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Postprint

This is the accepted version of a paper published in *IEEE Transactions on Smart Grid*. This paper has been peer-reviewed but does not include the final publisher proof-corrections or journal pagination.

Citation for the original published paper (version of record):

Yuan, Z., Reza Hesamzadeh, M. [Year unknown!]

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IEEE Transactions on Smart Grid

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A Modified Benders Decomposition Algorithm to Solve Second-Order Cone AC Optimal Power Flow

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Abstract—This paper proposes to speed up solving large-scale second-order cone AC optimal power flow (SOC-ACOPF) problem by decomposition and parallelization. Firstly, we use spectral factorization to partition large power network to multiple subnetworks connected by tie-lines. Then a modified Benders decomposition algorithm (M-BDA) is proposed to solve the SOC-ACOPF problem iteratively. Taking the total power output of each subnetwork as the complicating variable, we formulate the SOC-ACOPF problem of tie-lines as the master problem and the SOC-ACOPF problems of the subnetworks as the subproblems in the proposed M-BDA. The feasibility and optimality (preserving the original optimal solution of the SOC-ACOPF model) of the proposed M-BDA are analytically and numerically proved. A GAMS grid computing framework is designed to compute the formulated subproblems of M-BDA in parallel. The numerical results show that the proposed M-BDA can solve large-scale SOC-ACOPF problem efficiently. Accelerated M-BDA by parallel computing converges within few iterations. The computational efficiency (reducing computation CPU time and computer RAM requirement) can be improved by increasing the number of partitioned subnetworks.

Index Terms—Optimal power flow, Network partitioning, Modified Benders decomposition, Feasibility and optimality proof, Parallel computing.

NOMENCLATURE

Sets:

- N Nodes or Buses.
- L Lines.
- K Networks.
- N_k Nodes at subnetwork k .
- L_k Lines at subnetwork k .
- τ Tie-lines (lines connecting different subnetworks).
- N_τ Nodes at the ends of tie-lines.
- J Iterations.

Parameters:

- A_{nl}^+, A_{nl}^- Node to line incidence matrix.
- X_l Reactance of line l .
- R_l Resistance of line l .
- G_n Shunt conductance at bus n .
- B_n Shunt susceptance at bus n .
- P_{dn} Active power demand at bus n .
- Q_{dn} Reactive power demand at bus n .
- K_l Bound of active power loss of line l .
- c_l Partitioning cost of cutting line l .
- C Cost matrix of network partition.
- $\bar{n}, \bar{k}$ Cardinality of set N and set K .
- $C_{dn}^{P+}, C_{dn}^{P-}, C_n^v, C_n^\theta$ Penalty cost parameters.

Variables:

- p_n Active power generation at bus n .
- q_n Reactive power generation at bus n .
- p_{sl} Active power flow at the sending end of line l .
- q_{sl} Reactive power flow at the sending end of line l .
- p_{ol} Active power loss of line l .
- q_{ol} Reactive power loss of line l .
- v_n Voltage magnitude at bus n (lower case).
- v_{sl} Voltage at the sending end of line l (lower case).
- v_{rl} Voltage at the receiving end of line l (lower case).
- V_n Voltage magnitude square at bus n (upper case).
- V_{sl} Voltage at the sending end of line l (upper case).
- V_{rl} Voltage at the receiving end of line l (upper case).
- θ_l Voltage phase angle difference of line l .
- θ_{sl} Voltage phase angle at the sending end of line l .
- θ_{rl} Voltage phase angle at the receiving end of line l .
- M Network partition matrix.
- $Cost^M$ Objective of master problem.
- $Cost_{k,j}^S$ Objective of subnetwork k at iteration j .
- $P_{k,j}^{sum}$ Total active power generation of subnetwork k at iteration j .
- $Q_{k,j}^{sum}$ Total reactive power generation of subnetwork k at iteration j .

I. INTRODUCTION

POWER network is, either historically or technically, partitioned into several zones or subnetworks to reduce the complexity of operation, control or planning tasks [1], [2]. For example, Nord Pool operates the electricity market of Sweden over four well-defined bidding areas from the north to the south [3]. Ten regions (California, MISO, New England, New York, Northwest, PJM, Southeast, Southwest, SPP and Texas) of the USA power system are separately operated by the corresponding independent system operators (ISOs) [4]. One fundamental technical reason of operating the power system by network partitions is that the dimension of the Hessian matrix and Jacobian matrix during the iterations of the optimization is reduced. It is then easier for the solver to address small-scale optimization problems. Another advantage of decentralized operation is that the optimal power flow (OPF) problems of multiple subnetworks can be solved in a parallel manner. In facing growing power network expansions and accelerating penetrations of distributed energy resources (DERs) which may exceed the capability of centralized operation, parallel and decentralized operation could be a feasible solution [5].

Early research about the feasibility, applicability and comparison of decentralized OPF algorithms using auxiliary problem principle (APP), the predictor-corrector proximal multiplier method and alternating direction method are proposed or

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demonstrated by [6]–[8]. Decentralized approaches to solve the OPF problem also include Lagrangian relaxation [9], Benders decomposition (BD) [10], Dantzig-Wolfe decomposition [11] and Alternating Direction Method of Multipliers (ADMM) [12]. A comprehensive summary of distributed algorithms for optimization and control of power system can be found in [1]. Lagrangian relaxation approach relaxes the coupling constraints between the subnetworks and generally only approximated solutions can be guaranteed [9]. BD and Dantzig-Wolfe decomposition require to firstly formulate the master problem and subproblem, and then iterate until the solutions (of the master problem and subproblem) converge [10], [11]. BD is widely used to solve the security constrained unit commitment (SCUC) problem and transmission expansion planning (TEP) problem which are expanded applications of OPF [13]–[15]. In SCUC and TEP, mostly the integer variables are taken as the complicating variables to formulate the master problem and subproblem of BD [13]–[15]. Message exchanges among the subnetworks are required by ADMM [12]. Generally, when more subnetworks are partitioned, more iterations are required for ADMM to converge [12]. Reference [16] investigates three ADMM-based decentralized DCOPF solution algorithm with different communication strategies. It shows numerically that the convergence performance can be improved by enhancing the data exchange with the central controller (or coordinator). The network partitioning approach is also important for the convergence performance [16]. Authors in [17] solve the stochastic second-order cone programming (SOCP) based ACOPF by ADMM for radial distribution networks. The updates of the variables and multipliers are decomposed by each node and each scenario. As a result, it requires large number of iterations to converge (over 3000 iterations are required for a 50-node test case with 500 scenarios) [17]. Using ADMM, a comprehensive investigation of decomposing nonconvex ACOPF down to the individual node level is conducted by [5]. The results show that convergence speed of ADMM largely depends on test cases. The nonconvexity of ACOPF also requires a suitable selection of the penalty factors of ADMM to guarantee the convergence [5]. Reference [18] proposes a parametric quadratic programming approach to solve the regional correction equation in the proposed fully distributed interior point method (F-DIPM) to solve ACOPF. The power network is partitioned to several regions geographically. Then boundary variables associated with the tie-lines are duplicated for each region. A unidirectional ring communication is employed to transmit the information about boundary variables during each Newton-Raphson iteration. Various test cases show the robust convergence of F-DIPM. Authors in [19] use ADMM to solve the semi-definite programming (SDP) based relaxed ACOPF model for the formulated unbalanced microgrid. The fast convergence of ADMM over sub-gradient based method is demonstrated by test cases of the IEEE37-node feeder partitioned to four areas and a 10-node microgrid partitioned to three areas in [19]. As a improvement, we consider much larger power networks in this paper. By deriving and proving the closed form solutions for the OPF subproblems, [20] speed up the convergence of ADMM for radial distribution networks. Both mesh and radial power

networks are addressed in this paper.

