The mod p Cohomology of the Projective Unitary Group

ISAK JOHANSSON
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ISAk JOHANSSON
Abstract

We begin with an introduction to spectral sequences, in particular, we present how a spectral sequence can arise from an exact couple, state and construct the Serre spectral sequence, mention some of the properties of the mod $p$ cohomology and state the dual Eilenberg-Moore spectral sequence. Fiber bundles, together with the concept of pullback bundles, principal bundles, classifying spaces and Chern classes are also discussed to lay a foundation for our results. We compute the mod $p$ cohomology of the projective unitary group. Finally, we compute the mod 3 cohomology of the classifying space of the projective unitary group of order 3.
Sammanfattning

Acknowledgements

First and foremost, I would like to thank my supervisor, Tilman Bauer, for nice discussions and for always being a source of inspiration. I could not have done it without you.

I would also like to thank my family for their continuous support and my partner Annika for always being there.
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Introduction

The story about how spectral sequences were invented is quite fascinating. The French born mathematician Jean Leray invented them during his time as a prisoner in a concentration camp during the second world war in his studies of sheaves. However, it was not until 1951 when Jean-Pierre Serre in his doctoral thesis[18] applied spectral sequences to algebraic topology that spectral sequences became a popular tool for computations. Today, spectral sequences is one of the most powerful tools in algebraic topology. The spectral sequence that we in this thesis call the Serre spectral sequence sometimes go by the name Leray-Serre spectral sequence to acknowledge the earlier work done by Leray.

First, an introduction to spectral sequence is given in chapter 1. In chapter 2, we discuss fiber bundles which we wish to apply spectral sequences to. In chapter 3 and 4 we use the theory from the previous chapters to come up with results on the mod p cohomology of the projective unitary group and the mod 3 cohomology of its classifying space.

\( R \) will denote a commutative ring with unit.
Chapter 1

Spectral Sequences

We will here introduce the important concept of a spectral sequence that we will use throughout the thesis. One should pay close attention to the Serre spectral sequence, for which we will use in most of our computations. The main reference to this chapter has been [10], an introduction to spectral sequence can also be found in [4].

In section 1.1, we give a way of constructing a spectral sequence from an exact couple, but for now let us present some definitions without giving any further idea for how a spectral sequences arise.

At a first glance, spectral sequences will seem messy and not very intuitive. We will here aim to demonstrate that spectral sequences are not so terrifying and once one has got his mind right about the indices, one can really enjoy working with them.

Definition 1. A differential bigraded module is a bigraded module $E^{p,q}$ with a map $d$ with bidegree $(-r,-r+1)$ or $(r,r-1)$ such that

$$d_r : E^{p,q} \mapsto E^{p-r,q-r+1}$$

or

$$d_r : E^{p,q} \mapsto E^{p+r,q+r-1}.$$  

and $d \circ d = 0$, such a map $d$ is called a differential.

Definition 2. A spectral sequence is a family of differential bigraded modules $\{E^{r,*}_r\}$ together with maps $d_r$ of bidegree $(r,-r+1)$ or $(-r,r-1)$ such that $d_r \circ d_r = 0$ and $E^{r,*}_{r+1} = H(E^{r,*}_r, d_r)$.

We will mainly be interested in the second differential with bidegree $(r,1-r)$, a spectral sequence with such a differential is called a spectral sequence.
of cohomology type. A spectral sequence with bidegree \((-r,r-1)\) is called a spectral sequence of homology type.

The definition on its own may not give a lot of intuition for why spectral sequences is such a powerful tool, but it turns out that they are and in fact also easy to visualize. A spectral sequence can be thought of as a book with pages of planes. On page \(r\) and coordinate \((p,q)\) we find the \(E^{p,q}_r\) term. On page 3, we find \(E^{0,2}_3\) as

\[
\begin{array}{cccccc}
5 & & & & & \\
4 & & & & & \\
3 & & & & & \\
2 & & & & & \\
1 & & & & & \\
0 & 1 & 2 & 3 & 4 & 5
\end{array}
\]

and a differential from \(E^{0,2}_3\) to \(E^{3,0}_3\) can be seen as

\[
\begin{array}{cccccc}
5 & & & & & \\
4 & & & & & \\
3 & & & & & \\
2 & & & & & \\
1 & & & & & \\
0 & 1 & 2 & 3 & 4 & 5
\end{array}
\]

Notice that the differentials will always go from a white route to a grey route or vice versa. We should also note that the homology is well-defined at every object in our spectral sequences, this is implied by the condition that says that \(d_r \circ d_r = 0\). Therefore, it makes sense to define the next page as the homology of the previous page with respect to the map \(d_r\). Here it is time for a remark.

We want to emphasize the fact that knowing \(E^{*,*}_r\) and \(d_r\) determines \(E^{*,*}_{r+1}\) but not the differentials \(d_{r+1}\).

A natural question would be to ask: what happens as we go to page infinity? Well, if the spectral sequence converges, that is that for some \(i\) and for all \(j > i\), \(E^{*,*}_j = E^{*,*}_i\), we define the page at infinity to be just the same as page \(i\), i.e \(E^{*,*}_\infty = E^{*,*}_i\). Spectral sequences that does not converge is out of scope of this thesis, all spectral sequences in this thesis converges.

The Serre spectral sequence is a spectral sequence of algebras, such a spectral sequence is a spectral sequence with the extra structure of a multiplication.

**Definition 3.** A spectral sequence of algebras over \(R\) is a spectral sequence, \(\{E_r\}\) together with differentials \(\{d_r\}\) that satisfies the Leibniz rule, and multiplications \(\phi_r : E_r \otimes E_r \to E_r\) such that the \(\phi_{r+1}\) can be written as the composition

\[
\phi_{r+1} : E_{r+1} \otimes E_{r+1} \xrightarrow{\sim} H(E_r) \otimes H(E_r) \to H(E_r \otimes E_r) \xrightarrow{H(\phi_r)} H(E_r) \xrightarrow{\sim} E_{r+1}
\]
and where the middle map is given by $[u] \otimes [v] \rightarrow [u \otimes v]$.

That $d$ satisfies the Leibniz rule means that for $d : \bigoplus_{p+q=n} E_{p}^{s,q} \rightarrow \bigoplus_{r+s=n+1} E_{r}^{s,q}$ and for $x$ and $y$ in $E_{p}^{s,q}$,

$$d(x \otimes y) = d(x) \otimes y + (-1)^{p+q}x \otimes d(y).$$

We are now ready to move on and see how a spectral sequence can arise from an exact couple.

### 1.1 Exact Couples

Spectral sequences arise naturally in many different setups. The most common examples are via a filtration or an exact couple. In this section we treat the case of an exact couple and later we will show how we can use this machinery in the construction of Serre’s spectral sequence.

**Definition 4.** An exact couple is a collection $\{D, E, i, j, k\}$, where $D$ and $E$ are $R$-modules and $i, j, k$ are maps such that we have the following diagram,

$$
\begin{array}{ccc}
D & \xrightarrow{i} & D \\
\downarrow{k} & & \downarrow{j} \\
E & \xrightarrow{j} & E
\end{array}
$$

with $\text{im}(i) = \ker(j)$, $\text{im}(j) = \ker(k)$ and $\text{im}(k) = \ker(i)$.

We like to think of an exact couple as a long exact sequence that just goes around and around. From an exact couple it is possible to extend to something we will call a derived couple.

**Definition 5.** A derived couple is obtained from an exact couple $\{D, E, i, j, k\}$ by defining: $D' = \text{im}(i)$, $i' = i|_{i(D)}$, $j' = j \circ i^{-1}$, $k'(x + j \circ k(E)) = k(x)$ for $x$ in $E$, $E' = H(E, j \circ k)$

$$
\begin{array}{ccc}
D' & \xrightarrow{i'} & D' \\
\downarrow{k'} & & \downarrow{j'} \\
E' & \xrightarrow{j'} & E'
\end{array}
$$

It can be shown that the derived couple is also an exact couple.
Lemma 1. A derived couple \( \{D', E', i', j', k'\} \) from an exact couple \( \{D, E, i, j, k\} \) is exact.

