Estimation of the Elastic Moduli of Porous Materials using Analytical Methods, Numerical Methods, and Image Analysis

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November 2007
Abstract

The effective bulk modulus and effective shear modulus of porous materials having various types of pore shapes are investigated, using both analytical and numerical methods. These solutions, and the scaling laws that are derived with the aid of these solutions, are then used to make predictions of the effective elastic moduli of some sandstones and ceramics, based on two-dimensional images of the pore space.

The complex variable method is used to find the hydrostatic and shear compliances of a large family of pores that have $N$-fold rotational symmetry, and which have at most four terms in their conformal mapping function. This solution is validated using boundary element (BEM) calculations, and is also used to test two scaling laws that estimate the compliances based on the area and perimeter of the pore.

The boundary perturbation method is used to study the effect of small-scale roughness on the compressibility and shear compliance of a nominally circular pore. The solution is carried out to fourth order in the roughness parameter for the case of hydrostatic loading, and to second order for shear loading. These solutions allow one to judge the scale of roughness that can safely be ignored when obtaining images of the pores.

Predictions are then made of the elastic moduli of some porous materials – two sandstones and a ceramic. Starting with scanning electron micrographs, image analysis software is used to isolate and extract each pore from the host material. The bulk and shear compliances are estimated using both BEM and the two scaling laws. Areally-weighted mean values of these compliances are calculated for each material, and the differential effective medium scheme is used to obtain expressions for the moduli as functions of porosity. These predictions agree well with the experimental values found in the literature.
Acknowledgements

This PhD degree thesis was produced in the Royal Institute of Technology (KTH) in Stockholm, within the Research Group of Engineering Geology and Geophysics, in the Department of Land and Water Resources, during the period of December 2004 – November 2007.

My special thanks go to my supervisor, Prof. Robert Zimmerman, for his guidance and support during my study period at KTH. His support, constant encouragement, and invaluable suggestions, are highly appreciated.

I also express my sincere gratitude to Assoc. Prof. Herbert Henkel, my supervisor during my masters degree period. His encouragement and kindness are highly acknowledged.

Special thanks go to Staffan Molin, Emmanuel David, Alireza Baghbanan, Tomofumi Koyama, and Mimmi Arvidsson, for their comments and encouragements offered, especially when difficult situations occurred.

I also express my gratitude to the other members of the Engineering Geology and Geophysics Group: Lanru Jing, Joanne Fernlund, Katrin Grünfeld, and Solomon Tafesse, for their help and friendship. Finally, I give my sincere thanks to Britt Chow and Aira Saarelainen, for their for effective and kind help with administrative matters.

And, last but not least, I would like to thank my family members: my father, my mother, and my brothers.
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References
Papers

This thesis, although written in monograph format, consists essentially of the following papers:


Ekneligoda, T.C., and Zimmerman, R.W. Effect of small-scale roughness on the shear compliance of a nominally circular pore. To be submitted. [thesis chapter 5]

1 Introduction

1.1 Background to problem

Most natural geological materials such as rocks or soils are porous to one extent or another. This is also true of many man-made materials such as ceramics or concrete (Rice, 1998), not to mention biological materials such as bone (Cowin, 2001). The mechanical properties of these materials are controlled by the geometry and structure of the pore space.

Traditionally, most attempts to understand the effect of pore structure on mechanical properties such as the elastic moduli have assumed that the pores can be modeled as being ellipsoids (Eshelby, 1957), or special cases of ellipsoids such as spheres (Budiansky, 1965), elliptical cylinders (Walsh et al., 1965), circular cylinders (Biot, 1956), penny-shaped cracks (Warren, 1973), elliptical cracks (Budiansky & O’Connell, 1975), etc. Moreover, although models based on ellipsoidal pores have often been used in an inverse sense to infer pore geometry data based on measured modulus (Kuster & Toksöz, 1974; Zimmerman, 1991), such models have rarely if ever been used in a direct sense to estimate the moduli based on observation of the pore structure.

Nevertheless, scanning electron micrographs show that the pore shapes in materials such as sandstones or ceramics are never as simple as circles or ellipses. But the use of real pore shapes in the modeling process has been hindered by the lack of analytical solutions for these shapes. The main goal of this thesis is to develop methods to analyze the effect of non-ellipsoidal pores on the elastic moduli, and then to use these results to make predictions of the effective bulk and shear modulus of some porous materials based on information obtained from two-dimensional pore space images.

1.2 Previous research work on micromechanical modeling

The vast literature on the effective moduli problem that has been based on ellipsoidal pores has been reviewed extensively elsewhere, and will not be discussed further here (Mura, 1987; Christensen, 1991; Zimmerman, 1991; Mavko et al., 1998; Grimvall, 1999; Nemat-Nasser & Hori, 1999).

The complex variable method of Kolosov and Muskhelishvili (1963) had been used for years to investigate stress concentration problems around two-dimensional non-elliptical pores, but not in conjunction with the effective medium problem. Zimmerman (1986) used this approach to calculate the compressibility of a two-dimensional hypotrochoidal pore, and thereby find the effective bulk modulus of a material that contains a dilute dispersion of such pores. Jasiuk et al. (1994) and Kachanov et al. (1994) used the complex variable method to study quasi-polygonal holes, under both hydrostatic and shear loading. Tsukrov & Kachanov (1993) and...
Tsukrov & Novak (2000) continued in this vein of using the complex variable method to study pores of non-ellipsoidal shapes. Much of this work makes use of the $H$-tensor, which is the tensor that relates the applied far-field stress to the excess strain caused by the presence of a pore or inclusion.

