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Citation for the original published paper (version of record):

Rumpler, R., Göransson, P. (2017)
An assessment of two popular Padé-based approaches for fast frequency sweeps of
time-harmonic finite element problems
Proceedings of Meetings on Acoustics, 30: 022003
https://doi.org/10.1121/2.0000649

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Citation: Proc. Mtgs. Acoust. 30, 022003 (2017); doi: 10.1121/2.0000649
View online: https://doi.org/10.1121/2.0000649
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An assessment of two popular Padé-based approaches for fast frequency sweeps of time-harmonic finite element problems.

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Several Padé-based computational methods have been recently combined with the finite element method for the efficient solution of complex time-harmonic acoustic problems. Among these, the component-wise approach, which focuses on the fast-frequency sweep of individual degrees of freedom in the problem, is an alternative to the projection-based approaches. While the former approach allows for piecewise analytical expressions of the solution for targeted degrees of the freedom, the projection-based approaches may offer a wider range of convergence. In this work, the two approaches are compared for a range of problems varying in complexity, size and physics. This includes for instance the modelling of coupled problems with non-trivial frequency dependence such as for the modelling of sound absorbing porous materials. Conclusions are drawn in terms of computational time efficiency, implementation, and suitability of the methods depending on specific scientific problems of interest.
INTRODUCTION

In order to limit the high computational burden associated with parametric solutions using the Finite Element (FE) method, such as for the frequency-sweep of large structural-acoustic problems, several approaches have been developed. Modal methods, widely used for several decades, have been able to provide both numerical and experimental support to the field, despite some limitations in frequency range and for specific problems. They for instance become increasingly inefficient with increasing damping modelled in the form on non-proportional damping. In these situations, the use of Padé approximants in various forms has proved to be very effective and has received considerable attention for fast frequency sweeps.

In the subset of Padé-based methods suited for the general case of non-polynomial, explicit, parametric dependence of the FE linear system of equations, two complementary approaches are presented in this contribution. The objective is to provide a comparison and to highlight their respective properties on realistic problems. The first Padé-based method considered, subsequently referred to as the component-wise approach, relies on the expansion of each component of interest in the solution vector, in the form of a Padé approximant of the parameter of interest.\textsuperscript{1–4} The second approach focuses instead on the construction of a reduced sequence of vectors spanning a subspace on which the FE system can be projected, thus leading to a parametric sweep involving a reduced-size system.\textsuperscript{5–8} While the former approach provides piecewise analytical expressions of the solution components, has been extended to multivariate problems,\textsuperscript{9} and is very well suited for partial-field solutions, the latter method is potentially much more efficient, as also shown in the present contribution.

In the following Section, the theoretical foundation for the two methods considered is summarized. Then, a Section focussing on applications provides a comparison of these two methods on a realistic acoustic problem, both conservative and with substantial acoustic treatment in the form of porous material. A short Section highlighting the main observations made from the examples considered then concludes the paper.

PADÉ APPROXIMANTS FOR FAST FREQUENCY SWEEP

THE COMPONENT-WISE EXPANSION IN PADÉ APPROXIMANTS

The starting point of the component-wise Padé univariate sweep, as detailed in Ref. [4], is given by a linear system of \( N \) equations (and \( N \) Degrees Of Freedom (DOFs)) in the following form,

\[
Z(x)U(x) = F(x),
\]

where \( x \) is the independent variable corresponding to the parametric problem of interest, e.g. the angular frequency \( \omega \) for the examples discussed in the present work. In an FE problem, \( Z(x) \), \( U(x) \), and \( F(x) \) respectively represent the system matrix of the discretized problem, the solution vector and the vector of externally applied loads.