Proof. We start with proving exactness at the left \( D' \).

\[
im(k') = k'(\ker(j \circ k)/\im(j \circ k)) = k(\ker(j \circ k)) = k(k^{-1}(\ker(j))) = \ker(j) \cap \im(k)
\]

so we have exactness at the left \( D' \). For exactness at the right \( D' \), note first that \( D'/\ker(i) \) and consider

\[
\ker(j') = j^{-1}(\im(j \circ k))/\ker(i) = (j^{-1} \circ j(\im(k)))/\ker(i)
\]

\[
= (\ker(j) + \im(k))/\ker(i) = (\im(i) + \ker(i))/\ker(i)
\]

\[
= \im(i(\im(i))) = \im(i')
\]

Finally, the exactness at \( E' \) follows from

\[
\ker(k') = \ker(k)/\im(j \circ k) = \im(j)/\im(j \circ k) = \im(j').
\]

The process of derived couples can therefore be continued to the exact couple

\[
\begin{array}{ccc}
D_r & \xrightarrow{i_r} & D_r \\
\downarrow{k_r} & & \downarrow{j_r} \\
E_r & \xleftarrow{k_r} & D_r
\end{array}
\]

The meticulous reader now notice that we have something that begin to look as the pages of a spectral sequence but without any bigradings or differentials. So let us now assume that \( D \) and \( E \) are bigraded and that the maps \( i, j \) and \( k \) have bidegrees \((-1, 1), (1, 0)\) and \((0, 0)\). This give rise to a spectral sequence, let us present this as a theorem.

Theorem 1. Let \( D^{\ast \ast} \) and \( E^{\ast \ast} \) be bigraded modules over some ring \( R \) and let \( (D^{\ast \ast}, E^{\ast \ast}, i, j, k) \) be an exact couple with bidegrees of \( i, j \) and \( k \) as \((-1, 1), (1, 0)\) and \((0, 0)\). Then this gives rise to a spectral sequence \( \{E_r, d_r\} \) with \( E_r \) being the \((r - 1)\)-st derived module in the process of derived couples and \( d_r = j^r \circ k^r \) where \( j^r \) and \( k^r \) is the \((r - 1)\)-st derived maps.

Proof. We refer to [10] for the full proof. The only thing that needs to be shown is that \( d_r \) has the right bidegree, clearly \( d_r \circ d_r = 0 \) is satisfied. \(\square\)
1.2 The Serre Spectral Sequence

After having become acquainted with the definition of a spectral sequence we go on to the Serre spectral sequence, this is the spectral sequence that we are going to rely heavily on in this work, much because of its structural properties. We will begin by stating the theorem and then present the idea behind the theorem in section 1.2.1.

**Theorem 2.** Let \( F \rightarrow E \xrightarrow{\pi} B \) be a fibration of topological spaces, with \( B \) simply-connected and \( F \) connected. Then there exists a first quadrant spectral sequence \( \{E^r_{p,q}, d_r\} \) converging to \( H^*(E; R) \) as an algebra, with

\[
E_2^{p,q} \cong H^p(B; H^q(F; R)).
\]

The cup-product and the product on \( E_2^{*,*} \) satisfy \( u \cdot v = (-1)^{pq}u \sim v \) for \( u \) in \( E_2^{p,q} \) and \( v \) in \( E_2^{p',q'} \). Furthermore, the edge homomorphism from \( E_2^{k,0} \rightarrow E_\infty^{k,0} \) is the map \( \pi^* \) induced by the fiber map \( \pi^* \).

So basically, the Serre spectral sequence gives us a way of computing the cohomology of a topological space \( E \) if we can find a fibration involving \( E \) with \( B \) simply connected and \( F \) connected. It is not just the fact that the Serre spectral sequence converges that makes this result so astonishing, it is also because of the relationship between the product on \( E_2^{*,*} \) and the cup product structure on the cohomology algebra. With these things taking into account, the Serre spectral sequence is a very powerful tool.

For the Serre spectral sequence to make any sense, we need the notion of a fibration.

**Definition 6.** A map \( \pi : E \rightarrow B \) is said to be a fibration if it satisfies the homotopy lifting property with respect to any space \( X \), i.e. for a homotopy \( G \) and a map \( g \) there exists a map \( \tilde{G} \) such that the following diagram commutes

\[
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{g} & E \\
\downarrow{i} & & \downarrow{\pi} \\
X \times I & \xrightarrow{G} & B
\end{array}
\]

For \( b \) in \( B \), we call the \( \pi^{-1}(b) \) the fiber over \( b \). If \( B \) is path connected, all fibers are homotopy equivalent so we denote them by just \( F \) and call it the fiber of the fibration. There are standardized names for the other spaces as well, \( E \) will be refereed to as the total space and \( B \) as the base space. A fibration is a generalization of a fiber bundle, which chapter 3 is devoted to.

Another thing that makes the Serre spectral sequence enjoyable to work with is that the expression for the \( E_2^{*,*} \) often can be simplified even further. Since \( E_2 \)
is a bigraded algebra, we have a map $E^{p,q}_2 \otimes E^{p',q'}_2 \to E^{p+p',q+q'}_2$. If we assume that $B$ is simply connected, $F$ is connected and $H^p(B; R)$ and $H^q(F; R)$ are free $R$-modules of finite type and then apply the universal coefficient theorem we get an isomorphism $E^{p,0}_2 \otimes E^{0,q}_2 \cong E^{p,q}_2$. This a very important tool so we will present it as a theorem.

**Theorem 3.** If $B$ is simply connected, $F$ is connected and $H^p(B; R)$ and $H^q(F; R)$ are free $R$-modules of finite type then $E^{p,q}_2 = H^p(B; R) \otimes H^q(F; R)$.

**Proof.** By a version of the universal coefficient theorem stated in [5], we have the short exact sequence

$$0 \to H^p(B; R) \otimes H^q(F; R) \to H^p(B; H^q(F; R)) \to Tor(H^{p+1}(B; R), H^q(F; R)) \to 0$$

and since $H^p(B; R)$ and $H^q(F; R)$ are free $R$-modules,

$$H^p(B; R) \otimes H^q(F; R) \cong H^p(B; H^q(F; R)).$$

When doing calculations with a first quadrant spectral sequence such as the Serre spectral sequence one often comes down to the case where there is a "last chance" differential, meaning that if the differential is zero, the element will survive. For $E^{n,n-1}_n$, we call $d_n$ the transgression. More formally, the transgression is often defined as a map $\tau$,

$$\tau: (\delta^{-1}(im(p_n^*))) \subset H^{q-1}(F) \to H^q(B)/j^*(ker p_n^*)$$

$$\tau(x) = j^*(y) + j^*(ker p_n^*)$$

where $x$ is in $\delta^{-1}(im(p_n^*))$ and $y$ is in $H^q(B, *)$ such that $p^*(y + ker(p_n^*)) = \delta x$.

The maps are defined in the following diagram

$$\begin{array}{cccccc}
& & H^{q-1}(*) & \delta & H^q(B, *) & j^* & H^q(B) & \to & H^q(*) & \to \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & H^{q-1}(F) & \delta & H^q(E, F) & j^* & H^q(E) & \to & H^q(F) & \\
\end{array}$$

The diagram is constructed from the pairs $(B, *)$ and $(E, F)$ and together with naturality we have the above diagram in cohomology commutes.

Using the formal definition of a transgression we can actually prove that this map is the "last chance" differential.
Lemma 2. The transgression is \( \tau: (\delta^{-1}(\mathrm{im}(p_0^*))) \subset H^{n-1}(F) \to H^n(B)/j^*(\ker p_0^*) \) is exactly the map \( d_n: E_{n-1}^0 \to E_{n}^0 \).

Proof. There is a proof of the homology case in McCleary[10]. It holds for the dual as well. \( \square \)

1.2.1 The Construction of the Serre Spectral Sequence

This section will serve as a sketch of Theorem 2 and we will here try to illustrate the idea behind it.

Let \( F, E, B \) and \( \pi \) be as in Theorem 2. To simplify the idea a bit, suppose \( B \) has a CW-complex and that \( B^s \) is the \( s \)-skeleton of \( B \) and define \( J^s = \pi^{-1}(B^s) \).

