d \)

We now turn to Bergman spaces in \( \mathbb{C}^d \) for \( d \geq 2 \). In this section, we write \( z = (z_1, \ldots, z_d) \) for a point in \( \mathbb{C}^d \), and we let \( dV \) denote the Lebesgue measure on \( \mathbb{C}^d \). Finally, we set
\[ z \cdot w = z_1 \overline{w_1} + \cdots + z_d \overline{w_d}; \]
the norm corresponding to the standard Euclidean inner product will be denoted by \( \|z\| \). The \( d \)-dimensional polydisk is the set
\[ D^d = \{ z \in \mathbb{C}^d : |z_i| < 1, \; i = 1, \ldots, d \} \]
and the unit ball in \( \mathbb{C}^d \) is
\[ B^d = \{ z \in \mathbb{C}^d : \|z\|^2 < 1 \}. \]

Let \( \Omega \) be a domain in \( \mathbb{C}^d \). We shall consider the Lebesgue spaces \( L^2(\Omega) \) of functions \( f \) that are measurable with respect to \( dV \) and have finite norm in the sense that
\[ \|f\|^2 = \int_{\Omega} |f(z)|^2 dV(z) < +\infty. \tag{2.6} \]
It is clear that \( L^2(\Omega) \) is a Hilbert space with inner product given by
\[ \langle f, g \rangle = \int_{\Omega} f(z) \overline{g(z)} dV(z). \tag{2.7} \]
More generally, we need weighted \( L^2 \)-spaces, where the norm is defined by expressions of the type
\[ \|f\|_\omega^2 = \int_{\Omega} |f(z)|^2 \omega(z) dV(z), \tag{2.8} \]
for some reasonable weight function $\omega$. Just as before, the *weighted Bergman spaces* $A^2_\omega(\Omega)$ are the subspaces consisting of holomorphic functions. In paper III, for instance, we study a weighted Bergman space in the unit bidisk with norm

$$\|f\|_{A^2_\omega(\Omega)}^2 = \int_{D^2} |f(z_2, z_2)|^2 |1 - \overline{z_1} z_1 |^{2\theta} |z_1 - z_2|^{2\beta} dA_\alpha(z_1) dA_\beta(z_2). \quad (2.9)$$

In the unweighted case, we will simply write $A^2(\Omega)$ instead of $A^2_1(\Omega)$.

We again find that the point-evaluation functional

$$\Phi_w : A^2(\Omega) \ni f \to f(w)$$

is bounded at each point $w \in \Omega$, and this ensures the existence of a kernel function $k^\Omega_w$ such that

$$f(w) = \langle f, k^\Omega_w \rangle$$

for each $f \in A^2(\Omega)$. The function $k^\Omega(z; w) = k^\Omega_w(z)$ is called the *Bergman kernel* of $\Omega$; it has properties analogous to those of the kernel function in the one-dimensional case. At this point, we should mention that the Bergman kernel is a particular instance of the more general concept of reproducing kernel; we refer the reader to the book [21] for a more extensive treatment of such kernels.

It is of great interest to understand the properties of the Bergman kernel of a domain $\Omega$. The kernel function is a useful tool in the study of function spaces on $\Omega$ as well as in the theory of biholomorphic mappings (we suggest the reader consult the book [16] for more information on these topics). In principle, we can always write down an expression for $k^\Omega$ in terms of an orthonormal basis $\{e_i\}_{i=1}^\infty$ for the space $A^2(\Omega)$:

$$k^\Omega(z; w) = \sum_{i=1}^\infty e_i(z) \overline{e_i(w)}, \quad (2.10)$$

and this series converges uniformly on compact subsets of $\Omega$. We note that the representation (2.10) is actually independent of the choice of orthonormal system.