A component-wise expansion of the solution vector may be sought as Padé approximants in the form

\[
u(x_0 + \Delta x) \approx \hat{u}(x_0 + \Delta x) = \frac{P_L(\Delta x)}{Q_M(\Delta x)},
\]

where the solution vector \( U(x_0) \), of which \( u(x_0) \) is a component, is assumed to be known after solving the system in Eq. (1) for \( x = x_0 \). \( P_L(\Delta x) \) and \( Q_M(\Delta x) \) are two truncated power series in the variable
\[ \Delta x = (x - x_0), \text{ to the orders } L \text{ and } M \text{ respectively, and given by} \]
\[ P_L(\Delta x) = \sum_{k=0}^{L} p_k(\Delta x)^k, \quad (3a) \]
\[ Q_M(\Delta x) = \sum_{k=0}^{M} q_k(\Delta x)^k. \quad (3b) \]

The expansion in Eq. (2) being done for a component \( u(x) \) of \( U(x) \) around \( x = x_0 \), the subsequent scalar expressions in this Section also correspond to scalar components of the solution vector \( U(x) \). In previous works, it was shown\(^1\) that the coefficients of the power series in Eqs. (3) may be determined from the coefficients of the Taylor series expansion at order \((L + M)\)

\[ u(x_0 + \Delta x) \approx A_{L+M}(\Delta x) = \sum_{k=0}^{L+M} a_k(\Delta x)^k, \quad (4) \]

where

\[ a_k = \frac{u^{(k)}(x_0)}{k!}, \text{ with } u^{(0)}(x_0) = u(x_0) = a_0. \quad (5) \]

These coefficients \( p_k \) and \( q_k \) are indeed solutions of the system of linear equations resulting from equating the Padé approximant in Eq. (2) to the Taylor series expansion Eq. (4), such that

\[ P_L(\Delta x) - A_{L+M}(\Delta x)Q_M(\Delta x) = 0. \quad (6) \]

After developing Eq. (6), a set of \((L + M + 1)\) equations emerges from the identification of the coefficients of equal order in \( \Delta x \), and normalizing the zero-order denominator coefficient, i.e. \( q_0 = 1 \),

\[
\begin{bmatrix}
  p_0 = a_0 \\
p_1 - a_0 q_1 = a_1 \\
m_p - a_{L-1} q_1 - \cdots - a_0 q_p = a_L \\
- a_L q_1 - a_{L-1} q_2 - \cdots - a_{L-M+1} q_M = a_{L+1} \\
- a_{L+M-1} q_1 - a_{L+M-2} q_2 - \cdots - a_L q_M = a_{L+M}
\end{bmatrix}, \quad (7)
\]

where

\[
\begin{cases}
  a_k = 0 \text{ if } k < 0 \\
  q_k = 0 \text{ if } k > M.
\end{cases}
\]

This system of linear equations may then be solved in two steps. First, the denominator coefficients \( q_k \) may be solved for with a system consisting of the lower subset of equations in Eqs. (7),

\[
\begin{bmatrix}
a_L & \cdots & a_{L-M+1} \\
\vdots & \vdots & \vdots \\
a_{L+M-1} & \cdots & a_L
\end{bmatrix}\begin{bmatrix}
  q_1 \\
  \vdots \\
  q_M
\end{bmatrix} = - \begin{bmatrix}
a_{L+1} \\
\vdots \\
a_{L+M}
\end{bmatrix} \quad \text{with } a_i = 0 \text{ if } i < 0. \quad (8)
\]
The numerator coefficients may subsequently be determined in a second step by simple algebraic operations,

\[ p_k = \sum_{i=0}^{M} q_i a_{(k-i)}, \]  

(9)

with

\[
\begin{align*}
    k &= 0 \ldots L \\
    a_j &= 0 \quad \text{if} \quad j < 0.
\end{align*}
\]

The calculation of these Padé coefficients, being dependent on the Taylor coefficients \( a_k \) in Eqs. (4)-(5), relies on the ability to efficiently calculate the \( L + M \) successive partial derivatives of the solution vector, at the reference point \( x = x_0 \). This may be achieved via a recursive scheme, using a Leibniz formula resulting from the differentiation of Eq. (1) with respect to \( x \), at order \( k \), in \( x_0 \),

\[
\sum_{j=0}^{k} \binom{k}{j} Z^{(k-j)}(x_0) U^{(j)}(x_0) = F^{(k)}(x_0), \quad \text{for} \quad k = 1, \ldots, (L + M),
\]