Use of the complex variable method necessarily entails performing a conformal mapping to map the unit circle into the pore shape of interest. Consequently, one of the main disadvantages of the complex variable and conformal mapping approach is that, as the pore shape becomes more irregular and “non-circular”, a larger and larger number of terms are needed on the mapping function. In order to overcome this difficulty, Tsukrov & Novak (2000, 2002) and Tsukrov et al. (2005) used two methods: numerical conformal mapping for irregular shapes, and finite elements, which is a purely numerical approach that does not require a mapping. In the numerical conformal mapping technique, a toolbox developed in Matlab was used to carry out the numerical mapping. The boundary of the hole is approximated with an $N$-sided polygon with vertices on the boundary of the hole. This technique can be readily used to handle arbitrarily irregular shapes. But the mapping procedure is only the start of the complex variable method, and the subsequent calculations required to find the complex potentials are very cumbersome. An alternative approach was to use finite elements (Gudehus, 1977; Venturini, 1983).

A different formalism for studying the effect of non-circular pores on the effective moduli was developed by Jasiuk (1995). The real-valued Airy stress function, in the form of an infinite series of logarithmic, power-law and trigonometric terms, was used to represent the displacement and stresses around a pore in an infinite body, under a prescribed far-field stress, such as shear or hydrostatic loading. The effect of the pore on the overall elastic moduli was then expressed in terms of one specific coefficient in the Airy function.

A different approach to the problem of quantifying the effect if an irregular pore on the elastic moduli was originated by Zimmerman (1986). Recognizing the difficulty of treating irregular shapes, he proposed an approximate scaling law based on the area and perimeter of the pore, to express the compressibility of that pore relative to that of a circular pore. This approach is similar in spirit to the hydraulic radius approximation for the hydraulic conductance of an irregular pore tube (Scheidegger, 1974). This scaling law was tested against some regular shapes, with reasonable accuracy, but not against any “real” pores.

Arns et al. (2002) used the microtomographic images to predict the effective elastic properties of porous materials. They focused on a suite of Fontainebleau sandstones with porosities varying from 7.5-22%. Three-dimensional information of the sample was obtained from X-ray computed microtomographic images (see Auserais et al., 1996). Their image analysis technique represents the original image in gray scale. Based on color separation, a binary solid-and-pore image was created. Numerical methods were then used to solve the elasticity equations within the mineral phase of the reconstructed porous rock, and good agreement was obtained with the experimentally measured moduli.
A final issue that must be addressed when modeling the effective moduli of a porous material is that the results discussed above strictly apply only to isolated pores that are infinitely separated from each other. Hence, some method must be devised to account for “pore-pore interactions”, which is to say, to extend the results to finite porosities. Although it might be thought that progress might be made on this problem by solving the elasticity problem of a body containing a pair of pores, such solutions are extremely unwieldy (Ling, 1948; Duan et al., 1986; Zimmerman 1988). Hence, several approximate schemes have been devised that require only the solution to the “one-pore” problem. Some of these methods have been reviewed by Christensen (1991) and Zimmerman (1991), and will be briefly described now.

The basic method is the “no-interaction” method, which assumes that each pore indeed behaves as if it were isolated, in which case the overall compliance of the pore space is simply the sum of the individual pore compliances. Unfortunately, this method is known to over-estimate the moduli for porosities above about 10%, since its predictions exceed the Hashin-Shtrikman upper bounds (Hashin, 1983). The so-called “self-consistent” method uses the results for an isolated pore, but assumes that the moduli of the material surrounding the pore are those of the effective medium rather than those of the matrix material (Budiansky, 1965). The method of Mori & Tanaka (1973) also utilizes the single pore solutions, but assumes that the stress at infinity is equal not to the actual applied stress, but to the mean stress in the matrix phase (Benveniste, 1987). The differential scheme (Norris, 1985) considers the Nth inclusion to be placed in the material whose elastic moduli are those of the material with N-1 inclusions, using the single-pore solution at each incremental stage. The generalized-self consistent method (Christensen, 1990) is based on the analysis of a single pore surrounded by a concentric shell of matrix material, imbedded in a material whose elastic moduli are those of effective medium. Despite years of study and debate regarding the merits of these methods, there is as yet no consensus on the preferred method, or even agreement as to whether or not one method is more accurate in all cases.

### 1.3 Outline of the present thesis

In this study, the effective bulk modulus and effective shear modulus of porous materials having various types of pore shapes are determined. Both analytical and numerical methods are used. These solutions, and the scaling laws that are derived with the aid of these solutions, are then used to make predictions of the effective elastic moduli of some sandstones and ceramics, based on two-dimensional images of the pore space.

In Chapter 2, the complex variable method is used to find the hydrostatic compressibility of pores that have \(N\)-fold rotational symmetry, and which have at most four terms in their conformal mapping function. As the solution is obtained for arbitrary (allowable) values of the mapping coefficients, this allows a large range of pore shapes to be studied. This solution is used to test the aforementioned scaling law proposed by Zimmerman (1986).

