Constructing Multidimensional Dynamical Systems with Positive Lyapunov Exponents

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Abstract

In this thesis we adapt Marcelo Viana’s general construction of smooth transformations exhibiting non-uniform expansion in several dimensions. In our construction we, instead of coupling with a quadratic map, as Viana did, derive the expansion from a mapping with a cubic critical point. Further, we discuss how the argument can be extended and expansion derived from certain mappings with a critical point of arbitrarily high (odd) degree. We also discuss the issues in trying to use Viana’s method with the double standard map as our source of non-uniform expansion.
Konstruktion av Flerdimensionella Dynamiska System med Positiva Lyapunovexponenter

Sammanfattning

I denna uppsats anpassar vi Marcelo Vianas generella konstruktion av glatta transformationer som uppvisar icke-likformig expansion i flera dimensioner. I vår konstruktion, istället för att koppla ihop vårt system med en kvadratisk avbildning, får vi expansionen från en avbildning med en kubisk kritisk punkt. Dessutom så diskuterar vi hur argumentationen kan utvidgas och expansionen fås av vissa typer av avbildningar med en kritisk punkt av godtyckligt hög (udda) grad. Vi diskuterar också problemen i att försöka använda Vianas metod med den så kallade double standard-avbildningen som källan till den icke-likformiga expansionen.
Acknowledgements

I want to thank my supervisor Kristian Bjerklöv for all of his support, guidance and great enthusiasm for the subject.
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1 Introduction

Say we have a differentiable map $f$ from a manifold $M$ to itself. We may refer to this as a dynamical system, and throughout this thesis we will assume $M$ is smooth. We could ask ourselves, what can we say about the orbits of this dynamical system? By this we mean the sets $\{x, f(x), f^2(x), \ldots \}$, sometimes even $\{\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots \}$, if $f$ is a diffeomorphism. This turns out to often be a difficult question, since the orbits easily become very complicated. Perhaps we can relent a little, and disregard sets with zero measure, for some reasonable measure. This helps, and from a measure-theoretic viewpoint, is quite prudent. Perhaps we might even say it is enough only to know how the orbits behave asymptotically, say if they are attracted to or occupy only a small subset of $M$ after enough iterations. Either that, or we could ask ourselves what measurements along the orbits are on average, for instance do the limits

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(f^j(x)),$$

for some continuous function $g$, exist? If so, how do they behave? It is also of interest to know what kind of dynamical behavior will persist if we perturb the system. This is connected to the structural stability of a system, which says that other systems that are close to it have similar dynamics. To be able to talk about systems that are close we need a notion of closeness of two dynamical systems, and one that will be used throughout this thesis is the $C^r$-distance between two $C^r$ functions $f$ and $g$, written $||f - g||_{C^r}$. This is given by the supremum of the absolute value of the differences between the partial derivatives up to order $r$. For instance if $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ are two $C^1$ functions, where

$$f(x, y) = (f_1(x, y), f_2(x, y))$$
$$g(x, y) = (g_1(x, y), g_2(x, y)),$$

then

$$||f - g||_{C^1} = \sup_{(x, y) \in \mathbb{R}^2} \{|f_i - g_i|, |\partial_x f_i - \partial_x g_i|, |\partial_y f_i - \partial_y g_i|, \text{ for } i = 1, 2\}.$$

This is very important when we use dynamical systems to model real world systems, because more often than not, we do not have the exactly right model at hand. There are measurement errors and approximations involved, and it is problematic if small changes to the system could dramatically alter the
behavior of a dynamical system. These are important questions that we want to answer when examining dynamical systems.

One way of trying to find answers to these questions is by looking at the derivative of $f$ at points in the orbits. We can have conditions of varying strength on the derivative, but in this thesis we focus on how the derivative behaves on average, by examining the limits

$$\lim_{n \to \infty} \frac{1}{n} \log ||D(f^n(x))v||, \quad \text{where } x \in M \text{ and } v \in T_x M. \quad (1)$$

By $T_x M$ we mean the tangent space at $x$. For a point $x$ and non-zero vectors $v \in T_x M$, we refer to the limits as the Lyapunov exponents of $f$ at $x$. Should the limit above not exist, we can replace it with $\liminf$ or $\limsup$, such that vectors for which $\liminf > 0$ imply a positive Lyapunov exponent, and those for which $\limsup < 0$ imply a negative Lyapunov exponent. Given $U \subset M$ such that $f(U) \subset \text{int}(U)$, we say $f$ has non-uniformly expanding behavior on $U$ if almost every $x \in U$ have tangent vectors such that the limit (or $\liminf$ if the limit does not exist) above is positive (in other words, at least one positive Lyapunov exponent). This also implies that if a point $x \in M$ has more than one positive Lyapunov exponent, there is non-uniform expansion in more than one dimension. Later in this thesis we will discuss what these positive Lyapunov exponents can imply for a dynamical system.

Let us first note that it can be quite difficult to determine if we have non-uniform expansion if we have regions in $U$ where $||D(f(x))v||$ can get arbitrarily close to zero, for some non-zero $v \in T_x M$. Because looking at the limit

$$\lim_{n \to \infty} \frac{1}{n} \log ||D(f^n(x))v|| = \lim_{n \to \infty} \frac{1}{n} \log \left( \prod_{i=0}^{n-1} Df^i(x) \right) v,$$

we realize that we will need to know if a typical orbit is prone to enter these critical regions, where $D(f(x))v$ can be close to zero. Another complicating factor is that there can be rotations of the tangent space, and expanding directions can be interchanged with contracting directions.

In 1997 Marcelo Viana in a novel way constructed smooth dynamical systems with several positive Lyapunov exponents at almost every point, a multidimensional non-uniform expansion. He also showed that this expansion persists in systems close in $C^3$-distance. Viana’s strategy was to couple non-uniformly expanding quadratic maps $x \mapsto a_0 - x^2$ with suitable systems that have an everywhere large expansion. More precisely, he begins with the system $\psi_\alpha : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$, given by

$$(\theta, x) \mapsto (d\theta \mod 1, a_0 + \alpha \sin(2\pi \theta) - x^2),$$

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where $S^1 = \mathbb{R}/\mathbb{Z}$ and $d \geq 2$ is an integer. Then for certain values of $a_0 \in (1, 2)$ and a compact interval $I_0 \subset (-2, 2)$ such that $\psi_\alpha(S^1 \times I_0) \subset \text{int}(S^1 \times I_0)$ for small enough $\alpha$, he proved the following:

**Theorem 1.1** (Viana [18]). If $d$ is large enough, in this case $d \geq 16$, the following holds. For every $\alpha > 0$ sufficiently small, the map $\psi_\alpha$ has two positive Lyapunov exponents at Lebesgue almost every point $(\theta, x) \in S^1 \times I_0$. Moreover, the same holds for every map $\psi$ such that $\|\psi_\alpha - \psi\|_{C^3}$ is sufficiently small.

In his argument, the fact that the critical point of $a_0 - x^2$ is quadratic is important. The main result of this thesis will be an extension of his argument to other systems, where we in the construction replace the quadratic map with a class of maps with instead a cubic critical point. We also have a few extra assumptions on our systems, similar to ones Viana has in the first part of his general proof. Otherwise we try to mirror the construction and method, to show our result. We will also describe how to extend our result if the critical point has higher order, for certain classes of systems. We will also discuss the problems in trying to replace the quadratic map above with the so called double standard map, $x \mapsto 2x + \frac{1}{\pi} \sin 2\pi x \mod 1$, from $S^1$ to $S^1$.

Dynamics that are characterized by expansion or contraction through the derivative are often called hyperbolic, and before coming to our main results we will go through some of the history and theory of hyperbolic dynamics.

## 2 Uniform hyperbolicity

As stated above, hyperbolic dynamics are characterized by the presence of expanding and contracting directions for the derivative. When this is the case the differential can give a lot of information about topological and measurable aspects of the dynamics. [8].

In particular uniformly hyperbolic dynamics has been a historically fruitful object of research. Uniform hyperbolicity is a stronger condition on these expanding and contracting directions of the derivative, than the kind of expansion (or contraction) we can derive from looking at Lyapunov exponents. More precisely, it often means the following: [8]

**Definition 2.1.** Suppose $f : M \to M$ is a diffeomorphism. We say that $f$ is uniformly hyperbolic or an Anosov diffeomorphism if for every $x \in M$ there is a splitting of the tangent space $T_x M = E^s(x) \bigoplus E^u(x)$, where $E^s$ and $E^u$
are $Df$-invariant subspaces, and there are constants $C > 0$ and $\lambda \in (0,1)$ such that for every $n \in \mathbb{N}$ one has

$$||Df^n(v)|| \leq C\lambda^n||v|| \quad \text{for } v \in E^s(x) \text{ and}$$

$$||Df^n(v)|| \leq C\lambda^{-n}||v|| \quad \text{for } v \in E^u(x).$$

We call $E^s(x)$ and $E^u(x)$ the stable and unstable subspaces at $x$. Importantly, the constant $C$ is uniformly chosen for all $x \in M$.

**Example 2.2.** Famous examples of uniformly hyperbolic systems are toral automorphisms. These are dynamical systems where

$$M = \mathbb{R}^k / \mathbb{Z}^k = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1,$$

and the transformation is given by $x \mapsto Ax \mod 1$. Here $A$ is a matrix with integer coefficients such that $\det A = 1$ and $A$ has no eigenvalues on the unit circle. For example, if $k = 2$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

then $Df^n = A^n$. Also $A$ has two eigenvectors $v_1, v_2$ with eigenvalues $\lambda_1 = \frac{3-\sqrt{5}}{2} \in (0,1)$ and $\lambda_2 = \frac{3+\sqrt{5}}{2}$. In the notation above then, $E^s(x)$ is the subspace generated by $v_1$ and $E^u$ is generated by $v_2$.

In dynamical systems, the notion of $C^r$ structural stability is important. This means that a system should be equivalent to other systems in a $C^r$ neighborhood, for some $0 < r \in \mathbb{N}$. Two dynamical systems $f$ and $g$ on a manifold $M$ are equivalent if there exists a homeomorphism $h : M \to M$ such that the diagram below commutes.

$$\begin{array}{ccc}
M & \xrightarrow{f} & M \\
\downarrow{h} & & \downarrow{h} \\
M & \xrightarrow{g} & M
\end{array}$$

**Example 2.3.** The quadratic map $F_4 : [0,1] \to [0,1]$ defined by $x \mapsto 4x(1-x)$, is topologically conjugated to the tent map $T : [0,1] \to [0,1]$, given by $T(x) = 2x$ when $0 \leq x \leq 1/2$ and $T(x) = 2(1-x)$ when $1/2 \leq x \leq 1$. The homeomorphism $h$ of the interval is given by $h(x) = \sin^2(x)$.

Structural stability implies topological properties of the system are preserved under perturbation. One reason why this is important is if we wish to model some phenomena with a dynamical system, our model is unlikely to
exactly mirror the phenomena, but maybe some perturbation of the system actually does. Then we would not want dynamically interesting properties of our model to disappear if we change the system a little bit. For instance:

The system in Example 2.3 is part of the quadratic family $F_a = ax(1-x)$, and if $a > 4 + \sqrt{5}$, the map, this time from $\mathbb{R}$ to $\mathbb{R}$, is $C^1$ structurally stable [11]. If we know $f$ is a diffeomorphism, there is a very tight connection between uniformly hyperbolic systems and structural stability. Work done by Robbin, de Melo, Robinson and Mañé led to this important theorem: [5, Theorem 1.5]

**Theorem 2.4.** A $C^1$ diffeomorphism on a compact manifold is $C^1$ structurally stable if and only if it is uniformly hyperbolic and verifies the strong transversality condition.

We need not talk about what exactly strong transversality means, but the point is structural stability of a diffeomorphism is closely connected to uniform hyperbolicity.

A dynamical system can also have uniformly hyperbolic dynamics on a part of the state space $M$. A hyperbolic set is defined to be an invariant (under the action of $f$) and compact set $\Lambda \subset M$ such that every $x \in \Lambda$ allows a splitting of the tangent space as above. Of particular interest are hyperbolic sets which are also attractors. A compact $f$-invariant set $\Lambda$ is called an attractor if there is a neighborhood $U$ of $\Lambda$ such $\lim_{k \to \infty} \cap_{k \geq 0} f^k(U) \subset \Lambda$. Such a set $\Lambda$ is called an Axiom A attractor if it is also a hyperbolic set. If we say an attractor is $\Lambda$ is irreducible, then we mean it cannot be written as the union of two disjoint attractors. An $f$-invariant probability measure $\mu$ is a probability measure for which given any measurable set $A$, $\mu(f^{-1}(A)) = \mu(A)$. Sinai, Ruelle, Bowen showed [5, Theorem 1.7]

**Theorem 2.5.** Let $f$ be a $C^2$ diffeomorphism with an irreducible Axiom A attractor $\Lambda$. Then there is a unique $f$-invariant probability measure $\mu$ on $\Lambda$ such that there exists a set $V \subset U$ having full Lebesgue measure (with regards to $U$) with the property that for every continuous function $\varphi : U \to \mathbb{R}$ we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \int \varphi \, d\mu,$$

for every $x \in V$.

This can be interpreted to say that the behavior of orbits of typical points in the open set $U$ are completely determined on the statistical level.
[5], at least with regards to the measure \(\mu\). This can be contrasted with Birkhoff’s ergodic theorem (see section 4), which could only give us the same result for points on the attractor itself. This unique measure \(\mu\) is called the Sinai-Ruelle-Bowen measure (SRB-measure). There are several special things about this measure. One is that it in some sense is observable, meaning that the measure of subsets by \(\mu\) can be approximated by observing the fraction of the time the orbit of a typical point in the basin spends there. For this reason it is sometimes referred to as a physical measure [5]. By \(Df \mid E^u\) we mean \(Df\) the restriction of \(Df\) as a function from \(E^u \rightarrow E^u\).

**Theorem 2.6.** The SRB-measure is also characterized by the fact that
\[
h_\mu(f) = \int |\det(Df \mid E^u)| \, d\mu,
\]
where \(h_\mu(f)\) is the metric entropy of \(f\).

The metric entropy of \(f\), in a sense, is a measure of the randomness in a dynamical system [20]. Let us take a moment and introduce this measure, because it is also connected to Lyapunov exponents.

