Multi-period portfolio optimization
given a priori information on signal
dynamics and transactions costs

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Abstract

Multi-period portfolio optimization (MPO) has gained a lot of interest in modern portfolio theory due to its consideration for inter-temporal trading effects, especially market impacts and transactions costs, and for its subtle reliability on return predictability. However, because of the heavy computational demand, portfolio policies based on this approach have been sparsely explored. In that regard, a tractable MPO framework proposed by N. Gărleanu & L. H. Pedersen has been investigated. Using the stochastic control framework, the authors provided a closed form expression of the optimal policy. Moreover, they used a specific, yet flexible return predictability model. Excess returns were expressed using a linear factor model, and the predicting factors were modeled as mean reverting processes. Finally, transactions costs and market impacts were incorporated in the problem formulation as a quadratic function.

The elaborated methodology considered that the market returns dynamics are governed by fast and slow mean reverting factors, and that the market transactions costs are not necessarily quadratic. By controlling the exposure to the market returns predicting factors, the aim was to uncover the importance of the mean reversion speeds in the performance of the constructed trading strategies, under realistic market costs. Additionally, for the sake of comparison, trading strategies based on a single-period mean variance optimization were considered. The findings suggest an overall superiority in performance for the studied MPO approach even when the market costs are not quadratic. This was accompanied with evidence of better usability of the factors’ mean reversion speed, especially fast reverting factors, and robustness in adapting to transactions costs.

Keywords: multi-period portfolio optimization, portfolio selection, mean-variance optimization, return predictability, mean reverting processes, transactions costs, market impacts, stochastic optimal control.
Sammanfattning


Den utarbetade metodiken ansåg att marknadens avkastningsdynamik styrs av snabba och långsamma återhämtningsfaktorer, och att kostnaderna för marknads transaktioner inte nödvändigtvis är kvadratiska. Genom att reglera exponeringen mot marknaden återspeglar förutsägande faktorer, var målet att avslöja viken av de genomsnittliga omkastningshastigheterna i utförandet av de konstruerade handelsstrategierna, under realistiska marknadskostnader. Dessutom, för jämförelses skull, övervägdes handelsstrategier baserade på en enstaka genomsnittlig variansoptimering. Resultaten tyder på en överlägsen överlägsenhet i prestanda för det studerade MPO-tillvägagångssättet, även när marknadsutgifterna inte är kvadratiska. Detta åtföljdes av bevis för bättre användbarhet av faktorernas genomsnittliga återgångshastighet, särskilt snabba återställningsfaktorer och robusthet vid anpassning till transaktionskostnader.
Résumé

L’optimisation multi-période de portefeuille (OMP) a gagné de l’intérêt dans la théorie moderne de portefeuille, et ce grâce à sa prise en considération des effets inter-temporels de trading, notamment les impacts du marché et les coûts de transactions, ainsi que sa dépendance subtile de la prédicibilité des rendements. Cependant, l’usage de cet approche dans la construction des portefeuille a été faiblement exploré, et cela à cause d’une grande nécessité en puissance de calcul. Dans ce contexte, l’étude d’un modèle OMP proposé par N. Gărleanu & L. H. Pedersen est suggérée. Les auteurs ont utilisé des résultats de la théorie du contrôle stochastique pour trouver la solution explicite. En outre, ils ont utilisé un modèle spécifique mais flexible pour modéliser la prédicibilité des rendements. Les rendements en excès ont été basé sur un modèle à facteurs linéaire, et ces facteurs ont été modélisé par des processus auto-régressifs. Enfin, les coûts de transactions ont été représenté sous la forme d’une fonction quadratique.

Dans la méthodologie élaborée, les rendements du marché sont gouvernés par des facteurs à décadence, soit rapide, soit lente, envers leur moyenne. Les coûts de transactions quant à eux, ils ne sont pas forcément de nature quadratique. Par ailleurs, l’observation des facteurs est contrôlé, et ce dans le but d’élucider l’impact de leur vitesse de décadence sur la performance des stratégies de trading considérées. En plus, d’autres stratégies se basant sur un modèle d’optimisation de portefeuille à une seule période ont été utilisé dans le but de faire des comparaisons. Les résultats ont conclue que l’approche MPO étudiée est globalement supérieur même quand les coûts de transactions ne sont pas quadratiques. D’autre part, l’approche semble faire preuve d’un meilleur usage des vitesses caractérisants les facteurs, en particulier les facteurs à décadence rapide, ainsi que d’une robustesse face aux coûts de transactions même si ceux-la ne sont pas quadratiques.
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A.7 Combination strategies $DT_{comb}$, $MW_{comb}$ and $E_6MW_{comb}$ evolution with respect to quadratic costs.
Chapter 1

Introduction

1.1 Motivation

In the problem of portfolio selection, the investor is faced with the decision to make the best investment possible, that maximizes his profit over a set of assets. Due to the random nature of asset returns in the financial market, the problem becomes somewhat difficult to model and formulate. Additional aspects such as transaction costs even further the complexity of the task, and renders the problem difficult to solve. Figure 1.1 illustrates the portfolio selection process, thereby sketching a problem description, and figure 1.2 illustrates the transaction costs aspect of the problem.

The standard approach to tackle the problem is undoubtedly the mean variance optimization (MVO) approach, which was pioneered by the work of Harry Markowitz on the theory of portfolio selection [12]. In this approach, expectation and covariance of returns are used respectively to situate the unknown payoffs in the future and quantify the risk of investment. Then, using these measures, an optimization problem is formulated. That being said, this approach can be theoretically elegant and intuitive, however it has some shortcomings when it comes to practice, such as incorporating transaction costs and market impacts, being sensitive to the returns’ moments estimation among others as highlighted in [9]. These shortcomings were constantly addressed and some new directions emerged during the last decades one of which is the multi-period portfolio optimization (MPO) approach [3].
CHAPTER 1. INTRODUCTION

Figure 1.1 – Illustration of the portfolio selection problem. The investor seeks to make the best investment possible at time $t$, by choosing the optimal amount of shares to hold in the assets he wishes to invest in, in order to maximize his profit at time $t + 1$. Because the returns are unknown after time $t$, the investor needs to predict future returns at $t + 1$, thus there will be a risk to his investment that needs also to be considered.

Figure 1.2 – Illustration of transaction costs in the portfolio selection problem. Once the new portfolio has been decided, the investor must now trade his way out from the old portfolio, that was held throughout the period $t - 1$ to $t$. In other words, he buys and sells shares in the asset that he invests in, to reach the new portfolio. As a consequence of his tradings or transactions, costs are incurred. In fact, the current trading decisions can affect even future outcomes and cause more costs.
1.1. MOTIVATION

The MPO approach consist of expanding the mean variance framework over multiple periods while incorporating the transaction costs at the same time. The motivation behind this approach lies in the fact that transaction costs and market impacts can affect the future outcomes of a current trade as illustrated in figure 1.2. Therefore, this approach is generally viewed and argued in literature as being better than a single period approach [5, 13]. Nonetheless, the MPO approach has been poorly used in practice, mainly due to the formulations being intractable and computationally demanding [3]. Figures 1.3 and 1.4 give an illustration of both a single period optimization approach and an MPO approach.

In this thesis, the work on MPO published by N. Gårleanu & L. H. Pedersen in [7] was of interest. In their work, the authors derived a closed form solution that was expressed as a weighted linear combination between a current portfolio and an aim portfolio. Return predictability played an important role in this expression, specifically in the construction of the aim portfolios and the weights of this combination. Transaction costs on the other hand only affected the latter. These findings alongside further considerations lead the authors to conclude that their optimal strategy followed two rules, first, to aim in front of the Markowitz portfolio, and second, to trade partially towards the aim. Figure 1.5 illustrates the two principles at work. That being said, the main innovation of N. Gårleanu & L. H. Pedersen’s work was that they presented a tractable MPO framework that accounted for transaction costs and considered a flexible return predictability model, thereby reducing the computational demands and opening wide clear numerous doors for investigating their approach.
CHAPTER 1. INTRODUCTION

Figure 1.3 – Illustration of a single period portfolio selection approach. The illustration shows that in a single portfolio portfolio selection approach, the investor tries to predict returns one period ahead to find the optimal portfolio choice.

Figure 1.4 – Illustration of an MPO approach. In an MPO approach, the investor expand his portfolio selection to multiple period. By recreating the portfolio selection process over a horizon $T - t$, the investor seeks to find the best trading decisions while accounting for transaction costs. Then, only the first trading decision at the time of investment $t$ is carried. Note that return predictability here is essential and has more impact on the performance of the investment outcome.
1.2 Objectives

The general thesis purpose was to investigate N. Gârleanu & L. H. Pedersen’s framework. First, an understanding of how the explicit optimal solution was derived, needed to be provided in the light of what the stochastic control theory offers.

Second, return predictability in the framework was based on a factor model where the factors were considered to be mean reverting processes. In other words, the stochastic returns were written as a linear combination of a set of predicting factors, and these factors were model as processes that have a tendency to return to their mean. The authors proved under certain conditions that the optimal solution can be directly linked to the factors’ strengths, specifically their mean reversion speed. A question of interest was to see how these factors’ characteristics impact the performance of the strategy. Hence, an objective was to design a methodology that sheds some light on this point.

Finally, N. Gârleanu & L. H. Pedersen used a quadratic cost function in their framework. While they argued their choice to be a valid one, it remains that realistic cost functions have a different form \[3, 11, 9\]. Therefore, a last objective

Figure 1.5 – Illustration of the optimal dynamic trading strategy. Illustration of the dynamic trading strategy (blue trajectory) trading partially towards the aim portfolios red trajectory. As seen in the plot, the dynamic trading strategy tend to follow the aim trajectory. The illustration is a reproduction of N. Gârleanu & L. H. Pedersen’s plot in \[7\].
was to test the limitations of such a consideration in the framework at hand.

1.3 Context

Lynx asset management is a hedge fund company based in Stockholm that aims to provide high risk-adjusted returns thought investments. Lynx is characterized by being a model-based asset management firm. They leverage the use of quantitative mathematical approaches to better analyze patterns and identify trends in the financial markets, as well as to construct systematic and complex investment models.

This thesis was carried within the research department of Lynx under the supervision of Anders Blomqvist and co-supervision of Mats Brodén and Ola Backman. The work was research oriented and concerned multi-period portfolio optimization with the aim of potential future use. The research methodology was purely experimental, and access to computational resources was provided to carry heavy experiments. Lynx has changed its locals by the end of November but the transition was smooth and did not affect the work flow anyhow.

