Stochastic regularity of general quadratic observables of high frequency waves

G. Malenová and O. Runborg

Abstract

We consider solutions to the wave equation with uncertain initial data and medium, whose wavelength is short compared to the distance traveled by the wave. We are interested in the statistics of the observables, i.e. functionals of the wave solution. Computation by direct methods gets very expensive or outright non-feasible as the wavelength decreases. To address the difficulties, we proposed a method consisting of the Gaussian beam method to treat the high frequencies and the sparse stochastic collocation method to remedy the curse of dimensionality in the stochastic space. For the latter method to converge, we need the observables to satisfy certain stochastic regularity conditions. The main contribution of this work is to show this regularity for a set of quadratic observables obtained by the Gaussian beam approximation of the wave solution.

1 Introduction

Many physical phenomena can be described by propagation of high-frequency waves with stochastic parameters. For instance, an earthquake where seismic waves with uncertain epicenter travel through the layers of the Earth with uncertain soil characteristics represents one such problem stemming from geophysics. Similar problems arise e.g. in optics, acoustics or oceanography. By high frequency we understand that the wavelength is very short compared to the distance traveled by the wave.
As a simplified model of the wave propagation, we use the scalar wave equation

$$u^\varepsilon_{tt}(t,\mathbf{x},\mathbf{y}) = c(\mathbf{x},\mathbf{y})^2\,\Delta u^\varepsilon(t,\mathbf{x},\mathbf{y}), \qquad \text{in } [0,T]\times\mathbb{R}^n\times\Gamma, \tag{1a}$$

$$u^\varepsilon(0,\mathbf{x},\mathbf{y}) = B_0(\mathbf{x},\mathbf{y})\,e^{i\,\varphi_0(\mathbf{x},\mathbf{y})/\varepsilon}, \qquad \text{in } \mathbb{R}^n\times\Gamma, \tag{1b}$$

$$u^\varepsilon_t(0,\mathbf{x},\mathbf{y}) = \varepsilon^{-1}B_1(\mathbf{x},\mathbf{y})\,e^{i\,\varphi_0(\mathbf{x},\mathbf{y})/\varepsilon}, \qquad \text{in } \mathbb{R}^n\times\Gamma, \tag{1c}$$

with highly oscillatory initial data, represented by the small wavelength $\varepsilon \ll 1$, and a stochastic parameter $\mathbf{y}\in\Gamma\subset\mathbb{R}^N$ which models the uncertainty. The dimension N can be fairly large. Two sources of uncertainty are considered: the local speed, $c = c(\mathbf{x},\mathbf{y})$, and the initial data, $B_0 = B_0(\mathbf{x},\mathbf{y})$, $B_1 = B_1(\mathbf{x},\mathbf{y})$, $\varphi_0 = \varphi_0(\mathbf{x},\mathbf{y})$. The solution is therefore also a function of the random parameter, $u^\varepsilon = u^\varepsilon(t,\mathbf{x},\mathbf{y})$.
The focus of this work is on the uncertainty quantification of such a model, more precisely on statistics of a family of quantities of interest (QoI)

$$\mathcal{Q}^{p,\boldsymbol{\alpha}}(\mathbf{y}) = \varepsilon^{2(p+|\boldsymbol{\alpha}|)}\int_{\mathbb{R}}\int_{\mathbb{R}^n} g(t,\mathbf{x},\mathbf{y})|\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u^\varepsilon(t,\mathbf{x},\mathbf{y})|^2\psi(t,\mathbf{x})\,d\mathbf{x}\,dt, \tag{2}$$

with $\psi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^n)$ and $g\in C^\infty(\mathbb{R}\times\mathbb{R}^n\times\Gamma)$. The article generalizes the results in [6] where the QoI

$$\widetilde{\mathcal{Q}}(t,\mathbf{y}) := \widetilde{\mathcal{Q}}^{0,\mathbf{0}}(t,\mathbf{y}) = \int_{\mathbb{R}^n}|u^\varepsilon(t,\mathbf{x},\mathbf{y})|^2\psi(t,\mathbf{x})\,d\mathbf{x}, \tag{3}$$

was studied.
The simplest case in (2),

$$\mathcal{Q}(\mathbf{y}) := \mathcal{Q}^{0,\mathbf{0}}(\mathbf{y}) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} |u^\varepsilon(t,\mathbf{x},\mathbf{y})|^2 \psi(t,\mathbf{x})\, d\mathbf{x}\, dt, \tag{4}$$

represents the weighted average intensity of the wave. More precisely, if the solution u^ε to (1) describes the pressure, then $\mathcal{Q}$ represents the acoustic potential energy.
Another significant example is the weighted energy of the wave,

$$E(\mathbf{y}) = \varepsilon^2 \int_{\mathbb{R}} \int_{\mathbb{R}^n} (|u_t^\varepsilon(t,\mathbf{x},\mathbf{y})|^2 + c^2(\mathbf{x},\mathbf{y})|\nabla u^\varepsilon(t,\mathbf{x},\mathbf{y})|^2)\psi(t,\mathbf{x})\, d\mathbf{x}\, dt,$$

which can be decomposed into terms of type (2), or a weighted and averaged version of the Arias intensity,

$$I(\mathbf{y}) = \varepsilon^4 \int_{\mathbb{R}} \int_{\mathbb{R}^n} |u_{tt}^\varepsilon(t,\mathbf{x},\mathbf{y})|^2 \psi(t,\mathbf{x})\, d\mathbf{x}\, dt,$$

which represents the total energy per unit weight stored by a set of undamped oscillators at the end of an earthquake, see [4].
When computing the QoI and its statistical moments, we are facing two difficulties: high frequencies and large dimensions of the stochastic space. It was suggested in [7] to treat the first issue by means of the *Gaussian beam method* (GB) and the latter by the *sparse stochastic collocation technique.*
More precisely, due to the high-frequencies, the direct solution to (1) is computationally demanding as the cost is of order $O(\varepsilon^{-n-1})$. We will therefore employ asymptotic methods based on geometrical optics: the GB method is particularly versatile. A big advantage of the GB method is that it approximates the solution to the PDE (1) by an ansatz obtained via a solution to a set of ε-independent ODEs instead.
Furthermore, the dimension N of the stochastic space can be large which renders the computation of the statistical moments of QoIs intricate. The sparse stochastic collocation method provides a remedy to the curse of dimensionality, see [1, 3]. A necessary condition for this method to converge is that the QoIs are regular in the stochastic space. That is, we require that for all compact $\Gamma_c \subset \Gamma$ and all $\boldsymbol{\sigma} \in \mathbb{N}_0^N$,

$$\sup_{\mathbf{y}\in\Gamma_c} \left| \frac{\partial^{\boldsymbol{\sigma}} \mathcal{Q}^{p,\boldsymbol{\alpha}}(\mathbf{y})}{\partial \mathbf{y}^{\boldsymbol{\sigma}}} \right| \le C_{\boldsymbol{\sigma}}, \qquad \forall \boldsymbol{\sigma} \in \mathbb{N}_0^N, \tag{5}$$

for some constants $C_{\boldsymbol{\sigma}}$, uniformly in ε.
The GB approximation $\tilde{u}$ to u^ε features two modes, $\tilde{u} = \tilde{u}^+ + \tilde{u}^-$, satisfying two different sets of ODEs. In certain cases, it is possible to approximate u^ε by one of the modes only, i.e. either $\tilde{u} = \tilde{u}^+$ or $\tilde{u} = \tilde{u}^-$. In those cases, we can examine a QoI that is, in contrast to (2), only integrated in space,

$$\widetilde{\mathcal{Q}}^{p,\boldsymbol{\alpha}}(t,\mathbf{y}) = \varepsilon^{2(p+|\boldsymbol{\alpha}|)} \int_{\mathbb{R}^n} g(t,\mathbf{x},\mathbf{y}) |\partial_t^p \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u^\varepsilon(t,\mathbf{x},\mathbf{y})|^2 \psi(t,\mathbf{x})\, d\mathbf{x}, \tag{6}$$

with $\psi \in C_c^\infty(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R} \times \mathbb{R}^n \times \Gamma)$, and show a stronger regularity condition,

$$\sup_{\substack{\mathbf{y}\in\Gamma_c \\ t\in[0,T]}} \left| \frac{\partial^{\boldsymbol{\sigma}} \widetilde{\mathcal{Q}}^{p,\boldsymbol{\alpha}}(t,\mathbf{y})}{\partial \mathbf{y}^{\boldsymbol{\sigma}}} \right| \le C_{\boldsymbol{\sigma}}, \qquad \forall \boldsymbol{\sigma} \in \mathbb{N}_0^N, \tag{7}$$

uniformly in ε. The simplest case (3) was considered in [6], including the proof of the regularity of the type (7) for u^ε approximated by a one-mode GB solution $\tilde{u}$.

In this work, we extend the results of [6] by proving the regularity of a one-mode QoI in (7) with u^ε replaced by $\tilde{u}$, for any $p \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^N$. This will then serve as a stepping stone for the proof of regularity of the two-mode QoI, i.e. we will show (5) for $\mathcal{Q}^{p,\boldsymbol{\alpha}}$ as in (2) with $\tilde{u} = \tilde{u}^+ + \tilde{u}^-$ as the GB approximation to u^ε.
The layout of this article is as follows: we briefly introduce our assumptions in Section 2 and then present the Gaussian beam method in Section 3. The one-mode QoI with u^ε approximated by $\tilde{u}$ is regarded in Section 4 with the stochastic regularity (7) shown in Theorem 4.2. Finally, stochastic regularity (5) of a general two-mode QoI approximated by the GB method is proved in Theorem 5.2 in Section 5.

2 Assumptions and preliminaries

Let us consider the Cauchy problem (1). By $t \in [0, T] \subset \mathbb{R}$ we denote the time, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is the spatial variable and the uncertainty in the model is described by the random variable $\mathbf{y} = (y_1, \dots y_N) \in \Gamma$ where $\Gamma \subset \mathbb{R}^N$ is an open set.
We make the following precise assumptions.

(A1) Strictly positive, smooth and bounded speed of propagation,

$$c \in C^\infty(\mathbb{R}^n \times \Gamma), \qquad 0 < c_{\min} \le c(\mathbf{x}, \mathbf{y}) \le c_{\max} < \infty, \qquad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall \mathbf{y} \in \Gamma.$$

and for each multi-index pair $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ there is a constant $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that

$$\left|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} c(\mathbf{x}, \mathbf{y})\right| \le C_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \qquad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall \mathbf{y} \in \Gamma.$$

(A2) Smooth and (uniformly) compactly supported initial amplitudes,

$$B_\ell \in C^\infty(\mathbb{R}^n \times \Gamma), \qquad \operatorname{supp} B_\ell(\,\cdot\,, \mathbf{y}) \subset K_0, \qquad \ell = 0, 1, \quad \forall \mathbf{y} \in \Gamma,$$

where $K_0 \subset \mathbb{R}^n$ is a compact set.

(A3) Smooth initial phase with non-zero gradient,

$$\varphi_0 \in C^\infty(\mathbb{R}^n \times \Gamma), \qquad |\nabla \varphi_0(\mathbf{x}, \mathbf{y})| > 0, \qquad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall \mathbf{y} \in \Gamma.$$

(A4) High frequency,

$$0 < \varepsilon \le 1.$$

(A5) Smooth and compactly supported QoI test function,

$$\psi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n), \qquad \operatorname{supp} \psi \subset [0, T] \times K_1,$$

where $K_1 \subset \mathbb{R}^n$ is a compact set.

Let us now introduce the following notation used throughout this paper.

- $B_\mu := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \le \mu\}$.
- $\mathcal{X} := C^\infty(\mathbb{R} \times \mathbb{R}^n \times \Gamma \times \mathbb{R}^n)$.
- $\mathcal{Y} := C^\infty(\mathbb{R} \times \mathbb{R}^n \times \Gamma \times \mathbb{R}^n \times \mathbb{R}^n)$.

Note that $B_\infty = \mathbb{R}^n$.

3 Gaussian beam approximation

Solving (1) directly requires a substantial number of numerical operations when the wavelength ε is small. In particular, to maintain a given accuracy for a fixed $\mathbf{y}$, we need $O(\varepsilon^{-n})$ discretization points in $\mathbf{x}$ and $O(\varepsilon^{-1})$ time steps resulting into the computational cost $O(\varepsilon^{-n-1})$. To avoid the high cost we employ asymptotic methods arising from the geometrical optics. In particular, the Gaussian beam (GB) method provides a powerful tool, see [2, 5, 9, 10, 11].
Individual Gaussian beams are asymptotic solutions to the wave equation (1) that concentrate around the central ray in space-time. Rays are bicharacteristics of the wave equation (1). They are denoted by $(\mathbf{q}^\pm, \mathbf{p}^\pm)$ where $\mathbf{q}^\pm(t, \mathbf{y}, \mathbf{z})$ represents the position and $\mathbf{p}^\pm(t, \mathbf{y}, \mathbf{z})$ the direction, respectively, and $\mathbf{z} \in K_0$ is the starting point so that $\mathbf{q}^\pm(0, \mathbf{y}, \mathbf{z}) = \mathbf{z}$ for all $\mathbf{y} \in \Gamma$. From each $\mathbf{z}$, the ray folds in opposite directions, here distinguished by the superscript $\pm$. Therefore, the GB solution also features two versions with either of the $\pm$ superscript. We denote the k-th order Gaussian beam starting at $\mathbf{z} \in K_0$ by $v_k^\pm(t, \mathbf{x}, \mathbf{y}, \mathbf{z})$ and define it as

$$v_k^\pm(t, \mathbf{x}, \mathbf{y}, \mathbf{z}) = A_k^\pm(t, \mathbf{x} - \mathbf{q}^\pm(t, \mathbf{y}, \mathbf{z}), \mathbf{y}, \mathbf{z}) e^{i\Phi_k^\pm(t, \mathbf{x} - \mathbf{q}^\pm(t,\mathbf{y},\mathbf{z}), \mathbf{y}, \mathbf{z})/\varepsilon}, \tag{8}$$

where

$$\Phi_k^\pm(t, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \phi_0^\pm(t, \mathbf{y}, \mathbf{z}) + \mathbf{x}^T \mathbf{p}^\pm(t, \mathbf{y}, \mathbf{z}) + \frac{1}{2} \mathbf{x}^T M^\pm(t, \mathbf{y}, \mathbf{z}) \mathbf{x} + \sum_{|\boldsymbol{\beta}|=3}^{k+1} \frac{1}{\boldsymbol{\beta}!} \phi_{\boldsymbol{\beta}}^\pm(t, \mathbf{y}, \mathbf{z}) \mathbf{x}^{\boldsymbol{\beta}}, \tag{9}$$