In Chapter 3, the boundary perturbation method is used to study the effect of small-scale roughness on the compressibility of a nominally circular pore. The actual
problem solved is that of a rigid circular inclusion with sinusoidal roughness of "small" amplitude, $\varepsilon$. The results of Jasiuk (1995) are used to convert this solution into one applicable to a pore of the same shape. The solution is carried out to fourth order in the roughness parameter, $\varepsilon$. Although the calculations are tedious, the final result for the pore compressibility is given by only one of the coefficients in the Airy stress function. This solution allows one to judge the scale of roughness that can be ignored when obtaining images of the pores.

In Chapter 4, the same family of rotationally symmetric pores that were studied in Chapter 2 under hydrostatic loading are studied under far-field shear loading, again using the complex variable approach. The results are quantified in terms of a "pore shearability" parameter that is analogous to the pore compressibility, and is also closely related to the coefficient $H_{1212}$ of the $H$-tensor. Extensive discussion is given to the issue of pores that are "isotropic" with respect to shear, as previously discussed by Eroshkin & Tsukrov (2005). A scaling law, similar to that proposed by Zimmerman (1986) for pore compressibility, is proposed and tested.

In Chapter 5, the boundary perturbation problem is used to study the rough-walled pore under far-field shear loading. In contrast to Chapter 3, the solution is taken only to second order in $\varepsilon$, rather than fourth order. The solution is compared to the results obtained by boundary element calculations.

In Chapter 6, predictions are made of the elastic moduli of some porous materials – two sandstones and a ceramic. Starting with scanning electron micrographs, Image Analysis software such as Idrisi and Cartalinx are used to isolate and extract each pore from the host material. The separation between pore and host material is carried out based on the color difference. After identifying the pores, the boundary of each pore is digitized, to create the grid for a boundary element calculation. The area and perimeter of each pore are measured, to be used as input to the two scaling laws. Both methods (BEM, scaling law) are used to estimate the bulk and shear compliances of the pores. An areally-weighted mean value of these compliances is calculated for each material. The differential effective medium scheme is then used, along with the values of the moduli of the non-porous host material, to yield expressions for the moduli as functions of porosity, which are then compared to measured values from the literature.

Finally, Chapter 7 contains a brief summary and discussion of the results, along with a few suggestions for future work.
2 Compressibility of Two-Dimensional Symmetric Pores

2.1 Introduction

As was discussed in Chapter One, the modeling of elastic properties in porous materials is usually carried out assuming that the pores can be represented as ellipsoids or spheroids. Although ellipsoidal pore models offer the advantage of mathematical tractability, examination of images of real materials reveals that pores are in many cases more irregular. Pores in ceramics often seem, at least visually in two-dimensional sections, to be better modeled as being polygonal. Pores in sedimentary rocks are much more irregular, and are often not convex. Analysis of the influence that such irregular three-dimensional pores would have on the elastic moduli is thus far intractable analytically, and only slight progress has been made numerically (Burnley & Davis, 2004). But if the pores are modeled as two-dimensional, pores of essentially any shape can be analyzed, using the complex variable methods developed by Muskhelishvili (1963) and others. These complex variable methods are quite powerful, and have been used not only for linear elastic problems involving cavities, but can also be used, for example, for elastic inclusions (Chang & Conway, 1968; Jasiuk 1995), problems involving finite elasticity (Ru et al., 2004) and thermopiezoelectric problems involving cavities (Qin et al., 1999).

The complex variable method involves mapping the interior of the unit circle into the exterior of the hole in the physical domain. Using this approach, Savin (1961) analyzed several holes of polygonal and quasi-polygonal shape, but with emphasis on calculating the stress concentrations, rather than on the displacements or on the effect that the pores have on the macroscopic elastic moduli. With regards to the effect of pores on the macroscopic elastic properties, some results for quasi-polygons represented by three and four terms of the Schwarz–Christoffel mapping function have been obtained by Zimmerman (1986), Jasiuk et al. (1994) and Kachanov et al. (1994). In the present work, explicit expressions are developed for holes that have an \( N \)-fold axis of rotational symmetry, and which can be represented by up to four terms in the mapping function. These results are more general than those of Jasiuk et al. (1994) or Kachanov et al. (1994), in that we give explicit results for arbitrary (allowable) values of the mapping coefficients, rather than only for the case of the quasi-polygons represented by the Schwarz–Christoffel mapping.

However, unlike Jasiuk et al. (1994) and Kachanov et al. (1994), only the case of hydrostatic loading will be considered here; the case of shear loading is treated in Chapter 4. Specifically, in this chapter the pore compressibility parameter, \( C_{pp} \), as defined by Zimmerman (1991), will be calculated for an isolated pore. This parameter represents the fractional change in pore area due to a hydrostatic pressure of unit magnitude acting along the pore walls. This parameter appears directly in equations such as the fluid diffusion equation for porous media, where it is added to the fluid compressibility in the denominator of the hydraulic diffusivity term (Bear, 1967). It is also directly related to the effective macroscopic compressibility, by
$C_{\text{eff}} = C_0 + \phi(C_0 + C_{\text{pp}})$, where $C_0$ is the two-dimensional compressibility of the non-porous host material, and $\phi$ is the porosity (Zimmerman, 1991). Hence, the calculation of the pore compressibility is intimately related to the prediction of the effective macroscopic elastic moduli.

\[ \gamma(z) = \omega(\zeta) \]

Figure 2.1. Conformal mapping of the interior of the unit circle $\gamma$ in the $\zeta$-plane into the region exterior to the contour $\Gamma$ in the $z$-plane.