A *measure preserving* transformation, or map, \(f\), of a probability space is one for which \(\mu(f^{-1}A) = \mu(A)\). Given such a transformation \(f\) of a probability space \((M, B, \mu)\), we let \(\alpha = \{A_1, A_2, \ldots, A_n\}\) to be measurable partition of \(M\). Then for \(n \leq m\), set \(\alpha_n = f^{-n}\alpha \bigvee \cdots \bigvee f^{-m}\alpha\), where \(\alpha \bigvee \beta\) is defined as \(\{A \cap B, A \in \alpha \text{ and } B \in \beta\}\). Continuing, we define
\[
H(\alpha) = -\sum p_i \log p_i \quad \text{where} \quad p_i = \mu(A_i),
\]
and finally the *entropy* of a transformation \(f\), \(h_\mu(f)\), is defined as:
\[
h_\mu(f) = \sup_{\alpha} h(f; \alpha),
\]
where
\[
h(f; \alpha) = \lim_{n \to \infty} \frac{1}{n} H(\alpha_{0}^{n-1}).
\]
For proof that the above limit actually exists we refer to Walter’s chapter on entropy in [19].

**Example 2.7.** The entropy of a rotation of the unit circle \(\mathbb{S}^1\) is 0.
3 One-dimensional maps

The field of one-dimensional dynamics have occupied a special place in the theory of non-linear dynamics [9]. The transformations are relatively simple, but the dynamical behaviour very rich. It has also served as a model for the study of dynamical systems in higher dimensions, and we will take a moment to talk about recent developments in the field. In particular, the quadratic family \( f_a : [0, 1] \to [0, 1] \) given by

\[
f_a : x \mapsto ax(1 - x), \quad a \in [1, 4],
\]

has been a leading example in the area. It is an example of more general class of maps, called unimodal, which are maps from a closed interval \( I \) to itself, which has only one critical point in its interior. For technical reasons, we assume one of the endpoints of \( I \) is the preimage of the other one, which is also a repelling fixed point. If the number of critical points in the interior is \( \geq 1 \) we say the map is multimodal. Let us say a uni- or multimodal map is hyperbolic, or regular, if there is a periodic attractor in \( I \) whose basin of attraction is \( I \setminus K \), and \( K \) an expanding Cantor set of zero length. Expanding meaning that there are \( C > 0 \) and \( \sigma > 1 \) such that \( |Df^n(x)| \geq C\sigma^n \) for every \( x \in K \). This implies that almost every orbit converges to the periodic attractor, and so has quite a simple behaviour. We say a map is of mixing type if:

There is a cycle of intervals \( f^k(J), k = 0, 1, \ldots, p - 1 \), where

\[
\text{int}(f^n(J)) \cap \text{int}(f^m(J)) = \emptyset
\]

for \( 0 \leq m < n < p \) and \( f^p(J) \subset J \), and such that the return map \( f^p : J \to J \) is topologically mixing. A dynamical system \( f : M \to M \) is topologically mixing if for every pair of non-empty and open subsets \( A, B \subset M \), there exists an integer \( N > 0 \) such that for every \( n \geq N \), \( f^n(A) \cap B \neq \emptyset \). To be of mixing type, we also need of the maps that \( I \setminus \bigcup_{n \geq 0} f^{-n}(J) \) is an expanding Cantor set of zero length.

A big problem in one-dimensional dynamics have been whether or not the hyperbolic maps are dense among multimodal maps. It was first shown in 1997, by Graczyk together with Swiatek, and independently by Lyubich, that in the quadratic family, hyperbolic maps are dense. This result was then carried further to unimodal maps, and then finally Kozlovski, Shen and Van Strien showed in 2004 that [10]

**Theorem 3.1.** Hyperbolic maps are dense in the space of \( C^k \) maps of the compact interval or the circle, \( k = 1, 2, \ldots, \infty \).
But the picture is quite interesting and complicated, because for instance, among the unimodal maps we have the following result [5, Theorem 2.11]:

**Theorem 3.2.** For an open class of families of $C^2$ unimodal maps, including the quadratic family, the set of parameters for which the map has an absolutely continuous invariant probability measure $\nu$ has positive Lebesgue measure.

Recall that a measure $\nu$ is absolutely continuous (with respect to the ordinary Lebesgue measure $\mu$), if $\mu(A) = 0$ implies $\nu(A) = 0$. A map with an absolutely continuous invariant probability measure as above is called stochastic. In this setting $\nu$ will be a SRB-measure, meaning as earlier that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k x) = \int \varphi \, d\nu.$$ 

Then the dynamics are not necessarily very simple, but we have a very good probabilistic understanding of the dynamics. Lyubich proved that in the quadratic family, almost every real quadratic map is either hyperbolic or stochastic (meaning for almost every parameter $a \in [1, 4]$ in the definition of $F_a$). After this, Lyubich and Avila extended this result [12]:

**Theorem 3.3.** In any nontrivial real analytic family of unimodal maps with quadratic critical point, almost every map is either hyperbolic or stochastic.

We see much important work began with the quadratic family and then extended to the general unimodal or even multimodal case. A further generalisation is of course to families of maps of higher degree, for instance of form $f_{c,d} : z \mapsto z^d + c$. As late as 2011, Avila, Lyubich and Shen showed that a typical polynomial $f_{c,d}$ of mixing type is stochastic (and from Theorem 3.1 we know hyperbolic maps are dense in these families).

## 4 Lyapunov exponents

In the history of dynamical systems, the study of uniform hyperbolicity has been very important for developing techniques and insights, but it is in many ways not a wide enough class of systems. For instance not every manifold admits an Anosov diffeomorphism [15].

Much work has been done to relax the conditions of uniform hyperbolicity, to allow a wider class of systems, but still potentially salvage some the insights and results from the study of uniform hyperbolicity. One way of doing this is instead of prescribing uniform bounds on the contraction and expansion by
the derivative, we instead look at the asymptotic behavior of orbits; can the tangent space be decomposed into subspaces that either contract or expand on average? One way of formulating this is through Lyapunov exponents.

Let the setting now be that \( f: M \to M \) is a differentiable map, and \( M \) a smooth compact manifold. Given \( x \in M \) and \( v \in T_M(x) \), let us consider \( \lambda(x,v) \) defined as

\[
\lambda(x,v) = \lim_{n \to \infty} \frac{1}{n} \log ||D(f^n(x))v||.
\]

If there is no such limit one can can use instead \( \overline{\lambda}(x,v) \) and \( \underline{\lambda}(x,v) \), that is, \( \limsup \) or \( \liminf \) of the above. As stated in the introduction, these limits are referred to as Lyapunov exponents. If \( \lambda(x,v) \) is positive, then \( ||Df^n(x)v|| \) grows exponentially and we can interpret it to mean an exponential divergence of nearby orbits.

An important theorem in the theory of Lyapunov exponents is Oseledec’s multiplicative theorem, which tells us when the limit above actually exists. It says that if \( \mu \) is an \( f \)-invariant probability measure, then almost everywhere there exists for each \( x \), numbers

\[
\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_r(x)
\]

with multiplicities \( m_1(x), m_2(x), \ldots, m_r(x) \) such that

1. For every nonzero vector \( v \in T_xM \), \( \lambda(x,v) = \lambda_i(x) \) for some \( 1 \leq i \leq r(x) \).
2. The sum \( \sum_{i \leq r(x)} m_i(x) = \dim(M) \).
3. The sum \( \sum_{i \leq r(x)} \lambda_i(x)m_i(x) = \lim_{n \to \infty} \frac{1}{n} \log ||\det(Df^n(x))|| \).

Should \( f \) also happen to be a diffeomorphism, the tangent space is decomposed into \( Df \)-invariant subspaces \( E_1(x), \ldots, E_r(x) \). We define \( a^+ := \max\{a, 0\} \), and then if \( f \) also is \( C^2 \) we have the following fascinating connection with the entropy \( h_\mu \) (recall the definition from the section on Uniform hyperbolicity) of a system:

**Theorem 4.1.** [22] Let \( f : M \to M \) be a \( C^2 \) diffeomorphism and \( \mu \) an \( f \)-invariant probability measure with compact support. Then the following holds:

- In general
  \[
  h_\mu(f) \leq \int \sum_i \lambda^+_i \dim(E_i) \, d\mu
  \]
• If \( \mu \) is equivalent to the Lebesgue measure, then

\[
    h_\mu(f) = \int \sum \lambda_i^+ \dim(E_i) \, d\mu
\]

• When \( \lambda_1 > 1 \), the equality in (ii) holds if and only if \( \mu \) is an SRB measure.

Above then is an example how results on uniformly hyperbolic systems, Theorems 2.5 and 2.6, have analogues in a non-uniform setting.

### 4.1 Oseledec’s Theorem

Since Oseledec’s Theorem is central in the theory of Lyapunov exponents, let us state it properly and give a proof of it. We are working with Lyapunov exponents, and thus interested in how the differential operator \( D \) acts on \( f^n(x) \), but the setting for the theorem is more general than that.

We assume first that \((X, \mathcal{B}, \mu)\) is a probability space, and let \( T : X \to X \) be a measure preserving transformation. We recall that a measure preserving transformation \( T : X \to X \) is one for which \( \mu(T^{-1}A) = \mu(A) \) for every \( A \in \mathcal{B} \). Then we let \( A : X \to GL(m, \mathbb{R}) \) be a measurable mapping. Let us define \( A^n := A(T^{n-1}x) \cdots A(x) \). Let is also assume that \( \int \log^+ ||A|| \, d\mu < \infty \).

**Theorem 4.2** (Oseledec’s Theorem [14]). Let \( (T, \mu; A) \) be as above. Then at \( \mu \)-a.e \( x \), there exists a filtration of subspaces

\[
\{0\} = V_0(x) \subset V_1(x) \subset \cdots \subset V_r(x) = \mathbb{R}^m
\]

and numbers \( \lambda_1(x) < \cdots < \lambda_r(x) \) s.t.

1. \( \forall v \in V_i(x) - V_{i-1}(x), \lambda_+(x, v) = \lambda_i(x) \)
2. \( \lim_{n \to \infty} \frac{1}{n} \log |\det A^n(x)| = \sum \lambda_i(x) \cdot [\dim V_i(x) - \dim V_{i-1}(x)] \).

The functions \( x \to r(x), \lambda_i(x) \) and \( V_i(x) \) are measurable.

The proof for the full theorem is quite technical, and we will only prove it in the two dimensional case. To do this we will need Birkhoff’s ergodic theorem and also Kingman’s subadditive ergodic theorem. We will state these two without proofs. Recall that a measure is ergodic if for every \( A \in \mathcal{B}, T^{-1}(A) = A \implies \mu A = 0 \) or 1. Then
Theorem 4.3 (Birkhoff’s Ergodic Theorem [4]). Let $T : X \to X$ be as above, and let $\varphi \in L^1(\mu)$. Then there exists $\varphi^* \in L^1(\mu)$ such that

$$\frac{1}{n} \sum_{0}^{n-1} \varphi \circ T^i \to \varphi^* a.e.$$  

Moreover, $\varphi^* \circ T = \varphi^* a.e.$ and $\int \varphi^* \, d\mu = \int \varphi \, d\mu$. It is also the case that if $(T, \mu)$ is ergodic, then $\varphi^* = \int \varphi \, d\mu$.

Theorem 4.4 (Kingman’s subadditive ergodic theorem [20]). Let $T : (X, \mathcal{B}, \mu)$ be a measure preserving transformation. For $n = 1, 2, \ldots$, let $\varphi_n : X \to \mathbb{R}$ be a sequence of measurable functions s.t.

1. $\int \varphi_n^+ < \infty$
2. $\varphi_{m+n} \leq \varphi_m + \varphi_n \circ T^m$ a.e. for $m, n \geq 1$.

Then there exists $\varphi^* : X \to \mathbb{R} \cup \{-\infty\}$ with $\varphi^* \circ T = \varphi^* a.e.$ and $\int (\varphi^*)^+ < \infty$ such that

$$\frac{1}{n} \varphi_n \to \varphi^* a.e.$$  

The following proof will be based on the exposition by Young in [20]:

Proof. First we recall that any given matrix $A \in GL(2, \mathbb{R})$, $A$ has a singular value decomposition [16, Theorem 17.1], meaning that $A = O_2DO_1$, where $O_1$ and $O_2$ are orthonormal matrices and $D$ is diagonal matrix with

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

and $d_2 \geq d_1$. Since the $O_1$, $O_2$ are orthogonal, there are unit vectors $u_1$ and $u_2$ for which $O_1u_1 = e_1$ and $O_1u_2 = e_2$ and thus minimally and maximally stretched by $A$. We also know that $Au_1 = O_2(d_1e_1)$ and $Au_2 = O_2(d_2e_2)$ are orthogonal.

To prove the theorem we will apply the subadditivity theorem to the functions

$$\varphi_n^{(k)}(x) = \log \left[ \sup_{W \subset \mathbb{R}^2} |\det(A^n(x) \mid W)| \right],$$

where $W$ is a $k$-dim subspace of $\mathbb{R}^2$, $k \in \{1, 2\}$, corresponding to subspaces spanned by subsets of $\{u_1, u_2\}$, so that $\det(A^n(x) \mid W)$ makes sense. The first hypothesis follows by our assumption on $A$, that $\int \log^+ A \, d\mu < \infty$, together
with the fact that \( X \) is a probability space. The second hypothesis is also easy since

\[
\varphi^{(k)}_{m+n} = \log \left[ \sup_{W \subset \mathbb{R}^2} |\det(A^{m+n}(x) \mid W)| \right]
\]

\[
= \log \left[ \sup_{W \subset \mathbb{R}^2} |\det(A(T^{m+n-1}(x) \cdots A(x) \mid W)| \right]
\]

\[
\leq \log \left[ \sup_{W \subset \mathbb{R}^2} |\det(A(T^{m+n-1}(x) \cdots A(T^m(x)) \mid W)| \right] + \log \left[ \sup_{W \subset \mathbb{R}^2} |\det(A(T^{m-1}(x) \cdots A(x) \mid W)| \right]
\]

\[
= \varphi_n^{(k)} \circ T^m + \varphi_n^{(k)}.
\]