For general domains $\Omega$ it is rather difficult to find an explicit orthonormal basis, and explicit expressions for the kernel function are known only in a limited number of cases. For instance, in [16] it is shown that the Bergman kernel for the unit ball is

$$k^{B^d}(z; w) = \frac{d!}{\pi^d} \frac{1}{(1 - z \cdot w)^{d+1}}.$$
and that the Bergman kernel for the bidisk is given by the expression
\[ k^D_d(z; w) = \frac{1}{\pi^d} \prod_{i=1}^{d} \frac{1}{(1 - z_i w_i)^2}. \]

As one might expect, it is tricky to compute the Bergman kernel function for weighted Bergman spaces, even in the cases of such simple domains as the polydisk or the unit ball. For very special kinds of weight functions that match the symmetry of the domain \( \Omega \), it is sometimes possible to compute the kernel functions (see [20] for radial weight functions in the unit ball) without too much effort. Paper III of this thesis is devoted to the problem of computing kernel functions for weighted Bergman spaces using a technique introduced in the paper [12].

### 2.3 Projections and restrictions

Associated with the Bergman kernel \( k^\Omega \) is the orthogonal projection of \( L^2_\omega(\Omega) \) onto \( A^2_\omega(\Omega) \) given by
\[
P[f](z) = \int_{\Omega} f(w) k^\Omega(z; w) \omega(w) dV(w);
\]

It holds true that any closed subspace \( \mathcal{N}(\Omega) \) of \( A^2_\omega(\Omega) \) admits a reproducing kernel \( k^\Omega_{\mathcal{N}} \), and this kernel in turn induces another projection operator \( P_{\mathcal{N}} \) of the form
\[
P_{\mathcal{N}}[f](z) = \int_{\Omega} f(w) k^\Omega_{\mathcal{N}}(z, w) \omega(w) dV(w)
\]
that maps \( L^2_\omega(\Omega) \) onto \( \mathcal{N}(\Omega) \). This provides us with another motivation to study reproducing kernels: an understanding of the kernel function yields information about projection operators.

Now, let \( p \) be a (nontrivial) polynomial in \( d \) variables and consider the variety \( \mathcal{V} = \{ z \in \Omega : p(z) = 0 \} \). For instance, returning to the weighted Bergman space in the bidisk, we take \( p(z_1, z_2) = z_1 - z_2 \) and obtain the diagonal \( \{ z \in \mathbb{D}^2 : z_1 = z_2 \} \) as our subvariety. In what follows, we shall in addition assume that the gradient of \( p \) does not vanish along \( \mathcal{V} \). The subspaces \( \mathcal{N}_N(\Omega) \) of functions that are divisible by \( p^N \) then provide us with examples of closed subspaces of the reproducing kernel space \( A^2_\omega(\Omega) \). Next, we consider the difference space
\[ \mathcal{M}_N(\Omega) = \mathcal{N}_N(\Omega) \oplus \mathcal{N}_{N+1}(\Omega), \]
which is a closed subspace of $\mathcal{N}_N$ and hence a reproducing kernel space. We denote the projection operator associated with the kernel of $\mathcal{M}_N$ by $Q_N$.

No nontrivial holomorphic function can be divided by $p$ an infinite number of times. Hence, we can decompose the space $A^2_\omega(\Omega)$ as a direct sum of the spaces $\mathcal{M}_N$:

$$A^2_\omega(\Omega) = \bigoplus_{N=0}^{\infty} \mathcal{M}_N(\Omega),$$

and this fact leads to a decomposition of the norm:

$$\|f\|_{2,\omega}^2 = \sum_{N=0}^{\infty} \|Q_N[f]\|_{\omega}^2$$

as well as a series representation of the kernel function:

$$k_\Omega(z; w) = \sum_{N=0}^{\infty} k_\Omega^{\mathcal{M}_N}(z; w).$$

If the variety $V$ has certain nice properties, it may be possible to find an expression for the kernel function when its arguments are restricted to the variety. We then try to identify the restriction of the kernel function with the kernel function of a Hilbert space of functions on the variety. Let us denote this space by $\mathcal{H}$. In the case of the bidisk, the restricted kernel function coincides the the usual kernel function of a weighted Bergman space of type $A^2_\alpha(D)$ for an appropriate choice of $\alpha$. We introduce the operator $\circ$ which restricts a function $f \in A^2_\omega(\Omega)$ to the subvariety $V$. The theory of reproducing kernels now provides us with an embedding of restricted functions into the space $\mathcal{H}$ via the inequality

$$\| \circ [f]\|_\mathcal{H}^2 \leq \|f\|_{2,\omega}^2.$$  

(2.13)

