Nonuniform Bandpass Sampling in Radio Receivers

Yi-Ran Sun

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Abstract

As an interface between radio receiver front-ends and digital signal processing blocks, sampling devices play a dominant role in digital radio communications. Based on different sampling theorems (e.g., classic Shannon’s sampling theorem, Papoulis’ Generalized sampling theorem, bandpass sampling theory), signals are processed by the sampling devices and then undergo additional processing. It is a natural goal to obtain the signals at the output of the sampling devices without loss of information.

In conventional radio receivers, all the down-conversion and channel selection are realized in analog hardware. The associated sampling devices in A/D converters are based on the classic Shannon’s sampling theorem. Driven by the increased speed of microprocessors, there is a tendency to use mixed-signal/digital hardware and software to realize more functions (e.g., down-conversion, channel selection, demodulation and detection) in a radio communication system. The new evolution of radio receiver architecture is Software Defined Radio (SDR). One design goal of SDR is to put the A/D converter as close as possible to the antenna. BandPass Sampling (BPS) enables one to have an interface between the higher IF and the A/D converter by a sampling rate of $2B$ or more ($B$ is the information bandwidth), and it might be a solution to SDR.

A signal can be uniquely determined from the samples by NonUniform Sampling (NUS) such that NUS has the potential to suppress harmful signal spectrum aliasing. BPS makes use of the signal spectrum aliasing to represent the signal uniquely at any band position. A harmful aliasing of signal spectrum will cause a performance degradation. It is of great benefit to use NUS scheme in BPS system. However, a signal cannot be recovered from its nonuniform samples by using only an ideal lowpass filter (or the classic Shannon’s reconstruction function). The reconstruction of the samples by NUS is crucial for the implementation of NUS. Besides the harmful signal spectrum aliasing, noise aliasing and timing jitter are other two sources of performance degradation in a BPS system. Noise aliasing is the direct consequence of lower sampling rate of subsampling. With the increase of input frequency by directly sampling a signal at higher IF, the timing error of the sampling clock causes large jitter effects on the sampled-data signal.

In this thesis work, first, a filter generalized by a certain Reconstruction Algorithm (RA) is proposed to reconstruct the signal from its nonuniform samples. A
general reconstruction formula in terms of a basis-kernel (BK) is used to describe the algorithm. The corresponding reconstruction performance, computational complexity and implementation of these RAs are discussed. Second, three sources of performance degradation in a BPS system, harmful signal spectrum aliasing, noise aliasing and timing jitter, are studied. In the light of noise aliasing, a Generalized Quadrature BPS (GQBPS) algorithm is proposed to suppress the noise aliasing. Theoretical analyses show that GQBPS might be a potential way to reduce the noise aliasing at the cost of a more complicated reconstruction algorithm, although it is sensitive to large timing jitter. Then, aliasing-free sampling by NUS is studied in theory and verified by simulations. Thermal noise and timing errors are always present in real circuit implementations. Finally, the performance of additive noise and jitter on RAs in BPS is evaluated and discussed.
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List of Abbreviations

2/2.5/3 G the second/second and half/third Generation
AA Anti-Aliasing
A/D Analog-to-Digital
ARS Additive Random Sampling
AWGN Additive White Gaussian Noise
BER Bit Error Rate
BK Basis-Kernel
BPF BandPass Filter
BPS BandPass Sampling
BS cellular system Base Station
CDMA Code Division Multiple Access
CMOS Complementary Metal Oxide Semiconductor
CSMA-CA Carrier Sense Multiple Access/Collision Avoidance
CT Continuous-Time
CT-scan Computerized Tomography-scan
D/A Digital-to-Analog
DC Direct Current
DCF Density Compensation Factor
DCS Digital Cellular System
DECT Digital European Cordless Telephone
DL Down-Link
DPD Digital Product Detector
DT Discrete-Time
DTFT Discrete-Time Fourier Transform
DFT Discrete Fourier Transform
DSB Double-SideBand
DSSS Direct Sequence Spread Spectrum
EDS Energy Density Spectrum
EDGE Enhanced Data rates for Global/GSM Evolution
FHSS Frequency-Hopping Spread Spectrum
FIR Finite Impulse Response
FPO Floating Point Operation
GSM Global System for Mobile communications
GQBPS: Generalized Quadrature BandPass Sampling
iid: independent, identically distributed
I/Q: In-phase/Quadrature
IEEE: Institute of Electrical and Electronics Engineers
IF: Intermediate Frequency
IIR: Infinite Impulse Response
IRF: Image-Rejection Filter
ISI: Inter-Sample Interval
ISM: Industrial, Scientific and Medical
JS: Jitter Sampling
LO: Local Oscillator
LPF: LowPass Filter
LPS: LowPass Sampling
LSR: Least Square Reconstruction
LTI: Linear Time-Invariant
MRI: Magnetic Resonance Imaging
NB: Narrow Band
NUS: NonUniform Sampling
OFDM: Orthogonal Frequency Division Multiplexing
PDC: Personal Digital Cellular
PSD: Power Spectral Density
RF: Radio Frequency
RA: Reconstruction Algorithm
SC: Switched-Capacitor
SDR: Software Defined Radio
S/H: Sample-and-Hold
SNDR: Signal-to-Noise-and-Distortion Ratio
SNR: Signal-to-Noise Ratio
SSB: Single-SideBand
SVD: Singular-Value Decomposition
TDMA: Time Division Multiple Access
UE: User Equipment for cellular terminal
UL: Up-Link
UMTC: Universal Mobile Telecommunication System
US: Uniform Sampling
WB: Wide Band
W-CDMA: Wideband Code-Division Multiple-Access
WSS: Wide-Sense Stationary
List of Notations

\( B \) The bandwidth of lowpass information signal
\( f_c \) The carrier frequency
\( f_{in} \) The frequency of input signal
\( \delta(t) \) The Dirac delta function
\( \delta[m - n] \) The Kronecker delta function
\( f_s, F_s \) The sampling rate
\( T_s \) The sampling interval and \( T_s = 1/f_s \)
\( k(t, t_n) \) The Basis Kernel (BK) of Reconstruction Algorithm (RA)
\( \text{Re}\{\cdot\} \) The real part of complex signal
\( \langle a, b \rangle \) The inner product operator
\( \lfloor \cdot \rfloor \) The floor operator
\( \hat{x}(t) \) The reconstructed result of \( x(t) \)
\( \ast \) The complex conjugate operator
\( \ast \) The convolution operator
\( \in \) An element of
\( \sigma_\tau \) The standard deviation of sampling jitter
\( E[\cdot] \) The expectation operator
\( \text{Mean}[\cdot] \) The mean value
\( \lim_{x \to a} f(x) \) The limit of function \( f(x) \)
\( \text{rect}(\cdot) \) The rectangular function
\( p(\tau) \) Probability Density Function (PDF)
\( p(\tau_n, \tau_m) \) Joint PDF
\( \otimes \) The ideal sampling operator, i.e., the process of multiplying
\( B_{eff} \) The effective bandwidth of noise
\( x(t_n) \) The sampled-data of NUS
\( x(nT_s) \) The sampled-data of US
\( x(t_s(n)) \) The sampled-data of random sampling
\( r_{xx}(\gamma) \) The autocorrelation function of \( x(t) \)
\( R_{xx}(f) \) The Fourier transform of \( r_{xx}(\gamma) \)
\( f_0 \otimes f_c \) The notation of a signal with an information bandwidth of \( f_0 \) centered at \( f_c \)
\( f_l \) The lower frequency of bandpass signal
\( f_u \) The upper frequency of bandpass signal
\( U[a, b] \) Uniform distribution of a random variable
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<td>$\Pi$</td>
<td>The symbol of product</td>
</tr>
<tr>
<td>$\sum$</td>
<td>The symbol of sum</td>
</tr>
<tr>
<td>$\cup$</td>
<td>The symbol of union</td>
</tr>
<tr>
<td>$|\bullet|$</td>
<td>The symbol of norm</td>
</tr>
<tr>
<td>$\neq$</td>
<td>The symbol of inequality</td>
</tr>
<tr>
<td>$\approx$</td>
<td>The symbol of approximately equal</td>
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<td>$\infty$</td>
<td>The symbol of infinity</td>
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<td>$F{\bullet}$</td>
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Chapter 1

Introduction

The conventional radio receiver architecture, superheterodyne, has existed for almost one century since Edwin H. Armstrong proposed it in the 1910s. Many variations were proposed afterwards based on the theme, such as single-IF and dual-IF receivers [1]. Superheterodyne receivers create a beat frequency defined by the difference between the output of a Local Oscillator (LO) and the input signal frequency to realize frequency down-conversion. However, the signal located at the “image band” which is the mirror to the information band with respect to the output of the LO will also be inevitably present at the beat frequency. This signal is called image of the expected information signal. Normally an Image-Reject Filter (IRF) is used to suppress the image prior to the mixer. In order to more efficiently suppress the image, a special receiver family called image-reject receiver was developed based on the superheterodyne. Two typical architectures of an image-reject receiver are the Hartley architecture and the Weaver architecture [1]. The other way to suppress the image is to directly down-convert the RF spectrum to baseband without IF. The corresponding receiver architecture is called homodyne, “zero-IF” or “direct-conversion” receiver [1].

In general, a single narrow channel of the RF signal is translated to baseband before the digitization in an A/D converter. Oversampling is normally used to reduce the requirements on the dynamic range of the A/D converter. By moving the A/D converter to the IF, a signal is digitized at IF, and the demodulation and the detection are realized in the digital domain. The corresponding receiver is called digital-IF receiver. For these two cases, the sampling in the A/D converter is a LowPass Sampling (LPS) based on the Shannon’s sampling theorem. It is known that the frequency translation could also be realized by subsampling (or undersampling). Replacing the lowpass sampling with a BandPass Sampling (BPS), the corresponding receiver architecture is the so-called subsampling receiver. In this case, a continuous-time (CT) signal at IF will be represented in discrete-time (DT) at a lower IF or baseband.

The receiver architectures mentioned above are mostly designed for single stan-
standard, narrow band radio communications. For a specific communication standard, only a limited spectrum is allocated to each user, e.g., 200 kHz for GSM and 30 kHz for IS-54/-136 (more details are contained in Table 1.1). A single channel is normally selected prior to the sampling in the A/D converter. Alternatively, the channel can be selected as late as possible. This is introduced in a family of WideBand (WB) receivers, e.g., wide-band IF receivers and generic wide-band receivers [2] [3].

Table 1.1: Wireless Radio Communication Standards

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<th>Access Method</th>
<th>Frequency Band (MHz)</th>
<th>Carrier Spacing (MHz)</th>
<th>Data Rate (Mbps)</th>
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<td>IS-54/-136</td>
<td>UL:824-849 DL:869-894</td>
<td>TDMA</td>
<td>25</td>
<td>0.03</td>
<td>0.048</td>
</tr>
<tr>
<td>IS-95</td>
<td>UL:824-849 DL:869-894</td>
<td>CDMA</td>
<td>25</td>
<td>1.25</td>
<td>1.228</td>
</tr>
<tr>
<td>GSM</td>
<td>UL:890-915 DL:935-960</td>
<td>TDMA</td>
<td>25</td>
<td>0.2</td>
<td>0.2708</td>
</tr>
<tr>
<td>DCS 1800 (EDGE)</td>
<td>UL:1710-1785 DL:1805-1880</td>
<td>TDMA</td>
<td>75</td>
<td>0.2</td>
<td>0.2708</td>
</tr>
<tr>
<td>PCS 1900 (EDGE)</td>
<td>UL:1856-1915 DL:1930-1990</td>
<td>TDMA</td>
<td>60</td>
<td>0.2</td>
<td>0.2708</td>
</tr>
<tr>
<td>PDC</td>
<td>UL:340-956 DL:810-826</td>
<td>TDMA</td>
<td>16</td>
<td>0.025</td>
<td>0.042</td>
</tr>
<tr>
<td>DECT</td>
<td>1880-1900</td>
<td>TDMA</td>
<td>20</td>
<td>1.728</td>
<td>1.152</td>
</tr>
<tr>
<td>IEEE 802.11a</td>
<td>5180-5350</td>
<td>CSMA-CA</td>
<td>200</td>
<td>OFDM: 20</td>
<td>6-54</td>
</tr>
<tr>
<td>IEEE 802.11b</td>
<td>ISM:2400-2483.5</td>
<td>CSMA-CA</td>
<td>83.5</td>
<td>FHSS:1 DSSS:25</td>
<td>1-2</td>
</tr>
<tr>
<td>Bluetooth</td>
<td>ISM:2400-2483.5</td>
<td>TDMA</td>
<td>83.5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>DCS 1800 (W-CDMA)</td>
<td>UE:1710-1785 BS:1805-1880</td>
<td>CDMA</td>
<td>75</td>
<td>5</td>
<td>3.84</td>
</tr>
<tr>
<td>PCS 1900 (W-CDMA)</td>
<td>UE:1850-1915 BS:1930-1990</td>
<td>CDMA</td>
<td>60</td>
<td>5</td>
<td>3.84</td>
</tr>
<tr>
<td>UMTS(3G)</td>
<td>UE:1920-1980 DL:2110-2170</td>
<td>CDMA/TDMA</td>
<td>60</td>
<td>5</td>
<td>3.84</td>
</tr>
</tbody>
</table>

With the evolution of radio communications, a combination of multi-band antennas and RF conversions, wide-band A/D converters and their implementations at IF results in multi-mode multi-band radio receivers. Both Software Defined Radio (SDR) receivers and homodyne receivers support this multi-mode multi-band radio communications [4] [5]. The design key of SDR is the placement and design techniques of A/D converters whose goal is to put the A/D converter as close as possible to the antenna. The big revolution of SDR receivers compared to the conventional receivers is that SDR technology replaces many tasks in the radio receiver with digital processing, including down-conversion, filtering (mode- or band- selection), demodulating, decoding and converting to the desired data format. SDR receivers make use of digital signal processing (DSP) to implement the complex tasks in today’s communication systems, and are easy to be extended for more complex systems. By using homodyne receivers, the mode-selection can be realized.
by an external digital controller, and the hardware in the analog part is shared by different communication bands or modes as much as possible. However, the extension to a complex communication system is hard. In addition, all the associated problems of the homodyne receiver are also present.

In this chapter, traditional superheterodyne and homodyne receiver architectures are shown and compared with two WB receiver architectures: the wide-band IF receiver and the generic wide-band receiver. After that, the SDR receiver is shown and compared with the homodyne receiver with respect to multi-mode multi-band radio communications. The SDR receiver with different data acquisition technologies are also discussed. Then an outline of technical problems in SDR receivers by using the BPS technique is presented, and previous work on BPS is reviewed. Finally, the contributions in the following chapters are summarized.

1.1 Superheterodyne receivers

In the literature there is usually no distinction between heterodyne and superheterodyne architectures. To “heterodyne” means to mix two frequencies and produce a beat frequency defined by either the difference or the sum of the two. To “superheterodyne” is only to produce a beat frequency defined by the difference of the two frequencies. Two stages of down-conversion (dual-IF) based on the theme of superheterodyne is mostly used in today’s RF receivers (see Fig. 1.1). This receiver translates the signal to a low frequency band by two stages of down-conversion mixing and relaxes the requirement on the $Q$-factor of the channel-select filter. The

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/dual-IF-superheterodyne.png}
\caption{Conventional dual-IF superheterodyne receiver architecture}
\end{figure}
CHAPTER 1. INTRODUCTION

first IF might be between 70 and 250 MHz for 2G, 2.5G and 3G applications [6]. For narrow-channel standards, the second IF is often equal to 455 kHz, but for wide-channel application such as DECT, it may be several megahertz. These numbers vary a lot in present systems. If the second IF of a dual-IF receiver is equal to zero, the second down-conversion normally separates the signal to $I$ (in-phase) and $Q$ (quadrature) components for Single-SideBand (SSB) communication systems or frequency-/phase-modulated signals, and the corresponding demodulation and detection are performed at baseband. This down-conversion is realized by two LOs which have a $90^\circ$ phase shift between each other. Any offset from the nominal $90^\circ$ phase shift and the amplitude mismatches between $I$ and $Q$ components will raise the Bit Error Rate (BER). If the second IF is not equal to zero, the receiver becomes a digital-IF receiver. The IF bandpass signal is directly processed by an A/D converter, and the $I/Q$ mismatch can be avoided. After that, IF demodulation and detection are processed in the digital domain.

Both down-conversion schemes entail the image problem. The choice of two IFs faces the trade-off between the image rejection (or sensitivity) and channel-selection (or selectivity). If the IF is high, the image band appears far way from the information band such that the image can be easily suppressed by an IRF. However, the channel selection filter will require a high $Q$-factor to select a narrow channel at a high IF. On the contrary, if the IF is low, the design of the channel selection filter becomes easier but the image band is so close to the information band that it becomes difficult to achieve a proper image suppression by a BandPass Filter (BPF). More than one stage of down-conversion makes the trade-off easily achieved. In a dual-IF superheterodyne receiver, the first IF is selected high enough to efficiently suppress the image, and the second IF is selected low enough to relax the requirement on the channel selection filter. The selectivity and sensitivity of the superheterodyne makes it a dominant choice in RF receiver architectures. Unfortunately, the high $Q$-factors of the discrete-components in the superheterodyne receiver make it difficult to fully integrate the whole front-end on a single chip.

1.2 Homodyne receivers

In a homodyne receiver, no IF stage exists between RF and baseband. The input of the A/D converter is located at baseband (see Fig. 1.2). The channel selection filter is just a lowpass filter prior to the A/D converter. The homodyne receiver has two advantages compared to superheterodyne receiver. First, the architecture is simpler. Second, the image problem can be avoided due to zero IF (i.e., $f_{IF} = 0$) such that no IRF is needed.

The homodyne receiver allows a higher level of integration than the superheterodyne receiver as the number of discrete components are reduced. However, this receiver inevitably suffers from the problems of LO leakage and DC-offset. The output of the LO may leak to the input of the mixer or the LNA due to improper isolation. The leaked signal will be mixed with the output of the LO (i.e., the origin
1.3 Wideband IF Receivers

A superheterodyne receiver with an RF channel-select frequency synthesizer and an IF or baseband channel-select filter is a NarrowBand (NB) receiver. An alternative architecture, called Wide Band (WB) IF receiver (see Fig. 1.3), postpones the channel-select frequency synthesizer to IF and channel-select filter to baseband [2].
CHAPTER 1. INTRODUCTION

1.3 Wideband IF Receiver Architecture with Double Down-Conversion

Figure 1.3: Wideband IF receiver architecture with double down-conversion

The entire frequency band of information signal located at RF is translated to IF by multiplying the output of the LO with a fixed frequency. The IF signal passes through a LPF such that the frequency components above the IF band are removed. One channel out of the entire band is first translated to DC by a tunable LO and then fed into a LPF. The selected lowpass channel signal is processed further by an A/D converter which is the same as in the superheterodyne and homodyne receivers. Compared to the traditional superheterodyne receiver, this receiver architecture is well-suited for full integration, and it has also the potential to be implemented for multi-band multi-mode radio communications.

1.4 Generic Wideband Receivers

It is advantageous to use conventional receiver architectures for single-mode NB radio communications since the technologies are mature and it is also easy to fulfill the system performance requirements. Nevertheless, driven by the increased speed of microprocessors and the high performance of A/D converters, a WB radio architecture has drawn more and more attention for the support of multi-band multi-mode radio communications. From a general aspect, the wideband IF receiver mentioned in section 1.3 is not a real WB receiver since the input of the A/D converter is still NB. A generic wideband receiver was depicted in [3]. In this architecture, the whole band centered at a RF corresponding to a specific communication standard is selected by a tunable LO. Then the complete signal spectra within the band is translated to baseband and digitized. The channel is selected in the digital domain by a digital channelizer which is much easier to realize than in the analog domain.
For multi-mode WB operation, the frequency of the tunable LO is adapted to fit a particular standard.

![Generic wideband receiver architecture](image)

**Figure 1.4**: Generic wideband receiver architecture

### 1.5 Software Defined Radio Receivers

The concept of Software Defined Radio (SDR) was originally conceived for military applications. It consists of a single radio receiver to communicate with different types of military radios using different frequency bands and modulation schemes [7]. This concept is starting to be introduced into commercial applications. SDR means a radio where functionality is extensively defined in software, and it supports multi-band multi-mode radio communications. It constitutes the second radio evolution since the radio systems migrated from analog to digital in 1970s and 1980s. Modern radio designs mix analog hardware, digital hardware and software technologies.

