Mode Matching Analysis of One-Dimensional Periodic Structures

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Abstract

In this thesis, we analyze the electromagnetic wave propagation in waveguide-structures exhibiting periodical geometry, including glide symmetry. The analysis is performed using a mode matching technique which correlates the different mode coefficients from separate but, connected regions in the structure. This technique is used to obtain the dispersion diagrams for two one-dimensional periodic structures: a glide-symmetric corrugated metasurface and a coaxial line loaded with periodic holes. The mode matching formulation is presented in Cartesian and cylindrical coordinate system for the former and the later, respectively. The mode matching results are compared to simulated results obtained from the Eigenmode Solver in CST Microwave Studio and are found to agree very well.

Key words: waveguide, wave propagation, electromagnetic fields, dispersion relation, mode matching, glide symmetry, periodic structures
Sammanfattning

I detta examensarbete, analyseras elektromagnetisk vågutbredning i periodiska vågledarstrukturer som uppvisar glid symmetri. Analysen genomfördes genom en mod matchnings-teknik som korrelerar de olika mod-koefficienterna från separate-rade regioner i strukturen med varandra. Denna teknik används för att ta fram dispersionsrelationen för två endimensionella periodiska strukturer: en glid symmetrisk korrugerad meta-yta och en koaxial ledare belagd med periodiskt urgröpda håligheter. Mod matchnings-formuleringen presenteras i Kartesiska och cylindriska koordinatssystem respektive för de ovan nämnda fallen. Mod matchnings-resultaten jämförs med data-simulerade resultat erhållna från CST Microwave Studio och de överensstämmer väl med varandra.

Nyckelord: Vågledare, Vågutbredning, elektromagnetiska fält, dispersionsrelation, mod matchning, glid symmetri, periodiska strukturer
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Chapter 1

Introduction

The quest for controlled wave propagation in waveguides has led to the introduction of metamaterials and metasurfaces [1], [2], [3] (metamaterial with one spatial dimension at sub-wavelength scale). These are artificial material designed to have properties not found in naturally occurring material [2].

Recently, the progress in metamaterial and metasurface research has generated an increasing interest in the concept of higher symmetries in periodic structures [4], [5], [6]. Two special cases of higher symmetries are glide symmetry and twist symmetry [7]. In the case of glide symmetry, the structure remains invariant under a translation and a reflection [8], while for twist symmetry, the structure remains invariant under a translation followed by a rotation [7], [9]. The treatment of periodic structures has allowed concepts from Solid State Physics such as unit cell and Brillouin zone (see e.g. [10] for a refresh) to be integrated with the theory of electromagnetic fields when developing the mathematical tools needed to describe the wave propagation within these structures. It has been shown that periodic structures exhibiting glide and twist symmetry show almost zero frequency dispersion for the first mode of propagation [8], [9]. This feature allows realizing ultra-wide band flat lens antennas based on high-symmetric periodic structures [11], [12], [13], [14]. Recently, a mode matching technique has been used to analyze wave propagation through one-dimensional and two-dimensional glide symmetric periodic structures [15], [16], [17], [18]. Comparing the analytically obtained dispersion diagrams (by the mode matching method) to the diagrams obtained from the CST Microwave Studio is done to show the accuracy of the method.

Obtaining the dispersion diagrams using this method is not only faster than CST but also more instructive about which modes are dominating within the structure.

Chapters 2 and 3 will provide basic background knowledge about electromagnetic field theory and waveguides. In chapter 4 we analyze a corrugated 1D-periodic structure exhibiting glide symmetry by using mode matching technique in Cartesian coordinate, while in chapter 5, we present mode matching formulation in cylindrical coordinates to analyze a periodical coaxial structure.
Chapter 2

Background to Electromagnetic Theory

This chapter will contain a brief background to the electromagnetic field theory needed when performing this thesis. For the more experienced reader (when it comes to electromagnetic field theory), this chapter can safely be skipped.

2.1 Maxwell’s Equations

The basic equations making up the foundation of the theory of electromagnetism is the Maxwell’s equations. These were stated by the Scottish mathematician James Clerk Maxwell \[19\] and hence bare his name. The perhaps most common way of writing Maxwell’s equations are

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1}
\]

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.2}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.3}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{2.4}
\]

which relates the electric field, \( \mathbf{E} \), and the magnetic flux density, \( \mathbf{B} \), to the charge density \( \rho \) and the current density \( \mathbf{J} \). The equations in (2.1)-(2.4) are Maxwell’s equations on differential form. However, they can also be given on integral form, where their physical meaning can be more easily seen. Equation (2.1) is called
Chapter 2. Background to Electromagnetic Theory

Faraday’s law [20] and by integrating it over a surface, we can use Stoke’s theorem on the LHS to obtain the integral equation

$$\int_{\partial A} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_A \mathbf{B} \cdot d\mathbf{A}. \quad (2.5)$$

Equation (2.5) states that the change (in time) in magnetic flux through a surface will induce a loop voltage across the closed contour $\partial A$ enclosing $A$. Performing the same trick with equation (2.3), we obtain

$$\int_{\partial A} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_A \mathbf{J} \cdot d\mathbf{A} + \mu_0 \epsilon_0 \frac{d}{dt} \int_A \mathbf{E} \cdot d\mathbf{A}. \quad (2.6)$$

This equation states that the induced magnetic field around a closed loop is equal to $\mu_0$ times the total current through the enclosing surface plus the time rate of change of electric field. In the static case we recognize equation (2.6) as the Ampere’s law [20]

$$\int_{\partial A} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad (2.7)$$

while the last term in (2.6) and/or (2.3) is called the displacement current and due to the extremely small factor $\mu_0 \epsilon_0 = 1/c^2$ (where $c$ is the speed of light), this term is very hard to detect in the laboratory and it would require a quite large time derivative to compete with the ”normal” current term [20].

If we integrate equations (2.2) and (2.4) over a volume $V$, we can use Gauss’s theorem on the LHS:s in order to obtain

$$\int_A \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} \int_V \rho dV \quad (2.8)$$

and

$$\int_A \mathbf{B} \cdot d\mathbf{A} = 0. \quad (2.9)$$

We recognize

$$\int_V \rho dV = Q_{\text{enc}}$$

as the total enclosed charge and hence (2.2) is seen to be Gauss’s law,

$$\int_A \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0}, \quad (2.10)$$

the electric flux through a surface is proportional to the total enclosed charge. While the same should be true for equation (2.9), we see that this equation states that there are no magnetic monopoles, at least not on a macroscopic level. However, on a microscopic level such objects (with a magnetic charge) can exist within so called spin ice material [21].
2.1. Maxwell’s Equations

2.1.1 The D and H Fields

The Maxwell’s equations stated in (2.1)-(2.4) are only valid in Vacuum. When considering Maxwell’s equations in some other medium than vacuum, one has to take into account how the medium response to the presence of the fields. The charged particles that make up the atoms and/or molecules in the material (electrons and protons), will naturally feel the external electric field and, due to their different charge, orient themselves as tiny dipole configurations. The medium will hence exhibit an electric polarization, \( P \). This is a vector quantity that represents the number of dipole moments per unit volume [19]. The effect from the polarization gives rise to, so called, bound charge distributions in the medium given by [19]

\[
P \cdot \hat{n} = \sigma_b \quad (2.11)
\]

and

\[
-\nabla \cdot P = \rho_b \quad (2.12)
\]

where \( \sigma_b \) is the bound surface charge density and \( \rho_b \) the bound volume charge density. Hence the total charge density can be written as the sum of the free and bound charge densities \( \rho = \rho_f + \rho_b \) and using this together with (2.12) into (2.2) one obtains

\[
\epsilon_0 \nabla \cdot E = \rho_f - \nabla \cdot P. \quad (2.13)
\]

Combining the two divergence terms then yields an expression as

\[
\nabla \cdot (\epsilon_0 E + P) = \nabla \cdot D = \rho_f \quad (2.14)
\]

where the \( D \)-field, or electric displacement, has been defined as

\[
D \equiv \epsilon_0 E + P. \quad (2.15)
\]

Similarly, when a medium is exposed to a magnetic field, the matter will be magnetized. Hence on the microscopic level, the material will be made up of tiny magnetic dipoles [20]. The magnetization (similar to electric polarization) is the magnetic dipole moment per unit volume and will be denoted by \( M \). Just as the electric polarization gave rise to bound charge densities, so does the magnetization give rise to a bound volume current density [20]

\[
J_b = \nabla \times M \quad (2.16)
\]

and a surface current density [20]

\[
K_b = M \times \hat{n}. \quad (2.17)
\]

Hence we can separate the total current density into a free and a bound current density \( J = J_f + J_b \). However, there is here a third contribution. Because much like a change in an electric field gives rise to a current (displacement current), a
change in electric polarization will also induce a bound current density as well \[20\]. Taking the time derivative of (2.12) yields

\[-\nabla \cdot \frac{\partial P}{\partial t} = \frac{\partial \rho_b}{\partial t} = -\nabla \cdot J_p\] (2.18)

where the continuity equation has been used in the last step and we see that

\[J_p = \frac{\partial P}{\partial t}\] (2.19)

and we obtain for the total current

\[J = J_f + J_b + J_p\]

Using this result, (2.19) and (2.16) into (2.3) yields

\[\nabla \times B = \mu_0 J_f + \mu_0 \nabla \times M + \mu_0 \frac{\partial P}{\partial t} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}\] (2.20)

Rearranging some terms, (2.20) can be written

\[\nabla \times \left( \frac{B}{\mu_0} - M \right) = J_f + \frac{\partial}{\partial t} (\epsilon_0 E + P)\] (2.21)

where we recognize the definition of the \(D\)-field in the last parenthesis and where the expression in the first parenthesis will be denoted by the letter \(H\) and is referred to as the magnetic field

\[H \equiv \frac{B}{\mu_0} - M.\] (2.22)

Putting it all together, we can rewrite Maxwell’s equations in (2.1)-(2.4) as

\[\nabla \times E + \frac{\partial B}{\partial t} = 0\] (2.23)

\[\nabla \cdot D = \rho_f\] (2.24)

\[\nabla \times H - \frac{\partial D}{\partial t} = J_f\] (2.25)

\[\nabla \cdot B = 0\] (2.26)

These are Maxwell’s equations in matter. The four fields are also related through certain constitutive relations depending on the medium i.e. the magnetization \(M = M[B, H]\) and the polarization \(P = P[E, D]\) can be written as functions of the fields. When solving Maxwell’s equations, such relations need to be decided together with the appropriate boundary conditions for the fields.
2.2 Boundary Conditions for the Fields

In order to solve Maxwell’s equations in a closed region, one needs a set of boundary conditions. It is therefore very convenient to investigate how the fields behave at boundaries between different media and interfaces sustaining different types of sources.

2.2.1 Conductor Boundaries

One of the most important boundary to consider is the one at the surface of a conductor. Consider a conductor in an external electric field as illustrated in Figure 2.1. The external field will give rise to a ”bound” charge separation that in turn sets up an equal but opposite (to the external E-field) electric field inside the conductor. Hence, the electric field inside a conductor will always be zero, and following from Gauss’s law, so will the charge density. Any net charge will reside on the surface of the conductor [20] and the electric field will always be perpendicular to the surface because, if it piled up somewhere, it would immediately flow around to kill off the tangential component [20]. Defining the two unit vectors \( \hat{n} \) (normal to the conductor surface) and \( \hat{\tau} \) (tangential to the conductor surface), we must have at a conductor surface

\[
\hat{\tau} \cdot \mathbf{E} = 0
\]

or

\[
\hat{n} \times \mathbf{E} = 0.
\]

![Figure 2.1](image) Figure 2.1. When a conductor (rectangle) is exposed to an external electric field (\( E_0 \)), the induced charge separation gives rise to an equal but opposite electric field (\( E_1 \)). Picture taken from [20].
2.2.2 Interfaces

Considering an interface between two regions containing a free surface charge density, \( \sigma_f \), and a free surface current, \( \mathbf{K}_f \). The fields will show a discontinuous behaviour across the interface. Integrating equations (2.24) and (2.26) over a Gaussian pillbox volume with an infinitesimally height extending on both sides of the interface one will obtain [20]

\[
D_1^\perp - D_2^\perp = \sigma_f \tag{2.27}
\]

and

\[
B_1^\perp - B_2^\perp = 0 \tag{2.28}
\]

where the unit normal is positive from region 2 to 1.