### 2.2 Methodology

The stresses and displacements under two-dimensional plane strain or plane stress conditions can be represented in terms of two complex potential functions, $\phi(z)$ and $\psi(z)$, as follows (Godfrey, 1959; Muskhelishvili, 1963; England, 1971):

\[ 2G(u_x + iu_y) = \kappa \phi(z) - z\phi'(z) - \psi(z), \]  

(2.1)

where $\kappa = 3 - 4\nu$ for plane strain, and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress. It is sometimes convenient to refer to the “complex displacement”, $U = u_x + iu_y$. Consider an infinite elastic body containing a single, simple closed contour $\Gamma$, with no stresses acting at infinity, and with a uniform hydrostatic pressure of magnitude $p$ acting along $\Gamma$. For problems in which the tractions are specified along the contour $\Gamma$, the boundary condition for these potentials along $\Gamma$ can be written as

\[ \phi(z) + z\phi'(z) + \psi(z) = f_x + if_y \equiv F, \]  

(2.2)

where $F$ is equal to $i$ times the integral of the complex traction vector, $t_x + it_y$, along the boundary contour, starting from some arbitrary point $z_0$ on $\Gamma$.

The solution of this problem using the complex potential method proceeds as follows. First, the region outside of $\Gamma$ in the $z$-plane is mapped into the interior of the unit circle $\gamma$ in the $\zeta$-plane by a conformal mapping of the form (Fig. 2.1)

\[ z = \omega(\zeta) = \frac{1}{\zeta} + \sum_{k=1}^{\infty} a_k \zeta^k. \]  

(2.3)

If only one term in the series is taken, i.e., $\omega(\zeta) = \zeta^{-1} + a_n \zeta^n$, the hole is a hypotrochoid, which is a quasi-polygon having $n + 1$ equal “sides”. In order for the mapping to be single-valued, and for $\Gamma$ not to contain any self-intersections, $a_n$ must
satisfy the restriction $0 \leq a_n \leq 1/n$. The choice $a_n = 0$ gives a circle, whereas $a_n = 1/n$ gives a pore with $n + 1$ pointed cusps. For the particular choice $a_n = 2/(n(n+1))$, the mapping coincides with the first two terms of the Schwarz-Christoffel mapping for an $(n+1)$-sided equilateral polygon, and resembles a polygon with slightly rounded corners.

If the hole contour possesses an $(n+1)$-fold axis of symmetry, only powers that differ by $(n+1)$ will appear in the mapping function, i.e.,

$$z = \omega(\zeta) = \frac{1}{\zeta} + a_n\zeta^n + a_{2n+1}\zeta^{2n+1} + \ldots.$$  \hfill (2.4)

The problem of the hypotrochoidal hole (i.e., the two-term mapping function) under uniform normal traction has been solved previously (Zimmerman, 1986). Some special cases, such as holes represented by the first three or four terms in the Schwarz-Christoffel mapping for an $(n+1)$-sided equilateral polygon, have been discussed by Jasiuk et al. (1994) and Kachanov et al. (1994). In the present work, the case is considered of holes having $(n+1)$-fold rotational symmetry, and which can be represented by the first three or four terms in this mapping function. The solution will be derived in detail for the three-term case, whereas for space considerations the solution will be presented without detailed derivation for the four-term case, which poses no fundamental additional difficulties aside from increased algebraic complexity. To simplify the notation slightly, the mapping function is written as

$$\omega(\zeta) = \frac{1}{\zeta} + m_1\zeta^n + m_2\zeta^{2n+1} + m_3\zeta^{3n+2}. \hfill (2.5)$$

In order for the mapping to be conformal, and for the contour not to have any self-intersections, it must never be the case that $\omega'(\zeta) = 0$ along the contour. This poses restrictions on the allowable range of values for the $m_i$ coefficients. If $\omega'(\zeta) = 0$ for some value of $\zeta$ on the unit circle, as the $m_i$ values increase, this will first occur at $(n+1)$ equally-spaced points that include the point corresponding to $\zeta = 1$. Hence, the restrictions for the $m_i$ can be found by setting $\omega'(1) = 0$. For the two-term mapping, this leads to the restriction $m_1 \leq 1/n$. For the three-term mapping, the condition is $nm_1 + (2n+1)m_2 \leq 1$, and for the four-term mapping, the condition is $nm_1 + (2n+1)m_2 + (3n+2)m_3 \leq 1$.