is the k-th order phase function and

$$A_k^\pm(t, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{j=0}^{\lceil \frac{k}{2} \rceil - 1} \varepsilon^j \sum_{|\boldsymbol{\beta}|=0}^{k-2j-1} \frac{1}{\boldsymbol{\beta}!} a_{j,\boldsymbol{\beta}}^\pm(t, \mathbf{y}, \mathbf{z}) \mathbf{x}^{\boldsymbol{\beta}}, \tag{10}$$

is the k-th order amplitude function. The higher the order k, the more accurately $v_k^\pm$ approximates the solution to (1) in terms of ε. The variables $\phi_0^\pm, \mathbf{q}^\pm, \mathbf{p}^\pm, M^\pm, \phi_{\boldsymbol{\beta}}^\pm, a_{j,\boldsymbol{\beta}}^\pm$ are given by a set of ODEs, the simplest ones being

$$\dot{\phi}_0^\pm = 0, \tag{11a}$$

$$\dot{\mathbf{q}}^\pm = \pm c(\mathbf{q}^\pm) \frac{\mathbf{p}^\pm}{|\mathbf{p}^\pm|}, \tag{11b}$$

$$\dot{\mathbf{p}}^\pm = \mp \nabla c(\mathbf{q}^\pm) |\mathbf{p}^\pm|, \tag{11c}$$

$$\dot{M}^\pm = \mp(D^\pm + (B^\pm)^T M^\pm + M^\pm B^\pm + M^\pm C^\pm M^\pm), \tag{11d}$$

$$\dot{a}_{0,\mathbf{0}}^\pm = \pm \frac{1}{2|\mathbf{p}^\pm|} \left(-c(\mathbf{q}^\pm) \mathrm{Tr}(M^\pm) + \nabla c(\mathbf{q}^\pm)^T \mathbf{p}^\pm + \frac{c(\mathbf{q}^\pm)(\mathbf{p}^\pm)^T M^\pm \mathbf{p}^\pm}{|\mathbf{p}^\pm|^2} \right) a_{0,\mathbf{0}}^\pm, \tag{11e}$$

where

$$B^\pm = \frac{\mathbf{p}^\pm \nabla c(\mathbf{q}^\pm)^T}{|\mathbf{p}^\pm|}, \qquad C^\pm = \frac{c(\mathbf{q}^\pm)}{|\mathbf{p}^\pm|} - \frac{c(\mathbf{q}^\pm)}{|\mathbf{p}^\pm|^3} \mathbf{p}^\pm (\mathbf{p}^\pm)^T, \qquad D^\pm = |\mathbf{p}^\pm| \nabla^2 c(\mathbf{q}^\pm).$$

For the ODEs determining $\phi_{\boldsymbol{\beta}}^\pm$ and $a_{j,\boldsymbol{\beta}}^\pm$ other than the leading term we refer the reader to [9, 11].
As motivated above, the sign ambiguity corresponds to GBs moving in opposite directions which means that they are governed by two different sets of ODEs. We will say that GBs following the same set of ODEs (i.e. all either described by v_k^+ or v_k^- in (8)) constitute *the same mode*.

Single beams from the same mode with their starting points in K_0 are summed together to form the k-th order one-mode solution $u_k^{\pm}(t, \mathbf{x}, \mathbf{y})$,

$$u_k^{\pm}(t, \mathbf{x}, \mathbf{y}) = \left(\frac{1}{2\pi\varepsilon}\right)^{n/2} \int_{K_0} v_k^{\pm}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}) \varrho_\eta(\mathbf{x} - \mathbf{q}^{\pm}(t, \mathbf{y}, \mathbf{z})) d\mathbf{z}. \tag{12}$$

where the integration in $\mathbf{z}$ is over the support of the initial data $K_0 \subset \mathbb{R}^n$. Since the wave equation is linear, the superposition of beams is still an asymptotic solution. The function $\varrho_\eta \in C^\infty(\mathbb{R}^n)$ is a real-valued *cutoff* function with radius $0 < \eta \leq \infty$,

$$\varrho_\eta(\mathbf{x}) = \begin{cases} 1, & \text{if } |\mathbf{x}| \leq \eta, \quad \text{for } 0 < \eta < \infty, \\ 0, & \text{if } |\mathbf{x}| \geq 2\eta, \quad \text{for } 0 < \eta < \infty, \\ 1, & \qquad\qquad\quad\ \text{for } \eta = \infty. \end{cases} \tag{13}$$

For the first order GBs, $k = 1$, one can choose $\eta = \infty$, i.e. no ϱ_η. Each GB $v_k^{\pm}$ requires initial values for all its coefficients. An appropriate choice makes $u_k^{\pm}(0, \mathbf{x}, \mathbf{y})$ converge asymptotically as $\varepsilon \to 0$ to the initial conditions in (1). As shown in [5], the initial data are to be chosen as follows:

$$\mathbf{q}^{\pm}(0, \mathbf{y}, \mathbf{z}) = \mathbf{z}, \tag{14a}$$

$$\mathbf{p}^{\pm}(0, \mathbf{y}, \mathbf{z}) = \nabla\varphi_0(\mathbf{z}, \mathbf{y}), \tag{14b}$$

$$\phi_0^{\pm}(0, \mathbf{y}, \mathbf{z}) = \varphi_0(\mathbf{z}, \mathbf{y}), \tag{14c}$$

$$M^{\pm}(0, \mathbf{y}, \mathbf{z}) = \nabla^2\varphi_0(\mathbf{z}, \mathbf{y}) + i\, I_{n\times n}, \tag{14d}$$

$$\phi_{\boldsymbol{\beta}}^{\pm}(0, \mathbf{y}, \mathbf{z}) = \partial_{\mathbf{x}}^{\boldsymbol{\beta}}\varphi_0(\mathbf{z}, \mathbf{y}), \qquad |\boldsymbol{\beta}| = 3, \dots, k+1, \tag{14e}$$

$$a_{0,\mathbf{0}}^{\pm}(0, \mathbf{y}, \mathbf{z}) = \frac{1}{2}\left(B_0(\mathbf{z}, \mathbf{y}) \pm \frac{B_1(\mathbf{z}, \mathbf{y})}{ic(\mathbf{z}, \mathbf{y})|\nabla\varphi_0(\mathbf{z}, \mathbf{y})|}\right), \tag{14f}$$

where $I_{n\times n}$ denotes the identity matrix of size n. The initial data for the higher order amplitude coefficients are given in [5]. The following proposition shows that all these variables are smooth and $a_{j,\boldsymbol{\beta}}^{\pm}$ remain supported in K_0 for all times t and random variables $\mathbf{y} \in \Gamma$.

Proposition 3.1. *Under assumptions (A1)–(A3), the coefficients* $\phi_0^{\pm}, \mathbf{q}^{\pm}, \mathbf{p}^{\pm}, M^{\pm}, \phi_{\boldsymbol{\beta}}^{\pm}, a_{j,\boldsymbol{\beta}}^{\pm} \in C^\infty(\mathbb{R}\times \Gamma \times \mathbb{R}^n)$ *and*

$$supp(a_{j,\boldsymbol{\beta}}^{\pm}(t, \mathbf{y}, \cdot)) \subset K_0, \quad \forall t \in \mathbb{R},\ \mathbf{y} \in \Gamma.$$

Consequently, $\Phi_k^{\pm} \in \mathcal{X}$.

Proof. Properties (P1) and (P2) of Proposition 1 in [6]; the proof is in [8]. □

Finally, the k-th order GB superposition solution is defined as a sum of the two modes in (12),

$$u_k(t, \mathbf{x}, \mathbf{y}) = u_k^{+}(t, \mathbf{x}, \mathbf{y}) + u_k^{-}(t, \mathbf{x}, \mathbf{y}). \tag{15}$$

Approximating u^ε with u_k we can define the GB quantity of interest corresponding to (2) as

$$\mathcal{Q}_{\mathrm{GB}}^{p,\boldsymbol{\alpha}}(\mathbf{y}) = \varepsilon^{2(p+|\boldsymbol{\alpha}|)} \int_{\mathbb{R}} \int_{\mathbb{R}^n} g(t, \mathbf{x}, \mathbf{y}) |\partial_t^p \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u_k(t, \mathbf{x}, \mathbf{y})|^2 \psi(t, \mathbf{x}) d\mathbf{x}\, dt, \tag{16}$$

where ψ is as in (A5) and $g \in C^\infty(\mathbb{R} \times \mathbb{R}^n \times \Gamma)$. It was argued in [6] that $\mathcal{Q}^{p,\boldsymbol{\alpha}}$ is well-approximated by $\mathcal{Q}_{\mathrm{GB}}^{p,\boldsymbol{\alpha}}$ as u_k is a good approximation of u^ε.

Notably, in the 1D case with constant speed $c(x, y) = c(y)$ the GB QoI (16) is exact, $\mathcal{Q}_{\text{GB}}^{p,\alpha} = \mathcal{Q}^{p,\alpha}$, provided we choose $u^\varepsilon(0, x, y)$ in (1) identical to $u_k(0, x, y)$, and $u_t^\varepsilon(0, x, y) = 0$. This is because $p^\pm, M^\pm, a_{j,\beta}^\pm, \phi_\beta^\pm$ are constant, and $q^\pm = z \pm c(y)\text{sign}(p^\pm)t$ due to (11).
Before considering the QoI (16) it is advantageous to first focus on its one-mode counterpart with u_k consisting of either $u_k = u_k^+$ or $u_k = u_k^-$ only. This is partly due to the one-mode QoI being a stepping stone in the two-mode QoI analysis. Moreover, its examination is important in its own right as in certain cases, a one mode solution suffices to approximate the full solution u^ε in (1). For example, in $n = 1$ with a constant speed, the d'Alembert solution to the wave equation is a superposition of a left and a right going wave. In addition, if B_1 in (1) is chosen such that u^ε essentially propagates in one direction, then merely one mode, either u_k^+ or u_k^-, is needed to approximate u^ε. For a more detailed discussion and examples, see Section 5.1.

4 One-mode quantity of interest

In this section, we will only consider a one-mode GB solution, so either $u_k = u_k^+$ or $u_k = u_k^-$ in (15). It is not important which one we choose as long as the choice is consistent. We henceforth omit superscripts of all variables.
Let us now define the GB-approximated version of the QoI in (6),

$$\widetilde{\mathcal{Q}}_{\text{GB}}^{p,\boldsymbol{\alpha}}(t, \mathbf{y}) = \varepsilon^{2(p+|\boldsymbol{\alpha}|)} \int_{\mathbb{R}^n} g(t, \mathbf{x}, \mathbf{y}) |\partial_t^p \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u_k(t, \mathbf{x}, \mathbf{y})|^2 \psi(t, \mathbf{x}) d\mathbf{x}, \tag{17}$$

with $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R} \times \mathbb{R}^n \times \Gamma)$. To introduce the terminology used in this Section, we will need the following proposition.

Proposition 4.1. *Let us assume (A1)–(A3) hold. Then for all $T > 0$, beam order k and compact $\Gamma_c \subset \Gamma$, there is a GB cutoff width $\eta > 0$ and constant $\delta > 0$ such that for all $\mathbf{x} \in B_{2\eta}$,*

$$\operatorname{Im} \Phi_k(t, \mathbf{x}, \mathbf{y}, \mathbf{z}) \geq \delta |\mathbf{x}|^2, \quad \forall t \in [0, T], \, \mathbf{y} \in \Gamma_c, \, \mathbf{z} \in K_0. \tag{18}$$

For the first order GB, $k = 1$, we have $\eta = \infty$ and (18) is valid for all $\mathbf{x} \in \mathbb{R}^n$.

Proof. Property (P4) in Proposition 1 in [6], the proof is in [8]. □

Note that η is the width of the cutoff function ϱ_η in (13) used in the GB superposition (12).

Definition 1. The cutoff width η used for the GB approximation is called admissible for a given T, k and Γ_c if it is small enough in the sense of Proposition 4.1.

We will prove the following main theorem.

Theorem 4.2. *Let us assume that (A1)–(A5) hold and consider a one-mode GB solution. Moreover, let η be admissible in the sense of Definition 1 for $T > 0$, k and a compact $\Gamma_c \subset \Gamma$. Then for all $p \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^N$, there exist $C_{\boldsymbol{\sigma}}$ such that*

$$\sup_{\substack{\mathbf{y} \in \Gamma_c \\ t \in [0,T]}} \left| \frac{\partial^{\boldsymbol{\sigma}} \widetilde{\mathcal{Q}}_{GB}^{p,\boldsymbol{\alpha}}(t, \mathbf{y})}{\partial \mathbf{y}^{\boldsymbol{\sigma}}} \right| \leq C_{\boldsymbol{\sigma}}, \quad \forall \boldsymbol{\sigma} \in \mathbb{N}_0^N,$$

where $C_{\boldsymbol{\sigma}}$ is independent of ε but depends on T, k and Γ_c.

The proof of Theorem 4.2 is presented in Section 4.3.

4.1 Known results

Let us now recall the known results regarding the simplest version of the QoI (17),

$$\widetilde{\mathcal{Q}}_{\mathrm{GB}} := \widetilde{\mathcal{Q}}_{\mathrm{GB}}^{0,\mathbf{0}} = \int_{\mathbb{R}^n} |u_k(t,\mathbf{x},\mathbf{y})|^2 \psi(t,\mathbf{x}) d\mathbf{x}, \tag{19}$$

obtained in [6].

Theorem 4.3 ([6, Theorem 1])**.** *Let us assume that (A1)–(A5) hold and consider a one-mode GB solution. Moreover, let η be admissible in the sense of Definition 1 for $T > 0$, k and a compact $\Gamma_c \subset \Gamma$. Then there exist $C_{\boldsymbol{\sigma}}$ such that*

$$\sup_{\substack{\mathbf{y}\in\Gamma_c \\ t\in[0,T]}} \left| \frac{\partial^{\boldsymbol{\sigma}} \widetilde{\mathcal{Q}}_{GB}(t,\mathbf{y})}{\partial \mathbf{y}^{\boldsymbol{\sigma}}} \right| \le C_{\boldsymbol{\sigma}}, \quad \forall \boldsymbol{\sigma} \in \mathbb{N}_0^N,$$

where $C_{\boldsymbol{\sigma}}$ is independent of ε but depends on T, k and Γ_c.