(10)

where the zero-order derivatives correspond to the non-differentiated functions, and the binomial coefficients are given by

\[
\binom{k}{j} = \frac{k!}{j!(k-j)!}.
\]

The recursive expression for \( U^{(k)}(x_0) \) follows from extracting the highest-order term from the summation in Eq. (10),

\[
Z(x_0) U^{(k)}(x_0) = F^{(k)}(x_0) - \sum_{j=0}^{(k-1)} \binom{k}{j} Z^{(k-j)}(x_0) U^{(j)}(x_0), \quad \text{for} \quad k = 1, \ldots, (L + M),
\]

(12)

This implies that the successive derivatives of \( U \) with respect to \( x \), required for the determination of the Padé approximations, can be efficiently calculated as the solution of a full-sized system of equations, with multiple right-hand sides.

**MOMENT-MATCHING METHOD, A PROJECTION-BASED REDUCED-ORDER MODEL**

The projection-based approach follows the Taylor expansion of the solution vector as in Eq. (4), where the Taylor coefficients imply the calculation of the successive derivatives of the solution vector according to Eq. (12). This vector sequence of successive derivatives may then be used to form a suitable basis for the reference problem Eq. (1) around \( x_0 \) rather than being used for the explicit construction of Padé approximants. This is the idea of the Galerkin Asymptotic Waveform Evaluation\(^{10}\) (GAWE), where the \( U^{(k)} \) vectors, \( k = 0, \ldots, (L + M) \), are orthonormalized into a basis \( W_N \), with \( N = L + M + 1 \), subsequently used as a reduced-order model for the transformation of system Eq. (1) such that

\[
U(x) \approx \hat{U}(x) = W_N \alpha(x),
\]

(13)

\( \alpha \) being a vector of \( N = L + M + 1 \) generalized coordinates. This transformation leads to a reduced system corresponding to Eq. (1), solving for the generalized coordinates in \( \alpha(x) \),

\[
W_N^T Z(x) W_N \alpha(x) = W_N^T F(x).
\]

(14)
However, the sequence to generate the basis vectors rapidly leads to an inherently ill-conditioned transformation matrix, which translates into a lack of robustness and stagnation in the convergence upon increasing the size of the subspace spanned. This limits the range of suitable approximation for the solution of the reduced system Eq. (14). The well-conditioned AWE (WCAWE) method proposed by Slone et al.\textsuperscript{5} overcomes this issue by including a modified Gram-Schmidt process to orthonormalize the basis vectors generated at each step of the sequential procedure (12), together with some correction terms in the RHS of the sequential problems to be solved. In the context of the generation of an orthonormalized basis associated with a transformation of the type of Eq. (13), the well-conditioned procedure, in $x = x_0$, is given by the following multiple right-hand-side sequence,

$$
\begin{align*}
Z^{(0)}\nu_1 &= F^{(0)} \\
\text{Normalization} \quad \nu_1 \rightarrow \nu_1 \\
Z^{(0)}\nu_2 &= F^{(1)}e_1^TP_{Q_1}(2,1)e_1 - Z^{(1)}\nu_1 \\
\text{Othonormalization} \quad \nu_2 \rightarrow \nu_2 \\
\vdots \\
Z^{(0)}\nu_n &= \left( \sum_{j=1}^{(n-1)} \left( F^{(n)}e_1^TP_{Q_1}(n,j)e_{n-j} \right) \right) - Z^{(1)}\nu_{n-1} - \sum_{j=2}^{(n-1)} \left( Z^{(j)}\nu_{n-j}P_{Q_2}(n,j)e_{n-j} \right) \\
\text{Othonormalization} \quad \nu_n \rightarrow \nu_n \\
\vdots \\
Z^{(0)}\nu_N &= \left( \sum_{j=1}^{(N-1)} \left( F^{(N)}e_1^TP_{Q_1}(N,j)e_{N-j} \right) \right) - Z^{(1)}\nu_{N-1} - \sum_{j=2}^{(N-1)} \left( Z^{(j)}\nu_{N-j}P_{Q_2}(N,j)e_{N-j} \right) \\
\text{Othonormalization} \quad \nu_N \rightarrow \nu_N
\end{align*}
$$