### 2.3 Determination of the complex potentials

If all the terms in the boundary condition (2.2) are expressed in terms of $\zeta$, this equation becomes

$$\phi[z(\sigma)] + \omega(\sigma)\frac{\overline{\phi[z(\sigma)]}}{\omega'(\sigma)} + \overline{\psi[z(\sigma)]} = F(\sigma), \hfill (2.6)$$

where the variable $\sigma$ is used to denote values of $\zeta$ along the unit circle $\gamma$ in the $\zeta$-plane. To avoid a cumbersome notation, hereafter the notation $\phi[z(\sigma)] = \phi(\sigma)$ and $\psi[z(\sigma)] = \psi(\sigma)$ is used, in which case eq. (2.6) becomes
If the traction acting on the boundary is a hydrostatic pressure of magnitude $p$, it can easily be shown (Sokolnikoff, 1956) that $F = -pz = -p\omega(\sigma)$. Taking $p$ to be of unit magnitude, eq. (2.7) takes the form:

$$\phi(\sigma) + \omega(\sigma) \frac{\phi'(\sigma)}{\omega'(\sigma)} + \psi(\sigma) = F(\sigma). \tag{2.7}$$

For a problem with no body forces and no stresses acting at infinity, both potentials will be represented (Sokolnikoff 1956, p. 279) by power series that converge for $\zeta \leq 1$, i.e.,

$$\phi(\zeta) = b_0 + b_1 \zeta + b_2 \zeta^2 + b_3 \zeta^3 + ... + b_k \zeta^k + ... \tag{2.9}$$

$$\psi(\zeta) = c_0 + c_1 \zeta + c_2 \zeta^2 + c_3 \zeta^3 + ... + c_k \zeta^k + ... \tag{2.10}$$

The second term in eq. (2.8) is now expanded in a power series in $\sigma$. Considering first the case of a three-term mapping function, the following results are necessary:

$$\omega(\sigma) = \frac{1}{\sigma} + m_1 \sigma^n + m_2 \sigma^{2n+1}, \tag{2.11}$$

$$\omega'(\sigma) = -2 \sigma^{-2} + \frac{nm_1}{\sigma^{n-1}} + \frac{(2n+1)m_2}{\sigma^{2n}}, \tag{2.12}$$

$$\phi'(\sigma) = \overline{b_1} + \frac{2\overline{b_2}}{\sigma} + \frac{3\overline{b_3}}{\sigma^2} + ... + \frac{k\overline{b_k}}{\sigma^{k-1}} + ..., \tag{2.13}$$

in which case

$$\omega(\sigma) \frac{\phi'(\sigma)}{\omega'(\sigma)} = \left( \frac{1}{\sigma} + m_1 \sigma^n + m_2 \sigma^{2n+1} \right) \left[ -2 \sigma^{-2} + \frac{nm_1}{\sigma^{n-1}} + \frac{(2n+1)m_2}{\sigma^{2n}} \right]^{-1} \left[ \frac{\overline{b_1} + \frac{2\overline{b_2}}{\sigma} + \frac{3\overline{b_3}}{\sigma^2} + ... + \frac{k\overline{b_k}}{\sigma^{k-1}}}{\sigma^n} + \frac{m_1 \sigma^n + m_2 (2n+1)}{\sigma^{2n}} \right]^{-1} \tag{2.14}$$

After inserting eq. (2.14) into eq. (2.8), the coefficients of each power on $\sigma$ can be equated on both sides of the equation. This procedure yields

$$b_0 = b_1 = ... = b_{n-1} = 0, \quad b_n = \frac{-m_1}{1 - m_2 n}. \tag{2.15}$$
In principal, this procedure could yield all the coefficients $b_k$, even those for $k > n$, but it would require generating all terms in the series expansion (2.14), which is not practical. So, to find the additional non-zero terms in $\phi$, if any, both sides of eq. (2.8) are divided by $2\pi i (\sigma - \zeta)$, and then integrate around the unit circle $\gamma$:

$$\frac{1}{2\pi i} \int_{\gamma} \frac{\phi(\sigma)}{\sigma - \zeta} d\sigma + \frac{m_1 m_2 n}{(1 - m_2 n)} \frac{1}{2\pi i} \int_{\gamma} \frac{\sigma^n}{\sigma - \zeta} d\sigma + \frac{1}{2\pi i} \int_{\gamma} \frac{\psi(\sigma)}{\sigma - \zeta} d\sigma$$

$$= -\frac{1}{2\pi i} \int_{\gamma} \left( -\frac{1}{\sigma} + m_2 \sigma + m_2 \sigma^{2n+1} \right) \frac{1}{\sigma - \zeta} d\sigma. \quad (2.16)$$

Evaluation of the integrals, ignoring any resulting constant terms (which correspond to rigid-body displacements), yields

$$\phi(\zeta) = \frac{-m_1}{1 - m_2 n} \zeta^n - m_2 \zeta^{2n+1}. \quad (2.17)$$

If four terms are taken in the mapping function, the same procedure as used above eventually leads to

$$\phi(\zeta) = B_1 \zeta^n + B_2 \zeta^{2n+1} - m_3 \zeta^{3n+2}, \quad (2.18)$$

where

$$B_1 = \frac{-m_1 - m_2 m_3 (2n + 1)}{1 - nm_2 - n^2 m_1 m_3 - n(2n + 1) m_3^2}, \quad (2.19)$$

$$B_2 = \frac{-m_2 + nm_2^2 + n^2 m_1 m_2 m_3 - nm_1 m_3}{1 - nm_2 - n^2 m_1 m_3 - n(2n + 1) m_3^2}. \quad (2.20)$$