We can illustrate this by

\[
\begin{array}{cccccccc}
E & \supset & J^s & \supset & J^{s-1} & \cdots & \supset & J^0 = 0 \\
\pi & \downarrow & \pi & \cdots & \pi & & \pi \\
B & \supset & B^s & \supset & B^{s-1} & \cdots & \supset & B^0 = 0
\end{array}
\]

The short exact sequence

\[
0 \to J^{s-1} \to J^s \to J^s/J^{s-1} \to 0,
\]

give rise to the long exact sequence in cohomology

\[
\ldots \to H^n(J^s; R) \to H^n(J^{s-1}; R) \to H^{n+1}(J^s, J^{s-1}; R) \to H^{n+1}(J^s; R) \to \ldots
\]

This long exact sequence could also be viewed as an exact couple if we think of the cohomologies as bigraded algebras. Using the notation from section 1.1 we have that \( D^{p,q} = H^{p+q}(J^p) \) and \( E^{p,q} = H^{p+q}(J^p, J^{p-1}) \), this can be viewed as

\[
\begin{array}{ccc}
H^{p+q}(J^p; R) & \xrightarrow{d_1} & H^{p+q}(J^{p-1}; R) \\
\downarrow & & \downarrow \\
H^{p+q+1}(J^p; R) & & H^{p+q+1}(J^p, J^{p-1}; R)
\end{array}
\]

This indeed determines a spectral sequence with the first page \( E_1^{p,q} = H^{p+q}(J^p, J^{p-1}; R) \) with the differential \( d_1 \) being the induced inclusion map composed with the coboundary map.
Now consider the case when $(J^p, J^{p-1}) \cong (B^p, B^{p-1}) \times F$ and when all $H^i(F; R)$ modules are free. Recall that we assumed that $B$ is simply connected. Applying the Kunneth formula in cohomology gives

$$E_1^{p,q} = \bigoplus_{r+s=p+q} H^r(B^p, B^{p-1}; R) \otimes H^s(F; R)$$

$$E_1^{p,q} = H^p(B^p, B^{p-1}; R) \otimes H^q(F; R).$$

After an application of the universal coefficient theorem, we get $E_2^{p,q} = H^p(B; H^q(F; R)).$

By prop. 2.6 in McCleary[10], the spectral sequence converges and

$$E_\infty^{p,q} = F^p H^{p+q}(E)/F^{p-1} H^{p+q}(E),$$

where

$$F^p H^{p+q}(E) = \text{im}(H^{p+q}(J^p) \xrightarrow{\text{inc}} H^{p+q}((E))).$$

To see that the spectral sequence has the multiplicative structure that is stated in the theorem, we refer to [10].

1.2.2 Examples

The only way to get acquainted with spectral sequence is to work through a lot of examples. In this section we will present some well-known results that illustrate how one can apply Serre’s spectral sequence to make the computations far more easy. The idea of the computations of these examples is to find a fibration that we can apply Serre’s spectral sequence to and obtain the cohomology of the total space. Both examples involve the unitary group, the unitary group is denoted $U(n)$ and is defined as

$$U(n) := \{ A \in M(n, \mathbb{C}); A^t A^* = I \}.$$

**Example:** $H^*(U(n); R) = \wedge(x_1, x_3, ..., x_{2n-1})$, where $x_i$ is the generator for $H^i(u(n))$ and $\wedge$ denotes the exterior algebra. We will show this by induction. This is easily seen to be true for $u(1) = S^1$, this is our base case. Proceeding our induction we have the following fiber bundle, which is a fibration, see chapter 2.

$$u(n) \to u(n + 1) \to S^{2n+1}.$$
The Serre spectral sequence gives us $E_2^{*,*}$ as

$$x_{2n-1} \bullet \bullet \bullet \bullet$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\bullet$$

$$0$$

$$\bullet$$

$$x_{2n+1}$$

where the elements of products $x_ix_j$ are omitted for $i, j < 2n + 1$. From this it is easy to see that the differentials from each generator of $H^*(u(n))$ has to be zero on each page. Therefore, $E_1^{*,*} = E_2^{*,*} = \wedge(x_1, x_3, ..., x_{2n-1}) \otimes \wedge(x_{2n+1}) = \wedge(x_1, x_3, ..., x_{2n-1}, x_{2n+1})$ where $x_{2n+1}$ is the generator of degree $2n + 1$. This is the desired result.

To get a more solid understanding of the calculations in the previous example, we go on to the computation of the cohomology of the Stiefel Manifold $V_k(\mathbb{C}^n) = u(n)/u(n - k)$.

**Example:** $H^*(V_k(\mathbb{C}^n); R) = \wedge(x_{2(n-k)+1}, x_{2(n-k)+3}, ..., x_{2n-1})$, where $x_i$ is the generator for $H^i(u(n))$.

We notice that the following fibration looks interesting,

$$U(n - k)/U(n - k - 1) \to U(n)/U(n - k - 1) \to U(n)/U(n - k),$$

this is equivalent to

$$S^{2(n-k)-1} \to V_{k+1}(\mathbb{C}^n) \to V_k(\mathbb{C}^n).$$

We also note that $V_1(\mathbb{C}^n) = S^{2n-1}$, so the formula holds for $k = 1$. Proceeding by induction and applying Serre’s spectral sequence to this fibration we get a spectral sequence that again converge at the second page since the differential from the generator of the cohomology of the fiber has nowhere else than zero to go to.
The following diagram shows how $E_2^{2,*}$ look like for $n = 6$ and $k = 4$:

![Diagram showing $E_2^{2,*}$]

where the elements obtained from multiplication of the generators on the horizontal axis are omitted.

### 1.2.3 The Gysin Sequence

When dealing with computational problems for spectral sequences one is often interested in reducing the problem to something easier or something already known. For example, in the Serre spectral sequence, if we known that the fibre is an $n$-sphere in homology we can make use of the Gysin sequence. This sequence will be of much importance for us later, so we introduce it as a lemma.

**Lemma 3.** Let $F \rightarrow E \xrightarrow{\pi} B$ be a fibration with $B$ simply connected, $F$ connected and $H^*(F) = H^*(S^n)$ for some $n \geq 1$, then we have the following exact sequence

$$
\cdots \rightarrow H^k(B; R) \xrightarrow{\pi_*} H^{n+k+1}(B; R) \xrightarrow{\pi^*} H^{n+k+1}(E; R) \rightarrow H^{k+1}(B; R) \rightarrow \cdots,
$$

where $z$ is an element of $H^{n+1}(B; R)$.

**Proof.** Since the fiber has the same homology as some sphere, we get the second
where we have two stripes of nonzero elements and everything else just zero. This gives rise to an interesting pattern, we see that all differentials except those on page $n + 1$ will be zero, this gives us

$$E_2 \cong E_3 \cong \ldots \cong E_{n+1}$$

$$E_{n+2} \cong E_{n+3} \cong \ldots \cong E_{\infty}.$$

Therefore the cohomology of the total space is determined by $E_{n+2}$.

We have the following exact sequence

$$0 \to E^{k,n}_\infty \to E^{k,n}_2 \xrightarrow{d_{n+1}} E^{k+n+1,0}_2 \to E^{k+n+1,0}_\infty \to 0.$$

Also note that we have $E^{p,n}_\infty = H^{k+n}(E; R)/E^{k+n,0}_\infty$, so we have the short exact sequence

$$0 \to E^{k+n,0}_\infty \to H^{k+n}(E; R) \to E^{k,n}_\infty \to 0.$$

With these new insights we can put together a diagram
From this diagram we obtain a long exact sequence by using composition on the

\[ E^k,n \rightarrow E^{k+1,0} \rightarrow H^{k+n+1}(E; R) \rightarrow E^{k+1,n} \rightarrow \]

Note that this does not give us the properties stated in theorem about the maps. If we let \( h \) be a generator for \( H^n(F; R) \), we obtain that

\[ E^{*,*}_n = (H^*(B; R) \otimes 1) \oplus (H^*(B; R) \otimes h) \]

Now let \( z \) be the element in \( H^{n+1}(B; R) \) such that \( d_{n+1}(h \otimes 1) = 1 \otimes z \). Let us now consider \( d_{n+1} \) acting on \( x \otimes h \) in \( E^{k,n}_n = E^{k,n}_2 \) where \( x \) is an element of \( H^k(B; R) \).

\[ d_{n+1}(x \otimes h) = (-1)^{n \cdot \deg(x)}d_{n+1}((1 \otimes h) \sim (x \otimes 1)) \]
\[ = (-1)^{n \cdot \deg(x)}(d_{n+1}(1 \otimes h) \sim (x \otimes 1) + (-1)^n(1 \otimes h) \sim d_{n+1}(x \otimes 1)) \]
\[ = (-1)^{n \cdot \deg(x)}((z \otimes 1) \sim (x \otimes 1)) = (-1)^{n \cdot \deg(x)}(z \sim x \otimes 1) \]

Now we can replace \( E^{k,n}_2 \) with \( H^k(B; R) \), \( E^{k+n+1,0}_2 \) with \( H^{k+n+1}(B; R) \) and \( d_{n+1} \) with the map \( \gamma(x) = z \sim x \). The second map is the edge homomorphism, which in this case is the map induced by the fiber map. This gives us the Gysin sequence.
1.3 The mod $p$ Cohomology

For the mod $p$ cohomology, i.e. the cohomology with coefficients in a field with characteristic $p$, where $p$ is a prime, one can introduce the Steenrod Algebra. This algebra was first defined by Steenrod\[19\] for $p = 2$, and later generalized to odd primes. We will be interested in the case of odd primes for the sake of chapter 4.