(15)

where a modified Gram-Schmidt orthonormalization step is performed between each vector generation by the multiple RHS systems in Eq. (15), and where

- $e_k$ is a unitary standard basis vector associated with the $k^{th}$ component of the solution vector,
- $\nu_k$ is the non-orthonormalized vector generated in the $k^{th}$ iteration of the procedure,
- $\nu_k$ is the basis vector orthonormalized against $\nu_{k-1}$, generated after the $k^{th}$ iteration of the procedure,
- $P_{Q_\omega}(\alpha, \beta)$, $\omega = 1, 2$, corresponds to the RHS correction terms, chosen to be associated with the modified Gram-Schmidt orthonormalisation process.\textsuperscript{5}

The orthonormalized and non-orthonormalized bases, $V_N$ and $\nu_N$ respectively, are related by

$$
V_N = \nu_N Q^{-1},
$$

(16)

where $Q$ is an $N \times N$ upper triangular, nonsingular matrix containing the modified Gram-Schmidt coefficients. More precisely, column $k$ of $Q$ contains the successive coefficients resulting from the projection of partially orthonormalized $\nu_k$ on the orthonormalized vectors $\nu_j$, $j < k$, and $Q_{k,k}$ corresponds to the norm of $\nu_k$ before its normalization. The correction terms $P_{Q_\omega}(\alpha, \beta)$, $\omega = 1, 2$, are given by the following product of block matrices extracted from $Q$,

$$
P_{Q_\omega}(\alpha, \beta) = \prod_{t=\omega}^{\beta} Q_{t:\alpha-\beta+t-1, t:\alpha-\beta+t-1}^{-1}
$$

(17)

Further discussions on the choice of the RHS correction coefficients other than associated with the Gram-Schmidt coefficients, may be found in Ref. [11].
Similarly to Eq. (14), the WCAWE approach leads to a reduced set of equations to be solved at each frequency, involving a preliminary projection of the system matrix at each step,

$$V_N^T Z(x)V_N \alpha(x) = V_N^T F(x). \quad (18)$$

The approximated solution is subsequently evaluated at all DOFs from the generalized coordinates vector $\alpha$ as

$$U(x) \approx \hat{U}(x) = V_N \alpha(x), \quad (19)$$

The cost to solve for the small systems of Eqs. (18) is higher than to solve for the Padé coefficients Eq. (7) once for a whole frequency interval of convergence. Furthermore, the projection-based approach loses the elegance of a Padé expansion, which provides a piecewise analytical expression of the frequency-dependent solution, thus not being affected by the choice of frequency sampling. However, the substantial improvements in range of convergence expected from the WCAWE procedure may more than compensate for the extra cost and its dependency on the choice of frequency sampling.

**CONVERGENCE AND ERROR ESTIMATION**

The solutions for the examples below are evaluated in terms of the Sound Pressure Level (SPL), after a solution vector in terms of acoustic pressure fluctuation, such that

$$S = 10 \log \left( \frac{U^2}{U_0^2} \right), \quad (20)$$

where $U_0$ is the reference acoustic pressure, 20 $\mu$Pa.