Graph theory shows that the network-partitioning problem is NP-hard [21], [22]. Accordingly, various heuristic approaches such as geometric approach and flow-based approach have been proposed to solve the network-partitioning problem [21]. Authors in [23] define the electrical distance based on the network admittance parameter and then use it as a measure to distinguish strongly connected buses from weakly connected buses. The electrical distance between the buses within each partitioned zone are minimized while the electrical distance between buses of different partitioned zones are maximized in the multi-attribute network partitioning problem [23]. Reference [2] improves the Modularity Index in the community-detection based network-partition algorithm such that both network topology and reactive power capability are taken into account. The goal in [2] is to control the zonal voltage of distribution network using a parallel processing approach. Authors in [24] show promising advantages of solving ACOPF by decomposing the optimality conditions. The effects of network partitioning on the computation efficiency are also investigated in [24]. However, the proposed intelligent network partitioning method in [24] requires to first solve the ACOPF problem. Considering the complexity of different network-partitioning methods, we use spectral factorization [25] to partition the power networks in this paper.

Two challenges can be identified to solve the large-scale SOC-ACOPF problem in a decentralized way: (1) How to decompose the problem efficiently? The problem-decomposition algorithm should be computationally fast; (2) How to efficiently coordinate the objectives of the decomposed subproblems such that the final solutions converge to global optimality? Here, the global optimality means the SOC-ACOPF problem is solved in a single programming model. Accordingly, the main contributions of this paper are:

- 1) A modified Benders decomposition algorithm (M-BDA) based on network partitions is proposed for solving large-scale SOC-ACOPF problem. The computation is accelerated by parallel computing;
- 2) The feasibility of the proposed M-BDA is analytically and numerically proved. Since the proposed M-BDA is modified from the original Benders decomposition, the feasibility proof of M-BDA is provided. We also prove that the original optimal solution of the SOC-ACOPF model is preserved by the proposed M-BDA.

The proposed solution algorithm (based on network partitions, M-BDA and parallel computing) provides an efficient framework to speed up large-scale SOC-ACOPF computations. As an important contribution, there is no message exchange requirement among the subnetworks in the proposed approach. Our decomposition approach also requires few number of iterations to converge for the test cases in this paper and it is robust to the number of partitioned subnetworks. The rest of this paper is organized as follows. Section II explains the SOC-ACOPF model. Section III presents the network partitioning approach based on spectral factorization. Section IV details the formulations of M-BDA. The feasibility and optimality of the formulated master problem and subproblem in M-

BDA is analytically proved. The parallel computing structure to accelerate the proposed M-BDA is also designed in this section. Section V discusses numerical results for various IEEE test cases and power network partitions. The power network partitions based on spectral factorization are also plotted. Section VI concludes.

II. THE SOC-ACOPF MODEL

The SOC-ACOPF model, as a convex relaxation of the nonconvex ACOPF model, is summarized here in (1) [26]. The convexity, accuracy and applicability of this model have been proved by our work in [26]. The objective function (1a) can be any convex function of the decision variables.

$$\text{Minimize}_{\Omega} \quad f(p_n, q_n, p_{o_l}, q_{o_l}) \quad (1a)$$

subject to

$$p_n - P_{d_n} = \sum_l (A_{nl}^+ p_{s_l} - A_{nl}^- p_{o_l}) + G_n V_n, \quad \forall n \in N \quad (1b)$$

$$q_n - Q_{d_n} = \sum_l (A_{nl}^+ q_{s_l} - A_{nl}^- q_{o_l}) - B_n V_n, \quad \forall n \in N \quad (1c)$$

$$K_l \geq p_{o_l} \geq \frac{p_{s_l}^2 + q_{s_l}^2}{V_{s_l}} R_l, \quad \forall l \in L \quad (1d)$$

$$p_{o_l} X_l = q_{o_l} R_l, \quad \forall l \in L \quad (1e)$$

$$V_{s_l} - V_{r_l} = 2R_l p_{s_l} + 2X_l q_{s_l} - R_l p_{o_l} - X_l q_{o_l}, \quad \forall l \in L \quad (1f)$$

$$z_{h_l} = X_l p_{s_l} - R_l q_{s_l}, \quad \forall l \in L \quad (1g)$$

$$z_{h_l} \geq v_{m_l}^{\min} z_{\theta_l} + z_{\theta_l}^{\min} v_{m_l} - v_{m_l}^{\min} z_{\theta_l}^{\min}, \quad \forall l \in L \quad (1h)$$

$$z_{h_l} \geq v_{m_l}^{\max} z_{\theta_l} + z_{\theta_l}^{\max} v_{m_l} - v_{m_l}^{\max} z_{\theta_l}^{\max}, \quad \forall l \in L \quad (1i)$$

$$z_{h_l} \leq v_{m_l}^{\min} z_{\theta_l} + z_{\theta_l}^{\max} v_{m_l} - v_{m_l}^{\min} z_{\theta_l}^{\max}, \quad \forall l \in L \quad (1j)$$

$$z_{h_l} \leq v_{m_l}^{\max} z_{\theta_l} + z_{\theta_l}^{\min} v_{m_l} - v_{m_l}^{\max} z_{\theta_l}^{\min}, \quad \forall l \in L \quad (1k)$$

$$v_{m_l} \geq v_{s_l}^{\min} v_{r_l} + v_{r_l}^{\min} v_{s_l} - v_{s_l}^{\min} v_{r_l}^{\min}, \quad \forall l \in L \quad (1l)$$

$$v_{m_l} \geq v_{s_l}^{\max} v_{r_l} + v_{r_l}^{\max} v_{s_l} - v_{s_l}^{\max} v_{r_l}^{\max}, \quad \forall l \in L \quad (1m)$$

$$v_{m_l} \leq v_{s_l}^{\min} v_{r_l} + v_{r_l}^{\max} v_{s_l} - v_{s_l}^{\min} v_{r_l}^{\max}, \quad \forall l \in L \quad (1n)$$

$$v_{m_l} \leq v_{s_l}^{\max} v_{r_l} + v_{r_l}^{\min} v_{s_l} - v_{s_l}^{\max} v_{r_l}^{\min}, \quad \forall l \in L \quad (1o)$$

$$z_{\theta_l} \geq \cos\left(\frac{\theta_l^{\max}}{2}\right)(\theta_l + \frac{\theta_l^{\max}}{2}) - \sin\left(\frac{\theta_l^{\max}}{2}\right), \quad \forall l \in L \quad (1p)$$

$$z_{\theta_l} \leq \cos\left(\frac{\theta_l^{\max}}{2}\right)(\theta_l - \frac{\theta_l^{\max}}{2}) + \sin\left(\frac{\theta_l^{\max}}{2}\right), \quad \forall l \in L \quad (1q)$$

$$V_n \geq v_n^2, \quad \forall n \in N \quad (1r)$$

$$V_n \leq (v_n^{\max} + v_n^{\min})v_n - v_n^{\max} v_n^{\min}, \quad \forall n \in N \quad (1s)$$

$$V_n^{\min} \leq V_n \leq V_n^{\max}, \quad \forall n \in N \quad (1t)$$

$$p_n^{\min} \leq p_n \leq p_n^{\max}, \quad \forall n \in N \quad (1u)$$

$$q_n^{\min} \leq q_n \leq q_n^{\max}, \quad \forall n \in N \quad (1v)$$

Where $\Omega = \{p_n, q_n, p_{s_l}, q_{s_l}, p_{o_l}, q_{o_l}, V_n, \theta_l\} \in \mathfrak{R}$ is the set of decision variables. z_{h_l} , z_{θ_l} and v_{m_l} are the introduced auxiliary variables to derive the SOC-ACOPF model. Equations (1b) and (1c) represent the active and reactive power balance. A_{nl}^+ and A_{nl}^- are incidence matrix of the network with $A_{nl}^+ = 1$, $A_{nl}^- = 0$ if n is the sending end of line l and $A_{nl}^+ = -1$, $A_{nl}^- = 1$ if n is the receiving end of line l . Constraints (1d)-(1e) represent active power and reactive power loss. Constraint (1d) is the convex relaxation of the active power loss by rotated cone. The left side of (1d) bounds