To calculate the second potential function, $\psi$, the conjugate of eq. (2.8), is taken, making use of the fact that $\phi$ is now known. After expanding out the product, one obtains

$$\overline{\phi(\sigma)} - \frac{nm_1}{\omega(\sigma)(1 - nm_2)} \sigma^n - \frac{(2n + 1)m_2}{\omega(\sigma)} \sigma^{2n+1} = \frac{nm_1^2}{(1 - nm_2) \omega(\sigma) \omega'(\sigma)} - \frac{(2n + 1)m_2}{\omega'(\sigma)} \sigma^n + \psi(\sigma) = -\sigma - \frac{m_1}{\sigma^n} - \frac{m_2}{\sigma^{2n+1}}. \quad (2.21)$$

Both sides of this equation are divided by $2\pi i (\sigma - \zeta)$, and integrated around the unit circle in the $\zeta$-plane. As the mapping is single-valued, $\omega'(\sigma)$ can never vanish on or within $\gamma$. Most of the integrals therefore contain only simple poles. After ignoring any resulting constants, the evaluation of these integrals yields
\[
\frac{-nm_1 \zeta^n}{(1-nm_2)\omega'(\zeta)} - \frac{m_2(2n+1)\zeta^{2n+1}}{\omega'(\zeta)} - \frac{m_1^2 n}{(1-nm_2)\omega(\zeta)\zeta} - \frac{(2n+1)m_1m_2\zeta^n}{\omega'(\zeta)} \\
- \frac{m_2^2(2n+1)}{\omega(\zeta)\zeta} - \frac{nm_1m_2}{(1-nm_2)} \frac{1}{2\pi i} \int_\gamma \frac{1}{\omega'(\sigma)\sigma^{n+2}(\sigma - \zeta)} d\sigma + \psi(\zeta) = -\zeta. \tag{2.22}
\]

The remaining integrand has poles at \(\sigma = \zeta\) and \(\sigma = 0\). To find the residue at \(\sigma = 0\), the denominator is expanded as a power series in the numerator, to yield

\[
\left(\frac{m_1m_2n}{1-m_2n}\right) \frac{1}{2\pi i} \int_\gamma \frac{1}{\sigma^n(\sigma - \zeta)} \left[1 + nm_1\sigma^{n+1} + (2n+1)m_2\sigma^{2n+2} + \ldots\right] d\sigma. \tag{2.23}
\]

The residue at \(\sigma = 0\) is evaluated first. Including the terms that multiply this integral in eq. \(2.22\), we find a contribution of \(-nm_1m_2\zeta^{-n}/(1-nm_2)\). Similarly, the residue associated with the simple pole at \(\zeta = \sigma\) leads to a contribution \(-m_1m_2n\zeta^{-(n+2)}/(1-m_2n)\omega'(\zeta)\). Therefore, the second potential is

\[
\psi(\zeta) = -\zeta + \frac{nm_1}{1-m_2n} \frac{\zeta^n}{\omega'(\zeta)} + \frac{m_2(2n+1)}{\omega'(\zeta)} \zeta^{2n+1} + \frac{m_1^2 n}{(1-m_2n)\omega'(\zeta)\zeta} + \frac{m_2^2(2n+1)}{\omega'(\zeta)\zeta^n} + \frac{\left(\frac{m_1m_2n}{1-m_2n}\right)\frac{1}{\omega'(\zeta)}}{\left(\frac{m_1m_2n}{1-m_2n}\right)} \frac{1}{\zeta^{n+2}}. \tag{2.24}
\]

The full expression for the second potential associated with the four-term mapping function is similarly found to be

\[
\psi(\zeta) = -\zeta + \frac{nm_1B_1}{\zeta^n} + \frac{nm_2B_1}{\zeta^{2n+1}} + \frac{nm_1m_2B_1}{\zeta^n} + \frac{(2n+1)m_3B_2}{\zeta^n}
\]

\[
\begin{bmatrix}
  -nB_1\zeta^n - B_2(2n+1)\zeta^{2n+1} - m_3(3n + 2)\zeta^{3n+2} + m(2n+1)B_2\zeta^n \\
  + m_1m_3(3n+2)\zeta^{2n+1} - m_2m_3(3n+2)\zeta^n - \frac{nm_1B_1}{\zeta} - \frac{(2n+1)m_2B_2}{\zeta} \\
  -nm_2B_1 + nm_3B_1 + \frac{(2n+1)m_3B_2}{\zeta^{n+2}} - \frac{(3n+2)m^2B_2}{\zeta^n}
\end{bmatrix}
\]

where the \(B_1\) terms are defined in eqs. \(2.19\) and \(2.20\). Although these expressions seem to have singularities at \(\zeta = 0\), algebraic manipulation shows that these apparent singularities are removable, and \(\psi(\zeta)\) is indeed analytic inside the unit circle, as originally claimed.
2.4. Determination of the pore compressibility

Two pore compressibility parameters can be defined for a porous material: \( C_{pc} \), the fractional decrease in the area of the pore due to an external hydrostatic confining pressure of unit magnitude, and \( C_{pp} \), the fractional increase in the area of the pore due to an internal hydrostatic pore pressure of unit magnitude (Zimmerman, 1991; Detournay & Cheng, 1992). Simple superposition arguments show that these two parameters are related by

\[
C_{pc} = C_{pp} + (\kappa - 1)/2G,
\]

where \( (\kappa - 1)/2G \) is the two-dimensional areal bulk compressibility (Zimmerman, 1986). For plane strain, this coefficient is \((1 - 2\nu)/G\).