In order to compare the two approaches, and assess their convergence properties with respect to the reference solution, the same error indicator is used for both methods. The convergence comparison relies on a monotonic convergence of a sequence of increasing order of approximation. This translates, for the WCAWE, into an increase of the size of the projection basis. For the component-wise Padé approximant expansion, however, the monotonic convergence relies on the uniform convergence of a series of Padé approximants with a fixed degree of denominator polynomial (i.e. $M$ is fixed), a convergence ensured by the Montessus de Ballore theorem.\(^\text{12}\)

The error indicator then relies on the relative response between two consecutive approximation orders. Such an indicator may be expressed pointwise as the difference between the sound pressure levels, i.e. for the component-wise Padé approach,

$$\epsilon_{pad}^{(L+M+1)}(x) = \hat{S}_{(L+1)/M}(x) - \hat{S}_{(L)/M}(x), \quad (21)$$

where $\hat{S}_{(J/K)}(x)$ refers to the approximated SPL with the component-wise approach for Padé approximants with numerator polynomials of order $J$ and denominator polynomials of order $K$, corresponding to an approximation basis of $N = (J + K + 1)$ vectors with the WCAWE approach. The error indicator for the WCAWE, corresponding to the same order as Eq. (21) would therefore be

$$\epsilon_{proj}^{(N)}(x) = \epsilon_{proj}^{(L+M+1)}(x) = \hat{S}_{(L+M+2)}(x) - \hat{S}_{(L+M+1)}(x). \quad (22)$$

The range of convergence of the approximation by either method $\alpha \in \{pad, proj\}$ at order $N$ is subsequently given by the interval $[x_\min, x_\max]$, such that

$$\epsilon_\alpha^{(N)}(x) \leq \epsilon_{\max}, \quad \forall x \in [x_\min, x_\max], \quad (23)$$

$\epsilon_{\max}$ being the chosen tolerance.
RESULTS AND COMPARISON

The improvements in convergence offered by the WCAWE approach are illustrated on two average-sized problems, consisting of the interior cavity of a passenger train section. Two configurations are modelled, a conservative problem where the interior cavity is governed by the Helmholtz equation, and a configuration with a 15-cm layer of sound absorbing porous material at the top surface of the cavity. A time-harmonic point source is defined at a corner of the cavity, and frequency sweeps are performed. All boundary walls are considered as rigid walls, except from the porous boundary in the second configuration. The porous boundary is modelled by an equivalent fluid formulation, consisting of a modified Helmholtz equation where the equivalent speed of sound is complex and frequency-dependent, given by

$$\tilde{c}_p = \frac{c_0}{\sqrt{1 - i\Phi \rho_0 \omega}},$$  \hspace{1cm} (24)

where $\tilde{()}$ denotes a complex-valued quantity, $\rho_0 = 1.21$ kg.m$^{-3}$ is the ambient density of the air saturating the pores and $c_0 = 343$ m.s$^{-1}$ the speed of sound in the air. $\Phi$ is the static flow resistivity, associated with the viscous dissipation in the porous material, and chosen to be such that $\Phi = 25000$ N.s.m$^{-4}$.

After a standard expression of the problem in its weak form and subsequent discretization by a Galerkin method, the finite element problem has the general form

$$\left( K_a - \frac{\omega^2}{c_0^2} M_a + K_p - \frac{\omega^2}{\tilde{c}_p^2} M_p \right) \tilde{U} = F,$$  \hspace{1cm} (25)

where $\square_a$ correspond to air cavity global matrices, and $\square_p$ to porous global matrices. The right-hand-side vector $F$, associated with the time-harmonic acoustic excitation is in practice only non-zero at a few DOFs. The FE problem Eq. (25) is evidently of the form of generic problem Eq. (1), thus suitable for the two ROMs described in the previous section. The conservative problem is simply deduced from Eq. (25) by replacing the complex-valued speed of sound $\tilde{c}_p$ associated with the porous material, by the air property $c_0$ (the global matrices for the air cavity or the porous layer both result from the same operators in the Helmholtz equation, the constitutive difference between the two media is thus carried by the speed of sound only). The FE problems are implemented in FreeFem++, and solved with a Matlab interface to the linear solver MUMPS.

The reference solutions in the range [50, 110] Hz, together with snapshots of the acoustic pressure solution fields close to the cavity mode, at 63 Hz, are plotted in Fig. 1 and Fig. 2 respectively.