p_{o_l} (which equivalently bounds capacity of line l). $V_{s_l} = v_{s_l}^2$ and $V_{r_l} = v_{r_l}^2$ are voltage magnitude squares. $\theta_l = \theta_{s_l} - \theta_{r_l}$ is the voltage phase angle difference of line l . Constraints (1g)-(1s) are convex hulls of the nonconvex AC power flow constraint $v_{s_l} v_{r_l} \sin \theta_l = X_l p_{s_l} - R_l q_{s_l}$. (1g)-(1s) are valid for $0 < \theta_l^{\max} < \frac{\pi}{2}$ and positive $V_n^{\min}$, $V_n^{\max}$ parameters. Note the term s in $p_{s_l}, q_{s_l}, v_{s_l}, V_{s_l}$ is not an index but only to imply the meaning of sending end of line l . The term r in v_{r_l}, V_{r_l} is not an index but only to imply the meaning of receiving end of line l . The term d in P_{d_n}, Q_{d_n} is not an index but only to imply the meaning of power demand. Similar reasoning holds for the term o in p_{o_l}, q_{o_l} which is to denote the meaning of power loss, the term h in z_{h_l} which is to denote the meaning of convex hull, the term θ in z_{θ_l} which is to denote the meaning of auxiliary variable related to phase angle, and the term m in v_{m_l} which is to denote the meaning of auxiliary variable related to voltage. Compared with the SOCP-based ACOPF model in [27] which is valid for only radial power networks (since the model does not include constraints related with voltage phase angle variables), our model (1) is valid for both mesh and radial power networks. In section V of this paper, we give numerical comparisons and more explanations of our SOC-ACOPF model and the model in [27].

III. POWER NETWORK PARTITIONING

Power network topology can be always equivalently represented by a graph $G = (N, L)$ with a vertex set N denoting the nodes or buses and an edge set L denoting the lines or branches. This implies we can use graph-partitioning algorithms to partition a power network. A $\bar{k}$ partition of N defines $\bar{k}$ disjoint subsets of N as $P = \{N_1, N_2, \dots, N_{\bar{k}}\}$. If the partitioning cost of cutting line l (allocating the ends of the line to two separate subnetworks) is c_l , the total cost of partition P is $C(P) = \sum_{l \in \tau} c_l$ where τ is the set of lines with nodes belonging to different subnetworks i.e. the set of tie-lines. Higher c_l means higher possibility that line l is to be kept in one subnetwork (if we minimize the total cost of partitioning). There can be various strategies to set c_l . If all c_l parameters are equal, the network partitioning algorithm will result in least number of tie-lines. In this paper, we set $c_l = 1$ for all lines in order to obtain minimal number of tie-lines after the network partitioning. The network partitioning problem is formulated in (2) [25]:

$$\text{Maximize}_M \quad \text{trace}(M^T C M) \quad (2a)$$

$$\text{subject to} \quad \|M\|_F = \sqrt{\bar{n}} \quad (2b)$$

$$M^T M \leq \epsilon I_{\bar{k}\bar{k}} \quad (2c)$$

$$M_{nk} \in \{0, 1\}, \quad n = 1, \dots, \bar{n}, \quad k = 1, \dots, \bar{k} \quad (2d)$$

where M is $\bar{n} \times \bar{k}$ orthogonal $\bar{k}$ -partition matrix with $M_{nk} = 1$ if $n \in N_k$ and $M_{nk} = 0$ if $n \notin N_k$. $\|M\|_F = \sqrt{\text{trace}(M^T M)}$ is Frobenius norm. Constraint (2b) is valid because M is a $\bar{k}$ -partition matrix if and only if each row of M is the canonical basis of $R^{\bar{k}}$ [25]. C is the $\bar{n} \times \bar{n}$ cost matrix of the network. $C_{in} = c_l$ if i and n are the connecting nodes of line l . $C_{in} = 0$ if i and n are not the connecting nodes of line l . The $I_{\bar{k}\bar{k}}$ is $\bar{k} \times \bar{k}$ unit matrix. Constraint

(2c) models an ϵ -bounded partition (the maximum number of nodes of all subnetworks to be partitioned is ϵ). $C, I_{\bar{k}\bar{k}}$ and ϵ are the parameters of optimization problem (2). The objective is to minimize the total cost of partitioning. This is valid because $C(P) = I_{\bar{n}}^T C I_{\bar{n}} - \text{trace}(M^T C M)$ [25]. The network partitioning problem (2) can be solved approximately by the spectral factorization algorithm proposed in [25]. We show in Section V that even by using the approximated solutions of network partitioning problem (2), the computation efficiency of SOC-ACOPF can still be improved. Please note by using this power network partitioning algorithm, we are not re-organizing the current operational setting of geographically partitioned power network which already exists. This power network partitioning algorithm is executed in the computer and the aim is to solve large-scale SOC-ACOPF problem more efficiently when centralized solution approach requires much computational time and computer RAM capacity.

IV. PROPOSING M-BDA AND THE PARALLEL COMPUTING

The SOC-ACOPF model explained in Section II is used here to formulate the M-BDA. The key contribution is that we decompose SOC-ACOPF by taking the total power generation of each subnetwork as the complicating variable in formulating the proposed M-BDA. This formulation shows very fast convergence performance. We first decompose the large-scale power network to $\bar{k}$ subnetworks using the power network partitioning algorithm described in Section III. The SOC-ACOPF of each subnetwork is taken as a subproblem in the proposed M-BDA. The subproblem k of the proposed M-BDA is formulated in (3).

$$Cost_{k,j}^S = \text{Minimize} \sum_{\forall n \in N_k, l \in L_k} f(p_n, q_n, p_{ol}, q_{ol}) \quad (3a)$$

$$\text{subject to } (1b) - (1v), \forall n \in N_k, l \in L_k \quad (3b)$$

$$\sum_{n \in N_k} p_n = P_{k,j}^{sum} : \mu_{k,j}^P, \forall k \in K, j \in J \quad (3c)$$

$$\sum_{n \in N_k} q_n = Q_{k,j}^{sum} : \mu_{k,j}^Q, \forall k \in K, j \in J \quad (3d)$$

Where (3b) refers to the power flow constraints for all lines and nodes located in subnetwork k . N_k and L_k are the sets of nodes and lines located in the subnetwork k . $P_{k,j}^{sum}$ and $Q_{k,j}^{sum}$ are the solutions of subnetwork total power generation from the master problem of the proposed M-BDA in iteration j . $\mu_{k,j}^P$ and $\mu_{k,j}^Q$ are the dual variables for corresponding constraints used for constructing Benders cuts. To guarantee the feasibility of all the subproblems in M-BDA, we allow load increments or decrements for all the nodes in the network. Thus the power balance constraints (1b)-(1c) in (3b) are modified as:

$$p_n - P_{dn} = \Delta P_{dn}^+ - \Delta P_{dn}^- + \sum_{l \in L_k} (A_{nl}^+ p_{sl} - A_{nl}^- p_{ol}) + G_n V_n, \forall n \in N_k, l \in L_k \quad (3e)$$

$$q_n - Q_{dn} = \Delta Q_{dn}^+ - \Delta Q_{dn}^- + \sum_{l \in L_k} (A_{nl}^+ q_{sl} - A_{nl}^- q_{ol}) - B_n V_n, \forall n \in N_k, l \in L_k \quad (3f)$$