The Steenrod algebra denoted $A_p$ is a graded Hopf-algebra over a field with characteristic $p$ generated by the Steenrod’s reduced $p$-th powers $P^i$ and the Bockstein homomorphism $\beta$. $P^i$ and $\beta$ are stable cohomology operations, see \[10\] and \[21\].

**Theorem 4.** Steenrod’s reduced $p$-th powers $P^i$ and the Bockstein homomorphism $\beta$, satisfies

- $\quad a) \quad P^i : H^*(X; \mathbb{Z}_p) \rightarrow H^{*+2i(p-1)}(X; \mathbb{Z}_p), \quad i \geq 0$
- $\quad b) \quad P^0 = Id$
- $\quad c) \quad If \ x \in H^{2n}(X; \mathbb{Z}_p) \ and \ k \geq 2n, \ then \ P^k x = 0$
- $\quad d) \quad For \ x, y \in H^*(X; \mathbb{Z}_p): \ P^k(x \smile y) = \sum_{j=0}^{k} P^j x \smile P^{k-j} y$
- $\quad e) \quad For \ x, y \in H^*(X; \mathbb{Z}_p): \ \beta(x \smile y) = \beta x \smile y + (-1)^{deg(x)} x \smile \beta y$

where $X$ is a topological space.

Using this new structure, Kudo\[22\], was able to come up with a transgression theorem for the mod $p$ cohomology.

**Theorem 5.** If $x$ in $E_2^{0,2k} \cong H^{2k}(F; \mathbb{Z}_p)$ has the property that $d_2(x) = d_3(x) = ... = d_{2k}(x) = 0$ and transgresses to $y$ in $E_2^{2k+1,0} \cong H^{2k+1}(B; \mathbb{Z}_p)$, then $y \otimes x^{p-1}$ transgresses to $-\beta P^k y$ and $P^k x = x^p$ transgresses to $P^k y$. 

![Diagram](image-url)
1.4 The Dual Eilenberg-Moore Spectral Sequence

In some cases, given a fibration $F \to E \to B$ where we know the cohomology of the fiber and the total space, it can be very hard or impossible to work our way backwards using the Serre spectral sequence and determine the cohomology of the base space. Instead of working our way backwards, there exist another spectral sequence called the Dual Eilenberg-Moore spectral sequence. This spectral sequence arises from the Eilenberg-Moore spectral sequence, which is used to determine the homology of the base space. However, in this thesis, we will be interested in computing the cohomology of the classifying space of the topological group $PU(n)$. See the notion of classifying space in chapter 2.

Therefore, we need a spectral sequence that converges to the classifying space of a topological space, but first we present the following theorem which is often referred to as the homology Eilenberg-Moore spectral sequence.

**Theorem 6.** If $G$ is a topological connected group, then there is a spectral sequence of coalgebras with

$$E^2 \cong \text{Tor}^H_*(G;k)(k, k)$$

and which converges to $H_*(BG; k)$ as a coalgebra.

**Proof.** The proof can be found in the classical paper by Rothenberg-Steenrod[16].

For the above theorem to make any sense we need the notions of coalgebras, comodule, cotensor product and the functor cotor.

A coalgebra $C$ is a vector space over a field $k$ together with $k$-linear maps $\psi$ and $\epsilon$, such that the following diagrams commute:

$$\begin{array}{ccc}
C & \xrightarrow{\psi} & C \otimes C \\
\downarrow{\psi} & & \downarrow{\psi \otimes 1} \\
C \otimes C & \xrightarrow{1 \otimes \psi} & C \otimes C \otimes C
\end{array} \quad \quad \begin{array}{ccc}
C & \xrightarrow{\psi} & C \otimes C \\
\downarrow{\psi} & & \downarrow{1} \\
C \otimes C & \xrightarrow{\epsilon \otimes 1} & C
\end{array}$$

where $\epsilon : C \to k$ and $C \otimes k \cong C \cong k \otimes C$. The map $\psi$ is called a comultiplication and $\epsilon$ a counit.

A right comodule $M$ over a coalgebra $C$ with comultiplication $\psi$ and counit $\epsilon$ is a vector space over a field $k$ together with a linear map $\lambda$ making the following diagrams to commute:
In particular, $k$ as a field is also a module over itself and $k^* = k$ is a comodule over $k$. From this it is also possible to define the cotensor product for a right comodule $A$ and a left comodule $B$ over $C$ as

$$A \square_C B = \ker(\lambda_A \otimes 1 - 1 \otimes \lambda_B),$$

where,

$$(\lambda_A \otimes 1 - 1 \otimes \lambda_B) : A \otimes B \to A \otimes C \otimes B$$

and $\lambda_A, \lambda_B$ are comultiplications for $A$ and $B$. Using the cotensor product we can define a corresponding Tor functor but for comodules, the cotor functor. To do this consider an injective resolution $I_*$ of comodules over $C$ as

$$0 \to A \to I_0 \to I_1 \to I_2 \to ...,\n$$

now consider

$$I_0 \square_C B \to I_1 \square_C B \to I_2 \square_C B \to ...$$

It now make sense to define

$$\text{Cotor}^C_n(A, B) := H^n(I_0 \square_C B \to I_1 \square_C B \to I_2 \square_C B \to ...).$$

For the cotor functor and the Tor functor, we have the following relationship[15]:

$$\text{coTor}^C_n(k^*, k^*) = (\text{Tor}_n^A(k, k))^*,$$

where $k$ is a field and $C$ is a coalgebra. Note, as mentioned earlier, $k^* = k$ is a coalgebra over itself. In [10], we have that the dual $C^*$ of an algebra $C$ over $k$ is a coalgebra over $k$, so the above expression is well-defined and can further be simplified to

$$\text{coTor}^A_n(k, k) = (\text{Tor}_n^A(k, k))^*,$$

We can now dualize Eilenberg-Moore spectral sequence and obtain the following theorem.

**Theorem 7.** If $G$ is a topological connected group, then there is a spectral sequence of algebras with

$$E_2 \cong \text{Cotor}^{H^*(G; k)}(k, k)$$

which converges to $H^*(BG; k)$ as an algebra.
The fact that the dual of $H_*(BG; k)$ is $H^*(BG; k)$ follows from an application of the universal coefficient theorem. The universal coefficient theorem gives us the following short exact sequence

$$0 \to \text{Ext}^1_k(H_{p-1}(BG; k), k) \to H^p(BG; k) \to \text{Hom}_k(H_p(BG; k), k) \to 0$$

since $k$ is an injective $k$-module, $\text{Ext}^1_R(H_{p-1}(BG; k), k) = 0$ and we get that

$$H^p(B; k) \cong \text{Hom}_R(H_p(BG; k), k) \cong H_p(BG; k)^*.$$
Chapter 2

Fiber Bundles

In this chapter we give the definition of a fiber bundle and present some of its main properties that will come in handy for the rest of the thesis. A fiber bundle is a special case of a fibration, see theorem 8. This reveals why we are interested in them, once we have a fibration we can release our machinery of spectral sequences. The main references to this chapter has been [13], [6] and [10].

Seldom does the geometric intuition of a fiber bundle help one in understanding the fiber bundle, therefore we will go directly to the formal definition of a fiber bundle.

Definition 7. A fiber bundle is a map \( p : E \to B \), where \( E \) is the total space and \( B \) is the base space. The fiber \( F \) is a space such that for every point \( b \) in \( B \) there exists an open neighborhood \( U \) of \( b \) and a homeomorphism \( \phi : U \times F \to p^{-1}(U) \) such that the following diagram commutes

\[
\begin{array}{ccc}
U \times F & \xrightarrow{\phi} & p^{-1}(U) \\
\downarrow{\text{proj}} & & \downarrow{p} \\
U & \to & \\
\end{array}
\]

We will often refer to a fiber bundle \( E \xrightarrow{p} B \) as \( E \) over \( B \). We will also call a fiber bundle with \( p^{-1}(x) \) isomorphic to a vector space for all \( x \) in \( U \) a vector bundle, if there is a linear transformation \( T \), such that \( T(v) = \phi(x, v) \) is an isomorphism between the vector space and \( p^{-1}(x) \). If this vector space has rank 1 we sometimes call the vector bundle a line bundle.

A first simple example of a fiber bundle is the Mobius band over the circle with the fiber of an interval. An example which is more relevant to our work is the
fibre bundle over a protective space. The following two examples are standard examples.