Next we will examine how the singular values of \( A^n \) grows. We define \( d_1^{(n)}(x) \leq d_2^{(n)}(x) \) to be the singular values of \( A^n(x) \), \( u_1^{(n)}(x) \) and \( u_2^{(n)}(x) \) be unit vectors such that \(|A^n(x)u_1^{(n)}(x)| = |d_1^{(n)}(x)|\) and \(|A^n(x)u_2^{(n)}(x)| = |d_2^{(n)}(x)|\). We claim that the limits

\[
\lim_{n \to \infty} \frac{1}{n} \log d_1^{(n)}(x)
\]

and

\[
\lim_{n \to \infty} \frac{1}{n} \log d_2^{(n)}(x)
\]

exists, almost everywhere. That the second one exists follows from the fact that \( \frac{1}{n} \log d_2^{(n)} = \frac{1}{n} \log \|A^n\| = \frac{1}{n} |A^n u_2^{(n)}| = \frac{1}{n} \varphi_n^{(1)}(x) \), and using the subaddivity. Then since \( \frac{1}{n} \log d_1^{(n)}d_2^{(n)} = \frac{1}{n} \log |\det A^n| \) converges by the ergodic theorem, \( \frac{1}{n} d_1^{(n)} = \frac{1}{n} \log |\det A^n| - \frac{1}{n} \log d_2^{(n)} \) converges too. Let us denote the limits of \( d_1^{(n)} \) and \( d_2^{(n)} \) by \( d_1(x) \) and \( d_2(x) \). Now let an \( x \) be typical such that the limits \( d_1 \) and \( d_2 \) exists. If \( d_1 = d_2 \) then \( r(x) = 1 \), \( \lambda_1 = d_2 \) and \( V_1 = \mathbb{R}^2 \). Also (2) is true since

\[
\lim_{n \to \infty} \frac{1}{n} \log |\det A^n(x)| = \lim_{n \to \infty} \frac{1}{n} \log |d_1^{(n)}d_2^{(n)}| = d_2 \cdot 2.
\]

From now on then, assume \( d_1 < d_2 \). We will begin by showing now that \( u_1^{(n)} \) converges. Recall that each pair \( (u_1^{(n)}, u_2^{(n)}) \) defines an orthonormal basis in \( \mathbb{R}^2 \). Let us then write \( u_1^{(n)} = v_1 + v_2 \) with regards to the splitting \( u_1^{(n+1)} \ominus u_2^{(n+1)} \), and \( \psi_n \) be the angle between vectors \( u_1^{(n)} \) and \( u_1^{(n+1)} \). Then \( v_2 = (u_1^{(n)} \cdot u_2^{(n+1)})u_2^{(n+1)} \), and \( |u_1^{(n)} \cdot u_2^{(n+1)}| = |\cos(\pi/2 - \psi)| = |\sin \psi_n| \). Now let \( h(n) \) be a function going to 0 as \( n \to \infty \), dependent only on \( x \). Let also \( c(n) \) denote a constant close to 1 dependent on \( n \). The preceding gives us that

\[
|A^{n+1}v_2| = |\sin \psi_n||A^{n+1}u_2^{(n+1)}| = |\sin \psi_n||d_2^{(n+1)}| = c|\psi_n d_2^n| + h(n).
\]
We also know
\[
|A^{n+1}v_2| = |A^{n+1}(u_1^{(n)} - v_1)| \leq |A^{n+1}u_1^{(n)}| = |A(T^n x)A^n(x)u_1^{(n)}| \\
\leq ||A(T^n x)|| \cdot d_1^{(n)} \leq ||A(T^n x)|| \cdot (d_1^n + h(n)).
\]

The first inequality follows since \(v_1\) is the direction of minimum stretch of \(A^{n+1}\). Putting the two inequalities above together, we have, for arbitrary \(\epsilon\),
\[
|\psi_n d_2^n| \leq ||A(T^n x)|| (d_1^n + h(n)) \implies \\
|\psi_n| \leq ||A(T^n x)|| (d_1^n + h(n)) \implies \\
|\psi_n| \leq (1 + \epsilon^n) |d_1^n |
\]
for large enough \(n\). This is because \(||A|| \in L^1\), and the ergodic theorem then gives \(\frac{1}{n} A^0 T^n \rightarrow 0\) a.e.. For small enough \(\epsilon\) then, \(\psi_n\) is a converging geometric sequence, and \(u_1^{(n)}\) converges. We can also see that the angle between \(u_1\) and \(u_1^{(n)}\) is less than \(|\psi| \leq \text{const} (1 + \epsilon) \left| \frac{d_1}{d_2} \right| \).

Now we will first show that \(\frac{1}{n} \log |A^n u_1| \rightarrow \log d_1\). Let \(c\) be a constant close to 1. We write \(u_1 = w_1 + w_2\) with regards to the splitting \(u_1^{(n)} \oplus u_2^{(n)}\). Then \(A^n w_1 = cd_1^{(n)} = cd_1^n + h(n)\). Also, similar as before, \(A^n w_2 \leq \text{const} (1 + \epsilon^n) d_2^{(n)} \leq \text{const} (1 + \epsilon^n) d_2^n h(n) = \text{const} (1 + \epsilon^n) d_2^n + h(n)\). Using also that \(A^n w_1\) is orthogonal to \(A^n w_2\), we get that
\[
\log d_1 + h(n) \leq \frac{1}{n} \log |A^n u_1| \leq \log(1 + \epsilon) + \log d_1 + h(n),
\]
and since \(\epsilon \rightarrow 0\) as \(n \rightarrow \infty\), we are done.

We let \(V_1(x)\) be the space spanned by \(u_1^{n}\). Now instead we look at \(v \in V_2(x) = \mathbb{R}^2 \setminus V_1(x)\). Again we write \(v = v_1 + v_2\) in the splitting \(u_1^{(n)} \oplus u_2^{(n)}\).

Since we now know that \(u_1^{(n)}\) converges, for sufficiently large \(n\), there is a constant \(c > 0\) such that \(|v_2| \geq d\). Then \(A^n v_1 \leq d_1^n + h(n)\) and \(A^n v_2 \geq cd_2^n\), which implies \(\frac{1}{n} \log |A^n v| \rightarrow \log d_2\), which was the last piece in proving the theorem.

\[\square\]

5 Results and comments on choice of system

Recall the setup behind and the conclusions in Theorem 1.1, given in the introduction. Viana begins his main argument by adding a few more assumptions on the systems \(\psi\) close to \(\psi_t\). He assumes first that every \(\psi\) is
of skew-product form, meaning of the form \( \psi(\theta, x) = (g(\theta), f(\theta, x)) \). He also assumes that \( \partial_x f(\theta, x) = 0 \) if and only if \( x = 0 \). In this first part, he further only assumes that \( \psi \) is \( C^2 \) and \( ||\psi - \psi_\alpha||_{C^2} \leq \alpha \).

Before we come to our choice of system, some comments are in order. When starting work on this thesis, we tried at first to look for a similar set of \( C^2 \) functions close to \( \phi_\alpha : S^1 \times S^1 \to S^1 \times S^1 \), given by

\[
\phi_\alpha(\theta, x) = (d\theta \mod 1, 2x + \alpha \sin(2\pi \theta) + \frac{1}{\pi} \sin(2\pi x) \mod 1)
\]

The reason being that the critical point \( x = 1/2 \) of \( x \mapsto 2x + \frac{1}{\pi} \sin(2\pi x) \) is cubic, and there was good reason to believe one could get similar results as in Viana’s section on building expansion \([13]\). However, the fact that the critical point was not quadratic gave some qualitatively different issues in the method. Without going into details now, Viana in \([18, \text{Lemma 2.5}]\), seemingly, only needed that when orbits tend to be a certain distance away from the critical point, there is an average expansion greater than one, but in the ways we have tried to adapt the proof the actual magnitude of the expansion becomes important.

After some experimenting with different reasonable systems, we instead based the system to couple with on \( \tilde{h} : S^1 \to S^1 \), defined by

\[
\tilde{h}(x) = \begin{cases} 
  kx \mod 1, & x - \frac{1}{2} \leq -\epsilon \\
  \frac{k}{3x^2}(x - \frac{1}{2})^3 \mod 1, & |x - \frac{1}{2}| < \epsilon \\
  kx + D \mod 1, & x - \frac{1}{2} \geq \epsilon,
\end{cases}
\]

where \( k > 2 \) and \( D \) are chosen so that the system is continuous,\(^1\), and \( \epsilon > 0 \). See Figure 1 for a plot of \( \tilde{h} \) with \( \epsilon = 0.1 \), and note that \( k \) approaches 2 as \( \epsilon \to 0 \). Although \( \tilde{h}''(x) \) is not defined when \( |x - 1/2| = \epsilon \), \( \tilde{h}' \) is Lipschitz continuous with Lipschitz constant \( \frac{2k}{\epsilon} \).

Then we can find a \( C^3 \) map \( h : S^1 \to S^1 \) such that the following holds: If \( \epsilon/2 \leq |x - 1/2| \leq 2\epsilon \):

(i)  \( |\tilde{h} - h| \leq \alpha \)

(ii)  \( |\tilde{h}' - h'| \leq \alpha \)

(iii)  \( |\tilde{h}'' - h''| \leq \frac{2k}{\epsilon} \), where \( \tilde{h}'' \) is defined,

and \( \tilde{h} = h \) otherwise. Both \( \epsilon > 0 \) and \( \alpha > 0 \) are small constants, and just how small will be determined through considerations in the main proof. The constant \( \alpha \) can be much smaller than \( \epsilon \).

\(^1\)We let \( k = \frac{1}{\epsilon - \frac{4}{\pi}} \) and \( D = -k \left( \frac{1}{2} + \frac{2\epsilon}{\pi} \right) \).
Similar then to how Viana began his proof with the additional assumptions on $\psi$ above, we look at systems $\varphi$ sufficiently close to $\varphi_\alpha : S^1 \times S^1 \to S^1 \times S^1$, defined by:

$$\varphi_\alpha(\theta, x) = (d\theta \mod 1, f_\alpha(\theta, x)),$$

where

$$f_\alpha(\theta, x) = h(x) + \alpha \sin(2\pi \theta) \mod 1 \quad (*)$$

and $d$ an integer greater than 16. The assumptions on the systems $\varphi$ close to $\varphi_\alpha$ we have to make are:

1. The systems are of skew-product form, so $\varphi(\theta, x) = (g(\theta), f(\theta, x))$.

2. We assume $\partial_x f = \partial_{xx} f = 0$ if and only if $x = 1/2$.

The main result of this thesis is then the following:

**Theorem 5.1.** Let $\varphi_\alpha : S^1 \times S^1 \to S^1 \times S^1$ be as defined above. Then if $\epsilon$ and $\alpha$ are small enough and $d \geq 16$, every $\varphi$ such that $\|\varphi - \varphi_\alpha\|_{C^3} \leq \alpha$, with the added assumptions on $\varphi$ as stated above, has two positive Lyapunov exponents Lebesgue almost everywhere.

Hopefully the extra assumptions can be removed in a similar way as Viana removed his assumptions above on $\psi$. We believe that the fact we need that
\[ \| \varphi - \varphi_\alpha \|_{C^3} \leq \alpha \text{ and not only } \| \varphi_\alpha - \varphi \|_{C^2} \leq \alpha \text{ at this stage of the proof is a consequence of having a cubic critical point.} \]

We also do not see any issues with extending Theorem 5.1, making a very similar construction but replacing \( \tilde{h}(x) \) with

\[
\tilde{h}(x) = \begin{cases} 
\hat{k}x \mod 1, & x - \frac{1}{2} \leq -\epsilon \\
\frac{\hat{k}}{n\epsilon - \pi}(1 - x)^n \mod 1, & \epsilon < x - \frac{1}{2} \\
\hat{k}x + \hat{D} \mod 1, & x - \frac{1}{2} \geq \epsilon,
\end{cases}
\]

where \( n = 2m + 1 > 3 \), and \( m \) is a positive integer. This is also reminiscent of work done on one-dimensional systems by Thunberg in [17].

### 6 Overview of proof

What we will essentially do is use and adapt Viana’s method where necessary. Exactly as in Viana’s case, it is clear that every point in \( \mathbb{S}^1 \times \mathbb{S}^1 \) has at least one positive Lyapunov exponent since \( (g^n)' \) grows exponentially fast as \( n \to \infty \).

\[
\lim_{n \to \infty} \left\| D\varphi^n(\theta_0, x_0) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \right\| = \lim_{n \to \infty} \left\| \left( \prod_{j=0}^{n-1} A(\theta_j, x_j) \right) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \right\| \geq |g'|^n \geq (d - \alpha)^n,
\]

where \( (\theta_j, x_j) = \varphi^j(\theta_0, x_0) \), \( (\theta_0, x_0) \in \mathbb{S}^1 \times \mathbb{S}^1 \) and

\[
A(\theta_j, x_j) = \begin{bmatrix}
g'(\theta_j) & 0 \\
\partial_\theta f(\theta_j, x_j) & \partial_x f(\theta_j, x_j)
\end{bmatrix}.
\]

Then we know at least that the lim inf, \( \lambda \left( (\theta, x), \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \right) \), is positive for every \( (\theta, x) \in \mathbb{S}^1 \times \mathbb{S}^1 \), and we have one positive Lyapunov exponent. The task we are set with, and the challenge, is to show that

\[
\lim_{n \to \infty} \left\| D\varphi(\theta_0, x_0)^n \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \right\| = \prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)| \geq e^c
\]

for some \( c > 0 \) and almost every \( (\theta, x) \in \mathbb{S}^1 \times \mathbb{S}^1 \). Again, this is difficult because we will need to have a good amount of control on how often and how close the \( x_j \)’s get to the critical point \( 1/2 \), since there we can have an infinite contraction. An important part of this strategy is the use of something we (and Viana) call admissible curves.
Definition 6.1. We say $\hat{X} \subset \mathbb{S}^1 \times \mathbb{S}^1$ is an admissible curve if $\hat{X} = \text{graph}(X)$, where $X : \mathbb{S}^1 \to \mathbb{S}^1$, and is such that $X$ is $C^2$ except maybe discontinuous on the left at $\theta_0$, the fixed point of $g$. Also, $|X'| \leq \alpha$ and $|X''| \leq \alpha$.

Given an admissible curve $\hat{X}_0$, we let $\hat{X}_j(\theta) = \varphi^j(\theta, X_0(\theta))$. In the main proof we will show that, if $n$ is large enough, then

$$\left| D(\varphi^n(\hat{X}_1(\theta))) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right| = \prod_{j=1}^n |\partial_x f(\hat{X}_j(\theta))| \geq e^{cn}.$$ 

Or almost. It will be true everywhere except on a set $E_n$ of values of $\theta$, with Lebesgue measure $m(E_n) \leq D_1 e^{-\gamma \sqrt{n}}$, for some $D_1, \gamma > 0$. Then we can see that the set $E = \cap_{n \geq 1} \cup_{k \geq n} E_k$ will have measure 0, since

$$m(E) \leq \sum_{k \geq n} D_1 e^{-\gamma \sqrt{k}} = D_1 e^{-\gamma \sqrt{n}} \sum_{k \geq n} e^{-\gamma (\sqrt{k} - \sqrt{n})}$$

$$= D_1 e^{-\gamma \sqrt{n}} \sum_{k \geq 0} e^{-\gamma \sqrt{k}} \leq D_2 e^{-\gamma \sqrt{n}},$$

for some constant $D_2 > 0$. This expression goes to 0 as $n \to \infty$, meaning $m(E) = 0$. If we can show all this, then all but a subset of $\mathbb{S}^1 \times \mathbb{S}^1$ of measure 0 will have two positive Lyapunov exponents, since the admissible curve $\hat{X}_0$ was arbitrary.