Remark. This version of Theorem 1 in [6] has been changed in the following minor ways:

1. We assumed Γ to be the closure of a bounded open set in [6]. Here, we instead consider compact subsets Γ_c of an open set Γ. This does not affect the proof.
2. We considered ψ to be a function of $\mathbf{x}$ only in [6]. It is however easy to check that the estimate is valid for more general $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$.
3. Originally, we only considered $\sup_{\mathbf{y}\in\Gamma_c} \left|\partial_{\mathbf{y}}^{\boldsymbol{\sigma}} \widetilde{\mathcal{Q}}_{\mathrm{GB}}(t,\mathbf{y})\right|$ at a fixed time t. However, all estimates in [6] are also uniform in t and we can therefore generalize it to $\sup_{\substack{\mathbf{y}\in\Gamma_c \\ t\in[0,T]}} \left|\partial_{\mathbf{y}}^{\boldsymbol{\sigma}} \widetilde{\mathcal{Q}}_{\mathrm{GB}}(t,\mathbf{y})\right|$.

4.2 Preliminaries

We start with a note on the case $\eta = \infty$, which is sometimes an admissible cutoff width in the sense of Proposition 4.1. In particular, it is always admissible when $k = 1$. It amounts to removing the cutoff functions ϱ_η in (12) altogether. This is convenient in computations, but there are some technical issues with having $\eta = \infty$ in the proofs below. We note, however, that, in any finite time interval $[0, T]$, the Gaussian beam superposition (15) with no cutoff is identical to the one with a large enough cutoff, because of the compact support of the test function $\psi(t, \mathbf{x})$. We can therefore, without loss of generality, assume that $\eta < \infty$. Indeed, suppose supp $\psi(t, \cdot) \subset B_R$, for $t \in [0, T]$. Then for $|\mathbf{x}| \le R$ we have

$$|\mathbf{x} - \mathbf{q}(t,\mathbf{y},\mathbf{z})| \le |\mathbf{x}| + |\mathbf{q}(t,\mathbf{y},\mathbf{z})| \le R + |\mathbf{q}(t,\mathbf{y},\mathbf{z})|, \qquad \forall t \in [0,T],\ \forall \mathbf{y} \in \Gamma,\ \forall \mathbf{z}, \in K_0.$$

Hence, for $\bar{\eta} = R + \sup_{t\in[0,T],\mathbf{y}\in\Gamma,\mathbf{z}\in K_0} |\mathbf{q}(t,\mathbf{y},\mathbf{z})|$ we will have

$$\psi(t,\mathbf{x}) = \varrho_{\bar{\eta}}(\mathbf{x} - \mathbf{q}(t,\mathbf{y},\mathbf{z}))\varrho_{\bar{\eta}}(\mathbf{x} - \mathbf{q}(t,\mathbf{y},\mathbf{z}'))\psi(t,\mathbf{x}), \qquad \forall t \in [0,T],\ \forall \mathbf{y} \in \Gamma,\ \forall \mathbf{z},\mathbf{z}' \in K_0.$$

We will use the following preliminaries to prove Theorem 4.2. Let us define a shorthand for the following sets:

- $\mathcal{P}_\infty := \{P \in \mathcal{X} : P(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{|\boldsymbol{\alpha}|=0}^{M} a_{\boldsymbol{\alpha}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\, \mathbf{x}^{\boldsymbol{\alpha}},\ \text{ where }\ a_{\boldsymbol{\alpha}} \in \mathcal{X},\ \forall \boldsymbol{\alpha}\}$.

- $\mathcal{P}_\mu := \{P \in \mathcal{P}_\infty : \operatorname{supp} a_{\boldsymbol{\alpha}}(t,\cdot,\mathbf{y},\mathbf{z}) \subset B_{2\mu},\ \forall \boldsymbol{\alpha},\ t \in \mathbb{R},\ \mathbf{y} \in \Gamma,\ \mathbf{z} \in \mathbb{R}^n\}, \quad \text{for } 0 < \mu < \infty.$
- $\mathcal{S}_\mu := \{f \in \mathcal{X} : f(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{j=0}^{L} \varepsilon^j P_j(t,\mathbf{x},\mathbf{y},\mathbf{z}) e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon}, \ \text{where } \ P_j \in \mathcal{P}_\mu,\ \forall j\}.$

The phase Φ_k in the definition of $\mathcal{S}_\mu$ is as in (9). By Proposition 3.1, it can be written as $\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{|\boldsymbol{\alpha}|=0}^{k+1} d_{\boldsymbol{\alpha}}(t,\mathbf{y},\mathbf{z})\,\mathbf{x}^{\boldsymbol{\alpha}}$, with $d_{\boldsymbol{\alpha}} \in C^\infty(\mathbb{R}\times\Gamma\times\mathbb{R}^n)$ and hence $\Phi_k \in \mathcal{P}_\infty$. The following properties hold for the sets defined above.

Lemma 4.4. *Let $R \in \mathcal{P}_\infty$, $P_1, P_2 \in \mathcal{P}_\mu$ and $w_1, w_2 \in \mathcal{S}_\mu$. Then:*

1. *$P_1 + P_2 \in \mathcal{P}_\mu$.*
2. *$w_1 + w_2 \in \mathcal{S}_\mu$.*
3. *$RP_1 \in \mathcal{P}_\mu$.*
4. *$Rw_1 \in \mathcal{S}_\mu$.*
5. *$\partial_s P_1 \in \mathcal{P}_\mu$, for $s \in \{t, x_\ell,\ \ell = 1,\dots n\}$.*
6. *$\varepsilon\partial_s w_1 \in \mathcal{S}_\mu$, for $s \in \{t, x_\ell,\ \ell = 1,\dots n\}$.*

Proof. We will denote

$$P_m(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{|\boldsymbol{\alpha}|=0}^{M_m} a_{m,\boldsymbol{\alpha}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\,\mathbf{x}^{\boldsymbol{\alpha}}, \qquad w_m(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{j=0}^{L_m} \varepsilon^j Q_{m,j}(t,\mathbf{x},\mathbf{y},\mathbf{z}) e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon},$$
$$R(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{|\boldsymbol{\gamma}|=0}^{M} c_{\boldsymbol{\gamma}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\,\mathbf{x}^{\boldsymbol{\gamma}}, \qquad m \in \{1,2\}.$$

Let us assume without loss of generality that $M_2 \geq M_1$ and $L_2 \geq L_1$.

1. The sum $P_1 + P_2$ can be rewritten as $P_1 + P_2 = \sum_{|\boldsymbol{\beta}|=0}^{M_2} b_{\boldsymbol{\beta}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\,\mathbf{x}^{\boldsymbol{\beta}}$, where $b_{\boldsymbol{\beta}}$ is such that
$$b_{\boldsymbol{\beta}} = \begin{cases} a_{1,\boldsymbol{\beta}} + a_{2,\boldsymbol{\beta}}, & \text{for } |\boldsymbol{\beta}| \leq M_1, \\ a_{2,\boldsymbol{\beta}}, & \text{for } M_1 < |\boldsymbol{\beta}| \leq M_2, \end{cases}$$
and hence $b_{\boldsymbol{\beta}} \in \mathcal{X}$ and $\operatorname{supp} b_{\boldsymbol{\beta}}(t,\cdot,\mathbf{y},\mathbf{z}) \subset B_\mu$, for all $t \in \mathbb{R},\ \mathbf{y} \in \Gamma,\ \mathbf{z} \in \mathbb{R}^n$. We therefore have $P_1 + P_2 \in \mathcal{P}_\mu$.

2. The sum $w_1 + w_2$ can be rewritten as $w_1 + w_2 = \sum_{j=0}^{L_2} \varepsilon^j R_j(t,\mathbf{x},\mathbf{y},\mathbf{z}) e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon}$, where R_j is such that
$$R_j = \begin{cases} Q_{1,j} + Q_{2,j}, & \text{for } j \leq L_1, \\ Q_{2,j}, & \text{for } L_1 < j \leq L_2. \end{cases}$$
By point 1 we have that $R_j \in \mathcal{P}_\mu$ for all j and therefore $w_1 + w_2 \in \mathcal{S}_\mu$.

3. We have
$$R(t,\mathbf{x},\mathbf{y},\mathbf{z})P_1(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{|\boldsymbol{\gamma}|=0}^{M} c_{\boldsymbol{\gamma}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\mathbf{x}^{\boldsymbol{\gamma}} \sum_{|\boldsymbol{\alpha}|=0}^{M_1} a_{1,\boldsymbol{\alpha}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\mathbf{x}^{\boldsymbol{\alpha}} = \sum_{|\boldsymbol{\delta}|=0}^{M_1+M} d_{\boldsymbol{\delta}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\mathbf{x}^{\boldsymbol{\delta}},$$
where $d_{\boldsymbol{\delta}} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\gamma}=\boldsymbol{\delta}} a_{1,\boldsymbol{\alpha}} c_{\boldsymbol{\gamma}} \in \mathcal{X}$ and since $\operatorname{supp} a_{1,\boldsymbol{\alpha}}(t,\cdot,\mathbf{y},\mathbf{z}) \subset B_\mu$, we also have $\operatorname{supp} d_{\boldsymbol{\delta}}(t,\cdot,\mathbf{y},\mathbf{z}) \subset B_\mu$ for all $t \in \mathbb{R},\ \mathbf{y} \in \Gamma,\ \mathbf{z} \in \mathbb{R}^n$ and therefore $RP_1 \in \mathcal{P}_\mu$.

4. We have

$$R(t,\mathbf{x},\mathbf{y},\mathbf{z})w_1(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{j=0}^{L_1} \varepsilon^j R(t,\mathbf{x},\mathbf{y},\mathbf{z})Q_{1,j}(t,\mathbf{x},\mathbf{y},\mathbf{z})e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon},$$

where $RQ_{1,j} \in \mathcal{P}_\mu$ by point 3 for all j. Therefore $Rw_1 \in \mathcal{S}_\mu$.

5. The time derivative of P_1 reads $\partial_t P_1(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{|\boldsymbol{\alpha}|=0}^{M_1} \partial_t a_{1,\boldsymbol{\alpha}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\mathbf{x}^{\boldsymbol{\alpha}}$, and since $\operatorname{supp} \partial_t a_{1,\boldsymbol{\alpha}}(t,\cdot,\mathbf{y},\mathbf{z}) \subset B_\mu$ for all $t \in \mathbb{R}$, $\mathbf{y} \in \Gamma$, $\mathbf{z} \in \mathbb{R}^n$, we have $\partial_t P_1 \in \mathcal{P}_\mu$. Secondly, the derivative of P_1 with respect to x_ℓ reads

$$\partial_{x_\ell} P_1(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \underbrace{\sum_{|\boldsymbol{\alpha}|=0}^{M_1} \partial_{x_\ell} a_{1,\boldsymbol{\alpha}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\,\mathbf{x}^{\boldsymbol{\alpha}}}_{①} + \underbrace{\sum_{|\boldsymbol{\alpha}|=0}^{M_1} a_{1,\boldsymbol{\alpha}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\alpha_\ell\,\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{e}_\ell}}_{②}.$$

Since $\operatorname{supp} \partial_{x_\ell} a_{1,\boldsymbol{\alpha}}(t,\cdot,\mathbf{y},\mathbf{z}) \subset B_\mu$ for all $t \in \mathbb{R}$, $\mathbf{y} \in \Gamma$, $\mathbf{z} \in \mathbb{R}^n$, we have $① \in \mathcal{P}_\mu$. For ②, there exist $c_{\boldsymbol{\gamma}} \in \mathcal{X}$ such that $② = \sum_{|\boldsymbol{\gamma}|=0}^{M_1-1} c_{\boldsymbol{\gamma}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\,\mathbf{x}^{\boldsymbol{\gamma}}$ with $\operatorname{supp} c_{\boldsymbol{\gamma}}(t,\cdot,\mathbf{y},\mathbf{z}) \subset B_\mu$ for all $t \in \mathbb{R}$, $\mathbf{y} \in \Gamma$, $\mathbf{z} \in \mathbb{R}^n$ and hence $② \in \mathcal{P}_\mu$. By point 1, $\partial_{x_\ell} P_1 = ① + ② \in \mathcal{P}_\mu$.

6. The derivative $\partial_s w_1$ with respect to either of $s \in \{t, x_\ell,\ \ell = 1, \dots n\}$ reads

$$\begin{aligned}&\partial_s w_1(t,\mathbf{x},\mathbf{y},\mathbf{z})\\ &= \underbrace{\sum_{j=0}^{L_1} \varepsilon^j \partial_s Q_{1,j}(t,\mathbf{x},\mathbf{y},\mathbf{z})e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon}}_{①} + \underbrace{\sum_{j=0}^{L_1} i\varepsilon^{j-1}\partial_s\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})Q_{1,j}(t,\mathbf{x},\mathbf{y},\mathbf{z})e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon}}_{②}.\end{aligned}$$

We have $\varepsilon① = \sum_{j=0}^{L_1+1} \varepsilon^j R_j(t,\mathbf{x},\mathbf{y},\mathbf{z})e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon}$, with

$$R_j = \begin{cases} 0, & \text{for } j = 0,\\ \partial_s Q_{1,j-1}, & \text{otherwise.}\end{cases}$$

By point 5, $R_j \in \mathcal{P}_\mu$, and we therefore obtain $\varepsilon① \in \mathcal{S}_\mu$. Since $\Phi_k \in \mathcal{P}_\infty$, we have by point 5 that $\partial_s\Phi_k \in \mathcal{P}_\infty$ and therefore $\varepsilon② \in \mathcal{S}_\mu$ by point 4. By point 2, we finally arrive at $\varepsilon\partial_s w_1 = \varepsilon① + \varepsilon② \in \mathcal{S}_\mu$.

□

As a consequence, we obtain the following corollary.

Corollary 1. *If $w \in \mathcal{S}_\mu$, all scaled mixed derivatives $\varepsilon^{p+|\boldsymbol{\alpha}|}\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} w \in \mathcal{S}_\mu$.*

Proof. Apply point 6 of Lemma 4.4 repeatedly. □

4.3 Proof of theorem 4.2

We will consider the following sets for $\mu < \infty$ in the course of the proof:

- $\Lambda_\mu = \Lambda_\mu(t,\mathbf{y},\mathbf{z},\mathbf{z}') := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{q}(t,\mathbf{y},\mathbf{z})| \le 2\mu \quad \text{and} \quad |\mathbf{x} - \mathbf{q}(t,\mathbf{y},\mathbf{z}')| \le 2\mu\}$.
- $\mathcal{T}_\mu := \{f \in \mathcal{Y} : \operatorname{supp} f(t,\cdot,\mathbf{y},\mathbf{z},\mathbf{z}') \subset \Lambda_\mu(t,\mathbf{y},\mathbf{z},\mathbf{z}'),\ \forall t \in \mathbb{R},\ \mathbf{y} \in \Gamma,\ \mathbf{z},\mathbf{z}' \in \mathbb{R}^n\}$.