Two key issues of SDR are the placements of the A/D converters and the performance of DSP coping with the large number of samples [4]. One goal of SDR is to put the A/D converter as close as possible to the antenna. The sampling function block of an A/D converter can be either classic LPS (oversampling) or BPS (undersampling). By LPS, the sampling rate of an IF bandpass signal is high, and the performance requirements, e.g., linearity, noise floor and dynamic range, on the A/D converter are stringent. The IF WB bandpass signal can also be sampled by BPS with a sampling rate which is only slightly larger than twice the information bandwidth. The lower sampling rate alleviates the requirements on the
CHAPTER 1. INTRODUCTION

following A/D converter. In addition, BPS can realize down-conversion through the intentional signal spectral folding such that the input IF signal is sampled to discrete-time (DT) at a lower IF or baseband at the output of the BPS. The conventional mixer for down-conversion is redundant when BPS is used, and the A/D converter can be moved further forward to the antenna. The receiver architecture of SDR by BPS (thereafter called BPS receiver) is shown in Fig. 1.5. The part of dashed-line box in Fig. 1.4 is also called the equivalent LPS system of BPS, an ideal image-reject mixer followed by a lowpass sampler. From the view of multi-band multi-mode communications and the placement of the A/D converter, both the homodyne receiver and the BPS receiver are candidates for the SDR implementation.

![Diagram showing the software defined radio receiver architecture by BPS](image)

Many single-mode wideband homodyne receivers were designed for W-CDMA (Wideband Code-Division Multiple-Access) [8] [9] [10] [11]. A design of a multi-band multi-mode receiver for four standards in homodyne was presented in [5]. All the problems associated with the homodyne receiver, e.g., LO leakage, DC-offset, I/Q mismatch and flicker noise, inevitably happen and are treated in many different ways. The selection among different standards is realized by an external digital controller and the hardware is shared as much as possible by different standards.

Bandpass sampling theory has been studied for more than half a century. Cauchy [12], Nyquist [13], Gabor [14] and Kohlenberg [15] did the earliest contributions. A late comprehensive introduction on the theory of BPS can be found in [16]. In recent years, radio receiver front-ends have been implemented by BPS [17] [18] [19] [20] [21]. Harmful signal spectrum folding (or aliasing), noise aliasing and timing jitter are three associated problems in the BPS receiver. The A/D converter in
the BPS receiver could directly digitize the received RF signal and process the RF demodulation and detection in the digital domain, provided that all the associated problems were solved. Even though the design technology of homodyne receiver is mature, the basic receiver architecture is fixed and designers could only find the solutions to the associated problems at the circuit level. Orienting the design goal toward SDR, the BPS receiver is easily extended and used for a more complicated communication. It would be more advantageous to study the BPS receiver and present solutions to the associated problems.

In summary, Table 1.2 provides a high level comparison among the above six receiver architectures.

Table 1.2: High Level Comparison of Receiver Architectures

<table>
<thead>
<tr>
<th>Receiver Architecture</th>
<th>Full Integration</th>
<th>A/D Converter</th>
<th>Potential for Multi-mode</th>
<th>Selecting Filter</th>
<th>Image Rejection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Superheterodyne</td>
<td>low</td>
<td>NB</td>
<td>low</td>
<td>IF BPF</td>
<td>BPF IR mixer</td>
</tr>
<tr>
<td>Homodyne</td>
<td>high</td>
<td>NB/WB</td>
<td>high</td>
<td>LPF</td>
<td>N/A</td>
</tr>
<tr>
<td>Digital IF</td>
<td>high</td>
<td>WB</td>
<td>high</td>
<td>IF BPF</td>
<td>BPF DSP</td>
</tr>
<tr>
<td>Wideband IF</td>
<td>high</td>
<td>NB</td>
<td>high</td>
<td>LPF</td>
<td>IR mixer</td>
</tr>
<tr>
<td>Generic wideband</td>
<td>high</td>
<td>WB</td>
<td>high</td>
<td>IF BPF</td>
<td>IR mixer</td>
</tr>
<tr>
<td>BPS receiver</td>
<td>high</td>
<td>WB</td>
<td>high</td>
<td>IF BPF</td>
<td>BPF or N/A</td>
</tr>
</tbody>
</table>

1.6 Outline of Technical Problems

BPS realizes frequency down-conversion by intentional signal spectral folding instead of mixing. However, harmful signal spectrum folding, noise aliasing and sampling timing error (or jitter) are the main causes of performance degradation in BPS system.

A bandpass signal is expressed as

\[ y(t) = \text{Re}\{x(t)e^{j2\pi f_c t}\}, \tag{1.1} \]

where \( f_c \) is the carrier frequency, \( x(t) \) is the equivalent lowpass information signal or the complex envelope of \( y(t) \). \( y(t) \) is band-limited within \([-f_u, -f_l]\) \[\cup\] \([f_l, f_u]\), where \( f_l \) and \( f_u \) are respectively the lower and upper frequencies of the bandpass signal in the positive frequency band. The situation of the signal spectrum aliasing depends on the band position which is defined as the fractional number of bandwidths from the origin at which the lower band edge resides [16]. As shown in Fig. 1.6, a fractional number \( r \) represents the band position of \( y(t) \) and \( r = f_l/B \). The classic bandpass sampling theory for Uniform Sampling (US) states that [16]

The signal can be reconstructed if the sampling rate is at least \( f_{s_{\min}} = 2f_u/n \), where \( n \) is the largest integer within \( f_u/B \), denoted by \( n = \lfloor f_u/B \rfloor \).
To avoid harmful signal spectrum folding by uniform BPS, the minimum sampling rate can only be used for the special case when \( y(t) \) has an integer band position (i.e., \( r = \lfloor f_l/B \rfloor = f_l/B \) and \( r = n - 1 \), where \( \lfloor \cdot \rfloor \) denotes a floor operator). In Jerri's tutorial review [22], it was mentioned that

**Unless the signal is band-limited to \((-2\pi W; 2\pi W)\) there will always be an aliasing error when we sample at the required Nyquist rate. So if there is any alias free sampling it must be based on a rate different from that of the Nyquist rate or in other words sampling at unequally spaced instants of time.**

Nonuniform BPS might have also the potential to suppress harmful signal spectrum aliasing for any band position [15] [23]. However, a signal cannot be reconstructed from its nonuniform samples by using a conventional lowpass filtering process. Reconstruction Algorithm (RA) of NonUniform Sampling (NUS) which is extensively used in image processing are proposed and studied for our implementation in radio receivers.

The noise combined in each of the Nyquist bands within the effective bandwidth \( B_{eff} \) of introduced thermal noise (e.g., \( kT/C \) noise) in BPS causes performance degradation. This is the so-called noise aliasing. Noise aliasing is a consequence of decreasing sampling rate by BPS as compared to lowpass sampling (LPS). For a certain band position, the signal-to-noise ratio (SNR) of a sampled-data signal by BPS depends on the ratio of \( B_{eff} \) and \( f_s \) significantly (\( f_s \) is the sampling rate of BPS). The lower the value of \( f_s \), the worse the SNR performance [24].

Under the effects of jitter, the samples become randomly distributed. It is known that jitter effects depend on both the standard deviation of random jitter and the input frequency of the signal [25]. Small jitter noise can be approximately assumed as sampled-data Additive White Gaussian Noise (AWGN) [26]. For large jitter, this assumption is not valid anymore [24]. With the increase of input frequency of BPS, jitter becomes a crucial problem. For the same sampling rate, the jitter effects in BPS are larger than in the equivalent LPS system. The noise power of BPS corresponding to the same normalized standard deviation of jitter \( \sigma_T/T_s \) is larger than for LPS due to the higher input signal frequency of BPS.
1.7 Overview of Previous Work on BPS

Under the design tendency of low cost, low power consumption and full integration, CMOS techniques have been extensively used in RF designs. Especially with the growing of CMOS technology, it can perform well on a very high frequency signal today and the cost is also lower than other techniques (e.g., GaAs, BiCMOS). Recently, many successful RF CMOS designs based on the concept of SDR and BPS have been presented.

Switched-capacitor (SC) circuits are often used in these designs and the analog signal is processed in the DT domain. Sampling can be performed in both conventional voltage-mode and current-mode. The work in [18] is based on voltage-mode sampling. A 2.4-GHz CMOS RF sampling receiver front-end was designed for the IEEE standard 802.11b focusing on an RF subsampling mixer. This design integrates the subsampling mixer, clock generator, DT down-conversion filter and output buffers into a single chip. The sampling rate was chosen to be about 43 times the channel spacing, around half the input signal carrier frequency. An accurate clock is generated by a specific generation scheme and the measured sampling jitter is around 0.54 ps. In [20], a DT Bluetooth receiver using BPS is designed in current-mode CMOS technology. The principle of corresponding multi-tap direct sampling mixer is shown in [21]. The input voltage signal is first converted to current by a transconductance amplifier. The input frequency of the direct sampling mixer is 2.4 GHz and the effective data rate at the output is $f_o/N$, where the sampling takes place on the input signal at the rate of $f_o = 2.4$ GHz and $N$ is a decimation factor.

In the existing BPS implementations in voltage-mode [18] [27] [28] [29], uniform BPS is normally used and a thorough frequency plan is needed to avoid wrong signal spectral folding and image problems. Nevertheless, the performance degradation due to noise aliasing is still present. Therefore, the ratio of $f_o/2B$ is still large for attaining a certain SNR performance. As a consequence of high input frequency of sampling devices, jitter is still a crucial problem.

Charge sampling is a new sampling technology. It avoids the voltage settling problem. Combining with IIR filtering performed by a cyclic charge readout, the noise aliasing can be suppressed to a certain degree [20] [21]. Both voltage sampling and charge sampling might be the potential way to implement BPS. This thesis work is mostly based on voltage-mode sampling, but the basic theory of charge sampling will be given in Chapter 2 in comparison to voltage-mode sampling.

1.8 Summary of Contributions

In this thesis work, NUS and reconstruction are mainly studied.

- In chapter 2, two kinds of deterministic sampling techniques, Uniform Sampling (US) and NonUniform Sampling (NUS) are shown. A filter generalized by a certain RA for LPS is introduced. A general reconstruction formula in
terms of a basis-kernel (BK) is used to describe the algorithms. The computational complexity and implementation of these RAs are evaluated and compared.

- In chapter 3, BPS technique is introduced. The associated problems, available sampling selection, noise aliasing and jitter are studied and numerically analyzed.

- In chapter 4, a brief overview of the Papoulis’ generalized sampling theorem, which is the extension of Shannon’s sampling theorem, is shown. A Generalized Quadrature BPS (GQBPS) algorithm based on the Papoulis’ sampling theorem is proposed for suppressing the noise aliasing. Theoretical analyses show that GQBPS might be a potential way to reduce the noise aliasing at the cost of a more complicated RA.

- In chapter 5, two classes of nonuniform random sampling, Jitter Sampling (JS) and Additive Random Sampling (ARS), are studied and analyzed by Power Spectral Density (PSD). The conditions of aliasing-free sampling are verified in simulations.

- In chapter 6, the reconstruction performances by different RAs of lowpass case in the presence of AWGN and jitter are evaluated and compared in a modeled BPS system.

- In chapter 7, the thesis is concluded and some future work is proposed.

Through the thesis, all the signals involved in the theoretical analyses are assumed ideal band-limited, which means that their Fourier transforms are zero for $|f| > B$, although this is not ideally realizable in practice.
Chapter 2

Sampling and Reconstruction

With the launch of digital radio communications, A/D and D/A converters become important devices as the interface between RF conversions and digital signal processing. A natural signal, such as speech, music, image and electromagnetic wave, is generally an analogue signal in a continuous-time (CT) domain. To process a signal digitally, it has to be represented as a digital format in a discrete-time (DT) domain. It is required that this digital format is fixed, and uniquely represents all the features of the original analogue signal. The reconstructed CT signal from this digital format may not be exactly the same as the original analogue signal, but it is a goal to decrease the difference as much as possible.

The two basic operations of an A/D converter are sampling and quantization. Sampling is to convert a CT analogue information signal into a DT representation by measuring the value of the analogue signal at regular or irregular intervals. Quantization is to convert a value or range of values into a digital value. The quantization level determines the resolution of the A/D converter (in bits per sample). In this chapter, two ideal sampling methods, voltage sampling and charge sampling, are introduced. Regular sampling and irregular sampling are compared. A filter generalized by a Reconstruction Algorithm (RA) is proposed and studied in terms of a Basis-Kernel (BK). Nine RAs are evaluated and compared based on their performance, computational complexity and hardware implementation.

2.1 Sampling

Nowadays the sampling theorem plays a crucial role in signal processing and communications. The selecting of a time sequence $x(t_n)$ to represent a CT function $x(t)$ is known as sampling.

Sampling methods in electrical unit include *voltage sampling* and *charge sampling*. Voltage sampling is a conventional method that is realized by the sample-and-hold (S/H) circuit. It tracks an analog signal and stores its value as a voltage across a sampling capacitor for some length of time. Charge sampling does not
track the signal voltage but integrates the signal current within a given time window [30]. An analog signal in voltage mode is first converted to current mode by a transconductor before charge sampling. As compared to voltage sampling, charge sampling has the advantage that the bandwidth of the charge sampling device only relies on the sampling duration but not on the switch-on resistance so that a wideband sampler design is more feasible [31]. BPS can also be performed by a charge sampling [17] [20] [21] besides a voltage sampling.

Whether the sampled-data signal uniquely represents the original signal or not depends on the sampling patterns and their implementations. Referring to the sampling period (or interval), sampling can be ideally divided into two categories, Uniform Sampling (US) and NonUniform Sampling (NUS). It is justified to assume that the sampling set is uniformly distributed in many applications, i.e., the samples are acquired at the regular time instants. However, in many realistic situations, the data is known only in an irregularly spaced sampled set. This irregularity is a fact of life and prevents the standard methods of Fourier analysis. For example in communication systems, when data from a uniformly distributed samples is lost, the obtained result is generally nonuniformly distributed, the so-called missing data problem. Scratching a CD is also such kind of a problem. On the contrary, it may be of advantage to use NUS patterns for some special cases (e.g., an aliasing-free sampling) [32] [22]. For NUS, there are four general sampling scenarios: generalized nonuniform sampling [33], Jitter sampling [34], Additive random sampling [32], and Predetermined nonuniform sampling. Without any specifications, the NUS mentioned in this chapter is predetermined and each sampling instant is known with high precision.

2.1.1 Voltage sampling and Charge sampling

Voltage sampling A voltage sampling process can be modeled as an input CT signal \( x(t) \) multiplied by a sampling function \( s(t) \) (see Fig. 2.1). The CT sampled-data signal \( x_s(t) \) is given by

\[
x_s(t) = x(t)s(t).
\]  

(2.1) 

For ideal voltage sampling process, \( s(t) = \sum_{n=-\infty}^{\infty} \delta(t - t_n) \), where \( \{t_n\} \) represents the set of sampling instants. The multiplication in time domain corresponds to the convolution in frequency domain, i.e.,

\[
X_s(f) = (X \ast S)(f),
\]  

(2.2)
where \( \star \) represents an convolution operation, \( X(f) \) and \( S(f) \) are the Fourier transforms of \( x(t) \) and \( s(t) \), respectively.

**Charge sampling** Charge sampling integrates charge within a time window \( [t_n, t_n + \Delta t] \) instead of storing the voltage value across a sampling capacitor. It is modeled as an input CT signal \( x(t) \) convolved with a sampling function \( s(t) \) (see Fig. 2.2). The sampled-data signal \( x_s(t) \) is given by

\[
x_s(t) = (x \star s)(t) = \sum_{n=-\infty}^{\infty} \int_{t_n}^{t_n + \Delta t} x(\xi)s_n(t - \xi)d\xi.
\]

(2.3)

In frequency domain,

\[
X_s(f) = X(f)S(f).
\]

(2.4)

The sampling function \( s(t) \) is a series of pulses with a duration of \( \Delta t \). For ideal charge sampling, \( s(t) = \sum_n s_n(t) = \sum_n \mu(t - t_n) - \mu(t - t_n - \Delta t) \), where \( \mu(t) \) is a Heaviside’s step function.

S/H circuit by voltage sampling has been extensively applied in common data acquisition systems (e.g., speech communication, music, image processing) and it has a good performance. Due to the tendency of moving the A/D converter as close as possible to the antenna for SDR in wireless radio communications, both sampling methods might be the potential way to implement BPS. Without any specification, the sampling process in the following will use the voltage sampling.

### 2.1.2 Uniform sampling and Nonuniform sampling

**Uniform Sampling (US)** For an ideal US process, \( t_n = nT_s \). Starting from eq. (2.1), the sampled-data signal is given by

\[
x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s),
\]

(2.5)

and the Fourier transform of \( x_s(t) \) can be expressed as

\[
X_s(f) = \int_{-\infty}^{\infty} x_s(t)e^{-j2\pi ft}dt = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j2\pi fnT_s}.
\]

(2.6)
This is the well-known Discrete-Time Fourier Transform (DTFT). The Discrete Fourier Transform (DFT) is a special case of DTFT, which is defined to be the DTFT evaluated at equally spaced frequencies over the Nyquist interval $[0, 2\pi)$. The $N$-point DFT of a length $M$ signal is defined as

$$X(k) = \sum_{m=0}^{M-1} x(m)e^{-j2\pi km/N}, \quad k = 0, 1, \ldots, N.$$  (2.7)

By using Poisson summation formula [35], eq. (2.6) can be written as

$$X_s(f) = f_s \sum_{m=-\infty}^{\infty} X(f - mf_s),$$  (2.8)

where $f_s = 1/T_s$. Obviously, the frequency spectrum of a sampled-data signal is a series of copies of the original CT signal and $X_s(f)$ is a periodic function with period $f_s$ (see Fig. 2.3).

Figure 2.3: a) Original CT band-limited signal, $B = 50$ and samples by US, $f_s = 200$; b) The corresponding frequency spectrum of CT signal; c) The corresponding frequency spectrum of sampled-data signal.
A band-limited signal can be completely determined by a US sequence with the sampling rate of at least twice the maximum frequency $B$ (critical- or over-sampling) according to the Shannon’s sampling theorem \[36\]:

**Theorem 1:** If a function $f(t)$ contains no frequencies higher than $W$ cps (in cycles per second), it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart.

When $f_s < 2B$ (undersampling), the frequency components above $B$ will be aliased back into the Nyquist band $[-f_s/2, f_s/2]$ such that the original signal cannot be uniquely reconstructed from the sampled-data signal. For a LPS process, the input signal is regarded as a lowpass signal with a bandwidth consisting of the maximum frequency component, and critical- or over-sampling is normally used to avoid the harmful signal spectrum aliasing. The input signal of BPS is, however, always a bandpass signal such that BPS can make use of a harmless signal spectrum aliasing by tactically selecting the undersampling rate. More discussion about BPS will be presented in Chapter 3.

**Nonuniform Sampling (NUS)** For an ideal NUS process,

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - t_n), \quad (2.9)$$

and $t_n \neq nT_s$. The frequency spectrum of $x_s(t)$ is not necessarily periodic and the Fourier transform becomes

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(t_n)e^{-j2\pi ft_n}. \quad (2.10)$$

The corresponding Energy Density Spectrum (EDS) of an ideal NUS is the magnitude squared of the Fourier transform \[37\], i.e.,

$$P_s(f) = |X_s(f)|^2 = \left[\cdots + x(t_0) \cos 2\pi ft_0 + x(t_1) \cos 2\pi ft_1 \right. \left. + x(t_2) \cos 2\pi ft_2 + \cdots + x(t_n) \cos 2\pi ft_n + \cdots \right]^2$$

$$+ \left[\cdots + x(t_0) \sin 2\pi ft_0 + x(t_1) \sin 2\pi ft_1 \right. \left. + x(t_2) \sin 2\pi ft_2 + \cdots + x(t_n) \sin 2\pi ft_n + \cdots \right]^2. \quad (2.11)$$

NUS is well applied for obtaining oscillograms in oscilloscopes and spectrograms for spectral analysis \[38\]. The aperiodic property of the frequency spectrum enables NUS to suppress harmful signal spectrum aliasing.

As shown in Fig. 2.4, given a wanted signal $s(t) = \cos(2\pi \cdot 2t)$ (solid line) and an interference signal $i(t) = \cos(2\pi \cdot 3t)$ (dashed line), when $f_s = 5$, the
component of $f = 3$ is larger than $f_s/2$ and will be folded back to $f = 2$ (see a)). By intentionally introducing a random shift with a uniform distribution $U(-\alpha T_s, \alpha T_s)$ ($\alpha$ is a scale factor) on the equidistant US time instants, for instance $\alpha = 0.3$ for b) and 0.7 for c) in Fig. 2.4, the aliasing effect is reduced to a certain degree. Obviously, this aliasing can be overcome by using NUS. NUS relaxes the requirements on the anti-aliasing (AA) filter.

2.1.3 Nonideal sampling process

In an A/D converter, when the input signal is band-limited, the sampling process is realized by a two-step sampling, an ideal sampling followed by a pulse shaping filtering, as shown in Fig. 2.5. When the band is not limited, an AA filter prior to the sampling is needed to restrict the bandwidth of the input signal.