1.4 Outline

This thesis is structured as follows. Chapter 2 reviews some background on portfolio selection, times series analysis and stochastic control theory, that is of relevance to the scope of the thesis. Chapter 3 presents N. Gårleanu & L. H. Pedersen’s framework and provides an understanding of how the closed form expression was obtained. Chapter 4 describes and motivates how the framework was used, as well as the designed experiments and considered settings. Chapter 5 presents the obtained results accompanied with discussions and comments. Chapter 6 summarizes the obtained results and concludes the work of the thesis.
Chapter 2

Background

In this chapter, the relevant background for this thesis is presented. First, an overview of the portfolio selection problem will be provided highlighting the main concepts and mathematical representations relevant to multi-period portfolio optimization. Second, some concepts, definitions and tools from time series analysis used in finance will be provided. Finally, a theoretical framework for stochastic optimal control that is of relevance to the work of N. Gârleanu & L. H. Pedersen will be presented.

2.1 Portfolio selection

2.1.1 Preliminaries

Returns

In the portfolio selection problem, investors and practitioners focus on returns instead of prices to make their investment decisions. This is often explained by the fact that asset returns are stationary processes and provide nice theoretical features [14]. Consider the set of equally-spaced points in time $\mathcal{D}$. Under the scope of this thesis, the time interval between subsequent time points is one trading day. Now, consider the set of asset prices at these time dates

$$p_t, \quad t \in \mathcal{D}.$$

For convenience, we abuse notations and consider that $\mathcal{D}$ is mapped to the set of integers $\mathbb{Z}$.

Definition 2.1.1 (Linear Return) The linear return at time $t$ is defined, following [14], as:

$$L_t = \frac{p_t}{p_{t-1}} - 1.$$
Definition 2.1.2 (Log Return) The log return, also called compounded return, at time \( t \) is defined, following [14], as:

\[
C_t = \log \left( \frac{P_t}{P_{t-1}} \right).
\]

Let us note that linear returns are approximately the same as compounded returns when the price volatility is low or the time interval between subsequent prices is very small. Under the scope of this thesis this approximation will be considered as a valid one.

Risk

Asset returns are widely recognized to exhibit a random behaviour. Therefore, quantifying and incorporating risk in the process of making an investment is crucial to the success of the investment. There are different risk measures used in practice such as the portfolio volatility, the value at risk and the expected shortfall to name a few. An extensive study of these measures is beyond the scope of this thesis but further information could be found in [6] for the most curious.

2.1.2 Mean variance optimization

In this section we shall present the MVO approach pioneered by H. Markowitz [12] and that represents a fundamental part in modern portfolio theory [9]. First, let us start by providing some definitions and notations.

Consider \( \mathcal{A} \) a set of \( n \) assets that the investor wishes to invest in. Let \( r \) be a random vector that takes values in \( \mathbb{R}^n \) and that represent the uncertain future returns of the assets in \( \mathcal{A} \). A portfolio invested in the assets \( \mathcal{A} \) can be defined as a vector \( w \) that contains \( n \) weights \( w_1, w_2, \ldots, w_n \) each assigned to an asset in \( \mathcal{A} \). It is in general required that the vector of weights \( w \) verifies

\[
\sum_{i=1}^{n} w_i = 1.
\]

The goal of the investor is to chose the optimal portfolio \( w \) that maximizes his profits. We shall now provide an MVO formulation as described generally in literature (see e.g., [8, 9, 12, 14, 15]). Consider the expectation and covariance matrix of the vector of uncertain future returns \( r \) to be, respectively,

\[
\mu = \mathbb{E}(r) \quad \text{and} \quad \Sigma = \mathbb{E} \left( (r - \mu)(r - \mu)^T \right),
\]
2.1. PORTFOLIO SELECTION

where $E$ denotes the mathematical expectation operator. The MVO problem can be formulated as follows

$$\max_{w \in \mathcal{W}} w^T \mu - \frac{\gamma}{2} w^T \Sigma w,$$  \hfill (2.2)

where $\mathcal{W}$ is a subset of $\mathbb{R}^n$ that represents the feasible portfolios defined by the set of constraints that the investor chooses to impose and $\gamma$ is a real number that represents the risk aversion of the investor. The optimization problem is convex as long as the covariance matrix $\Sigma$ is positive definite and the feasible portfolios subset $\mathcal{W}$ is convex in practice. This is generally the case in practice. The return of a portfolio $w$ can be defined as

$$r_p(w) = w^T r.$$

The variance of a portfolio $w$ and its volatility are defined, respectively, by

$$V_p(w) = w^T \Sigma w \quad \text{and} \quad \sigma_p(w) = \sqrt{w^T \Sigma w}.$$

In the MVO problem (2.2) the portfolio variance quantifies the risk that the investor seeks to minimize. By varying the risk aversion parameter, the investor obtains different optimal portfolios each with a specific risk level. The investor is then faced with what is referred to in the modern portfolio theory [9] as the risk-return trade-off.

The leveraged portfolio

When leverage is allowed (i.e. money can be borrowed to achieve the investment), the initial wealth is no more a constraint and the condition 2.1 can be overlooked. Additionally, when no extra constraint is considered by the investor, $\mathcal{W} = \mathbb{R}^n$, the optimal portfolio $w^*$ that solves the unconstrained convex case of the problem 2.2 is

$$w^* = \gamma^{-1} \Sigma^{-1} \mu.$$

(2.3)

2.1.3 Transaction costs

Transactions costs can often be divided into direct and indirect costs [9, 17]. Broker fees, commissions, taxes, holding costs and bid-ask spreads are often considered as direct costs and are assumed to be easy to estimate. On the other hand, indirect costs are very hard to estimate. They are in general referred to as market impacts and are viewed as a reaction force of the market. They tend to move prices against the expectations of the investor in response to his transactions [1, 7, 17]. For instance, slippage is due to a random price change at the time of the transaction.
CHAPTER 2. BACKGROUND

Transaction costs depend on the traded asset and are observed to grow with the value amount of the traded shares in that asset. Consider \( x_i \) to be such a value amount for a particular asset \( i \) in the set of assets \( \mathcal{A} \) that the investors wishes to invest in. Transactions costs model used in literature revolve all around using three terms dependant on the traded shares \( z_i \): a linear term, a super-linear term and a quadratic term (see for e.g. [1, 3, 9] and references therein). Bearing this in mind, for each asset \( i \) in \( \mathcal{A} \), we define the transaction cost function \( TC_i \) as a real function that maps a traded amount of shares \( z_i \) in asset \( i \) to its incurred cost as follows

\[
TC_i(z_i) = a_i |z_i| + b_i |z_i|^{3/2} + c_i z_i^2,
\]

where \( a_i, b_i \) and \( c_i \) are estimated parameters that depend on asset \( i \). Let us note that these parameters are generally positive in which case the transaction cost function is convex \(^1\). Now, we define the transactions cost function \( TC : \mathbb{R}^n \to \mathbb{R} \) that maps to each vector of trades \( z \) in the set of assets \( \mathcal{A} \) to the total incurred costs by the trades

\[
TC(z) = \sum_{i=1}^{n} TC_i(z_i).
\] (2.4)

2.1.4 Multi-period portfolio optimization

In the MPO approach, the investor seeks to find an optimal portfolio at the time of the investment by also considering future outcomes given a priori knowledge on the market. In this section, we shall give a formulation somewhat generic of the MPO approach inspired by [3, 9, 10].

Let \( t \) denote the time or the period at which the investor has to make his investment decision. Let \( T \) be the horizon that denotes the number of future periods that the investor wishes to consider in his decision. Given the a priori knowledge he has at time \( t \), the investor predicts the future returns

\[
r_{t+1|t}, r_{t+2|t}, \ldots, r_{t+T|t},
\]

and their respective covariances

\[
\Sigma_{t+1|t}, \Sigma_{t+2|t}, \ldots, \Sigma_{t+T|t}.
\]

Throughout the periods \( t, t+1, \ldots, t+T \), consider the sequence of the investor’s holdings in the assets \( \mathcal{A} \)

\[
h_{t+s} \in \mathcal{X}_s, \quad s = 0, 1, \ldots, T,
\]

\(^1\)Transactions costs are not always represented by convex function, for instance in the case of imposed fees [11].
2.1. PORTFOLIO SELECTION

where $X_s$ is a subset in $\mathbb{R}^n$ that is defined by the constraints that the investor wishes to impose on the holdings at the period $s$. At the time of investment $t$, the initial vector of holdings $h_t$ is already known. The investor is interested in finding the sequence of trades that will help him attain the future holdings. Let the trades be denoted as

$$z_{t+s} \in C_s, \quad s = 0, 1, \ldots, T - 1,$$

where $C_s$ is a subset of $\mathbb{R}^n$ that is defined by the constraints that the investor wishes to impose on the trades at the period $s$. The relation between two subsequent holdings due to the change in the price between periods is given by

$$h_{t+s+1} = (h_{t+s} + z_{t+s}) \circ (1 + r_{t+s+1|t}), \quad s = 0, \ldots, T - 1. \quad (2.5)$$

where $\circ$ represents the element-wise product operation sign. When returns are very small compared to one, which is generally the case in practice, the relation \ref{eq:2.5} can be linearized (see for e.g. [3, 9]) to become

$$h_{t+s+1} = h_{t+s} + z_{t+s}, \quad s = 0, \ldots, T - 1. \quad (2.6)$$

The portfolio weights are obtained by normalizing holdings $h$ with their total value $1^T h$ where $1$ is the “ones”-vector. Consider the portfolio’s weights over the planning horizon

$$w_{t+s} \in W_s, \quad s = 0, 1, \ldots, T,$$

where $W_s$ is deduced by normalizing the set of feasible holdings $X_s$. The normalized trades are obtained by normalizing the trades $z$ by the total value of holdings $1^T h$. Consider the normalized trades

$$u_{t+s} \in U_s, \quad s = 0, 1, \ldots, T,$$

where $U_s$ is deduced by normalizing the set of feasible trades $C_s$. By normalizing the relation \ref{eq:2.6} we obtain

$$w_{t+s+1} = w_{t+s} + u_{t+s}, \quad s = 0, \ldots, T - 1. \quad (2.7)$$

To lighten notations, we consider the time of investment to be $t = 0$ without loss of generality. Also, we drop the notation $|t$ representing the dependency of the predicted quantities on the prior knowledge of the market available at time $t$. Now considering all the above, the MPO problem can be formulated as follows

$$\max_{u_0, \ldots, u_{T-1}} \sum_{s=1}^{T} w_s^T r_{s+1} - \gamma_s w_s^T \Sigma_s w_s - TC(u_{s-1})$$

s.t.

$$w_s = w_{s-1} + u_{s-1}, \quad s = 1, \ldots, T,$$

$$w_s \in W_s, \quad s = 1, \ldots, T,$$

$$u_s \in U_s, \quad s = 0, \ldots, T - 1,$$

$w_0$ given
where the cost function $TC : \mathbb{R}^n \to \mathbb{R}$ is a transaction costs function and the parameter $\gamma_s$ represents the investor’s risk aversion at the period $s$.