Let us regard the QoI (17),

$$\begin{aligned}\widetilde{\mathcal{Q}}^{p,\boldsymbol{\alpha}}_{\mathrm{GB}}(t,\mathbf{y}) &= \varepsilon^{2(p+|\boldsymbol{\alpha}|)}\int_{\mathbb{R}^n} g(t,\mathbf{x},\mathbf{y})\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u_k(t,\mathbf{x},\mathbf{y})^*\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u_k(t,\mathbf{x},\mathbf{y})\psi(t,\mathbf{x})d\mathbf{x}\\ &= \left(\frac{1}{2\pi\varepsilon}\right)^n\int_{K_0\times K_0} I(t,\mathbf{y},\mathbf{z},\mathbf{z}')\,d\mathbf{z}\,d\mathbf{z}',\end{aligned} \tag{20}$$

where

$$\begin{aligned}I(t,\mathbf{y},\mathbf{z},\mathbf{z}') = \varepsilon^{2(p+|\boldsymbol{\alpha}|)}\int_{\mathbb{R}^n} &\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(w_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z}))^*\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(w_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}'))\\ &\times g(t,\mathbf{x},\mathbf{y})\psi(t,\mathbf{x})\,d\mathbf{x},\end{aligned} \tag{21}$$

and

$$w_k(t,\mathbf{x},\mathbf{y},\mathbf{z}) = A_k(t,\mathbf{x},\mathbf{y},\mathbf{z})\varrho_\eta(\mathbf{x})e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon}. \tag{22}$$

The following lemma allows us to rewrite I in (21) in terms of $\mathcal{S}_\eta$ functions.

Lemma 4.5. *Let w_k be as in (22). Then for each $k\geq 1$, $p\geq 0$, $\boldsymbol{\alpha}\in\mathbb{N}_0^N$, there exists $s_k\in\mathcal{S}_\eta$ such that*

$$\varepsilon^{p+|\boldsymbol{\alpha}|}\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(w_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})) = s_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z}).$$

Proof. We note that from (10),

$$w_k(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{j=0}^{\lceil\frac{k}{2}\rceil-1}\varepsilon^j\sum_{|\boldsymbol{\beta}|=0}^{k-2j-1}\frac{1}{\boldsymbol{\beta}!}a_{j,\boldsymbol{\beta}}(t,\mathbf{y},\mathbf{z})\varrho_\eta(\mathbf{x})\,\mathbf{x}^{\boldsymbol{\beta}}e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon},$$

and since ϱ_η is supported in $B_{2\eta}$ then $w_k\in\mathcal{S}_\eta$. Let us first differentiate $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(w_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})) = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}w_k(t,\mathbf{x},\mathbf{y},\mathbf{z})\big|_{\mathbf{x}=\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z})}$ and note that by Corollary 1, $r_k := \varepsilon^{|\boldsymbol{\alpha}|}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}w_k\in\mathcal{S}_\eta$. Furthermore, the time derivative of $r_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})$ reads

$$\partial_t\left(r_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})\right) = \partial_t r_k(t,\mathbf{x},\mathbf{y},\mathbf{z}) - \partial_t\mathbf{q}(t,\mathbf{y},\mathbf{z})\cdot\nabla_{\mathbf{x}}r_k(t,\mathbf{x},\mathbf{y},\mathbf{z})\big|_{\mathbf{x}=\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z})}.$$

From points 2, 4 and 6 in Lemma 4.4 and Proposition 3.1, we have that $Fr_k\in\mathcal{S}_\eta$, where F is the operator $F=\varepsilon(\partial_t-\partial_t\mathbf{q}\cdot\nabla_{\mathbf{x}})$. Repeated differentiation of $r_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})$ subject to an appropriate scaling with ε thus yields repeated application of the F operator:

$$\varepsilon^p\partial_t^p\left(r_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})\right) = F^p r_k(t,\mathbf{x},\mathbf{y},\mathbf{z})\big|_{\mathbf{x}=\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z})}.$$

Since $s_k := F^p r_k\in\mathcal{S}_\eta$ the proof is complete. □

The function $s_k\in\mathcal{S}_\eta$ can be rewritten recalling the definition of $\mathcal{S}_\eta$ as $s_k(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{j=0}^{L}\varepsilon^j P_j(t,\mathbf{x},\mathbf{y},\mathbf{z})e^{i\Phi_k(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon}$ with $P_j\in\mathcal{P}_\eta$, for all j. Then using Lemma 4.5, the quantity (21) becomes

$$\begin{aligned}I(t,\mathbf{y},\mathbf{z},\mathbf{z}') &= \int_{\mathbb{R}^n} s_k^*(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})s_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}')g(t,\mathbf{x},\mathbf{y})\psi(t,\mathbf{x})\,d\mathbf{x}\\ &= \sum_{j,\ell=0}^{L}\varepsilon^{j+\ell}\int_{\mathbb{R}^n} h_{j\ell}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')e^{i\Theta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon}\,d\mathbf{x},\end{aligned}$$

where Θ_k is the k-th order GB phase

$$\Theta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') = \Phi_k(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}') - \Phi_k^*(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z}), \tag{23}$$

and

$$h_{j\ell}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') = P_j^*(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})\, P_\ell(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}')g(t,\mathbf{x},\mathbf{y})\psi(t,\mathbf{x}).$$

Let us use the definition of $\mathcal{P}_\eta$ and write $P_j(t,\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{|\boldsymbol{\alpha}|=0}^{M} a_{j,\boldsymbol{\alpha}}(t,\mathbf{x},\mathbf{y},\mathbf{z})\,\mathbf{x}^{\boldsymbol{\alpha}}$, with $\operatorname{supp} a_{j,\boldsymbol{\alpha}}(t,\cdot,\mathbf{y},\mathbf{z}) \subset B_{2\eta}$ for all $j,\boldsymbol{\alpha}$, $t\in\mathbb{R}$, $\mathbf{y}\in\Gamma$, $\mathbf{z}\in\mathbb{R}^n$. We note that

$$h_{j\ell}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') = \sum_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|=0}^{M} c_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')(\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}))^{\boldsymbol{\alpha}}(\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}'))^{\boldsymbol{\beta}},$$

where $c_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') = a_{j,\boldsymbol{\alpha}}^*(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})a_{\ell,\boldsymbol{\beta}}(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}')g(t,\mathbf{x},\mathbf{y})\psi(t,\mathbf{x})$ implying that $c_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathcal{T}_\eta$. To summarize, the quantity (21) can be thus written as

$$I(t,\mathbf{y},\mathbf{z},\mathbf{z}') = \sum_{j,\ell=0}^{L} \varepsilon^{j+\ell} \sum_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|=0}^{M} I_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}}(t,\mathbf{y},\mathbf{z},\mathbf{z}'),$$

with

$$I_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}}(t,\mathbf{y},\mathbf{z},\mathbf{z}') = \int_{\mathbb{R}^n} c_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')(\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}))^{\boldsymbol{\alpha}}(\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}'))^{\boldsymbol{\beta}} e^{i\Theta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon}\, d\mathbf{x},$$

such that $c_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathcal{T}_\eta$. We will now utilize the following theorem.

Theorem 4.6. *Assume that (A1)–(A5) hold. Let η be admissible in the sense of Definition 1 for $T>0$, k and a compact $\Gamma_c \subset \Gamma$. We define*

$$I_0(t,\mathbf{y},\mathbf{z},\mathbf{z}') = \int_{\mathbb{R}^n} f(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')(\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}))^{\boldsymbol{\alpha}}(\mathbf{x}-\mathbf{q}(t,\mathbf{y},\mathbf{z}'))^{\boldsymbol{\beta}} e^{i\Theta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon} d\mathbf{x}, \tag{24}$$

where Θ_k is as in (23) *and $f \in \mathcal{T}_\eta$. Then there exist $C_{\boldsymbol{\sigma},\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that*

$$\sup_{\substack{\mathbf{y}\in\Gamma_c\\ t\in[0,T]}} \left(\frac{1}{2\pi\varepsilon}\right)^n \int_{K_0\times K_0} \left|\partial_{\mathbf{y}}^{\boldsymbol{\sigma}} I_0(t,\mathbf{y},\mathbf{z},\mathbf{z}')\right| d\mathbf{z}\, d\mathbf{z}' \le C_{\boldsymbol{\sigma},\boldsymbol{\alpha},\boldsymbol{\beta}},$$

for all $\boldsymbol{\sigma}\in\mathbb{N}_0^N$ and $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}_0^n$, where $C_{\boldsymbol{\sigma},\boldsymbol{\alpha},\boldsymbol{\beta}}$ is independent of ε but depends on T, k and Γ_c.

Proof. The proof is essentially the same as the proof of Theorem 1 in [6]. We include it in the Appendix. □

Since $I_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}}$ is of the form (24), we can use Theorem 4.6 (replacing the constant $C_{\boldsymbol{\sigma},\boldsymbol{\alpha},\boldsymbol{\beta}}$ with $C_{\boldsymbol{\sigma},j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}}$ to illustrate its dependence on j and ℓ as well). Then recalling (20) and (A4) we get

$$\begin{aligned}
\sup_{\substack{\mathbf{y}\in\Gamma_c\\ t\in[0,T]}} \left|\frac{\partial^{\boldsymbol{\sigma}} \widetilde{\mathcal{Q}}_{\mathrm{GB}}^{p,\boldsymbol{\alpha}}(t,\mathbf{y})}{\partial \mathbf{y}^{\boldsymbol{\sigma}}}\right| &\le \sup_{\substack{\mathbf{y}\in\Gamma_c\\ t\in[0,T]}} \left(\frac{1}{2\pi\varepsilon}\right)^n \int_{K_0\times K_0} \left|\frac{\partial^{\boldsymbol{\sigma}} I(t,\mathbf{y},\mathbf{z},\mathbf{z}')}{\partial \mathbf{y}^{\boldsymbol{\sigma}}}\right| d\mathbf{z}\, d\mathbf{z}' \\
&\le \sup_{\substack{\mathbf{y}\in\Gamma_c\\ t\in[0,T]}} \left(\frac{1}{2\pi\varepsilon}\right)^n \sum_{j,\ell=0}^{L} \varepsilon^{j+\ell} \sum_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|=0}^{M} \int_{K_0\times K_0} \left|\frac{\partial^{\boldsymbol{\sigma}} I_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}}(t,\mathbf{y},\mathbf{z},\mathbf{z}')}{\partial \mathbf{y}^{\boldsymbol{\sigma}}}\right| d\mathbf{z}\, d\mathbf{z}' \\
&\le \tilde{C} \sup_{j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}} C_{\boldsymbol{\sigma},j,\ell,\boldsymbol{\alpha},\boldsymbol{\beta}} \\
&\le C_{\boldsymbol{\sigma}},
\end{aligned}$$

where $C_{\boldsymbol{\sigma}}$ depends on $\eta, T, k, \Gamma_c, L, M$, but is independent of ε, for all $\boldsymbol{\sigma}\in\mathbb{N}_0^N$. This concludes the proof of Theorem 4.2.

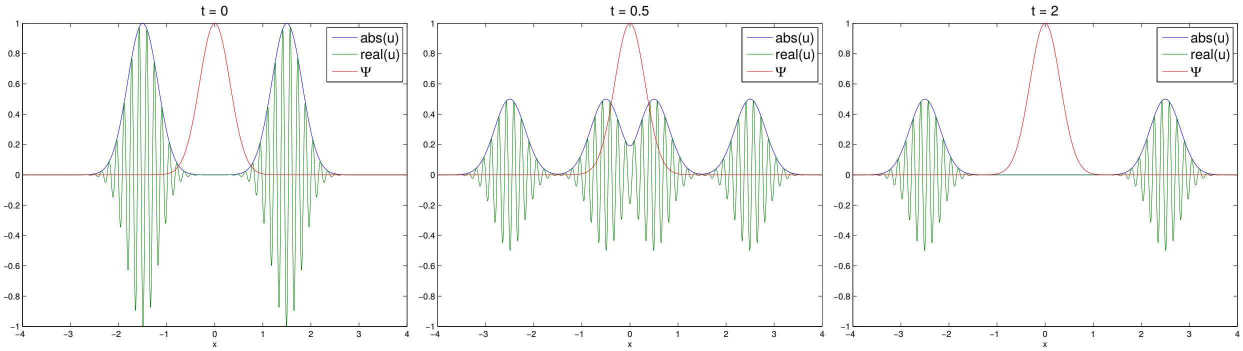

Figure 1: d'Alembert solution with initial data (25) and (27).

5 Two-mode quantity of interest

Let us consider a wave superposed of both forward and backward propagating modes as defined in (15). In this case, Theorem 4.2 for the QoI (17) is no longer necessarily true as $\widetilde{\mathcal{Q}}_{\mathrm{GB}}^{p,\boldsymbol{\alpha}}$ can be highly oscillatory. We will therefore have to look at a slightly different QoI where the averaging is also done in time, not just in space.