For the two-step sampling case with an ideal arbitrary sampling scheme, the
2.2. RECONSTRUCTION

output in the time domain is given by

\[ y(t) = x_s \ast h(t) = \int_{-\infty}^{\infty} x_s(\tau)h(t-\tau)d\tau. \]  

(2.12)

Substituting eq.(2.9) into eq.(2.12),

\[
y(t) = \sum_{n=-\infty}^{\infty} x(t_n)\delta(t-t_n)h(t-t_n). 
\]

(2.13)

The corresponding frequency domain expression of \( y(t) \) is expressed as

\[
Y(f) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(t_n)h(t-t_n)e^{-j2\pi ft}dt \\
= \sum_{n=-\infty}^{\infty} x(t_n)H(f)e^{-j2\pi ft_n} \\
= H(f)X_s(f),
\]

(2.14)

where \( X_s(f) \) is given by eq.(2.8) and eq.(2.10) for US and NUS, respectively. The form of eq. (2.13) may be interpreted as the convolution summation of the modulated delta-function series \( x(t_n) \) with the filter weighting function \( h(t) \). A step function (or zero-order holding) is mostly used in the A/D converter as a filter,

\[
h(t) = \begin{cases} 
1, & t_{n-1} \leq t < t_n \\
0, & \text{otherwise}
\end{cases},
\]

(2.15)

which is equivalent to the sample-and-hold interpolant. Eq. (2.14) also indicates a complete reconstruction of signal \( x(t) \) as long as the sampling rate satisfies the Nyquist criterion and \( H(f) \) is well designed.

2.2 Reconstruction

Depending on the context, “reconstruction” has different definitions. Image reconstruction is defined in imaging technology wherein data is gathered through
methods such as computerized tomography-scan (CT-scan) and magnetic resonance imaging (MRI), and then reconstructed into viewable images [39]. In analog signal processing, reconstruction mostly means that a continuous-time signal is obtained from the DT data by an interpolation filter or some other filtering processes.

Although modern data processing always uses a DT version of the original signal that is obtained by a certain sampling pattern on a discrete set, reconstruction to a continuous version of sampled data is also needed for some specific applications. In Hi-Fi applications such as digital audio, to maintain high quality in the resulting reconstructed analog signal, a very high quality analog reconstruction filter (postfilter) is required. Reconstruction from one discrete set to another is also useful in the non-fractional sampling rate alternation in digital signal processing. Additionally, in radio receiver front-ends, if the output of the sampling process is not uniformly distributed, a reconstruction process is needed to reconstruct the nonuniform samples to uniform distributed samples prior to the quantizer.

According to the Shannon’s sampling theorem [36], a band-limited signal can be exactly reconstructed from its samples by US. The perfect reconstruction formula derived by Whittaker [40] for a critical uniform sampling is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc}[2B(t - nT_s)], \quad (2.16)$$

where \(x(nT_s)\) represents samples at the series of equidistant sample instants, \(T_s = 1/(2B)\), and \(\text{sinc}(x) = \sin(\pi x)/(\pi x)\). The reconstruction of the input signal is realized by a convolution summation of uniform distributed samples \(x(nT_s)\) with a \(\text{sinc}\) function which is equivalent to ideal low-pass filtering.

In practice, the CT signal reconstruction is enhanced by first passing the sampled-data signal \(x_s(t)\) through a holding circuit with the function of eq. (2.15), and then feeding into a LPF or other RAs. The reconstruction discussed in the thesis is only realized by a certain RA without any enhancement from the zero-order holding.

For NUS, even if there is a large number of samples, only few of them possess a uniform distribution property with respect to the average sampling rate. The expansion of \(X(f)\) does not consist of periodic replicas of the fundamental spectrum. Consequently, the signal cannot be determined uniquely by the samples with only a lowpass filter. Based on the Fourier series expansion, \(X(f)\) can be generally expanded as

$$X(f) = \sum_{n=-\infty}^{\infty} c_n e^{-j2\pi f \tau_n}, \quad (2.17)$$

where \(\tau_n\) is the set of sampling instants either uniformly or nonuniformly distributed. Using inverse Fourier transform, we obtain the general reconstruction
2.2. RECONSTRUCTION

formula:

\[ x(t) = \int_{-B}^{B} \left( \sum_{n=-\infty}^{\infty} c_n e^{-j2\pi f\tau_n} \right) e^{j2\pi ft} df \]

\[ = 2B \sum_{n=-\infty}^{\infty} c_n \text{sinc}[2B(t - \tau_n)]. \quad (2.18) \]

For US \( \tau_n = nT_s \), \( c_n = x(nT_s) / 2B \) and eq. (2.18) is exactly the same as eq. (2.16). However, for NUS, since \( \tau_n = t_n \) and \( c_n \neq x(t_n) / 2B \) except when \( t_n = nT_s \), the reconstruction formula of eq. (2.18) cannot directly represent the original signal \( x(t) \) unless \( c_n \) is determined. RAs are expected to accurately predict the original signal \( x(t) \) from the nonuniform samples \( x(t_n) \).

In biomedical image processing, CT-scan and MRI frequently use the NUS pattern in the frequency domain. Four sampling patterns are shown in Fig. 2.6. The

![Sampling patterns of nonuniform sampling](image)

Figure 2.6: Sampling patterns of nonuniform sampling [41]. (Top-left): Polar sampling grid; (Top-right): Spiral sampling grid; (Bottom-left): variable-density nonuniform sampling grid; (Bottom-right): general nonuniform sampling grid.

sampled data of CT-scan and MRI are measured in the Fourier frequency domain.
CHAPTER 2. SAMPLING AND RECONSTRUCTION

The RA is needed to derive the Cartesian US grid (see Fig. 2.7) from the acquired data prior to the inverse Fourier transform operation. Inspired by the applications in biomedical image processing, some RAs extensively used in image reconstructions are proposed for the applications of radio communications.

However, the reconstruction process in radio communications is different from that in biomedical image processing. In radio communications, both sampling and reconstruction are in the time domain while they are in the frequency domain in image processing. Additionally, in radio communications, the RA can be used to reconstruct a set of unknown data at a regular time set from the NUS sequence. Then the reconstructed result can be directly fed into the following digital signal processing block (e.g., A/D converter). It is also possible to convert the samples by NUS to a CT signal when an analog signal is needed (e.g., in Hi-Fi) in the processing steam, which is different from the reconstruction in image processing.

2.3 Basis-Kernel (BK)

It is known that \( \{ e^{j2\pi ft_n} \} \) is a complete basis for \( X(f) \) within the bandwidth \([-B, B]\) and that \( \{ \text{sinc}[2B(t-t_n)] \} \) forms a complete basis for \( x(t) \) in \( t \in (-\infty, \infty) \), given in eq. (2.17) and eq. (2.18). In [42], another sampling basis \( k(t, t_n) \) which is the unique reciprocal basis of \( \{ g(t, t_n) \} = \text{sinc}[2B(t-t_n)] \) was introduced. An expression in terms of Kronecker delta function \( \delta[m-n] \) is given by

\[
\langle k(t, t_m), g(t, t_n) \rangle = \delta[m-n],
\]

where \( \langle a, b \rangle \) denotes the inner product of \( a \) and \( b \) which is given by \( \langle a, b \rangle = \int_{-\infty}^{\infty} a(t)b(t)dt \). In \( t \in (-\infty, \infty) \), \( \{ k(t, t_n) \} \) is also a complete basis-kernel for \( x(t) \).
Therefore, \( x(t) \) can be given either by

\[
 x(t) = \sum_{n=-\infty}^{\infty} \langle x(\bullet), k(\bullet, t_n) \rangle g(t, t_n) \\
 = 2B \sum_{n=-\infty}^{\infty} c_n g(t, t_n)
\]

(2.20)

in terms of \( c_n \) or by

\[
 x(t) = \sum_{n=-\infty}^{\infty} \langle x(\bullet), g(\bullet, t_n) \rangle k(t, t_n) \\
 = \sum_{n=-\infty}^{\infty} x(t_n) k(t, t_n)
\]

(2.21)

in terms of the nonuniform samples \( x(t_n) \). It was also mentioned in [42] that this method is appropriate in the case of \( L^2 \) signals only. In other words, the CT function \( x(t) \) has to satisfy \( \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \) [43]. According to Parseval’s equation [35], \( \int_{-B}^{B} |X(f)|^2 df < \infty \). This CT function has a finite energy, and this method is only suitable for a band-limited signal. However, the only complete orthonormal sampling basis for \( \chi \) are of the form \( \{ g(t, t_n) \} = \{ g(t, nT_s) \} \) (where \( \chi \) is a subspace of \( L^2 \)-space in the time domain). Obviously, Higgins sampling theorem includes the Shannon’s sampling theorem as a special case: For US \( t_n = nT_s \),

\[
 k(t, mT_s) = g(t, mT_s) \\
 = \text{sinc}[2B(t - mT_s)] \\
\langle k(t, mT_s), g(t, nT_s) \rangle = \text{sinc}[2B(n - m)T_s] \\
= \delta[m - n].
\]

A close form of the basis kernel (BK) \( k(t, t_n) \) is needed for the reconstruction of NUS, and \( k(t, t_n) \neq g(t, t_n) \).

### 2.4 Reconstruction Algorithms (RAs)

A filter generalized by a certain RA is expected to reconstruct the signal as close as possible to the original from the nonuniformly distributed samples. The selection of the BK \( k(t, t_n) \) determines the reconstruction performance. The reconstruction filter can be in either CT or DT. Based on eq. (2.21), a new sampling paradigm with RAs is proposed as shown in Fig. 2.8.

Frequently used RAs for NUS include

- **Low-pass Filtering (LPF) [interpolation]**,
Figure 2.8: Identity elements of (a) interpolation reconstruction with a CT filter; (b) interpolation reconstruction with a DT filter; (c) iterative reconstruction [44]

- Lagrange Interpolating Polynomial [interpolation],
- Spline Interpolating [interpolation],
- Gridding Algorithm [interpolation],
- Least Square Reconstruction (LSR) Algorithm [svd],
- Iterative Algorithms [iterative],
- Yen’s Interpolations [interpolation],
- Coefficient $c_n$ Determination Reconstruction Algorithm [svd],
- “Minimum-Energy” Signals [svd].

These methods can be simply classified into three types: interpolation, iterative and svd methods. The conventional FIR filter design with a constant data rate is normally based on Interpolation. Iterative methods are extensively used in image processing. They consist of three steps: orthogonal projection, iteration and procedure convergence. svd is an important element of many numerical matrix algorithms. If the matrix of eigenvectors of a given matrix is not a square matrix, the matrix of eigenvectors has no matrix inverse, and the given matrix does not have an eigen decomposition. The standard definition for the matrix inverse fails. By svd, it is possible to obtain a pseudoinverse which is defined as

$$A^{-1} = (A^*A)^{-1}A^* = VDU^T,$$

(2.22)
where \( A = UDV^T \) is a given \( m \times n \) real matrix, \( U \) and \( V \) are \( m \times m \) and \( n \times n \) unitary matrices (i.e., \( U^* = U^{-1}, V^* = V^{-1} \)), \( D \) is a \( m \times n \) diagonal matrix and the elements in the diagonal consist of the singular values of \( A \) and zeros, \( \{ \bullet \}^T \) denotes a matrix transpose operator. All the RAs involving matrix inverse operations are classified within the family of svd methods, e.g., LSR algorithm and coefficient \( c_n \) determination.

As we discussed in section 2.3, a signal can be reconstructed either in terms of the sampled-data signal by a BK \( k(t, t_n) \) (see eq. (2.21)) or in terms of the coefficient \( c_n \) by a sinc function (see eq. (2.20)). Most of the RAs start from eq. (2.20) except for the coefficient \( c_n \) determination method.

Note that all these RAs are directly applicable for LPS but not for BPS. In order to be used in BPS implementations, a carrier-modulated BK is needed (see chapter 4). In this thesis work, the above nine RAs are studied and compared from the aspects of reconstruction performance, computational complexity and hardware implementation. The BKs of four among these nine which are based on interpolation will be discussed and compared in both the time domain and the frequency domain.

**Low-pass Filtering (LPF)**

LPF technique directly treats nonuniform samples \( x(t_n) \) with the reconstruction method to uniform sampling by using a lowpass filter. The algorithm can be written as eq. (2.21) with

\[
k(t, t_n) \approx \text{sinc}[2B(t - t_n)] \tag{2.23}
\]

**Lagrange Interpolating Polynomial [42]**

When nonuniform sampling instants deviate from the equivalent uniform Nyquist sample instants by no more than \( T_s/4 \) (\( T_s \) is the equivalent US interval), \( k(t, t_n) \) can be approximated by the Lagrange interpolation function

\[
k(t, t_n) \approx \frac{P(t)}{P'(t_n)(t - t_n)} , \tag{2.24}
\]

where

\[
P(t) = (t - t_0) \prod_{m \neq 0} (1 - \frac{t}{t_m}), m \in (-\infty, +\infty)
\]

\[
P'(t_n) = \prod_{m \neq 0} (1 - \frac{t_n}{t_m}) + \frac{t_0 - t_n}{t_n} \prod_{m \neq 0, m \neq n} (1 - \frac{t_n}{t_m}).
\]

In eq. (2.24), \( k(t, t_n) \) is equal to zero at every sampling point except for the \( n \)th where it is equal to one. If \( t_0 = 0 \), eq. (2.24) can be simplified to

\[
k(t, t_n) \approx \prod_{m \neq n} \frac{t - t_m}{t_n - t_m} . \tag{2.25}
\]
Spline Interpolation

A spline function is a piecewise polynomial that has a simple form locally but is flexible globally. Cubic spline is one kind of spline with a third-order polynomial passing through a series of mesh points between any two fixed points. Assuming that \( x(t_n) \) is the ordinate of \( t_n \), a cubic spline is given by

\[
S_c(t) = a_n + b_n t + c_n t^2 + d_n t^3 \tag{2.26}
\]

in \( t \in (t_{n-1}, t_n) \), where the 4 coefficients \( a_n, b_n, c_n \) and \( d_n \) are determined by the following conditions: i) \( S_c(t_n) = x(t_n) \); ii) the first and second derivatives of \( S_c(t) \) are continuous at \( t_n \); iii) the second derivative of \( S_c(t) \) is zero at the endpoints (for "natural" cubic spline) \[45\]. Therefore, the reconstructed signal when using a cubic spline can be expressed as

\[
\hat{x}(t) = \sum_{n=-\infty}^{\infty} S_c(t)[\mu(t-t_{n-1}) - \mu(t-t_n)] \tag{2.27}
\]

where \( \mu(t) \) is a Heaviside’s step function, being 1 for \( t \geq 0 \) and 0 otherwise. It is difficult to write the spline interpolation in the form of eq. (2.21) in terms of \( x(t_n) \) and \( k(t, t_n) \) directly.

Gridding Reconstruction Algorithm [41]

Gridding reconstruction algorithm is suitable for a nonuniform sampling pattern but with a uniform measurement space. A Density Compensation Factor (DCF) is used, which is the same as in the adaptive weights method [46]. DCF is inversely proportional to the local sampling density.

The samples are weighted by DCF and then convolved with a kernel. For one-dimensional case, \( \omega_n = (t_{n+1} - t_{n-1})/(2\Delta t) \) is a simple and rather good definition for DCF, where \( \Delta t \) is the average sampling interval. The algorithm can be written as eq. (2.21) with

\[
k(nT_s, t_n) \approx h(nT_s, t_n)\omega_n \tag{2.28}
\]

where \( nT_s \) represents the uniform measurement space with an interval of \( T_s \), \( h(t, t_n) \) is a kernel which can be either Gaussian, a sinc or some other small finite windows [41]. As shown in Fig. 2.8 (b), the reconstruction filter comprising \( k(nT_s, t_n) \) is a DT filter. According to eq. (2.16), a reconstructed CT signal \( \hat{x}(t) \) will be obtained if the reconstructed DT signal \( \hat{x}(nT_s) \) is filtered by an ideal lowpass filter.

Least Square Reconstruction (LSR) Algorithm

Assume that \( \chi = [\hat{x}(t_m)] \) denotes an \( M \times 1 \) vector of the reconstructed signal, \( A = [h(t_m, t_n)] \) is an \( N \times M \) sampled approximation matrix of a kernel \( \{h(t, t_n)\} \)
and \( \bar{x} = [x(t_n)] \) is an \( N \times 1 \) vector of sampled data. Then there exists a linear expression between \( \bar{x} \) and \( \chi \),

\[
\bar{x} = A\chi,
\]

which can be regarded as reconstructing the known samples \( \bar{x} \) in terms of the unknown signal \( \chi \). However, we can only find an approximate basis-kernel matrix \( A \) and consequently introduce \( \epsilon = |\bar{x} - A\chi| \). The least-square error is minimized by solving for \( \chi \) in

\[
\min_{\chi} ||\bar{x} - A\chi||^2.
\]

Here, \( \bar{x} \) consists of the sampled data and \( \chi = (A^*A)^{-1}(A^*)\bar{x} \) (see also eq. (2.21)). \( A^* \) denotes the Hermitian transpose (or complex conjugate transpose) of the matrix. This linear system should be overdetermined for getting a unique reconstructed signal. It implies that the number of samples \( x(t_n) \) should be greater or equal than the number of reconstructed points. For \( \chi = [\hat{x}(mT_s)] \), where \( m \in [1, M] \) and \( M \leq N \), \( \hat{x}(mT_s) \) can also be filtered by an ideal lowpass filter according to the same rules as the gridding algorithm, and a CT signal reconstruction is achieved.

**Iterative Methods** [46] [47]

The iterative method first gets an initial reconstruction function \( \sum_{n=-\infty}^{\infty} x(t_n)\Theta_n \) from \( \{x(t_n)\} \) by using an indicator function \( \Theta_n \) which could be sample-and-hold interpolation function within \( [t_{n-1}, t_n] \):

\[
\Theta_n(t) = \mu(t - t_{n-1}) - \mu(t - t_n)
\]

or the midpoints of subsequent intervals, or some other interpolation function, and then goes through projection, correction and accumulation iteratively, expressed by the following equations

\[
\begin{align*}
    x_0(t) &= P\{\sum_{n=-\infty}^{\infty} x(t_n)\Theta_n\} \\
    \Delta_1(t) &= P\{x_0(t) - \sum_{n=-\infty}^{\infty} x_0(t_n)\Theta_n\}, \\
    \Delta_i(t) &= P\{\Delta_{i-1}(t) - \sum_{n=-\infty}^{\infty} \Delta_{i-1}(t_n)\Theta_n\}
\end{align*}
\]

and finally \( \hat{x}(t) = x_0(t) + \Delta_1 + \cdots + \Delta_i(i = 1, 2, \ldots, \infty) \). \( P\{\bullet\} \) denotes an orthogonal projection which projects a given signal onto the space of band-limited signals in the frequency range \( f \in [-B, B] \) (see Fig. 2.8 (c)). The projector could be a lowpass filter or a convolution using a sinc kernel. It is obvious that the filtering process destroys the pointwise interpolation property of the approximation procedure. The number of iterative procedures \( i \) depends on the convergence rate of \( \Delta_i \).
Yen’s Interpolation (THEOREM I in [48])

As mentioned in [49], the interpolation function for the reconstruction from the samples by NUS should: i) be band-limited to \([-B, B]\); ii) take on correct values at sampling instants. LPF based on the sinc kernel is band-limited, but it cannot take on correct values at nonuniform sampling instants (see Fig. 2.13 Bottom). A modification interpolation function based on LPF was proposed by Yen such that both requirements for the reconstruction of NUS are attainable.

THEOREM I in [48] states that if a finite number of uniform sample points in a uniform distribution are migrated to new distinct positions thus forming a new distribution denoted by \( t_n = t_n \), the band-limited signal \( x(t) \) remains uniquely defined. The reconstruction of the signal is obtained as

\[
\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(t_n) \Psi_n(t),
\]

with the composing function \( \Psi_n(t) \) given by

\[
\Psi_n(t) = \Psi_{n1}(t) = \frac{\prod_{q=1}^{N}(t - t_q) \prod_{q=1}^{N} [n/(2B) - n_q/(2B)]}{\prod_{q=1}^{N}(t - n_q/(2B)) \prod_{q=1}^{N} [n/(2B) - t_q]} \cdot \frac{(-1)^n \sin 2\pi B t}{\pi(2Bt - n)}
\]

for \( t_n = n/(2B) \neq n_q/(2B) \), while

\[
\Psi_n(t) = \Psi_{n2}(t) = \frac{\prod_{q=1,\neq p}^{N}(t - t_q) \prod_{q=1}^{N} [t_p - n_q/(2B)]}{\prod_{q=1}^{N}(t - n_q/(2B)) \prod_{q=1,\neq p}^{N} (t_p - t_q)} \cdot \frac{\sin 2\pi B t}{\sin 2\pi B t_q}
\]

for \( t_n = t_q \).