The comparison between the component-wise Padé expansion and the WCAWE projection approach is done on a single interval in the frequency ranged considered, with a reference point for the approximation arbitrarily chosen at 88 Hz. First, Fig. 3 illustrates the convergence limitation associated with the component-wise expansion. Both Figs. 3a and 3b, having a fixed denominator degree of $nM = 7$, are aiming at an approximation over 7 poles closest to 88 Hz. While the increase of the numerator polynomial degree from $nL = 8$ to $nL = 12$ extends the range of convergence, these approximations do not cover the range of 7 poles as theoretically expected. This known limitation was recently improved with a procedure proposed by the authors for conservative problems. The component-wise approach remains however limited in range by the inherent ill-conditioned nature of the procedure associated with the calculation of the solution derivatives, thus requiring a specific multi-interval strategy.

Fig. 4 illustrates a comparison for the component-wise approach between the conservative and damped problems, for a higher order of approximant: 9 poles ($nM = 9$), and a numerator polynomial degree of order 10. The convergence limitation is further illustrated in Fig. 4a as practically no improvement is observed compared to Fig. 3. Then, the smoothness of the damped solution, in Fig. 4b, allows for a wider
Figure 1: Reference solutions (SPL at a single output point) for the acoustic problems, forced response, point source at the lower back corner: (a) Conservative acoustic cavity problem (b) Damped acoustic cavity problem. Vertical lines: location of the full solution snapshots of Fig. 2.

Figure 2: Reference solutions SPL distribution, forced response at 63 Hz, scale 40 – 110 dB: (a) Conservative acoustic cavity problem (b) Damped acoustic cavity problem.

range of convergence for a given Padé approximant. A general observation in both Fig. 3 and Fig. 4 regarding the error indicator chosen in Section 2.3, is that occasionally, two consecutive orders of approximation may be diverging in very similar ways, thus implying a too optimistic range of convergence. Although marginal, this undesirable behaviour may be avoided by considering a sequence of 3 consecutive orders for convergence, or using a more costly residual-error-based estimator.

In Fig. 5, the convergence behaviour of the WCAWE is plotted for both the conservative and damped acoustic cases. The first plots, Fig. 5a and Fig. 5b, involving 17 vectors in the WCAWE basis, correspond to the same order of approximation as the plots in Fig. 3 for the component-wise Padé expansion approach. The well-conditioned procedure of the WCAWE clearly enables to overcome the convergence limitations observed with the component-wise approach. The convergence analysis highlights a good approximation of the reference solution with a WCAWE basis of 36 vectors for the conservative problem, and 28 vectors for the damped problem. The ability to approximate these frequency responses in a single interval makes the WCAWE approach far superior to the component-wise approach, provided that the additional operations involved in the generation of the basis (correction terms and orthogonalization), the ROM frequency sweep
Figure 3: Approximation with the component-wise Padé expansion, convergence for a fixed denominator degree $n_M = 7$: (a) Padé approximant $n_M = 7$; $n_L = 8$ (b) Padé approximant $n_M = 7$; $n_L = 12$; (c)-(d) error indicator using the pointwise difference between two consecutive orders of Padé approximant.

Figure 4: Approximation with the component-wise Padé expansion for a Padé approximant $n_M = 9$; $n_L = 10$: (a) Conservative acoustic cavity problem (b) Damped acoustic cavity problem.

(as opposed to a simple evaluation of the Padé approximant), do not become overly costly when compared to a multi-point strategy (see some further comments about computational cost below).

Fig. 5c is chosen as an intermediate step of the convergence sequence for the conservative problem, specifically for its illustration of the error indicator potential break-down already mentioned. In this case,
Figure 5: Approximation with the WCAWE projection method, conservative (left) and damped (right) acoustic problems: (a)-(b) WCAWE basis with 17 vectors; (c)-(d) WCAWE basis with 21 vectors; (e) WCAWE basis with 36 vectors; (f) WCAWE basis with 28 vectors.

the 2 subsequent bases used span very close subspaces, translating into very close approximate responses which may result in misleading error indicator values for Eq. (22), as has already been observed for the
component-wise approach. Similarly to the component-wise Padé approach, the smoothness of the damped response also leads to a faster convergence, the damped problem requiring a 28-vector basis when the conservative problem requires a 36-vector basis.