5.1 What could go wrong?

Since $\widetilde{\mathcal{Q}}_{\mathrm{GB}}$ in (17) is a good approximation of $\widetilde{\mathcal{Q}}$ in (6), it is oscillatory if and only if the other one is. We will first show a simple example where $\widetilde{\mathcal{Q}}$ in (3) is oscillatory. Let us consider a 1D case with spatially constant speed $c(x,y) = c(y)$. The initial data to (1),

$$u^\varepsilon(0,x,y) = B_0(x,y)e^{i\varphi_0(x,y)/\varepsilon}, \qquad u_t^\varepsilon(0,x,y) = 0, \tag{25}$$

generate the d'Alembert solution

$$u^\varepsilon(t,x,y) = u^+(t,x,y) + u^-(t,x,y), \qquad u^\pm(t,x,y) = \frac{1}{2}B_0(x \mp c(y)t, y)e^{i\varphi_0(x\mp c(y)t,y)/\varepsilon}. \tag{26}$$

The QoI (3) therefore reads

$$\begin{aligned}
\widetilde{\mathcal{Q}}(t,y) &= \int_{\mathbb{R}} |u^+(t,x,y) + u^-(t,x,y)|^2 \psi(t,x)\,dx \\
&= \int_{\mathbb{R}} \left(|u^+(t,x,y)|^2 + |u^-(t,x,y)|^2 + 2\,\mathrm{Re}(u^+(t,x,y)^* u^-(t,x,y))\right)\psi(t,x)\,dx \\
&=: \widetilde{Q}_+(t,y) + \widetilde{Q}_-(t,y) + \widetilde{Q}_0(t,y).
\end{aligned}$$

The first two terms of $\widetilde{\mathcal{Q}}$ yield

$$\widetilde{Q}_\pm(t,y) = \int_{\mathbb{R}} |u^\pm(t,x,y)|^2\psi(t,x)\,dx = \frac{1}{4}\int_{\mathbb{R}} B_0^2(x \mp c(y)t, y)\psi(t,x)\,dx,$$

where the integrand is smooth, compactly supported and independent of ε, including all its derivatives in y. Therefore, the terms $\widetilde{Q}_\pm$ satisfy Theorem 4.2. The last term $\widetilde{Q}_0$ reads

$$\widetilde{Q}_0(t,y) = \frac{1}{2}\int_{\mathbb{R}} \cos\left(\frac{\varphi(t,x,y)}{\varepsilon}\right) B_0(x + c(y)t, y)B_0(x - c(y)t, y)\psi(t,x)\,dx,$$

where $\varphi(t,x,y) := \varphi_0(x + c(y)t, y) - \varphi_0(x - c(y)t, y)$.
The $\widetilde{Q}_0$ could conceivably be problematic, depending on the choice of B_0 and φ_0. Notably, the selection

$$B_0(x,y) = e^{-5(x+s)^2} + e^{-5(x-s)^2}, \qquad \varphi_0(x,y) = x, \qquad \psi(t,x) = e^{-5x^2}, \tag{27}$$

produces two symmetric pulses centered at $\pm s$, each splitting into two waves traveling in opposite directions, see Figure 1 where we set $s = 1.5$ and $c = 2$. The test function ψ is compactly supported in x for numerical purposes. Let us also choose the speed $c(y) = y$ be the stochastic variable. Then $\varphi(t,x,y) = 2yt$ and $\widetilde{Q}_0$ includes an oscillatory prefactor $\cos\left(\frac{2yt}{\varepsilon}\right)$ that does not depend on x and hence cannot be damped by the test function ψ. Moreover, an $\varepsilon^{-\sigma}$ term is produced when differentiating $\partial_y^\sigma \widetilde{\mathcal{Q}}(t,y)$. Thus $\widetilde{\mathcal{Q}}$ does not satisfy Theorem 4.2.
The QoI (3) along with its first and second derivative in y is depicted in Figure 2, left column, for varying $\varepsilon = (1/40, 1/80, 1/160)$. The plots display oscillations of growing amplitude with increasing σ and decreasing ε as predicted. Here, we chose $y \in [1.5, 2]$, $s = 3$ and $t = 2$. In general, for odd-order polynomial φ_0, there is a cosine prefactor including a constant in x term in $\widetilde{Q}_0$ which induces oscillations in ε of the QoI (3).

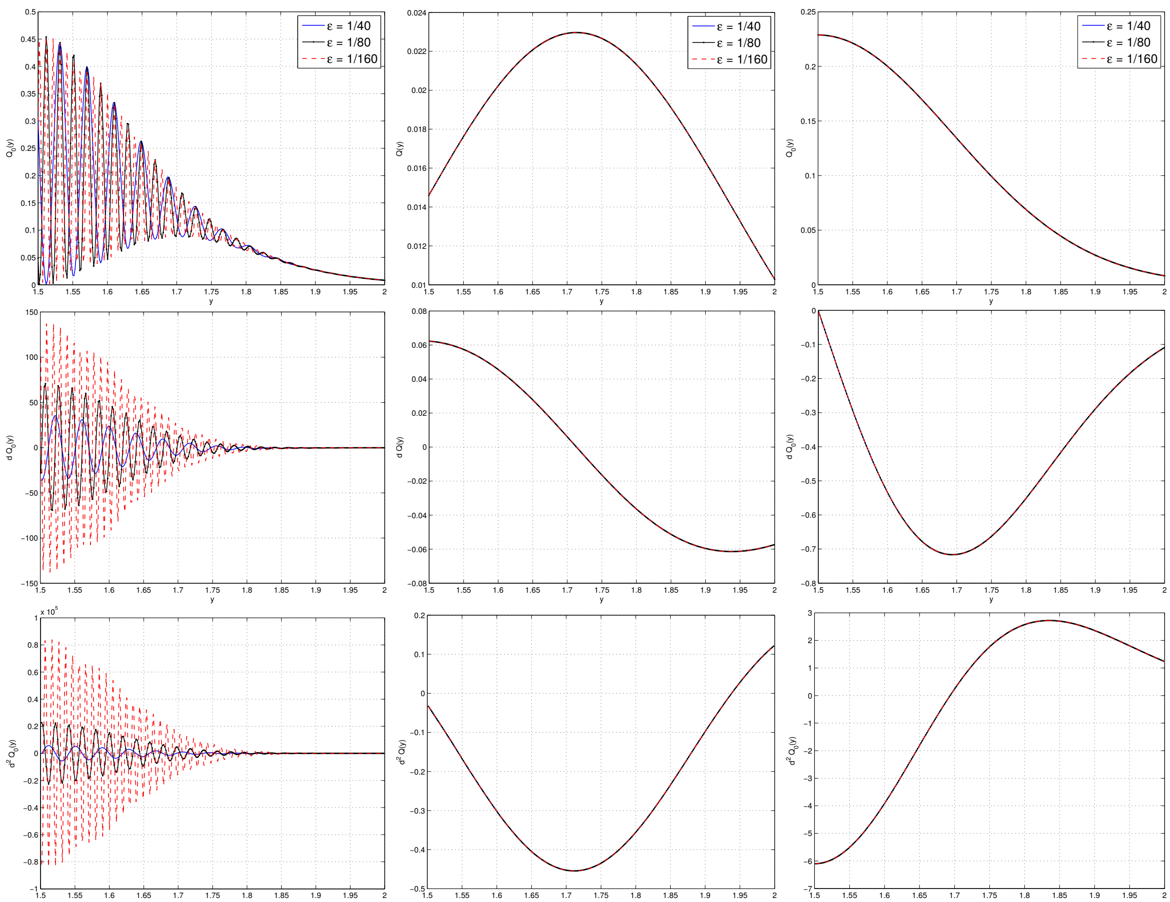

Figure 2: Left column: QoI (3) with $\varphi_0(x,y) = x$, and its first and second derivative in y. Central column: QoI (3) with $\varphi_0(x,y) = x^2$. Right column: QoI (4) with $\varphi_0(x,y) = x$.

Note that even-order in x polynomials φ_0 are not oscillatory for the example above. For instance, $\varphi_0(x,y) = x^2$ gives $\varphi(t,x,y) = 4xyt$. By the non-stationary phase lemma, for all compact $\Gamma_c \subset \Gamma$

there exist c_s independent of ε such that

$$\sup_{\substack{y\in\Gamma_c\\ t\in[0,T]}} \left| \int_{\mathbb{R}} \cos\left(\frac{4xyt}{\varepsilon}\right) B_0(x+yt,y)B_0(x-yt,y)\psi(x)\,dx \right| \leq c_s\varepsilon^s,$$

for all s as $\varepsilon \to 0$, and the same holds for its derivatives with respect to y.
The QoI (3) for $\varphi_0(x,y) = x^2$ and its first and second derivatives in y are plotted in Figure 2, central column, utilizing the same parameters as the previous example. No oscillations can be observed in the plot.
The different behavior of $\varphi_0(x,y) = x$ and $\varphi_0(x,y) = x^2$ in (27) does not come as a surprise if one looks at the GB approximation (19) of (3). Note that the left-going wave u^- in (26) is approximated solely by u_k^- in (12). This is because all GBs v_k^- in (8) move along the rays (q^-, p^-) whose initial data are $q^-(0,y,z) = z$ and $p^-(0,y,z) = 1$ by (14). From (11) this implies that $p^-(t,y,z) = 1$ and $q^-(t,y,z) = -yt+z$. Finally, as $y > 0$ we therefore have $q^- \to -\infty$ irrespectively of the starting point z; in other words all v_k^- move to the left. Similarly, u^+ is approximated merely by u_k^+. Therefore, the waves moving towards the origin (where the test function is supported) are from two different GB families. Hence, as conjectured, a two-mode solution can give highly oscillatory QoIs.
In contrast, for $\varphi_0(x,y) = x^2$ we obtain $p^\pm(0,y,z) = p^\pm(t,y,z) = 2z$ and hence $q^\pm(t,y,z) = \pm y\frac{z}{|z|}t + z$. Therefore, both q^+ and q^- can move in either direction depending on the starting point z. For our example, this implies that the two waves moving towards the origin belong to the same GB mode, u_k^-, and the two waves moving away belong to u_k^+. Since the test function ψ is compactly supported around the origin, only u_k^- will substantially contribute to the QoI (19). Finally, by Theorem 4.3, the QoI (19) consisting of one GB mode solution is regular.

Remark. Generally, a phase $\varphi_0 = \varphi_0(x)$ whose derivative changes sign on $\mathbb{R}$ allows for two waves approximated by the same mode moving in two different directions. In particular, this is true for even-order polynomials. Technically, φ_0 is not allowed to attain local extrema due to (A3). In practice however, it is enough to make sure that the support of B_0 and B_1 does not include the stationary point.

5.2 New quantity of interest

To evade the oscillatory behavior, we introduce a new QoI (4) consisting of (3) integrated not only in $\mathbf{x}$ but also in time t, with $\psi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^n)$. Let us first apply this QoI to the 1D oscillatory example from Section 5.1 with $\varphi_0(x,y) = x$,

$$\begin{aligned}
\mathcal{Q}(y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} |u^+(t,x,y) + u^-(t,x,y)|^2\psi(t,x)\,dx\,dt,\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(|u^+(t,x,y)|^2 + |u^-(t,x,y)|^2 + 2\,\mathrm{Re}(u^+(t,x,y)^* u^-(t,x,y))\right)\psi(t,x)\,dx\,dt\\
&=: Q_+(y) + Q_-(y) + Q_0(y).
\end{aligned}$$

Again, the first two terms yield

$$Q_\pm(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} |u^\pm(t,x,y)|^2\psi(t,x)\,dx\,dt = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} B_0^2(x\mp yt,y)\psi(t,x)\,dx\,dt,$$

where the integrand is smooth, compactly supported in both t and x and independent of ε, including all its derivatives in y. The last term reads

$$Q_0(y) = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \cos\left(\frac{2yt}{\varepsilon}\right) B_0(x+yt,y)B_0(x-yt,y)\psi(t,x)\,dx\,dt,$$

and since the phase of $\cos\left(\frac{2yt}{\varepsilon}\right)$ has no stationary point in t, we can utilize the non-stationary phase lemma in t. Due to ψ being compactly supported in both t and x, we obtain the desired regularity: for all compact $\Gamma_c \subset \Gamma$, $\sup_{y\in\Gamma_c} |\mathcal{Q}(y)| \leq c_s\varepsilon^s$ for all s as $\varepsilon \to 0$, where c_s is independent of ε and similarly for differentiation in y.
To confirm this numerically, we use the initial data from the previous section and set

$$\psi(t,x) = e^{-5x^2-300(t-t_s)^2},$$

where $t_s = 1.75$. The right column of Figure 2 shows the QoI (4) and its first and second derivatives with respect to y for $\varepsilon = (1/40, 1/80, 1/160)$. The oscillations are apparently eliminated and the QoI is independent of the wavelength.

5.3 Stochastic regularity of $\mathcal{Q}^{p,\boldsymbol{\alpha}}$

We consider the QoI $\mathcal{Q}^{p,\boldsymbol{\alpha}}$ in (2) with ψ as in (A5). Let us define its GB approximated version as

$$\mathcal{Q}^{p,\boldsymbol{\alpha}}_{\mathrm{GB}}(\mathbf{y}) = \varepsilon^{2(p+|\boldsymbol{\alpha}|)} \int_{\mathbb{R}}\int_{\mathbb{R}^n} g(t,\mathbf{x},\mathbf{y})|\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u_k(t,\mathbf{x},\mathbf{y})|^2\psi(t,\mathbf{x})d\mathbf{x}\,dt. \tag{28}$$

As before, we assume that u_k is a good approximation to u^ε and hence so is $\mathcal{Q}^{p,\boldsymbol{\alpha}}_{\mathrm{GB}}$ to $\mathcal{Q}^{p,\boldsymbol{\alpha}}$. We start off by defining the admissible cutoff parameter for the case of two-mode solutions.

Proposition 5.1. *Let us assume (A1)–(A3) hold. For all $T > 0$, beam order k and compact $\Gamma_c \subset \Gamma$, there is a GB cutoff width $\eta > 0$ and constant $\delta > 0$ such that for all $\mathbf{x} \in B_{2\eta}$,*

$$\operatorname{Im}\Phi_k^{\pm}(t,\mathbf{x},\mathbf{y},\mathbf{z}) \geq \delta|\mathbf{x}|^2, \quad \forall t \in [0,T],\ \mathbf{y} \in \Gamma_c,\ \mathbf{z} \in K_0. \tag{29}$$

Proof. By Proposition 4.1, for every Γ_c there exist $\delta^+ > 0$ and $\eta^+ > 0$ such that for all $\mathbf{x} \in B_{2\eta^+}$ we have $\operatorname{Im}\Phi_k^+(t,\mathbf{x},\mathbf{y},\mathbf{z}) \geq \delta^+|\mathbf{x}|^2$, and analogously for $\operatorname{Im}\Phi_k^-$ with δ^- and η^-. Then choosing $\delta = \min\{\delta^+,\delta^-\}$ and $\eta = \min\{\eta^+,\eta^-\}$ yields the relation (29) for all $\mathbf{x} \in B_{2\eta}$. □

Definition 2. The cutoff width η used for the GB approximation is called admissible for a given T, k and Γ_c if it is small enough in the sense of Proposition 5.1.

Remark. As in Section 4.2, we assume that $\eta < \infty$ without loss of generality. We note that also for the two-mode solutions, the Gaussian beam superposition (15) with no cutoff is identical to the one with a large enough cutoff, because of the compact support of the test function $\psi(t,\mathbf{x})$.

Let us now prove that QoI (28) is indeed non-oscillatory.

Theorem 5.2. *Let us assume that (A1)–(A5) hold. Moreover, let η be admissible in the sense of Definition 2 for $T > 0$, k and a compact $\Gamma_c \subset \Gamma$. Then there exist $C_{\boldsymbol{\sigma}}$ such that*

$$\sup_{\mathbf{y}\in\Gamma_c}\left|\frac{\partial^{\boldsymbol{\sigma}}\widetilde{\mathcal{Q}}^{p,\boldsymbol{\alpha}}_{GB}(\mathbf{y})}{\partial\mathbf{y}^{\boldsymbol{\sigma}}}\right| \leq C_{\boldsymbol{\sigma}}, \quad \forall\boldsymbol{\sigma} \in \mathbb{N}_0^N,$$

where $C_{\boldsymbol{\sigma}}$ is independent of ε but depends on T, k and Γ_c.