Starting from eq. (2.21), \( \Psi_n(t) \) corresponds to \( k(t, t_n) \). These nonuniformly sampled points consist of two sets: \( t_n = nT_s \) and \( t_n = t_q \) (\( \pi / T_s \) is not an integer). Then the reconstruction of \( x(t) \) can be written in another way:

\[
\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \Psi_{n1}(t) + \sum_{q=1}^{N} x(t_q) \Psi_{n2}(t).
\]

Sinc kernel \( \text{sinc}[2B(t - t_n)] \) has a zero at every \( t = mT_s \) but \( n \neq m \) for \( t_n = nT_s \). However, it does not vanish at \( t_n = t_q \) (see Fig. 2.13 Bottom). We must provide zeros for each \( t_q \) by \( \prod_{q=1}^{N} (t - t_q) \) in \( \Psi_n(t) \). In addition, it is also necessary to remove the zeros from the denominator factor of \( \Psi_n(t), \prod_{q=1}^{N} [t - n_q/(2B)] \), when \( t = n_q/(2B) \) for which the samples are unknown.

**Coefficient \( c_n \) Determination Reconstruction Algorithm**

As shown in eq. (2.20), a signal can also be perfectly reconstructed from nonuniformly sampled points by using sinc kernel provided that the coefficient \( c_n \) is de-
termined. For a special case of US, eq. (2.20) is written as

\[ \hat{x}(t) = 2B \sum_{m=-\infty}^{\infty} c_m g(t, mT_s). \]  

Substituting the time instants \( t_n \) of NUS and its corresponding sampled data \( x(t_n) \) by \( t \) and \( x(t) \) in eq. (2.35) respectively, there exists a linear expression:

\[ X = CA, \]  

where \( X = [x(t_n)] \ (n = 1, 2, \ldots, N) \) denotes an \( 1 \times N \) vector of nonuniform sampled data, \( A = [g(t_n, mT_s)] \) is an \( M \times N \) \((M \leq N)\) sampled approximation matrix of kernel \( g(t, mT_s) \) and \( mT \) represents a set of uniform sampling time instants which satisfies the Nyquist criterion \((1/T_s \geq 2B)\), and \( C = [2Bc_m] \) represents a \( 1 \times M \) vector of coefficients. \( C \) can be easily obtained by \( XA^{-1} \), where \( A^{-1} \) is obtained by svd. By substituting \( C \) into \([2Bc_m]\) in eq. (2.35), the signal reconstruction is achieved.

**“Minimum-Energy” Signals (THEOREM IV in [48])**

As we have known, \( \{\text{sinc}[2B(t-t_n)]\} \) form a complete basis for \( x(t) \) in \( t \in (-\infty, \infty) \) based on eq. (2.18). However, in most cases, the time variable \( t \) is not infinite but limited over an interval \( T_o \). When one does not wish to specify the time interval explicitly, for critical sampling, \( 2BT_o \) arbitrarily-located samples can be used to define uniquely a “minimum-energy” signal according to Yen’s THEOREM IV.

THEOREM IV in [48] states that if the sampled values at a finite set of arbitrarily distributed sample points \( t = t_n, \ n = 1, 2, \ldots, N \) are given, a signal \( x(t) \) with no frequency component above \( B \) is defined uniquely under the condition that the “energy” of the signal \( \int_{-\infty}^{\infty} |f(t)|^2 dt \) is a minimum. Moreover, the reconstruction of the signal is

\[ \hat{x}(t) = \sum_{n=1}^{N} x(t_n) \Psi_n(t), \]  

where

\[ \Psi_n(t) = \sum_{m=1}^{N} a_{mn} \frac{\sin 2\pi B(t-t_m)}{2\pi B(t-t_m)}. \]  

The coefficients \( a_{mn} \) are the coefficients of the inverse of a matrix whose elements are

\[ \frac{\sin 2\pi B(t_n-t_m)}{2\pi B(t_n-t_m)}, \quad n, m = 1, 2, \ldots, N. \]  

It is observed that \( a_{mn} \) has the same form as matrix \( A \) in eq. (2.36) with the only exception that \( t_m = mT_s \) in matrix \( A \). As shown in Fig. 2.9, both CT and DT signal reconstruction can be achieved by algorithms based on svd depending on the specification of the reconstruction filter.
2.5 Performance Evaluation of RAs

To evaluate the properties of these RAs, a sinusoidal signal with two periods is sampled. Then the sampled data is reconstructed by using these RAs. The sample distributions by US and NUS are shown in Fig. 2.10.

2.5.1 Reconstruction performance

A reconstruction error curve $e(t)$ can characterize the reconstruction performance, where

$$e(t) = x(t) - \hat{x}(t),$$  \hspace{1cm} (2.40)

$x(t)$ and $\hat{x}(t)$ are the original input signal and the reconstructed signal, respectively. The error curves of nine RAs are evaluated for NUS and shown in Fig. 2.11 and Fig. 2.12.

It is observed that for Lagrange interpolating polynomial, Spline interpolation, LSR algorithm, Yen’s interpolation, $c_n$ determination and “Minimum-energy” signals, correct values are taken at the sampled points by these RAs. Lagrange Polynomial Interpolating shows the best reconstruction performance. LPF has a good reconstruction performance for US except at the ends due to the truncation error of the sinc function [44]. The finite convolution of the sinc function will contribute to sidelobes and consequently cause a poor reconstruction performance. The general truncation error was defined by THEOREM 5 in [50]. Using the BK of LPF defined by eq. (2.23), we obtain the expression for reconstructing one point $x(t_n)$ by LPF [44]:

$$\hat{x}(t_n) = x(t_n) + \sum_{m=-\infty, m\neq n}^{\infty} x(t_m)\text{sinc}[2B(t_n - t_m)],$$  \hspace{1cm} (2.41)
where the second term represents the sidelobe effects of truncated sinc function. For NUS, however, the sinc kernel cannot pick up correct values at the sampled points such that the reconstruction performance by a LPF is degraded. The sinc kernel is also used in Gridding algorithm and the corresponding reconstruction performance is predetermined by the sinc kernel. It is observed that introducing DCF in the Gridding algorithm does not improve the reconstruction performance significantly in this simulation. In the iterative algorithm, a convolution by a sinc kernel is used as a projector and the number of iterative procedures is 10. The reconstruction performance of sampled points is improved to a certain degree by the iterative algorithm compared to LPF.

The BK of RAs based on interpolation are shown in Fig. 2.13 and Fig. 2.14. They are symmetric at the origin for US but asymmetric for NUS.

The SNDR is normally used to numerically evaluate the accuracy of reconstruction which is defined as [51]

\[
SNDR = \frac{\sum_{i=1}^{L} x_i^2}{\sum_{i=1}^{L} (x_i - \hat{x}_i)^2},
\]  

(2.42)
where \( i = [1, L] \) denotes the evaluated points, normally \( L > N \), \( x_i \) and \( \hat{x}_i \) represent the points from the original and reconstructed signal, respectively. The SNDR in dB is evaluated for the reconstruction performance of sampled and interpolated points of NUS respectively by different RAs (see Table 2.1).

2.5.2 Computational complexity

The computational complexity of a reconstruction filter depends on the accuracy requirement of the simulation model. For the interpolation reconstruction filter, the approximation error between \( x(t) \) and \( \hat{x}(t) \) can be decreased by increasing the length of the filters or the degree of the interpolation. For the iterative reconstruction, increasing the order of iteration is also helpful for reducing the error when the repeating procedure is convergent. Based on the simulations that we have done and the approximation errors shown in Fig. 2.11 and Fig. 2.12, the computational...
Figure 2.12: Reconstruction error curves ("•" represents the reconstruction of sampled point) (cont.).
Figure 2.13: The basis-kernels of interpolations in time domain.
Figure 2.14: The basis-kernels of interpolations in frequency domain.
Table 2.1: SNDR (in dB) comparison of different algorithms (N=34, L=201) for the NUS pattern shown in Fig. 2.10

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Nonuniform sampling</th>
<th>Interpolated points</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sampled points</td>
<td>Interpolated points</td>
</tr>
<tr>
<td>Lowpass filtering (LPF)</td>
<td>17.33</td>
<td>17.97</td>
</tr>
<tr>
<td>Lagrange interpolating polynomial</td>
<td>$\infty$</td>
<td>$\infty^1$</td>
</tr>
<tr>
<td>Spline interpolation</td>
<td>$\infty^1$</td>
<td>69.22</td>
</tr>
<tr>
<td>Gridding algorithm</td>
<td>18.49</td>
<td>19.22</td>
</tr>
<tr>
<td>LSR algorithm</td>
<td>$\infty^1$</td>
<td>39.54</td>
</tr>
<tr>
<td>Iterative algorithm</td>
<td>18.63</td>
<td>13.89</td>
</tr>
<tr>
<td>Yen’s interpolation</td>
<td>$\infty^1$</td>
<td>40.22</td>
</tr>
<tr>
<td>$c_n$ determination</td>
<td>$\infty^1$</td>
<td>39.54</td>
</tr>
<tr>
<td>$^\dagger$Minimum-energy$^\dagger$ signals</td>
<td>$\infty^1$</td>
<td>37.30</td>
</tr>
</tbody>
</table>

$^1$ It is a reasonable assumption that SNDR is approximated by $\infty$ when SNDR $> 100$. Complexity is only determined by the different BK functions. Here all the RAs were divided into two groups, sinc-based and nonsinc-based, and the number of floating point operations (FPOs) was evaluated by using Matlab 5.3 for each RA (see Table 2.2). We find that spline interpolation is the most expensive technique. Repeating the procedure many times causes a large number of FPOs for the iterative algorithm.

### 2.6 Implementations of RAs

Those RAs (i.e., gridding and iterative algorithm) which are extensively used in image processing cannot immediately be used for radio communications. One important difference between radio communication and image processing is that the former requires data processing on-line but the latter does not. These RAs together with those based on svd (i.e., LSR algorithm, coefficient determination and “minimum-energy” signals) have to be applied to blocks of data while the other methods (i.e., Lagrange interpolating polynomial, spline interpolation, Yen’s interpolation) can be applied on a sample-by-sample basis.

For NUS, Lagrange interpolating polynomial has a rather good reconstruction performance, the computation is also not very complex (see Table 2.2). However, it is observed from eq. (2.25) that the input samples intercept the interpolating filter impulse response at different time instants. This implies that it has a time-varying characteristic. Currently, Lagrange fractional delay filtering [52] and time-invariant filterbank [53] with synthesis filters generalized to Lagrange interpolating polynomial are two feasible methods for implementation of Lagrange interpolating polynomial.
Table 2.2: Computational complexity comparison of different algorithms by evaluating the number of FPOs (floating point operations) [44]

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Sinc-based</th>
<th>Nonsinc-based</th>
</tr>
</thead>
<tbody>
<tr>
<td>LPF</td>
<td>55,141</td>
<td>–</td>
</tr>
<tr>
<td>Lagrange</td>
<td>–</td>
<td>697,072</td>
</tr>
<tr>
<td>Spline</td>
<td>–</td>
<td>8.088 $\times$ 10^9</td>
</tr>
<tr>
<td>Gridding</td>
<td>64,307</td>
<td>–</td>
</tr>
<tr>
<td>LSR</td>
<td>–</td>
<td>1,131,518</td>
</tr>
<tr>
<td>Iterative</td>
<td>1,160,311</td>
<td>–</td>
</tr>
<tr>
<td>Yen’s</td>
<td>–</td>
<td>1,376,353</td>
</tr>
<tr>
<td>“Minimum-energy”</td>
<td>610,676</td>
<td>–</td>
</tr>
<tr>
<td>Coefficient</td>
<td>996,532</td>
<td>–</td>
</tr>
</tbody>
</table>

2.6.3 Fractional delay filtering [54]

Discrete-time linear time-invariant (LTI) systems can be classified into FIR (finite impulse response) and IIR (infinite impulse response) systems. For an $M$th-order interpolation by FIR filtering, the output $y(n)$ is given by

$$y(n) = \sum_{m=0}^{M} h(m)x(n - m)$$

(2.43)

in a direct form of convolution, where $h(m)$ represents the impulse response coefficients (or filter taps), and the set of $\{x(n - m)\}$ represents the past $M$ samples. The corresponding transposed FIR structure is shown in Fig. 2.15 [37]. For a LTI system $H_T$, the set of filter taps $\{h_T(m)\}$ which are the response of the system to a series of unit pulse $\{\delta[n - m]\}$ are constants. The input delay will cause the same time shift at the output, and normally this delay is an integer. The $z$ transform of $h_T(m)$ is given by

$$H_T(z) = \sum_{m=0}^{M} h_T(m)z^{-m}.$$  

(2.44)
For a time-varying system $H_\Delta$, for instance, a filter generalized to Lagrange interpolating polynomial based on eq. (2.25), each coefficient $h_\Delta(n)$ is not a constant but a polynomial in terms of the delay parameter $\Delta$ and

$$h_\Delta(n) = \sum_{m=0}^{M} c_m(n) \Delta^m = \prod_{p=0, \neq n}^{N} \frac{\Delta - t_p}{t_n - t_p}, \quad n = 0, 1, 2, \cdots, N. \quad (2.45)$$

The $z$ transform of $h_\Delta(n)$ is given by

$$H_\Delta(z) = \sum_{n=0}^{N} h_\Delta(n) z^{-n} = \sum_{n=0}^{N} \left( \sum_{m=0}^{M} c_m(n) \Delta^m \right) z^{-n} = \sum_{m=0}^{M} \left( \sum_{n=0}^{N} c_m(n) z^{-n} \right) \Delta^m = \sum_{m=0}^{M} C_m(z) \Delta^m. \quad (2.46)$$

As compared to eq. (2.44) and referred to Fig. 2.15, we can get the structure of fractional delay filtering (see Fig. 2.16). For example if the output $y(n) = \hat{x}(n - \Delta)$, then

$$\sum_{m=0}^{M} C_m(z) \Delta^m = z^{-\Delta}, \quad \Delta = 1, 2, \cdots, M. \quad (2.47)$$

and consequently the set of coefficients $\{C_m(z)\}$ (or the $N$th order FIR transfer function) can be obtained by solving $M$ equations.

2.6.4 Filterbank processing

Reconstruction from nonuniform samples is considerably more complex than reconstruction from uniform samples. Based on the Papoulis’ generalized sampling theorem [33], a band-limited signal can be uniquely determined from uniformly distributed samples of the outputs of $N$ LTI systems with the signal as their input sampled at one-$N$th of the Nyquist rate. Any set of nonuniform samples of a band-limited signal can be divided into $N$ groups such that the complete set of sampling points can be expressed as

$$t_p + nT, \quad p = 0, 1, \cdots, N - 1, n = 0, \pm 1, \pm 2, \cdots \quad (2.48)$$
where \( T = N/2B \) and \( B \) is the bandwidth of the band-limited input signal. For the general case, there is only one sample in each group and \( N \) is very large. As shown in Fig. 2.17, a CT band-limited signal \( x(t) \) can be reconstructed by a CT reconstruction filterbank with the synthesis filters generalized to a suitable reconstruction algorithm. An alternative implementation using a bank of DT filters can be obtained by using an interpolation identity defined in [53].

![Diagram](image-url)

Figure 2.17: Reconstruction from generalized uniform samples using a CT filterbank [53].

The above two methods of implementations are useful for all the RAs. For some RAs, an approximation of the algorithm is necessary for generating a suitable filter.
Chapter 3

Uniform Bandpass Sampling

Signals can be categorized as lowpass versus bandpass in terms of the center frequency. In the transmission of signal information over a communication channel, we always encounter bandpass signals. The modulation at the transmitter generates the bandpass signal, and the corresponding center frequency is not equal to zero. The demodulation at the receiver recovers the information-bearing signal located at DC (baseband) from the bandpass signal through frequency down conversion. With respect to the bandwidth of the equivalent lowpass signal, carrier-modulated signals can be classified into Single-SideBand (SSB) signals and Double-SideBand (DSB) signals. A modulated bandpass signal with an SSB equivalent complex lowpass signal can be expressed as

\[ y(t) = \text{Re}\{x(t)e^{j2\pi f_c t}\} = a(t)\cos(2\pi f_c t) - b(t)\sin(2\pi f_c t), \] (3.1)

where \( f_c \) is the carrier frequency, \( x(t) \) is the equivalent complex lowpass signal, \( x(t) = a(t) + j b(t) \), \( a(t) \), \( b(t) \) are called the quadrature (I/Q) components of the bandpass signal, and \( b(t) \) is the Hilbert transform of \( a(t) \) [55].

The modulated signals that satisfy the condition that their bandwidth is much smaller than the carrier frequency are termed narrowband bandpass signals and otherwise wideband bandpass signals. For a bandpass signal, it could be sampled either by LowPass Sampling (LPS) process or BandPass Sampling (BPS). BPS is a technique for undersampling a modulated signal to realize frequency down conversion through intentional aliasing with the sampling rate of being down to only twice the information bandwidth \( B \) (\( B << f_c \)), i.e., \( F_s \geq 2B \). LPS is based on the Shannon’s sampling theorem, and \( f_s \geq 2f_c + B \) (see Fig. 2.3). When \( f_c >> B, f_s >> F_s \). An example of sampled-data signal spectrum of LPS and BPS is shown in Fig. 3.1. The randomly generated real band-limited signal is modulated by a sinusoidal signal \( \cos(2\pi f_c t) \) (i.e., \( y(t) = a(t)\cos(2\pi f_c t) \)) and \( f_c = 500 \). The minimum sampling rate, \( f_s = 2f_c + B \) (\( B = 50 \)) and \( F_s = 2B \), are used for LPS and BPS, respectively. Obviously, the output signal spectrum of LPS is the periodic replica of the original modulated bandpass signal with period of \( f_s \), which
Figure 3.1: a) Original modulated bandpass signal with a bandwidth $B = 50$ and $f_c = 500$; b) The corresponding frequency spectrum of sampled-data signal by LPS, $f_s = 2f_c + B$; c) The corresponding frequency spectrum of sampled-data signal by BPS, $F_s = 2B$.

is similar as shown in Fig. 2.3. The output spectrum of BPS is equivalent to a periodic replica of a lowpass signal with a bandwidth of $B$ in the period of $F_s$. The sampled-data signal at the output is at baseband. The BPS technique shrinks the Nyquist interval $[-f_s/2, f_s/2]$ based on the first Nyquist criterion to the new narrow interval $[-F_s/2, F_s/2]$ and realizes a frequency down conversion at the same time. With a goal of low power consumption, BPS is more and more attractive to mixed-signal system design. It has been extensively studied in optics, radar, sonar, communications and general instrumentation, etc.

The concept of software defined radio (SDR) has been paid more and more attention for its support of multi-mode wideband radio communications. One key technology of SDR is the placement and design technique of A/D converter in the channel processing stream, and it is a goal to put the A/D converter as close as possible to the antenna (see Fig. 1.5). By using conventional LPS, the sampling rate would be too high to be achieved by current design technology. BPS may be a solution for SDR by using a much lower sampling rate.

Besides the advantage of lower sampling rate, BPS has also limitations in real
implementations. The BPS rate has to be carefully chosen in order to avoid harmful signal spectrum aliasing. Noise aliasing is a direct consequence of lower sampling rate as compared to the highest frequency component of the input bandpass signal. The input signal frequency of a BPS is still high even though the sampling rate is low. It was shown in [56] that the jitter effects depend on both the variance of the random jitter and the input frequency such that the performance is degraded at the output of BPS as compared to the equivalent LPS system, an ideal image-rejecting mixer followed by an ideal lowpass sampler.

3.1 Sampling Rate Selection

The classic bandpass sampling theory states that for uniform sampling the signal can be reconstructed if the sampling rate is at least twice the information bandwidth. Feldman & Bennett [57] and Kohlenberg [15] showed that for uniform sampling, the minimum BPS rate is only valid for integer band position [16] [23], where

$$r = \lfloor f_l/B \rfloor = f_l/B$$

(see Fig. 1.6). The definition of band position has been given in section 1.6.

For uniform BPS, the determination of $F_s$ depends significantly on the band position which represents how far away the information band is from DC. To minimize the transmission bandwidth, it is important to know the minimum sampling rate for different band positions.

Assume that a DSB bandpass signal is located at $[f_l, f_u] \cup [-f_u, -f_l]$ as shown in Fig. 3.2 with a fractional band position, i.e., $|f_l/B| \neq f_l/B$, where $f_l$ and $f_u$ are given by $f_l = f_c - B$, $f_u = f_c + B$. The minimum sampling rate corresponds to the maximum number of foldings. For easily representing the signal spectral folding, we introduce the folding triangle in the width of $F_s$ (by dotted line in figures below).