Where non-negative variables ΔP_{dn}^+ and ΔQ_{dn}^+ are the load increments. Non-negative variables ΔP_{dn}^- and ΔQ_{dn}^- are the load decrements. The load increments and decrements are penalized in the objective function using the penalty parameters $C_{dn}^{P+}, C_{dn}^{P-}, C_{dn}^{Q+}$ and C_{dn}^{Q-} . Note the solutions of tie-line variables are obtained by solving the formulated master problem (4) of M-BDA. To force the solutions of tie-line voltage variables to be same as the solutions from the master problem (4), we also include penalty terms for the tie-line voltage variables in the objective function of the subproblem. C_n^v, C_n^θ are positive penalty parameters. Minimizing quadratic objective function over a convex feasible region is a convex optimization problem. This is formulated as:

$$\begin{aligned} Cost_{k,j}^S = \text{Minimize} \quad & \sum_{\forall n \in N_k, l \in L_k} f(p_n, q_n, p_{ol}, q_{ol}) \\ & + \sum_{n \in N_k} (C_{dn}^{P+} \Delta P_{dn}^+ + C_{dn}^{P-} \Delta P_{dn}^- \\ & + C_{dn}^{Q+} \Delta Q_{dn}^+ + C_{dn}^{Q-} \Delta Q_{dn}^-) \\ & + \sum_{n \in N_k \cap N_\tau, j \in J} [C_n^v (v_n - v_{n,j}^M)^2 \\ & + C_n^\theta (\theta_n - \theta_{n,j}^M)^2] \end{aligned} \quad (3g)$$

Where $v_{n,j}^M, \theta_{n,j}^M$ are the voltage and phase angle solutions from the master problem (4) at iteration j . Although some load increments or decrements may exist at the beginning of the iterations, the final solution of the proposed M-BDA does not have these increments or decrements. This is because the cost of these increments and decrements are very high and they will iteratively converge to zero. Our simulations show that this method is more efficient to guarantee the feasibility of the subproblems than using the feasibility cut approach in the original Benders decomposition (the MOSEK solver can not converge after several hours for the test cases in our paper when using the original feasibility cut approach. we believe the numerical failure of MOSEK using the original Benders feasibility cuts approach is because the infeasible region of the SOC-ACOPF is very complex and hard to be removed by the feasibility cuts).

The master problem of the proposed M-BDA is formulated in (4):

$$\text{Minimize } Cost^M = \sum_{k \in K} Cost_k^S \quad (4a)$$

$$\text{subject to } (1b) - (1v), \forall l \in \tau \quad (4b)$$

$$\begin{aligned} Cost_k^S \geq Cost_{k,j-1}^S + \mu_{k,j-1}^P (P_{k,j}^{sum} - P_{k,j-1}^{sum}) \\ + \mu_{k,j-1}^Q (Q_{k,j}^{sum} - Q_{k,j-1}^{sum}), \forall k \in K, j \in J \end{aligned} \quad (4c)$$

Where (4b) refers to the power flow constraints of all the tie-lines. $P_{k,j-1}^{sum}$ and $Q_{k,j-1}^{sum}$ are the decisions of the previous iteration which are considered as parameters in the current iteration. The decisions of $P_{k,j}^{sum}$ and $Q_{k,j}^{sum}$ are made in the master problem (4) by considering the expanding Benders cuts (4c) and tie-line constraints (4b). We model each subnetwork as a single virtual node in the master problem. This is conceptually illustrated in Fig. 2, 3 and 4 in Section V. Constraints (4c) are Benders cuts from the subproblems.

$\mu_{k,j-1}^P$ and $\mu_{k,j-1}^Q$ are dual variable solutions of equations (3c)-(3d) in subnetwork k at the previous iteration $j-1$. As the iterations proceed, more Benders cuts from solving the subproblems are iteratively included into the master problem. After solving the master problem, all the subproblems can be solved in parallel. The proposed parallel computing structure using the proposed M-BDA is illustrated in Fig. 1. The master problem is responsible for giving solutions of tie-line power flows and lower bound of the objective function. The solutions of subnetwork power flows and upper bound of the objective function are given by the subproblems. There is no communication requirement between the subproblems. In each iteration, firstly the master problem is solved and then all the subproblems are solved in parallel. Since we assign the same partitioning cost c_l to all lines, the number of tie-lines of the network partitions are minimized which in turn minimizes the size of the master problem. We summarize the parallel computing management algorithm in Algorithm 1. The parallelization of the subproblems include one Parallel Loop which generates threads of all the subproblems and one Collect Loop which repeatedly checks the status of the threads and stores the solutions when available. To avoid overloading of the computer disk capacity, the subproblem thread Thread- k is released after the solutions are collected.

Fig. 1. The proposed parallel computing structure using M-BDA

Theorem 1. *If the original SOC-ACOPF model (without decomposition) is feasible, the formulated master problem (4) of the proposed M-BDA is feasible.*

Proof. We prove theorem 1 by mathematical induction. We prove firstly the formulated master problem (4) is feasible at iteration $j = 1$. Afterwards, we prove for any iteration $j' \in J$, if the formulated master problem (4) is feasible for $j = j'$ then it is feasible for the next iteration $j = j' + 1$.

Step 1: The formulated master problem (4) is feasible for $j = 1$. We prove this by constructing one feasible solution for the formulated master problem (4) of the proposed M-BDA. Assume $\Omega_0 = \{p_{n,0}, q_{n,0}, p_{s1,0}, q_{s1,0}, p_{ol,0}, q_{ol,0}, v_{n,0}, \theta_{l,0}\} \in \mathbb{R}$ is one feasible solution of the original SOC-ACOPF model, Ω_0 is also feasible for constraints (4b) in the master problem (4) of the proposed M-BDA. Note it is not necessarily required that the feasible solution Ω_0 is optimal for the original SOC-ACOPF model.

We construct the feasible solution $P_{k,j,0}^{sum}, Q_{k,j,0}^{sum}, P_{k,j-1,0}^{sum}$

Algorithm 1: Parallel Computing Management Algorithm

```

Initialization:  $j = 1$ ;
do
    Generate SOC-ACOPF model (4) for the tie-lines:
        SOC-ACOPF-Master;
    Assign SOC-ACOPF-Master to Thread- $m$ ;
    Execute Thread- $m$ ;
    Broadcast  $P_{k,j}^{sum}$  and  $Q_{k,j}^{sum}$  to Thread- $k$  for  $k \in K$ ;
     $k = 1$ ;
    do
        Generate SOC-ACOPF model (3) for the
            subnetwork- $k$ : SOC-ACOPF- $k$ ;
        Assign SOC-ACOPF- $k$  to Thread- $k$ ;
         $k = k + 1$ ;
    while  $k < k^{max}$ ;
    do
        if thread- $k$  is ready then
            Collect Solutions from Thread- $k$ ;
            Send  $Cost_{k,j}^S, \mu_{k,j}^P$  and  $\mu_{k,j}^Q$  to Thread- $m$ ;
            Release Thread- $k$ ;
        while SOC-ACOPF thread is nonempty;
         $j = j + 1$ ;
    while  $\frac{\sum_{k \in K} Cost_{k,j}^S - Cost^M}{\sum_{k \in K} Cost_{k,j}^S} > 2\%$  and  $j < j^{max}$ ;
    Release Thread- $m$ ;

```

and $Q_{k,j-1,0}^{sum}$ as:

$$P_{k,j,0}^{sum} = \sum_{n \in N_k} p_{n,0} \quad (5a)$$

$$Q_{k,j,0}^{sum} = \sum_{n \in N_k} q_{n,0} \quad (5b)$$

$$P_{k,j-1,0}^{sum} = \sum_{n \in N_k} p_{n,0} \quad (5c)$$

$$Q_{k,j-1,0}^{sum} = \sum_{n \in N_k} q_{n,0} \quad (5d)$$

If we use $\Omega_0, P_{k,j,0}^{sum}, Q_{k,j,0}^{sum}, P_{k,j-1,0}^{sum}$ and $Q_{k,j-1,0}^{sum}$, then constraint (4c) becomes:

$$Cost_k^S \geq Cost_{k,j-1}^S + \mu_{k,j-1}^P (P_{k,j,0}^{sum} - P_{k,j-1,0}^{sum}) + \mu_{k,j-1}^Q (Q_{k,j,0}^{sum} - Q_{k,j-1,0}^{sum}), \forall k \in K \quad (5e)$$

Which is:

$$Cost_k^S \geq Cost_{k,j-1}^S, \forall k \in K \quad (5f)$$

No matter what values $\mu_{k,j-1}^P$ and $\mu_{k,j-1}^Q$ are chosen at $j = 1$, constraint (5f) is always feasible since we do not have upper bound for $Cost_k^S$. The feasible objective solution is:

$$Cost^{M,0} = \min[\sum_{k \in K} Cost_k^S] = \sum_{k \in K} Cost_{k,j-1}^S \quad (5g)$$

Which is actually the lower bound of the non-negative term $\sum_{k \in K} Cost_{k,j-1}^S$. Thus we successfully construct a feasible solution of the master problem as $\Omega_0, P_{k,j,0}^{sum}, Q_{k,j,0}^{sum}$ with objective value of $Cost^{M,0}$. This means the master problem is feasible for $j = 1$.