In fact, we find that equality holds precisely when $f$ belongs to the subspace $\mathcal{M}_0(\Omega)$. Hence, using the projection operator $Q_0$, we may rewrite the above inequality as an equality:

$$\| \circ [f]\|_\mathcal{H}^2 = \|Q_0[f]\|_{\omega}^2.$$  

(2.14)

This means that if $f$ does not vanish on $V$, then we can compute the norm of the function in the space $A^2_\omega(\Omega)$ by computing a norm in a Hilbert space
\( \mathcal{H} \) on a lower-dimensional subvariety of \( \Omega \). Of course, if \( f \in A^2_\omega(\Omega) \) vanishes along \( \mathcal{V} \), we have strict inequality in (2.13) since the left-hand side is zero. Thus, taking the restriction to \( \mathcal{V} \) does not give us anything. Instead, we project the function \( f \) onto \( \mathcal{M}_1(\Omega) \), we divide the result by the polynomial \( p \), apply the operator \( \otimes \) and try to repeat the procedure. In practice, we want to express the resulting operation in terms of derivatives and restrictions of the original function \( f \).

In the end, this repeated procedure leads to an expression for the norm of the full space \( A^2_\omega(\Omega) \) in terms of a series of norms computed in suitable Hilbert spaces on \( \mathcal{V} \). Analogously, the kernel function \( k^\Omega \) will be given by a series expansion in terms of restricted kernel functions. Typically, each term in the series expansion of the norm involves the original function differentiated with respect to one of the variables and restricted to the variety. The usefulness of the method is unfortunately somewhat limited by several factors; one of these is that we have to be able to identify the necessary restriction Hilbert spaces on the variety. Depending on the function space we consider, the computations required to express the projected and divided function in terms of derivatives and restrictions of the original function may be quite involved. The ideas we have presented here are developed in greater detail in paper III of this thesis.

### 3 Estimating the universal means spectrum

In this final section, we provide a sketch of how Bergman space techniques can be applied to the study of universal means spectra of conformal mappings.

#### 3.1 Bloch spaces and Bergman spaces

The **Bloch space** \( \mathcal{B}(\mathbb{D}) \) consists of functions \( f \) that are holomorphic in the unit disk and satisfy

\[
\|f\|_{\mathcal{B}(\mathbb{D})} = \sup \{ (1 - |z|^2)|f'(z)| : z \in \mathbb{D} \} < +\infty.
\]  

The **little Bloch space** \( \mathcal{B}_0(\mathbb{D}) \) is the subspace of \( \mathcal{B}(\mathbb{D}) \) consisting of functions with

\[
\lim_{|z| \to 1^{-}} (1 - |z|^2)|f'(z)| = 0
\]

The idea here is to develop techniques for estimating the norm of the full space \( A^2_\omega(\Omega) \) in terms of a series of norms computed in suitable Hilbert spaces on \( \mathcal{V} \).
The expression (3.1) defines a semi-norm on $B(D)$ that can be changed into a norm by taking
\[ \|f\| = |f(0)| + \|f\|_{B(D)}. \] (3.3)

When supplied with this norm, $B(D)$ is a Banach space and the little Bloch space is the closure of the set of polynomials. Moreover, we note that the Bloch space contains the space $H^\infty$ and that $B(D)$ is in $A^2_\alpha(D)$ for every $\alpha$. The first chapter of [11] contains more material on the Bloch space, as does the book [9].

Let us see how the Bloch space relates to univalent functions and the study of the universal means spectrum. It is well-known that if $\varphi \in S$, then the function $f = \log \varphi'$ belongs to the Bloch space and $\|f\|_{B(D)} \leq 6$. Conversely, if $\|f\|_{B(D)} \leq 1$ holds for a Bloch function $f$, then there exists a function $\varphi \in S$ such that $f = \log \varphi'$.