If we instead integrate equations (2.23) and (2.25) over a small rectangular loop with an infinitesimally height extending on both sides of the interface one will obtain [20]

\[
E_1^\parallel - E_2^\parallel = 0 \tag{2.29}
\]

and

\[
H_1^\parallel - H_2^\parallel = \mathbf{K}_f \times \mathbf{n}. \tag{2.30}
\]

Hence the parallel component in \( \mathbf{E} \) and the perpendicular component in \( \mathbf{B} \) are continuous over the interface while the perpendicular component in \( \mathbf{E} \) and the parallel component in \( \mathbf{B} \) are discontinuous.

2.3 Electromagnetic Waves

If one study Maxwell’s equations in free space or vacuum where the current density \( \mathbf{J} \) and volume charge density \( \rho \) are absent. We set

\[
\rho = 0 \quad , \quad \mathbf{J} = 0
\]

in (2.1)-(2.4) and obtain:

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.31}
\]

\[
\nabla \cdot \mathbf{E} = 0 \tag{2.32}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.33}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{2.34}
\]

Taking the curl of (2.31) and (2.33) we obtain

\[
\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} \tag{2.35}
\]
and

\[ \nabla \times \nabla \times B = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \times E. \]  

(2.36)

Inserting equations (2.31) and (2.33) in the RHS of (2.35) and (2.36) while using the vector identity

\[ \nabla \times \nabla \times A = \nabla (\nabla \cdot A) - \nabla^2 A \]

on the LHS we get

\[ \nabla (\nabla \cdot E) - \nabla^2 E = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \]  

(2.37)

and

\[ \nabla (\nabla \cdot B) - \nabla^2 B = -\mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}. \]  

(2.38)

Since the divergence of both E and B are zero (due to no sources), (2.37) and (2.38) reduce to

\[ \nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \]  

(2.39)

and

\[ \nabla^2 B - \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2} = 0. \]  

(2.40)

These are recognized as the wave equation. Hence both the E-field and the B-field are describing electromagnetic waves, propagating at the speed \( \frac{1}{\sqrt{\mu_0 \epsilon_0}} \) seen from the above equations. In vacuum, this is of course the speed of light

\[ c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}. \]

If we consider free space, a solution to (2.39) and (2.40) are propagating plane waves

\[ \mathbf{E}(r, t) = E_0 e^{i(k \cdot r - \omega t)} \hat{e} \]  

(2.41)

\[ \mathbf{B}(r, t) = B_0 e^{i(k \cdot r - \omega t)} \hat{b} \]  

(2.42)

where the \( \hat{e} \) and \( \hat{b} \) are the direction of the fields and \( E_0 \) and \( B_0 \) are the amplitudes. However, using (2.31) and (2.41), one can express \( B_0 \) and \( \hat{b} \) in (2.42) as

\[ \hat{b} = \hat{k} \times \hat{e} \]

\[ B_0 = \frac{E_0}{c} \]  

(2.43)

and we immediately find that

\[ \mathbf{B} = \frac{1}{c} \hat{k} \times \mathbf{E}. \]

Hence forth, we will always assume a time dependence as \( e^{-i\omega t} \) for the fields as \( \mathbf{E}(r, t) = \mathbf{E}(r)e^{-i\omega t}. \) This will simplify equations (2.39) and (2.40) as

\[ \nabla^2 \mathbf{E}(r) + k^2 \mathbf{E}(r) = 0 \]  

(2.44)
\[ \nabla^2 \mathbf{B}(\mathbf{r}) + k^2 \mathbf{B}(\mathbf{r}) = 0 \] (2.45)

where \( k = \omega/c \). Next we will consider confined waves or guided waves in so called waveguides.
Chapter 3

Waveguides

A waveguide is (as the name states) a structure designed for guiding waves. In microwave applications, electromagnetic waves can be guided using a hollow structure made up of conducting material in order to transport energy from one place to another [22]. The fields inside the waveguide are obtained by solving Maxwell’s equations together with suitable boundary conditions specified from the geometry of the structure. In this chapter we will investigate the metallic waveguide phenomenon by considering the two simplest and most common cross section geometries.

3.1 Decomposition with respect to a fixed direction

Figure 3.1. The cross section of a waveguide. The three spatial coordinates are split into one longitudinal direction (z-direction) and one transversal (within the cross section \( A \)). Picture taken from [22].
Often when analyzing a waveguide structure with a certain cross section it is very helpful to decompose the fields into a longitudinal and a transversal part with respect to (w.r.t) the cross section. The longitudinal direction (or direction of propagation) will be chosen to be $\hat{z}$. We will call the cross section of the waveguide $A$ and the boundary enclosing it $\Gamma$, see the Figure 3.1. Hence, a vector field $A$ is divided as

$$A = A_t + A_z \hat{z}$$

and the nabla-operator as

$$\nabla = \nabla_t + \hat{z} \frac{\partial}{\partial z}$$

(3.1)

where $\nabla_t$ is the transversal nabla operator, operating on the transversal components only.

As it will be seen, this will transform the cumbersome problem of solving a vector equation, such as (2.39) and/or (2.40), into a much more nicer scalar equation problem for solely the z-components of the fields, $E_z$ and $H_z$. Assuming a z-dependency as $e^{\pm jk_z z}$, the other components can then be generated from the z-components using Maxwell’s equations [22]

$$E_t = \frac{-j}{k^2_t} \left[ k_z \nabla_t E_z - k \hat{z} \times \nabla_t H_z \right]$$

(3.2)

$$H_t = \frac{-j}{k^2_t} \left[ k_z \nabla_t H_z + \frac{k}{\eta} \hat{z} \times \nabla_t E_z \right]$$

(3.3)

where $\eta = \sqrt{\mu/\epsilon}$ is the wave impedance and the three wave-numbers fulfill the Pythagorean relation

$$\frac{\omega^2}{c^2} = k^2 = k^2_t + k^2_z.$$  

(3.4)

If the whole space is filled with one material, the $H$-field might be scaled with the wave impedance $\eta$ as

$$\eta H \rightarrow H.$$  

Hence, considering the scaling of the $H$-field equations (3.2) and (3.3) become

$$E_t = \frac{-j}{k^2_t} \left[ k_z \nabla_t E_z - k \hat{z} \times \nabla_t H_z \right]$$

(3.5)

$$H_t = \frac{-j}{k^2_t} \left[ k_z \nabla_t H_z + k \hat{z} \times \nabla_t E_z \right]$$

(3.6)

Consider the waveguide cross section shown in Figure 3.1. The structure is filled with isotropic material and supports propagating waves along the $z$-axis. We want to solve the wave equation(s) in (2.44) and (2.45) inside the structure with suitable boundary conditions for the fields on the enclosing metallic surface. Due to the
3.1. Decomposition with respect to a fixed direction

propagation along the z-axis this can be turned into a two-dimensional problem by using (3.1) and setting

$$E(\mathbf{r}) = E(\rho)e^{\pm jk_z z}, \quad \rho \in \mathcal{A}$$

and similar for the magnetic field. We obtain

$$\nabla_t^2 E(\rho) + k_t^2 E(\rho) = 0$$
$$\nabla_t^2 H(\rho) + k_t^2 H(\rho) = 0$$

and since we can generate the transverse components from the longitudinal (z-component) it will be enough to consider the z-component in the above equations

$$\nabla_t^2 E_z(\rho) + k_t^2 E_z(\rho) = 0 \quad \text{(3.7)}$$
$$\nabla_t^2 H_z(\rho) + k_t^2 H_z(\rho) = 0 \quad \text{(3.8)}$$

where $k_t^2 = k^2 - k_z^2$. Before solving the above equations we need to decide boundary conditions on the two field components $E_z$ and $H_z$. Since the walls are metallic (conducting material), the parallel components (to the surface) of the electric field must vanish at the boundary walls. This can be seen from section 2.2.1 where

$$\hat{n} \times E = \hat{n} \times E_t + \hat{n} \times \hat{z} E_z = 0 \quad \text{(3.9)}$$

Since the the first term in (3.9) only has a z-component and the second term only has a transverse component, they both has to be zero. Hence, we demand

$$E_z = 0, \quad \rho \in \Gamma$$

as a boundary condition for (3.7). To find a similar condition for $H_z$ is little more tricky. We have [22]

$$\hat{n} \cdot \mathbf{B} = 0$$

which only states (using $\mathbf{B} = \mu \mathbf{H}$)

$$\hat{n} \cdot \mathbf{H} = \hat{n} \cdot \mathbf{H}_t = 0, \quad \text{(3.10)}$$

However, using (2.33) (with a time dependence $e^{-j\omega t}$) and operating with $\hat{z} \times$ on both sides yields

$$\hat{z} \times (\nabla \times \mathbf{H}) = -j \omega \epsilon \hat{z} \times \mathbf{E}.$$ 

Separating the above equation into z-component and transversal component yields

$$\nabla_t H_z - \partial_z H_t = -j \omega \epsilon \hat{z} \times \mathbf{E}_z \quad \text{(3.11)}$$

and taking the scalar product $\hat{n} \cdot$ (3.11) and using (3.9) and (3.10) finally yields

$$\hat{n} \cdot \nabla_t H_z = 0.$$
We now have boundary conditions for the \( z \)-components of both the fields and we obtain an eigenvalue problem as

\[
(\nabla^2_z + k_t^2)E_z(\rho) = 0 \quad \rho \in A \tag{3.12}
\]

\[
E_z(\rho) = 0 \quad \rho \in \Gamma \tag{3.13}
\]

for \( E_z \) and

\[
(\nabla^2_z + k_t^2)H_z(\rho) = 0 \quad \rho \in A \tag{3.14}
\]

\[
\hat{n} \cdot \nabla_i H_z(\rho) = 0 \quad \rho \in \Gamma \tag{3.15}
\]

for \( H_z \).

### 3.2 The Rectangular Metallic Waveguide

Within a waveguide the field solutions can be divided up into two linearly independent set of modes [22]. Those are the TE-modes (transversal electric modes) and TM-modes (transversal magnetic modes). The TE-modes assume \( H_z = 0 \) and are hence calculated using (3.12) together with (3.13). While for the TM-modes it is assumed \( E_z = 0 \) and hence equation (3.14) together with (3.15) need to be used. There are also a third set of modes called TEM-modes (transversal electromagnetic modes). These modes assume \( E_z = H_z = 0 \) and are only present if the interior of the waveguide-cross section is bounded by more than one conductor [22].

![Figure 3.2. The cross section of a rectangular waveguide. As a standard, it is always assumed that \( a \geq b \).](image)

#### 3.2.1 TM-modes

Perhaps the most simplest waveguide cross section to consider is the rectangular one. The setup is shown in Figure 3.2.
3.2. The Rectangular Metallic Waveguide

In order to calculate the TM-modes of the rectangular waveguide, equation (3.12) needs to be solved along with the boundary condition in (3.13) within the cross section seen in Figure 3.2. Assuming $E_z = E_z(x, y)$ we obtain:

$$(\partial_x^2 + \partial_y^2 + k_t^2)E_z(x, y) = 0$$  \hspace{1cm} (3.16)

$$E_z(x = 0, y) = E_z(x, y = 0) = E_z(x = a, y) = E_z(x, y = b) = 0$$  \hspace{1cm} (3.17)

This is an easy second order PDE-problem and a solution can be found by assuming a separable ansatz as $E_z(x, y) = X(x)Y(y)$ in (3.16) to obtain the two (separate) equations

$$(\partial_x^2 + k_x^2)X(x) = 0 \hspace{2cm} (\partial_y^2 + k_y^2)Y(y) = 0$$

with the solutions \[23]

$$X(x) \sim \sin (k_x x) \hspace{0.5cm} Y(y) \sim \sin (k_y y)$$

where the eigenvalues $k_x$ and $k_y$ fulfill $k_t^2 = k_x^2 + k_y^2$ and, from the boundary conditions in (3.17), are given by

$$k_x = \frac{m \pi}{a} \hspace{0.5cm} k_y = \frac{n \pi}{b} \hspace{0.5cm} m, n \in 1, 2, 3...$$  \hspace{1cm} (3.18)

Hence, for the TM-case, the z-component of the electric field is given by

$$E_{zmn}(x, y) = A_{mn} \sin (k_{xmn}x) \sin (k_{yn}y)$$  \hspace{1cm} (3.19)

where the $A_{mn}$:s are arbitrary (complex) constants. The rest of the field components can then be calculated from (3.19) by using (3.5) and (3.6) with $H_z = 0$.