Step 2: If the formulated master problem (4) is feasible for $j = j'$, then it is feasible for $j = j' + 1$. We prove this step by showing that there is at least one feasible solution for iteration $j = j' + 1$ which can always be constructed by using the feasible solution for $j = j'$. Suppose one feasible solution for iteration $j = j'$ is $\Omega_{j'}, P_{k,j',0}^{sum}, Q_{k,j',0}^{sum}$ with master problem and subproblem objective values as $Cost_k^{M,j',0}, Cost_k^{S,j',0}$. We construct the feasible solution for $j = j' + 1$ as:

$$P_{k,j'+1,0}^{sum} = P_{k,j',0}^{sum} \quad (5h)$$

$$Q_{k,j'+1,0}^{sum} = Q_{k,j',0}^{sum} \quad (5i)$$

The added Benders cut at iteration $j = j' + 1$ is:

$$Cost_k^S \geq Cost_{k,j'}^S + \mu_{k,j'}^P (P_{k,j'+1,0}^{sum} - P_{k,j',0}^{sum}) + \mu_{k,j'}^Q (Q_{k,j'+1,0}^{sum} - Q_{k,j',0}^{sum}), \forall k \in K \quad (5j)$$

Or equivalently:

$$Cost_k^S \geq Cost_{k,j'}^S, \forall k \in K \quad (5k)$$

We show in following that no matter what value $Cost_{k,j'}^S$ takes, we can always construct a feasible solution for the master problem (4). If there exists:

$$Cost_{k,j'}^S \geq Cost_k^{S,j',0}, \forall k \in K \quad (5l)$$

The feasible solution of $Cost_k^S$ for $j = j' + 1$ is $Cost_{k,j'}^S$. Accordingly, feasible objective solution is $Cost_k^{M,j'+1,0} = \sum_{k \in K} Cost_{k,j'}^S$.

Otherwise, if there exists $k \in K' \subset K$ such that:

$$Cost_{k,j'}^S \leq Cost_k^{S,j',0}, \forall k \in K' \subset K \quad (5m)$$

We replace these $Cost_k^S$ by $Cost_k^{S,j',0}$ ($\forall k \in K' \subset K$). The feasible solution of $Cost_k^S$ is $\{Cost_{k,j'}^S, \forall k \notin K' \subset K\} \cup \{Cost_k^{S,j',0}, \forall k \in K' \subset K\}$. Accordingly, the feasible objective solution of the master problem (4) for iteration $j = j' + 1$ is:

$$Cost_k^{M,j'+1,0} = \sum_{k \notin K' \subset K} Cost_{k,j'}^S + \sum_{k \in K' \subset K} Cost_k^{S,j',0} \quad (5n)$$

Combining **Step 1** and **Step 2**, we have proven that the formulated master problem (4) is always feasible as long as the original SOC-ACOPF model is feasible. $\square$

Theorem 2. *If the original SOC-ACOPF model (without decomposition) is feasible, the necessary and sufficient condition for the feasibility of the formulated subproblem (3) of the proposed M-BDA is:*

$$\sum_{n \in N_k} p_n^{min} \leq P_{k,j}^{sum} \leq \sum_{n \in N_k} p_n^{max}, \forall k \in K, j \in J \quad (6a)$$

$$\sum_{n \in N_k} q_n^{min} \leq Q_{k,j}^{sum} \leq \sum_{n \in N_k} q_n^{max}, \forall k \in K, j \in J \quad (6b)$$

Proof. We firstly prove (6) is necessary for the feasibility of subproblem (3) i.e. if subproblem (3) is feasible then (6) holds. Suppose the feasible solution of subproblem (3) at iteration $j \in J$ is $\Omega_j = \{p_{n,j}, q_{n,j}, p_{s_{l,j}}, q_{s_{l,j}}, p_{o_{l,j}}, q_{o_{l,j}}, v_{n,j}, \theta_{l,j}\} \in \mathcal{R}$, from (3c)-(3d) of (3) we have:

$$\sum_{n \in N_k} p_{n,j} = P_{k,j}^{sum}, \forall k \in K, j \in J \quad (7a)$$

$$\sum_{n \in N_k} q_{n,j} = Q_{k,j}^{sum}, \forall k \in K, j \in J \quad (7b)$$

Ω_j is feasible for constraints (4b) of (4), so:

$$p_n^{min} \leq p_{n,j} \leq p_n^{max}, \forall n \in N_k \quad (7c)$$

$$q_n^{min} \leq q_{n,j} \leq q_n^{max}, \forall n \in N_k \quad (7d)$$

Obviously,

$$\sum_{n \in N_k} p_n^{min} \leq \sum_{n \in N_k} p_{n,j} \leq \sum_{n \in N_k} p_n^{max} \quad (7e)$$

$$\sum_{n \in N_k} q_n^{min} \leq \sum_{n \in N_k} q_{n,j} \leq \sum_{n \in N_k} q_n^{max} \quad (7f)$$

From (7a)-(7b) and (7e)-(7f), the expression (6) holds.

Next, we prove (6) is sufficient for the feasibility of subproblem (3) i.e. if (6) holds, subproblem (3) is feasible. Again, we assume $\Omega_0 = \{p_{n,0}, q_{n,0}, p_{s_{l,0}}, q_{s_{l,0}}, p_{o_{l,0}}, q_{o_{l,0}}, V_{n,0}, \theta_{l,0}\} \in \mathcal{R}$ is one feasible solution of the original SOC-ACOPF model. Obviously, $\{p_{s_{l,0}}, q_{s_{l,0}}, p_{o_{l,0}}, q_{o_{l,0}}, v_{n,0}, \theta_{l,0}\}$ is feasible for constraints (1d)-(1v) in constraint (3b) of (3). The remaining constraints are (1b)-(1c) (which are modified as (3e)-(3f)) and (3c)-(3d). We construct the feasible solution of p_n, q_n at iteration j as $p_{n,j}, q_{n,j}$:

$$p_{n,j} = p_{n,0} + \Delta P_{d_n}^+ - \Delta P_{d_n}^-, \forall n \in N_k, l \in L_k \quad (7g)$$

$$q_{n,j} = q_{n,0} + \Delta Q_{d_n}^+ - \Delta Q_{d_n}^-, \forall n \in N_k, l \in L_k \quad (7h)$$

Substitute (7g)-(7h) in (3e)-(3f), we have:

$$p_{n,0} + \Delta P_{d_n}^+ - \Delta P_{d_n}^- - P_{d_n} = \Delta P_{d_n}^+ - \Delta P_{d_n}^- + G_n V_{n,0} + \sum_{l \in L_k} (A_{nl}^+ p_{s_{l,0}} - A_{nl}^- p_{o_{l,0}}), \forall n \in N_k, l \in L_k \quad (7i)$$

$$q_{n,0} + \Delta Q_{d_n}^+ - \Delta Q_{d_n}^- - Q_{d_n} = \Delta Q_{d_n}^+ - \Delta Q_{d_n}^- - B_n V_{n,0} + \sum_{l \in L_k} (A_{nl}^+ q_{s_{l,0}} - A_{nl}^- q_{o_{l,0}}), \forall n \in N_k, l \in L_k \quad (7j)$$