For the positive frequency components, assume that the distance between $f_c$ and
the left boundary of the closest folding triangle is \( x \). Then we obtain

\[
(f_c - x) - nF_s = 0, \quad F_s = \frac{f_c - x}{n}
\]

(3.2)

It is also true that

\[
\begin{aligned}
\frac{B + x}{2} &\leq F_s \\
x &\geq B
\end{aligned}
\]

(3.3)

Substituting eq. (3.3) in eq. (3.2), the acceptable minimum sampling rate for fractional band position is obtained in the range of

\[
\frac{f_u}{n + 1} \leq F_s \leq \frac{f_l}{n}
\]

(3.4)

where \( n \) is the maximum number of folding triangles in \([0, f_l]\), \( n = \lfloor f_l/2B \rfloor \). The sampled data is still a bandpass signal located at \([-F_s, 0] \cup [0, F_s]\). A demodulation is needed to get the equivalent lowpass signal.

The minimal \( F_s \) for directly getting the equivalent lowpass signal can be obtained by the same way.

\[
(f_c - \frac{F_s}{2}) - nF_s = \frac{F_s}{2}, \quad F_s^{\text{min}} = \frac{f_c}{n + 1}
\]

where \( n = \max_{m \in \mathbb{Z}^+} \{f_c/(m + 1) \geq 2B\} \). As an example of a bandpass signal with \( B = 267 \) and \( f_c = 5000 \), it has a fractional band position and \( n = \max_{m \in \mathbb{Z}^+} \{5000/(m + 1) \geq 534\} = 8 \), then \( F_s^{\text{min}} \approx 556 \) (see Fig. 3.4 a)). The acceptable sampling rate for getting a bandpass signal in \([-F_s, 0] \cup [0, F_s]\]

is
3.1. SAMPLING RATE SELECTION

Figure 3.4: Examples of fractional band position and folding bands defined by different $F_s$, a) $F_s = 556$; b) $F_s = 585$; c) $F_s = 592$.

$585 < F_s < 592$ and $n = 8$ (see Fig. 3.4 b) and c)). Eq. (3.5) is also useful for a signal with an integer band position where $F_s^{\text{min}}$ is equal to $2B$.

The above discussion is based on a DSB bandpass signal. For an SSB bandpass signal with $f_l = f_c - B/2$ and $f_u = f_c + B/2$ (see Fig. 3.5), the acceptable uniform BPS rates have been obtained as [58] [16]

$$\frac{2f_u}{n} \leq f_s \leq \frac{2f_l}{n-1}, \quad (3.6)$$

Figure 3.5: An example of SSB bandpass signal with a fractional band position located at $\left[f_l, f_u\right] \cup [-f_u, -f_l]$
where $n$ is the integer given by

$$1 \leq n \leq \lfloor \frac{f_u}{B} \rfloor. \quad (3.7)$$

The minimum acceptable sampling rate corresponds to $n = \lfloor f_u/B \rfloor$. The DSB signal requests twice the channel bandwidth of the equivalent lowpass signal for transmission. The transmission bandwidth of the SSB signal is only half of the DSB signal.

For uniform BPS, the selection of acceptable sampling rate depends on the band position. The conditions of acceptable uniform BPS rate for SSB signals was depicted graphically by [57] and [16], where [57] only showed the minimum sampling frequency and [16]'s was an extension of [57]'s for all the cases. Brown [59] pointed out that for symmetric DSB signals, the spectra can be “folded over” each other without loss of information. This is exactly the case shown in Fig. 3.4 a), and the DSB of sampled-data signal is overlapped at the baseband. The corresponding sampled rates falls over the dashed-line within the disallowed area in the Fig. 3.6. However, it is difficult to adjust the BPS rate exactly to $F_{s_{\text{min}}}$. Any small sampling rate variation will cause $F_s$ move into the disallowed area such that an incorrect folding of signal spectrum happens.

![Diagram](image_url)

**Figure 3.6**: The allowed and disallowed (shaded area) uniform sampling rates versus the band position. $F_s$ is BPS rate, $B$ is the bandwidth, and the information band is located at $[f_l, f_u] \cup [-f_u, -f_l]$ [16].

It is observed from Fig. 3.6 that the set of allowable BPS rates consists of $n$ disconnected segments within $[2B, \infty)$. To do sampling efficiently, a lower sampling
rate is more attractive. With the increase of $f_u/B$ (or increasing $f_c$ of information band), $n$ is increased and consequently the gap between any two segments in the area of lower $F_s$ becomes narrower and narrower. Even a small error in the sampling rate might cause $F_s$ to fall into a disallowed area. The efficient selection of sampling rate becomes more and more difficult.

### 3.2 Noise Spectrum Aliasing

Additive noise is one common cause of performance degradation in telecommunication systems. Physically, the additive noise process may arise from electronic components and amplifiers at the receiver of the communication system or from the interference encountered in transmission. The noise at the input signal of BPS can be assumed as an Additive White Gaussian Noise (AWGN), i.e., having a delta-function autocorrelation with a flat power spectral density (PSD).

BPS technique can perform frequency down conversion by sampling a signal at an IF stage and shift the frequency to a lower IF stage or baseband through intentional signal spectral folding (also called downsampling mixer). However, the resulting SNR of BPS will be lower than that of the equivalent LPS system in the presence of thermal noise in sampling devices.

As shown in Fig. 3.7, the model of heterodyning a bandpass signal $y(t)$ to baseband in order to apply conventional LPS is shown in (a) as compared to the equivalent BPS shown in (b), where $y(t)$ is a bandpass signal and $x(t)$ is the equivalent lowpass signal of $y(t)$. In the LPS system shown in (a), a lowpass filter is used as an Anti-Aliasing (AA) filter prior to the lowpass sampler. Generally, a BandPass Filter (BPF) is needed as an AA filter prior to the bandpass sampler in the BPS system shown in (b). These AA filters can only reduce the out-of-band noise prior to the sampler.
It is known that a resistor charging a capacitor gives rise to a total thermal noise with power \( kT/C \) [1], where \( k \) is Boltzmann constant, \( T \) is the absolute temperature and \( C \) is the capacitance. The on-resistance of the switch will introduce thermal noise at the output. The noise is stored on the capacitor along with the instantaneous value of the input voltage when the switch turns off. As shown in Fig. 3.8, the resistor \( R_{on} \) and sampling capacitor \( C \) is an LPF with a transfer function of

\[
H(f) = \frac{1}{1 + j2\pi f R_{on} C}.
\]  

(3.8)

The PSD of thermal noise introduced by a resistor can be given by \( S_{in}(f) = 4kTR_{on} \). The corresponding PSD of noise at the output of LPF is given by

\[
S_{out}(f) = S_{in}(f)|H(f)|^2 = 2kTR_{on} \frac{1}{1 + 4\pi^2 f^2 R_{on}^2 C^2}
\]  

(3.9)

by a two-sided representation, and the total noise power is obtained as

\[
P_{out} = \int_{-\infty}^{\infty} S_{out}(f) df = \frac{kT}{C}.
\]  

(3.10)

The effective noise bandwidth of the sampling device \( B_{eff} \) depends on the on-resistance of the switch and the sampling capacitance, and it is normally larger than the maximum frequency of the input signal. Besides the capacitor switching noise (\( kT/C \) noise), op-amp wide-band noise and op-amp \( 1/f \) noise are two other noise sources with minor weights in practical SC circuits [60]. To simplify the following analysis, the dominant capacitor switching noise is regarded as the only noise source in the sampling device.

Based on Fig. 3.7 (b), assume that a bandpass signal is first fed into an ideal AA filter whose passband is located at \([-f_0 - B/2, -f_0 + B/2] \cup [f_0 - B/2, f_0 + B/2]\) \((f_0 \) is the center frequency of the bandpass signal), and then sampled by critical sampling (i.e., \( F_s = 2B \)). The sampled-data signal is located at baseband. For the equivalent system (ES) as shown in Fig. 3.7 (a), all the noise at the output is in-band, including the introduced thermal noise. Under the assumption that the
introduced thermal noise in a sampling device is an AWGN with zero-mean and the PSD of the noise is a constant $N_0$, the corresponding SNR is given by

$$SNR_{ES} = \frac{P_s}{P_{N_s} + P_{N_{Th}}} = \frac{P_s}{P_{N_i}},$$

(3.11)

where $P_s$ is the signal power, $P_{N_s}$ is the input signal noise power after the AA filter, $P_{N_{Th}}$ is the introduced thermal noise power and $P_{N_{Th}} = N_0 \cdot B$, $P_{N_i}$ denotes the total in-band noise power which is the sum of $P_{N_s}$ and $P_{N_{Th}}$. For BPS, the SNR is given by

$$SNR_{BPS} = \frac{P_s}{P_{N_i} + (M - 1)P_{N_{Th}}},$$

(3.12)

where $M = B_{eff}/B$ is the total number of $F_s$ bands within $[-B_{eff}, B_{eff}]$, $(M - 1)P_{N_{Th}}$ represents the total out-of-band thermal noise power. This SNR of BPS is consistent with the result obtained by Vaughan (see eq. (63) in [16]). When $P_{N_i} \gg P_{N_{Th}}$, SNR degradation is only loosely dependent on the effects of noise aliasing. However, when $P_{N_i} \approx P_{N_{Th}}$,

$$SNR_{BPS} \approx \frac{P_s}{M \cdot P_{N_i}},$$

(3.13)

and the SNR degradation in dB between BPS system and the equivalent LPS system is expressed as

$$SNR_{deg} \approx 10 \log_{10} M = 10 \log_{10} \frac{B_{eff}}{B}.$$ 

(3.14)

For more general case when $F_s > 2B$,

$$SNR_{BPS} \approx \frac{P_s}{M \cdot \frac{B}{F_s/2} \cdot P_{N_i}},$$

(3.15)

and the SNR degradation in dB becomes

$$SNR_{deg} \approx 10 \log_{10} \frac{B_{eff}}{B} \cdot \frac{B}{F_s/2} = 10 \log_{10} \frac{2B_{eff}}{F_s}.$$ 

(3.16)

Obviously, all out-of-band noise in BPS will be combined into each of the bands of width $F_s$. The higher the BPS rate, the lower $2B_{eff}/F_s$ and hence the lower SNR degradation.

It is known that an ideal uniform BPS is equivalent to an ideal uniform LPS followed by a decimation operation [24] (see Fig. 3.9) provided that the BPS rate $F_s = 1/T_s \geq 2B$ and the LPS rate $M/T_s \geq 2f_c + B$, where $M$ is the decimation factor.

The effects of noise aliasing can be graphically interpreted by the PSD spectrum. As shown in Fig. 3.10, a BPS is replaced by a LPS followed by a decimation and the noise aliasing in BPS is illustrated step by step. To avoid the noise aliasing
in LPS, the LPS rate is larger than or equal to $2B_{\text{eff}}$. The PSD of LPS from $-f_s/2$ to $f_s/2$ is shown in Fig. 3.10 (a). Assume that the minimum sampling rate $2B$ is used for BPS and $B_{\text{eff}}$ is an integer multiple $M$ of $B$. By doing an $M$-fold decimation on the output of LPS, the sampling rate will be reduced to the rate of BPS. Decimation is one of the most basic operations in multirate digital signal processing. It is also called decimator, downsampler or sampling rate compressor. For the $M$-fold decimation, the expression of the output PSD $Y_d(f_N)$ in terms of the input PSD $X(f_N)$ is given by [61]

$$Y_d(f/f_s) = \frac{1}{M} \sum_{k=0}^{M-1} X((f/f_s - k)/M),$$  \hspace{1cm} (3.17)

where

$$X(f) = \begin{cases} 2kTR_s, & -B_{\text{eff}} \leq f \leq B_{\text{eff}} \\ 0, & \text{others}, \end{cases}$$  \hspace{1cm} (3.18)

and $f/f_s$ is the normalized frequency. It can be interpreted as three steps: (i) stretch $X(f/f_s)$ by a factor $M$ to obtain $X(f/(M \cdot f_s))$, (ii) create $M - 1$ copies of

Figure 3.9: Identity elements of ideal uniform BPS

Figure 3.10: Illustration of noise aliasing in BPS
3.2. NOISE SPECTRUM ALIASING

this stretched version by shifting it uniformly in successive amounts of 1, and (iii) add all these shifted stretched versions to the unshifted stretched version \( X(f/f_s) \), and divided by \( M \). The final PSD spectrum of the \( M \)-fold decimation is shown in Fig. 3.10 (b). The noise PSD from \(-F_s/2\) to \(F_s/2\) is increased by \( M \) such that the output SNR by BPS is degraded as compared to the equivalent system shown in Fig. 3.7 (a).

![Figure 3.10: The effects of noise aliasing.](image1)

Figure 3.11: Demonstration of noise aliasing. (Top): a) Decimated sampled-data signal of LPS by factor 50 with a BPF, \( f_s = 25000 \), SNR\( \approx \) 54.4 dB; b)BPS by \( F_s = 500 \), SNR\( \approx \) 26.5 dB. (Bottom): LPS by \( f_s = 25000 \), SNR\( \approx \) 41.5 dB [24].

The effects of noise aliasing can also be demonstrated by simulations using the MATLAB \texttt{psd} function. A sinusoidal carrier signal with a carrier frequency of 5000 is modulated by a randomly generated band-limited signal with \( B = 50 \). A band-limited AWGN is added as the introduced thermal noise and \( B_{eff} = 12500 \). Assume that \( P_{N_i} = 0 \) such that \( P_{N_i} = P_{N_{Th}} \). To see the effects of noise aliasing, we choose \( F_s \) equal to 10\( B \) for uniform BPS (undersampling). As a reference, we choose \( f_s = 2B_{eff} \) for uniform LPS (oversampling). The sampled-data output signal by BPS is directly available at baseband. The resulting PSD spectra for
these two cases are shown in Fig. 3.11. To avoid the effects of the transition band of the AA filter, the passband and stopband frequencies are tactically selected as shown in Fig. 3.11. The SNR is evaluated by

\[
SNR = \frac{(\text{Avg}[R_i] - \text{Avg}[R_o]) \cdot B}{\text{Avg}[R_o] \cdot f_s/2},
\]

where \(R_i\) and \(R_o\) represents the in-band and out-of-band PSD, \(\text{Avg}[\bullet]\) denotes the average value in a given band of frequencies. The periodogram spectrum by averaging 20 power spectra is shown in Fig. 3.11. The SNR of BPS and LPS are about 26.8 dB and 42.5 dB, respectively (see Fig. (3.11)) and hence the degradation of SNR is about 15.7 dB. By eq. (3.16), \(\text{SNR}_{\text{deg}} \approx 17\) dB. The difference of 1.3 dB between the theoretical and simulated result is probably due to the transition band of the AA filter, a forth order Butterworth BPF. Sampling rate can be converted by either decimation (\(\downarrow f_s\)) or interpolation (\(\uparrow f_s\)). The sampling rate of LPS \(f_s = 2B_{\text{eff}}\) can be converted to 10B (the same as \(F_s\) of BPS) by decimation with a factor of 50. The resulting PSD spectrum is exactly the same as that of BPS due to the same noise aliasing (see Fig. 3.11 Top a)). If the sampled-data signal of LPS is first fed into a BPF and then decimated, the out-of-band noise is suppressed by the BPF and hence SNR is increased as compared to that of BPS. Note that when using BPS, the out-of-band noise cannot be suppressed by a filter.

### 3.3 Jitter Effects

The intention of sampling systems is to obtain a sample value at the corresponding time instant for an input signal. Based on sampling theorems, it is expected to uniquely determine the input signal by the sampled data information. The effects of random errors on the nominal sampling time instant are commonly called timing jitter. As shown in Fig. 3.12, the random error \(\tau_n\) which is a time offset from the nominal time instant \(t_n\) causes a random error \(\varepsilon_{\tau}(n)\) in the amplitude. The effect
of jitter on the spectrum of the signal may give rise to new discrete components and produce frequency selective attenuation [34].

The noise power due to jitter is given by

$$N_{\tau}(n) = E[\varepsilon_{\tau}^2(n)] - E[\varepsilon_{\tau}(n)]^2 = E[\varepsilon_{\tau}^2(n)]$$ (3.20)

under the assumption that jitter noise has a zero-mean, where $$\varepsilon_{\tau}(n) = y(t_n + \tau_n) - y(t_n)$$. The approximate normalized average noise power in the time domain is expressed as

$$\bar{N}_{\tau} = E\left\{ \lim_{K \to \infty} \frac{1}{K} \sum_{n=0}^{K-1} [\varepsilon_{\tau}^2(n)] \right\}. \quad (3.21)$$

For a sinusoidal signal $$y(t) = A\sin(2\pi f_{in}t)$$ and $$2\pi f_{in}\tau_n << 1$$, the error between input and output of a sampling system is given by

$$\varepsilon_{\tau}(n) \approx \tau_n \frac{dy(t)}{dt} = 2\pi f_{in}\tau_n A \cos(2\pi f_{in}t_n)$$ (3.22)

and the corresponding average noise power is approximately given by

$$\bar{N}_{\tau} \approx 2\pi^2 f_{in}^2 \sigma_{\tau}^2 A^2$$, \quad (3.23)

where $$\sigma_{\tau}^2 = E[\tau_n^2] - E[\tau_n]^2$$. However, when the jitter is larger such that $$2\pi f_{in}\tau_n << 1$$ is not satisfied, the average noise power becomes [25]

$$\bar{N}_{\tau} = A^2(1 - e^{-2\pi^2 f_{in}^2 \sigma_{\tau}^2})$$. \quad (3.24)

Note that $$\bar{N}_{\tau}$$ is independent of the sampling sequence $$t_n$$ but depends on $$f_{in}$$. The higher the value of $$f_{in}$$, the more noise power $$\bar{N}_{\tau}$$ and hence the larger jitter effects. Under the assumption of $$2\pi f_{in}\sigma_{\tau} << 1$$, eq. (3.24) reduces to eq. (3.23). Eq. (3.23) applies to all jitter distributions while eq. (3.24) assumes a Gaussian distributed jitter.

To study the jitter effects, a random jitter with Gaussian distribution $$N(0, \sigma_{\tau})$$ is applied to the real sinusoidal signal $$y(t) = \sin(2\pi f_{in}t)$$ with $$f_{in} = 10$$ and 500, respectively, where $$\sigma_{\tau}$$ is the standard deviation of jitter and $$\sigma_{\tau} = \alpha T_s$$ ($$\alpha$$ is a scale factor, $$\alpha = [1.15 \times 10^{-3}, 12 \times 10^{-3}]$$). The LPS rate is 5 $$f_{in}$$. The theoretical Signal-to-Noise-and-Distortion Ratio $$\text{SNDR}_t$$ and simulated $$\text{SNDR}$$ [44] are calculated using

$$\text{SNDR}_t = \frac{A^2/2 \bar{N}_{\tau}}{f_{in}} = \begin{cases} \frac{1}{(4\pi^2 f_{in}^2 \sigma_{\tau}^2)}, & f_{in} = 10 \\ \frac{1}{2(1 - e^{-2\pi^2 f_{in}^2 \sigma_{\tau}^2})}, & f_{in} = 500 \end{cases}$$ (3.25)

and

$$\text{SNDR} = \frac{\sum_{i=1}^{L} x_i^2}{\sum_{i=1}^{L} (x_i - \hat{x}_i)^2}.$$ (3.26)

$$\text{SNDR}$$ is normally used to measure the signal reconstruction error where $$x_i$$ and $$\hat{x}_i$$ denote the points from the original and reconstructed signal, respectively, $$i = [1, L]$$.
is the index of evaluated points and normally $L > N$ ($N$ is the number of sampled points). In this simulation, $x_i$ and $\hat{x}_i$ are used to represent the sampled points without jitter and with jitter respectively, and $L = N$.

It is observed from Fig. 3.13 that the theoretical SNDR is in agreement with the simulation result by LPS very well for both $f_{\text{in}} = 10$ and 500. However, when the real sinusoidal signal with $f_{\text{in}} = 10$ is frequency-shifted to $f_c = 500$ and then sampled by BPS with $F_s = 50$, the corresponding simulated SNDR is lower than that of the equivalent LPS system (see Fig. 3.13 Left). With the increase of $\sigma_\tau$ from $0.23 \times 10^{-4}$ to $2.4 \times 10^{-4}$, the SNDR difference varies from 13.5 dB to 27.6 dB. This difference is only due to the large jitter in BPS which is different from the SNR$_{\text{deg}}$ due to noise aliasing discussed in section 3.2. Jitter effects depend on both the standard deviation of random jitter and the input frequency (see eq. (3.25)). With the increase of input frequency by using BPS, jitter becomes a more crucial problem than in the equivalent LPS system.