It is seen from (3.18) that the eigenvalue $k_t$ is actually a set of eigenvalues

$$k_{tmn} = \sqrt{\left(\frac{m \pi}{a}\right)^2 + \left(\frac{n \pi}{b}\right)^2} \hspace{0.5cm} m, n \in 1, 2, 3...$$  \hspace{1cm} (3.20)

and the complete set of fields belonging to a certain eigenvalue $k_{tmn}$ ($m$ and $n$) is called a waveguide mode \[22].

3.2.2 TE-modes

For the TE-modes we have by definition $E_z = 0$, so for these solutions we set up the eigenvalue problem for the $H_z$-component instead using (3.14) together with (3.15)

$$(\partial_x^2 + \partial_y^2 + k_t^2)H_z(x, y) = 0$$  \hspace{1cm} (3.21)

$$\frac{\partial H_z}{\partial x}(x = 0, y) = \frac{\partial H_z}{\partial y}(x, y = 0) = \frac{\partial H_z}{\partial x}(x = a, y) = \frac{\partial H_z}{\partial y}(x, y = b) = 0.$$  \hspace{1cm} (3.22)
The calculations are analogue to the TM-case above, but due to the boundary conditions in (3.22) we obtain cosine instead of sine functions

\[ H_{zmn}(x, y) = B_{mn} \cos(k_{xm}x) \cos(k_{yn}y) \]

\[ \begin{cases} (m, n) \in \mathbb{N} \\ (m, n) \neq (0, 0) \end{cases} \]  

(3.23)

where again the \( B_{mn} \)'s are arbitrary (complex) constants. The rest of the field components can then be calculated from (3.23) by using (3.5) and (3.6) with \( E_z = 0 \).

### 3.2.3 The Cut-off Frequency

An important concept in a waveguide is the so called cut-off frequency. This frequency (if existing) decides whether a waveguide-mode will propagate or not in the structure. Each mode is characterized by a longitudinal wave number \( k_z \) and the factor

\[ E(r) \propto \exp(\pm jk_z z). \]

When \( k_z \) is real, we have propagating modes and when it is imaginary, \( k_z \to -j|k_z| \) and we obtain so called evanescent modes characterized by

\[ E(r) \propto \exp(\pm |k_z| z) \]

which are not propagating. The cut-off frequency is the transition frequency where a mode switches between propagating and evanescent [22] \( i.e. \) when \( k_z = 0 \) and from equation (3.4) this yields

\[ k = k_t \]  

(3.24)

and with perfectly conducting walls, the transversal wave number is always real [22] and determined by the waveguide structure. For the rectangular waveguide, \( k_t \) is given by eq. (3.20). Using \( k = 2\pi f \sqrt{\frac{\epsilon \mu}{2}} = 2\pi f/v \) and (3.20) in (3.24) we obtain the cut-off frequencies in a rectangular waveguide to be

\[ f_{c, mn} = \frac{v}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \]

\[ \begin{cases} (m, n) \in \mathbb{N} \\ (m, n) \neq (0, 0) \end{cases} \]  

(3.25)

where \( v \) is the propagation speed in the medium filling the waveguide.

### 3.3 The Circular Metallic Waveguide

Another common waveguide geometry is the circular wave guide. The cross section is illustrated in Figure 3.3. Since the cross section is simply-connected there are no TEM-modes present.
3.3.1 TM-modes

For TM-modes, $H_z = 0$ and the eigenvalue problem to be solved is the one in (3.12) and (3.13). In cylinder coordinates, $(\rho, \varphi, z)$, we obtain

$$\left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + k_t^2 \right) E_z(\rho, \varphi) = 0$$

(3.26)

$$E_z(\rho = a, \varphi) = 0.$$  

(3.27)

We also demand periodicity in the $\varphi$-direction

$$E_z(\rho, \varphi + 2\pi) = E_z(\rho, \varphi).$$

This problem can be solved by a separation ansatz $E_z(\rho, \varphi) = R(\rho)F(\varphi)$. This yields the two equations

$$\begin{cases} \partial_\varphi^2 F + m^2 F = 0 \\ F(\varphi + 2\pi) = F(\varphi) \end{cases}$$

(3.28)

$$\begin{cases} \rho^2 \partial_\rho^2 R + \rho \partial_\rho R + (k_t^2 \rho^2 - m^2) R = 0 \\ R(a) = 0 \end{cases}$$

(3.29)

where $m$ is a separation constant. For (3.28) we obtain the familiar trigonometric solutions

$$F(\varphi) \sim \begin{cases} \sin (m\varphi) \\ \cos (m\varphi) \end{cases}, \quad m = 0, 1, 2...$$

(3.30)
while the general solution for equation (3.29) are given by the Bessel functions of the first and second kind [23]

\[ R(\rho) \sim \begin{cases} J_m(k_t \rho) \\ Y_m(k_t \rho) \end{cases} \]

However, since \( R(\rho) \) has to be bounded as \( \rho \to 0 \), the \( Y_m(k_t \rho) \) is not a valid solution. The boundary condition, \( R(a) = 0 \), then yields the solution

\[ R(\rho) \sim J_m(k_{tmn} \rho) \quad (3.31) \]

where

\[ k_{tmn} = \frac{\xi_{mn}}{a} \quad (3.32) \]

and \( \xi_{mn} \) are the \( n:\text{th} \) zero to \( J_m \). Putting it together we obtain the eigenfunctions in the TM-case to be

\[ E_{zm\rho}(\rho, \varphi) = A_{mn} \begin{cases} \sin(m\varphi) \\ \cos(m\varphi) \end{cases} J_m \left( \frac{\xi_{mn} \rho}{a} \right) \quad (3.33) \]

where \( m = 0, 1, 2... \) and \( n = 1, 2, 3... \) are integers while \( A_{mn} \) are arbitrary (complex) constants.

### 3.3.2 TE-modes

For the TE-modes in the circular waveguide, the eigenvalue problem to solve is

\[ \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + k_t^2 \right) H_z(\rho, \varphi) = 0 \quad (3.34) \]

\[ \frac{\partial H_z}{\partial \rho} (\rho = a, \varphi) = 0. \quad (3.35) \]

This differs only from the TM-case in the boundary condition (3.35). Hence, analogue calculations yields

\[ \begin{cases} \partial^2_{\varphi} F + m^2 F = 0 \\ F(\varphi + 2\pi) = F(\varphi) \end{cases} \quad (3.36) \]

\[ \begin{cases} \rho^2 \partial^2_{\rho} R + \rho \partial_{\rho} R + (k_t^2 \rho^2 - m^2) R = 0 \\ \frac{\partial R}{\partial \rho} (\rho = a) = 0 \end{cases} \quad (3.37) \]

to be solved. The solution for (3.36) is given by (3.30) while the solution, fulfilling (3.37) is

\[ R(\rho) = A_{mn} J_m \left( \frac{\zeta_{mn} \rho}{a} \right) \quad (3.38) \]
where we this time instead have

\[ k_{\text{TE}}^{\text{mn}} = \frac{\zeta_{mn}}{a} \]  

(3.39)

and \( \zeta_{mn} \) are the \( n \):th zero to \( J'_m \). Putting it all together we find the eigenfunctions for the TE-modes to be

\[ H_{znm}(\rho, \varphi) = B_{mn} \begin{cases} \sin (m\varphi) \\ \cos (m\varphi) \end{cases} J_m \left( \zeta_{mn} \frac{\rho}{a} \right) \]  

(3.40)

where \( m = 0, 1, 2, \ldots \) and \( n = 1, 2, 3, \ldots \) are integers while \( B_{mn} \) are arbitrary (complex) constants.

### 3.3.3 The Cut-off Frequency

With the transversal wave number decided by the cylindrical geometry, the longitudinal wave number can be decided from (3.4). For TM-modes

\[ k_z = \sqrt{k^2 - \left( \frac{\xi_{mn}}{a} \right)^2} \]

and for TE-modes

\[ k_z = \sqrt{k^2 - \left( \frac{\zeta_{mn}}{a} \right)^2} . \]

The cut-off frequencies are then obtained when \( k_z = 0 \) i.e. when \( k = k_t \). We obtain for TM-modes

\[ f_{c, mn} = \frac{v \xi_{mn}}{2\pi a} \]  

(3.41)

and for TE-modes

\[ f_{c, mn} = \frac{v \zeta_{mn}}{2\pi a} \]  

(3.42)

where \( v \) is the phase velocity in the homogeneous material filling the waveguide. The zeros \( \xi_{mn} \) and \( \zeta_{mn} \) can be found in many mathematical tables such as e.g. [24].

### 3.4 The Coaxial Cable

The two previous waveguide geometries do not contain any TEM-modes. This is because TEM-modes do only propagate in structures containing two or more conducting surfaces. In fact, there are \( N - 1 \) linearly independent TEM-modes for a number of \( N \) separate conductors at different potentials [22]. A good example of a geometry sustaining TEM-modes is the coaxial cable. Its geometry can be seen in Figure 3.4. As seen, the cross section now consist of \( N = 2 \) conductor surfaces hence there should exist one set of TEM-mode solution.
For TEM-modes we have $E_z = H_z = 0$. This means $\mathbf{E} = \mathbf{E}_t$ and $\mathbf{H} = \mathbf{H}_t$ and Maxwell’s equation becomes

\[
\nabla \times \mathbf{E}_t = -jk\mathbf{H}_t, \quad \nabla \times \mathbf{H}_t = \mathbf{E}_t
\]  

(3.43)

and if we divide the nabla-operator into transversal and longitudinal components we obtain

\[
(\nabla_t + \hat{z}\partial_z) \times \mathbf{E}_t = -jk\mathbf{H}_t.
\]  

(3.44)

Since $\nabla_t \times \mathbf{E}_t$ only has a $z$-component and $\mathbf{H}_t$ has none, we obtain from (3.44)

\[
\nabla_t \times \mathbf{E}_t = 0.
\]  

(3.45)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{coaxial_cable.png}
\caption{Cross section of the geometry of a coaxial cable.}
\end{figure}

Combining the two equations in (3.43) as done in section 2.3 one can easily obtain the equations

\[
(\partial_z^2 + k^2) \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]  

(3.46)

which yield a $z$-dependence

\[
\begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} = \begin{pmatrix} \mathbf{E}_t(\rho)e^{\pm jkz} \\ \mathbf{H}_t(\rho)e^{\pm jkz} \end{pmatrix}.
\]  

(3.47)

This states that for TEM-modes $k = k_z$ and hence there are no cut-off frequencies and TEM-modes propagate for all frequencies [22]. This make TEM-modes (if present) very dominating compared to the TE- and TM-modes. Equation (3.45)
suggests that the transversal electric field can be expressed as the gradient of a scalar potential

\[ E_t(\rho) = -\nabla_t V(\rho) \] (3.48)

which together with \( \nabla \cdot E = \nabla_t \cdot E_t = 0 \) yields an electrostatic boundary value problem:

\[ \nabla_t^2 V(\rho) = 0 \quad \rho \in \mathcal{A} \] (3.49)

Solving (3.49) for the coaxial cable in Figure 3.4, we first assume due to symmetry of the geometry \( V(\rho) \equiv V(\rho) \). We then obtain

\[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) = 0 \] (3.50)

which has the solution

\[ V(\rho) = C_1 + C_2 \ln \frac{\rho}{a}. \] (3.51)