In this section, we will use the following notation: let $\Sigma_\mu, \mathcal{W}_\mu$, for $\mu < \infty$, denote the spaces

- $\Sigma_\mu = \Sigma_\mu(t,\mathbf{y},\mathbf{z},\mathbf{z}') = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{q}^+(t,\mathbf{y},\mathbf{z})| \leq 2\mu \quad \text{and} \quad |\mathbf{x} - \mathbf{q}^-(t,\mathbf{y},\mathbf{z}')| \leq 2\mu\}$.

- $\mathcal{W}_\mu = \{f \in \mathcal{Y} : \operatorname{supp} f(t,\cdot,\mathbf{y},\mathbf{z},\mathbf{z}') \subset \Sigma_\mu(t,\mathbf{y},\mathbf{z},\mathbf{z}'),\ \forall t \in \mathbb{R},\ \mathbf{y} \in \Gamma,\ \mathbf{z},\mathbf{z}' \in \mathbb{R}^n\}$.

The space Σ_μ is similar to Λ_μ introduced in Section 4.3. Instead of containing $\mathbf{x}$ that are close enough to two beams from the same mode, it contains $\mathbf{x}$ that lie at a distance at most 2μ from two beams from different modes. We also note that there exist two spaces $\mathcal{S}_\mu^\pm$ as defined in Section 4.2 since we have two modes of $\Phi_k^\pm$ and that Lemma 4.4 holds for both.
For the remainder of this section we fix the compact $\Gamma_c \subset \Gamma$ and select $\eta < \infty$ admissible in the sense of Definition 2. The proof of Theorem 5.2 uses the following two lemmas.

Lemma 5.3 (Non-stationary phase lemma)**.** *Suppose $\Theta \in C^\infty(\mathbb{R})$ and $f \in C_c^\infty(\mathbb{R})$ with supp$f \subset [0,T]$. If $\partial_t\Theta(t) \neq 0$ for all $t \in [0,T]$ then the following estimate holds true for all $K \in \mathbb{N}_0$,*

$$\left|\int_{\mathbb{R}} f(t)e^{i\Theta(t)/\varepsilon}dt\right| \leq C_K(1+\|\Theta\|_{C^{K+1}([0,T])})^K\varepsilon^K \sum_{m\leq K}\int_{\mathbb{R}} \frac{|\partial_t^m f(t)|}{|\partial_t\Theta(t)|^{2K-m}}e^{-\operatorname{Im}\Theta(t)/\varepsilon}\,dt,$$

where C_K depends on K but is independent of ε, f, Θ, T, and

$$\|\Theta\|_{C^{K+1}([0,T])} = \sum_{k=0}^{K+1}\sup_{t\in[0,T]}\left|\Theta^{(k)}(t)\right|.$$

Lemma 5.4. *Define*

$$I(\mathbf{y},\mathbf{u}) = f(\mathbf{y},\mathbf{u})e^{i\Theta(\mathbf{y},\mathbf{u})/\varepsilon},$$

for $f,\Theta \in C^\infty(\Gamma\times\mathbb{R}^d)$, where supp $f(\mathbf{y},\cdot) \subset D \subseteq \mathbb{R}^d$, $\forall \mathbf{y} \in \Gamma$. Then there exist functions $f_{j\boldsymbol{\sigma}} \in C^\infty(\Gamma\times\mathbb{R}^d)$ with supp $f_{j\boldsymbol{\sigma}}(\mathbf{y},\cdot) \subset D$, $\forall \mathbf{y} \in \Gamma$ such that,

$$\frac{\partial^{\boldsymbol{\sigma}} I(\mathbf{y},\mathbf{u})}{\partial \mathbf{y}^{\boldsymbol{\sigma}}} = \sum_{j=0}^{|\boldsymbol{\sigma}|}\varepsilon^{-j}f_{j\boldsymbol{\sigma}}(\mathbf{y},\mathbf{u})e^{i\Theta(\mathbf{y},\mathbf{u})/\varepsilon}. \tag{30}$$

Proof. We will carry out the proof by induction. For $\boldsymbol{\sigma} = \mathbf{0}$, we choose $f_{\mathbf{00}} = f$ and the lemma holds. Let us assume (30) is true for a fixed $\boldsymbol{\sigma}$. Then for $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \mathbf{e}_k$ where $\mathbf{e}_k$ is the k-th unit vector we have

$$\begin{aligned}\frac{\partial^{\tilde{\boldsymbol{\sigma}}} I(\mathbf{y},\mathbf{u})}{\partial \mathbf{y}^{\tilde{\boldsymbol{\sigma}}}} &= \frac{\partial}{\partial y_k}\sum_{j=0}^{|\boldsymbol{\sigma}|}\varepsilon^{-j}f_{j\boldsymbol{\sigma}}(\mathbf{y},\mathbf{u})e^{i\Theta(\mathbf{y},\mathbf{u})/\varepsilon}\\ &= \sum_{j=0}^{|\boldsymbol{\sigma}|}\varepsilon^{-j}\left(\frac{\partial f_{j\boldsymbol{\sigma}}(\mathbf{y},\mathbf{u})}{\partial y_k} + f_{j\boldsymbol{\sigma}}(\mathbf{y},\mathbf{u})\frac{i}{\varepsilon}\frac{\partial\Theta(\mathbf{y},\mathbf{u})}{\partial y_k}\right)e^{i\Theta(\mathbf{y},\mathbf{u})/\varepsilon}.\end{aligned}$$

Hence we can take

$$f_{j\tilde{\boldsymbol{\sigma}}} = \begin{cases}\frac{\partial f_{0\boldsymbol{\sigma}}}{\partial y_k}, & j=0,\\ \frac{\partial f_{j\boldsymbol{\sigma}}}{\partial y_k} + if_{j-1\boldsymbol{\sigma}}\frac{\partial\Theta}{\partial y_k}, & 1\leq j\leq|\tilde{\boldsymbol{\sigma}}|-1,\\ if_{j-1\boldsymbol{\sigma}}\frac{\partial\Theta}{\partial y_k}, & j=|\tilde{\boldsymbol{\sigma}}|.\end{cases}$$

Clearly, we have $f_{j\tilde{\boldsymbol{\sigma}}} \in C^\infty(\Gamma\times\mathbb{R}^d)$ with supp $f_{j\tilde{\boldsymbol{\sigma}}}(\mathbf{y},\cdot) \subset D$ for all $\mathbf{y} \in \Gamma$, hence the proof is complete. ☐

Recalling the definition of u_k in (15), $\mathcal{Q}_{\text{GB}}^{p,\boldsymbol{\alpha}}$ in (28) becomes

$$\begin{aligned}\mathcal{Q}_{\text{GB}}^{p,\boldsymbol{\alpha}}(\mathbf{y}) &= \varepsilon^{2(p+|\boldsymbol{\alpha}|)}\int_{\mathbb{R}}\int_{\mathbb{R}^n} g(t,\mathbf{x},\mathbf{y})\left|\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u_k^+(t,\mathbf{x},\mathbf{y})+\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u_k^-(t,\mathbf{x},\mathbf{y})\right|^2\,\psi(t,\mathbf{x})d\mathbf{x}\,dt\\ &= \varepsilon^{2(p+|\boldsymbol{\alpha}|)}\int_{\mathbb{R}}\int_{\mathbb{R}^n} g(t,\mathbf{x},\mathbf{y})[|\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u_k^+(t,\mathbf{x},\mathbf{y})|^2+|\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u_k^-(t,\mathbf{x},\mathbf{y})|^2\\ &\qquad\qquad +2\,\text{Re}(\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u_k^+(t,\mathbf{x},\mathbf{y})^*\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u_k^-(t,\mathbf{x},\mathbf{y}))]\,\psi(t,\mathbf{x})d\mathbf{x}\,dt\\ &=: Q_1(\mathbf{y})+Q_2(\mathbf{y})+2\,\text{Re}(Q_3(\mathbf{y})),\end{aligned}\tag{31}$$

where $\psi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^n)$ is as in (A5) and $g\in C^\infty(\mathbb{R}\times\mathbb{R}^n\times\Gamma)$. The first two terms of (31), Q_1 and Q_2, possess the required stochastic regularity as a consequence of Theorem 4.2,

$$\sup_{\mathbf{y}\in\Gamma_c}\left|\partial_{\mathbf{y}}^{\boldsymbol{\sigma}}Q_1(\mathbf{y})\right| \le \int_0^T \sup_{\substack{\mathbf{y}\in\Gamma_c\\ t\in[0,T]}}\left|\partial_{\mathbf{y}}^{\boldsymbol{\sigma}}\widetilde{\mathcal{Q}}_{\text{GB}}^{p,\boldsymbol{\alpha}}(t,\mathbf{y})\right|\,dt \le TC_{\boldsymbol{\sigma}},\tag{32}$$

and analogously for Q_2.

We will now prove that Q_3 satisfies the same regularity condition owing to the absence of stationary points of the phase. Let us examine the quantity

$$\begin{aligned}\partial_{\mathbf{y}}^{\boldsymbol{\sigma}}Q_3(\mathbf{y}) &= \varepsilon^{2(p+|\boldsymbol{\alpha}|)}\partial_{\mathbf{y}}^{\boldsymbol{\sigma}}\int_{\mathbb{R}}\int_{\mathbb{R}^n} g(t,\mathbf{x},\mathbf{y})\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u_k^+(t,\mathbf{x},\mathbf{y})^*\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u_k^-(t,\mathbf{x},\mathbf{y})\,\psi(t,\mathbf{x})d\mathbf{x}\,dt\\ &= \left(\frac{1}{2\pi\varepsilon}\right)^n\int_{K_0\times K_0}\int_{K_1}\partial_{\mathbf{y}}^{\boldsymbol{\sigma}}I(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')\,d\mathbf{x}\,d\mathbf{z}\,d\mathbf{z}',\end{aligned}\tag{33}$$

where

$$\begin{aligned}I(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') = \varepsilon^{2(p+|\boldsymbol{\alpha}|)}\int_{\mathbb{R}}&\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}w_k^+(t,\mathbf{x}-\mathbf{q}^+(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})^*\partial_t^p\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}w_k^-(t,\mathbf{x}-\mathbf{q}^-(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}')\\ &\times g(t,\mathbf{x},\mathbf{y})\psi(t,\mathbf{x})\,dt,\end{aligned}$$

with

$$w_k^\pm(t,\mathbf{x},\mathbf{y},\mathbf{z}) = A_k^\pm(t,\mathbf{x},\mathbf{y},\mathbf{z})\varrho_\eta(\mathbf{x})e^{i\Phi_k^\pm(t,\mathbf{x},\mathbf{y},\mathbf{z})/\varepsilon}.$$

Recalling Lemma 4.5, we can find $s_k^\pm\in\mathcal{S}_\eta^\pm$ such that

$$\begin{aligned}I(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') &= \int_{\mathbb{R}} s_k^+(t,\mathbf{x}-\mathbf{q}^+(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})^*s_k^-(t,\mathbf{x}-\mathbf{q}^-(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}')g(t,\mathbf{x},\mathbf{y})\psi(t,\mathbf{x})\,dt\\ &= \sum_{\ell=0}^{L_1}\sum_{m=0}^{L_2}\varepsilon^{\ell+m}\int_{\mathbb{R}} a_{\ell m}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')\psi(t,\mathbf{x})e^{i\vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon}\,dt,\end{aligned}$$

where

$$a_{\ell m}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') = g(t,\mathbf{x},\mathbf{y})P_\ell^+(t,\mathbf{x}-\mathbf{q}^+(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})^*P_m^-(t,\mathbf{x}-\mathbf{q}^-(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}'),$$

with $P_\ell^+, P_m^-\in\mathcal{P}_\eta$, and

$$\vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') = \Phi_k^-(t,\mathbf{x}-\mathbf{q}^-(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}') - \Phi_k^+(t,\mathbf{x}-\mathbf{q}^+(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z})^*.\tag{34}$$

By Proposition 3.1, we have $\vartheta_k\in\mathcal{Y}$, and $a_{\ell m}\in\mathcal{W}_\eta$ because both P_ℓ^+, P_m^- are supported in the ball $B_{2\eta}$. Therefore, by Lemma 5.4, there exist functions $f_{\ell mj\boldsymbol{\sigma}}\in\mathcal{W}_\eta$ such that

$$\partial_{\mathbf{y}}^{\boldsymbol{\sigma}}I(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') = \sum_{j=0}^{|\boldsymbol{\sigma}|}\sum_{\ell=0}^{L_1}\sum_{m=0}^{L_2}\varepsilon^{\ell+m-j}\int_{\mathbb{R}} f_{\ell mj\boldsymbol{\sigma}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')\psi(t,\mathbf{x})e^{i\vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon}dt.\tag{35}$$

The following proposition shows that ϑ_k has no stationary points in $t \in [0,T]$ for all $\mathbf{x} \in \Sigma_\mu$ with a small enough μ. Note that this is true even for $\mathbf{z} = \mathbf{z}'$.