Now the function $f(\theta, x)$ is such that $\partial_x f \approx k > 2$ for most values of $x$, it is only close to the critical point $x = 1/2$ that we can lose expansion when we iterate $f$. What we will do is essentially two things then. We will show that orbits that get extremely close to $\mathbb{S}^1 \times \{1/2\}$ will be a very rare, and orbits that only get very close will do so seldom enough so that the expansion will be enough to offset the contracting effects close to $\mathbb{S}^1 \times \{1/2\}$. What we mean by extremely and very close will be made precise later in the proof.

7 Admissible curves

In this section we will prove some very similar results about our admissible curves, as Viana’s (very similar since the curves are very similarly defined). First we will create a set of Markov partitions on $\mathbb{S}^1$. A Markov partition of $\mathbb{S}^1$ is simply a partition into intervals $I_j$, such that $g|_{I_j}(\theta)$ is a homeomorphism onto a union of other intervals in the partition. In our case we have a partition $P_j$ for each $j \geq 0$, and we build them up recursively. We let $\theta_0$ be the fixed point of $g(\theta)$ close to 0. Let $\{\tilde{\theta}_0, \tilde{\theta}_1, \ldots, \tilde{\theta}_{d-1}\} = g^{-1}(\theta_0)$, where the pre-images
are ordered according to the orientation of $S^1 = \mathbb{R}/\mathbb{Z}$ inherited from $\mathbb{R}$. Then we define

$$\mathcal{P}_1 := \{[\tilde{\theta}_j, \tilde{\theta}_j], \text{for } j = 1, 2, \ldots, d \text{ where } \tilde{\theta}_0 = \tilde{\theta}_0\}$$

and recursively

$$\mathcal{P}_{n+1} = \{\text{All the connected components of } g^{-1}(\omega), \text{ for each } \omega \in \mathcal{P}_1\}.$$ 

For example, if $g(\theta) = d\theta \mod 1$, then $\{\tilde{\theta}_0, \tilde{\theta}_1, \ldots, \tilde{\theta}_{d-1}\} = \{0, \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}\}$, and $\mathcal{P}_1 = \{(0, \frac{1}{2}), \ldots, [\frac{d-1}{d}, 0]\}$.

Given any $\omega \in \mathcal{P}_k$ and admissible curve $\hat{X}$, we define $\hat{X} \mid \omega = \text{graph}(X \mid \omega)$. The following fact will be very useful

**Lemma 7.1.** If $\omega \in \mathcal{P}_m$, and $\hat{X}$ is an admissible curve, then so is $\varphi^n(\hat{X} \mid \omega)$

**Proof.** It is quite clear, by how we have defined the markov partitions $\mathcal{P}_m$, that the set $\varphi^n(\hat{X} \mid \omega)$ properly defines a $C^2$ function over $S^1$ with possibly a discontinuity at $\tilde{\theta}_0$. Let $Y : S^1 \to S^1$ be defined by $Y(g(\theta)) = f(\theta, X(\theta))$.

Then we just need to show that $|Y'| \leq \alpha$ and $|Y''| \leq \alpha$, because then we know this property is preserved on each iteration of $\varphi$. So

$$Y'(g(\theta))g' = \partial_\theta f + \partial_x f(\theta, X(\theta))X'$$

and since

$$g' > 15, \quad |\partial_\theta f| \leq \alpha 2\pi |\sin(2\pi \theta)| + \alpha \leq 8\alpha, \quad |\partial_x f| \leq 3,$$ 

we know

$$|Y'| \leq \frac{1}{15} 12\alpha \leq \alpha.$$ 

Also

$$Y''(g')^2 + Y' g'' = \partial_{\theta \theta} f + \partial_{\theta x}^2 f + \partial_{xx}^2 f(\theta, X)X'^2 + \partial_x f(\theta, X)X'',$$

and similarly, since

$$|g''| \leq \alpha, \quad |\partial_{\theta \theta}^2| \leq \alpha (2\pi)^2 |\sin(2\pi \theta)| + \alpha \leq 50\alpha, \quad |\partial_{xx}^2 f| \leq \alpha, \quad \partial_{xx}^2 f \leq \frac{2k}{\epsilon} + \alpha,$$ 

we know $|Y''| \leq \left(\frac{1}{15}\right)^2 100\alpha \leq \alpha$.

The above then proves that $\varphi^n(\hat{X} \mid \omega)$ too will be an admissible curve. \hfill \Box

In the next lemmas we want to, given an admissible curve $\hat{X}$ and an interval $I \subset S^1$, find a bound on the measure of values of $\theta$ that upon iteration of $\varphi$ on $\hat{X}$ has $x$-values in $I$. This will be important to find a bound on the measure of values of $\theta$ such that $\varphi^n(\hat{X})$ are extremely close to $S^1 \times \{1/2\}$.

First the following fact about the metric distortion of $g^n$, where we define the metric distortion of a function $h$ as $\sup_{\|f\|} \frac{\|f\|}{\inf_{\|f\|}}$. 

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Lemma 7.2. The metric distortion of $g^n$ is bounded on $\omega_n \in P_n$ for all $n$.

Proof. Since $d - \alpha \leq g' \leq d + \alpha$ and $-\alpha \leq g'' \leq \alpha$, the distortion of $g^n$ on $\omega$ is bounded by

$$\frac{\sup (g^n)'(\theta)}{\inf (g^n)'(\theta)} \leq \prod_{i=1}^{n} \frac{g'(x_i) + m(\omega_i)\alpha}{g'(x_i) - m(\omega_i)\alpha} \leq \prod_{i=1}^{n} \left(1 + 2m(\omega_i)\alpha\right) \leq \prod_{i=1}^{n} \left(1 + \frac{\alpha}{2i}\right),$$

where $x_i \in \omega_i$, and since $m(\omega_i) \leq \left(\frac{1}{d - \alpha}\right)^i \leq \frac{1}{2^{2i}}$. But

$$\log \prod_{i=1}^{n} 1 + 2m(\omega_i)\alpha = \sum_{i=1}^{n} \log \left(1 + \frac{\alpha}{2i}\right) \leq \sum_{i=1}^{\infty} D \frac{\alpha}{2i} < C,$$

for some constants $D$ and $C$, the latter then being a bound for the distortion. \qed

Lemma 7.3. Given an admissible curve $\hat{X}$, let $\hat{X}(\theta) = (\theta, X(\theta))$, and let $\hat{Z}(\theta) = \varphi(\hat{X}(\theta)) = \varphi(\theta, X(\theta)) = (g(\theta), Z(\theta))$. Then given any interval $I \in S^1$, $m(\{\theta \in S^1 : \hat{Z}(\theta) \in S^1 \times I\}) \leq \frac{2|I|}{\alpha} + 2\sqrt{\frac{|I|}{\alpha}}$.

Proof. Here properties of $\sin(2\pi \theta)$ and the particular bounds on $|X'|, |X''|$ will be important. First we partition $S^1$ by setting $A_1 = \{\theta \in S^1 \mid \sin(2\pi \theta) \leq 1/3\}$, and $A_2$ to be the complement of $A_1$. Note that both $A_1$ and $A_2$ has two connected components. Now by the trigonometric one, $\cos(2\pi \theta) \geq 11/12$ for $\theta \in A_1$. Then if $\theta \in A_1$,

$$|Z'| = |\partial_\theta f + \partial_x f(X(\theta)) X'| \geq |\alpha 2\pi \cos(2\pi \theta) - \alpha| = 3\alpha \geq \frac{11\alpha}{2} - 4\alpha \geq \alpha$$

By use of the mean value theorem it follows that, for a connected component $[a, b] \in A_1$,

$$|Z(b) - Z(a)| \geq \alpha |b - a| \text{ which implies that }$$

$$|b - a| \leq \frac{|Z(b) - Z(a)|}{\alpha}.$$ 

Since there are two connected components in $A_1$ and $|Z(b) - Z(a)| \leq |I|$, we get

$$m(\{\theta \in A_1 : Z(\theta) \in I\}) \leq \frac{2|I|}{\alpha}.$$

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Now instead assume that $\theta \in A_2$, so $\sin(2\pi \theta) > 1/3$. Then
\[
|Z''| = |\partial_{\theta \theta} f + 2\partial_{\theta x} f(\theta, X)X' + \partial_{xx} f(\theta, X)(X')^2 + \partial_x f(\theta, X)X''|
\geq \alpha \left( \frac{(2\pi)^2}{3} - \alpha \right) - 2\alpha^2 - \left( \frac{2k}{\epsilon} + 2\alpha \right) \alpha^2 - 3\alpha \geq 6\alpha.
\]

Then we claim that
\[
m(\{ \theta \in A_2 : Z(\theta) \in I \}) \leq 2\sqrt{2} \sqrt{|I|/(6\alpha)} \leq 2 \sqrt{|I|/\alpha}
\]

One way to see this is the following. Worst case is that $|Z'(\theta)| = 0$ for some $\xi$ in a connected component $[a, b]$ of $A_2$. Using taylor expansion around that point we see that $|Z(\xi + h) - Z(\xi)| \geq \left| \frac{Z''(\xi)h^2}{2} \right| \geq \frac{6\alpha h^2}{2}$, for $|h| \geq (b - a)/2$. The result follows.

**Corollary 7.4.** There is an $C_1 > 0$ such that given any admissible curve $\hat{X}_0$, and an interval $I \subset S^1$ with $|I| \leq \alpha$, the following holds:
\[
m(\{ \theta \in S^1 : \hat{X}_j(\theta) \in S^1 \times I \}) \leq C_1 \sqrt{\frac{|I|}{\alpha}} m(\omega)
\]

**Proof.** Again we will use properties of the Markov partitions. Let $\omega \in \mathcal{P}_{j-1}$, and set $\hat{X}_\omega = \varphi^{j-1}(\hat{X}_0)$ and $\tilde{Z}_\omega = \varphi(\hat{X}_\omega)$. Then
\[
m(\{ \theta \in S^1 : \hat{Z}_\omega(\theta) \in S^1 \times I \}) \leq 2 \sqrt{\frac{|I|}{\alpha}} \leq 4 \sqrt{\frac{|I|}{\alpha}}
\]

since $\alpha \leq |I|$. Now by construction, $\hat{X}_j(\theta) = \tilde{Z}_\omega(g^{j-1}(\theta))$. Then by Lemma 7.3,
\[
m(\{ \theta \in S^1 : \hat{X}_j(\theta) \in S^1 \times I \}) \leq C_* 4 \sqrt{\frac{|I|}{\alpha}} m(\omega).
\]

The prove the last inequality, let us argue more generally. Let $h(\theta) := g^n(\theta)$ and $\omega \in \mathcal{P}_n$, $A$ be a subset of $S^1$ and $w'$ the inverse image of $A$. Then
\[
1 = m(S^1) \leq m(\omega) \sup h'
\]
and
\[
m(A) \geq m(\omega') \inf h'.
\]

This implies that
\[
m(\omega') \sup h' m(A) \geq m(\omega'),
\]
which is the same as
\[
m(\omega') \leq C_* m(A) m(\omega).
\]

\[\square\]
8 Building expansion

This section contains similar results as in the likewise named section in Viana’s paper [18], where he shows what kind of expanding behavior we can expect, depending on the distance of the $x$-variable to the critical point $1/2$.

Given a point $(\theta_0, x_0)$, we let $(\theta_j, x_j) = \varphi^j(\theta_0, x_0)$. Throughout this section, let $d_j$ denote $|x_i - \frac{1}{2}|$. We introduce two constants. One will be $C \propto \epsilon^2$ and dependent on $h$, and $C'$ which is fixed with respect to $\epsilon$, but otherwise dependent on $h$. We also introduce the small constant $\eta = \eta(\alpha) > 0$. How small will be determined later in the proof.

Lemma 8.1. There exists an integer $N = N(\alpha) > 0$ such that
\[
\prod_{j=0}^{N-1} |\partial_x f(\theta_j, x_j)| \geq |x_0 - \frac{1}{2}|^2 \alpha^{-1+\eta} \quad \text{whenever} \quad |x_0 - \frac{1}{2}| \leq 6\alpha^{1/3}.
\]

Proof. Now $|f(\theta, x)| \leq C(d_0^3 + \alpha)$, and given $d_0 < 6\alpha^{1/3}$, $f(\theta, x) \leq C\alpha$. Then since $|x_i| = (k + \alpha)|x_{i-1}| + C'\alpha$ we see that
\[
|x_i| \leq (1 + (k + \alpha) + (k + \alpha)^2 + \cdots + (k + \alpha)^i)C\alpha \leq \frac{(k + \alpha)^{i+1} - 1}{k - 1} C\alpha \leq (k + \alpha)^i C\alpha.
\]

Let $\tilde{N}$ be the smallest integer such that $C\alpha(k + \alpha)^{\tilde{N}} > \frac{1}{C}$. This implies that
\[
\tilde{N} > \frac{1}{\log(k + \alpha)} \left( 2\log \frac{1}{C} + \log \frac{1}{\alpha} \right) > \log \frac{1}{C} + \log \frac{1}{\alpha}
\]

Setting $N = \tilde{N} + 1$, we can estimate that
\[
\prod_{j=0}^{N-1} |\partial_x f(\theta_j, x_0)| \geq |\partial_x f(\theta_0, x_0)|(k - \alpha)^{\tilde{N}} \geq \frac{1}{C} d_0^2 \alpha^{-1} \geq \alpha^{2/3} \alpha^{-1+\eta},
\]

for $\alpha$ chosen small enough, and the smaller $\alpha$ is, the smaller we can make $\eta(\alpha)$. We also used that since for $x = 1/2$, $\partial_x f = \partial_{xx} f = 0$, taylor expansion gives that $|\partial_x f(\theta_0, x_0)| \geq \text{const} (x_0 - 1/2)^2$.

Lemma 8.2. There is a constant $C_2 > 0$ such that for a given $\delta$, less than but close to 1, and all $(\theta_0, x_0) \in S^1 \times S^1$, with $d_0, d_1, d_2, \ldots, d_{a-1} \geq \alpha^{1/3}$,
\[
\prod_{j=0}^{a-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \alpha^{2/3} (\delta k)^a.
\]

In addition, if $d_a < \epsilon$, then $\prod_{j=0}^{a-1} |\partial_x f(\theta_j, x_j)| \geq C_2 (\delta k)^a$. 