The constants can be decided from the condition that the potential difference between the inner and outer conductor is \( V_0 \)

\[ V(\rho) = V_0 \frac{\ln \left( \frac{\rho}{a} \right)}{\ln \left( \frac{b}{a} \right)} \]

The electric field component is then decided through (3.48) and the \( H_t \)-components from the relation

\[ H_t = \hat{z} \times E_t. \] (3.52)

Hence, the electric field then becomes

\[ E_t^{TEM} = \left( C^+ e^{-j k z} + C^- e^{+j k z} \right) \frac{\hat{\rho}}{\rho} \] (3.53)

and the H-field is

\[ H_t^{TEM} = \left( C^+ e^{-j k z} + C^- e^{+j k z} \right) \frac{\hat{\phi}}{\rho} \] (3.54)

where \( C^\pm \) are constants.
Chapter 4

Mode Matching in Cartesian Coordinates

4.1 The Mode Matching Method

Mode matching technique is a method for treating transitions (discontinuities) in systems of waveguides [22], if the waveguide at the discontinuous surface can be divided into separate regions with a simpler (easier) geometry each containing a different set of waveguide-modes. The modes of the two separate regions can be matched at the surface (connecting the two regions) using the boundary conditions for the electromagnetic fields $E$ and $H$. The mode-coefficients of the two regions are then connected through a so called scattering matrix [22]

$$
\begin{pmatrix}
a_1 \\
b_1
\end{pmatrix} =
\begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix}
\begin{pmatrix}
a_2 \\
b_2
\end{pmatrix}
$$

where the mode coefficients ($a$ and $b$) and scattering parameters ($S_{ij}$) are also matrices. However, in this thesis we’re interested to find the dispersion diagrams for a certain structure using the mode matching method, so the procedure includes the following steps:
1. Solve Maxwell’s equations in the different regions.
2. Obtain an equation system by imposing boundary condition for the transversal field components.
3. Testing the different equations (over the boundary) with suitable eigenfunctions in order to write the whole system on (square) matrix form.
4. Set the determinant of the matrix to zero in order to find the dispersion relation.
4.2 The 1D Glide-Symmetric Corrugated Metasurface

In this section we will analyze a one dimensional periodic structure using the mode matching method. The aim is to find the dispersion relation for wave propagating along the $x$-direction (see Figure 4.1) and compare it with the results from CST simulations. The (general) geometry of the unit cell can be seen in Figure 4.1. It is made up of three regions and the period of the cell is denoted by $p$.

The structure is invariant along the $y$-direction. Hence, for simplicity, we only consider wave propagation along the $x$-direction and because of the invariance we can set the wave number along the $y$-direction ($k_y$) equal to zero. The medium is assumed to be lossless with the total wavenumber $k = 2\pi f \sqrt{\varepsilon \mu}$ and the H-field scaled with the wave impedance $\eta H \rightarrow H$ (which will always be the case from now on).

4.2.1 The Field Expressions

The fields will be decomposed along the $y$-direction and equations (3.5) and (3.6) becomes

$$E_t = -\frac{j}{k_t^2} [k_y \nabla_t E_y - k \hat{y} \times \nabla_t H_y]$$

(4.2) and

$$H_t = -\frac{j}{k_t^2} [k_y \nabla_t H_y + k \hat{y} \times \nabla_t E_y].$$

(4.3)
4.2. The 1D Glide-Symmetric Corrugated Metasurface

Hence, the $x$ and $z$ components can be obtained from the $y$ components of the fields. The geometry of the three different regions in Figure 4.1 are rectangular and, due to the lower cut-off frequencies of the TE-modes (see section 3.2), only TE$_y$-modes will be considered [15]. For TE$_y$-modes, $H_y \neq 0$ and $E_y = 0$, and together with $k_y = 0$, (4.2) and (4.3) we have that

$$k^2 = k_t^2 = k_x^2 + k_z^2,$$

(4.4)

$$E_t = \frac{j}{k} \hat{y} \times \nabla_t H_y,$$

(4.5)

and

$$H_t = 0.$$  

(4.6)

Hence the only non zero field components are $H_y$, $E_x$ and $E_z$.

The field components are obtained by solving Maxwell’s equations in the three separate regions in Figure 4.1. This means $H_y$ should fulfill Helmholtz equation

$$(\nabla_t^2 + k_t^2)H_y(x, z) = 0$$

with suitable boundary conditions (different for each region). However, we can utilize the analysis from section 3.2. Region 1 and 3 of Figure 4.1 are basically rectangular waveguides but with one side ”open”. Hence, making the separable ansatz $H_y(x, z) = X(x)Z(z)$ with boundary conditions $\partial_x H_y = 0$ on the conductor surfaces, it is easy to see that we must have a standing wave along the $x$-direction (represented by a cosine [25]) and propagating waves in the $z$-direction (represented by exponential functions with pure imaginary argument) and a solution for region 1 can then be written as

$$H_y^{(1)} = k \sum_{m=0}^{\infty} \left( a_m e^{-jk_{zm}z} + a_m^* e^{jk_{zm}z} \right) \cos \left[ k_{xm}(x + L/2) \right] \quad m = 0, 1, 2...$$

(4.7)

$$E_x^{(1)} = \sum_{m=0}^{\infty} k_{zm} \left( a_m e^{-jk_{zm}z} - a_m^* e^{jk_{zm}z} \right) \cos \left[ k_{xm}(x + L/2) \right] \quad m = 0, 1, 2...$$

(4.8)

where $k_{xm} = m \pi / L$ and $k_{zm} = \sqrt{k^2 - k_{xm}^2}$ are obtained from (4.4).

Similar reasoning for region 3, we obtain

$$H_y^{(3)} = k \sum_{r=0}^{\infty} \left( c_r^+ e^{-jk_{xr}(z-h)} + c_r^- e^{jk_{xr}(z-h)} \right) \cos \left[ k_{xr}(x - d + L/2) \right] \quad r = 0, 1, 2...$$

(4.9)

$$E_x^{(3)} = \sum_{r=0}^{\infty} k_{xr} \left( c_r^+ e^{-jk_{xr}(z-h)} - c_r^- e^{jk_{xr}(z-h)} \right) \cos \left[ k_{xr}(x - d + L/2) \right] \quad r = 0, 1, 2...$$

(4.10)

where $h = h_1 + h_2 + g$ and $k_{xr} = r \pi / L$ while $k_{zm} = \sqrt{k^2 - k_{xr}^2}$ is obtained from (4.4). In region 2, we will still have propagating wave-solutions in the $z$-direction (or
for \( Z(z) \). However, due to the periodicity along \( x \), the separable ansatz function, \( X(x) \), have to fulfill the condition \( X(x) = X(x + p) \) along with the Helmholtz equation. The solution(s) are Floquet modes \([7]\)

\[
H_y^{(2)} = k \sum_{n=-\infty}^{\infty} (b_n^+ e^{-jk_{zn}z} + b_n^- e^{jk_{zn}z}) e^{-jk_{zn}x} \quad n \in \mathbb{Z} \tag{4.11}
\]

\[
E_x^{(2)} = \sum_{n=-\infty}^{\infty} k_{zn} (b_n^+ e^{-jk_{zn}z} - b_n^- e^{jk_{zn}z}) e^{-jk_{zn}x} \quad n \in \mathbb{Z} \tag{4.12}
\]

where \( k_{xn} = k_{x0} + 2n\pi/p \) and \( k_{zn} = \sqrt{k^2 - k_{xn}^2} \) is obtained from (4.4). Note here that the \( k_{x0} \) is an unknown eigenvalue that will be decided through the dispersion diagram sought. The \( E_x \)-components in (4.8), (4.10) and (4.12) are obtained from eq. (4.5). The \( E_z \)-components can, if desired, also be obtained from (4.5). However, since we only going to impose boundary conditions on the transverse (to the matching surface) field components, that component is of no interest here.

### 4.2.2 The Matching

Before we perform the matching of the transversal field components between the regions, we can reduce the number of unknown (set) of constants from six to four by considering that

\[
E_x^{(1)}(x, z = 0) = 0 \tag{4.13}
\]

and

\[
E_x^{(3)}(x, z = h) = 0. \tag{4.14}
\]

Conditions (4.13) and (4.14) yields \( a_m^+ = a_m^- = a_m \) and \( c_r^+ = c_r^- = c_r \). Hence, we obtain for region 1,

\[
H_y^{(1)} = 2k \sum_{m=0}^{\infty} a_m \cos (k_{zm} z) \cos [k_{xm}(x + L/2)] \quad m = 0, 1, 2... \tag{4.15}
\]

\[
E_x^{(1)} = -2j \sum_{m=0}^{\infty} a_m k_{zm} \sin (k_{zm} z) \cos [k_{xm}(x + L/2)] \quad m = 0, 1, 2... \tag{4.16}
\]

for region 2,

\[
H_y^{(2)} = k \sum_{n=-\infty}^{\infty} (b_n^+ e^{-jk_{zn}z} + b_n^- e^{jk_{zn}z}) e^{-jk_{zn}x} \quad n \in \mathbb{Z} \tag{4.17}
\]

\[
E_x^{(2)} = \sum_{n=-\infty}^{\infty} k_{zn} (b_n^+ e^{-jk_{zn}z} - b_n^- e^{jk_{zn}z}) e^{-jk_{zn}x} \quad n \in \mathbb{Z} \tag{4.18}
\]
4.2. The 1D Glide-Symmetric Corrugated Metasurface

and for region 3

\[ H_y^{(3)} = 2k \sum_{r=0}^{\infty} c_r \cos [k_{zr}(z - h)] \cos [k_{xr}(x - d + L/2)] \quad r = 0, 1, 2... \] (4.19)

\[ E_x^{(3)} = -2j \sum_{r=0}^{\infty} c_r k_{zr} \sin [k_{zr}(z - h)] \cos [k_{xr}(x - d + L/2)] \quad r = 0, 1, 2... \] (4.20)

The matching of the three regions will occur at the two dashed horizontal lines in Figure 4.1 at \( z = h_1 \) and \( z = h_1 + g \). The \( E_x \)-field components will be matched over the whole unit cell (period \( p \)) while the \( H_y \)-field components are just matched over the ”hole”. We obtain the following:

\[ E_x^{(2)}(x, z = h_1) = \begin{cases} 0 : & -p/2 < x < -L/2 \\ E_x^{(1)}(x, z = h_1) : & -L/2 < x < L/2 \\ 0 : & L/2 < x < p/2 \end{cases} \] (4.21)

\[ H_y^{(2)}(x, z = h_1) = H_y^{(1)}(x, z = h_1) : -L/2 < x < L/2 \] (4.22)

\[ E_x^{(2)}(x, z = h_1 + g) = \begin{cases} 0 : & -p/2 + d < x < -L/2 + d \\ E_x^{(3)}(x, z = h_1 + g) : & -L/2 + d < x < L/2 + d \\ 0 : & L/2 + d < x < p/2 + d \end{cases} \] (4.23)

\[ H_y^{(2)}(x, z = h_1 + g) = H_y^{(3)}(x, z = h_1 + g) : -L/2 + d < x < L/2 + d \] (4.24)

Multiplying (4.21) and (4.23) with \( e^{jk_{xn}x} \) and integrating from \(-p/2\) to \( p/2\), using the expressions in (4.16), (4.18) and (4.20), yields the two equations

\[ k_{zn} (b_n^+ e^{-jk_{zn}h_1} - b_n^- e^{jk_{zn}h_1})p = -2j \sum_{m=0}^{\infty} a_m k_{zm} \sin (k_{zm}h_1) C_{nm} \] (4.25)

\[ k_{zn} (b_n^+ e^{-jk_{zn}(h_1+g)} - b_n^- e^{jk_{zn}(h_1+g)})p = 2j \sum_{r=0}^{\infty} c_r k_{zr} \sin (k_{zr}h_2) D_{nr} \] (4.26)

where

\[ C_{nm} = \int_{-L/2}^{L/2} \cos [k_{xm}(x + L/2)] e^{jk_{xn}x} dx = \frac{j k_{zn} [(-1)^m e^{jk_{zn}L/2} - e^{-jk_{zn}L/2}]}{k_{zn}^2 - k_{xm}^2} \] (4.27)

and

\[ D_{nr} = \int_{-L/2+d}^{L/2+d} \cos [k_{xr}(x - d + L/2)] e^{jk_{xn}x} dx = e^{jk_{zn}d} C_{nr}. \] (4.28)
Multiplying (4.22) with \( \cos \left[ k_{xm}(x + L/2) \right] \) and (4.24) with \( \cos \left[ k_{xm}(x - d + L/2) \right] \) while integrating (4.22) from \(-L/2\) to \(L/2\) and (4.24) from \(-L/2 + d\) to \(L/2 + d\), using the expressions in (4.15), (4.17) and (4.19), yields

\[
\sum_{n=-\infty}^{\infty} \left( b_n^+ e^{-jk_{xn}h_1} + b_n^- e^{jk_{xn}h_1} \right) C_{mn}^* = a_m (1 + \delta_{0m}) L \cos (k_{zm}h_1) \quad (4.29)
\]

\[
\sum_{n=-\infty}^{\infty} \left( b_n^+ e^{-jk_{xn}(h_1+g)} + b_n^- e^{jk_{xn}(h_1+g)} \right) D_{rn}^* = c_r (1 + \delta_{0r}) L \cos (k_{zr}h_2) \quad (4.30)
\]

where \( C_{mn}^* \) and \( D_{rn}^* \) are the complex conjugations of (4.27) and (4.28).