Or equivalently:

$$p_{n,0} - P_{d_n} = \sum_l (A_{nl}^+ p_{s_{l,0}} - A_{nl}^- p_{o_{l,0}}) + G_n V_{n,0}, \forall n \in N_k, l \in L_k \quad (7k)$$

$$q_{n,0} - Q_{d_n} = \sum_l (A_{nl}^+ q_{s_{l,0}} - A_{nl}^- q_{o_{l,0}}) - B_n V_{n,0}, \forall n \in N_k, l \in L_k \quad (7l)$$

Which are feasible since Ω_0 is a feasible solution. To construct the feasible solutions of the power load increments or

decrements variables, we consider feasibility of the constraints (3c)-(3d):

$$\begin{aligned} \sum_{n \in N_k} p_{n,j} &= \sum_{n \in N_k} (p_{n,0} + \Delta P_{d_n}^+ - \Delta P_{d_n}^-) \\ &= P_{k,j}^{sum}, \forall k \in K, j \in J \end{aligned} \quad (7m)$$

$$\begin{aligned} \sum_{n \in N_k} q_{n,j} &= \sum_{n \in N_k} (q_{n,0} + \Delta Q_{d_n}^+ - \Delta Q_{d_n}^-) \\ &= Q_{k,j}^{sum}, \forall k \in K, j \in J \end{aligned} \quad (7n)$$

Since (6) holds, we can express $P_{k,j}^{sum}$ and $Q_{k,j}^{sum}$ as:

$$P_{k,j}^{sum} = \lambda_{k,j}^P \sum_{n \in N_k} p_n^{max} + (1 - \lambda_{k,j}^P) \sum_{n \in N_k} p_n^{min} \quad (7o)$$

$$Q_{k,j}^{sum} = \lambda_{k,j}^Q \sum_{n \in N_k} q_n^{max} + (1 - \lambda_{k,j}^Q) \sum_{n \in N_k} q_n^{min} \quad (7p)$$

Where $0 \leq \lambda_{k,j} \leq 1$. Similarly, we can express $p_{n,0}, q_{n,0}$ as:

$$p_{n,0} = \lambda_{n,0}^P p_n^{max} + (1 - \lambda_{n,0}^P) p_n^{min} \quad (7q)$$

$$q_{n,0} = \lambda_{n,0}^Q q_n^{max} + (1 - \lambda_{n,0}^Q) q_n^{min} \quad (7r)$$

Where $0 \leq \lambda_{n,0} \leq 1$. By which we can construct the feasible solution of $\Delta P_{d_n}^+, \Delta P_{d_n}^-, \Delta Q_{d_n}^+$ and $\Delta Q_{d_n}^-$. If $\lambda_{k,j}^P \geq \lambda_{n,0}^P$ and $\lambda_{k,j}^Q \geq \lambda_{n,0}^Q$, we have:

$$\Delta P_{d_n}^+ = (\lambda_{k,j}^P - \lambda_{n,0}^P)(p_n^{max} - p_n^{min}), \forall n \in N_k \quad (7s)$$

$$\Delta P_{d_n}^- = 0, \forall n \in N_k \quad (7t)$$

$$\Delta Q_{d_n}^+ = (\lambda_{k,j}^Q - \lambda_{n,0}^Q)(q_n^{max} - q_n^{min}), \forall n \in N_k \quad (7u)$$

$$\Delta Q_{d_n}^- = 0, \forall n \in N_k \quad (7v)$$

Otherwise If $\lambda_{k,j}^P < \lambda_{n,0}^P$ or $\lambda_{k,j}^Q < \lambda_{n,0}^Q$, then:

$$\Delta P_{d_n}^+ = 0, \forall n \in N_k \quad (7w)$$

$$\Delta P_{d_n}^- = (\lambda_{n,0}^P - \lambda_{k,j}^P)(p_n^{max} - p_n^{min}), \forall n \in N_k \quad (7x)$$

$$\Delta Q_{d_n}^+ = 0, \forall n \in N_k \quad (7y)$$

$$\Delta Q_{d_n}^- = (\lambda_{n,0}^Q - \lambda_{k,j}^Q)(q_n^{max} - q_n^{min}), \forall n \in N_k \quad (7z)$$

(7s)-(7z) guarantee the non-negativity of variables $\Delta P_{d_n}^+, \Delta P_{d_n}^-, \Delta Q_{d_n}^+, \Delta Q_{d_n}^-$, the feasibility of (3c)-(3d) (equality) as well as the feasibility of $p_{n,j}$ and $q_{n,j}$ expressed in (7g)-(7h) (satisfying the constraints (1u)-(1v)). $\square$

Theorem 3. *If the original SOC-ACOPF model (without decomposition) is feasible and condition (6) holds, the optimal solution of the original SOC-ACOPF model is preserved by the proposed M-BDA.*

Proof. Since the convergence of M-BDA is guaranteed by the convexity of the SOC-ACOPF model which has been proved by [10], the remaining task is to prove the convergent optimal solution of M-BDA is exactly the optimal solution of the original SOC-ACOPF model. The proof of Theorem 3 is straightforward based on the Theorem 1 and Theorem 2. Note we have actually proved in Theorem 1 that any feasible solution of the original SOC-ACOPF model is also feasible for the formulated master problem (4) of M-BDA. We denote the optimal solution of the original SOC-ACOPF model as $\Omega_* = \{p_{n,*}, q_{n,*}, p_{s1,*}, q_{s1,*}, p_{o1,*}, q_{o1,*}, V_{n,*}, \theta_{l,*}\} \in \mathcal{R}$. Since

Ω_* is also feasible for the original SOC-ACOPF model, we can set the feasible solutions Ω_0 used in the proof of Theorem 1 as:

$$\Omega_0 = \Omega_* \quad (8a)$$

Again, we can use mathematical induction to prove Ω_* is always feasible for the master problem (4). Firstly, we prove Ω_* is feasible for the master problem (4) at iteration $j = 1$. According to the **Step 1** of the proof for Theorem 1, if we set $\Omega_0 = \Omega_*$, we can construct the feasible solution $P_{k,j,*}^{sum}, Q_{k,j,*}^{sum}, P_{k,j-1,*}^{sum}$ and $Q_{k,j-1,*}^{sum}$ as:

$$P_{k,j,*}^{sum} = \sum_{n \in N_k} p_{n,*} \quad (8b)$$

$$Q_{k,j,*}^{sum} = \sum_{n \in N_k} q_{n,*} \quad (8c)$$

$$P_{k,j-1,*}^{sum} = \sum_{n \in N_k} p_{n,*} \quad (8d)$$

$$Q_{k,j-1,*}^{sum} = \sum_{n \in N_k} q_{n,*} \quad (8e)$$

Same reasoning through the **Step 1** of the proof for Theorem 1, we can prove $\Omega_*, P_{k,j,*}^{sum}, Q_{k,j,*}^{sum}$ is feasible for the master problem (4) at iteration $j = 1$. Then we prove Ω_* is feasible for the master problem (4) at iteration $j = j' + 1$ if Ω_* is feasible at iteration $j = j'$. Since Ω_* is feasible at iteration $j = j'$, at **Step 2** of the proof for Theorem 1, we can set:

$$\Omega_{j'} = \Omega_* \quad (8f)$$

$$P_{k,j',0}^{sum} = P_{k,j'+1,0}^{sum} = \sum_{n \in N_k} p_{n,*} \quad (8g)$$

$$Q_{k,j',0}^{sum} = P_{k,j'+1,0}^{sum} = \sum_{n \in N_k} q_{n,*} \quad (8h)$$

Same reasoning through **Step 2** of the proof for Theorem 1, we can prove Ω_* is feasible for the master problem (4) at iteration $j = j' + 1$. Up to now, we have proved Ω_* is always feasible for the master problem (4).