The issue of smoothness and monotony of the convergence upon increasing the size of the WCAWE basis is further detailed in Fig. 6. For both the conservative problem (Fig. 6a) and the damped problem (Fig. 6b), the range of convergence estimated from the error indicator Eq. (22) is plotted for each addition of a vector in the basis, up to a fully converged solution over the frequency range of interest with 36 basis vectors. The overall monotony and smoothness of the convergence is only challenged by a few exceptions, in particular for the conservative problem (e.g. 14, 21, 24-26 vectors in the basis). On the contrary, for the damped problem, the smoothness of the solution itself translates into a smooth convergence when adding components to the WCAWE basis. From these plots, it is also very clear that the approximation converges much faster for the damped problem than for the conservative problem, an interesting observation contrasting with the convergence of modal-based methods with a high level of damping. 16, 17

This difference in convergence rate, by increasing the size of the basis, is further illustrated in Fig. 7 for the two acoustic test cases. While the slope of the trend lines is partly dependent on the max tolerance for the error indicator, \( \epsilon_{\text{max}} \) (see Eq. (23), chosen to be corresponding to a 1 dB difference max here), the convergence for the damped problem is both smoother and faster. The error indicator for the damped problem is obviously less prone to the inaccuracies occasionally observed with the conservative problem.

Although efficiency measure may mostly be qualitative, Fig. 8 presents such an analysis for a single interval of approximation, relying on the factorization-normalized CPU times (the horizontal dotted line represents the CPU time associated with the factorization of the system matrix at the start of the vector-generation procedure). The red and blue surface areas correspond to the WCAWE projection approach, with and without accounting for the sweeping time, respectively. Obviously, the WCAWE basis generation procedure rapidly becomes more costly than the sweep itself, the latter being only marginally impacted by an increase of the basis. However, even with a basis consisting of 36 vectors, the factorization step still remains the most costly step of the complete approximation procedure. This highlights indirectly the efficiency of the WCAWE, projection-based approach. Indeed, although the component-wise expansion approach is increasingly faster when the order of approximation increases, the broader range of convergence of the WCAWE implies a reduced number of approximation intervals, and thus fewer factorizations to be
Figure 7: Comparison of the convergence trend (size of converged interval) between the conservative and damped acoustic problems.

Figure 8: Comparison of the relative CPU time between the two methods as a function of the order of approximation, normalized with respect to the initial factorization step.

performed overall.
CONCLUSION

Two Padé-based methods were compared, a component-wise Padé expansion of the solution vector, and a projection-based approach leading to a reduced-size system matrix. Both methods rely on an initial sequential generation of full-sized vectors, differing slightly in their construction. The direct generation of solution derivatives involved in the component-wise approach, although allowing for a straightforward calculation of the coefficients of Taylor series, leads to an ill-conditioned problem, thus limiting the range of convergence of the method. The modified generation sequence in the projection-based approach leads to an orthonormal, well-conditioned basis matrix, suitable for an extended parametric range.

In both cases, the smoothness of a highly damped response enables a faster convergence than for a conservative case, i.e., an extended range of convergence for a given order of approximation. From a global efficiency point of view, the sequential generation of vectors (and in particular the system matrix factorization involved at the start of this procedure) being the most costly step of the approximation within one interval (see Fig. 8), the projection-based approach would perform better in virtually all situations. Indeed, its extended range of convergence implies a need for fewer main reconstruction points in the parametric space, thus limiting the number of intervals and their associated vector-generation sequence.

ACKNOWLEDGEMENTS

The financial support from the Swedish Research Council (Vetenskapsrådet VR Grant 2015-04925 for the first author) and the Centre for ECO2 Vehicle Design (VINNOVA Grant 2016-05195) are gratefully acknowledged.

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