2.2 Time series

Financial data often presents itself as time dependant data and therefore the natural choice for representing it is time series. In this section, some of the tools in time series analysis relevant to the scope of the thesis will be presented.

2.2.1 Moving average

In time series analysis, a moving average process $\{X_t\}$ of order $q$ can be defined, following [4], as

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q},$$

where $\{Z_t\}$ is a white noise of variance $\sigma^2$ and $\theta_1, \ldots, \theta_q$ are constants. Such a definition is meant for modelling purposes. However, it is common to use moving averages in other contexts where the purpose is to smoothen or filter signals out of noise for easy interpretations. In which case, given a time series $\{X_t\}$, its moving average, $\{Y_t\}$, can be defined by

$$Y_t = \theta_0 X_t + \theta_1 X_{t-1} + \cdots + \theta_q X_{t-q},$$

where $\theta_0, \theta_1, \ldots, \theta_q$ are constants and $q$ is a lag parameter. The constants and the lag parameter gives multiple choices of moving averages. Two broadly used families in finance are the simple moving average (SMA) and the exponential moving average (EMA). These appear often in financial market’s news (Wall Street Journal, Bloomberg, Yahoo finance, ...) as technical indicators of market indices’ directions.

Simple moving average

For a given time series $\{X_t\}$, its SMA, $\{Y_t\}$, with a lag parameter $q$ can be defined as follows

$$Y_t = \frac{1}{q+1} (X_t + X_{t-1} + \cdots + X_{t-q}), \quad \forall t \in \mathbb{Z}.$$ 

By construction, the random variable $Y_t$ has a lower variance than $X_t$ for all $t \in \mathbb{Z}$. In other words, the resulting time series $\{Y_t\}$ is much smoother and less noisy than $\{X_t\}$. Therefore, the SMA can be an appropriate tool for identifying trends in a time series.
Exponential moving average

For a given time series \( \{X_t\} \), the EMA, \( \{Y_t\} \), with a decay factor \( \alpha \in ]0, 1[ \) can be defined by the following recursive formula

\[
Y_t = \alpha X_t + (1 - \alpha) Y_{t-1}, \quad \forall t \in \mathbb{Z}
\]

or equivalently, expressed by previous points, as

\[
Y_t = \alpha (X_t + (1 - \alpha)X_{t-1} + (1 - \alpha)^2X_{t-2} + \ldots ),
\]

In finance and other engineering applications, EMA are often defined in terms of a lag parameter instead of a decay factor by using the following relation

\[
\alpha = \frac{2}{q + 1};
\]

where \( q \) is the lag parameter. In comparison to the SMA, the EMA puts more weights on recent points. Therefore, it allows for a faster reaction to recent changes in a times series trend.

2.2.2 Mean reverting processes

Mean reversion is a behaviour often observed in financial time series. The behaviour corresponds to the tendency of processes to revert or return to their mean over time. Such phenomena can be modelled using First-order Autoregression or AR(1). Using the definition in [1], a times series \( \{X_t\} \) is a First-Order Autoregression if it verifies, for all \( t \),

\[
X_t = \phi X_{t-1} + Z_t,
\]

where \( \{Z_t\} \) is a white noise of variance \( \sigma^2 \), \(|\phi| < 1 \) and \( Z_t \) is uncorrelated with \( X_s \) for all \( s < t \). then \( \{X_t\} \) is weakly stationary, the expectation and autocovariance function \( \gamma_X(.) \) are respectively

\[
\mathbb{E}X_t = 0 \quad \text{and} \quad \gamma_X(h) = \frac{\phi^h \sigma^2}{1 - \phi^2}.
\]

The mean reversion in the above model is toward the value 0. It can be easily generalized to any value \( \mu \) by replacing \( X_t \) with \( Y_t - \mu \) for all \( t \), where \( \{Y_t\} \) becomes the new mean reverting time series. A typical description of a mean reverting process under this model is expressed by rearranging (2.9) as

\[
\Delta X_t = -\hat{\phi} X_t + Z_t,
\]

(2.10)
where $\tilde{\phi} = 1 - \phi$. The parameter $\tilde{\phi}$ is often called the mean reversion speed and represents how fast the process modelled by the time series $\{X_t\}$ reverts to its mean. Alternatively, the half life time $t_{1/2}$ is also used to describe the mean reversion speed. It represents the time at which the process decays in expectation by half toward its mean. Its relation with the mean reversion speed parameter can be given by

$$t_{1/2} = \frac{\log(0.5)}{\log(|1 - \tilde{\phi}|)}.$$ 

The AR(1) model under the form (2.10) resembles the well known Ornstein-Uhlenbeck process used for modelling mean reverting processes in a continuous time setting. N. Garleanu & L. H. Pedersen’s used the former in a multivariate form to model mean reverting signals [7].

### 2.3 Linear stochastic optimal control

In this section, a standard stochastic control problem of relevance to the scope of the thesis will be described since it represents a fundamental part in the work published by N. Garleanu & L. H. Pedersen in modeling and solving their MPO problem.

#### 2.3.1 Formulation

Let us consider a formulation of a stochastic control problem similar to the one described in [2] in the case of a complete state information. A more similar description of the given problem is described in [10] and argued to be equivalent to the standard stochastic control problem where there is no cross terms in the loss function. Consider that the dynamics of the system are governed by the following stochastic difference equation

$$\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t + \tilde{B}\tilde{u}_t + \tilde{\varepsilon}_t \quad (2.11)$$

where $\{\tilde{x}_t \in \mathbb{R}^n, t \in \mathbb{Z}\}$ is a sequence of $n$-dimensional state vectors, $\{\tilde{u}_t \in \mathbb{R}^p, t \in \mathbb{Z}\}$ is a sequence of $p$-dimensional control vector variables and $\{\tilde{\varepsilon}_t \in \mathbb{R}^n, t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed (iid) random variables with zero mean and covariance

$$\text{cov}(\tilde{\varepsilon}_t, \tilde{\varepsilon}_t) = \Sigma,$$

where $\tilde{A} \in \mathbb{R}^{n \times n}$ and $\tilde{B} \in \mathbb{R}^{n \times p}$. 

Given an initial state $x_0$, the goal is to find a control sequence $\{\tilde{u}_t \in \mathbb{R}^p, t \in \mathbb{Z}^+\}$ that minimizes the expected discounted loss over an infinite horizon

$$\min_{\tilde{u}_0(\tilde{x}_0), \tilde{u}_1(\tilde{x}_1), \ldots} \mathbb{E} \left[ \sum_{t=0}^{+\infty} \beta^t \left( \tilde{x}_t^T Q \tilde{x}_t + \tilde{u}_t^T R \tilde{u}_t + 2 \tilde{u}_t^T H \tilde{x}_t \right) \right]$$

s. t. $\tilde{x}_{t+1} = \tilde{A} \tilde{x}_t + \tilde{B} \tilde{u}_t + \tilde{\varepsilon}_t, \quad t = 0, 1, \ldots \tag{2.12}$

where $\beta$ is in $]0, 1[$ and represents a discount parameter, $Q \in \mathbb{R}^{n \times n}$ in $R \in \mathbb{R}^{p \times p}$ are positive definite matrices and $H \in \mathbb{R}^{p \times n}$.

### 2.3.2 The Bellman functional equation

The problem (2.12) can be solved using dynamic programming. First, the value function needs to be introduced, following [2, 10], it can be defined as

$$V(\tilde{x}_\tau, \tau) = \min_{\tilde{u}_\tau, \tilde{u}_{\tau+1}, \ldots} \mathbb{E} \left[ \sum_{t=\tau}^{+\infty} \beta^{t-\tau} \left( \tilde{x}_t^T Q \tilde{x}_t + \tilde{u}_t^T R \tilde{u}_t + 2 \tilde{u}_t^T H \tilde{x}_t \right) \right] \tag{2.13}$$

Bellman’s principle is used here by considering that the optimal controls $u_\tau, u_{\tau+1}, \ldots$ depend only on the resulting state $x_\tau$ from the previous controls $\ldots, u_{\tau-2}, u_{\tau-1}$ no matter what their values were. This comes naturally from the fact that the dynamics of $x_t$ are expressed by a stochastic difference equation, as well as the mathematical properties of the expectation and the considered noise distributions, presented more in detail in [2]. The Bellman functional equation is then deduced from the definition above as follows

$$V(\tilde{x}_\tau, \tau) = \min_{\tilde{u}_\tau} \mathbb{E} \left[ \left( \tilde{x}_\tau^T Q \tilde{x}_\tau + \tilde{u}_\tau^T R \tilde{u}_\tau + 2 \tilde{u}_\tau^T H \tilde{x}_\tau \right) + \beta V(\tilde{x}_{\tau+1}, \tau+1) \right] \tag{2.14}$$

### 2.3.3 The algebraic Riccati equation and the optimal control

Under the formulation (2.12), also known as the linear quadratic regulator problem, the value function that solves the problem is proven (see [2, 10]) to be a quadratic function

$$V(\tilde{x}, \tau) = \tilde{x}^T P \tilde{x} + S. \tag{2.15}$$

where $P$ is positive definite matrix and $S$ is a scalar. The so-called Riccati equations that must be verified by the matrices $P$ and the scalar $S$ can be obtained by
plugging the quadratic form (2.15) in the Bellman equation (2.14). Following our specification and based on the work of Lars Ljungqvist and Thomas J. Sargent in [10], the Riccati equations are

\[
S = \beta \left( tr(P\Sigma) + S \right),
\]

\[
P = (Q + \beta \tilde{A}^T P \tilde{A}) - (H + \beta \tilde{B}^T P \tilde{A})^T (R + \beta \tilde{B}^T P \tilde{B})^{-1} (H + \beta \tilde{B}^T P \tilde{A}),
\]

(2.16)

A sufficient condition for the Riccati equation to admit a positive definite solution \( P \) is that the matrices \( R \) and \( Q \) must be positive definite. In which case, the optimal controls are given by

\[
u^*(\tilde{x}, \tau) = -(R + \beta \tilde{B}^T P \tilde{B})^{-1}(H + \beta \tilde{B}^T P \tilde{A}) \tilde{x}_\tau.
\]

(2.17)
Chapter 3

Model

This chapter presents the MPO approach proposed by N. Gâteanu & L. H. Pedersen. First, an overview of the model will be presented and described including the optimal control formulation and its explicit form solution. Second, an understanding of how the problem was solved in the light of the standard stochastic optimal control framework will be provided.