We have here utilized the orthogonality relations

\[
\int_0^p e^{-j(k_{xn} - k_{xn'})x} dx = p\delta_{nn'}
\]

and

\[
\int_{-L/2}^{L/2} \cos (k_{xm}x) \cos (k_{xm'}x) dx = \frac{\delta_{mm'}L}{2 - \delta_{0m}}
\]

where \( \delta_{mn} \) is the kronecker delta.

We now have obtained four equations for the four unknown mode constants \( a_m, b_n^+, b_n^- \) and \( c_r \). Equations (4.25), (4.26), (4.29) and (4.30) can be written in matrix form but, before doing so, we must truncate the number of modes as \( \max\{m\} = M \), \( \max\{r\} = R \) and \( \max\{|n|\} = N \). We obtain the following matrix equation system

\[
\begin{bmatrix}
A_1 & B_1^+ & B_1^- & 0 \\
0 & B_2^+ & B_2^- & C_2 \\
A_3 & B_3^+ & B_3^- & 0 \\
0 & B_4^+ & B_4^- & C_4 \\
\end{bmatrix} \begin{bmatrix}
a_m \\
b_n^+ \\
b_n^- \\
c_r \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \quad (4.31)
\]

where

\[
a_m = [a_0, a_1, \ldots, a_M]^T,
\]

\[
b_n^+ = [b_n^+, \ldots, b_N^+]^T
\]

and

\[
c_r = [c_0, c_1, \ldots, c_R]^T
\]

are the set of mode constants while the elements of the matrix in (4.31) are in turn (smaller) matrices given by

\[
(A_1)_{nm} = 2jk_{zm} \sin (k_{zm}h_1) C_{nm}
\]

\[
B_1^\pm = \pm \text{diag} \left\{ pk_{zn} e^{\mp jk_{xn}h_1} \right\}
\]

\[
B_2^\pm = \pm \text{diag} \left\{ pk_{zn} e^{\mp jk_{xn}(h_1+g)} \right\}
\]
4.2. The 1D Glide-Symmetric Corrugated Metasurface

\[
\begin{align*}
(C_2)_{nr} & = -2j k_{zr} \sin (k_{zm} h_2) D_{nr} \\
A_3 & = \text{diag} \{(1 + \delta_{0m}) L \cos (k_{zm} h_1)\} \\
(B_{3}^{\pm})_{mn} & = e^{\mp j k_{zn} h_1} C_{mn}^* \\
(B_{4}^{\pm})_{rn} & = e^{\mp j k_{zn} (h_1 + g)} D_{rn}^*
\end{align*}
\]

and

\[
C_4 = \text{diag} \{(1 + \delta_{0r}) L \cos (k_{zr} h_2)\}.
\]

The dispersion diagram(s) are then obtained by searching for the values of frequency, \( f \), and unknown eigenvalue, \( k_{x0} \), when setting the determinant of the matrix in (4.31) to zero

\[
D(f, k_{x0}) = \text{det} \left( \begin{array}{cccc}
A_1 & B_{1}^{+} & B_{1}^{-} & 0 \\
0 & B_{2}^{+} & B_{2}^{-} & C_2 \\
A_3 & B_{3}^{+} & B_{3}^{-} & 0 \\
0 & B_{4}^{+} & B_{4}^{-} & C_4
\end{array} \right) = 0.
\]

(4.32)

4.2.3 Results

The dispersion diagrams for the structure in Figure 4.1 can be seen for different parameter values in Figure 4.2, 4.3 and 4.4. For all cases, the number of modes has been set to \( N = 3 \), \( M = 4 \) and \( R = 4 \).

![Dispersion diagram](image)

**Figure 4.2.** Dispersion diagram obtained by both the mode matching method and *CST Microwave Studio* of the structure in Figure 4.1 with parameter values of \( p = 4 \text{ mm}, L = 3 \text{ mm}, d = 2 \text{ mm}, h_1 = 1.5 \text{ mm}, h_2 = 1.5 \text{ mm} \) and \( g = 0.15 \text{ mm} \).
Figure 4.3. Dispersion diagram obtained by both the mode matching method and CST Microwave Studio of the structure in Figure 4.1 with parameter values of $p = 4$ mm, $L = 3$ mm, $d = 0$ mm, $h_1 = 1.5$ mm, $h_2 = 1.5$ mm and $g = 0.15$ mm.

Figure 4.4. Dispersion diagram obtained by both the mode matching method and CST Microwave Studio of the structure in Figure 4.1 with parameter values of $p = 4$ mm, $L = 3$ mm, $d = 0$ mm, $h_1 = 1.5$ mm, $h_2 = 0$ mm and $g = 0.15$ mm.
Chapter 5

Mode Matching in Cylindrical Coordinates

5.1 Coaxial Cable with Periodic Holes

In this section we will use the mode matching method to find the dispersion diagram of a coaxial structure with the unit cell illustrated in Figure 5.1. Figure 5.2 shows a side view of the unit cell and the two separate regions that will be considered. In this case, the matching will be over an area connecting the gap between the conductors and the ”hole” (excavation) in the inner conductor.

![Figure 5.1. Two unit cells of the periodic structure considered. The chosen unit cell is indicated with the letter p.](image-url)
5.1.1 The General Field Expressions

Because of the cylindrical geometry, the fields will be expressed in cylindrical coordinates ($\rho$, $\varphi$, $z$). The unit cell is divided into two regions. Region 1 is the "Floquet" region, carrying propagating waves down the structure along the $z$-direction (see Figure 5.3). Region 2 is the "hole" on the inner conductor which has similar geometry to region 1 but, with PEC (perfect electric conductor) surfaces at 0 and $d$, forcing a standing wave in the $z$-direction and "reversed" (to region 1) boundary conditions in the radial direction (compare Figure 5.3 and 5.4). We decompose the fields along the fixed $z$-direction and only match the transversal (to the dashed boundary in Figure 5.2) field-components. The total fields will consist of both TE- and TM-modes. No TEM-modes will be present because of the periodicity along the $z$-direction. This stems from section 3.4 where we have that TEM-modes must fulfill $k = k_z$ and hence $k_t = 0$. This is not valid in this case since both regions have propagating waves along the radial (transversal) direction.
5.1. Coaxial Cable with Periodic Holes

Figure 5.3. The geometry from which the region 1-fields will be calculated. The dashed circle is the boundary to region 2 and hence wave-solutions are free to propagate inwards.

Figure 5.4. The geometry from which the region 2-fields will be calculated. The dashed circle is the boundary to region 1 and hence wave-solutions are free to propagate outward.

Region 1

In region 1, we will have Floquet modes, hence the $z$-components will fulfill the periodic condition(s):

$$E_z(\rho, \varphi, z + p) = E_z(\rho, \varphi, z), \quad H_z(\rho, \varphi, z + p) = H_z(\rho, \varphi, z)$$  \hspace{1cm} (5.1)
Chapter 5. Mode Matching in Cylindrical Coordinates

The Floquet condition in (5.1), along with the Helmholtz equation gives us a $z$-dependence as

$$E_z(\rho, \varphi, z) = E_z(\rho, \varphi)e^{\pm j k_{zn} z} \quad (5.2)$$

and

$$H_z(\rho, \varphi, z) = H_z(\rho, \varphi)e^{\pm j k_{zn} z} \quad (5.3)$$

where $k_{zn} = k_{z0} + 2\pi n/p$. The transversal part will also fulfill the Helmholtz equation. For TM-modes we have

$$\begin{cases} \left( \nabla_t^2 + k_t^2 \right) E_z(\rho, \varphi) = 0 \\ E_z(\rho = b, \varphi) = 0 \\ E_z(\rho, \varphi + 2\pi) = E_z(\rho, \varphi) \end{cases} \quad (5.4)$$

while for TE-modes we have

$$\begin{cases} \left( \nabla_t^2 + k_t^2 \right) H_z(\rho, \varphi) = 0 \\ \partial \rho H_z(\rho = b, \varphi) = 0 \\ H_z(\rho, \varphi + 2\pi) = H_z(\rho, \varphi) \end{cases} \quad (5.5)$$

where

$$\nabla_t^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2}$$

is the transversal Laplace operator in cylindrical coordinates. The general solution to (5.4) and (5.5) has already been obtained in section 3.3 and we get

$$E_{z,mn} = [A_{mn}^+ J_m(k_t \rho) + A_{mn}^- Y_m(k_t \rho)] \begin{cases} \sin (m \varphi) \\ \cos (m \varphi) \end{cases} \quad (5.6)$$

and

$$H_{z,mn} = [B_{mn}^+ J_m(k_t \rho) + B_{mn}^- Y_m(k_t \rho)] \begin{cases} \sin (m \varphi) \\ \cos (m \varphi) \end{cases} \quad (5.7)$$

where $m = 0, 1, 2...$ and $n \in \mathbb{Z}$. The boundary condition of $E_z(\rho = b, \varphi) = 0$ in (5.4) and $\partial \rho H_z(\rho = b, \varphi) = 0$ in (5.5) then yields

$$A_{mn}^- = -A_{mn} J_m(k_t b) , \quad A_{mn}^+ = A_{mn} Y_m(k_t b)$$

and

$$B_{mn}^- = -B_{mn} J'_m(k_t b) , \quad B_{mn}^+ = B_{mn} Y'_m(k_t b)$$

where the prime represents $\Omega'_m \equiv \frac{d \Omega_m}{d (k_t \rho)}$ and $\Omega$ stands for any Bessel function and the transversal wave-number is given by $k_t^2 \equiv k_{tn}^2 = k^2 - k_{zn}^2$. Introducing the following short notations

$$R_{Em}(k_{tn} \rho; b) \equiv J_m(k_{tn} \rho) Y_m(k_{tn} b) - Y_m(k_{tn} \rho) J_m(k_{tn} b) \quad (5.8)$$
5.1. Coaxial Cable with Periodic Holes

\[ R_{Hm}(k_{tn}\rho; b) \equiv J_m(k_{tn}\rho) Y'_m(k_{tn}b) - Y_m(k_{tn}\rho) J'_m(k_{tn}b) \]  \hspace{1cm} (5.9)

we can write (5.6) and (5.7) as

\[ E_{z,mn}(\rho, \varphi) = A_{mn} R_{E_m}(k_{tn}\rho; b) \left\{ \sin (m\varphi) \begin{array}{c} \cos (m\varphi) \end{array} \right\} \]  \hspace{1cm} (5.10)