Next, we prove Ω_* is the optimal convergent solution of M-BDA by contradiction. Suppose the convergent solution of M-BDA is $\Omega' \neq \Omega_*$. Since Ω_* is the optimal solution of the original SOC-ACOPF model, we have:

$$\sum_{k \in K} Cost_k^{S*} < \sum_{k \in K} Cost_k^{S'} = Cost^{M'} \quad (8i)$$

Where $Cost_k^{S*}$ is the optimal objective solution for the subproblem (3) using Ω_* . $Cost_k^{S'}$ is the convergent optimal objective solution for the subproblem (3) using Ω' . $Cost^{M'}$ is the convergent optimal objective solution for the master problem (4). Because we have proved Ω_* is always feasible for the master problem (4) (and thus still feasible when M-BDA converges), and we know:

$$Cost^{M*} < Cost^{M'} \quad (8j)$$

Where $Cost^{M*}$ is the feasible objective solution for the master problem (4) using Ω_* . Relationship expressed by (8j) contradicts with the assumption that $Cost^{M'}$ is the optimal objective solution of the master problem (4) (Ω' is the convergent

optimal solution). Thus $\Omega_i \neq \Omega_*$ cannot hold. The convergent optimal solution must be Ω_* . Note it is not required that Ω_* is feasible for all iterations of the subproblem (3). As long as the master problem (4) can converge to Ω_* and gives $P_{k,j,*}^{sum} = \sum_{n \in N_k} p_{n,*}$, $Q_{k,j,*}^{sum} = \sum_{n \in N_k} q_{n,*}$ as parameters to the subproblem (3) (in the final iterations of M-BDA), the final convergent optimal solution is Ω_* . $\square$

V. NUMERICAL RESULTS

All computations are run on a computer with 2.4GHz CPU and 8GB RAM. We use the power network data from MATPOWER directly [28]. The network partitioning algorithm is implemented in MATLAB [25]. The GAMS Grid Computing Facility [29] is employed for implementing the proposed parallel computing.

A. Comparison of SOC-ACOPF models

To compare our SOC-ACOPF model with the SOCP-based ACOPF model in [27], we implement both models in GAMS and solve the models for various IEEE test cases by MOSEK solver. Note we implement the active power loss constraint (1d) in GAMS coding of the SOC-ACOPF model which is different from the implementation in our work in [26]. Accordingly, there are some minor numerical differences compared with the results in [26]. As benchmarks of the comparisons, we use the results from MATPOWER which gives local optimal solutions of the nonconvex ACOPF model and the results from LINDOGLOBAL solver in GAMS which can give global optimal solutions of the nonconvex ACOPF model. MATPOWER uses MATLAB built-in Interior Point Solver (MIPS) to solve nonconvex ACOPF. The LINDOGLOBAL solver employs branch-and-cut methods to find the global optimal solution. If a solution is not found, we denote the corresponding result as 'NA'.

The results are listed in Table I. For all test cases, our SOC-ACOPF model can give very close results compared with LINDOGLOBAL. Compared to MATPOWER and LINDOGLOBAL results, the objective value of IEEE14-bus from our SOC-ACOPF model is bit higher. The reason of this is that the voltage phase angle constraint $0 < \theta_i < \frac{\pi}{2}$ is included in our SOC-ACOPF model (this constraint is required by the convex envelopes expressed by constraint (1g) to (1q)) while this is not necessary for the nonconvex ACOPF model in MATPOWER and GAMS solved by LINDOGLOBAL (the reason can also be due to numerical accuracy tolerance differences of different solvers). For IEEE300-bus, 1354pegase and 2869pegase, LINDOGLOBAL cannot solve these test cases since the model scale exceed the limit of LINDOGLOBAL. In these test cases, we compare the results with MATPOWER which are still very close to our results. It is worth to mention that the results of mesh power networks from the model in [27] are much relaxed solutions since this model does not include constraints related to voltage phase angle. It can be seen from the results in Table I that lower objective values are obtained from the model in [27] compared with our SOC-ACOPF model. However, less AC feasibility can be guaranteed from the solutions of the model in [27]. In terms of computation

time, the model in [27] requires the least computation time since this model has the least number of constraints. LINDOGLOBAL requires the most the computation time to find a global optimal solution. Our model takes less computation time than MATPOWER. Because we have included the voltage phase angle constraints (1g) to (1q) in our SOC-ACOPF model to make it valid for both mesh and radial power networks, the computation time is larger than the model in [27].

B. The power network partitioning

Fig. 2. IEEE14-bus network partitions (two subnetworks) and the master problem representation

Fig. 3. IEEE57-bus network partitions (three subnetworks) and the master problem representation

Fig. 4. IEEE118-bus network partitions (four subnetworks) and the master problem representation

TABLE I
COMPARISON OF SOC-ACOPF MODELS

Test Case	Objective Value [\$]				Computation CPU Time [s]			
	SOC-ACOPF	Model in [27]	MATPOWER	LINDOGLOBAL	SOC-ACOPF	Model in [27]	MATPOWER	LINDOGLOBAL
IEEE14-bus	8092.32	8072.42	8081.53	8081.54	0.09	0.08	0.11	0.20
IEEE57-bus	41711.78	41673.10	41737.79	41737.93	0.17	0.11	0.12	2.31
IEEE118-bus	129376.00	129330.74	129660.70	129660.54	0.36	0.13	0.30	27.10
IEEE300-bus	718546.27	718091.78	719725.11	NA	0.64	0.25	0.48	NA
1354pegase [30]	74040.99	74006.84	74069.35	NA	2.50	0.64	8.58	NA
2869pegase [30]	133934.70	133866.95	133999.29	NA	9.72	1.23	18.66	NA

TABLE II
RESULTS OF POWER NETWORK PARTITIONING

Test Case	Partition	Number of Nodes in Subnetworks	Number of Lines in Subnetworks	Number of Tie-lines	CPU Time [s]
IEEE14-bus	2	6, 8	11, 6	3	0.02
	3	3, 5, 6	2, 7, 6	5	0.02
	4	4, 2, 3, 5	4, 1, 2, 7	6	0.03
	5	2, 2, 5, 3, 2	1, 1, 7, 2, 1	8	0.03
	6	3, 3, 2, 2, 2, 2	3, 3, 1, 1, 1, 1	10	0.03
	7	2, 1, 3, 2, 2, 1, 3	1, 0, 2, 1, 1, 0, 3	12	0.04
	8	3, 1, 2, 2, 1, 2, 2, 1	3, 0, 1, 1, 0, 0, 1, 0	14	0.05
IEEE57-bus	2	18, 39	19, 55	6	0.03
	3	12, 21, 24	12, 25, 35	8	0.03
	4	27, 11, 10, 9	38, 13, 10, 9	10	0.04
	5	11, 10, 7, 9, 20	11, 12, 6, 9, 29	13	0.04
	6	6, 7, 20, 8, 10, 6	6, 8, 29, 7, 12, 5	13	0.04
	7	7, 6, 10, 10, 9, 5, 10	7, 7, 12, 14, 7, 4, 10	19	0.05
	8	6, 9, 5, 7, 9, 4, 6, 11	5, 10, 6, 7, 9, 3, 5, 15	20	0.05
IEEE118-bus	2	79, 39	121, 60	5	0.03
	3	38, 35, 45	51, 55, 70	10	0.04
	4	25, 38, 37, 18	29, 60, 57, 23	17	0.04
	5	22, 16, 27, 19, 34	27, 17, 45, 23, 51	23	0.05
	6	23, 16, 18, 17, 30, 14	29, 23, 25, 21, 48, 18	22	0.05
	7	15, 12, 8, 18, 16, 16, 33	19, 16, 10, 27, 19, 19, 47	29	0.06
	8	21, 15, 6, 16, 15, 12, 16, 17	29, 20, 6, 19, 19, 16, 19, 25	33	0.06
IEEE300-bus	2	116, 184	159, 246	6	0.06
	3	116, 89, 95	159, 113, 127	12	0.06
	4	35, 87, 98, 80	37, 117, 132, 114	11	0.07
	5	80, 105, 45, 33, 37	105, 135, 62, 39, 52	18	0.07
	6	54, 52, 48, 33, 60, 53	74, 59, 65, 39, 84, 69	21	0.09
	7	34, 59, 35, 37, 81, 21, 33	47, 80, 37, 52, 113, 25, 39	18	0.09
	8	34, 33, 53, 35, 35, 21, 35, 54	47, 39, 77, 42, 50, 25, 37, 74	20	0.12
1354pegase [30]	2	828, 526	1211, 753	27	0.56
	3	585, 428, 341	830, 644, 479	38	0.59
	4	311, 253, 319, 471	481, 352, 443, 659	56	0.66
	5	178, 377, 297, 246, 256	253, 536, 460, 337, 355	50	0.65
	6	363, 87, 67, 217, 319, 301	510, 135, 85, 293, 446, 465	57	0.67
	7	120, 255, 72, 288, 152, 225, 242	180, 363, 97, 443, 211, 311, 322	64	0.72
	8	168, 151, 120, 243, 186, 102, 260, 124	242, 210, 180, 331, 274, 138, 354, 193	69	0.92
2869pegase [30]	2	1899, 970	2929, 1633	20	1.20
	3	600, 903, 1366	1026, 1526, 2010	20	1.25
	4	975, 457, 531, 906	1512, 647, 852, 1532	39	1.43
	5	377, 509, 528, 789, 666	536, 818, 849, 1213, 1108	58	1.51
	6	562, 355, 412, 392, 771, 377	944, 585, 644, 621, 1187, 536	65	1.45
	7	354, 312, 579, 322, 508, 580, 214	501, 485, 915, 535, 754, 972, 354	66	1.62
	8	324, 532, 295, 214, 312, 288, 554, 350	544, 791, 491, 354, 485, 464, 876, 494	83	2.10