3.1 Garleanu and Pederson’s framework

N. Gâteanu & L. H. Pedersen modelled the problem of portfolio selection in a multi-period setting as a stochastic control problem. Assuming a given return predictability and a quadratic transaction cost model, they derived a closed form solution for selecting the optimal portfolio.

3.1.1 Return predictability

The returns in excess of the risk free rate are defined in the model as

\[ r_{t+1} = p_{t+1} - (1 + r_f)p_t, \]  

where \( \{r_t\} \) and \( \{p_t\} \) take values in \( \mathbb{R}^n \) and denote respectively the sequences of excess returns and prices. The scalar \( r_f \) represents the risk free rate of return. N. Gâteanu & L. H. Pedersen used a factor model to describe the excess returns and considered that the factors are mean reverting processes were described by a multivariate AR(1) model. Then, the return predictability model is given as follows

\[ r_{t+1} = B f_t + \varepsilon^r_{t+1}, \]
\[ \Delta f_{t+1} = -\Phi f_t + \varepsilon^f_{t+1} \]  

(3.2)
where \( \{ f_t \} \) takes values in \( \mathbb{R}^k \) and represents the sequence of the predicting factors. \( \Delta \) is the backward difference operator. The matrix \( B \in \mathbb{R}^{n \times k} \) denotes the factors loading matrix and \( \Phi \in \mathbb{R}^{k \times k} \) denotes the factors mean reversion matrix. The terms \( \{ \varepsilon_t^r \} \) represent the unpredictable noise affecting the excess returns. They take values in \( \mathbb{R}^n \) and are a sequence of independent and identically distributed random variables with zero mean and covariance \( \Sigma \). The terms \( \{ \varepsilon_t^f \} \) represent the shocks affecting the factors. They take values in \( \mathbb{R}^k \) and are a sequence of independent and identically distributed random variables with zero mean and covariance \( \Omega \).

### 3.1.2 Decision variables

The decision variables in the framework at hand are the dimensionless amount of shares held by the investor throughout the periods and denoted by the sequence

\[
x_{-1}, x_0, \ldots, x_t, \ldots,
\]

that takes values in \( \mathbb{R}^n \). At the time of investment, \( t = -1 \), the previously held portfolio \( x_{-1} \) is given.

### 3.1.3 Transaction costs

The authors incorporated a quadratic transaction cost function \( TC : \mathbb{R}^n \to \mathbb{R} \) in their optimization problem, that is given by

\[
TC(\Delta x_t) = \frac{1}{2} \Delta x_t^T \Lambda \Delta x_t, \quad (3.3)
\]

where \( \Delta x_t \) are traded amount of shares. The matrix \( \Lambda \) represents trading costs coefficients and is an \( n \times n \) symmetric definite positive matrix. The authors argued that their transaction costs model is a valid choice, not only because it allows their formulation to be tractable, but also because there is experimental evidence to it.

### 3.1.4 The multi-period portfolio optimization formulation

Given the previously described return predictability (3.2) and transaction costs (3.3) models, N. Gârleanu & L. H. Pedersen’s extended the mean variance approach to an infinite horizon multi-period setting and formulated the optimization problem.
as follows
\[
\max_{x_0, x_1, \ldots} \mathbb{E} \left[ \sum_{t=0}^{+\infty} (1 - \rho)^{t+1} \left( x_t^T r_{t+1} - \frac{\gamma}{2} x_t^T \Sigma x_t \right) - (1 - \rho)^{t+1} \frac{1}{2} \Delta x_t^T \Lambda \Delta x_t \right],
\]
s.t. \( r_{t+1} = B f_t + \varepsilon_{t+1}^r, \quad t = 0, 1, \ldots \)
\( f_{t+1} = (I - \Phi) f_t + \varepsilon_{t+1}^f, \quad t = 0, 1, \ldots \)
\( x_t \in \mathbb{R}^n, \quad t = 0, 1, \ldots \)
\( f_t \in \mathbb{R}^k, \quad t = 1, \ldots \)
\( r_t \in \mathbb{R}^n, \quad t = 1, \ldots \)
x_{-1}, f_0, \quad \text{given}
\]
where \( \gamma \) denotes the risk aversion coefficient. The parameter \( \rho \) is the discount rate parameter that must be taken between 0 and 1 to insure the existence of a solution.

### 3.1.5 The closed form solution

The MPO problem (3.4), viewed as control problem, can be solved by stochastic dynamic programming. The closed form solution provided by N. Gărleanu & L. H. Pedersen is
\[
x_t = x_{t-1} + \Lambda^{-1} A_{xx} (\text{aim}_t - x_{t-1}),
\]
where
\[
\text{aim}_t = A_{xx}^{-1} A_{xf} f_t.
\]
and the matrices \( A_{xx}, A_{xf}, A_{ff} \) are obtained by solving the Riccati equations that arise from the problem, and were given as follows
\[
A_{xx} = \left( \bar{\rho} \gamma \Lambda^2 \Sigma \Lambda^2 + \frac{1}{4} (\rho^2 \Lambda^2 + 2 \rho \gamma \Lambda^2 \Sigma \Lambda^2 + \gamma^2 \Lambda^2 \Sigma \Lambda^{-1} \Sigma \Lambda^2) \right)^{\frac{1}{2}} - \frac{1}{2} (\rho \Lambda + \gamma \Sigma)
\]
\[
\text{vec}(A_{xf}) = \bar{\rho} (I_{KS} - \bar{\rho} (I_K - \Phi)^T \otimes (I_S - A_{xx} \Lambda^{-1}))^{-1} \text{vec}((I_S - A_{xx} \Lambda^{-1}) B)
\]
\[
\text{vec}(A_{ff}) = \bar{\rho} (I_{KS} - \bar{\rho} (I_K - \Phi)^T \otimes (I_K - \Phi)^T) \text{vec}(Q)
\]
where \( \text{vec}(. \) is the vectorization operator, and \( \otimes \) is the Kronecker product, and
\[
\bar{\rho} = 1 - \rho,
\]
\[
\bar{\Lambda} = \bar{\rho}^{-1} \Lambda.
\]

### 3.2 Standard stochastic optimal control framework

In the light of what we described in section 2.3, the problem (3.4) can be rewritten to match the formulation (2.12). First, let us proceed by considering the state,
control and noise processes that we denote, respectively, by

\[ \tilde{x}_t = \begin{bmatrix} f_t \\ x_{t-1} \end{bmatrix}, \quad \tilde{u}_t = \Delta x_t, \quad \text{and} \quad \tilde{\epsilon}_t = \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}, \]

then the matrices \( \tilde{A} \) and \( \tilde{B} \) describing the dynamics of the problem follow

\[ \tilde{A} = \begin{bmatrix} I - \Phi & 0 \\ 0 & I \end{bmatrix}, \quad \text{and} \quad \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \]

and finally, the cost function matrices \( Q, H \) and \( R \) are

\[ Q = \frac{1}{2} \begin{bmatrix} 0 & \beta B^T \\ \beta B & \beta \gamma \Sigma \end{bmatrix}, \quad H = \frac{1}{2} \begin{bmatrix} \beta B & \beta \gamma \Sigma \end{bmatrix} \quad \text{and} \quad R = \frac{1}{2} \left( \beta \gamma \Sigma + \Lambda \right), \]

where \( \beta = 1 - \rho \). Following the notations that the authors used in their formulation, we write the sought positive matrix \( P \) and scalar \( S \), respectively, as

\[ P = \frac{1}{2} \begin{bmatrix} A_{ff} & A_{xf}^T \\ A_{xf} & -A_{xx} \end{bmatrix} \quad \text{and} \quad S = A_0. \]

Assuming the positive definiteness of the matrices \( Q \) and \( R \), the arising Riccati equations from these formulation were then solved explicitly by N. Gărleanu & L. H. Pedersen in [7] where they obtained the closed form expression of the solution (3.7).
Chapter 4

Methodology

In this chapter, the aim is to describe the methodology that was followed to investigate N. Gărleanu & L. H. Pedersen’s framework, especially the return predictability and transaction costs aspects. In that regard, our methodology for the investigation was purely experimental and used only synthetic data. It is indeed true that real data is essential to validate a trading strategy but in any case focusing only on the former does not guarantee a conclusive investigation since, for instance, the results might be biased by the available data. While in contrast, using synthetic data, gives us the flexibility to isolate and narrow our study to specific aspects of the model. Our methodology will remain only complementary to one based on real data, nonetheless a great deal of care has been taken to maintain our experiments as realistic as possible.

That being said, we will first describe how N. Gărleanu & L. H. Pedersen’s framework was adapted and used. Then, we give an overview of the main procedure that will be followed for evaluating the model. Next, the main parts of the presented procedure will be detailed and motivated. Finally, the used values for generating synthetically the historical returns shall be presented.

4.1 Adaptation of the framework

In N. Gărleanu & L. H. Pedersen’s, the decision variables and the returns definitions are different from what is generally the case in the modern portfolio literature described in chapter 2. Proving the appropriateness of their proposed return predictability under their given returns definition is beyond the scope of this thesis. For the sake of convenience, returns will be considered as linear returns (see definition [2.1.1]). Additionally, assuming that leverage is allowed and that returns are
small (see section 2.1.4), the MPO problem will be reformulated as follows

$$\max_{w_0, w_1, \ldots} E \left[ \sum_{t=0}^{+\infty} (1 - \rho)^{t+1} \left( w_t^T r_{t+1} - \frac{\gamma}{2} w_t^T \Sigma x_t \right) - (1 - \rho)^{t+1} \frac{1}{2} \Delta w_t^T \Lambda \Delta w_t \right],$$

s.t. $r_{t+1} = Bf_t + v_{t+1}, \quad t = 0, 1, \ldots$

$f_{t+1} = (I - \Phi)f_t + \epsilon_{t+1}, \quad t = 0, 1, \ldots$

$w_t \in \mathbb{R}^n, \quad t = 0, 1, \ldots$

$f_t \in \mathbb{R}^k, \quad t = 1, \ldots$

$r_t \in \mathbb{R}^n, \quad t = 1, \ldots$

$w_{-1}, f_0, \text{ given}$

where our decision variables are weights $w_{-1}, w_0, \ldots, w_t, \ldots$

The closed form expression of the solution is still obtained following the results of N. Gârleanu & L. H. Pedersen as a linear combination of a current weight and the aim portfolio (see section 3.1.5)

$$w_t = w_{t-1} + \Lambda^{-1} A_{xx} (\text{aim}_t - w_{t-1}).$$

(4.2)

from the given description, the required parameters that needs to be estimated in order to use the closed form expression given by N. Gârleanu & L. H. Pedersen will be denoted by

$$\mathcal{E} = \{B, \Sigma, \Phi, \Omega, \Lambda\},$$

and the parameters that needs to be chosen by the investors will be denoted by

$$\mathcal{I} = \{\rho, \gamma\}.$$

4.2 Backtesting & Monte Carlo simulations

The standard approach to evaluate the performance of a trading strategy is to use back-testing. The approach, consists of running through real historical data and at each time period only looking back to make the trading decisions. The obtained outcomes over all the periods are then used for the evaluation.