and

\[ H_{z,mn}(\rho, \varphi) = B_{mn} R_{H_m}(k_{tn}\rho; b) \left\{ \sin (m\varphi) \begin{array}{c} \cos (m\varphi) \end{array} \right\} \]  \hspace{1cm} (5.11)

where \( A_{mn} \) and \( B_{mn} \) are arbitrary complex mode coefficients. The transversal components are obtained from equations (3.5) and (3.6)

\[ E_{TM}^{\rho}(\rho) = -\frac{j k_z}{k_t^2} \nabla_t E_z \]  \hspace{1cm} (5.12)

\[ H_{TM}^{\rho}(\rho) = -\frac{j k}{k_t^2} \hat{z} \times \nabla_t E_z \]  \hspace{1cm} (5.13)

\[ E_{TE}^{\rho}(\rho) = \frac{j k}{k_t^2} \hat{z} \times \nabla_t H_z \]  \hspace{1cm} (5.14)

\[ H_{TE}^{\rho}(\rho) = -\frac{j k_z}{k_t^2} \nabla_t H_z \]  \hspace{1cm} (5.15)

using (5.16) and (5.19) in (5.12)-(5.15) yields:

\[ E_{z,mn} = A_{mn} R_{E_m}(k_{tn}\rho; b) \left\{ \sin (m\varphi) \begin{array}{c} \cos (m\varphi) \end{array} \right\} e^{-jk_{zn}z} \]  \hspace{1cm} (5.16)

\[ E_{\varphi,mn} = -A_{mn} \frac{jm k_{zn}}{k_{tn}\rho} R_{E_m}(k_{tn}\rho; b) \left\{ \cos (m\varphi) \begin{array}{c} -\sin (m\varphi) \end{array} \right\} e^{-jk_{zn}z} \]  \hspace{1cm} (5.17)

\[ H_{\varphi,mn} = -A_{mn} \frac{j k}{k_{tn}} R'_{E_m}(k_{tn}\rho; b) \left\{ \sin (m\varphi) \begin{array}{c} \cos (m\varphi) \end{array} \right\} e^{-jk_{zn}z} \]  \hspace{1cm} (5.18)

TM-Field components

TE-Field components
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\[ H_{z,mn} = B_{mn} R_{Hm}(k_{tn} \rho; b) \left\{ \begin{array}{c} \sin (m\varphi) \\ \cos (m\varphi) \end{array} \right\} e^{-jk_{zn}z} \]  
(5.19)

\[ E_{\varphi,mn}^{TE} = B_{mn} \frac{jk}{k_{tn}} R'_{Hm}(k_{tn} \rho; b) \left\{ \begin{array}{c} \sin (m\varphi) \\ \cos (m\varphi) \end{array} \right\} e^{-jk_{zn}z} \]  
(5.20)

\[ H_{\varphi,mn}^{TE} = -B_{mn} \frac{jmk_{zn}}{k_{tn}^2} R_{Hm}(k_{tn} \rho; b) \left\{ \begin{array}{c} \cos (m\varphi) \\ -\sin (m\varphi) \end{array} \right\} e^{-jk_{zn}z} \]  
(5.21)

The radial field components are here left out because they will not be matched but can, of course be obtained from (5.12)-(5.15). The reason that we keep the cosine and sine solution together (with the curly bracket) instead of writing e.g. \( A_{mn} \cos (m\varphi) + B_{mn} \sin (m\varphi) \) will be evident later on when we do the matching.

Region 2

The field expressions in region 2 are obtained through similar calculations as the ones in Region 1. From Figure 5.4 we see that one difference is that the conducting surface is now on the inner cylinder instead of the outer. Also, seen from Figure 5.2, we must have a standing wave along the \( z \)-direction because of the conducting surfaces at \( z = 0 \) and \( z = d \). From [22], we know that for a waveguide mode the complete fields can be written

\[ E_{i}(r) = E_{ti}(\rho) \left( C_{i}^{+} e^{-jk_{zn}z} + C_{i}^{-} e^{jk_{zn}z} \right) + \hat{z} E_{zi}(\rho) \left( C_{i}^{+} e^{-jk_{zn}z} - C_{i}^{-} e^{jk_{zn}z} \right) \]

\[ H_{i}(r) = H_{ti}(\rho) \left( C_{i}^{+} e^{-jk_{zn}z} - C_{i}^{-} e^{jk_{zn}z} \right) + \hat{z} H_{zi}(\rho) \left( C_{i}^{+} e^{-jk_{zn}z} + C_{i}^{-} e^{jk_{zn}z} \right) \]

where \( C_{i}^{\pm} \) are mode coefficients. Since the total tangential electric field has to vanish at \( z = 0 \) we must have \( C_{i}^{+} = -C_{i}^{-} \) and we obtain the \( z \)-dependence’s

\[ jE_{i}(r) = E_{ti}(\rho) \sin (k_{z}z) + \hat{z} E_{zi}(\rho) \cos (k_{z}z) \]  
(5.22)

\[ H_{i}(r) = H_{ti}(\rho) \cos (k_{z}z) + \hat{z} j H_{zi}(\rho) \sin (k_{z}z) \]  
(5.23)

where \( E_{t} = 0 \) at \( z = d \) gives

\[ k_{z} = k_{zr} = \frac{r\pi}{d} \quad r = 0, 1, 2... \]  
(5.24)

The \( \rho \) and \( \varphi \) dependence for \( E_{z} \) and \( H_{z} \) are obtained with analogue calculations as for region 1, i.e. equations (5.6) and (5.7) but, with the boundary condition

\[ E_{z}(\rho = s, \varphi) = 0 \]

for TM-modes and

\[ \frac{\partial H_{z}}{\partial \rho} (\rho = s, \varphi) = 0 \]

for TE-modes. This yields the field expressions for Region 2:
5.1. Coaxial Cable with Periodic Holes

**TM-Field components**

\[ E_{z,lr} = \alpha_{lr} R_{El}(k_{tr}\rho; s) \left\{ \sin (l\varphi) \over \cos (l\varphi) \right\} \cos (k_{zr}z) \] (5.25)
\[ E_{\varphi,lr}^{TM} = -\alpha_{lr} j \frac{k_{zr}}{k_{tr}^{2}} R_{El}(k_{tr}\rho; s) \left\{ \frac{\cos (l\varphi)}{-\sin (l\varphi)} \right\} \sin (k_{zr}z) \] (5.26)
\[ H_{\varphi,lr}^{TM} = -\alpha_{lr} \frac{j k}{k_{tr}} R'_{El}(k_{tr}\rho; s) \left\{ \frac{\sin (l\varphi)}{\cos (l\varphi)} \right\} \cos (k_{zr}z) \] (5.27)

**TE-Field components**

\[ H_{z,lr} = \beta_{lr} R_{Hi}(k_{tr}\rho; s) \left\{ \sin (l\varphi) \over \cos (l\varphi) \right\} \sin (k_{zr}z) \] (5.28)
\[ E_{\varphi,lr}^{TE} = \beta_{lr} \frac{j k}{k_{tr}} R'_{Hi}(k_{tr}\rho; s) \left\{ \frac{\sin (l\varphi)}{\cos (l\varphi)} \right\} \sin (k_{zr}z) \] (5.29)
\[ H_{\varphi,lr}^{TE} = -\beta_{lr} \frac{j k_{zr}}{k_{tr}^{2}} R_{Hi}(k_{tr}\rho; s) \left\{ \frac{\cos (l\varphi)}{-\sin (l\varphi)} \right\} \cos (k_{zr}z) \] (5.30)

where \( l, r = 0, 1, 2, \ldots \), \( k_{tr}^{2} = k^{2} - k_{zr}^{2} \) and \( \alpha_{lr}, \beta_{lr} \) are (complex) arbitrary constants.

**5.1.2 The Matching**

The total field (in each region) is given by the sum of all TM- and TE-modes for each component. For region 1 we have:

\[ E_{z}^{(1)} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} R_{Em}(k_{tn}\rho; b) \left\{ \sin (m\varphi) \over \cos (m\varphi) \right\} e^{-jk_{zn}z} \] (5.31)
\[ H_{z}^{(1)} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} B_{mn} R_{Hm}(k_{tn}\rho; b) \left\{ \sin (m\varphi) \over \cos (m\varphi) \right\} e^{-jk_{zn}z} \] (5.32)
\[ E_{\varphi}^{(1)} = -\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} \frac{jm k_{zn}}{k_{tn}^{2}} R_{Em}(k_{tn}\rho; b) \left\{ \cos (m\varphi) \over -\sin (m\varphi) \right\} e^{-jk_{zn}z} \]
\[ + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} B_{mn} \frac{j k}{k_{tn}} R'_{Hm}(k_{tn}\rho; b) \left\{ \sin (m\varphi) \over \cos (m\varphi) \right\} e^{-jk_{zn}z} \] (5.33)
\[ H^{(1)}_\phi = -\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} \frac{j k}{k_{tn}} R'_{Em}(k_{tn}\rho; b) \begin{cases} \sin (m\phi) \\ \cos (m\phi) \end{cases} e^{-jk_{zn}z} \]

\[- \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} B_{mn} \frac{j m k_{zn}}{k_{tn}^2} R_{Hm}(k_{tn}\rho; b) \begin{cases} \cos (m\phi) \\ -\sin (m\phi) \end{cases} e^{-jk_{zn}z} \quad (5.34)\]

While for region 2:

\[ E_z^{(2)} = \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \alpha_{lr} R_{Ez}(k_{tr}\rho; s) \begin{cases} \sin (l\phi) \\ \cos (l\phi) \end{cases} \cos (k_{zr}z) \quad (5.35) \]

\[ H_z^{(2)} = \sum_{l=0}^{\infty} \sum_{r=1}^{\infty} \beta_{lr} R_{Hz}(k_{tr}\rho; s) \begin{cases} \sin (l\phi) \\ \cos (l\phi) \end{cases} \sin (k_{zr}z) \quad (5.36) \]

\[ E_\phi^{(2)} = -\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \alpha_{lr} \frac{j l k_{zr}}{k_{tr}^2} R_{E\phi}(k_{tr}\rho; s) \begin{cases} \cos (l\phi) \\ -\sin (l\phi) \end{cases} \sin (k_{zr}z) \]

\[ + \sum_{l=0}^{\infty} \sum_{r=1}^{\infty} \beta_{lr} \frac{j k}{k_{tr}} R'_{E\phi}(k_{tr}\rho; s) \begin{cases} \sin (l\phi) \\ \cos (l\phi) \end{cases} \sin (k_{zr}z) \quad (5.37) \]

\[ H_\phi^{(2)} = -\sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \alpha_{lr} \frac{j k}{k_{tr}} R'_{H\phi}(k_{tr}\rho; s) \begin{cases} \sin (l\phi) \\ \cos (l\phi) \end{cases} \cos (k_{zr}z) \]

\[ - \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \beta_{lr} \frac{j k_{zr}}{k_{tr}^2} R_{H\phi}(k_{tr}\rho; s) \begin{cases} \cos (l\phi) \\ -\sin (l\phi) \end{cases} \cos (k_{zr}z) \quad (5.38) \]

We impose the boundary conditions at the surface \( \rho = a \) and over \( 0 \leq \phi < 2\pi \) as follows:

\[ E_z^{(1)} = \begin{cases} E_z^{(2)} & 0 \leq z < d \\ 0 & d \leq z \leq p \end{cases} \quad (5.39) \]

\[ H_z^{(1)} = H_z^{(2)} \quad 0 \leq z \leq d \quad (5.40) \]

\[ E_\phi^{(1)} = \begin{cases} E_\phi^{(2)} & 0 \leq z < d \\ 0 & d \leq z \leq p \end{cases} \quad (5.41) \]

\[ H_\phi^{(1)} = H_\phi^{(2)} \quad 0 \leq z \leq d \quad (5.42) \]

The \( \mathbf{E} \)-field components are matched over the whole unit cell, \( 0 \leq z \leq p \) and \( 0 \leq \phi < 2\pi \), while the \( \mathbf{H} \)-field components are only matched over the ”hole”, \( 0 \leq z \leq d \) and \( 0 \leq \phi < 2\pi \). Note that (5.39)-(5.42) gives four equations but, generally we have eight unknown set of mode coefficients in the field expressions.
(5.31)-(5.38), one proportional to \( \sin (m\varphi) \) and one to \( \cos (m\varphi) \). However, making use of the orthogonality relations

\[
\int_0^{2\pi} \cos (mx) \cos (nx) dx = \begin{cases} \pi \delta_{mn} & m, n = 1, 2, 3... \\ 2\pi & (m, n) = (0, 0) \end{cases}
\] (5.43)

\[
\int_0^{2\pi} \sin (mx) \sin (nx) dx = \pi \delta_{mn} & m, n = 1, 2, 3... 
\] (5.44)

and also

\[
\int_0^{2\pi} \cos (mx) \sin (nx) dx = 0 & m, n = 0, 1, 2...
\] (5.45)

when testing the boundary conditions (see below), it turns out that the sine-coefficients (constants obtained when testing with \( \sin (m\varphi) \)) can be "absorbed" into the cosine-coefficients (constants obtained when testing with \( \cos (m\varphi) \)) and hence reducing the number of constants to four.