By definition, \( C_{pp} = (1/A)(dA/dp) \). In the context of linear elasticity, and bearing in mind the choice of \( p = 1 \), it follows that \( C_{pp} = \Delta A/A \), where \( A \) is the initial area of the pore. The initial area \( A \) can be expressed in terms of the mapping function, as follows (Zimmerman, 1986), using Green’s theorem:

\[
A = \iint dx dy = \frac{1}{2} \int xdy - ydx = -\frac{1}{2} \int [xy'(\alpha) - yx'(\alpha)] d\alpha
\]

\[
= -\frac{1}{2} \int \text{Im}[z(\alpha)z'(\alpha)] d\alpha,
\]

where the minus sign is introduced to ensure that the integration is performed in the anti-clockwise sense with respect to the hole in the \( z \)-plane. For the three-term mapping function, the functions needed for this last integral are

\[
z(\alpha) = e^{i\alpha} + m_1 e^{-i\alpha} + m_2 e^{-i(2n+1)\alpha}, \tag{2.27}
\]

\[
z'(\alpha) = -ie^{-i\alpha} + im_1 ne^{i\alpha} + im_2(2n + 1)e^{i(2n+1)\alpha}. \tag{2.28}
\]

Therefore, the initial area is

\[
A = -\frac{1}{2} \int_{0}^{2\pi} \text{Im}[e^{i\alpha} + m_1 e^{-i\alpha} + m_2 e^{-i(2n+1)\alpha}] [e^{-i\alpha} + im_1 ne^{i\alpha} + im_2(2n + 1)e^{i(2n+1)\alpha}] d\alpha
\]

\[
= \frac{1}{2} \int_{0}^{2\pi} \text{Im}[\pi(1 - nm_1^2 - (2n + 1)m_2^2) + \text{periodic terms}] d\alpha
\]

\[
= \pi[1 - nm_1^2 - (2n + 1)m_2^2]. \tag{2.29}
\]

Similarly, it can be shown that the original area of the pore represented by the four-term mapping function is given by

\[
A = \pi[1 - m_1^2 n - m_2^2 (2n + 1) - m_2^2 (3n + 2)]. \tag{2.30}
\]

The area change can be determined by integrating the normal component of the displacement around the pore boundary. The outward unit normal vector to the contour is given by \( n = [-y'(\alpha), x'(\alpha)]/|z'(\alpha)| \), the arclength of the contour is
\[ ds = |z'(\alpha)|d\alpha, \] the displacement vector is \( u = (u, v) \), and the normal component of the displacement is \( u_n = u \cdot n \), so

\[
\Delta A = \int_{r} u_n ds = \int u \cdot n |z'(\alpha)|d\alpha = \int_{\partial r} (u, v) \cdot \left[ \frac{-y'(\alpha), x'(\alpha)}{|z'(\alpha)|} \right] z'(\alpha) d\alpha
\]

\[
= \int_{\partial r} (u, v) \cdot [-y'(\alpha), x'(\alpha)] d\alpha = \int_{0}^{2\pi} [v x'(\alpha) - u y'(\alpha)] d\alpha
\]

\[
= -\text{Im} \int_{0}^{2\pi} \overline{U(\alpha)} z'(\alpha) d\alpha.
\] (2.31)

In vector notation the displacement is represented by \( u = (u, v) \), whereas in the complex plane this same displacement is written as \( U = u + iv \); both notations have been used in eq. (2.31), where appropriate.

Inserting the complex displacement eq. (2.1) into eq. (2.31) shows that the area change can be broken up into three parts, which are evaluated separately. In each case the integral can be evaluated by elementary means to give:

\[
2G\Delta A_1 = -\text{Im} \int_{0}^{2\pi} \kappa \phi(\alpha) z'(\alpha) d\alpha
\]

\[
= -\kappa \text{Im} \left[ \int \frac{-m_1}{(1 - m_2 n)} e^{-in\alpha} - m_2 e^{-i(2n+1)\alpha} \right] \left[ -ie^{-i\alpha} + m_1 ni e^{-in\alpha} + m_2 (2n + 1)i e^{i(2n+1)\alpha} \right] d\alpha
\]

\[
= 2\pi \kappa \left[ \frac{m_1^2}{1 - m_2 n} + m_2^2 (2n + 1) \right], \quad (2.32)
\]

\[
2G\Delta A_2 = \text{Im} \int_{0}^{2\pi} \frac{z(\alpha)}{z'(\alpha)} \phi'(\alpha) z'(\alpha) d\alpha
\]

\[
= \int_{0}^{2\pi} \left[ e^{i\alpha} + m_1 e^{-in\alpha} + m_2 e^{-i(2n+1)\alpha} \right] \left[ \frac{-m_1 ni}{1 - m_2 n} e^{in\alpha} - im_2 (2n + 1)e^{i(2n+1)\alpha} \right] d\alpha
\]

\[
= -2\pi \left[ \frac{m_1^2 n}{1 - m_2 n} + m_2^2 (2n + 1) \right], \quad (2.33)
\]

and similarly, but omitting the details,

\[
2G\Delta A_3 = \text{Im} \int \psi(\alpha) z'(\alpha) d\alpha = 2\pi \left[ 1 + \left( \frac{m_1^2 n}{1 - m_2 n} \right) + m_2^2 (2n + 1) \left( \frac{m_1^2 m_2 n^2}{1 - m_2 n} \right) \right]. \quad (2.34)
\]
Hence, the total area change of the “three-term” hole is given by

$$\Delta A = 2\pi \left\{ \kappa \left[ \frac{m_1^2 n}{1 - nm_2} + m_2^2 (2n + 1) \right] + \left( \frac{m_1^2 m_2 n^2}{1 - m_2 n} + 1 \right) \right\}. \quad (2.35)$$