The underlying hypothesis behind back-testing is that what happened in the past will happen in the future. This might not be always the case, thus the approach alone might lead to biased results. In our case, we will use the backtest procedure on simulated historical data. Due to the stochastic nature of the simulations, multiple backtest runs will be carried on in a Monte Carlo fashion in order to obtain an overall assessment that is reliable.
4.3. THE EXPERIMENTAL SETUP

Furthermore, back-testing can be computationally challenging, let alone with Monte Carlo simulations on top of it. Fortunately, the closed form solution given by N. Gărleanu & L. H. Pedersen’s solution reduces the computations but not enough since the solution and parameters estimation still requires heavy matrix computations (square root of a matrix, inverse of a matrix). To remedy to such an issue, the parameters that will need to be estimated will be fixed along the backtests. Additionally, to account for estimations errors an amount of noise will be added to the estimates when generating the synthetic data. Figure 4.1 illustrates the general steps of a single back-test run.

4.3 The experimental setup

Let us recall that our goals were to specifically investigate the return predictability impact on the performance of the obtained trading strategies by N. Gărleanu
& L. H. Pedersen’s framework as well as the transaction costs incorporation in the framework. Bearing that in mind, we shall describe how these aspects are considered in designing our experimental setup alongside the design itself.

### 4.3.1 The returns generating model

To investigate the return predictability part of N. Gărleanu & L. H. Pedersen’s framework, we propose to use the same dynamics (3.2) for our return predictability with some specific considerations. For simplicity, we will only consider two assets in our experiments. Each asset will have only two predicting factors, one that decays fast and another that decays slowly which we describe as follows

\[
\begin{bmatrix}
r_{1,t+1} \\
r_{2,t+1}
\end{bmatrix} =
\begin{bmatrix}
\beta_{1,fast} & \beta_{1,slow} & 0 & 0 \\
0 & 0 & \beta_{2,fast} & \beta_{2,slow}
\end{bmatrix}
\begin{bmatrix}
f_{1,fast,t} \\
f_{1,slow,t} \\
f_{2,fast,t} \\
f_{2,slow,t}
\end{bmatrix}
+ \varepsilon_{r,t+1},
\]

\[
\begin{bmatrix}
f_{1,fast,t+1} \\
f_{1,slow,t+1} \\
f_{2,fast,t+1} \\
f_{2,slow,t+1}
\end{bmatrix} =
\begin{bmatrix}
\phi_{1,fast} & 0 & 0 & 0 \\
0 & \phi_{1,slow} & 0 & 0 \\
0 & 0 & \phi_{2,fast} & 0 \\
0 & 0 & 0 & \phi_{2,slow}
\end{bmatrix}
\begin{bmatrix}
f_{1,fast,t} \\
f_{1,slow,t} \\
f_{2,fast,t} \\
f_{2,slow,t}
\end{bmatrix}
+ \varepsilon_{f,t+1}.
\]

(4.3)

By using this structure, we aim to investigate how the slow and fast decaying factors participate in the performance of the trading strategy. Further, we will consider that the two assets have approximately the same characteristics which means that

\[
\begin{align*}
\beta_{1,fast} & \approx \beta_{2,fast}; \\
\beta_{1,slow} & \approx \beta_{2,slow}; \\
\sigma_{r1}^f & \approx \sigma_{r2}^f;
\end{align*}
\]

and

\[
\begin{align*}
\phi_{1,fast} & \approx \phi_{2,fast}; \\
\phi_{1,slow} & \approx \phi_{2,slow}; \\
\sigma_{f1}^f & \approx \sigma_{f2}^f.
\end{align*}
\]

(4.4)

where \(\sigma_r\) and \(\sigma_f\) denote the returns respective volatilities, and \(\sigma_{f1}, \sigma_{f2}, \sigma_{1,slow}, \sigma_{2,slow}\) and \(\sigma_{2,fast}\) denote the factors respective standard deviations. Under such consideration, the returns will only differ due to the random realizations of the noise and chocks sequences, respectively, \(\{\varepsilon_{r,t}\}\) and \(\{\varepsilon_{f,t}\}\). This will allow us to focus our analysis only on how N. Gărleanu & L. H. Pedersen’s framework is utilizing the predicting factors’ strengths, namely their mean reversion speed, in making profit, and rule out any dependence of the strategies’ performance on major differences between the two assets’ characteristics such us their returns volatilities or factors loading matrix structure. Let us also note that it is common for investors to trade in assets that have similar volatilities.
4.3. THE EXPERIMENTAL SETUP

4.3.2 Noise addition

As we have mentioned before, an amount of noise will be added to the different parameters when generating the excess returns. Following the notations of the previous section, the concerned parameters are

\[ \beta_{\text{slow}}, \beta_{\text{fast}}, \phi_{\text{slow}}, \phi_{\text{fast}}, \sigma_r, \sigma_{\text{slow}}^f, \sigma_{\text{fast}}^f. \]

For each parameter \( \theta \) from the above, the amount of noise added is done in the following manner

\[ \theta_{\text{noised}} = \theta (1 + \epsilon), \]

where \( \epsilon \sim U(-g, g) \) and \( g \in [0, 1] \). This means that the added amount of noise does not exceed a percentage of \( g \) from the parameter to be estimated, and can be interpreted in a real case by having an estimator that does not deviate from the real value by \( g \).

4.3.3 Transaction costs

Following the general form of the market transaction costs described in section 2.1.3 and considering that the decisions variables of the investment strategies will be weights, the market transaction costs will be defined as follows for the two assets

\[
\begin{align*}
\text{TC}_1(u_1) &= a_1|u_1| + b_1|u_1|^{3/2} + c_1u_1^2, \\
\text{TC}_2(u_2) &= a_2|u_2| + b_2|u_2|^{3/2} + c_2u_2^2, \\
\text{TC}(u) &= \text{TC}_1(u_1) + \text{TC}_2(u_2),
\end{align*}
\]

(4.5)

where \( u_1 \) and \( u_2 \) represent respectively the normalized trade on asset 1 and asset 2, and \( u = [u_1 \ u_2]^T \). As in the previous section, the two assets have the same characteristics as well wish means that \( a_1 \approx a_2, b_1 \approx b_2 \) and \( c_1 \approx c_2 \). Let us note that by using normalized trades the superlinear and quadratic transaction costs coefficients, \( b \) and \( c \), will depend on the total wealth but since we assumed that leverage is allowed, these parameters will be considered as adjustable by the investor.

In N. Gârleanu & L. H. Pedersen’s framework, transaction costs are quadratic. Assuming that there is no spill-over effect\(^1\) the transaction costs matrix will be described as

\[
\Lambda = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix},
\]

\(^1\)The spillover effect is when incurred costs on a certain asset are due to trading on a different asset as N. Gârleanu & L. H. Pedersen refer to in their paper \([7]\).
CHAPTER 4. METHODOLOGY

To find the coefficients $\lambda_1$ and $\lambda_2$, we approximate the general transaction costs function described by [4.5] by solving the following minimization problems

$$\min_{\lambda_i} \int_{u \in J} \left( TC_i(u) - \frac{1}{2} \lambda_i u^2 \right)^2 du, \quad i = 1, 2,$$

where $J$ is the trading region in which we assume the normalized trades $u$ to fall in. These minimization problems can be interpreted as reducing the squared error between the approximated market transaction costs and the estimated quadratic costs, while assuming that the normalized trades are uniformly distributed over the interval $J$. Finally, obtaining the solution to these problems is straightforward and gives

$$\lambda_i = \frac{\int_{u \in J} 2u^2 TC_i(u) du}{\int_{u \in J} u^4 du}, \quad i = 1, 2,$$

4.3.4 The trading strategies

In order to better analyze and understand the portfolio selection approach proposed by N. Gârleanu & L. H. Pedersen, two other approaches that consists of a naive single based MVO approach and a filtered version of it were also used. And for each approach, considering the proposed return generating model, a set of strategies were developed in order to study the return predictability aspect of the problem.

Dynamic trading

The dynamic trading strategies are the strategies that uses the framework of N. Gârleanu & L. H. Pedersen under the formulation described in section 4.1 with the specifications used in the return generating model from section 4.3.1. This family of strategies will be referred to as $DT$. Now, to study how the model utilizes the predicting factors with different strengths, we will consider the following strategies:

- The strategy $DT_{fp}$ that observes both, slow and fast decaying factors. In other words, all the characteristics of returns will be used in solving the decision problem.

- The strategy $DT_p$ that observes only slow decaying factor (the subscript $p$ stands for persistent factor), in which case $\beta_{fast}$ will be set to zero.

- The strategy $DT_f$ that observes only fast decaying factor, in which case $\beta_{slow}$ will be set to zero.
4.3. THE EXPERIMENTAL SETUP

- The strategies $DT_{comb}$ that uses a weighted average of the obtained portfolios from $DT_p$ and $DT_f$.

$$w_{comb,\alpha} = \alpha w_f + (1 - \alpha)w_p,$$

where $w_f$, $w_p$ and $w_{comb,\alpha}$ are, respectively, the weights of the strategies $DT_f$, $DT_p$ and $DT_{comb,\alpha}$, and $\alpha \in [0, 1]$. The strategy $DT_{comb,best}$ shall denote the best performing strategy among the strategies $\{DT_{comb,\alpha}, \alpha \in [0, 1]\}$. These strategies are used as an attempt to study the contribution of the fast and slow decaying predicting factors in the portfolio construction, and potentially verify the optimality of the strategy $DT_{fp}$.

Markowitz

The Markowitz strategies are the strategies that uses the MVO approach as described in section 2.1.2. Using the leveraged Markowitz portfolio (2.3) and considering the return generating model specifications, the optimal Markowitz portfolios at a time $t$ can be expressed as

$$w_t^* = \gamma^{-1}\Sigma^{-1}Bf_t.$$  

This family of strategies will be referred to as $MW$. Let us note that these strategies does not account for transactions cost, they are used for the purpose of illustrating the relevance of incorporating transaction costs in N. Gárleanu & L. H. Pedersen’s framework if there is any. In a similar way to the dynamic trading approach, the different considered strategies for this approach are $MW_{fp}$, $MW_p$, $MW_f$ and $MW_{comb}$.