**Example:** $S^n \to \mathbb{R}P^n$ with the natural map is a fiber bundle with fiber $S^0$.

**Example:** $S^{2n+1} \to \mathbb{C}P^n$ is a fiber bundle with fiber $S^1$. Notice that $S^{2n+1}$ can be seen as the unit circle in $\mathbb{C}^{n+1}$ and our map $p$ from $S^{2n+1}$ onto $\mathbb{C}P^n$ is the natural map. To see that this is a fiber bundle consider $\phi_1 : p^{-1}(U) \to U \times S^1$, where $U$ is open in $\mathbb{C}P^n$ and $\phi(z_0, ..., z_n) = (z_0, ..., z_n) \times z_i/|z_i|$ for some $z_i \neq 0$. One can easily check that $\phi_1$ is a homeomorphism.

Fiber bundles are important to us since they are a special case of the more general concept of a fibration from Chapter 1. Fibrations are like fiber bundles except that the fibers of a fibration does not need to be homeomorphic. We will often try to solve our problems by obtaining a fiber, which is a fibration, and then apply the Serre spectral sequence to the fiber. We state this relationship as a theorem.

**Theorem 8.** A fiber bundle $F \to E \xrightarrow{p} B$, with $B$ paracompact[7] is a fibration.

**Proof.** The fiber bundle is locally trivial, meaning that for an open set $U$ in $B$, the following diagram commutes

$$
\begin{array}{ccc}
U \times F & \xrightarrow{\phi} & p^{-1}(U) \\
\downarrow{\text{proj}} & & \downarrow{p} \\
U & & \\
\end{array}
$$

where $\phi$ is a homeomorphism. If we set $g(x) = (g_1(x), g_2(x))$ then $G(x, t) = g_1(x)$ and setting $\tilde{G}(x, t) = g(x)$ gives the that the following diagram commutes,

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{g} & U \times F. \\
\downarrow{i} & & \downarrow{\text{proj}} \\
X \times I & \xrightarrow{\tilde{G}} & U \\
\end{array}
$$

where $X$ is a topological space. Therefore, a fiber bundle is locally a fibration, but then a fiber bundle over a paracompact space is a global filtration by Hurewicz theorem[8].

Now when we are familiar with the basic definition of a fiber bundle we want to go on to the very useful notion of principal fiber bundle. A principal bundle is a fiber bundle with extra structure and with the property that the fiber is
isomorphic to a so called structure group. Before presenting the definition of a principal bundle, we will introduce the notion of a structure group which tells us about the structure of the bundle.

**Definition 8.** A structure group $G$ for a fiber bundle $F \to E \to B$ is a topological group together with a right action $\mu$ on $F$ such that for $g_1, g_2$ in $G$ and $x$ in $F$

$$\mu(g_1, \mu(g_2, x)) = \mu(g_1 g_2, x)$$

$$\mu(e, x) = x$$

and if $\mu(g, x) = x$ for all $x$ in $F$, then $g = e$, where $e$ is the identity in the topological group $G$.

**Definition 9.** A principal $G$-bundle is a fiber bundle $p : E \to B$, with fiber $F$ and structure group $G = F$ such that the left action on the fiber coincides with the left-multiplication in the topological group.

Principal bundles are very important and has a special relationship to the classifying spaces, see section 2.2.

From a principal $G$-bundle $E \to B$, there is a way to construct another bundle with fiber $F$ if $G$ acts by a continuous left action on $F$ and a right action on $E$. By defining

$$E \times_G F = (E \times F)/\sim,$$

where the relations are $(xg, g^{-1}y) \sim (x, y)$, with $(x, y)$ in $E \times F$ and $g$ in $G$. The fiber map $\pi' : E \times_G F \to B$, with $\pi'(x, y) = \pi(x)$ gives a fiber bundle $E \times_G F \to B$ with fiber $F$. This bundle is the associated fiber bundle and will be denoted $E'$.

### 2.1 The Pullback-Bundle

We now move on to the for us very important concept of a pullback bundle. One is often equipped with the situation of having the diagram

$$\begin{array}{ccc}
E & \to & Y \\ \
\downarrow{\pi} & & \downarrow{f} \\ 
& & B,
\end{array}$$

where $E$ is a bundle over $B$. From this it turns out that there is a natural way of constructing a bundle over $Y$. Define $E_f$ with the subspace topology of $Y \times E$ as

$$E_f = \{(y, e) \in Y \times E \mid f(y) = \pi(e)\}.$$
It turns out that \( E_f \to Y \) with the projection map is actually a fiber bundle and furthermore, inherits the same fiber as the fiber bundle \( E \to B \). This is something that will be essential for us in the computation of \( H^\bullet(PU(n); \mathbb{Z}_p) \).

### 2.2 Classifying Spaces

A fundamental result in the theory of fiber bundles is that for each topological group \( G \) there exists a fiber bundle with a weakly contractible total space denoted \( EG \) with a free action of \( G \) on it and a base space \( BG \). The space \( BG \) is the classifying space. This result is crucial for us. For example we can then construct the fiber bundle

\[
G \to EG \to BG
\]

and apply Serre’s spectral sequence and try to work our way backwards to obtain results on either the cohomology of the base space knowing the cohomology of the fiber or the other way around.

In [12], Milnor gives us a way of constructing a classifying space with the help of an operation called join. Join is an operation of two topological spaces \( X \) and \( Y \) such that

\[
X * Y = (X \times Y \times I) / \sim
\]

where the relations are given by \( (x, y_1, 0) \sim (x, y_2, 0) \) and \( (x_1, y, 1) \sim (x_2, y, 1) \).

The idea is that with more and more join operations we kill the lower order homotopy group. For example for \( S^1 \), \( S^1 * S^1 = S^3 \) and \( \pi_1(S^3) \cong \mathbb{Z} \) but \( \pi_1(S^3) \cong 0 \). When taking the colimit \( G * G * ... * G \) on a topological group \( G \) we obtain a weakly contractible space, this is the space \( EG \), \( BG \) is the space \( EG/G \).

To see that this is a fiber bundle, note that we have the following sequence of maps

\[
G \to G * G \to ... \to G * G * ... * G
\]

this imply that

\[
G \to EG \to BG
\]

is a fiber bundle.

To call \( BG \) a classifying space may seem odd, but we do have a special relationship between this space and the principal \( G \)-bundles which clarifies the concept a bit. It turns out that there is a one-to-one correspondence between
homotopy classes of maps from a base space $X$ to $BG$ and isomorphism classes of $G$-principal bundles over $X$.

Two principal $G$-bundles over $X$ are isomorphic if the following diagram of maps commutes

\[
\begin{array}{ccc}
E_1 & \longrightarrow & E_2 \\
p_1 & \downarrow & \downarrow p_1 \\
X & \longrightarrow & X
\end{array}
\]

and the composition $\phi_2^{-1} \circ f \circ \phi_1$ is a homeomorphism.

\[
\{b\} \times G \xrightarrow{\phi_1} p_1^{-1}(b) \xrightarrow{f} p_2^{-1}(f_2(b)) \xrightarrow{\phi_2^{-1}} \{f(b)\} \times G
\]

Where $b$ is an element of $B$ and $\phi_1$ and $\phi_2$ are the charts in definition 6.

**Theorem 9.** If $G$ is a topological group, $BG$ its classifying space and $X$ a topological space, then there is a bijection between the homotopy classes of maps between $X$ and $BG$ and isomorphism classes of bundles

\[
[X, BG] \longleftrightarrow \text{Prin}_G(X)/\sim
\]

*Proof.* We will not go into the details of the full proof, the full proof can be found in Steenrod[20].

Further on, in chapter 3, we will work with the projective unitary group. This is the Lie group $U(n)$ quotient its center, see chapter 3. The following theorem will then be essential for us.

**Theorem 10.** For any Lie group $G$ and $H$ a closed subgroup of $G$, each one of three consecutive spaces in the following sequence is a fiber bundle

\[
H \rightarrow G \rightarrow G/H \rightarrow BH \rightarrow BG
\]

and if $H$ is a normal subgroup

\[
BH \rightarrow BG \rightarrow B(G/H)
\]

is a fiber bundle.

*Proof.* The first one is obviously a principal $H$-bundle. For $G/H \rightarrow BH \rightarrow BG$, we know that $G \rightarrow EG \rightarrow EG/G$ is a principal $G$-bundle but then $G/H \rightarrow EG/H \rightarrow EG/G$ is a fiber bundle. With $EG/H = BH$ we get the desired result.