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Proof. Since \(|\partial_x f(\theta_j, x_j)| \geq k - 2\alpha \geq \delta k| whenever \(d_j \geq \epsilon| and \(\alpha| is small enough, we need to check what happens if \(\alpha^{1/3} < d_j < \epsilon|\). Let \(t| be the least integer such that \(d_t < \epsilon|$. If \(x_s > \epsilon| for all \(s > t|, then for \(C_2| small enough (independent of \(\alpha|)

\[
\prod_{j=0}^{a-1} |\partial_x f(\theta_j, x_j)| \geq (C_2 d_t^2 - C'\alpha)(\delta k)^{a-1} \geq (C_2\alpha^{2/3} - C'\alpha)(\delta k)^{a-1}
\]

\[
\geq C_2\alpha^{2/3}(\delta k)^a.
\]

Suppose now there exists \(l > 0| such that \(d_t+l \leq \epsilon|$. We assume \(l| is the smallest such integer. Similar as in the proof of lemma 8.1, \(|x_{t+(i+1)}| \leq (k+\alpha)|x_{t+i}| + C'\alpha|$. Then

\[
|x_{t+i}| \leq (1 + (k + \alpha) + (k + \alpha)^2 + \cdots + (k + \alpha)^{i-1})C'\alpha + (k + \alpha)^iC d_t^3
\]

\[
\leq (k + \alpha)^i(C\alpha + C d_t^3) \leq (k + \alpha)^iC d_t^3,
\]

since \(d_t \geq \alpha^{1/3}|$. It is clear that \(l \geq l'|, where \(l'| is least integer such that \((k + \alpha)^l C d_s^3 \geq \frac{1}{\epsilon'}|$. Then

\[
l' \log(k + \alpha) - 2 \log(\epsilon) + 3 \log(d_t) \geq -\log C'
\]

\[
l' \geq \frac{1}{\log(k + \alpha)}(2 \log(\epsilon) - 3 \log(d_t) - \log C') \geq 2 \log(\epsilon) - 3 \log(d_t) - \log C',
\]

if \(k| is close enough to 2. We can estimate

\[
\prod_{j=t}^{t+l-1} |\partial_x f(\theta_j, x_j)| \geq C_2 d_t^2(k - \alpha)^l-1.
\]

Thus the question becomes is if \(l| is big enough so that

\[
C_2(k - \alpha)^l-1 d_t^2 \geq (\delta k)^l.
\]

But the above is equivalent to

\[
C_2 d_t^2 \left(\frac{k - \alpha}{k}\right)^l \geq (\delta)^l
\]

which is implied if

\[
C_2 d_t^2 \geq (\delta')^l,
\]

where \(0 < \delta' < \delta|, by making \(\alpha| small enough. But

\[
(\delta')^l \leq (\delta')^{l'} \leq \frac{1}{C' \epsilon^2}d_t^2.
\]
Then if \( \epsilon > d_s \) is small enough, the inequality holds.

This means that anytime we know that the system returns to the \( \epsilon \) neighborhood we gain enough expansion to prove the second part of the lemma. If we do not return, then we get the added factor \( C_2 \alpha^{2/3} \), as seen above. \( \square \)

**Remark 8.3.** The above proof we believe could be adapted to show that for a more general type of system, meaning other choices of \( h(x) \) in (*) that do not need to be so close to a linear function in large parts of \( S^1 \), we would get expansion \( \approx \delta \inf |h'(x)| \).

**Remark 8.4.** Notice that Lemma 8.2 implies that if \( (\theta, x) \) has an orbit that only intersects \( S^1 \times I \) a finite number of times, where \( I = \{ x : |x - 1/2| \geq \alpha^{1/3} \} \) (and not in such that \( x = 1/2 \)), \( (\theta, x) \) will have two positive Lyapunov exponents.

### 9 Technical lemma

This section is also based on Viana’s section Technical lemma, and now we come to the part of Viana’s proof the broke down when we tried to adapt it to our first choice of system. Let us comment on the issues as we come to them in the proof of the lemma.

Important sets in the proof will be \( J(r) \), which we define as

\[
J(r) := \{ x \in S^1 : |x - \frac{1}{2}| < \alpha^{1/3} e^{-r} \},
\]

where \( r \geq 0 \). We also define \( M \) to be the largest integer such that \( (k^p)^M \alpha \leq 1 \), where \( 1 < p < 2 \). We emphasize that this \( p \) and its relation to \( \delta \) will be important in a later discussion on generalising our main result, Theorem 5.1.

What we want to bound is the recurrence of orbits into sets of the form \( S^1 \times J(r) \), where we can have very high contraction.

**Lemma 9.1.** There are \( C_3 > 0 \) and \( \beta > 0 \) such that, given any admissible curve \( \hat{Y}_0 = \text{graph}(Y_0) \) and any \( r \geq (\frac{1}{6} - \eta) \log(\frac{1}{\alpha}) \),

\[
m(\{ \theta \in S^1 : \hat{Y}_M(\theta) \in S^1 \times J(r - 2) \}) \leq C_3 e^{-5\beta r}
\]

**Remark 9.2.** A comment on the condition \( r \geq (1/6 - \eta) \log \frac{1}{\alpha} \). Note that Lemmas 8.1 and 8.2 imply that it is precisely when \( r \geq (1/6 - \eta/2) \log \frac{1}{\alpha} \) that orbits which enter \( S^1 \times J(r) \) might have non-positive Lyapunov exponents.

Before proving the above lemma, we need another preliminary result. Given an admissible curve \( \hat{X} \) and \( 1 \leq j \leq d \), we set \( \tilde{Z}_j = \varphi(\hat{X} \mid [\tilde{\theta}_{j-1}, \tilde{\theta}_j]) = \text{graph}(Z_j) \), then
Lemma 9.3. There are $H_1, H_2 \subset \{1, \ldots, d\}$ with $\#H_1, \#H_2 \geq [d/16]$ such that $|Z_{j_1}(\theta) - Z_{j_2}(\theta)| \geq \alpha/100$ for all $\theta \in S^1$, $j_1 \in H_1$ and $j_2 \in H_2$.

Proof. As in Lemma 2.2, let $\tilde{Z}(\theta) = (g(\theta), Z(\theta)) = \varphi(\tilde{X}(\theta))$, but also $l = [d/16]$ and $\chi_1 < \chi_2$ be the fixed points of $Z(\theta)$. There are two and only two such points since $Z'(\theta) \geq \alpha$ in one of the two components of $A_1$, and $Z'(\theta) \leq -\alpha$ in the other, and $|Z''(\theta)| \geq 6\alpha$ on $A_2$. And of course $\chi_1, \chi_2 \in A_2$. Let $k_i$ for $i = 1, 2$ be such that $\chi \in (\tilde{\theta}_{k_i}, \chi_{k_i})$ and suppose now that neither $\chi_1$ nor $\chi_2$ is in $[1/4, 3/4]$ (1/4 and 3/4 are the fixed points of $\sin(2\pi \theta)$, so $\chi_1, \chi_2$ should be close to these points). Then we can set $H_1 = \{k_1 + 1, \ldots, k_1 + l\}$ and $H_2 = \{k_2 - l, \ldots, k_2 - 1\}$. Now we observe that $\tilde{\theta}_{k_1+l} < \frac{1}{4} + \frac{l+1}{d-\alpha} < \frac{1}{2} - \frac{1}{2\pi} \arcsin \frac{1}{3}$, where $\frac{l+1}{d-\alpha}$ is the minimal length of $l + 1$ intervals in $P_1$. The last inequality is a calculation, and follows since $\frac{2}{15} > \frac{l+1}{d-\alpha}$. Similarly, we have that $\tilde{\theta}_{k_2-l} > \frac{3}{4} - \frac{l-1}{d-\alpha} > \frac{1}{2} + \frac{1}{2\pi} \arcsin \frac{1}{3}$. What is this good for? Well, then we know that for $j = 1, \ldots, l$,

$$\sin \tilde{\theta}_{k_1+j} > \sin \left(\frac{1}{2} - \frac{1}{2\pi} \arcsin \frac{1}{3}\right) = \frac{1}{3},$$

and

$$\sin \tilde{\theta}_{k_2-j} < \sin \left(\frac{1}{2} + \frac{1}{2\pi} \arcsin \frac{1}{3}\right) = -\frac{1}{3},$$

meaning they are all separated by at least $\frac{1}{\pi} \arcsin \frac{1}{3}$. Since $Z$ is monotone decreasing on $[\tilde{\theta}_{k_1}, \tilde{\theta}_{k_2-1}]$ and $Z' \geq \alpha$ on $A_1$, we get

$$\inf Z | [\tilde{\theta}_{j_1-1}, \tilde{\theta}_{j_1}) - \sup Z | [\tilde{\theta}_{j_2-1}, \tilde{\theta}_{j_2}) \geq \frac{\alpha}{\pi} \arcsin \frac{1}{3} \geq \frac{\alpha}{100}$$

for every $j_1 \in H_1$ and $j_2 \in H_2$, which proves the lemma in this case.

Now suppose instead that $\chi_1 \geq 1/4$. Then we can choose $H_1 = \{k_1 - l, \ldots, k_1 - 1\}$ and $H_2 = \{k_2 - l, \ldots, k_2 - 1\}$. Then $\tilde{\theta}_{k_1-1} > \frac{1}{4} - \frac{l}{d-\alpha} > \frac{1}{2\pi} \arcsin \frac{1}{3}$ of course for every $j = 1, \ldots, l$, $\tilde{\theta}_{k_1-j} < 0$, so then the intervals given by $H_1$ and $H_2$ are separated by at least $\frac{1}{2\pi} \arcsin \frac{1}{3}$, which together with that $Z$ is monotone increasing on $[\tilde{\theta}_{k_2-1}, \tilde{\theta}_{k_2-1}]$ we get that

$$\inf Z | [\tilde{\theta}_{j_1-1}, \tilde{\theta}_{j_1}) - \sup Z | [\tilde{\theta}_{j_2-1}, \tilde{\theta}_{j_2}) \geq \frac{\alpha}{2\pi} \arcsin \frac{1}{3} \geq \frac{\alpha}{100}.$$ 

The case if $\chi_2 \leq 3/4$ is symmetrical to this, and so the lemma is proved. $\square$

Remark 9.4. The above result essentially means that on admissible curves, since they have such a low derivative, the effects of $\alpha \sin(2\pi \theta)$ is on the same order as $h(X(\theta))$. See Lemma 7.1. We do not expect the above result to be improved in any significant way to help us in other systems.
Now for the proof of Lemma 9.1:

Proof. Let $Y_0$ be an admissible curve, and denote the distance $|Y_j(\theta) - \frac{1}{2}|$ by $d_j(\theta)$. To prove the theorem, we can assume that for some $\tau \in S^1$, $d_M(\tau) \leq \alpha^{1/3}$, otherwise the lemma is obviously true. What we will show now is that this implies that $d_j(\theta) \geq \alpha^{1/3}$ for all $\theta \in S^1$, $j = 1, \ldots, M - 1$, and that $d_M(\theta) < \epsilon$, which will give us a high rate of expansion through Lemma 8.2. First we define

$$osc(Y_j) := \sup_{\theta, \tau \in S^1} |Y_j(\theta) - Y_j(\tau)|$$

Then $osc(Y_0) = \alpha$ and

$$osc(Y_j) \leq (\sup \partial_x f) osc(Y_{j-1}) + \alpha \leq (k + C') \alpha osc(Y_{j-1}) + \alpha \leq k osc(Y_{j-1}) + C' \alpha,$$

implying $osc(Y_j) \leq C' k^j \alpha$. Thus $osc(Y_M) \leq C' k^M \alpha \leq C' \alpha^{1-1/p} < \alpha^q$, for some $q > 0$. This gives us that

$$d_M(\theta) \leq C' \alpha^q < \epsilon \quad \text{for all} \quad \theta \in S^1,$$

if we make $\alpha$ much smaller than $\epsilon$. We also know that for $0 \leq j \leq M - 1$ and $1 \leq i \leq M - j$

$$|Y_{i+j}(\theta)| \leq C(k + \alpha)^i (d_j(\theta)^j + \alpha),$$

when $d_j(\theta) \leq \epsilon$, and

$$|Y_{i+j}| \leq C(k + \alpha)^i (|Y_j(\theta)| + \alpha)$$

when $d_j(\theta) > \epsilon$. To show that $d_j(\theta) \geq \alpha^{1/3}$ for $j = 1, \ldots, M - 1$, let us argue by contradiction and assume otherwise. Then $d_j(\theta) \leq \alpha^{1/3}$, and by a similar reasoning as in Lemma 8.1 and 8.2,

$$|Y_M(\theta)| \leq C(k + \alpha)^{M-j}(d_j(\theta)^j + \alpha) \leq (k + \alpha)^{M-j} C\alpha \leq C(k + \alpha)^M \alpha \leq \alpha^q.$$

The last inequality is true, since we can make $\alpha$ small enough such that for any fixed $1 < p < 2$, $k + \alpha \leq k^p$. But again, choosing $\alpha$ small enough, this will contradict (3). This means, through Lemma 8.2 that given $\hat{y} \in \hat{Y}_0$,

$$|\partial_x f^M(\hat{y})| \geq C_2 (\delta k)^M.$$

The next step is to find a uniform bound for the metric distortion in the $x$-direction upon iteration of an admissible curve $\hat{Y}_0$. More precisely, we will show that for $0 \leq j \leq M - 1$ and $1 \leq i \leq M - j$, we have that

$$\left| \frac{\partial_x f^j(\theta_j, x_j)}{\partial_x f^j(\tau_j, y_j)} \right| \leq \prod_{m=j}^{j+i-1} \left| \frac{\partial_x f(\theta_m, x_m)}{\partial_x f(\tau_m, y_m)} \right| \leq 2$$

(7)
where \((\theta_j, x_j), (\tau_j, y_j) \in \hat{Y}_j\). This is reasonable since we know points on the curve \(\hat{Y}_j\) are at a distance greater than \(\alpha^{1/3}\) from \(x = 1/2\), and \(\text{osc}(Y_j)\) has a good bound. To begin with, we first note that (4) gives us that if \(\alpha^{1/3} \leq d_j(\theta) \leq \epsilon\), then
\[
\frac{1}{C} \leq (k + \alpha)^{M-\epsilon}d_j(\theta)^3.
\]

Now if both \(x_m\) and \(y_m\) are at a distance greater than \(\epsilon\) from \(\frac{1}{2}\), then
\[
\left| 1 - \frac{\partial_x f(\theta_m, x_m)}{\partial_x f(\tau_m, y_m)} \right| \leq \left| 1 - \frac{k + C\alpha}{k - C\alpha} \right| = \frac{C\alpha}{k - C\alpha} < C\alpha^q.
\]