![Figure 3.13](image-url)  
Figure 3.13: Comparison of theoretical and simulated SNDR for $y(t) = \sin(2\pi f_{\text{in}} t)$. Left: for LPS, $f_{\text{in}} = 10$ and $f_s = 5f_{\text{in}} = 50$; for BPS, $f_c = 500$ and $F_s = 50$. Right: $f_{\text{in}} = 500$, $f_s = 5f_{\text{in}} = 2500$.

Additionally, jitter effects for a general input signal was also discussed in [25]. Time skewing problem in A/D converter system which is very similar to the jitter problem was also analyzed and compared in [25].
Chapter 4

Quadrature Bandpass Sampling

Shannon (1949) mentioned in [36] that any function limited to the bandwidth $B$ and the time interval $T$ can be specified by giving $2BT$ samples. These samples are unnecessarily evenly spaced, and the samples from the signal and its derivative at half the Nyquist rate at least can also uniquely determine the signal without loss of information. Later Papoulis (1977) established the generalized nonuniform sampling theorem [33] which is an expansion of classic Shannon’s sampling theorem. It states that a band-limited signal is uniquely determined by the samples on the outputs of $M$ linear systems with input of the signal at one-$M$th of the Nyquist rate at least for each. The Papoulis’ generalization of sampling theorem treats extensively the representation of the signal from (i) the samples of the signal and its derivatives, (ii) Recurrent nonuniform sampling [48], (iii) the samples of the signal and its Hilbert transform (e.g. quadrature sampling) [59], and some other functions.

In digital communications, the modulated signal is always expressed in terms of $I/Q$ formats or in quadrature. The main advantage of $I/Q$ modulation is the symmetric case of combining independent signal components into a single composite signal and later splitting such a composite signal into its independent component parts [62]. It is more attractive to use quadrature mixers or quadrature BPS to separate the signal to $I$ and $Q$ parts before baseband.

4.1 Generalized Nonuniform Sampling

The generalized sampling expansion was first introduced in [33]. As shown in Fig. 4.1, given $M$ linear systems with transfer functions of $\{H_k(\omega)\}$, $k = 1, 2, \cdots, M$, the responses of the linear systems to an input band-limited signal $f(t)$ is given by

$$g_k(t) = \int_{-\omega_0}^{\omega_0} F(\omega)H_k(\omega)d\omega, \quad (4.1)$$
where \( \omega_0 = 2\pi B \) (B is the bandwidth of the signal), and \( F(\omega) \) is the Fourier transform of \( f(t) \). Each of the \( M \) responses is sampled at least in one-
\( M \)-th Nyquist rate. Define \( M \) Linear Time-Invariant (LTI) functions \( \{ y_k(t) \} \) such that the input signal \( f(t) \) can be obtained at the output in terms of the samples \( \{ g_k(nT) \} \) and the LTI functions \( \{ y_k(t) \} \):

\[
f(t) = \sum_{n=-\infty}^{\infty} \left[ g_1(nT)y_1(t-nT)+g_2(nT)y_2(t-nT)+\cdots+g_M(nT)y_M(t-nT) \right]. \tag{4.2}
\]

where \( T = 1/f_s \) and \( \{ y_k(t) \} \) is given by

\[
y_k(t) = \frac{1}{\Delta\omega} \int_{-\omega_0}^{-\omega_0+\Delta\omega} Y_k(\omega, t) e^{j\omega t} d\omega,
\]

and \( \Delta\omega = 2\omega_0/M \), \( T = 2\pi/\Delta\omega \), \( M \) unknown functions \( \{ Y_k(\omega, t) \} \) are determined by \( M \) linear expressions:

\[
H_1(\omega)Y_1(\omega, t)+\cdots+H_M(\omega)Y_M(\omega, t) = 1;
\]

\[
H_1(\omega+\Delta\omega)Y_1(\omega, t)+\cdots+H_M(\omega+\Delta\omega)Y_M(\omega, t) = e^{j\Delta\omega t}
\]

\[
\cdots \cdots \cdots
\]

\[
H_1[\omega+(M-1)\Delta\omega]Y_1(\omega, t)+\cdots+H_M[\omega+(M-1)\Delta\omega]Y_M(\omega, t) = e^{j(M-1)\Delta\omega t}.
\tag{4.3}
\]

Eq. 4.3 can be easily written in matrix form.

Figure 4.1: Identity of signal representation by Papoulis’ generalized sampling theorem.
4.1.1 Example – Derivative sampling

Starting from the generalized sampling theorem, if

\[ H_k(\omega) = (j\omega)^{k-1}, \]

\[ g_k(t) = \int_{-\omega_0}^{\omega_0} F(\omega)(j\omega)^{k-1}d\omega \]

\[ = f^{<k-1>}(t) \]  

(4.4)

based on the property of derivative of Fourier transform. The responses of the linear system \( H_k(\omega) \) are derivatives of input signal. Starting from eq. (4.3), the \( M \) linear expressions for determining the \( M \) unknowns \( \{Y_k(\omega, t)\} \) can be expressed in matrix form as

\[
\begin{bmatrix}
1 & j\omega & \ldots & (j\omega)^{M-1} \\
ge^{j\omega t} & e^{j(\omega + \Delta \omega) t} & \ldots & e^{j(M-1)\Delta \omega t} \\
\vdots & \vdots & \ddots & \vdots \\
1 & j[\omega + (M-1)\Delta \omega] & \ldots & [j[\omega + (M-1)\Delta \omega]]^{M-1}
\end{bmatrix}
\begin{bmatrix}
Y_1(\omega, t) \\
Y_2(\omega, t) \\
\vdots \\
Y_M(\omega, t)
\end{bmatrix}
= \begin{bmatrix}
1 \\
\vdots \\
e^{j(M-1)\Delta \omega t}
\end{bmatrix}
\]  

(4.5)

It can be solved in a closed form, using Cramer’s rule and the Vandermonde determinant [63].

Generalized sampling with \( M \) branches is also called \( M \)th-order sampling. It could be either US or NUS depending on the distribution of sampling time instants from all the branches. For a special case of \( M = 1 \), the generalized sampling theorem is reduced to the classic Shannon’s sampling theorem.

Figure 4.2: Identity of signal representation of derivative sampling, where \( f^{<k-1>} \) denotes the \( k - 1 \) order derivative of \( f(t) \).
4.1.2 Example – Recurrent nonuniform sampling

Starting from the generalized sampling theorem and Fig. 4.1, suppose that the sampling time instant of one branch lags behind the previous one by $\alpha_k$ and $|\alpha_k| < T/2$, then we have

$$g_k(t) = f(t + \alpha_k)$$

and

$$H_k(\omega) = e^{j\alpha_k \omega}.$$
4.1. GENERALIZED NONUNIFORM SAMPLING

Such sampling is also referred to as bunched or interlaced sampling [63]. The $M$ unknowns $Y_k(\omega, t)$ are given by

$$
Y_k(\omega, t) = \frac{1}{e^{j\alpha_k \omega}} \sum_{l=1}^{M} e^{j\alpha_l [\omega + (M-1)\Delta \omega]} \cdots e^{j\alpha_k [\omega + \Delta \omega]} \cdots e^{j\alpha_2 [\omega + \Delta \omega]} e^{j\alpha_1 \omega}
$$

(4.6)

By using Vandermonde determinant, the closed form of the reconstructing function $y_k(t)$ is given by [eq. (4.47) in [63]]

$$
y_k(t) = \text{sinc} \left[ \frac{2B_N}{N} (t - \alpha_k) \right] \prod_{l=1, l \neq k}^{N} \frac{\sin \left[ \frac{2\pi B_N}{N} (t - \alpha_l) \right]}{\sin \left[ \frac{2\pi B_N}{N} (\alpha_k - \alpha_l) \right]}
$$

(4.7)

Yen also showed an expression of reconstruction function for recurrent nonuniform sampling which is similar to eq. (4.7) with only the replacement of $\text{sinc} \left[ \frac{2B_N}{N} (t - \alpha_k) \right]$ by $\frac{(-1)^m N}{2\pi B_N (t-\alpha_k - \frac{m}{N})}$ ($-\infty < m < \infty$ is the index of the sampled points) (eq. (9) in [48]).

4.1.3 Example – Quadrature sampling

A quadrature local oscillators (LOs) is normally used to split an input IF signal into $I$ and $Q$ parts and then fed into a quadrature sampler at a half-Nyquist rate ($M=2$). The quadrature LO realizes both frequency down-conversion and phase shift by $90^\circ$. As shown in Fig. 4.4, the filter $H_{tr}(\omega)$ is called Hilbert transformer. The frequency response of this filter is given by [55]

$$
H_{tr}(\omega) = \begin{cases} 
-j2\pi, & \omega > 0 \\
0, & \omega = 0 \\
-j2\pi, & \omega < 0 
\end{cases}
$$

(4.8)

This filter basically realizes a $90^\circ$ phase shift for all frequencies in the input signal. The response of $H_k(\omega) = \{1, H_{tr}(\omega)\}$ for an input signal has the same phase response as the output of a quadrature LO [1].

Starting from eq. (4.3), the corresponding reconstruction function for representing a signal by quadrature sampling is given by

$$
\begin{bmatrix}
1 \\
H_{tr}(\omega) \\
1
\end{bmatrix}
\begin{bmatrix}
Y_1(\omega, t) \\
Y_2(\omega, t)
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
\frac{e^{j\omega_0 t} - 1}{H_{tr}(\omega + \omega_0) - H_{tr}(\omega)} \\
1 - H_{tr}(\omega)
\end{bmatrix}
$$

(4.9)

in matrix form, where $\omega_0 = 2\pi B$. Hence,

$$
Y_1(\omega, t) = \frac{e^{j\omega_0 t} - 1}{H_{tr}(\omega + \omega_0) - H_{tr}(\omega)},
$$

$$
Y_2(\omega, t) = 1 - H_{tr}(\omega) \cdot \frac{e^{j\omega_0 t} - 1}{H_{tr}(\omega + \omega_0) - H_{tr}(\omega)}.
$$

(4.10)
CHAPTER 4. QUADRATURE BANDPASS SAMPLING

Figure 4.5: Model of second-order BPS based on the Kohlenberg’s sampling theorem, where \( x(t) = \text{Re}\{[f(t) + j\tilde{f}(t)]e^{j2\pi f_c t}\} \) is the bandpass signal with an SSB equivalent lowpass complex signal, \( f(t) \) is the real signal and \( \tilde{f}(t) \) represents the Hilbert transform of \( f(t) \).

Quadrature sampling is a special case of second-order sampling.

4.2 Quadrature Bandpass Sampling

As we discussed in section 3.1 that for uniform BPS, a minimum sampling rate \( 2B \) is only valid for a bandpass signal with an integer band position. Kohlenberg [15] showed that the minimum sampling rate in the form of an average can be applied for a bandpass signal independent of the band position by a second-order bandpass sampling. It was also stated that an SSB bandpass signal \( x(t) \) located at \((f_l, f_u) \cup (-f_u, -f_l)\) can be exactly represented by

\[
x(t) = \sum_{n=0}^{N-1} [x(nT)y(t - nT) + x(nT + \alpha)y(nT + \alpha - t)],
\]

(4.11)

where \( T = 1/B \) for the minimum BPS rate, \( y(t) \) is given by

\[
y(t) = \frac{\cos[2\pi f_u t - (r + 1)\pi B\alpha] - \cos[2\pi (rB - f_l)t - (r + 1)\pi B\alpha]}{2\pi Bt \sin[(r + 1)\pi B\alpha]} + \frac{\cos[2\pi (rB - f_l)t - r\pi B\alpha] - \cos[2\pi Bt - r\pi B\alpha]}{2\pi Bt \sin(r\pi B\alpha)},
\]

(4.12)

\( \alpha \) is the time lag between two sets of samples from the first and the second branches and it is arbitrarily selected except for the values which would make \( y(t) \) infinite, \( r \) is in the range of \([2f_l/B, 2f_l/B + 1) \) \( (r = 2f_l/B \) for integer band position and \( r = 2f_l/B + 1/2 \) for half integer band position). The model of second-order sampling for a bandpass signal based on the Kohlenberg’s sampling theorem is shown in Fig. 4.5. For each branch, the sampling rate is half of the equivalent BPS rate.

Assume that \( x(t) \) is an SSB bandpass signal and \( x(t) = \text{Re}\{x(t)e^{j2\pi f_c t}\} = f(t)\cos(2\pi f_c t) - \tilde{f}(t)\sin(2\pi f_c t) \), where \( \tilde{f}(t) \) is the Hilbert transform of \( f(t) \). The
input signals of samplers $x(t)$ and $x(t + \alpha)$ with an arbitrary time lag $\alpha$ are sampled at the same rate $1/T$. When $\alpha = 1/(4f_c) + m/(2f_c)(m = 0, \pm 1, \pm 2, \cdots)$ \[16\], the corresponding sampled-data signals are given by

$$x_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} [f(nT) \cos(2\pi f_c nT) - \tilde{f}(nT) \sin(2\pi f_c nT)]$$

$$x_s(t + \alpha) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} [f(nT + \alpha) \cos(2\pi f_c nT + 2\pi f_c \cdot \frac{1}{4f_c} + 2\pi f_c \cdot \frac{m}{2f_c})$$

$$- \tilde{f}(nT + \alpha) \sin(2\pi f_c nT + 2\pi f_c \cdot \frac{1}{4f_c} + 2\pi f_c \cdot \frac{m}{2f_c})]$$

$$= (-1)^{m+1} \sum_{n=-\infty}^{\infty} [f(nT + \alpha) \sin(2\pi f_c nT) + \tilde{f}(nT + \alpha) \cos(2\pi f_c nT)]$$

(4.13)

and there is a $90^\circ$ phase shift between $x_s(t)$ and $x_s(t + \alpha)$ (see Fig. 4.6). Therefore, this second-order sampling becomes the so-called quadrature BPS. For a special case
when $\alpha = T/2$, the second-order BPS becomes the conventional uniform BPS. The sampled-data signal at the output of each sampler is called $I$ (in-phase) and $Q$ (quadrature) component, respectively.

Without loss of generality, we assume that $\alpha = 1/(4f_c)$. It has been verified by simulations that two parallel uniform samples with a time shift of $1/(4f_c)$ are distinguished by the interpolation function $y(t)$ (see eq. (4.12)) such that the input signal can be reconstructed by these samples at the output. A band-limited SSB signal $x(t)$ is randomly generated as shown in Fig. 4.7, where $f_c = 100$, $(f_l, f_u) = (95, 105)$ with a half integer band position, $B = 10$, $\alpha = 1/4f_c = 0.0025$. It is sampled at $1/T = B$. The corresponding reconstructed result by eq. (4.11) and eq. (4.12) is shown in Fig. 4.7 (Center) and it is consistent with the original signal very well except the ends.

Besides the exact interpolation developed by Kohlenberg for a high-frequency band-limited function (see eq. (4.12)), Ries [64] also suggested a form of general reconstruction function derived from a lowpass reconstruction kernel. Shannon’s lowpass sampling theorem shows that a lowpass band-limited signal can be exactly reconstructed from its uniform samples by a sinc kernel. An alternative way to represent the bandpass signal by the samples is to use a carrier-modulated sinc function based on THEOREM 4.2 in [64]:

$$x(t) = \sum_{n=-\infty}^{\infty} x(t_n) s(t - t_n)$$

where

$$s(t) = \text{Re}\{\text{sinc}(2Bt)e^{2\pi f_c t}\}$$

(4.14)

and the set of $\{t_n\}$ consists of the samples from both $I$ and $Q$ branches. The corresponding reconstructed result is shown in Fig. 4.7 (Bottom). In general, eq. (4.14) could be extended to

$$s(t) = \text{Re}\{k(t)e^{2\pi f_c t}\},$$

(4.15)

for any BK discussed in Chapter 2 provided that the expression of $k(t)$ could be found.

It is observed that the reconstructed signal by eq. (4.11) is obtained at the original band position $(f_l, f_u)$. The frequency is not down-converted by BPS. An extra resampler is needed to digitize the reconstructed results before the A/D converter. By conventional LPS technique, quadrature lowpass signals are obtained by using a pair of analog multipliers or mixers prior to the sampler, for instance in homodyne architecture (see Fig. 1.2). The main advantage of quadrature BPS compared to the homodyne architecture is that the effect of DC-offset that occurs with quadrature mixers is removed.

### 4.3 Implementation of Quadrature BPS

Quadrature BPS as a special case of second-order BPS has been extensively studied. Rice and Wu [27] applied a single uniform sampling on a real bandpass analog
input signal and then obtained the equivalent low-pass quadrature components by a digital Hilbert transform and frequency translation. The corresponding block diagram is shown in Fig. 4.8. The innovations of this architecture include that (i) only a single sampling device is used for both I and Q branches instead of conventional double sampling devices; (ii) the quadrature component is computed via a digital Hilbert transform, which eliminates many problems of analog methods, such as temperature sensitivity and drifts in component values; (iii) the decimation by 2 on the samples is applied before the Hilbert transform by separating even samples and odd samples by a commutator. However, the bandpass signal is uniformly sampled such that a thorough receiver frequency plan is needed for avoiding harmful signal spectrum aliasing. The input bandpass analog signal is first frequency shifted by BPS to a lower IF, and then frequency translated to baseband by multiplying $e^{-j\pi n}$. It can be also done by decimating by 2 and modulating by $(-1)^n$ which is used in Pellon’s architecture [28].

Pellon also proposed an architecture of quadrature sampling with a double Nyquist Digital Product Detector (DPD) as shown in Fig. 4.9. The primary advantage of this architecture is the digital separation of the I and Q components by DPD such that the mismatch of I and Q parts from the analog devices can be avoided. The other property is the double Nyquist property. The input signal of DPD can be digitized into its I and Q components at the Nyquist rate. The sampling rate in an A/D converter is also the operation rate of digital filters. However, an extra mixer is needed to frequency down-convert to $f_{s1}/4$, where $f_{s1}$ is the sampling rate in the single A/D converter.

As a specific effect in BPS system, the noise aliasing cannot be avoided in a uniform BPS system. As discussed in chapter 2, NUS has the potential to suppress harmful signal spectrum aliasing by using a lower sampling rate. Considering the advantages and drawbacks of these two architectures, making use of NUS and RAs, a new Generalized Quadrature BPS (GQBPS) algorithm is proposed for suppressing the noise aliasing. As shown in Fig. 4.10, a real IF bandpass signal $x(t) = \text{Re}\{a(t)e^{2\pi f_c t}\}$ with an arbitrary band position is sampled by a second-order bandpass sampling, where

$$a(t) = i(t) + jq(t),$$

(4.16)

$i(t)$ and $q(t)$ represents the I and Q components of the equivalent lowpass complex signal $a(t)$. The uniform sampling period for each branch is $T_s$ and $T_s \leq 1/B$ such that the equivalent BPS rate is greater than or equal to $2B$. The samples from the second sampling branch lags behind those from the first by $\alpha$. A carrier-modulated sinc function $s(t)$ defined by eq. (4.14) is expected to obtain the reconstructed signal $\hat{x}(t)$ at the same band position as the input. It is well-known that the sinc function performs a lowpass filtering such that $s(t)$ has a property of BPF. This eliminates the need for an extra BPF to remove the out-of-band noise and unwanted image bands. A resampler is used to obtain the sampled-data signal of quadrature components $\hat{i}(mT'_s)$ and $\hat{q}(mT'_s)$ from $\hat{x}(t)$. Finally, $\hat{i}(mT'_s)$ and $\hat{q}(mT'_s)$ are quantized by corresponding A/D converters. The combined system consisting
of \( s(t) \) and the resampler can be realized digitally:

\[
c(t) = \sum_{m=-\infty}^{\infty} [s(mT_s') \delta(t - mT_s') + s(mT_s' + \alpha) \delta(t - mT_s - \alpha)],
\]

(4.17)

where \( 1/T_s' = f_c/N \) is a new data rate, \( N \) is a decimation factor and \( N \leq f_c/(2B) \) based on the Nyquist criterion. The resampler can also be built in the following A/D converters sampled at \( \{mT_s'\} \) and \( \{mT_s' + \alpha\} \), respectively.

### 4.3.1 Frequency domain analysis

In a previous study, it was found that the samples by NUS can determine the input signal uniquely. By using GQBPS algorithm, the input high frequency signal is unambiguously determined by the undersampled data and the selection of sampling rate is simple. It is also observed that the noise aliasing by GQBPS is unequally weighted and the noise gain due to aliasing is lower than by conventional BPS at some specific frequency bands. In the following subsections, the sampling rate selection, signal reconstruction and noise aliasing suppression are analyzed and illustrated in the frequency domain.