For the remaining fiber bundles, see Mimura[13].
2.3 Chern Classes

Much of the result in chapter 3 is based on the notion of Chern classes. A Chern class is something associated to a complex vector bundle that characterizes the bundle. If two bundles do not share the same Chern class we can be certain that they are not equivalent. Chern introduced this concept in his paper [3]. We should also mention that for real vector bundles, there is an analogous concept for the mod 2 cohomology called Stiefel-Whitney classes. Both Chern classes and Stiefel-Whitney classes are examples of characteristic classes of vector bundles, Milnor [11] is a classic text on this subject.

In order to define Chern classes, we need \( H^*(BU(n); R) \), recall that we calculated \( H^*(U(n); R) \) in 2.1.3. There is actually a way to construct \( BU(n) \) as the limit of Grassman manifold and obtain the interpretation of \( BU(n) \) as being the set of all \( n \)-dimensional subspaces of \( \mathbb{C}^\infty \), we will use this interpretation in chapter 3.

**Lemma 4.** \( H^*(BU(n); R) \cong R[c_1, c_2, \ldots, c_n] \), where \( \deg(c_i) = 2i \).

**Proof.** The proof idea is inspired of a proof of something slightly more general in [6]. We will prove that \( H^*(BU(n)) \cong R[c_1, c_2, \ldots, c_n] \) where \( \deg(c_i) = 2i \).

First we note that this is true for the base case \( n = 1 \), \( BU(1) \) is the complex projective space for which the statement is true.

A suitable fiber bundle for this is the following bundle

\[
U(n + 1)/U(n) \to BU(n) \to BU(n + 1),
\]

note that \( U(n + 1)/U(n) \cong S^{2n+1} \) so we can rewrite the fiber bundle above as

\[
S^{2n+1} \to BU(n) \to BU(n + 1),
\]

Applying Serre’s spectral sequence to this yields
in the above figure \( n = 4 \) and only the generators and not the multiplies of the generators on the horizontal axis are viewed. From the diagram we see that \( H^*(BU(n+1); R) \) at least must have the same polynomial generators \( c_1, c_2, \ldots, c_n \) as \( H^*(BU(n); R) \). We also note that we must have a polynomial generator \( z \) in \( H^{2n+2}(BU(n); R) \), which is not a multiple of the previous generators in order for the spectral sequence to converge.

We can easily see that we can not have any element in \( H^i(BU(n+1); R) \) when \( i \) is odd, the spectral sequence would not converge to \( H^*(BU(n); R) \) if that was the case. This determines all the differentials on page 2\( n + 2 \). Replacing \( z \) with \( c_{n+1} \) gives the desired result.

From this result, we can easily observe how each generator of \( H^*(U(n); R) \) transgresses in the following fiber bundle

\[
U(n) \to EU(n) \to BU(n).
\]

The Serre spectral sequence gives, with the multiples of the generators omitted

On \( E_2^{i\ast} \), the generator \( x_1 \) in degree one for \( H^*(U(n); R) \) has to disappear since the total space is contractible. This will imply some patterns and every element in the columns with no \( x_1 \)-factor will disappear. We quickly realize that this pattern has to continue, i.e. \( x_{2i-1} \) kills \( c_i \) and every element in the columns with no \( x_{2i-1} \) disappears. In other words, \( x_{2i-1} \) transgresses to \( c_i \). This is something that we will use in our derivation of \( H^*(PU(n); \mathbb{Z}_p) \).

After computing \( H^*(BU(n); R) \) and having observed the transgressions of the principal \( U(n) \)-bundle over \( BU(n) \), we are ready for the actual definition of a Chern class.

**Definition 10.** The \( i \)-th universal Chern class is the \( 2i \)-th generator of \( H^*(BU(n)) \) for \( i \geq 1 \), \( c_0 = 1 \) and \( c_i = 0 \) for \( i > n \).
CHAPTER 2. FIBER BUNDLES

Definition 11. The $i$-th Chern class of a complex vector bundle $E$ will be denoted $c_i(E)$ and it is the pullback of the generator $c_i$ in $H^*(BU(n))$.

For each Chern class, we have the following relation

$$c_i(f^* E) = f^* c_i(E).$$

where $f$ is the classifying map giving rise to a pullback bundle.

Taking the sum of all Chern classes gives rise to what we call the total Chern class, we will denote this as just $C(E)$. We state this as a definition.

Definition 12. The total Chern class $c(E)$ of a vector bundle $E$ is the sum of all its Chern classes, i.e

$$c(E) = c_0(E) + c_1(E) + c_2(E) + ...$$

For the total Chern class, similar to the previous relation for each Chern class $c_i$, we have the following relation

$$c(f^* E) = f^* c(E)$$

where $f$ is a classifying map giving rise to a pullback bundle.

For any complex line bundle $E$, $c_i(E) = 0$ for $i \geq 2$ for dimensional reasons.

In chapter 3, we want to use this fact about line bundles and from them build other vector bundles, this can be done via a Whitney sum.

If $E_1$ and $E_2$ are vector bundles over $B$ then we can create a vector bundle $E_1 \oplus E_2$ over $B$ with the fibers of each $x$ in $B$ being the direct sum $E_1^x \oplus E_2^x$, where $E_i^x$ is the fiber of $x$ in the fiber bundle $E_i$ over $B$. When having a total space being the direct sum of such spaces $E_1$ and $E_2$, we can make use of the Whitney product Lemma.

Lemma 5. If $E_1$ and $E_2$ are bundles over $B$, then for the bundle $E_1 \oplus E_2$ over $B$,

$$c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2).$$

In the next chapter, we will build fiber bundles from simpler line bundles and then use the previous lemma as a powerful tool to compute the resulting total Chern class.
Chapter 3

The mod p Cohomology of the Projective Unitary Group

In this chapter, we compute $H^*(PU(n); \mathbb{Z}_p)$ where $p$ is a prime and $PU(n)$ is the projective unitary group, i.e. $U(n)$ quotient by its center. We will do this using Chern classes, properties of the pullback for fiber bundles and spectral sequences. The main reference for this chapter has been an article by Browder[1], here we fill in the details that we think are necessary.

3.1 Preview

One can through some basic linear algebra arguments show that the center of $U(n)$ is exactly all the scalar matrices of $U(n)$ and in this case the subspace is homeomorphic to the circle $S^1$.

Maybe the most obvious candidate for a fiber bundle to apply Serre spectral sequence to would be

$$U(n) \to PU(n) \to BS^1,$$

where we actually know both the cohomology of the fiber and the base space. Note that the base space of the circle is just $CP^\infty$. However, this is not enough to directly determine the differentials on $E_2^{*,*}$. We will come back to this fiber bundle in later section of this chapter.

$SU(n)$ is the subspace of all matrices in $U(n)$ with determinant 1. It can be shown that $PU(n)$ is homeomorphic to $PSU(n)$. This can be shown with the
second isomorphism theorem for groups, consider $SU(n)$ as a subgroup of $U(n)$ and let $N$ be the center then

$$(SU(n)N)/N \cong SU(n)/(SU(n) \cap N)$$

which is equivalent to

$$PU(n) \cong PSU(n).$$

### 3.2 Determining the Total Chern Class

We start with the following maps of topological groups

$$S^1 \xrightarrow{\Delta_n} (S^1)^n \xrightarrow{i} U(n),$$

where $(S^1)^n$ is the subspace of $U(n)$ consisting of all diagonal matrices, $\Delta_n(\lambda) = \lambda I$, $\lambda \in S^1$, and $i$ is the inclusion map. One should have in mind that $S^1$ is the center of $U(n)$ and that we are now doing an inclusion into $U(n)$.

The construction of maps above give rise to the following maps of the corresponding classifying spaces,

$$\mathbb{C}P^\infty \xrightarrow{\Delta'_n} (\mathbb{C}P^\infty)^n \xrightarrow{i'} BU(n).$$

The composition $i'\Delta'_n$ is the induced inclusion map and induces the associated bundle $PU(n)'$ over $\mathbb{C}P^\infty$, see [14]. Note that $PU(n)'$ has the same homotopy type as $PU(n)$.