Testing boundary conditions (5.39)-(5.42) in the following way

\[
\frac{1}{ap} \int_0^p \int_0^{2\pi} (5.39) \cdot \cos (m\varphi) e^{jkz sigma z} d\varphi dz (5.46)
\]

\[
\frac{1}{ad} \int_0^d \int_0^{2\pi} (5.40) \cdot \cos (m\varphi) e^{jkz sigma z} d\varphi dz (5.47)
\]

\[
\frac{1}{ap} \int_0^p \int_0^{2\pi} (5.41) \cdot \cos (l\varphi) \sin (kz r z) d\varphi dz (5.48)
\]

\[
\frac{1}{ad} \int_0^d \int_0^{2\pi} (5.42) \cdot \cos (l\varphi) \cos (kz r z) d\varphi dz (5.49)
\]

yields:

(5.46):

\[
A_{mn}(1 + \delta_{0m})R_{Em}(ktna; b) = \frac{1}{p} \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \alpha_{lr} \frac{2\delta_{lm}}{2 - \delta_{0l}\delta_{0m}} R_{El}(ktr a; s) F_{nr}^c (5.50)
\]

(5.47):

\[
B_{mn}(1 + \delta_{0m})R_{Hm}(ktna; b) = \frac{1}{d} \sum_{l=0}^{\infty} \sum_{r=1}^{\infty} \beta_{lr} \frac{2\delta_{lm}}{2 - \delta_{0l}\delta_{0m}} R_{Hz}(ktr a; s) F_{nr}^s (5.51)
\]

(5.48):
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\[
- \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} \frac{mk_{zn}}{k_{tn}^{2}a_{p}} \delta_{lm} R_{Em}(k_{tn}a; b) F_{nr}^{ss} + \\
+ \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} B_{mn} \frac{k}{k_{tn}^{2}p}(2 - \delta_{0m}\delta_{0l}) R_{Hm}(k_{tn}a; b) F_{nr}^{ss} = \\
= -\alpha_{tr} \frac{l k_{zr}d}{2k_{tn}^{2}a_{p}} R_{El}(k_{tr}a; s) + \beta_{tr} \frac{kd}{(2 - \delta_{0l})k_{tn}p} R_{Hl}^{'}(k_{tr}a; s) \tag{5.52}
\]

(5.49):

\[
- \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} \frac{k}{k_{tn}d}(2 - \delta_{0m}\delta_{0l}) R_{Em}(k_{tn}a; b) F_{nr}^{cc} - \\
- \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} B_{mn} \frac{mk_{zn}}{k_{tn}^{2}d} \delta_{lm} R_{Hm}(k_{tn}a; b) F_{nr}^{cc} = \\
= -\alpha_{tr} \frac{k}{k_{tn}^{2}} \frac{1 + \delta_{0l}}{(2 - \delta_{0r})} R_{El}^{'}(k_{tr}a; s) - \beta_{tr} \frac{l k_{zr}}{2k_{tn}^{2}a} \frac{1}{2 - \delta_{0r}} R_{Hl}(k_{tr}a; s) \tag{5.53}
\]

where

\[
\left\{ F_{nr}^{ss}, F_{nr}^{cc} \right\} = \int_{0}^{d} \left\{ \sin(k_{zr}z), \cos(k_{zr}z) \right\} e^{jk_{zr}z} dz = \frac{1}{k_{zr}^{2} - k_{zn}^{2}} \left\{ \frac{k_{zr}}{k_{zr}^{2} - k_{zn}^{2}} \right\} \tag{5.54}
\]

and $F_{nr}^{ss}, F_{nr}^{cc}$ are the complex conjugates of (5.54). In order to write (and solve) (5.50)-(5.53) on matrix form we truncate the number of modes as max \{m\} = M, max \{|n|\} = N and max \{|r\} = R. We also reduce the number of indices down from four to two. Grouping the indices as

\[
i \in (m, n), \quad \lambda \in (l, r)
\]

we can define

\[
A_{i} = [A_{-N0}, ..., A_{00}, ..., A_{N0}, A_{-N1}, ..., A_{NM}]^{T}
\]

\[
B_{i} = [B_{-N0}, ..., B_{00}, ..., B_{N0}, B_{-N1}, ..., B_{NM}]^{T}
\]

\[
\alpha_{\lambda} = [\alpha_{00}, ..., \alpha_{0R}, \alpha_{10}, ..., \alpha_{1R}, ..., \alpha_{LR}]^{T}
\]

\[
\beta_{\lambda} = [\beta_{00}, ..., \beta_{0R}, \beta_{10}, ..., \beta_{1R}, ..., \beta_{LR}]^{T}
\]

and we get max \{|i\} = (M + 1)(2N + 1) and max \{|\lambda\} = (L + 1)(R + 1). With these definitions we can write the equation system in (5.50)-(5.53) on matrix form as

\[
\begin{pmatrix}
\Gamma_{\lambda_{1}} & 0 & \Gamma_{\alpha_{1}} & 0 \\
0 & \Gamma_{\beta_{2}} & 0 & \Gamma_{\beta_{2}}
\end{pmatrix}
\begin{pmatrix}
A_{i} \\
B_{i}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix} \tag{5.55}
\]
where

\[ \Gamma_{A_1} = \text{diag} \left\{ (1 + \delta_{0m}) R_{E_m}(k_{tn} a; b) \right\}_i \]

\[ (\Gamma_{\alpha_1})_{i\lambda} = -\frac{2\delta_{lm}}{(2 - \delta_{0l} \delta_{0m}) p} R_{E_l}(k_{tr} a; s) F_{nc}^c \]

\[ \Gamma_{B_2} = \text{diag} \left\{ (1 + \delta_{0m}) R_{H_m}(k_{tn} a; b) \right\}_i \]

\[ (\Gamma_{\beta_2})_{i\lambda} = -\frac{2\delta_{lm}}{(2 - \delta_{0l} \delta_{0m}) d} R_{H_l}(k_{tr} a; s) F_{nr}^s \]

\[ (\Gamma_{A_3})_{i\lambda} = -\frac{mk_{zn} k_{zn}^2}{k_{tn} d} \delta_{lm} R_{E_m}(k_{tn} a; b) F_{nr}^{s*} \]

\[ (\Gamma_{B_3})_{i\lambda} = \frac{k}{k_{tn} p} \frac{2\delta_{lm}}{(2 - \delta_{0l} \delta_{0m})} R_{H_m}'(k_{tn} a; b) F_{nr}^{s*} \]

\[ \Gamma_{\alpha_3} = \text{diag} \left\{ \frac{lk_{tr} d}{2k_{tn}^2 a p} R_{E_l}(k_{tr} a; s) \right\}_\lambda \]

\[ \Gamma_{\beta_3} = \text{diag} \left\{ -\frac{k d}{(2 - \delta_{0l}) k_{tr} p} R_{H_l}'(k_{tr} a; s) \right\}_\lambda \]

\[ (\Gamma_{A_4})_{i\lambda} = -\frac{k}{k_{tn} d} \frac{2\delta_{lm}}{(2 - \delta_{0n} \delta_{0l})} R_{E_m}'(k_{tn} a; b) F_{nr}^{c*} \]

\[ (\Gamma_{B_4})_{i\lambda} = -\frac{mk_{zn} k_{zn}^2}{k_{tn} d} \delta_{lm} R_{H_m}(k_{tn} a; b) F_{nr}^{c*} \]

\[ \Gamma_{\alpha_4} = \text{diag} \left\{ \frac{k}{k_{tr}} \frac{1 + \delta_{0l}}{(2 - \delta_{0r})} R_{E_l}'(k_{tr} a; s) \right\}_\lambda \]

\[ \Gamma_{\beta_4} = \text{diag} \left\{ \frac{lk_{tr} d}{2k_{tn}^2 a} \frac{1}{2 - \delta_{0r}} R_{H_l}(k_{tr} a; s) \right\}_\lambda \]

However, there is one more thing worth noting before solving the equation system (numerically) in (5.55). Since the structure exhibits rotational symmetry we will set \( M = L = 0 \) and hence only the \( m = l = 0 \) terms will survive the above equations. This will allow us to add more of the ”pure” radial modes (when solving numerically) which are more dominant in the structure and at the same time save a lot of computation time. Setting \( M = L = 0 \) in (5.55) we obtain the following equation system
Chapter 5. Mode Matching in Cylindrical Coordinates

\[
\begin{pmatrix}
\Gamma_A & 0 & \Gamma_{\alpha_1} & 0 \\
0 & \Gamma_B & 0 & \Gamma_{\beta_2} \\
0 & \Gamma_B & 0 & \Gamma_{\beta_3} \\
\Gamma_A & 0 & \Gamma_{\alpha_4} & 0
\end{pmatrix}
\begin{pmatrix}
A_{0n} \\
B_{0n} \\
\alpha_{or} \\
\beta_{0r}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\] (5.56)

where

\[
\Gamma_A = \text{diag}\{2R_{E_0}(k_t n a; b)\}_n
\]
\[
(\Gamma_{\alpha_1})_{nr} = -\frac{2}{p} R_{E_0}(k_t r a; s) F_{nr}^c
\]
\[
\Gamma_B = \text{diag}\{2R_{H_0}(k_t n a; b)\}_n
\]
\[
(\Gamma_{\beta_2})_{nr} = -\frac{2}{d} R_{H_0}(k_t r a; s) F_{nr}^s
\]
\[
(\Gamma_{B_3})_{\lambda i} = \frac{2k}{k_t n p} R'_{H_0}(k_t n a; b) F_{nr}^{s*}
\]
\[
\Gamma_{\beta_3} = \text{diag}\left\{-\frac{kd}{k_t r p} R'_{H_0}(k_t r a; s)\right\}_r
\]
\[
(\Gamma_{A_4})_{\lambda i} = -\frac{2k}{k_t n d} R'_{E_0}(k_t n a; b) F_{nr}^{c*}
\]
\[
\Gamma_{\alpha_4} = \text{diag}\left\{\frac{k}{k_t r (2 - \delta_{0r})} R'_{E_0}(k_t r a; s)\right\}_r
\]

The dispersion diagram(s) are then obtained by searching for the values of frequency, \(f\), and unknown eigenvalue, \(k_{\pm 0}\), when setting the determinant of the matrix in (5.56) to zero

\[
D(f, k_{\pm 0}) = \det \begin{pmatrix}
\Gamma_A & 0 & \Gamma_{\alpha_1} & 0 \\
0 & \Gamma_B & 0 & \Gamma_{\beta_2} \\
0 & \Gamma_B & 0 & \Gamma_{\beta_3} \\
\Gamma_A & 0 & \Gamma_{\alpha_4} & 0
\end{pmatrix} = 0.
\] (5.57)

5.1.3 Results

The dispersion diagrams for the periodic structure with the unit cell as in Figure 5.1 can be seen for three different parameter values in Figures 5.5, 5.6 and 5.7. For Figures 5.5 and 5.7, the number of modes was set to \(N = 2, M = 0, R = 2\) and \(L = 0\) which gives \((2N + 1)(M + 1) = 5 \times 1 = 5\) modes in region 1 and \((L + 1)(R + 1) = 1 \times 3 = 3\) modes in region 2. While, to get a desired result, the number of modes was set to \(N = 3, M = 0, R = 3\) and \(L = 0\) in Figure 5.6 which gives \((2N + 1)(M + 1) = 7 \times 1 = 7\) modes in region 1 and \((L + 1)(R + 1) = 1 \times 4 = 4\) modes in region 2.
5.1. Coaxial Cable with Periodic Holes

Figure 5.5. Dispersion diagram obtained by both the mode matching method (blue dots) and CST Microwave Studio (red line) of the periodic structure shown in Figure 5.1 with the parameter values as $p = 12 \text{ mm}$, $d = 2.4 \text{ mm}$, $s = 1 \text{ mm}$, $a = 1.5 \text{ mm}$ and $b = 2 \text{ mm}$. The number of modes was chosen to $N = R = 2$.