If both \(x_m\) and \(y_m\) are within distance \(\epsilon\) of \(\frac{1}{2}\) (this can also be handled in a similar way as in [18, Technical Lemma], by observing again that \(\partial_x f(\theta, x) \approx \text{const}(x - 1/2)^2\) when \(x\) is close to \(1/2\), then
\[
\left| 1 - \frac{\partial_x f(\theta_m, x_m)}{\partial_x f(\tau_m, y_m)} \right| \leq \frac{k/\epsilon^2(|(y_m - 1/2)^2 - (x_m - 1/2)^2| + C\alpha)}{k/\epsilon^2((y_m - 1/2)^2 - C\alpha)}.
\]

Now
\[
\frac{|(y_m - 1/2)^2 - (x_m - 1/2)^2| + C\alpha}{(y_m - 1/2)^2 - C\alpha} \leq \frac{|(y_m - 1/2)^2 - (y_m - 1/2 + 2(k + \alpha)^m\alpha^2)| + C\alpha}{C^{2/3}(k+\alpha)^{M-m/3} - C\alpha},
\]
the last inequality since we know \(|y_m - x_m| \leq (k + \alpha)^m\alpha\), and using equation (4). Continuing, the above is less than or equal to
\[
\frac{C(k + \alpha)^m\alpha}{C(k+\alpha)^{M-m/3} - C\alpha} \leq \frac{C(k + \alpha)^M\alpha}{1 - C(k + \alpha)^M\alpha} \leq \frac{C\alpha^q}{1/2} \leq C\alpha^q.
\]

Now assume that \((\theta_m, x_m) - 1/2 \geq \epsilon\) and \((\tau_m, y_m) - 1/2 < \epsilon\), then the worst case scenario is if:
\[
\left| \frac{\partial_x f(\theta_m, x_m)}{\partial_x f(\tau_m, y_m)} \right| \leq \frac{k + C\alpha}{k/\epsilon^2(\epsilon - 2k^m\alpha)^2 - C\alpha} = \frac{k + C\alpha}{k(1 - 4k^m\alpha/\epsilon - \alpha^2/\epsilon^2) - C\alpha} \leq \frac{k + C\alpha}{k - C\alpha^q}.
\]

Then
\[
\left| 1 - \frac{\partial_x f(\theta_m, x_m)}{\partial_x f(\tau_m, y_m)} \right| \leq \frac{C\alpha^q}{k - C\alpha^q} \leq C\alpha^q.
\]
Which implies that
\[
\frac{\partial_x f^i(\theta_m, x_m)}{\partial_x f^i(\tau_m, y_m)} \leq (1 + C\alpha^q)^{2i} \leq (1 + C\alpha^q + \frac{(C\alpha^q)^2}{2!} + \ldots)^{2M} \\
\leq e^{C\alpha^q M} \leq 2,
\]
since \(M \approx \log(\frac{1}{\alpha})\).
Now that we have a uniform bound for the metric distortion of \(f^i(\theta_j, x_j)\), we can fix an arbitrary \(\hat{y} \in \hat{Y}_0\), and derive a good bound by just looking at \(\partial_x f^i(\hat{y})\). What we want to show next is that, in a sense, the expansion is somewhat evenly distributed through the iterations of \(f^j(\theta, x)\). We begin by defining \(\lambda_j = \left| \partial_x f^{M-j}(\varphi^j(\hat{y})) \right|\). Now the result in (7) gives us that
\[
\frac{1}{2^i} \lambda_j \leq \left| \partial_x f^i(\theta_j, x_j) \right| \leq 2^i \lambda_j.
\]
We set \(K = 400e^2\), and define \(t_1 < t_2 < \cdots \leq M\) by \(t_1 = 1\) and \(t_{i+1} = \min\{s : t_i < s \leq M \text{ and } \lambda_{t_i} \geq 2K\lambda_s\}\). We assume \(r \geq (1/6 - \eta) \log(\frac{1}{\alpha})\), and we define \(k(r) = \max\{i : \lambda_i \geq \frac{2Ke^{-r}}{\alpha^{i/3}}\}\).

Let us pause and ponder here for a second, try to get our bearings. We know that \(\lambda_j \geq C_2(\delta k)^{M-j}\). The choice of the constant \(K\) is to a large degree a matter of convenience, the important part is the set \(\{t_1, t_2, \ldots\}\) which gives us a sense of how much expansion there is left of \(C_2(\delta k)^M\), and \(k(r)\) tell us how long we can expect to still have a lot of expansion. How much? Well more than \(\frac{2Ke^{-r}}{\alpha^{i/3}}\), which, multiplied with \(\frac{\alpha}{100}\) from Lemma 9.3 will be \(\approx \alpha^{1/3}e^{-r}\), which is a motivation for the specific definition of \(K\).

Let us continue with the proof. We set the average expansion \(\sigma = \delta k\). Note that \(\lambda_{t_i} \leq 2K\lambda_{t_i-1} \leq (k + C'\alpha)2K\lambda_{t_i+1} \leq 6K\lambda_{t_i+1}\), for every \(i\). If \(\sigma = \delta k\), this implies that \(C_2\sigma^{M-1} \leq \lambda_1 \leq (6K)^k\lambda_{t_{k+1}}\), and so \(\lambda_{t_{k+1}} \geq C_2\sigma^{M-1}(6K)^{-k}\).

By definition, \(\lambda_{t_{k+1}} \leq \frac{2Ke^{-r}}{\alpha^{k/3}}\), and putting these two inequalities together we
have that:

\[ C_2 \sigma^{M-1}(6K)^{-k} \leq \frac{2Ke^{-r}}{\alpha^{2/3}} \]

\[ \iff \]

\[ (M - 1) \log \sigma - k \log(6K) + C \leq -r + 2/3 \log \frac{1}{\alpha} \]

\[ \iff \]

\[ M \log(\sigma) + r - 2/3 \log \frac{1}{\alpha} + C \leq k \log(6K) \]

\[ \iff \]

\[ \frac{\log \sigma}{p \log k} \log \frac{1}{\alpha} + r - 2/3 \log \frac{1}{\alpha} \leq k \log(6K). \]

But the LHS is greater than

\[ r \left( 1 - \frac{2/3 - \log \frac{\delta k}{p \log k}}{1/6 - \eta} \right), \]

and if we can now show that \( 2/3 - \frac{\log \frac{\delta k}{p \log k}}{p \log k} < 1/6 - \eta \) we are done. Remembering how we defined the constant \( \eta(\alpha) \), we know that by making \( \alpha \) small we can make \( \eta \) arbitrarily small too. What we need then is that

\[ \frac{\log \delta k}{p \log k} > 1/2 \iff 2 \frac{\log \delta}{\log k} + 2 > p. \]

Since we can use an arbitrary \( \delta \in (0, 1) \) (by choosing \( \epsilon \) small enough), we see that any \( 1 < p < 2 \) will work. Then we have shown that for \( \eta \) small enough, \( k(r) \geq r \gamma \), for a constant \( \gamma > 0 \).

**Remark 9.5.** In a situation where we have a similarly defined system, but the critical point has degree \( \geq 2n + 1 \), for \( n \geq 2 \), what would happen in the proof? Can we generalise? Well, we would define \( J(r) := \{ x \mid |x - 1/2| \leq \alpha^{1/(2n+1)} e^{-r} \} \). An analogue to Lemma 8.1 would give us trouble when

\[ \left( \frac{1}{2n(2n + 1)} - \frac{\eta}{2n} \right) \log \frac{1}{\alpha} \geq r. \]

The earlier reasoning in this proof should carry through the same, and we would end up needing to show that

\[ \left( 1 - \frac{2n/(2n + 1) - \log \frac{\delta k}{p \log k}}{1/(2n + 1) - \eta} \right) > 0. \]
Which, since both $p$ and $\delta$ can be made arbitrarily close to 1, the proof should go through. We also see no reason why we would not get similar building expansion lemmas.

**Remark 9.6.** For the double standard map, where $\sigma > 1$ is the expansion factor given by Lemma 8.2, we end up needing to show that

$$0 < \left( 1 - \frac{2/3 - \frac{\log \sigma}{p \log \sup \partial_t f_n}}{1/6 - \eta} \right) = \left( 1 - \frac{2/3 - \frac{\log \sigma}{p \log \delta}}{1/6 - \eta} \right)$$

$$< \left( 1 - \frac{2/3 - \frac{\log 2}{p \log 4}}{1/6 - \eta} \right) \neq \left( 1 - \frac{2/3 - 1/2}{1/6 - \eta} \right),$$

which will not work.

The remainder of the proof is very similar to Viana’s corresponding proof, but we write it out for completion.

Now for each $\bar{l} = (l_1, \ldots, l_M) \in \{1, \ldots, d\}^M$ we denote with $\omega(\bar{l})$, the $\omega \in \mathcal{P}_m$ satisfying that $g_i - 1 (\omega) \subset [\bar{\theta}_{l_i - 1}, \bar{\theta}_{l_i}]$. Also for $1 \leq j \leq M$, $\hat{Y}_j(\bar{l}) = \text{graph}(Y_j(\bar{l})) = \varphi^j(\tilde{Y}_0 | \omega(\bar{l}))$. We call $\bar{l}$ and $\bar{m}$ incompatible if

$$|Y_M(\bar{l}, \theta) - Y_M(\bar{m}, \theta)| \geq 4e^{2-r} \alpha^{1/3}.$$

Now by the previous lemma, there are $H'_1, H''_1 \subset \{1, \ldots, d\}$, with $\#H'_1, \#H''_1 \geq [d/16]$, such that for every $l'_1 \in H'_1$ and $l''_2 \in H''_1$, we have that

$$|Y_1(l'_1, \ldots, l_M, \theta) - Y_1(l''_1, \ldots, l_M, \theta)| \geq \frac{\alpha}{100},$$

for $\theta \in g(\omega(l'_1, \ldots, l_M)) = g(\omega(l''_1, \ldots, l_M))$. These are equal since

$$g(\omega(l'_1, l''_1, \ldots, l_M) = \omega(l_2, \ldots, l_M) \in \mathcal{P}_M.$$

Then also

$$|Y_M(l'_1, \ldots, l_M, \theta) - Y_M(l''_1, \ldots, l_M, \theta)| \geq \frac{\lambda_1 \alpha}{2} \geq \frac{2K e^{-r} \alpha}{2^{2/3}} \geq e^{2-r} \alpha^{1/3}$$

So this means that all pairs $(l'_1, \ldots, l_M), (l''_1, \ldots, l_M)$ are incompatible. We can continue this line of argument for each of the successive $t_i$. For instance, given $L_i = (l_1, \ldots, l_{t_i-1})$, and any $l''_{t_i} \in H''_{t_i}$, $l''_{t_i+1} \in H''_{t_{i+1}}$,

$$|Y_{t_i}(L_i, l'_1, l_{t_i+1}, \ldots, l_M) - Y_{t_i}(L_i, l''_{t_i}, l_{t_i+1}, \ldots, l_M)| \geq \frac{\alpha}{100}.$$
for $\theta \in g^4(\omega(L_t, l_{t'}, l_{t+1}, \ldots, l_M, \theta)) = g^4(\omega(L_t, l_{t''}, l_{t+1}, \ldots, l_M, \theta))$. Then it follows that

$$|Y_M(L_t, l_{t'}, l_{t+1}, \ldots, l_M, \theta) - Y_M(L_t, l_{t''}, l_{t+1}, \ldots, l_M, \theta)|$$

as long as $k(r) \geq i$. This means that, looking at an arbitrary admissible curve $\hat{Y}_M(l_1, \ldots, l_k, \ldots, l_M)$, at each position $t_j$ when $j \leq k$, only elements from either $H'_j$ or $H''_j$ are possible, and since $\#H'_j, \#H''_j \geq [d/16]$, we can at most have $d^M - k(d - [d/16])^k$ of the admissible curves intersecting any segment $\{\theta\} \times J(r - 2)$. Also $\left|\left(g^M\right)'\right| \geq (d - \alpha)^M$, and since $M \leq \frac{1}{\alpha}$, we know $(\frac{d}{d^M})^M = (1 + \alpha)^M \leq ((1 + \alpha)^{\log \frac{1}{\alpha}})^\text{const} \leq ((1 + \alpha)^{\frac{1}{\alpha}})^\text{const} \leq \text{const}$ for $\alpha$ small enough. Add that $m(\omega) \leq \frac{1}{(g^M)^2}$, and it follows that

$$m \left( \{ \theta \in \hat{Y}_M(\theta) \in S_k \times J(r - 2) \} \right) \leq \# \{ \{ I \mid \hat{Y}_M(I) \cap S_k \times J(r - 2) \neq \emptyset \} \cdot \frac{1}{(d - \alpha)^M} \leq \frac{d^M((d - [d/16])/(d - \alpha)^M)}{((d - \alpha)^M)} \leq \text{const} \cdot \left(\frac{99}{100}\right)^{\eta r}.$$

Setting $\beta = \frac{1}{5} \log \frac{100}{99}$ the lemma is proved. We do not need it for this lemma but there is some room for improvement, because the fact that

$$|Y_M(L_t, l'_{t_i}, l_{t+1}, \ldots, l_M, \theta) - Y_M(L_t, l''_{t_i}, l_{t+1}, \ldots, l_M, \theta)| \geq \frac{\lambda_{t_i} \alpha}{2} \frac{\alpha}{100} \geq \frac{2K e^{-r}}{2\alpha^{2/3}} 100 \geq e^{2-r} \alpha^{1/3},$$

meaning $(L_t, l'_{t_i}, l_{t+1}, \ldots, l_M, \theta)$ and $(L_t, l''_{t_i}, l_{t+1}, \ldots, l_M, \theta)$ are incompatible also implies that every pair of the form $(l'_{t_1}, l_{t_2}, \ldots, l_{t_{j-1}}, l'_{t_j}, \ldots, l''_{t_M}), (l''_{t_1}, l_{t_2}, \ldots, l_{t_{j-1}}, l'_{t_j}, \ldots, l'_{t_M})$ are incompatible, where for $j \geq k$, the $l'_j, l''_j$ are arbitrary. To show this notice that

$$|Y_{t+1}(L_t, l'_{t}, l_{t+1}, \ldots, l_M, \theta) - Y_{t+1}(L_t, l''_{t}, l_{t+1}, \ldots, l_M, \theta)| \geq \frac{\lambda_{t_i} \alpha}{2} \frac{\alpha}{100} \geq 4e^2 \alpha.$$

and that

$$Y_{t+1}(L_t, l'_{t}, \ldots, l_{t+1}, \ldots, l_{t_M}, \theta) - Y_{t+1}(L_t, l''_{t}, \ldots, l_{t+1}, \ldots, l_{t_M}, \theta) \leq \text{osc}(\varphi(\hat{Y}_{t+1}(L_t, l'_{t}, l_{t+1}, \ldots, l_{t_M}))) \leq 8\alpha.$$
Since \( \tilde{Y}_{t+1}(L, l_{t+1}^1, \ldots, l_{t+1}^i, \ldots, l_{t+1}^r) \) is an admissible curve and \(|\text{osc}(Y_j)| \leq 2\alpha 4^j\).