**Deterministic Input Signal**

It is known that the carrier-modulated bandpass signal can be represented either as a double sideband (DSB) signal or a single sideband (SSB) signal depending on the definition of the equivalent lowpass signal \( a(t) \) [55]. A DSB signal requires twice the channel bandwidth of the equivalent lowpass signal for transmission. For saving transmission bandwidth, SSB signal is generally used in radio communications, and \( a(t) \) is defined as eq. (4.16). The Fourier transforms of the equivalent complex lowpass signal \( a(t) \) and its complex conjugate \( a^*(t) \) are shown in Fig. 4.11, where \( I(f) \) and \( Q(f) \) is the Fourier transform of \( i(t) \) and \( q(t) \), respectively. The spectrum of the corresponding bandpass signal \( x(t) \) is illustrated in Fig. 4.12 a).

Ideal quadrature sampling is equal to the continuous-time (CT) input signal multiplied by two infinite sequences of Dirac delta functions where the second sequence lags behind the first by \( \alpha \):

\[
x_s(t) = x(t) \left[ \sum_{n=-\infty}^{\infty} \delta(t - nT_s) + \sum_{n=-\infty}^{\infty} \delta(t - nT_s - \alpha) \right].
\]

(4.18)

The corresponding Fourier transform of \( x_s(t) \) is

\[
X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} (1 + e^{-j2\pi kf_s})X(f - kf_s),
\]

(4.19)

where \( X(f) \) is the Fourier transform of \( x(t) \) and

\[
X(f) = \frac{1}{2} \left[ A^*(f + f_c) + A(f - f_c) \right],
\]

(4.20)
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Then $X_s(f)$ is given by

$$X_s(f) = A_l(f) + A_r(f),$$

(4.21)

where

$$A_l(f) = \frac{1}{2T_s} \sum_{k=-\infty}^{\infty} (1 + e^{-j 2\pi kf_s\alpha}) A^*(f + f_c - kf_s)$$

$$A_r(f) = \frac{1}{2T_s} \sum_{k=-\infty}^{\infty} (1 + e^{-j 2\pi kf_s\alpha}) A(f - f_c - kf_s)$$

as shown in Fig. 4.12 b) and c). The carrier modulated sinc function is a rectangular function centered at $\pm f_c$ (see Fig.4.12 d)). A convolution between $s(t)$ and $x_s(t)$ in the time domain is equivalent to a multiplication between $S(f)$ and $X_s(f)$ in frequency domain, where

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j 2\pi f t} dt$$

$$= \frac{1}{2f_s} \left[ \text{rect} \left( \frac{f - f_c}{f_s} \right) + \text{rect} \left( \frac{f + f_c}{f_s} \right) \right]$$

(4.22)

and

$$\text{rect}(f) = \begin{cases} 1, & |f| < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

(4.23)

It is observed that the spectrum of $A_l(f)$ and $A_r(f)$ is the periodic replica of $A^*(f + f_c)$ and $A(f - f_c)$ in the period of $f_s$, respectively. Two criteria for sampling rate selection which are independent of the band position of $x(t)$ are used such that $x(t)$ can be reconstructed from the two sets of samples $\{x(nT_s)\}, \{x(nT_s + \alpha)\}$ by $s(t)$ in the proposed algorithm:

1. To avoid overlap between the adjacent folding spectra within the set of $A_l(f)$ and $A_r(f)$, the sampling rate has to satisfy $f_s \geq B$.

2. To avoid overlap between the set of $A_l(f)$ and $A_r(f)$, the ratio of $f_c$ to $f_s$ should be an integer or a half integer, and $f_s \geq 2B$.

In the analyses below, it is assumed that the ratio $f_c$ to $f_s$ is an integer, i.e., $f_s = f_c/i, i = 1, 2, \cdots$. It is seen from eq. (4.19) that there is a phase shift due to time-lag $\alpha$. Without loss of generality, it is assumed that $\alpha = 1/(4f_c)$:

$$1 + e^{-j\pi k/f_s} = 1 + e^{-j\pi k(\alpha/(4f_c))} = 1 + e^{-j\pi k/(2i)}. \quad (4.24)$$

When $k/i$ is even, $e^{-j\pi k/(2i)} = \pm 1$ such that

$$1 + e^{-j\pi k/(4f_c)} = \begin{cases} 2, & k/i = 4l, \\ 0, & k/i = 4l + 2, \quad l = 0, 1, 2, \cdots \end{cases} \quad (4.25)$$
The frequency spectra analysis is shown in Fig. 4.12. We expect that all other frequency bands are filtered out by $s(t)$ except for two located at $[-f_c - f_s/2, -f_c + f_s/2]$ and $[f_c - f_s/2, f_c + f_s/2]$. To obtain the reconstruction with $s(t)$, it is expected that the spectra located at $[-f_c - f_s/2, -f_c]$ and $[f_c, f_c + f_s/2]$ are the same as the CT input signal spectra multiplied by a gain factor while the spectra located at $[-f_c, f_c]$ are zero. It is always the case as long as the above two criteria are used. For the spectrum located at $[-f_c, -f_c + f_s/2]$ and $[f_c - f_s/2, f_c]$ which is the copy of $A(f - f_c)$ and $A^*(f + f_c)$ respectively by $k = 2f_c/f_s = 2i$ foldings, they are always zero, based on the second condition in eq. (4.25) for $l = 0$. However, the spectrum at $[-f_c - f_s/2, -f_c]$ and $[f_c, f_c + f_s/2]$ is just the copy of $A^*(f + f_c)$ and $A(f - f_c)$ with zero folding ($k = 0$). Based on the first condition in eq. (4.25) for $l = 0$, the weight factor $1 + e^{-j2\pi k f_s/\alpha}$ in eq. (4.19) is equal to 2 and the gain factor $2f_c$ in $X_s(f)$ will be balanced by $S(f)$ (see eq (4.22)) such that the signal reconstruction is realized.

**Stochastic Input Signal**

For a randomly generated input signal $x(t)$, the Power Spectral Density (PSD) of $x_s(t)$ is given by

$$R_{ss}(f) = \frac{1}{T_s^2} \sum_{k=-\infty}^{\infty} 4\cos^2(\pi kf_s\alpha)R_{xx}(f - kf_s), \quad (4.26)$$

where $R_{xx}(f)$ is the PSD of $x(t)$. It is observed that it is always the case that

$$\cos^2(\pi kf_s\alpha) = \begin{cases} 1, & k/i = 4l, \\ 0, & k/i = 4l + 2, \quad l = 0, 1, 2, \cdots \end{cases} \quad (4.27)$$

Based on the same process for deterministic signal reconstruction, the PSD spectrum of the stochastic $x(t)$ can be obtained by $s(t)$ without loss of information.

All the above analysis is based on an ideal sampling case, i.e., the input band-pass signal is sampled without any noise from the sampling device. However, the introduced thermal noise can never be avoided during the sampling process in real applications. Assume that the introduced noise $e(t)$ is Gaussian distributed with a zero mean and a constant PSD $N_0/2$. It is band-limited into $[-B_{eff}, B_{eff}]$. The time-varying autocorrelation function of $x_s(t)$ is given by

$$r_{ss}(t + \tau, t) = E[e_s(t + \tau)e_s^*(t)], \quad (4.28)$$

where $e_s(t)$ is the sampled-data signal of $e(t)$, $\tau$ is a time lag and $E[\bullet]$ represents an expectation operation. The time-average of $r_{ss}(\tau)$ over a single sampling period is defined as [65]

$$\tau_{ss}(\tau) = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} r_{ss}(t + \tau, t)dt. \quad (4.29)$$
Using this definition, we obtain

\[ r_{ss}(\tau) = r_{ee}(\tau) \frac{1}{T_s} \sum_{k=-\infty}^{\infty} 4 \cos^2(\pi k f_s \alpha) e^{j2\pi kf_s \tau} \]

The total noise power of \( e_s(t) \) within the fundamental noise bandwidth \([-B_{eff}, B_{eff}]\) is obtained when \( \tau = 0 \):

\[ P_e = N_0^2 \cdot \sum_{k=-M}^{M} 4 \cos^2(\pi k f_s \alpha), \quad (4.30) \]

where \( M = B_{eff}/f_s \) and \( k \) represents the different order of Nyquist bands in the period of \( f_s \). As compared to the \( P_e \) of conventional uniform BPS that is given by

\[ P_e = N_0^2 \cdot \sum_{k=-M}^{M} 2 = \frac{N_0^2}{2} \cdot f_s^2 \cdot 4M, \quad (4.31) \]

the noise aliasing by conventional uniform BPS is equally weighted but not by GQBPS. It is observed that neither of the cases can avoid noise aliasing. \( M \) represents the number of noise spectral foldings. The lower the sampling rate \( f_s \), the larger the value of \( M \) such that \( P_e \) is increased at the output of BPS system. The noise power within each of Nyquist bands is the same for conventional uniform BPS. However, for GQBPS, it is not constant but varying around a mean value

\[ \text{Mean} \left\{ \sum_{k=-M}^{M} 4 \cos^2(\pi k f_s \alpha) \right\} = 4M. \quad (4.32) \]

The factor of \( 4f_s^2 \) in both eq. (4.30) and (4.31) will be balanced by the gain of signal power (see eq. (4.26)). The gain of noise due to aliasing only depends on \( M \), which is consistent with eq. (3.16). As shown in Fig. 4.13, it is observed that the noise gain due to aliasing by GQBPS has a certain shape depending on the sampling parameters. It is the same as conventional uniform BPS at ±\( f_c \) but lower in the range \( R = ((4n+1)f_c, (4n+3)f_c) \), with the lowest at \( (4n+2)f_c \) (\( n = 0, \pm 1, \pm 2, \cdots \)).

With the increase of \( f_s \) or \( M \), the noise gain by GQBPS approaches to a constant value \( M \).

Obviously, the amplified in-band noise still survives from the process of the BPF \( s(t) \), but the out-of-band noise and other uninteresting image bands are filtered out. However, if we shift the passband of \( s(t) \) to the frequency bands with the lowest noise gain, the SNR of GQBPS algorithm will be larger than that by conventional uniform BPS. This advantage of performance improvement becomes more significant with the decrease of \( f_s \). As a consequence, the reconstruction becomes more complicated since the spectra located in \( R \) are not real anymore. The GQBPS might be a potential way to reduce the noise aliasing at the cost of a more
complicated reconstruction algorithm. We can combine this filter \( s(t) \) with the following resampler, and both the reconstruction and resampling can be performed by DSP (see eq. (4.17)). The thermal noise introduced in analog sampling devices will not present in the resampler realized digitally and consequently no noise aliasing happens. The output SNR of GQBPS algorithm is on average the same as by conventional uniform BPS if \( s(t) \) is defined by eq. (4.22).

### 4.3.2 Simulation results and discussions

The signal reconstruction and noise aliasing can be demonstrated by simulations using the MATLAB `psd` function. Assume that a 2.11 GHz RF signal is received at the antenna based on W-CDMA standard. The selected channel with 5 MHz bandwidth is centered at 700 MHz after the first mixer. Scaling down by \( 10^6 \), a randomly generated band-limited SSB signal \( a(t) \) with \( B = 5 \) is frequency-translated to \( f_c = 700 \) by multiplying with a sinusoidal carrier. A band-limited white Gaussian noise is added into such that \( P_N = P_{N_{Th}} \) (see eq. (3.12)) and \( B_{eff} = 10f_c \). Oversampling with respect to the BPS theorem is used to see the effects of noise aliasing and \( f_{s1} = f_c/2 = 350, f_{s2} = f_c/7 = 100 \). The simulation results are shown in Fig. 4.14.

It is observed that by GQBPS, the folded spectrum located at \([695, 700]\) disappears while a copy of the input signal spectrum corrupted by noise is located at \([700, 705]\). The signal reconstruction can be realized by the BPF \( s(t) \). It is in agreement with the analyses in section 4.3.1. An alternative \( s(t) \) with a narrower passband centered at \( \pm(f_c + B/2) \) can be also used. The SNR of the interesting band is decreased by around 5 dB for both conventional BPS and GQBPS when the sampling rate is decreased from \( f_{s1} \) to \( f_{s2} \). It is consistent with the theoretical evaluation result \( 10 \log_{10}(f_{s2}/f_{s1}) \approx 5.4 \) dB.

Quadrature processing inevitably encounters the I/Q mismatches in real implementations. The I/Q mismatches cause a signal component related to \( a^*(t) \) to appear also in the band of \( a(t) \) \([66]\), and vice versa. By GQBPS, both the wanted signal bands and “self-images” are sampled and processed. Due to the zero value of \( 1 + e^{-j2\pi f_s \alpha} \) or \( \cos^2(\pi f_s \alpha) \), both the wanted information bands and the associated “self-images” are transmitted to zero within the passband of \( s(t) \). The folded spectra of “self-images” will not overlapped with the wanted information bands. The “self-image” problem due to I/Q mismatches can be overcome by the proposed GQBPS algorithm. However, the presented approach is sensitive to phase shift of the sampling clock, e.g., by jitter in the sampling device. This limits the performance of GQBPS.
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Figure 4.7: Reconstruction of an SSB signal. (Top): input real bandpass signal, "×" and "○" represent the sampling positions of two parallel uniform samples by a quadrature bandpass sampling; (Center): reconstruction by Kohlenberg’s interpolation function defined by eq. (4.12); (Bottom): reconstruction by a modulated sinc function defined by eq. (4.14).
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Figure 4.8: Sequence of operations to obtain lowpass complex samples from uniform real real samples of a bandpass signal [27]

Figure 4.9: Digital baseband converter with digital product detector (DPD) [28]

Figure 4.10: Generalized Quadrature BPS architecture
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Figure 4.11: Illustration of SSB signal spectra.

Figure 4.12: Frequency spectra analysis of the proposed GQBPS algorithm.
Figure 4.13: Demonstration of noise gain due to noise aliasing by GQBPS based on eq. (4.30). (Top): $f_s = 100$, $f_c = 700$, $B_{eff} = 5f_c$, $M = 35$; (Bottom): $f_s = 700$, $f_c = 700$, $B_{eff} = 5f_c$, $M = 5$. 
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Figure 4.14: Comparison of normalized sampled-data spectra, (Top): by conventional BPS, (Bottom): by GQBPS, where $f_c = 700$, $B_{eff} = 10f_c$, $B = 5$, $f_{s1} = f_c/2 = 350$ (solid line) and $f_{s2} = f_c/7 = 100$ (dash-dotted line).
Chapter 5

Nonuniform Random Sampling

Nonuniform random sampling is very close to the realistic implementation. Under the effects of jitter, deterministic sampling becomes nonuniform random sampling. The NUS can be further classified into ideal nonuniform sampling (ideal NUS), jitter sampling (JS), and additive random sampling (ARS) [32]. Ideal NUS is a deterministic sampling. The other two are random sampling and the sampling time is not predetermined but is defined by the stochastic process of random jitter. JS is a common form in real life since the intentional US is generally used. ARS is equivalent to a nominal ideal NUS under the effects of jitter.

The sampling theorem defines the minimum sampling frequency $2B$, below which the reconstruction of a band-limited signal $x(t)$ is impossible ($B$ is the bandwidth of $x(t)$). From a practical point of view, the sampling rate $f_s$ must be many times greater than $2B$ to avoid aliasing. However, we neither encounter band-limited signals nor ideal filters in the real world. In addition, we have no possibility to obtain an infinite set of samples of the function. Finding a way to unambiguously determine a signal (either ideal or nonideal band-limited) by a finite number of samples is more of a challenge and interesting.

The NUS mentioned in section 2.1 is deterministic. It is known that NUS has the potential to suppress the harmful spectrum aliasing of sampled-data signal by a lower sampling rate even though the input signal is not ideal band-limited (see Fig. 2.4), and hence the requirements on the AA filter prior to the sampler is relaxed by using NUS. However, it is still a mystery to select the nonuniformly distributed sampling scheme such that the input nonideal band-limited signal is uniquely determined by the samples without the harmful effects of aliasing (i.e., alias-free sampling). Shapiro and Silverman [32], Beutler [67] and Marsy [68] successively gave or extended the definition and conditions for alias-free sampling. Wojtik [69] highlighted that alias terms can be suppressed by increasing jitter variance, and also showed that a jitter with a uniform distribution over $[-0.5T_s, 0.5T_s]$ has the potential to eliminate the discrete frequency components of the sampled-data signal except for the information signal. Shapiro [32] also showed that some random
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Sampling schemes (e.g., Poisson sampling) can eliminate aliasing and lead to an unambiguous determination of the PSD spectrum.

Random sampling and reconstruction are never a pair of things. Reconstruction requires a set of exact sampling times, but random sampling can never provide it. A stochastic process $x_s(t)$ by random sampling is not absolutely integrable, i.e., $\int_{-\infty}^{\infty} |x_s(t)| \neq \infty$, and the Fourier transform of $x_s(t)$ does not exist [37]. The study of PSD is a normal way to analyze random sampling.

5.1 Jitter Sampling

Jitter sampling is also called jittered periodic sampling [32] which is an ideal periodic sampling affected by timing jitter. The set of sampling time instants is of the form

$$t_s(n) = nT_s + \tau_n, \quad n = 0, \pm 1, \pm 2, \ldots, \quad (5.1)$$

where $T_s$ is an ideal US interval, $\tau_n$ are a family of independent, identically distributed (iid) Gaussian random variables with a zero-mean and a standard deviation $\sigma_\tau$. Normally $\sigma_\tau << T_s$. In [70], timing jitter are classified into readin jitters and readout jitters depending on the way to be introduced in the system. Readin jitters are introduced when the analog signal is being sampled, whereas readout jitter when the samples of the output of the digital filter are being read out for reconstruction back to an analog signal. In the present thesis work, only the case of readin jitters is considered. We assume that the effects of jitter are unknown. If they were known, the sampling theory for deterministic NUS could be used [63] [71].

Starting from eq. (2.1), the CT sampled-data signal by JS is given by

$$\tilde{x}_{js}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s - \tau_n) = \sum_{n=-\infty}^{\infty} x(t_s(n)) \delta(t - nT_s - \tau_n), \quad (5.2)$$

where the input signal $x(t)$ could be either deterministic or stochastic process and $\tilde{x}_{js}(t)$ is a stochastic process. The statistic process $\tau_n$ and $x(t)$ are independent.
The autocorrelation function of \( x_{j_s}(t) \) is given by

\[
r_{\tilde{x}\tilde{x}}(\gamma, t) = E_{x,\gamma} [\tilde{x}_{j_s}(t + \gamma)\tilde{x}_{j_s}(t)] \\
= E_{\gamma} [\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E_{x} [x(t + \gamma)x^*(t)] \cdot \delta(t + \gamma - mT_s - \tau_m)\delta(t - nT_s - \tau_n)] \\
= E_{\gamma} [\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} r_{xx}(\gamma) \cdot \delta(t + \gamma - mT_s - \tau_m)\delta(t - nT_s - \tau_n)] \\
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_{xx}(\gamma) \cdot \delta(t + \gamma - mT_s - \tau_m)\delta(t - nT_s - \tau_n) \\
p(\tau_m, \tau_n) d\tau_m d\tau_n \\
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_{xx}(\gamma) \cdot \delta(t + \gamma - mT_s - \tau_m)\delta(t - nT_s - \tau_n) \\
p(\tau_m) p(\tau_n) d\tau_m d\tau_n \\
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} r_{xx}(\gamma)p(t - mT_s - \tau_m)p(t - nT_s), \tag{5.3}
\]

for \( m \neq n \), where \( E[\bullet] \) represents an expectation operator, \( E_{x,\gamma} \) is the average over the product statistics of \( \tau_n \) and \( x(t) \), \( E_{\gamma} \) is over the statistics of \( \tau_n \) and \( E_{x} \) over the statistics of \( x(t) \), \( r_{xx}(\gamma) \) is the autocorrelation function of \( x(t) \), \( \gamma \) is a time-lag between any two variables of stochastic process \( x_{j_s}(t) \), \( p(\tau_m, \tau_n) \) is the joint probability density function (PDF) of \( \{ \tau_n \} \) and \( \{ \tau_m \} \). The random variables \( \tau_n \) and \( \tau_m \) are assumed independent such that \( p(\tau_m, \tau_n) = p(\tau_m)p(\tau_n) \), where \( p(x) \) is the PDF of stochastic process \( x \). When \( m = n, \tau_m = \tau_n \),

\[
r_{\tilde{x}\tilde{x}}(\gamma) = r_{xx}(0)\delta(\gamma), \tag{5.4}
\]

where \( r_{xx}(0) \) corresponds to the total input signal power. Assuming that \( x_{j_s}(t) \) is a wide-sense stationary (WSS) process and \( x_{j_s}(t), x_{j_s}(t + \gamma) \) are jointly ergodic, the time average may be used to replace the ensemble average. The autocorrelation function of \( x_{j_s}(t) \) is simplified by time-average over a single sampling period [65]:

\[
r_{\tilde{x}\tilde{x}}(\gamma) = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} r_{\tilde{x}\tilde{x}}(\gamma, t) dt \\
= \frac{1}{T_s} r_{xx}(\gamma) \left( \sum_{l=-\infty}^{\infty} r_{pp}(lT_s + \gamma) - r_{pp}(\gamma) + \delta(\gamma) \right), \tag{5.5}
\]

where \( r_{pp}(lT_s + \gamma) \) is the convolution of two PDF functions. Based on Wiener-Khintchine Theorem, the PSD of the WSS process \( x_{j_s}(t) \) can be obtained from the
Fourier transform of the autocorrelation function \( r_{xx}(\gamma) \) \[65],

\[
R_{\tilde{x}\tilde{x}}(f) = \frac{1}{T_s} \int_{-\infty}^{\infty} r_{xx}(\gamma) \left( \sum_{l=-\infty}^{\infty} r_{pp}(lT_s + \gamma) - r_{pp}(\gamma) + \delta(\gamma) \right) e^{-j2\pi f \gamma} d\gamma
\]

\[
= \frac{1}{T_s} F\{r_{xx}(\gamma)\} \ast F \left\{ \sum_{l=-\infty}^{\infty} r_{pp}(lT_s + \gamma) - r_{pp}(\gamma) + \delta(\gamma) \right\}
\]

\[
= \frac{1}{T_s} R_{xx}(f) \ast \left( \sum_{l=-\infty}^{\infty} R_{pp}(f)e^{j2\pi lT_s} - R_{pp}(f) + 1 \right)
\]

\[
= \frac{1}{T_s} R_{xx}(f) \ast \left( \frac{1}{T_s} R_{pp}(f) \sum_{k=-\infty}^{\infty} \delta(f - kf_s) + [1 - R_{pp}(f)] \right)
\]

\[
= \frac{1}{T_s^2} \sum_{k=-\infty}^{\infty} R_{pp}(kf_s) R_{xx}(f - kf_s) + \frac{1}{T_s} R_{xx}(f) \ast (1 - R_{pp}(f)), \quad (5.6)
\]

where \( \ast \) denotes the convolution operator, \( F\{\bullet\} \) is the Fourier transform operator, \( R_{xx}(f) \) and \( R_{pp}(f) \) is the Fourier transform of \( r_{xx}(\gamma) \) and \( r_{pp}(\gamma) \), respectively and \( f_s = 1/T_s \).