For $\varphi \in S$, we consider the function
\[ g_\tau = (\varphi')^{\tau/2} = \exp \left\{ \frac{\tau}{2} f \right\}, \]
that is, the exponential of the Bloch space function $f = \log \varphi'$. Estimating $\beta_\varphi(\tau)$ now amounts to finding an $\alpha$ such that $g_\tau$ belongs to the Bergman space $A^2_\alpha(D)$. The estimate $B_S(\tau) \leq \alpha + 1$ follows if the functions $g_\tau$ belong to $A^2_\alpha(D)$ for all $\varphi \in S$. We now need to find a way of determining whether this is indeed the case.

Let $M_f$ denote the operator between $A^2_\alpha(D)$ and $A^2_{\alpha+2}(D)$ given by multiplication by the function $f$ and let $\|M_f\|$ denote its operator norm. One way of establishing that $g_\tau$ is in the space $A^2_\alpha(D)$ is to use (2.4) to obtain an inequality of the form
\[ \|g_\tau\|_\alpha^2 = C(\alpha)\|(g_\tau)'\|_{\alpha+2}^2 + O(\|g_\tau\|_{\alpha+\theta}) \]
\[ = C(\alpha)\left\| \frac{\tau}{2} f' g_\tau \right\|_{\alpha+2}^2 + O(\|g_\tau\|_{\alpha+\theta}) \]
\[ \leq C(\alpha)\frac{|\tau|^2}{4}\|M_f\|\|g_\tau\|_\alpha^2 + O(\|g_\tau\|_{\alpha+\theta}). \] (3.4)

If it holds that
\[ 1 - C(\alpha)\frac{|\tau|^2}{4}\|M_f\| > 0, \]
and if we can show, using for instance pointwise estimates for the class $S$, that $g_\tau \in A^2_{\alpha+\theta}(D)$ holds uniformly in $S$, then we may conclude that
$B_S(\tau) \leq \alpha + 1$. Our objective is to choose $\alpha$ as small as possible and this requires good control of the multiplier norm $\|M_f\|$.

The above idea is exploited by Serguei Shimorin in his paper [22]. There, he considers second order derivatives of $g_{\tau}$ and uses an estimate of the multiplier norm of the Schwarzian derivative

$$S[\varphi](z) = f''(z) - \frac{1}{2}(f'(z))^2, \quad f = \log \varphi',$$

to obtain the estimates $B_S(-1) \leq 0.4195$ and $B_S(-2) \leq 1.246$.

### 3.2 Area-type estimates and norm expansions

In their paper [12], Hedenmalm and Shimorin develop a technique for estimating the universal means spectrum that is based on a similar idea. We provide the reader with a brief sketch of their approach. We start out with a version of an inequality of Prawitz that generalizes the area theorem, namely

$$\int_D |\varphi'(z)\left(\frac{z}{\varphi(z)}\right)^{\theta+1} - 1|^2 \frac{dA(z)}{|z|^{2\theta + 2}} \leq \frac{1}{\theta}, \quad 0 < \theta \leq 1, \quad (3.5)$$

and we then introduce a second variable $w$ into the above inequality using the Koebe transform. That is, we replace the function $\varphi$ by

$$\phi_w(\zeta) = \varphi\left(\frac{\zeta + w}{1 - w}\right) - \varphi'(w), \quad \zeta, w \in \mathbb{D}.$$ 

Next, some computations involving a change of variables yield the inequality

$$\int_D |\Phi_\theta(z, w) + L_\theta(z, w)|^2 \frac{dA(z)}{|z - w|^{2\theta}} \leq \frac{1}{\theta}(1 - |w|^2)^{-2\theta}. \quad (3.6)$$

Here, $\Phi_\theta$ and $L_\theta$ are explicit functions that involve on the original function $\varphi \in \mathcal{S}$ as well as the parameter $\theta$ (see [12] or paper I of this thesis). We note that the inequality is uniform in $\mathcal{S}$. Next, we multiply both sides of the inequality by a function $g \in A^2_{\alpha - 2\theta}(\mathbb{D})$ and integrate with respect to the measure $dA_{\alpha - 2\theta}(w)$. We then identify the resulting expression on the left-hand side with the norm of the function $\Phi_\theta + L_\theta$ in a weighted Bergman space on the bidisk of the type defined by (2.9). On the right-hand side we
obtain with a multiple the norm of $g$ in the space $A_{\alpha-2\theta}^2(\mathbb{D})$; the result is the inequality
\[ \|g\Phi_\theta + gL_\theta\|_{0,\alpha,\theta,0}^2 \leq \frac{\alpha + 1}{\theta(\alpha - 2\theta + 1)} \|g\|_{\alpha-2\theta}^2. \]