A filtered Markowitz

The filtered Markowitz strategies are the strategies that are constructed by applying a filter on the obtained $MW$ strategies portfolio signals. The chosen filter in our case is an exponentially moving average (see section 2.2.1). This family of strategies will be referred to as $E_qMW$ where $q$ denotes the the lag parameter. In a similar way to the dynamic trading approach, the considered strategies for this approach are $E_qMW_{fp}$, $E_qMW_p$, $E_qMW_f$ and $E_qMW_{comb}$.

4.3.5 Analysis and evaluation metrics

The most common performance measure used for evaluating portfolios is the Sharpe Ratio introduced by William F. Sharpe in 1966. We will use this measure as well as a modified version of it that accounts for transaction costs. Other metrics
CHAPTER 4. METHODOLOGY

will also be used to better analyze the strategies such as the portfolio volatility and average holding period.

Gross Sharpe ratio

The Sharpe ratio can be defined, following [16], as

\[
\frac{E r_p(w) - r_f}{\sigma_p(w)},
\]

where \( r_p(w) \) and \( \sigma_p(w) \) are, respectively, the portfolio return and volatility (see section 2.1.2), \( r_f \) is the risk free rate of return, and \( E \) is the mathematical expectation. This is generally referred to as the ex-ante Sharpe ratio. In practice, we will use the ex-post Sharpe ratio that uses the realized returns instead of expected return in the above definition. This measure is interpreted as the amount of return made per unit of risk.

Net Sharpe ratio

In the net Sharpe ratio we account for transaction costs by modifying the expression (4.7) as follows

\[
\frac{E r_p(w) - TC(u) - r_f}{\sigma_p(w)},
\]

where \( u \) is the normalized trade that leads to the portfolio \( w \) given an initial portfolio \( w_0 \), and \( TC \) is the market transaction costs function 2.4 that we defined earlier in section 4.3.3. We will also use the realized returns in practice.

Average holding period

The average holding period of a trading strategy over \( T \) trading days is defined as

\[
\frac{2 \sum_t^T \sum_i^n |w_{i,t}|}{\sum_t^T \sum_i^n |\Delta w_{i,t}|}.
\]

This metric is used to check whether the different strategies are comparable or not. In other words, we need to have similar average holding periods for the strategies that we wish to compare their Sharpe ratios. By using this metric we aim to see whether there is some intelligence in the way N. Gărleanu & L. H. Pedersen’s frameworks accounts for transaction costs.
4.4. THE DATA GENERATING VALUES

Portfolio volatility and scaling

The portfolio volatility (see section 2.1.2) can be used as a measure of risk as in the Sharpe ratio definition. In our case, to have meaningful results, we will maintain the strategies that we wish to compare, on the same risk level. In order to do so, a common practice is to scale the obtained weight signals post realization. However, to perform that in a way that does not compromise the approaches solutions, we will only scale the Markowitz based strategies, namely \(MW\) and \(EMW\), to match their correspondent \(DT\) strategies’ portfolio volatility. Let us note that in this case, scaling is equivalent to choosing a different risk aversion parameter.

4.4 The data generating values

In order to accomplish our investigation, a base case setting for the returns generating model and the market transaction costs needed to be provided. The parameter choices were based on typical market values.

4.4.1 Base case setting

The daily volatility of assets’ returns can typically range between 0.004 and 0.05. For our setting, the volatility of returns was chosen to be 0.01, and the correlation between the two assets to be 20%. The fast and slow decaying factors had, respectively, a half life of 7 and 56 days, which translates to the values 0.0943 and 0.0123 in the mean reversion matrix \(\Phi\). The standard deviations of the shocks affecting the factors were chosen to be approximately 100 times smaller than the returns volatility, and precisely equal to \(1.4 \times 10^{-4}\) and \(0.8 \times 10^{-4}\), respectively, for the slow and fast decaying factors. The factors loading coefficient were chosen to be 1. The risk free rate \(r_f\) will be assumed to be equal to 0 since it is generally very small.

As for the transaction costs coefficients, for a market volatility of 1 and a trade of size 1, a typically incurred cost would be about 0.05 when using a linear cost function. Because the chosen returns volatility is 0.01, the incurred cost will be \(5 \times 10^{-4}\). Now, we consider that 90% of the incurred cost on a trade is due to the linear term, 6% is due to the super-linear term, and 4% is due to the quadratic term. Hence, the coefficients \(a\), \(b\) and \(c\) are, respectively, \(4.5 \times 10^{-4}\), \(3 \times 10^{-5}\) and \(2 \times 10^{-5}\).
4.4.2 Intervals for some settings

To further our analysis, some parameters need to be varied to evaluate how the performance of our strategies will change. The parameters that were most of interest are the transaction costs coefficients $a$, $b$ and $c$. The intervals in which each parameter was changed are, respectively, $[5 \times 10^{-4}, 0.50]$, $[9 \times 10^{-3}, 9]$ and $[1.05 \times 10^{-2}, 10.5]$. These ranges were taken large enough so that the dependence can be clearly seen. Additionally, whenever a parameter was varied, the others were set to zero.

Other parameters were also considered such as the predicting factors correlation and the shocks standard deviation magnitude. The range of the latter is constructed by multiplying with a factor that spans the interval $[0.1, 10]$ and means that the resulting standard deviations will vary from being 1000 times smaller to being 10 times smaller than the returns volatility.
### Table 4.1 – This table shows the selected parameters for the base case setting.

<table>
<thead>
<tr>
<th></th>
<th>$\beta_{\text{slow}}$</th>
<th>$\beta_{\text{fast}}$</th>
<th>$\sigma^r$</th>
<th></th>
<th>$\Sigma_{\text{corr}}$</th>
<th>$\phi_{\text{fast}}$</th>
<th>$\phi_{\text{slow}}$</th>
<th>$\sigma_{f_{\text{fast}}}$</th>
<th>$\sigma_{f_{\text{slow}}}$</th>
<th>$\Omega_{\text{corr}}$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factors loading matrix $B$</td>
<td>$\beta_{\text{slow}}$</td>
<td>$\beta_{\text{fast}}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\Sigma_{\text{corr}}$</td>
<td>$\phi_{\text{fast}}$</td>
<td>$\phi_{\text{slow}}$</td>
<td>$\sigma_{f_{\text{fast}}}$</td>
<td>$\sigma_{f_{\text{slow}}}$</td>
<td>$\Omega_{\text{corr}}$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
</tr>
<tr>
<td>Conditional return’s covariance matrix $\Sigma$</td>
<td>$\sigma^r$</td>
<td>$1.0 \times 10^{-2}$</td>
<td>$\Sigma_{\text{corr}}$</td>
<td>$\begin{bmatrix} 1 &amp; 0.2 \ 0.2 &amp; 1 \end{bmatrix}$</td>
<td>$\phi_{\text{fast}}$</td>
<td>$0.0943$</td>
<td>$\phi_{\text{slow}}$</td>
<td>$0.0123$</td>
<td>$\sigma_{f_{\text{fast}}}$</td>
<td>$1.4 \times 10^{-4}$</td>
<td>$\sigma_{f_{\text{slow}}}$</td>
<td>$0.8 \times 10^{-4}$</td>
<td>$\Omega_{\text{corr}}$</td>
</tr>
</tbody>
</table>
Chapter 5

Results

This chapter provides the obtained results. First, the selected investment and experiment parameters that lead to the obtained results are shown. Second, preliminary results concerning a single backtest run are presented. Next, the obtained performances when running multiple backtests for the base case setting are then provided. Finally, more results are described and discussed when the market transaction cost function parameters are varied.

5.1 Parameter selection

5.1.1 The investment parameters

The risk aversion parameter $\lambda$ was set to the value 1 and kept unchanged throughout the backtest period. The discount parameter $\rho$ has been set to the value $2 \times 10^{-4}$ which corresponds to an annualized discount rate of 5%.

5.1.2 The experiment parameters

The backtest length $T$ is 1000 periods which corresponds approximately to a 4 years investment period. The parameter $g$ that controls the amount of noise $\epsilon$ added to the return generating parameters is about 15%. Let us note that for the sampled evaluation metrics to converge, especially the Sharpe ratios, the noise added can not exceed a certain amount. The made choice seemed to verify the convergence of the sampled metrics which should be a normal distribution for the Sharpe ratio by the central limit theorem. The initial weights $w_{-1}$ on the two assets, as well as the initial predicting factors values $f_0$ were set to be equal to $[0 \ 0]^T$. 

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5.2. TRADING SIGNALS OF A SINGLE BACKTEST SIMULATION

Figure 5.1 – Convergence of the Monte Carlo Simulations. The plots illustrate the convergence of the mean of the net Sharpe ratio for the dynamic trading strategy $DT_f$ with the number of simulations $N$. Figure (a) shows the evolution of the net mean Sharpe ratio with $N$, while figure (b) shows the evolution of the mean squared error of the net Sharpe ratio with $N$.

For estimating the transaction costs parameters $\Lambda$, the used interval $\mathcal{J}$ for the normalized trades was $[-1, 1]$. This choice was made after observing that the normalized trades in our tests fell almost always in that interval.

As of the number of simulations, $N$, $10^5$ simulations gave a satisfying level of confidence [5.1] with a mean squared error of the order of $10^{-5}$ for the Sharpe ratio mean. When varying the parameters, the number of simulations was set to $10^4$ to speed up the running time of the experiments, and the used sequence of seeds for the random generation was fixed to obtain a reliable observation.

5.2 Trading signals of a single backtest simulation

As one can see from figure [5.2], the weight signals on asset 1 of the strategy $DT_f$ has a much lower magnitude than the weight signals of the trading strategies $DT_{fp}$ and $DT_p$. On the other hand, the two latter seem to have fairly similar signals. Furthermore, the aim signals that N. Gărleanu & L. H. Pedersen’s framework would target are more volatile than the actual weights for all the different dynamic trading strategies $DT_{fp}$, $DT_p$ and $DT_f$. These observed behaviours stand as well for the weight signals of asset 2.

In the figure [5.3] the strategy $DT_{fp}$ acts as if it constructs its weight signals by smoothing or filtering the weight signals of the strategy $MW_{fp}$. The EMA
Figure 5.2 – **Illustration of the dynamic trading weight signals.** These figures show the weight signals on asset 1 using the strategies $DT_{fp}$ (blue), $DT_{p}$ (red) and $DT_{f}$ (yellow). The bottom figure is a zoomed part of the top figure. The solid lines represent the weight signals and the dash-dotted lines the aim signals.

Figure 5.3 – **Illustration of the EMA filtered trading strategies weight signals.** These figures show the weight signals on asset 1 using the strategies $DT_{fp}$ (solid black line), $MW_{fp}$ (solid blue line), $E_{2}MW_{fp}$ (dash-dotted blue line), $E_{6}MW_{fp}$ (dash-dotted red line), $E_{10}MW_{fp}$ (dash-dotted yellow line), $E_{14}MW_{fp}$ (dash-dotted violet line). The bottom figure is a zoomed part of the top figure.

filters seem to act in the same manner. In fact, the weight signal on asset 1 of the strategies $E_{6}MW_{fp}$ and $DT_{fp}$ are very close to each other. This suggests that the
two strategies should have a similar performance.