Figure 5.6. Dispersion diagram obtained by both the mode matching method (blue dots) and CST Microwave Studio (red line) of the periodic structure shown in Figure 5.1 with the parameter values as $p = 12 \text{ mm}$, $d = 4.8 \text{ mm}$, $s = 1 \text{ mm}$, $a = 1.5 \text{ mm}$ and $b = 2 \text{ mm}$. For this this case the number of modes was chosen to $N = R = 3$. 
5.1.4 Analyzing the Band Gap

Studying the result from the Figures 5.5, 5.6 and 5.7. We see that when changing the parameters (e.g. the "hole length" $d$ or the gap $g = b - a$) the band gap between the first and second mode changes. Increasing the hole length ($d$) a factor of two, we see that the band gap is increasing while when increase the gap ($g$) a factor of two, the band gap is instead decreasing. It make sense that the band gap is decreasing when the gap ($g$) is increased because, studying the limit where $g \gg h = a - s$, the structure tends to that of a regular coaxial cable (suppressing the effect of the hole). Considering the "hole"-length $d$, we can investigate the band gap as a function of it by simply fix the $k_{z0} = 1$ in (5.57) and instead vary $d$. We find the plots shown in Figure (5.8). It is seen from these plots that a maximum band gap is obtain at a hole length of $d = 0.55p$. The three different plots seen in Figure 5.8 displays the band gap at different $d$:s for three different quotients of the conductor gap ($g$) and hole height ($h$) and it is seen that when the "hole"-height is increased the band gap is increased. A similar analysis with the gap-parameter $g$ can here be done. Since $0 < h < b - s$, we can consider the limiting cases. If $h \to 0$ this gives $a \to s$ and the structure tends to that of a "regular" coaxial cable and hence the dispersion diagram tends to following the line of light. If instead $h \to b - s$, the structure will tend to a case with no wave propagation (along the $z$-direction), hence greater hole height ($h = a - s$) should give greater band gap, which is also observed in Figure 5.8.
5.2. Comparison between the 1D and 2D case

An interesting fact about the two structures considered in chapter 4 and 5 (1D and 2D) is that the 2D-case (see Figure 5.1) is the 1D-case (with parameters \(d = h_1 = 0\)) rotated \(2\pi\) Rad around the \(x\)-axis (see Figure 4.1). Therefore it can be instructive to compare the two cases for analogue parameters to see how the solution differs. Figures 5.9 and 5.10 depicts the mode matching method solution for the 1D-case (red squares) and 2D-case (blue circles) for two different sets of parameters.

In order to study for which parameters the two cases agree the best (or worst), we calculate the relative error in frequency

\[
\Delta f = \frac{|f_{1D} - f_{2D}|}{f_{2D}} \tag{5.58}
\]

at five different \(k_z p / \pi = [0.1, 0.3, 0.5, 0.7, 0.9]\) for a range of the three parameters \(d, g = b - a\) and \(h = a - s\). The following set of parameters are used as reference parameters \(p = 12\) mm, \(d = 0.2p\), \(s = 1\) mm, \(h = 0.5\) mm and \(g = 0.5\) mm and we consider the same number of modes in both the structures, 7 modes in region 1 and 4 modes in region 2. The relative error can be seen in Figures 5.11-5.16.

**Figure 5.8.** The band gap between the first and second mode for different "hole lengths" \((d)\) keeping the rest of the parameters (see Figure 5.1) fixed with \(p = 12\) mm for all cases while \(s = 1\) mm, \(h = a - s = 0.5\) mm, \(g = b - a = 0.5\) mm for the blue, \(s = 1\) mm, \(h = a - s = 0.5\) mm, \(g = b - a = 1\) mm for the red and \(s = 1\) mm, \(h = a - s = 1\) mm, \(g = b - a = 0.5\) mm for the yellow.
Figure 5.9. Dispersion diagrams obtained by the mode matching method for coaxial case seen in Figure 5.1 (blue circles) and analogue 1D Corrugated metasurface seen in Figure 4.1 (red squares). The parameter values used was (see Figures 5.1 and 4.1) \( p = 12 \text{ mm}, \ d = 4.8 \text{ mm}, \ s = 1 \text{ mm}, \ a = 1.5 \text{ mm} \) and \( b = 2 \text{ mm} \) for the coaxial case and \( p = 12 \text{ mm}, \ L = 4.8 \text{ mm}, \ g = 0.5 \text{ mm}, \ h_1 = 0.5 \text{ mm} \) and \( d = h_2 = 0 \text{ mm} \) for the 1D-case.

Figure 5.10. Dispersion diagrams obtained by the mode matching method for coaxial case seen in Figure 5.1 (blue circles) and analogue 1D Corrugated metasurface seen in Figure 4.1 (red squares). The parameter values used was (see Figures 5.1 and 4.1) \( p = 12 \text{ mm}, \ d = 2.4 \text{ mm}, \ s = 1 \text{ mm}, \ a = 1.5 \text{ mm} \) and \( b = 2.5 \text{ mm} \) for the coaxial case and \( p = 12 \text{ mm}, \ L = 2.4 \text{ mm}, \ g = 1 \text{ mm}, \ h_1 = 0.5 \text{ mm} \) and \( d = h_2 = 0 \text{ mm} \) for the 1D-case.
5.2. Comparison between the 1D and 2D case

Figure 5.11. The calculated relative error in frequency (see eq. (5.58)) for the first mode between the 2D-coaxial-case (see Figure 5.1) and the analogue 1D-case for three different hole lengths, \(d\) (keeping the rest of the parameters fixed), calculated at five different \(k_{z0}\).

Figure 5.12. The calculated relative error in frequency (see eq. (5.58)) for the second mode between the 2D-coaxial-case (see Figure 5.1) and the analogue 1D-case for three different hole lengths, \(d\) (keeping the rest of the parameters fixed), calculated at five different \(k_{z0}\).
Figure 5.13. The calculated relative error in frequency (see eq. (5.58)) for the first mode between the 2D-coaxial-case (see Figure 5.1) and the analogue 1D-case for three different conductor gaps, $g$ (keeping the rest of the parameters fixed), calculated at five different $k_{z0}$.

Figure 5.14. The calculated relative error in frequency (see eq. (5.58)) for the second mode between the 2D-coaxial-case (see Figure 5.1) and the analogue 1D-case for three different conductor gaps, $g$ (keeping the rest of the parameters fixed), calculated at five different $k_{z0}$. 
5.2. *Comparison between the 1D and 2D case*

Figure 5.15. The calculated relative error in frequency (see eq. (5.58)) for the first mode between the 2D-coaxial-case (see Figure 5.1) and the analogue 1D-case for three different hole heights, \( h \) (keeping the rest of the parameters fixed), calculated at five different \( k_{z0} \).

Figure 5.16. The calculated relative error in frequency (see eq. (5.58)) for the second mode between the 2D-coaxial-case (see Figure 5.1) and the analogue 1D-case for three different hole heights, \( h \) (keeping the rest of the parameters fixed), calculated at five different \( k_{z0} \).
Chapter 6

Conclusions and Future Work

We have analyzed the wave propagation in the structure illustrated in Figure 5.1 using a mode matching technique described in section 4.1. It has been seen that the mode matching method can produce the CST Microwave studio-result extremely well for both the first and second mode of propagation. The result could be obtained using no more than 7 modes in region 1 and 4 modes in region 2. Although, increasing the number of EM-modes \(N = \max \{|n|\}\) and/or \(R = \max \{r\}\) (see section 5.1.2) could yield even better results, it would at the same time increase the computation time. For the number of electromagnetic modes considered, the mode matching technique for the cylindrical case was around 10 times faster then the Eigenmode Solver in CST Microwave Studio while the 1D-case was observed to be up to 100 times faster. However, for the cylindrical case, the computation can be made even faster using Multidimensional Arrays but, because this report focuses on the mathematical theory and physical understanding of wave propagation in periodical structures, code enhancement was considered beyond the scope of this thesis. We find that a maximal band gap between the first and second mode (band) is obtained for a hole-length, \(d\), at 55\% of the period, \(p\). This is illustrated in Figure 5.8 and is found to be the case for all three different ratios of conductor gap, \(g\), and hole-height, \(h\) considered.

The coaxial-structure is compared to an analogue 1D-structure in section 5.2. The relative error between the two structures are plotted and compared for different parameter values. Since the 1D-case formulation is much simpler and faster in computation time it is instructive to find out for which parameter values \((h, g\) and \(d\)) the 1D-structure can safely be used to model the wave propagation in the 2D-structure (coax). From Figures 5.11, 5.13 and 5.15 the relative error (for the first band) is observed to generally be larger at the higher \(k_{z0}\)-values. This has most likely to do with that the dispersion from the line of light is largest there
for both the cases (see for example Figures 5.9 and 5.10 for comparison). It is seen that a small relative error is obtained with small hole-heights \( (h) \), small hole-lengths \( (d) \) and large conductor gaps \( (g) \). Worth noticing is that, in Figure 5.11 and Figure 5.13 there’s only a few percentage points difference in the relative error between the three parameter values considered and it remains roughly constant over the \( k_{z0} \)-range (except for \( g = 0.1 \) mm in Figure 5.13). However, Figure 5.15 show for \( h = 0.1 \) mm almost zero difference between the 1D and 2D case while using \( h = 1 \) mm, displays a relative error at more than 10\% for all \( k_{z0} \)-values. It seems that, for the 1D-case to be a good approximation of the coaxial-case, the "hole"-height \( h \) should be kept at least below 1 mm.

The same (relative error) analysis was carried out on the second mode (band). From Figures 5.12, 5.14 and 5.16 we see that the relative error basically stays below 4\% at all \( k_{z0} \):s for all parameter values considered except for the \( h = 1 \) mm case when \( k_{z0}p/\pi < 0.5 \). Hence, the 1D-case can be considered a good approximation for the second mode for all parameter ranges considered in this thesis.

The coaxial structure considered had a "hole" (excavation) on the inner conductor of 360° opening angle. Taking the hole at 360°, turns out to simplify and symmetrize the formulation for the matching a great deal. However, considering similar structures but, with an opening angle below 360° should also work. A structure having a hole with an opening angle of 180°, as illustrated in Figure 6.1, was attempted using the mode matching method in a similar way as performed in section 5.1.2. However, due to reasons that (for now) are unknown, a reasonable result were not obtained. Finding the dispersion relation for such a unit cell is left for the future.

Deriving the dispersion relation analytically using the mode matching method for a periodic structure with unit cell as (or similar to) Figure 6.1 would enable one to obtain and study the dispersion relation for twist symmetric-structures like the ones shown in Figures 6.2 and 6.3 by following the work of [18].

![Figure 6.1. The unit cell of a periodic coaxial structure with a hole with an opening angle of 180°.](image)
Figure 6.2. The unit cell of a periodic coaxial structure with two-fold twist-symmetric holes with an opening angle of 180°.

Figure 6.3. The unit cell of a periodic coaxial structure with four-fold twist-symmetric holes with an opening angle of 180°.
Bibliography