After some additional manipulations, the norm expansion techniques described earlier can be applied to the left-hand side of the inequality, and the norm for the Bergman space on the bidisk can be expressed in a series of norms computed in weighted Bergman spaces in the disk. In the end, Hedenmalm and Shimorin arrive at the inequality
\[ \sum_{N=0}^{\infty} \frac{1}{\sigma_N} \left\| b_N g^{(N+1)} \right\|_{\alpha-2\theta + 2N + 2}^2 \leq K(\alpha, \theta) \|g\|_{\alpha-2\theta}^2 + O \left( \|g\|_{\alpha-\theta}^2 \right), \quad (3.7) \]
where $\sigma_N$, $b_N$, $a_{k,N}$ and $K$ are certain explicit constants. The symbol $\odot$ denotes the restriction of a function to the diagonal of the bidisk.

### 3.3 Obtaining estimates near the origin

Hedenmalm and Shimorin apply their result to the function
\[ g_\tau = (\varphi')^{\tau/2} = \exp \left\{ \frac{\tau}{2} f \right\} \]
in order to obtain estimates of the universal means spectrum near the origin. If we pick $\alpha = \beta + 2\theta - 1$ and ignore all but the first term in the left-hand side of the inequality, we get the estimate
\[ \frac{1}{\sigma_0} \left\| c_1(\beta, \theta) g'_\tau + c_2(\beta, \theta) \frac{\varphi''}{\varphi'} g_\tau \right\|_{\beta+1}^2 \leq K(\beta, \theta) \|g_\tau\|_{\beta-1}^2 + O \left( \|g_\tau\|_{\beta-1+\theta}^2 \right), \]
where $c_1$ and $c_2$ are certain explicit constants. Using the fact that
\[ g'_\tau = \frac{\tau}{2} \frac{\varphi''}{\varphi'} g_\tau, \]
we obtain
\[ \frac{|c_3(\beta, \theta, \tau)|^2}{\sigma_0} \|g_\tau\|_{\beta+1}^2 \leq K(\beta, \theta) \|g_\tau\|_{\beta-1}^2 + O \left( \|g_\tau\|_{\beta-1+\theta}^2 \right) \]
for an explicit constant $c_3$. This is the kind of result we asked for in (3.4), and in order to obtain the desired estimate of $B_S(\tau)$, it remains to find values of $\theta$ and $\beta$ such that

$$\frac{|c_3(\beta, \theta, \tau)|^2}{\sigma_0} - K(\beta, \theta) > 0$$

and $g_\tau \in A^{2\alpha-1+\theta}(D)$ for every univalent function $\varphi \in S$. Choosing the free parameters suitably, Hedenmalm and Shimorin obtain

$$\limsup_{|\tau| \to 0} \frac{B_S(\tau)}{|\tau|^2} \leq \frac{1}{2}.$$ 

They then proceed with an analysis that takes into account two terms in the series expansion. In that case, the situation is not as simple as before, since the expressions appearing in the second term of the norm expansion can no longer immediately be identified as a derivative of $g_\tau$. Instead, one has to resort to Cauchy-Schwarz-type estimates to control the norms. An improved result using more terms in the norm expansion is presented in the forthcoming paper [13]; this is the aforementioned estimate (1.18).

In theory, we should be able to improve these estimates by considering more terms in the series expansion in (3.7). In practice, however, it is difficult to make good use of the information provided by these extra terms, at least if we want to proceed analytically. We shall discuss this matter further in the next section.

In the same paper [12], Hedenmalm and Shimorin develop an optimization scheme that can be used to estimate $B_S$ numerically. They implement this method with two terms in the series expansion. The problem of estimating the universal means spectrum then reduces (at least for real values of $\tau$) to checking whether certain intervals intersect. This numerical study yields the estimates $B_S(-1) \leq 0.403$ and $B_S(-2) \leq 1.218$.