The same argument applies to show
\[
Y_{t+1}(L, l_{t+1}^1, \ldots, l_{t+1}^i, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r) - Y_{t+1}(L, l_{t+1}^1, \ldots, l_{t+1}^i, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r) 
\leq \text{osc}(\varphi(\tilde{Y}_{t+1}(L, l_{t+1}^1, \ldots, l_{t+1}^r))) \leq 8\alpha.
\]

This means that, by use of the triangle inequality that
\[
|Y_{t+1}(L, l_{t+1}^1, \ldots, l_{t+1}, \ldots, l_{t+1}^r, \ldots, l_{t+1}) - Y_{t+1}(L, l_{t+1}^1, \ldots, l_{t+1}^i, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r)| \geq \frac{\lambda_{t+1}}{2},
\]
which is greater than or equal to
\[
\geq |Y_{t+1}(L, l_{t+1}, \ldots, l_{t+1}^i, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r)| - Y_{t+1}(L, l_{t+1}^1, \ldots, l_{t+1}^i, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r, \ldots, l_{t+1}^r) + \frac{\lambda_{t+1}}{2} \geq (4e^2 - 16)\alpha \frac{\lambda_{t+1}}{2}
\]
In this case as long as \( k(r) \geq i + 1 \).

\begin{remark}
Some comments on how to get a more general result, where \( \sigma \) only need to be larger than one.

The problem is in large part due to the loss we incur through situations like Lemma 8.1, when \( |x - 1/2| \) is very small. We get in trouble as soon as \( r \geq (1/6 - \eta) \log \frac{1}{\alpha} \). However, if we could handle the case \( (1/6 - \eta) \log \frac{1}{\alpha} \leq r \leq (2/3 - \eta) \log \frac{1}{\alpha} \) in some other way, we could easily adapt the proof and show that for all \( r \geq 2/3 - \eta \), the Technical lemma would work for any fixed \( \sigma > 1 \). This would give us much more freedom in choice of \( h(x) \) in (*), and we believe we would be able to get a similar result for a system where \( h(x) \) is instead the double standard map.
\end{remark}

10 Putting it all together

Now finally we have all the lemmas we need to prove the main result, and given the earlier results no novel difficulties arise in regards to Vianas method.
By Lemma 7.1, we know thus has a derivative bounded by $\frac{1}{n}$ after Viana, that Corollary 7.4 and see that

We fix an arbitrary admissible curve $\gamma$ is also an admissible curve over some $\omega_t$, and thus has a derivative bounded by $\alpha$. This lets us bound the diameter in the $x$-direction of this graph with

$$\alpha(d - \alpha)^{-1} = \alpha(d - \alpha)^M(d - \alpha)^{-m} \leq \left( \frac{1}{\alpha} \right)^{\text{const}} (d - \alpha)^{-m} \ll \alpha^{1/3} e^{-m},$$

since $n \gg \frac{1}{\alpha}$. This in turn implies that $\gamma \subset J(m - 1)$. Then we can use Corollary 7.4 and see that

$$m(\{ \theta \in S^1 \mid \text{some } 1 \leq v \leq n \text{ is a } II_n\text{-situation} \}) \leq nC_1 \sqrt{\frac{J(m - 1)}{\alpha}}$$

$$= nC_1 \sqrt{\frac{\alpha^{1/3} e^{m-1}}{\alpha}} \leq nC_1 \alpha^{-1/3} e^{m/2} \leq e^{\sqrt{m}/4}.$$  

Let us call this set of $\theta$'s $B_2(n)$. From now on will assume that any $\theta \in S^1$ do not come this extremely close to $S^1 \times \{1/2\}$, but only perhaps very close. More precisely, we say that $v$ is a $I_n$-situation for $\theta \in \omega_{v+1}$ if

$$\gamma \cap (S^1 \times J(0)) \neq \emptyset \text{ but } \gamma \cap (S^1 \times J(m)) = \emptyset.$$

Note, by remark earlier, that $I_n$- and $II_n$-situations are the only ways in which we can get contraction along the orbits.

Let then $1 \leq v_1 < \cdots < v_s \leq n$ be $I_n$-situations of $\theta$. If we recall how $N$ was defined in Lemma (8.1), then we know that $n \geq (s - 1)N$. And for each $v_i$, let us set $r_i$ equal to the minimal integer such that $\gamma \cap S^1 \times J(r_i) = \emptyset$. What we are interested then is how

$$|\partial_x f^n(X_1(\theta))| = \prod_{j=1}^n |\partial_x f(X_j(\theta))| = \prod_{j=1}^{v_1} |\partial_x f(X_j(\theta))|$$

$$\prod_{i=1}^s \left( \prod_{j=v_i}^{v_i+N-1} |\partial_x f(X_j(\theta))| \prod_{j=v_i+N}^{v_{i+1}} |\partial_x f(X_j(\theta))| \right) \prod_{v_s}^n |\partial_x f(X_j(\theta))|$$

(8)
Again we denote the expansion factor $\delta k$ by $\sigma$ to make the remaining proof easier to read. Then using Lemma 8.2 and 8.1, we get the following estimates:

\[ \prod_{j=1}^{v_1} |\partial_x f(X_j(\theta))| \geq C_2 \sigma^{v_1-1}, \]  
(9)

\[ \prod_{j=v_i}^{v_i+N-1} |\partial_x f(X_j(\theta))| \geq (\alpha^{1/3} e^{-r_i})^2 \alpha^{-1+\eta} = \alpha^{-1/3+\eta} e^{-2r_i} \]  
(10)

and

\[ \prod_{v_i+1}^{v_i+N} |\partial_x f(X_j(\theta))| \geq C_2 \sigma^{v_i+1-v_i-N} \text{ for } 1 \leq i < s. \]  
(11)

We also know

\[ \prod_{j=1}^{n} |\partial_x f(X_j(\theta))| \geq \text{const} |x_{v_s} - 1/2|^2 \sigma^{n-v_s} \geq \text{const} \alpha^{2/3} e^{-2r_s} \sigma^{n-v_2}. \]  
(12)

(The above since close to $1/2$, $\partial_x f = \text{const} |x_s - 1/2|^2$.) Above we see that (9) and (11) give us expansion in our orbit, and (10) and (12) are causing us trouble. Taking the log of (8) we estimate

\[
\log \prod_{j=1}^{n} |\partial_x f(X_j(\theta))| \geq \log \sigma^{v_1-1} + \sum_{i=1}^{s-1} \log \left( e^{-2r_i} \alpha^{-1+\eta} \right) + \sum_{i=1}^{s-1} \log \sigma^{v_i+1-v_i-N} + \log \alpha^{2/3} e^{-2r_s} \sigma^{n-v_2} - (s-1) \log C_2 \\
\geq (n - (s-1)N) \log \sigma + \sum_{i=1}^{s-1} (-2r_i + (-1/3 + \eta) \log \alpha) + 2/3 \log \alpha - 2r_s \\
- s \cdot \text{const} \geq (n - (s-1)N) \log \sigma + \sum_{i=1}^{s} \left( \left( \frac{1}{3} - \eta \right) \log \frac{1}{\alpha} \right) - 2r_i \\
- (1/3 - \eta) \log \frac{1}{\alpha} - 2/3 \log \frac{1}{\alpha} - s \cdot \text{const} \geq (n - (s-1)N) \log \sigma \\
+ \sum_{i=1}^{s} \left( \left( \frac{1}{3} - \eta \right) \log \frac{1}{\alpha} \right) - 2r_i \right) - \log \frac{1}{\alpha} - s \cdot \text{const}.
\]

The worst cases in trying to find a lower bound for the above is if any of the $r_i \geq (1/6 - \eta/2) \log \frac{1}{\alpha}$. If we let $G = \{ i \mid r_i \geq (1/6 - \eta) \log \frac{1}{\alpha} \}$ we can see that

\[
\sum_{i=1}^{s} \left( \left( \frac{1}{3} - \eta \right) \log \frac{1}{\alpha} - 2r_i \right) \geq \frac{sn}{2} \log \frac{1}{\alpha} - \sum_{i \in G} 2r_i \geq \gamma_2 N s - \sum_{i \in G} 2r_i;
\]

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for some $\lambda_2 > 0$ and independent of $\alpha$ and $n$. We used that $N \approx \log \frac{1}{\alpha}$. This leads us to

$$\log \prod_{j=1}^{n} \left| \partial_x f(\hat{X}_j(\theta)) \right| \geq (n - (s - 1)N) \log \sigma + \gamma_2 N s - \sum_{i \in G} 2r_i - s \text{ const} - \log \frac{1}{\alpha}$$

If we assume $\log \sigma > \gamma_2$, then

$$(n - (s - 1)N) \log \sigma + \gamma_2 N s \geq n \log \sigma + N(s - 1)(\gamma_2 - \log \sigma) \geq n \gamma_2$$

and if $\log \sigma < \gamma_2$

$$3cn - \sum_{i \in G} 2r_i - s \text{ const} - \log \frac{1}{\alpha} \geq 2cn - \sum_{i \in G} 2r_i,$$

using that $s \leq \frac{n}{N} + 1$ and $n \gg N \approx \log \frac{1}{\alpha}$. Here we deviate somewhat from Vianas proof, to be more explicit about some details in the proof. So observe that

$$2cn - \sum_{i \in G} r_i \geq 2c - m - \sum_{i \in G'} r_i \geq 2c'n - \sum_{i \in G'} r_i,$$

where $c'$ is a constant. This works since $r_i \leq m$ (and $m \approx \sqrt{n}$) and $G'$ is the same as $G$; except if there possibly is an $I_n$-situation for $1 \leq \nu \leq M$, we have assumed a maximal loss, and removed any such $r_i$ from $G$. This is a technical detail which we feel is necessary, but is not explicitly stated in the original paper. If we now define $B_1(n) = \{\theta \in \mathbb{S}^1 \mid \sum_{i \in G'} \geq c'n\}$, and and $E_n = B_1(n) \cup B_2(n)$, we know that for $\theta \in \mathbb{S}^1 \setminus E_n$,

$$\log \prod_{j=1}^{n} \left| \partial_x f(\hat{X}_j(\theta)) \right| \geq cn.$$

Thus if we can show $B_1(n) \leq e^{-\gamma \sqrt{n}}$, for some $\gamma > 0$, we will be done. We will now use a large deviation argument, to show that $B_1(n)$ is indeed a very rare occurrence, asymptotically. We begin by fixing $0 \leq q \leq m - 1$ and define the set $G_q = \{i \in G' \mid v_i \equiv q \mod m\}$, and we set $m_q = \max\{j \mid mj + q \leq n\}$. Now for each $0 \leq j \leq m_q$, we define $\hat{r}_j = r_i$ if $mj + q = v_i$, for some $i \in G_q$.
(no other $i$’s are possible, of course), and $\hat{r}_j = 0$ otherwise. An important set will now be

$$\Omega(\rho_0, \ldots, \rho_m) = \{ \theta \in S^1 \setminus B_2(n) \mid \hat{r}_j = \rho_j \text{ for } 0 \leq j \leq m_q \},$$

where not all the $\rho_j$’s are equal to 0. If we now consider $\omega_{mj+q+l} \in P_{mj+q+l}$, we know from Lemma 7.1 that $\hat{Y}_0 = \varphi^{mj+q+l}(\hat{X}_0 \mid \omega_{mj+q+l})$ is an admissible curve, and by the choice of $l$ we see that $mj + q + l = mj + q + m - M = m(j + 1) + q - M$, we are in a position to use Lemma 9.1 to deduce that

$$m(\{ \theta \in \omega_{mj+q+l} \mid \hat{r}_{j+1} = \rho \}) \leq C_3 e^{-5\beta \rho} m(\omega_{mj+q+l}), \forall \rho \geq \frac{1}{2} \left(\frac{1}{3} - \eta\right) \log \frac{1}{\alpha}.$$

Then using this for each successive 0 $\leq j \leq m_q$, we get first for $j = 0$ that:

$$m(\{ \theta \in \omega_{0+q+l} \mid \hat{r}_{0+1} = \rho \}) \leq C_3 e^{-5\beta \rho} m(\omega_{0+q+l}), \forall \rho \geq \frac{1}{2} \left(\frac{1}{3} - \eta\right) \log \frac{1}{\alpha},$$

and then for $j = 1$

$$m(\{ \theta \in \omega_{1+q+l} \mid \hat{r}_{2} = \rho \}) \leq C_3 e^{-5\beta \rho} m(\omega_{1+q+l}), \forall \rho \geq \frac{1}{2} \left(\frac{1}{3} - \eta\right) \log \frac{1}{\alpha}.$$

Now if we want to know $m(\theta \in S^1 \mid \hat{r}_1 = \rho_1, \hat{r}_2 = \rho_2)$, we know that since $\hat{r}_1$ is constant on each $\omega_{1+q+l} \in P_{1+q+l}$ (by construction), the union of all $\omega_{1+q+l}$ for which $\hat{r}_1 = \rho_1$ is smaller than $C_3 e^{-\beta \rho_1}$. Then we see that

$$m(\theta \in S^1 \mid \hat{r}_1 = \rho_1, \hat{r}_2 = \rho_2) \leq (C_3 e^{-5\beta \rho_1 + \rho_2}).$$

where $\tau$ is $\#\{\rho_j \neq 0, \ j \in \{1, 2\}\}$. Since we can get the same kind of bounds for all 0 $\leq j \leq m_q$, repeating the reasoning above yields

$$\Omega_q(\rho_1, \ldots, \rho_m) \leq (C_3 e^{-5\beta \sum \rho_j}) \leq (C_3 e^{-5\beta \rho})$$

where again $\tau$ is the total number of $\rho_j \neq 0$. This is essentially that our function is mixing. This bound on $\Omega(\rho_1, \ldots, \rho_m)$ will be enough to bound $B_1(n)$. The following reasoning will go just the same as in Viana. First we note that by the above bound, we get

$$\int e^{2\beta \sum_{i \in C_q} r_i} \leq \sum_{(\rho_1, \ldots, \rho_m)} C^\tau_4 e^{-3\beta \sum \rho_j} \leq \sum_{\tau, R} C^\tau_4 \xi(\tau, R) e^{-3\beta R},$$

where the integral is taken over all sets $\Omega_q(\rho_1, \ldots, \rho_m)$, and $\xi(\tau, R)$ is the number of integer solutions to $x_1 + \cdots + x_\tau = R$, satisfying $x_j \geq 1/2(1/3 - \eta_j)$.  