If we let $\xi_1$ be the canonical line bundle over $\mathbb{C}P^\infty$ and $p_i : (\mathbb{C}P^\infty)^n \to \mathbb{C}P^\infty$ be the projection on the $i$-th component we have the pullback bundle $p_i^*(\xi_1)$ over $(\mathbb{C}P^\infty)^n$ and the following diagram commutes

$$p_i^*(\xi_1) \quad \xrightarrow{p_i^*} \quad \xi_1$$

and

$$(\mathbb{C}P^\infty)^n \quad \xrightarrow{p_i} \quad \mathbb{C}P^\infty.$$

Then by doing a Whitney sum of the bundles, we get the following fiber bundle

$$p_i^*(\xi_1) \oplus \ldots \oplus p_n^*(\xi_1) \to (\mathbb{C}P^\infty)^n.$$

This is exactly the bundle that we obtain from the pullback of the tautological $n$-plane bundle $\eta$ over $BU(n)$. We can now once again take the pullback of the above bundle and obtain:
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\[ \Delta_n^*(p_1^*(\xi_1) \oplus \ldots \oplus p_n^*(\xi_1)) \longrightarrow p_1^*(\xi_1) \oplus \ldots \oplus p_n^*(\xi_1) \longrightarrow \eta \]

\[ \mathbb{C}P^{\infty} \xrightarrow{\Delta_n^*} (\mathbb{C}P^{\infty})^n \xrightarrow{i'} BU(n) \]

We have now obtained a fiber bundle which is a pullback of the tautological n-plane bundle over \( BU(n) \), this seems promising for determining the chern classes of the bundle.

A nice property of line bundles is that the total Chern class has no higher order than 1, this means that \( c(p_1^*(\xi_1)) = 1 + \alpha_i \). The fiber bundle \( p_1^*(\xi_1) \oplus \ldots \oplus p_n^*(\xi_1) \) can now be determined with the use of the Whitney product theorem to the product of the total Chern class of each component, i.e

\[
c(p_1^*(\xi_1) \oplus \ldots \oplus p_n^*(\xi_1)) = \prod_{i=1}^{n} (1 + \alpha_i).
\]

Our main concern is the total Chern class of \( \Delta_n^*(p_1^*(\xi_1) \oplus \ldots \oplus p_n^*) \). Using the relation

\[
c(\Delta_n^*(p_1^*(\xi_1) \oplus \ldots \oplus p_n^*)) = \Delta_n^* c(p_1^*(\xi_1) \oplus \ldots \oplus p_n^*(\xi_1)),
\]

together with

\[
\Delta_n^* c(p_1^*(\xi_1) \oplus \ldots \oplus p_n^*(\xi_1)) = \prod_{i=1}^{n} \Delta_n^*((1 + \alpha_i)),
\]

we see that the total chern class of \( \Delta_n^*(p_1^*(\xi_1) \oplus \ldots p_n^*) \) is determined by \( \Delta_n^*(\alpha_i) \).

Luckily, this can obtained, our maps

\[ \mathbb{C}P^{\infty} \xrightarrow{\Delta_n} ((\mathbb{C}P^{\infty})^n \xrightarrow{p_1} \mathbb{C}P^{\infty} \]

with the composition \( p_i \circ \Delta_n = Id \), induces the maps

\[ H^*(\mathbb{C}P^{\infty}) \xrightarrow{p_i^*} H^*((\mathbb{C}P^{\infty})^n) \xrightarrow{\Delta_n^*} H^*(\mathbb{C}P^{\infty}) \]

with \( \Delta_n^* \circ p_i^* = Id \). Therefore,

\[
c(\Delta_n^* \circ p_i^*(\xi_1)) = \Delta_n^* \circ p_i^*(c(\xi_1)) = c(\xi_1)
\]

for \( 1 \leq i \leq n \). This implies we can set all \( \Delta_n^*(\alpha_i) \) to just \( \alpha \) and finally obtain

\[
c(\Delta_n^*(p_1^*(\xi_1) \oplus \ldots p_n^*)) = (1 + \alpha)^n = \sum_{i=0}^{n} \binom{n}{i} \alpha^i.
\]
3.3 Determining the Transgressions

Recall from chapter 1 that the transgressions are the "last chance" differentials in the Serre spectral sequence. In the last section, we obtained the total Chern class of the pullback bundle with classifying map $\Delta_n \circ \iota'$, we are now ready to determine the transgressions of this bundle.

From McCleary[10], we have the following commuting diagram for the fiber bundle $EU(n) \to BU(n)$. This diagram is constructed from the pairs $(BU(n), \ast)$ and $(EU(n), U(n))$. The cohomology is taken with coefficients in $R$.

$$
\begin{array}{ccccccc}
\text{H}^{q-1}(\ast) & \to & \text{H}^{q}(BU(n), \ast) & \to & \text{H}^{q}(BU(n)) & \to & \text{H}^{q}(\ast) \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{H}^{q-1}(U(n)) & \to & \text{H}^{q}(EU(n), U(n)) & \to & \text{H}^{q}(EU(n)) & \to & \text{H}^{q}(U(n)) \\
\downarrow \tau_1 & & \downarrow & & \downarrow & & \\
\text{H}^{q-1}(U(n)) & \to & \text{H}^{q}(PU(n)', U(n)) & \to & \text{H}^{q}(PU(n)') & \to & \text{H}^{q}(U(n))
\end{array}
$$

where $\tau_1$ is the transgression. Similarly, we also have a commuting diagram for the fiber bundle $PU(n)' \to \mathbb{C}P^\infty$

$$
\begin{array}{ccccccc}
\text{H}^{q-1}(\ast) & \to & \text{H}^{q}(\mathbb{C}P^\infty, \ast) & \to & \text{H}^{q}(\mathbb{C}P^\infty) & \to & \text{H}^{q}(\ast) \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{H}^{q-1}(U(n)) & \to & \text{H}^{q}(PU(n)', U(n)) & \to & \text{H}^{q}(PU(n)') & \to & \text{H}^{q}(U(n)) \\
\downarrow \tau_2 & & \downarrow & & \downarrow & & \\
\text{H}^{q-1}(U(n)) & \to & \text{H}^{q}(PU(n)', U(n)) & \to & \text{H}^{q}(PU(n)') & \to & \text{H}^{q}(U(n))
\end{array}
$$

By naturality, we can combine the above diagrams into a diagram which commutes

$$
\begin{array}{ccccccc}
\text{H}^{q-1}(\ast) & \to & \text{H}^{q}(BU(n), \ast) & \to & \text{H}^{q}(BU(n)) & \to & \text{H}^{q}(\ast) \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{H}^{q-1}(U(n)) & \to & \text{H}^{q}(EU(n), U(n)) & \to & \text{H}^{q}(EU(n)) & \to & \text{H}^{q}(U(n)) \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{H}^{q-1}(U(n)) & \to & \text{H}^{q}(PU(n)', U(n)) & \to & \text{H}^{q}(PU(n)') & \to & \text{H}^{q}(U(n))
\end{array}
$$

Since the diagram commutes, we obtain the transgressions in the very front diagram by taking the composition of the first transgression together with the map from $H^*(BU(n))$ to $H^*(\mathbb{C}P^\infty)$. Therefore, a generator $z_i$ for $H^{2i-1}(U(n))$ transgresses to $\binom{n}{i}\alpha^i$, where $\alpha^i$ is a generator for $H^{2i}(\mathbb{C}P^\infty(n))$.

3.4 Main Results

If we now turn to the case where the cohomology is taken with coefficients in $\mathbb{Z}_p$, where $p$ is a prime we obtain some very interesting results. It turns out[1], that
the binomial coefficient \( \binom{n}{i} \) is divisible by \( p \) if \( n = n'p^r \) except if \( i = p^r \). This means that all the transgressions except the transgression from \( x_{p^r} \) to \( y^{p^r} \) is zero. Here \( x_i \) is a generator for \( H^*(U(n)) \) in degree \( 2i-1 \) and \( y \) is the generator for \( H^*(\mathbb{C}P^\infty) \) in degree 2. Summarizing this, we get the following result.

**Theorem 11.** If \( n = n'p^r \), then

\[
H^*(PU(n); \mathbb{Z}_p) \cong \wedge(x_1, x_2, ..., x_n) \otimes \mathbb{Z}_p[y]/(y^{p^r})
\]

where the generator \( x_{p^r} \) is omitted, \( \deg(x_i) = 2i - 1 \) and \( \deg(y) = 2 \).

In addition to the computation of \( H^*(PU(n); \mathbb{Z}_p) \), the coalgebra structure of \( H^*(PU(n); \mathbb{Z}_p) \) is also determined in [1]. The diagonal map \( \phi \) is given by

\[
\phi(x_i) = \sum_{j=1}^{i-1} \lambda_{ij}(x_{i-j} \otimes y^{j}), \quad i \geq 2, \lambda_{ij} \in \mathbb{Z}_p
\]

and the counit \( \epsilon \) is given by

\[
\epsilon(x_i) = \epsilon(y) = 0 \quad \forall i \geq 1
\]

\[
\epsilon(\lambda) = \lambda, \quad \lambda \in \mathbb{Z}_p.
\]

For the case when \( p \) does not divide \( n \), we have can make use of the following fiber bundle

\[
SU(n) \to PSU(n) \to BC_n,
\]

where \( C_n \) is the cyclic group. We can actually show that \( \hat{H}^*(BC_n; \mathbb{Z}_p) = 0 \) by considering the fiber bundle

\[
S^1 = B\mathbb{Z} \xrightarrow{n} B\mathbb{Z} \to BC_n
\]

induced by

\[
0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to C_n \to 0
\]

In the mod \( p \) homology we get that \( n^* \) is an isomorphism, i.e

\[
H_\ast(S^1; \mathbb{Z}_p) \cong H_\ast(S^1; \mathbb{Z}_p)
\]

and therefore \( \hat{H}_\ast(BC_n; \mathbb{Z}_p) = 0 \) and dualizing this gives \( \hat{H}^*(BC_n; \mathbb{Z}_p) = 0 \).