In the second paper of this thesis, we implement the optimization technique suggested by Hedenmalm and Shimorin using three terms in the series expansion (3.7), with the same choices of $g$ and $\alpha$ as before. In this case, the problem is to find three ellipses that do not intersect. Unfortunately, it seems that the use of three terms instead of two does not lead to any significantly improved estimates of the universal means spectrum. For instance, the work in paper II leads to the numerical estimates $B_S(-1) \leq 0.388$ and $B_S(-2) \leq 1.206$. 

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4 Discussion

The last part of the introduction is devoted to a discussion of the difficulties we encounter when we try to estimate the universal means spectrum. We describe some connections between the method we have applied to estimate the universal means spectrum near the origin and other techniques used to study the spectrum for bounded functions near the point two. We also suggest some possible directions for future research.

4.1 Obstructions

At present, we have not found a way to truly exhaust all the information contained in the fundamental inequality (3.7). If \( f \in A_2^2(\mathbb{D}) \), then (2.4) provides us with very precise information about how the norm of the derivative \( f^{(n)} \) in \( A_{2+2n}^2(\mathbb{D}) \) compares to the norm of \( f \) in the space \( A_2^2(\mathbb{D}) \). When we insert \( g_\tau = \exp(\tau f/2) \), with \( f = \log \varphi' \), into (3.7), the expressions that appear on the left-hand side of the fundamental inequality for \( N \geq 1 \) are unfortunately not “pure” derivatives of \( g_\tau \), at least not for general values of the parameters \( \beta, \theta \) and \( \tau \). Thus, we cannot immediately apply the equality (2.4). For instance, the second term in the series expansion can essentially be rewritten as a constant times the expression

\[
\| \{ f'' + \eta |f'|^2 \} g_\tau \|_{\beta+3},
\]

where \( \eta \) depends on \( \beta, \theta \) and \( \tau \). We note that if \( \eta = -1/2 \) (the case of the Schwarzian), then the expression in the norm can be thought of as a second derivative of \( g_\tau \), and we can proceed as before. In general however, we are stuck: we do not know how to compare (4.1) to \( \| g_\tau \|_{\beta-1} \). We encounter the same kind of difficulty with the subsequent terms in the series expansion. So far, there exists no general remedy to this problem. In some special cases, the Cauchy-Schwarz and Hölder inequalities can be applied to (4.1) in order to get the desired inequalities (see [12] and paper II), but this is a rather crude approach.

4.2 Extending the method

One possible direction for future research might be to try to understand the algebraic structure of the linear combinations of derivatives of \( \varphi \) in the left-hand side of (3.7), at least in the case when \( \tau \) and the parameter \( \theta \) are
close to zero. This option has been explored in the paper [13], where the authors formulate some conjectures concerning the algebraic structure of the involved so-called \( \varphi \)-forms. Since these expressions become very cumbersome to compute for large \( N \), this algebraic approach feels somewhat difficult.

Another, perhaps more promising, approach to the problem of controlling the higher-order norms in (3.7) is to study the connection between the Bloch space properties of \( f \) and the growth space properties of the expressions that appear on the left-hand side. Let us return to the example of the term

\[
\| \{ f'' + \eta [f']^2 \} g_r \|_{3+3}^2.
\]

We would like to know whether this norm can be very small while \( \| g_r \|_{3-1}^2 \) is large. For general \( \eta \), this may indeed be the case (see paper I for an example), and in that case our approach fails. A first step towards understanding the situation better might be to study the local behavior of \( f'' + \psi [f']^2 \) near the boundary of the unit disk and to see how this compares with the local behavior of the function \( f \). We refer the reader to paper I for a more technical treatment of this issue.