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\[
\eta \log \frac{1}{\alpha}. \text{ Then }
\]
\[
\xi(\tau, R) \leq \left( \frac{R + \tau}{\tau} \right) = \frac{(R + \tau)!}{R! \tau!} \leq \frac{(R + \tau)^{R + \tau}}{R^{R + \tau}} = \left( \frac{1 + \frac{\tau}{R}}{R} \right)^{R} \leq e^{\beta R}.
\]

The first inequality is just simple combinatorics, and the second follows from the fact that
\[
(R + \tau)^{R + \tau} = R^{R + \tau} + \cdots + \left( \frac{R + \tau}{R} \right)^{R} = \cdots + \tau^{R + \tau}
\]
The second equality follows directly, just multiply both sides with \( R \) to see this. The last inequality follows since each factor can be made arbitrarily close to 1 by taking \( \alpha \) small enough, so arbitrarily close to \( e^\beta \). Small enough \( \alpha \) also makes \( R/\tau \geq D \log \frac{1}{\alpha} \), since each \( x_j \geq \frac{1}{2}(1/3 - \eta) \log \frac{1}{\alpha} \), for the relevant indices \( j \). For the same reason we may also suppose \( C_{\tau}^4 \leq e^{\beta R} \). Because \( e^{\tau \log C_4} \leq e^{\beta R} \iff \tau \log C_4 \leq \beta R \iff \frac{\log C_4}{\beta} \leq \frac{R}{\tau} \). It follows that
\[
\int e^{2\beta \sum_{i \in G_q} r_i} \, d\theta \leq \sum_{\tau, R} e^{-\beta R} \leq \sum_{R} R e^{-\beta R} \leq 1,
\]
since \( \tau \leq R \) and \( R \geq \frac{1}{2}(1/3 - \eta) \log \frac{1}{\alpha} \gg 1 \) (since \( \tau \geq 1 \)). The last inequality is true since we can make \( R \) arbitrarily big by decreasing \( \alpha \). Now we can say that
\[
m \left( \left\{ \theta \mid \sum_{i \in G_q} r_i \geq c' n/m \right\} \right) \leq e^{-2\varepsilon n/m} \int e^{2\beta \sum_{i \in G_q} r_i} \, d\theta \leq e^{-2\varepsilon n/m}.
\]
Finally then, we are in the end zone, because surely if \( \theta \in B_1(n) \), then \( \sum_{i \in G_q} r_i \geq c'n/m \) for some \( 0 \leq q \leq m - 1 \), and then
\[
m(B_1(n)) \leq m e^{-2\varepsilon n/m} \leq e^{-\gamma \sqrt{n}}
\]
This proves Theorem 5.1

11 The double standard map

As stated in the introduction, before working on the above system we first tried to adapt Viana’s method on another dynamical system, namely
\[
\phi_\alpha(\theta, x) = (g_\alpha(\theta), f_\alpha(\theta, x)) = \left( d\theta \mod 1, 2x + \alpha \sin(2\pi \theta) + \frac{1}{\pi} \sin(2\pi x) \mod 1 \right),
\]

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defined from \( S^1 \times S^1 \rightarrow S^1 \times S^1 \). A very similar setup, but for the choice of system to couple with we use the double standard map \( \tilde{f}(x) = 2x + \frac{1}{x} \sin 2\pi x \).

Taylor expansion of \( \tilde{f} \) around the critical point \( x = 1/2 \) shows the cubic behaviour:

\[
\tilde{f}(1/2 + t) = 1 + 2t + \frac{1}{\pi} \sin(\pi + 2\pi t) = \frac{4\pi t^3}{3} + B(t)t^5.
\]

This family of double standards maps has been studied for instance in [13] and [3], and we believe it would be of interest if one could use this system to generate a multidimensional system with non-uniform hyperbolic behavior, as in Theorem 5.1. Let us go through some partial results in the adaptation of Viana’s proof, but this time with regards to the new map (13). We then assume \( \phi = (g(\theta), f(\theta, x)) \) close to \( \phi_\alpha \) and is such that \( ||\phi - \phi_\alpha||_{C^3} \leq \alpha \), and \( \phi \) that \( \partial_x f = \partial^2_{xx} f = 0 \) if and only if \( x = 1/2 \).

### 11.1 Admissible curves

The definitions of admissible curves and the markov partitions are exactly the same, and the results in the corresponding Section 7 are very similar.

### 11.2 Building expansion (again)

First we want a similar result to Lemma 8.1.

**Lemma 11.1.** There exists \( N = N(\alpha) \) such that \( \prod_{j=0}^{N-1} |\partial_x f(\theta, x)| \geq |x - 1/2|^2 \alpha^{-1+n} \) whenever \( |x - 1/2| \leq 6\alpha^{1/3} \).

**Proof.** Something very similar to [18, Lemma 2.4] we believe would work well here.

The next important building expansion lemma is the analogue to Lemma 8.2. Again let \( d_i = |x_i - 1/2| \), and \( x_i \) is such that \( \varphi^i(\theta_0, x_0) = (\theta_i, x_i) \), and reminding ourselves that in Viana’s paper, the exact value of \( \sigma \) is not important, as long as \( \sigma > 1 \). The proof method below is similar to ones used in [2, Lemma 1].

**Lemma 11.2.** There exists \( \sigma > 1, C_2, \epsilon > 0 \) such that for all \( (\theta, x) \in S^1 \times S^1 \), if \( d_0, d_1, \ldots, d_{a-1} \geq \alpha^{1/3} \) then \( \prod_{j=0}^{a-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \alpha^{3/3} \sigma^a \). If in addition \( d_a < \epsilon \), then \( \prod_{j=0}^{a-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \sigma^a \).
Before we prove this lemma, let us introduce a suitable homeomorphism $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates the double standard map $\tilde{f}(x)$ to something similar to the doubling map. From the definitions, we know

$$f(\theta, x) = \tilde{f}(x) + s(\theta, x),$$

where $|s(\theta, x)|, |s'(\theta, x)|, |s''(\theta, x)| \leq 2\alpha$. We then let $h(x) = x - \frac{1}{2\pi} \sin(2\pi x)$. Given a point $(x_0, \theta_0)$, let us define

$$F(\theta, x) := h^{-1}(\tilde{f}(h(x)) + s(\theta_0, x)).$$

Similarly, if $\varphi^j(\theta_0, x_0) = (\theta_j, x_j)$, we recursively define

$$F^k(\theta_0, x_0) = h^{-1}(f(h(F^{k-1}(\theta_0, x_0)) + s(\theta_{k-1}, x_{k-1}))).$$

Then we see that

$$f(\theta_0, x_0) = \tilde{f}(x_0) + s(\theta_0, x_0) = h(F(h^{-1}(\theta_0, x_0))),$$

and also

$$f^k(\theta_0, x_0) = h(F^k(\theta_0, h^{-1}(x_0))).$$

We claim that as long as $|x - 1/2|, |x| > \alpha^{1/3}, \partial_x F(x, \theta) \geq 3/2$. This is reasonable, and numerical experiments support it, but we leave aside the proof of this. We assume we know this, and show how we can use it to prove Lemma 11.2.

**Proof.** Some notes. Taylor expansion of $h(x)$ around $x = 0$:

$$h(t) = \frac{2\pi^2 t^3}{3} + O(t^5),$$

and taylor expansion around $x = 1/2$ gives

$$h(1/2 + t) = 2t + O(t^3).$$

First we want to show that if $|x|, |x - 1/2| \geq \alpha^{1/3}$, then the same is true for $F(x, \theta)$. Let us first assume $x$ is close to 0, but $|x| \geq \alpha^{1/3}$. Then

$$F(\theta, x) = h^{-1}(\tilde{f}(\frac{2\pi^2 x^3}{3} + O(x^5)) + s(\theta, x))
= h^{-1}(\frac{8\pi^2 x^3}{3} + O(x^5) + s(\theta, x)).$$

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Now by contradiction, if
\[ h^{-1}\left(\frac{8\pi^2 x^3}{3} + O(x^5) + s(\theta, x)\right) < \alpha^{1/3}, \]
then
\[ \frac{8\pi^2 x^3}{3} + O(x^5) + s(\theta, x) < h(\alpha^{1/3}) = \frac{2\pi^2 \alpha}{3} + O(\alpha^{5/3}), \]
which is not possible since we assumed \(|x| > \alpha^{1/3}\). Now we assume we are close to \(x = 1/2\), but still \(|x - 1/2| \geq \alpha^{1/3}\). Let \(t\) be the distance to \(1/2\). Then
\[
F(x, \theta) = h^{-1}(\tilde{f}(2t + O(t^3)) + s(\theta, x)) = h^{-1}(4t + O(t^3) + s(\theta, x)).
\]
Then again, by contradiction, if
\[ h^{-1}(4t + O(t^3) + s(\theta, x)) < \alpha^{1/3}, \]
then
\[ 4t + O(t^3) + s(\theta, x) < \frac{2\pi^2 \alpha}{3} + O(\alpha^{5/3}), \]
which is not possible since \(t \geq \alpha^{1/3}\). By induction the same is true for \(F^k(x, \theta)\). Now we can prove the theorem. Given \(x_0, \ldots, x_a\) with distances \(d_0, \ldots, d_a\), assume \(j = 0, \ldots, k\) are such that \(|x_j| \leq \alpha^{1/3}\) and let \(k\) be the first index such that \(x_k \geq \alpha^{1/3}\). Then \(h^{-1}(x_k) \geq \alpha^{1/3}\) and
\[ f^a = h \circ F^{a-k} \circ h^{-1} \circ f^k \]
implying
\[ \partial_x f^a \geq \frac{d}{dx} h^{-1}(F^{a-k} \circ h^{-1} \circ f^k) \cdot (3/2)^{a-k} \cdot 1/2 \cdot \partial_x f^k \geq \frac{d}{dx} h(F^{a-k} \circ h^{-1} \circ f^k) \cdot \sigma^a. \]
Now we can see that since \(F^{a-k} \circ h^{-1} \circ f^k \geq \alpha^{1/3}, h' \geq C_2 \alpha^{2/3}\) and if we know \(d_a\) is less than small enough \(\epsilon\), then \(h'\) is close to 2 which proves the result.

11.3 Technical lemma (again)

In this section, Lemma 9.3 will go through more or less exactly as earlier (and in Viana’s paper). Again, with the results we have with expansion above, it is when \(|x - 1/2| \leq \alpha^{1/3}\) the perturbation \(\alpha \sin(2\pi \theta)\) will have a large influence, so we have the same definition of \(J(r) = \{x \in S^1 : |x - \frac{1}{2}| < \alpha^{1/3}e^{-r}\}\). We also define \(M\) in a similar way as before. To get the large deviation argument to go through, we would need exactly Lemma 9.1 again for this system. The proof is very similar, since the lemmas in the section on Admissible curves would be very similar.
11.4 Sketch of attempted proof of the Technical lemma

Similar results in the beginning of the proof can be made. For instance, in the same way as before, \( \text{osc}(Y_j) \leq 2(\sup \partial_x f)\alpha = 2\alpha 4^i \), and the constant \( M \) can in a similar way be chosen so that \( \text{osc}(Y_M) < \epsilon \). Similar results for the metric distortion can also be had, so given \( 0 \leq j \leq M - 1 \) and \( 1 \leq i \leq M - j \), we could again get that

\[
\left| \frac{\partial_x f^i(\theta_j, x_j)}{\partial_x f^i(\tau_j, y_j)} \right| = \prod_{m=j}^{j+i-1} \frac{\partial_x f(\theta_m, x_m)}{\partial_x f(\tau_j, y_j)} \leq 2.
\]

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Similar definition of \( \lambda_j, K \), and \( k(r) \) would eventually again land us in trying to show \( k(r) \geq \gamma r \), through the inequality

\[
C_2\sigma^{M-1}(6K)^{-k} \leq \frac{2Ke^{-r}}{\alpha^{2/3}}
\]

with \( r \geq (1/6 - \eta) \log \frac{1}{\alpha} \). As before we would need that

\[
1 - \frac{2/3 - \frac{\log \sigma}{p\log 4}}{1/6 - \eta} > 0,
\]

but this time we can not hope to get \( \sigma \geq 2 \), so as in Remark 9.6 we are stuck. There might be ways to remedy this. As we mention in Remark 9.7, if we want to be completely independent of the value of \( \sigma > 1 \), we would need to handle the case \((1/6 - \eta) \log \frac{1}{\alpha} \leq r \leq (2/3 - \eta) \log \frac{1}{\alpha}\) separately in some way. The set of problematic values of \( r \) could be made smaller however, since we perhaps would be able to get \( \sigma \) very close to 2 through a good choice of homeomorphism as in Lemma 11.2.

Would we be able to prove this Lemma also, we believe the rest of the proof should go through as before and as in [18].

12 Possible implications of result

In [1] Alves proves the existence of SRB-measures in systems with non-uniform expansion in several dimensions. In fact, he proves it for the particular class of system in which Viana showed such an expansion. In the setting of systems with non-uniform expansion, he defines an SRB measure for a transformation \( \psi : M \to M \) to be a finite measure \( \mu \), for which \( \psi \) is measure preserving and there is a positive Lebesgue measure set \( B \subset M \) such that
for every continuous map \( f \) one has that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{0}^{n-1} f \circ \varphi^j(x) = \int_M f \, d\mu.
\]

More precisely, in [1], he shows, with respect to the \( \psi_\alpha \) used by Viana and described in the introduction:

**Theorem 12.1.** For \( d \geq 16 \) and \( \alpha \) sufficiently small, the map \( \psi_\alpha \) has a finite absolutely continuous (with respect to the bidimensional Lebesgue measure) invariant measure \( \mu^* \). Moreover, the same holds for every map \( \psi \) in a sufficiently small neighborhood of \( \psi_\alpha \) in the \( C^3(S \times I) \) topology.

For these very same \( \psi \) he also shows that it only has finitely many ergodic absolutely continuous measure preserving probability measures, which means any such measure \( \mu^* \) as in the theorem will be a linear combination of these ergodic ones and thus ergodic. Using Birkhoff’s ergodic theorem one then has that

**Theorem 12.2.** Every \( \psi \) in a sufficiently small neighborhood of \( \psi_\alpha \) as in the Theorem 9.1 has an SRB measure.

We express hope that perhaps the proof in that paper could be adapted to the type of systems we have worked with in this thesis.

**References**