It is observed that the PSD of JS is equivalent to the power spectrum of the original signal plus an “additive uncorrelated noise”. The first term of eq. (5.6) can be regarded as a discrete component while the second term a continuous component. The discrete component is a weighted sum of the periodically shifted copies of input spectrum \( R_{xx}(f) \) in the period of average sampling rate \( f_s \). It is not necessarily a periodic function except when \( R_{pp}(kf_s) \) is periodic. When the jitter is small, \( R_{pp}(kf_s) \) decreases slowly and the discrete component is almost periodic. For a special case where jitter \( \tau_n \) is zero, \( R_{pp}(f) \) reduces to \( \delta(\tau_n) \) and then \( R_{pp}(f) = 1 \), eq. (5.6) reduces to the average PSD of US:

\[
R_{\tilde{x}\tilde{x}}(f) = \frac{1}{T_s^2} \sum_{k=-\infty}^{\infty} R_{xx}(f - kf_s). \quad (5.7)
\]

However, for ARS \( t_n(n) \neq nT_s + \tau_n \), the corresponding PSD still consists of the power spectrum of the original signal plus an “additive uncorrelated noise”, but the additive part could be arbitrary.

The PSD of JS on a sinusoidal input signal with a random phase is shown in Fig. 5.2 and Fig. 5.3 for different jitter. The corresponding theoretical weights \( R_{pp}(f) \) and theoretical PSD evaluation based on eq. (5.6) are superimposed. The input frequency is 2 and the average sampling rate is 5. The jitter is assumed to have a uniform distribution \( U(\alpha T_s, \alpha T_s) \) where \( \alpha = 0, 0.1, 0.3, 0.5 \) is a scale factor defined by jitter and \( 1/T_s \) is the average sampling rate. All the theoretical weights \( R_{pp}(f) \) are shown in Fig. 5.1 for different jitter cases.

Without care of the continuous component (or the bias), the PSD of JS is a periodically shifted copies of input spectrum \( R_{xx}(f) \) in the period of average
5.2. ADDITIVE RANDOM SAMPLING

Due to the contribution of $nT_s$, the PSD of JS at $t_s(n) = nT_s + \tau_n$ still retains the periodic property in the period of $1/T_s$ such that the aliasing is still presented in JS. It is also observed that when jitter has a uniform distribution over $[-0.5T_s, 0.5T_s]$, sampling rate $f_s = 5$ shaped by the weight of $R_{pp}(f)$. From the simulation result (see Fig. 5.2 and Fig. 5.3), it is observed that the input spectrum is weighted, and it matches with the theoretical estimation very well (see Fig. 5.1). When $\alpha = 0$ (or ideal US), image spectra appear at higher order Nyquist bands (2nd order $[2.5, 7.5]$, 3rd order $[7.5, 12.5]$, ···). The corresponding weight is a flat straight line since the PDF $p(\tau) = \delta(\tau)$ in time domain. With the increase of $\alpha$, the amplitude of image spectra decreased with the increase of frequency and the peak level is shaped by the weight function. When $\alpha$ is increased to 0.5, all image spectra in higher order Nyquist bands disappear and the spectrum uniquely identifies the input signal. This simulation result is also consistent with the conclusion given by Wojtiuk [69]. The corresponding sampling scenario is one kind of alias-free sampling.

Figure 5.1: Theoretical weights $R_{pp}(f)$ based on eq. (5.6) for different jitter cases.
i.e., the samples of JS get rid of the characteristics of US and distribute completely irregularly, aliasing from higher order Nyquist bands are significantly suppressed.

Shapiro [32] first noticed this and introduced Additive Random Sampling which breaks up the regular property from JS. It was defined that the samples are located at

\[ t_n = t_{n-1} + \gamma_n, \]  

(5.8)

where \( t_{n-1} \) and \( t_n \) are two successive sampling time instants, \( \gamma_n \) is an iid stochastic process with a certain distribution. There exists an average \( T_s \) such that \( E[\gamma_n] = T_s \) but \( t_n - t_{n-1} \neq T_s \). The PDF of \( \{\gamma_n\} \) is equal to zero (i.e., \( p(\gamma_n) = 0 \)) for \( \gamma_n < 0 \). This condition corresponds to the requirement that a set of samples in a given set of indices should come successively in the time order.

This is equivalent to a nominal ideal NUS under the effects of jitter, since

\[ t_s(n) = t_n + \tau_n \]

\[ = t_{n-1} + \gamma_{n-1} + \tau_n \]

\[ = t_{n-1} + \gamma'_n, \]

where \( \{t_n\} \) is the set of sampling time instants of nominal ideal NUS, \( E[\tau_n] = 0 \) and \( E[\gamma'_n] = E[\gamma_n] = T_s \).

### 5.3 Alias-free Sampling

In [32], it was shown that the aliasing can be avoided if the sampling occurs in a Poisson process with an average rate of \( \rho \). For the given Poisson process \( \{\gamma_n\} \), the corresponding Poisson distribution in terms of the average rate is given by [32]

\[ p(\gamma) = \rho e^{-\rho \gamma}. \]  

(5.10)

The same sinusoidal input signal with a random phase that is used for presenting the PSD of JS is also used for simulating the PSD of ARS. The input frequency is 2 and the average sampling rate is 5. The inter-sample intervals (ISI) \( \{\gamma_n\} \) satisfy the Poisson process defined by eq. (5.10). The corresponding simulated PSD is shown in Fig. 5.4. The PDF of \( \{\gamma_n\} \) used in the simulation is shown and compared with the theoretical PDF of Poisson process, see Fig. 5.5. It is observed that only the frequency component of \( f = 2 \) in the PSD exists and aliasing effects from other Nyquist bands are completely avoided. However, the noise floor is significantly increased such that SNR is degraded. Compared to Fig. 5.3 (Bottom), the in-band noise power by this ARS is higher than that by JS with the jitter distribution \( U[-0.5T_s, 0.5T_s] \).

Random sampling which is under the effects of jitter usually causes performance degradation in radio communications. However, by making use of the random sampling, aliasing can be suppressed efficiently while the signal reconstruction becomes hardly achievable.
Figure 5.2: The PSD of JS on a sinusoid input signal with $f = 2$ for different jitter and $f_s = 5$. (Top): $\alpha = 0$; (Bottom): $\alpha = 0.1T_s$. 
Figure 5.3: The PSD of JS on a sinusoid input signal with $f = 2$ for different jitter and $f_s = 5$ (cont.). (Top): $\alpha = 0.3T_s$; (Bottom): $\alpha = 0.5T_s$. 
Figure 5.4: The PSD of ARS with Poisson process. The input frequency is 2 and the average sampling rate is 5.
Figure 5.5: The practical $p(\gamma)$ (in vertical bar) which is used for above simulation as compared to the theoretical $p(\gamma)$.
Chapter 6

Noise and Jitter Performance on RAs

In chapter 2, nine RAs are studied. Three among the nine RAs based on interpolation are possibly used by sample-by-sample basis for online radio communications. The BKs of the three RAs are studied and presented in both time and frequency domain. Although it is shown that jitter errors cannot be canceled by using the RAs but amplified by a large kernel, it is still of interest to study the sensitivity to jitter and SNR responses for these RAs. As also discussed in chapter 3 that the jitter effects depend on both the input frequency and standard deviation of random jitter. BPS provides an interface to a higher input frequency signal. The SNR by BPS is degraded as compared to the equivalent LPS system in the presence of same random jitter. However, this difference of SNR is due to the larger jitter effects in BPS but not noise aliasing as discussed in section 3.2.

In this chapter, a concise model of radio receiver front-end based on BPS is modeled in MATLAB. The sensitivity to jitter of these three RAs are studied by SNDR evaluation based on eq. (2.42). As we have shown in chapter 5 that the sampling under the effects of jitter can be classified into jitter sampling and additive random sampling based on the nominal US and NUS, respectively. The jitter effects are studied for JS and ARS, respectively. The corresponding simulation results are shown and compared in this chapter.

6.1 Modeling

For studying the signal reconstruction by RAs in the presence of AWGN and jitter in BPS, a simple system model of a radio receiver front-end by BPS is simulated in MATLAB. Considering a current wireless communication standard, Wideband Code-Division Multiple-Access (W-CDMA) is one of the main technologies of 3G cellular systems. The required frequency band is located at 1920 MHz - 1980 MHz and 2110 MHz - 2170 MHz for uplink and downlink, respectively. The channel
spacing is 5 MHz [72]. Most traditionally used radio receiver architecture is the conventional superheterodyne receiver architecture, which normally includes two mixers by using two local oscillators (LOs). With respect to the BPS technique and the concept of SDR, the first IF stage is directly followed by a bandpass sampler and the output of BPS is at baseband. We assume that a 2.11 GHz RF signal is received at the antenna based on W-CDMA standard. The selected channel with 5 MHz bandwidth is centered at 500 MHz after the first mixer. For conveniently analyzing AWGN and jitter effects on RAs performance, the sampled data signal is fed into a reconstruction filter which is generalized by a RA.

As shown in Fig. 6.1, scaling by $10^5$, a sinusoidal carrier signal \( \cos(2\pi f_c t) \) is modulated by a randomly generated band-limited signal \( x(t) \), where \( f_c = 5000 \) and \( B = 50 \) (50@5000). The passband of the bandpass signal is located at \([-f_c - B, f_c + B] \cup [f_c - B, f_c + B]\) (a DSB signal). The AA filter is a fourth-order Butterworth BPF whose passband is exactly the same as the passband of the input bandpass signal. The modulated signal \( y(t) = \text{Re}\{x(t)e^{j2\pi f_c t}\} \) is either uniformly or nonuniformly sampled with an average rate of \( f_s = 4B \). For JS and ARS, jitter \( \{\tau_n\} \) has a Gaussian distribution \( N(0, \sigma_\tau) \) (\( \sigma_\tau \) is the standard deviation of \( \{\tau_n\} \)). The reconstruction error is measured by SNDR (see eq. (2.42)), For easily tracking the discussion, the definition of SNDR is given here again:

\[
\text{SNDR} = \frac{\sum_{i=1}^{L} x_i^2}{\sum_{i=1}^{L} (x_i - \hat{x}_i)^2}, \tag{6.1}
\]

where \( x_i \) and \( \hat{x}_i \) denote the points from the original and reconstructed signal, respectively, and \( L > N \) (\( N \) is the number of sampled points). Three RAs based on eq. (2.21) and three approximate expressions of kernel \( k(t, t_n) \) are evaluated in the simulation: Low-pass filtering (LPF), Lagrange interpolating polynomial and Spline interpolation (see the algorithm description in section 2.2).

Based on the above specification of modeling and simulation, we sampled 201 points within a unit time period out of an approximate CT signal consisting of 40001 points. Only the middle range of \( t \in [0.4, 0.6] \) is evaluated due to the divergence of Lagrange interpolating polynomial at the interval ends. A sliding window at

![Figure 6.1: A simplified model of a BPS receiver.](image)
the input combined with a low order polynomial is a way to implement Lagrange interpolating polynomial in real applications.

6.2 Sensitivity of RAs to Jitter

In the presence of only jitter, Fig. 6.2 shows that when the band position is close to DC ([a] and [c] 50@1000), the reconstruction performance of all the RAs is good and stable. With the shift of band position to a higher frequency ([b] and [d] 50@5000), the SNDR has a dramatic degradation with the increase of $\sigma_\tau$ for both JS and ARS.

It is known that a sinc kernel has a bad reconstruction performance for NUS [44], although it is good for US. For small jitter, the jitter error is additive (see eq. (3.22)) such that the frequency spectrum of JS still retains the periodic property of US spectrum with only spreading at the tone-frequency and higher power levels at neighboring frequencies. This periodic nature does not exist for ARS such that LPF based on a sinc kernel always performs worse for ARS than for JS. The Lagrange interpolating polynomial has good reconstruction performance for both US and ideal NUS [44]. Comparing [a] and [c], the reconstruction performance of the Lagrange interpolating polynomial is not sensitive to jitter and is also independent of the sampling distribution (JS or ARS) when jitter is small. When jitter is large, all the RAs perform badly (see [b] and [d]). It is concluded that the RAs based on interpolation do not provide any immunity to a large jitter.

6.3 Sensitivity of RAs to AWGN

The SNR response to an input disturbed by noise is defined as

$$\frac{SNR_{out}}{SNR_{in}} = \frac{P_{out}/\sigma_{out}^2}{P_{in}/\sigma_{in}^2},$$

which is the inverse of the noise reduction ratio [35], where $P_{in}$ and $P_{out}$ are the power of noise-free input and output signal, $\sigma_{in}^2$ and $\sigma_{out}^2$ are the noise variance of input and output. In this simulation model, $P_{out} = P_{in}/2$. The $SNR_{in}$ and $SNR_{out}$ is evaluated at the input of BPS after the AA filter and the output of RAs, respectively.

Based on the same model, the SNR response of RAs is studied and two different noise effects are considered in BPS device, with only band-limited AWGN $e(t)$ and with both the band-limited AWGN and random jitter. As shown in Fig. 6.3, with only AWGN effects (in solid line), the SNR response at the output approximates the input SNR in the range of $SNR_{in} \in [0, 10]$ dB. With the increase of $SNR_{in}$, the performance of each RA shows no significant improvement but approaches a constant that is defined by the current RA. Under the effects of both AWGN and jitter (in dash-dotted line), $SNR_{out}$ of each RA decreases by a larger value compared to the corresponding jitter free case. With the increase of $SNR_{in}$, $SNR_{out}$ of each
Figure 6.2: SNDR evaluation for jitter effects on BPS and $B = 50$, $f_s = 4B$. [a] $f_c = 1000$ JS. [b] $f_c = 5000$ JS. [c] $f_c = 1000$ ARS. [d] $f_c = 5000$ ARS.
RA approaches a constant that is defined by the current RA under the effects of only jitter.

Comparing Fig. 6.3 (top) and (bottom), LPF shows the best SNR response while Lagrange interpolating polynomial shows the worst at the output of the reconstruction filter for both US and JS. For JS, all RAs have equal bad performance. Spline interpolation shows the highest performance while LPF has the worst for ideal NUS. For ARS, spline interpolation and Lagrange interpolating polynomial are equally bad, but the LPF is even worse than theirs. It is also observed that spline interpolation algorithm has around 1 dB enhancement on the SNR at the output for SNR in ∈ [0, 10] dB for both US and ideal NUS. It is also seen that the RAs can not provide any immunity to noise aliasing. On the contrary, when the SNR is high at the input, the performance of the RAs limits the SNR performance at the output.

6.4 Jitter Noise Effects

To study the jitter noise effects in the BPS, a random jitter with Gaussian distribution \( N(0, \sigma_r) \) is applied to the randomly generated bandpass signal, where \( \sigma_r = 5.8 \times 10^{-5} \). The center frequency is 5000 (50@5000). The bandpass signal is sampled by uniform and nonuniform BPS in the presence of jitter (i.e., JS and ARS). An oversampling rate of 10\( B \) is used to see the effects of jitter on both in-band and out-of-band. It is better to discuss noise effects by PSD rather than by a signal reconstruction performance. The PSD of the sampled-data signal by BPS is obtained by MATLAB \texttt{psd} function.

As shown in Fig. 6.4, the out-of-band noise power \( P_{N_o} \) is increased due to the large jitter effects. \( P_{N_o} \) is evaluated for \( f > 100 \) to avoid the effects from the transition band of the AA filter. The \( P_{N_o} \) of ARS and JS are about 11.1 dB and 23.6 dB higher than that of ideal NUS and US, respectively. The jitter mainly affects the out-of-band noise power, and the effects of in-band are negligible. Comparing the dotted lines in Fig. 6.4 (top) and (bottom), the noise power by ideal NUS is larger than for ideal US. Even though the \( P_{N_o} \) increment of ARS is much lower than that of JS, the output performance by ARS and JS is still in the same order.
Figure 6.3: Comparison of SNR responses of RAs for US and JS (top), NUS and ARS (bottom) in BPS, $B = 50$, $f_c = 100B = 5000$, $f_s = 4B$. 
6.4. JITTER NOISE EFFECTS

Figure 6.4: The PSD of sampled-data signal by BPS with $f_c = 5000$, $B = 50$ and $f_s = 10B$ for JS (top) and ARS (bottom).
Chapter 7

Conclusions and Future Work

In this thesis work, the current existing receiver architectures are reviewed and compared to SDR receiver. Basic sampling and reconstruction theory are studied. In practice the samples by US is never uniformly distributed due to the effects of clock jitter or power supply noise in sampling devices. It is of great benefit to study the NUS. However, a single ideal lowpass filter based on the Shannon’s sampling theory is not good enough to reconstruct the signal from the samples by NUS. Starting from a general reconstruction formula in terms of the nonuniform samples and a BK, nine RAs are investigated for reconstructing the input signal from the nonuniform samples. The performance of this RAs are evaluated and compared by simulations. Most of them are extensively used in off-line image processing, but some of them based on interpolation are also possibly used in on-line radio communications.

The design goal of SDR is to put the A/D converter as close as possible to the antenna. BPS realizes frequency down conversion on a modulated bandpass signal by undersampling. It enables one to have an interface between the higher IF and the A/D converter and might be a solution to SDR. Three main aspects in the BPS technique, the allowable uniform BPS rate selection, noise aliasing and timing jitter, are reviewed and studied in this thesis work as compared to the conventional LPS cases.

It is noticed that noise aliasing plays an important role in the BPS applications. Starting from the Papoulis’ generalized sampling theorem, a Generalized Quadrature BPS (GQBPS) algorithm is proposed to suppress the noise aliasing. It is shown that the out-of-band noise aliasing is suppressed significantly but in-band noise aliasing is still present in the process of GQBPS algorithm. However, the GQBPS might be a potential way to reduce the noise aliasing at both the in-band and out-of-band at the cost of a more complicated reconstruction algorithm.

BPS makes use of signal spectral folding (or aliasing) by undersampling. Harmful signal spectrum aliasing due to careless uniform BPS rate selection will cause loss of information. It was shown in literature that NUS has the potential to suppress
harmful signal spectrum aliasing. In this thesis work, the PSDs of sampled-data
signal by two random samplings (JS and ARS) which are special cases of NUS
are studied. The definitions and conditions of alias-free sampling are verified by
simulations.

It is of more interest to see the performance of noise and jitter in real BPS
applications. A simplified model of a BPS receiver is modeled and simulated. The
sensitivity of RAs to jitter and AWGN are studied based on the model.

Future work is still needed to establish a more efficient sampling architecture
or algorithm that possesses the properties of i) flexible sampling rate selection
for avoiding harmful signal spectrum aliasing; ii) noise aliasing suppression; iii)
tolerance or correction to jitter. A CMOS chip is expected after the idea becomes
mature.
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