Another possibility is to replace the initial inequalities (3.5) and (3.6) by some other, perhaps more appropriate, estimates and then to proceed as before, using the norm expansion technique. This is suggested in the paper [12]. One alternative that is mentioned there is to use an omitted-value transformation to introduce branching points into the Prawitz inequality. It is also possible to replace the Prawitz inequality altogether by some other, perhaps classical, area-type estimate that allows for additional parameters. The author of the present thesis previously attempted this in his master thesis using an integral inequality essentially due to James Rovnyak and Louis de Branges:

\[
\int_{B} \left| \varphi'(z) \left( \frac{\varphi(z)}{z} \right)^{\theta-1} - 1 \right|^2 \frac{dA(z)}{|z|^{2\theta+2\eta+2}} \leq 4 \sum_{k=1}^{\infty} \frac{(k-\theta)^2 [(1-2\theta)k-1]^2}{(k!)^2 (k-\theta-\eta)},
\]

However, the results that were obtained in this manner were rather complicated and have not been implemented in the study of the universal means spectrum. The Hedenmalm-Shimorin method can also be applied to the class \( \Sigma \). However, the author has been informed that this does not yield significant new results.
4.3 Different methods at different points of the spectrum

As we have seen, the method devised by Hedenmalm and Shimorin works rather well for obtaining estimates on the universal means spectrum for the class $S$ near the origin. If we want find good estimates of the universal means spectrum for bounded functions near the point $\tau = 2$, we have to resort to a different technique. This problem was recently considered by Anton Baranov and Hedenmalm; their work was inspired by the earlier paper [14] of Peter Jones and Nikolai Makarov.

Suppose $\varphi$ belongs to the class $S_b$ of bounded univalent functions. As a first step, Baranov and Hedenmalm establish the identity

$$\log \frac{z(\varphi(z) - \varphi(\zeta))}{(z - \zeta)\varphi(z)} - \zeta(1 - |\zeta|^2) \left( \frac{\varphi'(\zeta)}{\varphi(\zeta) - \varphi(z)} - \frac{1}{\zeta - z} \right) + \log(1 - \overline{\varphi(z)}) \frac{1 - |\zeta|^2}{1 - \overline{\zeta}} = \zeta^2 \int_D \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\zeta - w}{(1 - wz)^2} dA(w), \quad z, \zeta \in \mathbb{D}. \quad (4.2)$$

Taking the diagonal restriction of the identity and appealing to some classical pointwise estimates for the class $S$, we obtain

$$\log \left\{ \frac{z\varphi'(z)}{\varphi(z)} (1 - |z|^2) \right\} + O(1) = z^2 \int_D \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\zeta - w}{(1 - wz)^2} dA(w). \quad (4.3)$$

Next, we consider the Cauchy-type operator

$$C_{\varphi}[f](z) = \int_D \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\zeta - w}{1 - wz} f(w) dA(w). \quad (4.4)$$

Using the Hölder inequality and certain results on Marcinkiewicz-Zygmund integrals, one obtains a uniform Sobolev embedding for the operator $C_{\varphi}$:

$$\int_D \exp \left\{ |\lambda| \sup_{f \in \mathcal{B}(\mathcal{X}_\varphi)} |C_{\varphi}[f](z)|^{2 + \kappa} \right\} |\varphi'(z)|^2 dA(z) < +\infty, \quad 0 < \kappa < +\infty. \quad (4.5)$$

Here, $\mathcal{X}_\varphi$ is a certain weighted $L^p$-space in the unit disk. Also, the number $\lambda \in \mathbb{C}$ has to satisfy a condition in terms of $\kappa$. The above results can be then be used to derive estimates of $B_{S_b}(2 - \tau)$ for complex $\tau$ by establishing the
membership of the function \((\varphi')^{1-\tau/2}\) in a weighted Bergman space. This requires a clever choice of the function \(f\) in (4.5) and a convexity argument. Essentially, one should pick the function

\[ f_z(w) = \frac{z^2}{1 - wz}, \quad \text{with} \quad z \in \mathbb{D}, \]

and normalize it. In their paper, Baranov and Hedenmalm obtain a result similar to that of Jones and Makarov, namely, they show that

\[ B_{S_b}(2 - \tau) = 1 - \Re \tau + (c + o(1))|\tau|^2 \log \frac{1}{|\tau|}, \quad \text{as} \quad |\tau| \rightarrow 0, \quad (4.6) \]

for an explicit constant \(c\). In paper I of this thesis, we use the convexity of the function \(B_{S_b}\) to obtain estimates of \(B_{S_b}(1)\) by combining the above results with the numerical study carried out in paper II. This has yielded the numerical estimate \(B_{S_b}(1) \leq 0.4598\).