Proposition 5.5. *There exist $0 < \mu \leq 1$ and $\nu > 0$ such that for all $\mathbf{y} \in \Gamma_c$, $\mathbf{z} \in K_0$, $\mathbf{z}' \in K_0$, $t \in [0,T]$ and for all $\mathbf{x} \in \Sigma_\mu$,*

$$|\partial_t \vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')| \geq \nu. \tag{36}$$

Proof. Differentiating (34) with respect to t and using (9) and (11), we obtain

$$\partial_t \vartheta_k = -\partial_t \mathbf{q}^- \cdot \mathbf{p}^- + \partial_t \mathbf{q}^+ \cdot \mathbf{p}^+ + R_k = -c(\mathbf{q}^-,\mathbf{y})|\mathbf{p}^-| - c(\mathbf{q}^+,\mathbf{y})|\mathbf{p}^+| + R_k, \tag{37}$$

where $R_k = R_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')$ reads

$$\begin{aligned} R_k &= (\mathbf{x}-\mathbf{q}^-)\cdot\partial_t\mathbf{p}^- - (\mathbf{x}-\mathbf{q}^+)\cdot\partial_t\mathbf{p}^+ - \partial_t\mathbf{q}^-\cdot M^-(\mathbf{x}-\mathbf{q}^-) + \partial_t\mathbf{q}^+\cdot(M^+)^*(\mathbf{x}-\mathbf{q}^+) \\ &+ \frac{1}{2}(\mathbf{x}-\mathbf{q}^-)\cdot\partial_t M^-(\mathbf{x}-\mathbf{q}^-) + \frac{1}{2}(\mathbf{x}-\mathbf{q}^+)\cdot(\partial_t M^+)^*(\mathbf{x}-\mathbf{q}^+) \\ &+ \sum_{|\boldsymbol{\beta}|=3}^{k+1}\frac{1}{\boldsymbol{\beta}!}\left(\partial_t\phi^-_{\boldsymbol{\beta}}(\mathbf{x}-\mathbf{q}^-)^{\boldsymbol{\beta}} + \phi^-_{\boldsymbol{\beta}}\partial_t(\mathbf{x}-\mathbf{q}^-)^{\boldsymbol{\beta}}\right) - \sum_{|\boldsymbol{\beta}|=3}^{k+1}\frac{1}{\boldsymbol{\beta}!}\left(\partial_t\phi^+_{\boldsymbol{\beta}}(\mathbf{x}-\mathbf{q}^+)^{\boldsymbol{\beta}} + \phi^+_{\boldsymbol{\beta}}\partial_t(\mathbf{x}-\mathbf{q}^+)^{\boldsymbol{\beta}}\right)^*. \end{aligned}$$

Since $\mathbf{q}^\pm, \mathbf{p}^\pm, M^\pm, \phi^\pm_{\boldsymbol{\beta}}$ are smooth in all variables by Proposition 3.1, their time derivative is uniformly bounded in the compact set $[0,T]\times\Gamma_c\times K_0$. If $\mathbf{x} \in \Sigma_\mu$ for some $0 < \mu \leq 1$, then both $|\mathbf{x}-\mathbf{q}^-| \leq 2\mu$ and $|\mathbf{x}-\mathbf{q}^+| \leq 2\mu$ and we arrive at

$$|R_k| \leq C_k\mu,$$

with C_k depending on Γ_c, T, k, but independent of μ.
Next, we note that $H(\mathbf{p}^+,\mathbf{q}^+,\mathbf{y}) = c(\mathbf{q}^+,\mathbf{y})|\mathbf{p}^+|$ is conserved along the ray,

$$c(\mathbf{q}^+(t,\mathbf{y},\mathbf{z}),\mathbf{y})|\mathbf{p}^+(t,\mathbf{y},\mathbf{z})| = c(\mathbf{q}^+(0,\mathbf{y},\mathbf{z}),\mathbf{y})|\mathbf{p}^+(0,\mathbf{y},\mathbf{z})| = c(\mathbf{z},\mathbf{y})|\nabla\varphi_0(\mathbf{z},\mathbf{y})|,$$

and therefore by (A1) and (A3) we obtain a uniform lower bound on $c(\mathbf{q}^+,\mathbf{y})|\mathbf{p}^+|$, for all $t \in \mathbb{R}$, $\mathbf{y} \in \Gamma_c$ and $\mathbf{z} \in K_0$,

$$c(\mathbf{q}^+(t,\mathbf{y},\mathbf{z}),\mathbf{y})|\mathbf{p}^+(t,\mathbf{y},\mathbf{z})| \geq c_{\min}\inf_{\substack{\mathbf{z}\in K_0\\ \mathbf{y}\in\Gamma_c}}|\nabla\varphi_0(\mathbf{z},\mathbf{y})| \geq \gamma > 0,$$

and similarly, from the conservation of $H(\mathbf{p}^-,\mathbf{q}^-,\mathbf{y})$ we obtain $c(\mathbf{q}^-(t,\mathbf{y},\mathbf{z}'),\mathbf{y})|\mathbf{p}^-| \geq \gamma > 0$. Thus from (37) we get

$$|\partial_t\vartheta_k| \geq c(\mathbf{q}^-,\mathbf{y})|\mathbf{p}^-| + c(\mathbf{q}^+,\mathbf{y})|\mathbf{p}^+| - |R_k| \geq 2\gamma - C_k\mu \geq \nu > 0,$$

for all $\mathbf{x} \in \Sigma_\mu$ with a small enough μ. □

We are now ready to prove the main theorem.

5.4 Proof of Theorem 5.2

Let us first choose $0 < \mu \leq \eta$ such that Proposition 5.5 holds. Furthermore, note that the admissibility condition implies that for all $\mathbf{x}$ satisfying $|\mathbf{x}-\mathbf{q}^\pm| \leq 2\eta$ we have $\operatorname{Im}\Phi^\pm_k(t,\mathbf{x}-\mathbf{q}^\pm,\mathbf{y},\mathbf{z}) \geq \delta|\mathbf{x}-\mathbf{q}^\pm|^2$. We can therefore express $\operatorname{Im}\vartheta_k$ with ϑ_k as in (34) as

$$\begin{aligned} \operatorname{Im}\vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') &= \operatorname{Im}\Phi^-_k(t,\mathbf{x}-\mathbf{q}^-(t,\mathbf{y},\mathbf{z}'),\mathbf{y},\mathbf{z}') + \operatorname{Im}\Phi^+_k(t,\mathbf{x}-\mathbf{q}^+(t,\mathbf{y},\mathbf{z}),\mathbf{y},\mathbf{z}) \\ &\geq \delta|\mathbf{x}-\mathbf{q}^-(t,\mathbf{y},\mathbf{z}')|^2 + \delta|\mathbf{x}-\mathbf{q}^+(t,\mathbf{y},\mathbf{z})|^2, \end{aligned} \tag{38}$$

for all $\mathbf{x} \in \Sigma_\eta$.
To estimate $|\partial_{\mathbf{y}}^{\boldsymbol{\sigma}} Q_3|$ we recall (33),

$$\left|\partial_{\mathbf{y}}^{\boldsymbol{\sigma}} Q_3(\mathbf{y})\right| \le \left(\frac{1}{2\pi\varepsilon}\right)^n \int_{K_0\times K_0}\int_{K_1} \left|\partial_{\mathbf{y}}^{\boldsymbol{\sigma}} I(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')\right| \, d\mathbf{x}\, d\mathbf{z}\, d\mathbf{z}', \tag{39}$$

and by (35) and (A4) one has

$$\left|\partial_{\mathbf{y}}^{\boldsymbol{\sigma}} I(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')\right| \le \sum_{j=0}^{|\boldsymbol{\sigma}|}\sum_{\ell=0}^{L_1}\sum_{m=0}^{L_2} \varepsilon^{-|\boldsymbol{\sigma}|} \left| \int_{\mathbb{R}} f_{\ell m j \boldsymbol{\sigma}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')\psi(t,\mathbf{x}) e^{i\vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon}\, dt \right|. \tag{40}$$

Let us introduce the function

$$g_1(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') = \varrho_\mu(\mathbf{x}-\mathbf{q}^+(t,\mathbf{y},\mathbf{z}))\varrho_\mu(\mathbf{x}-\mathbf{q}^-(t,\mathbf{y},\mathbf{z}')),$$

so that $g_1 \in \mathcal{W}_\mu$. Then for $g_2 := 1 - g_1 \in \mathcal{Y}$ and $\operatorname{supp} g_2(t,\cdot,\mathbf{y},\mathbf{z},\mathbf{z}') \subset \mathbb{R}^n \setminus \Sigma_{\mu/2}$ for all $t,\mathbf{y},\mathbf{z},\mathbf{z}'$. We will now regard (40) one term at a time, and use the partition of unity $1 = g_1 + g_2$,

$$\int_{\mathbb{R}} f_{\ell m j \boldsymbol{\sigma}} \psi e^{i\vartheta_k/\varepsilon}\, dt = \int_{\mathbb{R}} f_{\ell m j \boldsymbol{\sigma}} \psi (g_1+g_2) e^{i\vartheta_k/\varepsilon}\, dt = ① + ②.$$

Let us first estimate the term ①. We have $\Sigma_{\mu/2} \cap \Sigma_\eta = \Sigma_{\mu/2}$, therefore for $g_{\ell m j \boldsymbol{\sigma}} := f_{\ell m j \boldsymbol{\sigma}} \psi g_1$ we have $\operatorname{supp} g_{\ell m j \boldsymbol{\sigma}}(\cdot,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') \subset [0,T]$, $\forall \mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}'$, and $\operatorname{supp} g_{\ell m j \boldsymbol{\sigma}}(t,\cdot,\mathbf{y},\mathbf{z},\mathbf{z}') \subset \Sigma_{\mu/2}(t,\mathbf{y},\mathbf{z},\mathbf{z}') \cap K_1$, $\forall t,\mathbf{y},\mathbf{z},\mathbf{z}'$. Since the gradient $\partial_t \vartheta_k$ does not vanish for $\mathbf{x} \in \Sigma_{\mu/2}$ by Proposition 5.5 we can employ the non-stationary phase lemma 5.3,

$$\begin{aligned} |①| &\le \left| \int_{\mathbb{R}} g_{\ell m j \boldsymbol{\sigma}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') e^{i\vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon}\, dt \right| \\ &\le C_K \varepsilon^K \sum_{q=0}^{K} \int_{\mathbb{R}} \frac{|\partial_t^q g_{\ell m j \boldsymbol{\sigma}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')|}{|\partial_t \vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')|^{2K-q}} e^{-\operatorname{Im}\vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon}\, dt, \end{aligned}$$

for every $K \in \mathbb{N}_0$, where C_K depends on K but is independent of ε and $g_{\ell m j \boldsymbol{\sigma}}$. Since $g_{\ell m j \boldsymbol{\sigma}} \in \mathcal{Y}$, its time derivatives are uniformly bounded: for all $t \in [0,T]$, $\mathbf{y} \in \Gamma_c$, $\mathbf{z},\mathbf{z}' \in K_0$,

$$\sup_{\mathbf{x}\in\Sigma_{\mu/2}\cap K_1} |\partial_t^q g_{\ell m j \boldsymbol{\sigma}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')| \le \sup_{\mathbf{x}\in K_1} |\partial_t^q g_{\ell m j \boldsymbol{\sigma}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')| \le C_{\ell m j \boldsymbol{\sigma} q}.$$

Therefore, using the fact that $\operatorname{Im}\vartheta_k \ge 0$ from (38) and recalling (36) we obtain

$$|①| \le C_K \varepsilon^K \sum_{q=0}^{K} \int_0^T \frac{C_{\ell m j \boldsymbol{\sigma} q}}{\nu^{2K-q}}\, dt \le \widetilde{C}_{K\ell m j \boldsymbol{\sigma}} \varepsilon^K,$$

where $\widetilde{C}_{K\ell m j \boldsymbol{\sigma}}$ also depends on $T,\mu,\eta,\Gamma_c,k,\nu,p,\boldsymbol{\alpha}$, but is independent of ε.
Secondly, let us estimate the term ②. Since $\operatorname{supp} g_2(t,\cdot,\mathbf{y},\mathbf{z},\mathbf{z}') \subset \mathbb{R}^n \setminus \Sigma_{\mu/2}(t,\mathbf{y},\mathbf{z},\mathbf{z}')$, ② is only nonzero for either $|\mathbf{x}-\mathbf{q}^+(t,\mathbf{y},\mathbf{z})| > 2\mu$ or $|\mathbf{x}-\mathbf{q}^-(t,\mathbf{y},\mathbf{z}')| > 2\mu$ (or both) and therefore by (38),

$$\operatorname{Im}\vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') > \delta\mu^2.$$

We utilize the fact that $h_{\ell m j \boldsymbol{\sigma}} := f_{\ell m j \boldsymbol{\sigma}} \psi g_1 \in \mathcal{Y}$ and hence for all $t \in [0,T]$, $\mathbf{y} \in \Gamma_c$, $\mathbf{z},\mathbf{z}' \in K_0$,

$$\sup_{\mathbf{x}\in\mathbb{R}^n\setminus\Sigma_{\mu/2}\cap K_1} |h_{\ell m j \boldsymbol{\sigma}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')| \le \sup_{\mathbf{x}\in K_1} |h_{\ell m j \boldsymbol{\sigma}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')| \le \widetilde{C}_{\ell m j \boldsymbol{\sigma}}.$$

Therefore, ② can be estimated as

$$|②| \le \int_0^T \left| h_{\ell m j \boldsymbol{\sigma}}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') \right| e^{-\operatorname{Im} \vartheta_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon} \, dt$$
$$\le T \widetilde{C}_{\ell m j \boldsymbol{\sigma}} e^{-\delta \mu^2/\varepsilon}.$$

Collecting ① and ② together, we obtain from (40)

$$\left| \partial_{\mathbf{y}}^{\boldsymbol{\sigma}} I(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') \right| \le \sum_{j=0}^{|\boldsymbol{\sigma}|} \sum_{\ell=0}^{L_1} \sum_{m=0}^{L_2} \varepsilon^{-|\boldsymbol{\sigma}|} \left(|①| + |②| \right)$$
$$\le \max_{j,\ell,m} \varepsilon^{-|\boldsymbol{\sigma}|} \left(\widetilde{C}_{K \ell m j \boldsymbol{\sigma}} \varepsilon^K + T \widetilde{C}_{\ell m j \boldsymbol{\sigma}} e^{-\delta \mu^2/\varepsilon} \right).$$

Finally, by (39) we have

$$\left| \partial_{\mathbf{y}}^{\boldsymbol{\sigma}} Q_3(\mathbf{y}) \right| \le (2\pi)^{-n} \varepsilon^{-|\boldsymbol{\sigma}|-n} |K_0|^2 |K_1| \max_{j,\ell,m} \left(\widetilde{C}_{K \ell m j \boldsymbol{\sigma}} \varepsilon^K + T \widetilde{C}_{\ell m j \boldsymbol{\sigma}} e^{-\delta \mu^2/\varepsilon} \right).$$

That is, choosing $K \ge n + |\boldsymbol{\sigma}|$, the first term is converging with ε. Since $\delta > 0$, the second terms decays fast as a function of ε for any $\boldsymbol{\sigma}$. Therefore, there exists an upper bound $C_{\boldsymbol{\sigma}}$ such that

$$\sup_{\mathbf{y} \in \Gamma_c} \left| \partial_{\mathbf{y}}^{\boldsymbol{\sigma}} Q_3(\mathbf{y}) \right| \le C_{\boldsymbol{\sigma}},$$

where $C_{\boldsymbol{\sigma}}$ depends on $T, \mu, \eta, \Gamma_c, k, \delta, L_1, L_2, p, \boldsymbol{\alpha}$, but is uniform in ε.
Recalling (31) and (32) we then arrive at

$$\sup_{\mathbf{y} \in \Gamma_c} \left| \partial_{\mathbf{y}}^{\boldsymbol{\sigma}} \mathcal{Q}_{\mathrm{GB}}^{p,\boldsymbol{\alpha}}(\mathbf{y}) \right| \le \sup_{\mathbf{y} \in \Gamma_c} \left| \partial_{\mathbf{y}}^{\boldsymbol{\sigma}} Q_1(\mathbf{y}) \right| + \sup_{\mathbf{y} \in \Gamma_c} \left| \partial_{\mathbf{y}}^{\boldsymbol{\sigma}} Q_2(\mathbf{y}) \right| + 2 \sup_{\mathbf{y} \in \Gamma_c} \left| \partial_{\mathbf{y}}^{\boldsymbol{\sigma}} Q_3(\mathbf{y}) \right| \le \widetilde{C}_{\boldsymbol{\sigma}},$$

with $C_{\boldsymbol{\sigma}}$ dependent on $T, \mu, \eta, \Gamma_c, k, K, \delta, \nu, L_1, L_2, p, \boldsymbol{\alpha}$, but independent of ε, which concludes the proof.