5.3 Strategies performance from repeated back-test simulations

Inspecting panel A and B in the Table 5.1, the difference between the portfolio volatilities figures of the scaled and unscaled version of the Markowitz based strategies and EMA filtered strategies is very thin. Therefore, it is safe to say that these strategies’ performances are not much compromised by the scaling procedure. That being said, to make sense of the comparisons, the holding periods of the strategies need to be reasonably similar. This is the case between the $E_6 MW$ and $DT$ strategies. As for the $MW$ strategies, they do not account for the transaction costs and as a result their realized holding period are slightly lower as it is usually the case.

Now, comparing the gross Sharpe ratios, The Markowitz based strategies $MW$ outperform the $DT$ and $E_6 MW$ strategies as one would expect. While, looking at the net Sharpe Ratios the $DT$ and $E_6 MW$ are showing the best performances. On the one hand, these results confirm the findings of N. Gârleanu & L. H. Pedersen’s when comparing the dynamic trading to the Markowitz based strategies, or at least under the given settings. On the other hand, there seems to be a competition between the $DT$ and $E_6 MW$ strategies. However, the $DT$ strategies could be considered superior in the sense that N. Gârleanu & L. H. Pedersen’s framework determines the trading decisions systematically in contrast to the $E_q MW$ strategies that requires to select the optimal lag parameter $q$.

Furthermore, comparing the strategies $DT_f$ and $DT_p$ respectively with $MW_f$ and $MW_p$ suggests that N. Gârleanu & L. H. Pedersen’s framework is better at utilizing the factors’ strengths, namely their mean reversion speed and standard deviation, in achieving a better performance. Additionally, the plots in Figure 5.4 show that the constructed strategies $DT_{comb}$ $MW_{comb}$ and $E_6 MW_{comb}$ defined in section 4.3.4 tend to approach the performance of $DT_{fp}$ when the weight $\alpha$ is well chosen. Interestingly, the performance of the strategy $DT_{comb,best}$ is better than $MW_{comb,best}$ and $E_6 MW_{comb,best}$ in terms of the net Sharpe ratio, with a weight $\alpha_{best}$ not fully put on the $DT_p$. One way to interpret these best performing combination based strategies is to consider the $\alpha_{best}$ obtained for all three types of strategies as a measure of how much contribution is allocated to each of the fast predicting and slow predicting factors.
Table 5.1 – Results table. The tables show the mean and standard deviation of the net and gross annualized Sharpe ratio, net and gross annualized portfolio volatility, and average holding periods for the dynamic trading based strategies (\(DT_f\), \(DT_p\) and \(DT_f\)), the Markowitz based strategies (\(MW_f\), \(MW_f\) and \(MW_p\)), and some EMA filtered Markowitz strategies (\(E_{2\,MW_f}\), \(E_{6\,MW_f}\), \(E_{10\,MW_f}\), \(E_{14\,MW_f}\), \(E_{2\,MW_f}\) and \(E_{6\,MW_p}\)). The Markowitz based strategies and EMA filtered strategies’ weights were scaled to match the dynamic trading strategies’ volatility. The best achieved Sharpe ratios are highlighted in bold.

Panel A: The Markowitz based strategies \(MW\) and EMA filtered strategies \(E_q\,MW\) were not scaled to match the gross portfolio volatility of their corresponding \(DT\) strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Net Sharpe Ratio</th>
<th>Gross Sharpe Ratio</th>
<th>Average holding Period</th>
<th>Net Portfolio Volatility</th>
<th>Gross Portfolio Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (Std)</td>
<td>Mean (Std)</td>
<td>Mean (Std)</td>
<td>Mean (Std)</td>
<td>Mean (Std)</td>
</tr>
<tr>
<td>(MW_f)</td>
<td>1.094 (0.539)</td>
<td>1.274 (0.531)</td>
<td>9.8 (1.3)</td>
<td>0.6859 (0.0956)</td>
<td>0.6859 (0.0956)</td>
</tr>
<tr>
<td>(MW_p)</td>
<td>1.107 (0.538)</td>
<td>1.247 (0.516)</td>
<td>12.4 (1.8)</td>
<td>0.6320 (0.0956)</td>
<td>0.6319 (0.0956)</td>
</tr>
<tr>
<td>(MW_f)</td>
<td>0.211 (0.518)</td>
<td>0.581 (0.532)</td>
<td>4.6 (0.2)</td>
<td>0.1478 (0.0159)</td>
<td>0.1478 (0.0159)</td>
</tr>
<tr>
<td>(E_{6,MW_f})</td>
<td>1.180 (0.535)</td>
<td>1.249 (0.533)</td>
<td>24.5 (3.5)</td>
<td>0.6655 (0.0956)</td>
<td>0.6655 (0.0956)</td>
</tr>
<tr>
<td>(E_{6,MW_p})</td>
<td>1.162 (0.536)</td>
<td>1.218 (0.534)</td>
<td>30.3 (4.5)</td>
<td>0.6191 (0.0956)</td>
<td>0.6191 (0.0956)</td>
</tr>
<tr>
<td>(E_{6,MW_f})</td>
<td>0.438 (0.522)</td>
<td>0.588 (0.521)</td>
<td>11.0 (0.7)</td>
<td>0.1302 (0.0159)</td>
<td>0.1302 (0.0159)</td>
</tr>
<tr>
<td>(DT_f)</td>
<td>1.180 (0.533)</td>
<td>1.253 (0.530)</td>
<td>23.4 (3.3)</td>
<td>0.6463 (0.0956)</td>
<td>0.6463 (0.0956)</td>
</tr>
<tr>
<td>(DT_p)</td>
<td>1.163 (0.534)</td>
<td>1.223 (0.531)</td>
<td>28.2 (4.2)</td>
<td>0.6080 (0.0956)</td>
<td>0.6080 (0.0956)</td>
</tr>
<tr>
<td>(DT_f)</td>
<td>0.434 (0.521)</td>
<td>0.594 (0.520)</td>
<td>10.3 (0.6)</td>
<td>0.1108 (0.0159)</td>
<td>0.1108 (0.0159)</td>
</tr>
</tbody>
</table>

Panel B: The Markowitz based strategies \(MW\) and EMA filtered strategies \(E_q\,MW\) were scaled to match the gross portfolio volatility of their corresponding \(DT\) strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Net Sharpe Ratio</th>
<th>Gross Sharpe Ratio</th>
<th>Average holding Period</th>
<th>Net Portfolio Volatility</th>
<th>Gross Portfolio Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (Std)</td>
<td>Mean (Std)</td>
<td>Mean (Std)</td>
<td>Mean (Std)</td>
<td>Mean (Std)</td>
</tr>
<tr>
<td>(MW_f)</td>
<td>1.096 (0.536)</td>
<td>1.275 (0.529)</td>
<td>9.8 (1.3)</td>
<td>0.6463 (0.0956)</td>
<td>0.6463 (0.0956)</td>
</tr>
<tr>
<td>(MW_p)</td>
<td>1.107 (0.536)</td>
<td>1.246 (0.530)</td>
<td>12.5 (1.8)</td>
<td>0.6080 (0.0956)</td>
<td>0.6080 (0.0956)</td>
</tr>
<tr>
<td>(MW_f)</td>
<td>0.217 (0.519)</td>
<td>0.583 (0.517)</td>
<td>4.6 (0.2)</td>
<td>0.1108 (0.0159)</td>
<td>0.1108 (0.0159)</td>
</tr>
<tr>
<td>(E_{2,MW_f})</td>
<td>1.147 (0.535)</td>
<td>1.271 (0.529)</td>
<td>13.9 (1.9)</td>
<td>0.6463 (0.0956)</td>
<td>0.6463 (0.0956)</td>
</tr>
<tr>
<td>(E_{6,MW_f})</td>
<td>1.180 (0.533)</td>
<td>1.249 (0.530)</td>
<td>24.6 (3.5)</td>
<td>0.6463 (0.0956)</td>
<td>0.6463 (0.0956)</td>
</tr>
<tr>
<td>(E_{10,MW_f})</td>
<td>1.175 (0.533)</td>
<td>1.227 (0.531)</td>
<td>32.4 (4.7)</td>
<td>0.6463 (0.0956)</td>
<td>0.6463 (0.0956)</td>
</tr>
<tr>
<td>(E_{14,MW_f})</td>
<td>1.164 (0.534)</td>
<td>1.208 (0.532)</td>
<td>38.8 (5.7)</td>
<td>0.6463 (0.0956)</td>
<td>0.6463 (0.0956)</td>
</tr>
<tr>
<td>(E_{6,MW_f})</td>
<td>1.162 (0.534)</td>
<td>1.217 (0.532)</td>
<td>30.4 (4.5)</td>
<td>0.6080 (0.0956)</td>
<td>0.6191 (0.0956)</td>
</tr>
<tr>
<td>(E_{6,MW_f})</td>
<td>0.442 (0.521)</td>
<td>0.592 (0.520)</td>
<td>11.0 (0.6)</td>
<td>0.1108 (0.0159)</td>
<td>0.1108 (0.0159)</td>
</tr>
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<td>0.6080 (0.0956)</td>
</tr>
<tr>
<td>(DT_f)</td>
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<td>0.594 (0.520)</td>
<td>10.3 (0.6)</td>
<td>0.1108 (0.0159)</td>
<td>0.1108 (0.0159)</td>
</tr>
</tbody>
</table>
5.4. PERFORMANCE SENSITIVITY TO TRANSACTION COSTS

The results from the previous section demonstrated a superiority of the DT strategies in getting better net Sharpe ratios. However, this superiority remained debatable when compared to the $E_qMW$ strategies. That being said, the figs. 5.6 to 5.8 illustrate that the $DT_{fp}$ remains the best performing strategy when varying the different transaction costs parameters, while the $E_qMW$ strategies performance depends on the selected lag parameter $q$.

As one can see from figs. A.2 to A.4 it is clear that the net Sharpe ratio tend to decrease as the transaction costs increase. However, this drop in performance is less aggressive for the DT strategies compared to the MW and $E_0MW$ strategies. This
could be explained by the accompanied increase in the average holding period for the $DT$ strategies. Moreover, one can also note how the $MW_{fp}$ and $E_6 MW_{fp}$ are outperformed respectively by $MW_p$ and $E_6 MW_p$ as the transaction costs increase, while the $DT_{fp}$ remains the best performing strategy. Therefore, one could say that N. Gărlăeanu & L. H. Pedersen’s model performs better by utilizing better the predicting factors’ strengths regardless of the transaction costs coefficients.