It should be noted that the initial equality (4.3) was obtained by taking the diagonal restriction of the identity (4.2). While that identity at first sight seems unrelated to the other results we have discussed in this thesis, there are actually some connections that are worth pointing out. A simpler version of (4.2) is the equality

\[ \log z(\varphi(z) - \varphi(\zeta)) + \log(1 - z\zeta) = \int_\mathbb{D} \frac{\varphi'(w) \zeta}{\varphi(w) - \varphi(z) 1 - wz} dA(w). \quad (4.7) \]

Differentiating this with respect to \(\zeta\) and \(z\), we obtain that

\[ \frac{\varphi'(*) \varphi'(z)}{(\varphi(*) - \varphi(z))^2} - \frac{1}{(z - \zeta)^2} = \int_\mathbb{D} \frac{\varphi'(w) \varphi'(z)}{(\varphi(w) - \varphi(z))^2 (1 - wz)^2} dA(w). \quad (4.8) \]

As Baranov and Hedenmalm note, the corresponding operator version of this equality can be seen as an integrated version of the classical strong Grunsky inequality (see chapter 4 of [6]). Similarly, the equality (4.3) can be viewed as the diagonal restriction of the integrated Grunsky identity (4.2).

The classical Grunsky inequalities arise as a consequence of the aforementioned area theorem (1.9) and the Prawitz inequality used as a starting point by Hedenmalm and Shimorin is also part of the area-method tradition. In the recent paper [10], Hedenmalm studies generalized Beurling transforms that involve a conformal mapping and a free parameter \(\theta\), and this leads to
more general Grunsky-type identities. An application of the Grunsky-type identity to a suitable function then yields the inequality

\[
\int \left| \frac{\lambda \varphi(z)}{z \varphi(\lambda)} \varphi'(z) \varphi'(\lambda) (\varphi(\lambda) - \varphi(z))^2 - \frac{1}{(\lambda - z)^2} \right| |z|^2 dA(z)
\]

\[
+ \theta \frac{\varphi'(z)}{\varphi(z)} \left[ \frac{\lambda \varphi(z)}{z \varphi(\lambda)} \frac{\varphi'(\lambda)}{\varphi(\lambda) - \varphi(z)} \right] \left( \frac{\theta}{z(\lambda - z)} \right)^2 |z|^2 dA(z)
\]

\[
\leq \frac{1}{(1 - |\lambda|^2)} - \frac{\theta}{1 - |\lambda|^2}.
\]

In particular, the choice \( \lambda = 0 \) essentially leads to the Prawitz inequality.

In this introduction, we have described two methods for estimating the universal means spectrum at different points. In both cases, we start out with an equality or a uniform inequality involving a conformal mapping \( \varphi \) that has an “area theorem” flavor. Also, in both cases, the restriction of variables to the diagonal appears at some point. The Hedenmalm-Shimorin method is, in some ways, more precise; unfortunately, we are not able to extract all the information it yields. It would be very satisfactory if one could find a way of merging these two techniques into a single technique that could be used to estimate the universal means spectra at all points. It is possible that a more extensive study of generalized Beurling transforms and their associated Grunsky-type identities could lead to interesting results in this direction.

### 4.4 Final comments

In this thesis, we have focused on how to obtain upper estimates of the universal means spectra using what we could call Hilbert space methods. One of the papers of this thesis is devoted to the technique of norm expansion, which is interesting in its own right. There exist other approaches to universal means spectra that have less of a functional analysis flavour, and rely more on the theory of harmonic measure. A discussion of these techniques is unfortunately beyond the scope of this thesis. A considerable amount of work has also been devoted to finding lower estimates of universal means spectra, and new results have recently been obtained by Ilgiz Kayumov, Dmitry Beliaev and Stanislav Smirnov.

At present, determining the universal means spectra of conformal mappings in the classes \( S \) and \( \Sigma \) remains a challenging problem.
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Bibliography