Going back to the bundle

\[
SU(n) \to PSU(n) \to BC_n,
\]

we see that after applying Serre’s spectral sequence, we will get that \( H^*(PSU(n); \mathbb{Z}_p) \cong H^*(SU(n); \mathbb{Z}_p) \). Note that \( H^*(SU(n); \mathbb{Z}_p) \) is well-known to be the same \( H^*(U(n); \mathbb{Z}_p) \) but with \( c_1 \) omitted. We state this as our second main result.
CHAPTER 3. THE MOD P COHOMOLOGY OF THE PROJECTIVE UNITARY GROUP

Theorem 12. If $p$ does not divide $n$, then

$$H^*(PU(n); \mathbb{Z}_p) \cong H^*(SU(n); \mathbb{Z}_p).$$

We can relate this result to the fiber bundle

$$U(n) \to PU(n) \to \mathbb{C}P^\infty.$$ 

If we apply the Serre spectral sequence we get the second page as

```
\begin{center}
\begin{array}{cccccccccccc}
10 & & & & & & & & & & & \\
9 & & & & & & & & & & & \\
8 & & & & & & & & & & & \\
7 & & & & & & & & & & & \\
6 & & & & & & & & & & & \\
5 & & & & & & & & & & & \\
4 & & & & & & & & & & & \\
3 & & & & & & & & & & & \\
2 & & & & & & & & & & & \\
1 & & & & & & & & & & & \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\end{center}
```

In the above diagram, $n = 3$. The result in theorem 12 tells us that we have to have a differential from $x_1$ to $y$ (if $p$ does not divide $n$), but we could not have drawn this conclusion about that differential by just looking at the second page of Serre spectral sequence applied to the fiber bundle $U(n) \to PU(n) \to BU(n)$. With this fact about the differential on the second page, we can see that all factors with a $x_1$ or $y$ factor disappear. The only thing we have left are the generators $x_3, x_5, ..., x_{2n-1}$, which indeed the theorem tells us. This illustrate the fact that combining spectral sequence is often a good idea.

3.5 Where to go next

After computing the main results in this chapter, one may wonder, where to go next. In chapter 4, we go on to further investigate $H^*(BPU(3); \mathbb{Z}_3)$. It is also possible to view the projective unitary group as a special case of the complex Stiefel Manifold. Recall that $V_{n-1}(\mathbb{C}^n) = U(n)/U(1)$ and therefore $V_{n-1}(\mathbb{C}^n) \cong PU(n)$. In [17], Ruiz generalizes the result in this chapter and compute the mod $p$ cohomology of the complex Stiefel manifold.
Chapter 4

The Classifying Space of the Projective Unitary Group

In chapter 3, we computed $H^*(PU(n); \mathbb{Z}_p)$, we now move on to investigate the classifying space of $U(n)$, $BU(n)$, using the results from chapter 3. We present two results in this chapter, the first result is for when the prime $p$ does not divide $n$ and the second result when $n = 3$ and $p = 3$.

4.1 If $p$ does not divide $n$

Recall from chapter 3 that $PU(n)$ and $PSU(n)$ are homeomorphic, so we can also try to compute $H^*(BPSU(n); \mathbb{Z}_p)$. To compute this, we notice that the fibration

$$BC_n \rightarrow BSU(n) \rightarrow BPSU(n)$$

which is induced by

$$C_n \rightarrow SU(n) \rightarrow PSU(n)$$

looks very suitable for our purpose. This realization may not be immediate, but we showed in chapter 3 that $\tilde{H}^*(BC_n; \mathbb{Z}_p) = 0$. Therefore, after applying Serre’s spectral sequence to the above fibration of the classifying spaces we obtain the following theorem.

**Theorem 13.** If $p$ does not divide $n$, then

$$H^*(BPU(n); \mathbb{Z}_p) \cong H^*(BSU(n); \mathbb{Z}_p).$$
4.2 \( \text{mod 3 Cohomology of BPU}(3) \)

The general expression for \( H^* (\text{BPU}(n); \mathbb{Z}_p) \) is actually not yet known and computing it has proven to be a difficult task and involves not just the application of the Serre spectral sequence, which has been the main focus in this thesis. In this section, we will restrict ourselves to compute \( H^* (\text{BPU}(3); \mathbb{Z}_3) \) through the use of both the Serre spectral sequence and the Eilenberg-Moore spectral sequence.

In [9], the computation of \( \text{CoTor}^{H^*(\text{PU}(3))} (\mathbb{Z}_3, \mathbb{Z}_3) \) is carried out through the use of the twisted tensor product[2] and the coalgebra structure of \( H^*(\text{PU}(n); \mathbb{Z}_p) \) presented in chapter 3. The result of the computations yields that

\[
\text{CoTor}^{H^*(\text{PU}(3))} (\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3[y_1,1, y_1,2, y_2,5, y_2,6, y_3,9]/I,
\]

where \( I \) is the ideal generated by \( (y_1,1 y_1,2, y_1,1 y_2,5, y_1,1 y_2,6 + y_1,2 y_2,5, y_1,2, y_2,6) \).

Recall from section 1.4 that the \( E_2 \) term in Eilenberg-Moore spectral sequence for computing \( H^*(\text{BU}(3); \mathbb{Z}_3) \) is \( \text{CoTor}^{H^*(\text{PU}(3))} (\mathbb{Z}_3, \mathbb{Z}_3) \), i.e

\[
E_2 = \mathbb{Z}_3[y_1,1, y_1,2, y_2,5, y_2,6, y_3,9]/I
\]

Due to the structure of the generators, we get the second page as

From the above diagram it is easy to see that \( y_{1,1} \) and \( y_{1,2} \) will remain and that for \( * < 7 \)

\[
H^*(\text{BPU}(3); \mathbb{Z}_3) = \mathbb{Z}_3[y_1,1, y_1,2]/(y_1,1 y_1,2).
\]

Of course, we can also apply the Serre spectral sequence to the fiber bundle

\[
\mathbb{C}P^\infty \cong BS^1 \rightarrow BU(3) \rightarrow \text{BPU}(3).
\]

This gives us
where we have omitted potential classes in degree 7 or more. For the above spectral sequence to converge we need to kill everything in degree 3, therefore x transgresses to $y_{1,2}$. We can now use the Kudo Transgression theorem to deduce that we need to have a nontrivial element $-P\beta y_{1,2}$ in degree 8 and a nontrivial element $\beta y_{1,2}$ in degree 7. Since we only have one generator in degree 7 and one in degree 8 for $\text{CoTor}^{H^*(U(3))}(\mathbb{Z}_3, \mathbb{Z}_3)$, $y_{2,5}$ and $y_{2,6}$ need to be elements of $H^*(BPU(3); \mathbb{Z}_3)$.

We draw this in the following diagram

We have now concluded that for $* < 12$ we must have

$$H^*(BPU(3); \mathbb{Z}_3) = \mathbb{Z}_3[y_{1,1}, y_{1,2}, y_{2,5}, y_{2,6}, y_{3,9}] / (y_2y_3, y_{1,1}y_{2,5}, y_{1,1}y_{2,6} + y_{1,2}y_{2,5}, y_{1,2}^2, y_{2,5}^2)$$

Since there is no element in degree 9 of $\text{CoTor}^{H^*(U(3))}(\mathbb{Z}_3, \mathbb{Z}_3)$, $y_{2,6}$ must be a permanent cycle. The same argument goes for $y_{3,9}$, there is no element in degree 13. This proves the following theorem.

**Theorem 14.**

$$H^*(BPU(3); \mathbb{Z}_3) \cong \mathbb{Z}_3[y_{1,1}, y_{1,2}, y_{2,5}, y_{2,6}, y_{3,9}] / I,$$
where I is the ideal generated by \((y_2y_3, y_1, y_2, y_1^2y_2, y_1^2y_2^2, y_1^2y_2^3, y_1^2y_2^4)\).
Bibliography