Furthermore, focusing on the strategy $DT_{fp}$, the figure 5.5 shows that the increase of the cost coefficients tend to increase the daily incurred cost and reduce the Sharpe ratios regardless of the market transaction cost form. However, this latter introduces some differences in the way the Sharpe ratios and incurred costs evolve jointly. First, considering the net Sharpe ratios, the trajectory corresponding to varying the quadratic cost coefficient $c$ is under the trajectory corresponding to varying the super-linear coefficient $b$ which in its turn is under the trajectory corresponding to varying the linear cost coefficient $a$, and the same goes for the gross Sharpe ratios. In other words, for the same incurred cost on average, the $DT_{fp}$ strategy Sharpe ratios are different depending on the market costs nature. In fact, quadratic transaction costs are more constraining the performance than super-linear transaction costs which are in their turn more constraining than linear transaction costs. Second, the divergence between the net and gross Sharpe ratios trajectories increase quite much when varying the linear and super-linear coefficients $a$ and $b$ compared to the case when the quadratic cost coefficient $c$ is varied. This indicates an increase in the loss of performance when the market transaction cost function is not quadratic. This could be due to the approximation used to obtain the quadratic cost matrix $\Lambda$ described in section 4.3.3 but in any case there will be always an underestimation or overestimation of the real costs coefficients.

Comparing the figs. A.5 to A.7, the best compositions fast-slow corresponding to the strategies $MW_{comb,best}$ and $E_6 MW_{comb,best}$ tend to use less the fast decaying factors as the transaction costs coefficients increase compared to $DT_{comb,best}$. This shows that the fast decaying factors are still of importance for the $DT$ strategies. Additionally, the risk contribution of the fast based portfolios in all the strategies $DT_{comb,best}$, $MW_{comb,best}$ and $E_6 MW_{comb,best}$ is very low, which explains the high weight put on these fast based strategies.
5.4. PERFORMANCE SENSITIVITY TO TRANSACTION COSTS

Figure 5.5 – Performance comparison with respect to the different costs forms. This figure shows the trajectories of the net (solid lines) and gross (dash-dotted lines) Sharpe ratios against the normalized average daily incurred cost for the strategy $DT_{fp}$ when the transaction costs coefficients $a$, $b$ and $c$ are varied.

Figure 5.6 – Performance sensitivity to linear costs for the $E_q MW_{fp}$ and $DT_{fp}$. These figures show the evolution of the mean for the annualized Sharpe ratios (SR), net portfolio volatility (PV) and average holding period (AHP) with respect to the linear cost coefficient $a$ for the strategies $DT_{fp}$, $E_2 MW_{fp}$, $E_6 MW_{fp}$, $E_{10} MW_{fp}$ and $E_{14} MW_{fp}$. The super-linear and quadratic cost coefficients $b$ and $c$ have been set to 0.
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Figure 5.7 – Performance sensitivity to super-linear costs for the $E_q MW_{fp}$ and $DT_{fp}$. These figures show the evolution of the mean for the annualized Sharpe ratios (SR), net portfolio volatility (PV) and average holding period (AHP) with respect to the super-linear cost coefficient $b$ for the strategies $DT_{fp}$, $E_2 MW_{fp}$, $E_6 MW_{fp}$, $E_{10} MW_{fp}$ and $E_{14} MW_{fp}$. The linear and quadratic cost coefficients $a$ and $c$ have been set to 0.
Figure 5.8 – **Performance sensitivity to quadratic costs for the $E_qMW_{fp}$ and $DT_{fp}$.**
These figures show the evolution of the mean for the annualized Sharpe ratios (SR), net portfolio volatility (PV) and average holding period (AHP) with respect to the quadratic cost coefficient $c$ for the strategies $DT_{fp}$, $E_2MW_{fp}$, $E_6MW_{fp}$, $E_{10}MW_{fp}$ and $E_{14}MW_{fp}$. The linear and super-linear cost coefficients $a$ and $b$ have been set to 0.
Chapter 6

Conclusion

In the previous chapter, the results clearly showed an overall better performance of the elaborated strategies DT based on N. Gårleanu & L. H. Pedersen’s framework under the proposed methodology. This performance was accompanied with evidence showing a better usability of the predicting factors strengths, specifically their mean reversion speed. That being said, the results did not refute the fact that fast decaying predicting factors decrease in importance as the transaction costs gets higher. Actually, the seemingly suggested conclusion is that this decrease in importance is more optimal compared to a naive single period MVO approach.

Furthermore, the elaborated procedure for estimating the quadratic costs matrix coefficients from the considered market transactions cost function, while remaining simple, provides a reasonably uncompromised performance of the studied framework. Indeed, the DT strategies remained the best performing ones. Nonetheless, the observed loss in performance that increases when the non quadratic cost terms increase could be due to huge estimation errors in our procedure. A possible improvement to minimize these would be to guess a better prior probability distribution for the normalized trades instead of a uniformly distribution as it was suggested in 4.3.3.

On an other note, the proposed strategies $E_qMW$ showed competition with the DT strategies. A shortcoming to the former is the necessity to fine-tune the lag parameters $q$. In the shown results, this was done manually, yet a good performance was obtained. A potential improvement to this strategy is to design some systematic way that dynamically selects the optimal lag parameter.
To conclude, the used methodology for investigating N. Gărleanu & L. H. Pedersen’s framework has proven to be successful in reconfirming some findings from the originating work, as well as providing insights on the model behaviour with respect to the return predictability and transactions cost aspects. Future work would be to use the framework in a more realistic scenario. The challenges would be to achieve accurate estimations of the required inputs for the framework, more specifically the return predictability and transaction costs estimates.
Appendix A

Figures
Figure A.1 – **Sampled distributions of the Sharpe ratio.** The figures show the histogram of the obtained ex-ante annualized net *(blue)* and gross *(red)* Sharpe ratios for the dynamic trading strategies, the Markowitz based strategies, and the EMA filtered Markowitz strategies. The figures (a), (b), (c), (d), (e), (f), (g), (h) and (i) correspond respectively to the strategies $MW_{fp}$, $MW_p$, $MW_f$, $E_6MW_{fp}$, $E_6MW_p$, $E_6MW_f$, $DT_{fp}$, $DT_p$ and $DT_f$. 
Figure A.2 – Performance sensitivity to linear costs for the strategies DT, MW and E₆MW. These figures show the mean and standard deviation of the annualized Sharpe ratios (SR), annualized net portfolio volatility (PV) and average holding period (AHP) of the strategies DT (blue), MW (red) and E₆MW (yellow) when the linear cost coefficient $a$ is varied in the interval $[5 \times 10^{-4}, 0.5]$. The super-linear and quadratic cost coefficients $b$ and $c$ were set to 0.
Figure A.3 – Performance sensitivity to super-linear costs for the strategies $DT$, $MW$ and $E_6 MW$. These figures shows the mean and standard deviation of the annualized Sharpe ratios (SR), annualized net portfolio volatility (PV) and average holding period (AHP) of the strategies $DT$ (blue), $MW$ (red) and $E_6 MW$ (yellow) when the super-linear cost coefficient $b$ is varied in the interval $[9 \times 10^{-3}, 9]$. The linear and quadratic cost coefficients $a$ and $c$ were set to 0.
Figure A.4 – Performance sensitivity to quadratic costs for the strategies $DT$, $MW$ and $E_6 MW$. These figures shows the mean and standard deviation of the annualized Sharpe ratios (SR), annualized net portfolio volatility (PV) and average holding period (AHP) of the strategies $DT$ (blue), $MW$ (red) and $E_6 MW$ (yellow) when the quadratic cost coefficient $c$ is varied in the interval $[1.05 \times 10^{-2}, 10.5]$. The linear and super-linear cost coefficients $a$ and $b$ were set to 0.
Figure A.5 – Combination strategies $DT_{comb}$, $MW_{comb}$ and $E_6 MW_{comb}$ evolution with respect to linear costs. Figure (a) shows the evolution of the composition slow-fast in percentage corresponding to the best strategy $DT_{comb,best}$ in terms of the net Sharpe ratio when the linear cost coefficient $a$ varies in the interval $[5 \times 10^{-4}, 0.5]$. Figure (b) and (c) describe the same evolution as in (a) for the strategies $MW_{comb,best}$ and $E_6 MW_{comb,best}$. Figure (d) shows the evolution of the contribution to risk in percentage of the composition slow-fast corresponding to the best strategy $DT_{fp}$ in net Sharpe ratio when the linear cost coefficient $a$ varies in the interval $[5 \times 10^{-4}, 0.5]$. Figure (e) and (f) describe the same evolution as in (d) for the strategies $MW_{comb,best}$ and $E_6 MW_{comb,best}$. 
Figure A.6 – Combination strategies $DT_{comb}$, $MW_{comb}$ and $E_0MW_{comb}$ evolution with respect to super-linear costs. Figure (a) shows the evolution of the composition slow-fast in percentage corresponding to the best strategy $DT_{comb,best}$ in terms of the net Sharpe ratio when the super-linear cost coefficient $b$ varies in the interval $[9 \times 10^{-3}, 9]$. Figure (b) and (c) describe the same evolution as in (a) for the strategies $MW_{comb,best}$ and $E_0MW_{comb,best}$. Figure (d) shows the evolution of the contribution to risk in percentage of the composition slow-fast corresponding to the best strategy $DT_{fp}$ in net Sharpe ratio when the super-linear cost coefficient $b$ varies in the interval $[9 \times 10^{-3}, 9]$. Figure (e) and (f) describe the same evolution as in (d) for the strategies $MW_{comb,best}$ and $E_0MW_{comb,best}$. 
Figure A.7 – Combination strategies $DT_{comb}$, $MW_{comb}$ and $E_6MW_{comb}$ evolution with respect to quadratic costs. Figure (a) shows the evolution of the composition slow-fast in percentage corresponding to the best strategy $DT_{comb, best}$ in terms of the net Sharpe ratio when the quadratic cost coefficient $c$ varies in the interval $[1.05 \times 10^{-2}, 10.5]$. Figure (b) and (c) describe the same evolution as in (a) for the strategies $MW_{comb, best}$ and $E_6MW_{comb, best}$. Figure (d) shows the evolution of the contribution to risk in percentage of the composition slow-fast corresponding to the best strategy $DT_{fp}$ in net Sharpe ratio when the quadratic cost coefficient $c$ varies in the interval $[1.05 \times 10^{-2}, 10.5]$. Figure (e) and (f) describe the same evolution as in (d) for the strategies $MW_{comb, best}$ and $E_6MW_{comb, best}$. 
Bibliography