A Proof of Theorem 4.6

To simplify the expressions, we first introduce the symmetrizing variables

$$\bar{\mathbf{q}} = \bar{\mathbf{q}}(t, \mathbf{y}, \mathbf{z}, \mathbf{z}') = \frac{\mathbf{q}(t, \mathbf{y}, \mathbf{z}) + \mathbf{q}(t, \mathbf{y}, \mathbf{z}')}{2}, \qquad \Delta\mathbf{q} = \Delta\mathbf{q}(t, \mathbf{y}, \mathbf{z}, \mathbf{z}') = \frac{\mathbf{q}(t, \mathbf{y}, \mathbf{z}) - \mathbf{q}(t, \mathbf{y}, \mathbf{z}')}{2}. \tag{41}$$

We also let Ω_μ, $\mathcal{U}_\mu$ and $\mathcal{V}$ denote the following spaces:

- $\Omega_\mu = \Omega_\mu(t, \mathbf{y}, \mathbf{z}, \mathbf{z}') := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \Delta\mathbf{q}| \leq 2\mu \quad \text{and} \quad |\mathbf{x} + \Delta\mathbf{q}| \leq 2\mu\}$,
- $\mathcal{U}_\mu := \{f \in \mathcal{Y} : \text{ supp } f(t, \cdot, \mathbf{y}, \mathbf{z}, \mathbf{z}') \subset \Omega_\mu(t, \mathbf{y}, \mathbf{z}, \mathbf{z}'), \forall t \in \mathbb{R}, \mathbf{y} \in \Gamma, \mathbf{z}, \mathbf{z}' \in \mathbb{R}^n\}$,
- $\mathcal{V} := C^\infty(\mathbb{R} \times \Gamma \times \mathbb{R}^n \times \mathbb{R}^n)$.

Then I_0 in (24) can be symmetrized as

$$I_0(t, \mathbf{y}, \mathbf{z}, \mathbf{z}') = \int_{\mathbb{R}^n} h(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}')(\mathbf{x} - \Delta\mathbf{q})^{\boldsymbol{\alpha}} (\mathbf{x} + \Delta\mathbf{q})^{\boldsymbol{\beta}} e^{i\Psi_k(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}')/\varepsilon} d\mathbf{x}, \tag{42}$$

where $\Psi_k(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') = \Theta_k(t, \mathbf{x} + \bar{\mathbf{q}}, \mathbf{y}, \mathbf{z}, \mathbf{z}')$ and $h(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') = f(t, \mathbf{x} + \bar{\mathbf{q}}, \mathbf{y}, \mathbf{z}, \mathbf{z}')$ so that $h \in \mathcal{U}_\eta$ since $f \in \mathcal{T}_\eta$. The following auxiliary lemma is a compilation of Lemma 3 and the differentiated version of Lemma 4 in [6].

Lemma A.1. *There exists $f_{\boldsymbol{\mu},\boldsymbol{\nu}} \in \mathcal{V}$ such that*

$$(\mathbf{x} - \Delta\mathbf{q})^{\boldsymbol{\alpha}} (\mathbf{x} + \Delta\mathbf{q})^{\boldsymbol{\beta}} = \sum_{|\boldsymbol{\mu}+\boldsymbol{\nu}|=|\boldsymbol{\alpha}+\boldsymbol{\beta}|} f_{\boldsymbol{\mu},\boldsymbol{\nu}}(t, \mathbf{y}, \mathbf{z}, \mathbf{z}')(\mathbf{z} - \mathbf{z}')^{\boldsymbol{\mu}} \mathbf{x}^{\boldsymbol{\nu}}.$$

For the k-th order symmetrized Gaussian beam phase Ψ_k, there exist $a_{\boldsymbol{\alpha},\boldsymbol{\beta},m} \in \mathcal{V}$ such that

$$\partial_{y_m} \Psi_k(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') = \sum_{2 \leq |\boldsymbol{\alpha}+\boldsymbol{\beta}| \leq k+1} a_{\boldsymbol{\alpha},\boldsymbol{\beta},m}(t, \mathbf{y}, \mathbf{z}, \mathbf{z}') (\mathbf{z} - \mathbf{z}')^{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\beta}}.$$

The following proposition is an update of [6, Proposition 3] adapted to our case.

Proposition A.2. *There exist functions $g_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\sigma},\ell} \in \mathcal{U}_\eta$ and $L_{\boldsymbol{\sigma}}, M_{\boldsymbol{\sigma}} \geq 0$ such that the derivatives of I_0 in (42) with respect to $\mathbf{y}$ read*

$$\partial_{\mathbf{y}}^{\boldsymbol{\sigma}} I_0(t, \mathbf{y}, \mathbf{z}, \mathbf{z}') = \sum_{\ell=-|\boldsymbol{\sigma}|}^{L_{\boldsymbol{\sigma}}} \sum_{|\boldsymbol{\mu}+\boldsymbol{\nu}|+2\ell=0}^{M_{\boldsymbol{\sigma}}} \varepsilon^\ell (\mathbf{z} - \mathbf{z}')^{\boldsymbol{\mu}} \int_{\mathbb{R}^n} \mathbf{x}^{\boldsymbol{\nu}} g_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\sigma},\ell}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') e^{i\Psi_k(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}')/\varepsilon} d\mathbf{x}. \tag{43}$$

Proof. Recalling Lemma A.1, (42) can be reformulated as

$$I_0(t, \mathbf{y}, \mathbf{z}, \mathbf{z}') = \sum_{|\boldsymbol{\mu}+\boldsymbol{\nu}|=|\boldsymbol{\alpha}+\boldsymbol{\beta}|} (\mathbf{z} - \mathbf{z}')^{\boldsymbol{\mu}} \int_{\mathbb{R}^n} \mathbf{x}^{\boldsymbol{\nu}} g_{\boldsymbol{\mu},\boldsymbol{\nu}}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') e^{i\Psi_k(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}')/\varepsilon} d\mathbf{x},$$

with $g_{\boldsymbol{\mu},\boldsymbol{\nu}}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') = h(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') f_{\boldsymbol{\mu},\boldsymbol{\nu}}(t, \mathbf{y}, \mathbf{z}, \mathbf{z}')$. Therefore, since $h \in \mathcal{U}_\eta$ and $f_{\boldsymbol{\mu},\boldsymbol{\nu}} \in \mathcal{V}$ we have $g_{\boldsymbol{\mu},\boldsymbol{\nu}} \in \mathcal{U}_\eta$. We will now prove (43) by induction. First, the statement is valid for $\boldsymbol{\sigma} = \mathbf{0}$ since we can choose $L_{\mathbf{0}} = 0$, $M_{\mathbf{0}} = |\boldsymbol{\alpha} + \boldsymbol{\beta}|$ and

$$g_{\boldsymbol{\mu},\boldsymbol{\nu},\mathbf{0},0} = \begin{cases} g_{\boldsymbol{\mu},\boldsymbol{\nu}}, & \text{for } |\boldsymbol{\mu} + \boldsymbol{\nu}| = |\boldsymbol{\alpha} + \boldsymbol{\beta}|, \\ 0, & \text{otherwise.} \end{cases}$$

For the induction step let $L_{\boldsymbol{\sigma}}, M_{\boldsymbol{\sigma}} \geq 0$ and $g_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\sigma},\ell} \in \mathcal{U}_\eta$ be such that (43) holds. Then for $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \mathbf{e}_m$, where $\mathbf{e}_m$ is the m-th unit vector, we have $\partial_{\mathbf{y}}^{\tilde{\boldsymbol{\sigma}}} I_0 = \partial_{y_m} \partial_{\mathbf{y}}^{\boldsymbol{\sigma}} I_0$. Using (43), we can write

$$\begin{aligned}\partial_{\mathbf{y}}^{\tilde{\boldsymbol{\sigma}}} I_0 &= \sum_{\ell=-|\boldsymbol{\sigma}|}^{L_{\boldsymbol{\sigma}}} \sum_{|\boldsymbol{\mu}+\boldsymbol{\nu}|+2\ell=0}^{M_{\boldsymbol{\sigma}}} \varepsilon^{\ell} (\mathbf{z}-\mathbf{z}')^{\boldsymbol{\mu}} \int_{\mathbb{R}^n} \mathbf{x}^{\boldsymbol{\nu}} \left(\partial_{y_m} g_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\sigma},\ell} + g_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\sigma},\ell}\, i\varepsilon^{-1} \partial_{y_m} \Psi_k \right) e^{i\Psi_k/\varepsilon} d\mathbf{x} \\ &= ① + ②.\end{aligned}$$

Since $\partial_{y_m} g_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\sigma},\ell} \in \mathcal{U}_\eta$, ① is of the form (43) with $L_{\tilde{\boldsymbol{\sigma}}} = L_{\boldsymbol{\sigma}}$, $M_{\tilde{\boldsymbol{\sigma}}} = M_{\boldsymbol{\sigma}}$ and

$$g_{\boldsymbol{\mu},\boldsymbol{\nu},\tilde{\boldsymbol{\sigma}},\ell} = \begin{cases} \partial_{y_m} g_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\sigma},\ell}, & \text{for } \ell \geq -|\boldsymbol{\sigma}|, \\ 0, & \text{for } \ell = -|\boldsymbol{\sigma}| - 1. \end{cases}$$

Regarding the remaining terms ②, let us express the derivative $\partial_{y_m} \Psi_k$ by Lemma A.1. Then ② reads

$$\sum_{\ell=-|\boldsymbol{\sigma}|}^{L_{\boldsymbol{\sigma}}} \sum_{|\boldsymbol{\mu}+\boldsymbol{\nu}|+2\ell=0}^{M_{\boldsymbol{\sigma}}} \sum_{|\boldsymbol{\gamma}+\boldsymbol{\delta}|=2}^{k+1} \varepsilon^{\ell-1} (\mathbf{z}-\mathbf{z}')^{\boldsymbol{\mu}+\boldsymbol{\gamma}} \int_{\mathbb{R}^n} \mathbf{x}^{\boldsymbol{\nu}+\boldsymbol{\delta}} h_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\gamma},\boldsymbol{\delta},\ell}\, e^{i\Psi_k/\varepsilon} d\mathbf{x}, \tag{44}$$

with $h_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\gamma},\boldsymbol{\delta},\ell} = i a_{\boldsymbol{\gamma},\boldsymbol{\delta},m}\, g_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\sigma},\ell} \in \mathcal{U}_\eta$ since $g_{\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\sigma},\ell} \in \mathcal{U}_\eta$ and $a_{\boldsymbol{\gamma},\boldsymbol{\delta},m} \in \mathcal{V}$. Each of the terms in (44) is therefore of the form

$$\varepsilon^{\tilde{\ell}} (\mathbf{z}-\mathbf{z}')^{\tilde{\boldsymbol{\mu}}} \int_{\mathbb{R}^n} \mathbf{x}^{\tilde{\boldsymbol{\nu}}} h_{\tilde{\boldsymbol{\mu}},\tilde{\boldsymbol{\nu}},\tilde{\ell}}(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')\, e^{i\Psi_k(t,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')/\varepsilon} d\mathbf{x},$$

where

$$-|\tilde{\boldsymbol{\sigma}}| \leq \tilde{\ell} = \ell - 1 \leq L_{\boldsymbol{\sigma}} - 1 =: L_{\tilde{\boldsymbol{\sigma}}},$$

and

$$0 \leq |\tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\nu}}| + 2\tilde{\ell} = |\boldsymbol{\mu}+\boldsymbol{\nu}| + 2\ell + |\boldsymbol{\gamma}+\boldsymbol{\delta}| - 2 \leq M_{\boldsymbol{\sigma}} + k - 1 =: M_{\tilde{\boldsymbol{\sigma}}},$$

which finalizes the induction argument and concludes Proposition A.2. □

The rest of the proof of [6, Theorem 1] can be used as it is. In particular, Lemma 5 and Lemma 6 are valid without any alteration. Ultimately, we are using the fact that $0 \leq |\boldsymbol{\mu}+\boldsymbol{\nu}| + 2\ell$ in (43) which is still the case due to Proposition A.2. Finally, since all estimates in [6] are uniform in t, the constant $C_{\boldsymbol{\sigma}}$ is uniform in $[0,T]$ as well. This completes the proof of Theorem 4.6.

References

[1] H.-J. Bungartz and M. Griebel. Sparse grids. *Acta Numer.*, 13:147–269, 2004.

[2] V. Cervený, M. M. Popov, and I. Pšenčík. Computation of wave fields in inhomogeneous media—Gaussian beam approach. *Geophys. J. R. Astr. Soc.*, 70:109–128, 1982.

[3] M. Griebel and S. Knapek. Optimized general sparse grid approximation spaces for operator equations. *Math. Comp.*, 78:2223–2257, 2009.

[4] Robert J Hansen. *Seismic design for nuclear power plants.* The MIT Press, Cambridge, 1970.

[5] H. Liu, O. Runborg, and N. M. Tanushev. Error estimates for Gaussian beam superpositions. *Math. Comp.*, 82:919–952, 2013.

[6] G. Malenová, M. Motamed, and O. Runborg. Stochastic regularity of a quadratic observable of high-frequency waves. *Research in the Mathematical Sciences*, 4(1):3, 2017.

[7] G. Malenova, M. Motamed, O. Runborg, and R. Tempone. A sparse stochastic collocation technique for high-frequency wave propagation with uncertainty. *SIAM/ASA J. Uncertainty Quantification*, 4(1):1084–1110, 2016.

[8] Gabriela Malenová. Uncertainty quantification for high frequency waves, 2016. Licentite thesis, KTH Royal Institute of Technology.

[9] J. Ralston. Gaussian beams and the propagation of singularities. *Studies in partial differential equations*, 23:206–248, 1982.

[10] O. Runborg. Mathematical models and numerical methods for high frequency waves. *Commun. Comput. Phys.*, 2:827–880, 2007.

[11] N. M. Tanushev. Superpositions and higher order Gaussian beams. *Commun. Math. Sci.*, 6(2):449–475, 2008.