The results of power network partitioning by spectral factorization are listed in Table II. For each test case, we partition the power network from two to eight subnetworks. The 'Partition' column in Table II lists the total number of subnetworks which is the parameter used in the power network partitioning problem (2). Subnetworks are formed such that collection of them constructs the original power network. For small power networks, when more subnetworks are partitioned, there can be only one bus for some partitioned subnetworks. Thus, there is no line inside these single-bus subnetworks. The spectral factorization algorithm is capable of partitioning all test cases

in reasonable time. Generally, the computation time increases when more subnetworks are partitioned. The CPU time of partitioning large power networks is higher than the one for partitioning small power networks. Fig. 2, Fig. 3 and Fig. 4 are plotted to visualize some representative partitions for IEEE14-bus, IEEE57-bus and IEEE118-bus test cases. We use different colors to distinguish different partitioned subnetworks in Fig. 2, Fig. 3 and Fig. 4. The corresponding master problem of the proposed M-BDA for each partitioned power network is also conceptually illustrated. Note we do not plot all the tie-lines in the representative master problems in Fig. 2, Fig. 3 and Fig.

TABLE III
ACCELERATED M-BDA BY GAMS PARALLEL COMPUTING

Test Case	Centralized Solution Approach		Accelerated M-BDA				
	Objective [\$]	CPU Time [s]	Partition	Upper Bound [\$]	Lower Bound [\$]	Relative Gap	CPU Time [s]
IEEE14-bus	8092.32	0.09	2	8156.28	8067.17	1.09%	0.16
			3	8115.84	8043.42	0.89%	0.20
			4	8107.18	8043.43	0.79%	0.14
			5	8102.82	8057.13	0.56%	0.20
			6	8127.95	8080.57	0.58%	0.28
			7	8099.13	7967.99	1.62%	0.31
			8	8119.83	8073.33	0.57%	0.28
IEEE57-bus	41711.78	0.17	2	41801.64	41647.12	0.37%	0.30
			3	41871.87	41428.57	1.06%	0.25
			4	41798.99	41624.31	0.42%	0.22
			5	41924.35	41675.16	0.59%	0.44
			6	41814.77	41445.73	0.88%	0.45
			7	41956.47	41376.34	1.38%	0.25
			8	41848.47	41518.08	0.79%	0.37
IEEE118-bus	129376.00	0.36	2	130025.49	129261.01	0.59%	0.33
			3	130847.44	128976.06	1.43%	0.76
			4	130066.45	129182.29	0.68%	0.39
			5	130201.72	128631.17	1.21%	0.63
			6	131015.94	129089.14	1.47%	0.36
			7	130725.29	128393.25	1.78%	0.39
			8	129620.15	128955.16	0.51%	0.59
IEEE300-bus	718546.27	0.64	2	722485.13	712303.53	1.41%	0.86
			3	721442.74	709941.27	1.59%	0.55
			4	724036.39	717468.50	0.91%	1.03
			5	724084.83	712925.47	1.54%	0.59
			6	724038.39	709888.48	1.95%	0.70
			7	719765.51	716539.87	0.45%	0.52
			8	720591.14	714182.86	0.89%	0.47
1354pegase [30]	74040.99	2.50	2	74041.62	74041.60	0.00%	3.98
			3	74198.12	74038.80	0.21%	4.17
			4	74040.36	74040.36	0.00%	1.51
			5	74040.91	74040.91	0.00%	1.51
			6	74040.42	74040.42	0.00%	1.30
			7	74040.82	74040.82	0.00%	1.61
			8	74041.01	74040.99	0.00%	0.91
2869pegase [30]	133934.70	9.72	2	133938.47	133886.16	0.04%	11.83
			3	134455.42	133409.29	0.78%	14.26
			4	133940.19	133869.50	0.05%	6.72
			5	134122.31	131618.32	1.87%	8.70
			6	133915.61	133915.57	0.00%	3.42
			7	133928.71	133928.70	0.00%	3.15
			8	135021.76	132710.79	1.71%	5.34

4. Detailed results about the tie-lines are listed in Table II.

C. Accelerated M-BDA using GAMS parallel computing

The MOSEK solver in GAMS is used to solve the SOC-ACOPF model, the master problem and subproblem in the proposed M-BDA. The results of accelerated M-BDA using GAMS parallel computing are listed in Table III. For comparison, the results of centralized solution approach (without decomposition) of SOC-ACOPF are also listed. The 'Relative Gap' column shows the gap between the upper bound and lower bound of M-BDA calculated as $\text{Relative Gap} = \frac{\text{Upper Bound} - \text{Lower Bound}}{\text{Upper Bound}} \times 100\%$. For all these test cases, M-BDA converges to very close solutions to single-stage SOC-ACOPF without decomposition. With the increase of partition depth (more subnetworks and fewer nodes in each subnetwork), the SOC-ACOPF problem complexity is decreasing. All test cases converge within few iterations. Compared with the computation time of the centralized solution approach, the computational efficiency improvement is more prominent in large test cases. Our results of improved computational efficiency by varying the power network partitions show

promising approach of using sufficient off-line simulations to identify the most efficient power network partitions for a given power network.

VI. CONCLUSION

An accelerated M-BDA using parallel computing is proposed to tackle the complexity of large-scale SOC-ACOPF problem. The formulation, feasibility and optimality proof, and fast convergence of the proposed M-BDA are the main contributions of the current paper. The numerical results show that the M-BDA accelerated by GAMS grid computing can reduce SOC-ACOPF problem scale (reduce the RAM requirement for the computer) as well as computation time. The advantage of solving SOC-ACOPF in a decomposed way is that we reduce the dimension of Hessian matrix and Jacobian matrix during the iterations of interior point method. This is very useful for large-scale power networks where the number of variables and constraints of the formulated SOC-ACOPF exceed the solver limit. Another advantage of the proposed decomposition is that, by keeping the boundaries between different subnetworks or zones in the power system, the data

privacy of each operation zone can be protected. A coordinator who is solving the master problem of the proposed M-BDA does not need to know the detailed network configuration of the subnetworks. All the required information (from the subnetworks) by the coordinator is communicated through the Benders cuts. We prove the feasibility and optimality of M-BDA analytically and numerically. The convergence of the proposed approach is guaranteed by the convexity of the SOC-ACOPF model [10]. Future research can be directed on examining the proposed approaches for larger power networks.

ACKNOWLEDGMENT

The authors would like to thank the support from SETS Erasmus Mundus Joint Doctorate Fellowship funded by the European Commission for this research project on optimal power flow. The editors' and reviewers' great efforts of reviewing this paper are highly appreciated by the authors.

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