The corresponding terms in the area change for the four-term mapping function are

$$\Delta A_1 = \kappa \left[ B_1 m_1 n + B_2 m_2 (2n + 1) - m_2^2 (3n + 2) \right], \quad (2.36)$$

$$\Delta A_2 = B_1 m_1 n + B_2 m_2 (2n + 1) - m_2^2 (3n + 2), \quad (2.37)$$

$$\Delta A_3 = 1 - n (2n + 1) m_1 m_3 B_2 - nm_1 B_1 - (3n + 2) m_3^2 - (2n + 1) m_2 B_2 - n^2 m_1 m_3 B_1 - n^2 m_1 m_2 B_1, \quad (2.38)$$

where $B_1$ and $B_2$ are as defined in eqs. (2.19) and (2.20).

Finally, the pore compressibility with respect to pore pressure is found from $C_{pp} = \Delta A / A$, where $A$ is given by eq. (2.29) or eq. (2.30), and $\Delta A$ is given by eq. (2.35) or eqs. (2.36)-(2.38). The pore compressibility with respect to the far-field stress, $C_{pc}$, is related to $C_{pp}$ by $C_{pc} = C_{pp} + C_o$, which is to say, $C_{pc} = C_{pp} + (1 - 2\nu) / G$. The pore compressibility $C_{pc}$ is related to the effective macroscopic compressibility by $C_{eff} = C_o + \phi C_{pc}$. So, to relate the present results to previous work on the effective moduli problem, the following discussion will be phrased in terms of $C_{pc}$.

### 2.5. Results and discussion

To validate the above results, they are compared to the compressibility values that have been obtained previously in several special cases, and also to the values obtained by boundary element calculations. The boundary element calculations are performed using a code developed by Martel & Muller (2000), which is a simplified version of the more general two-dimensional BEM code from Crouch & Starfield (1983) that is based on the displacement discontinuity method. Martel’s code is in a sense optimised for the problem of a single void or crack in an infinite elastic body, with many of the options included in the original code removed, thus rendering it easier to use for the present problem.

In these calculations, all far-field stresses are set to zero, as are the body forces. A uniform normal traction of unit magnitude is prescribed over the surface of the hole. The cavity boundary is discretized into a number of equal-length elements. The number of elements is always taken to be a multiple of $n + 1$, the number of “sides” of the pore, to ensure that two boundary elements meet precisely at each corner or cusp, so that the corners are not chopped off. It was found generally that roughly 300 boundary elements are sufficient to achieve convergence of the computed compressibilities.
As mentioned previously, the results for pores represented by a two-term mapping function (hypotrochoid) have been obtained by Zimmerman (1986), Jasiuk et al. (1994) and Kachanov et al. (1994): \( C_{pc} = 2(1 - \nu)(1 + nm_1^2)/G(1 - nm_1^2) \). The present results indeed reduce to these values identically when \( m_2 = m_3 = 0 \). Moreover, it can be shown by tedious algebraic manipulation of our solution that \( C_{pc} \) always depends on the elastic moduli of the host material only through the multiplicative factor \( (1 - \nu)/G \). Hence, the purely geometric effect of pore shape can be discussed in terms of the dimensionless pore compressibility, \( G C_{pc} / (1 - \nu) \). For a circular hole, this dimensionless compressibility is exactly 2.

Next, consider the quasi-polygons represented by the first three or four terms of the Schwarz-Christoffel mapping:

\[
\omega(\zeta) = \frac{1}{\zeta} + \frac{2}{n(n+1)} \zeta^n + \frac{n-1}{(2n+1)(n+1)^2} \zeta^{2n+1} + \frac{2n(n-1)}{3(3n+2)(n+1)^3} \zeta^{3n+2}. \quad (2.39)
\]

Table 2.1 shows the present results for the three-term and four-term holes, compared with the values obtained by Jasiuk et al. (1994) and Kachanov et al. (1994). With the exception of the result obtained by Kachanov et al. (1994) for the three-term quasi-triangle, all the values agree to three decimal places. A more sensitive test of the results is shown in Table 2.2, where the percentage changes in \( C_{pc} \) obtained by going from two to three terms, and then from three to four terms, are shown. The agreement between the present analytical results, the boundary element results, and the results reported by Jasiuk et al. (1994) are excellent.

### Table 2.1. Normalised pore compressibilities, \( G C_{pc} / (1 - \nu) \), of some quasi-polygonal holes represented by \( N \) terms in the Schwarz-Christoffel mapping, as calculated by various methods.

<table>
<thead>
<tr>
<th>Shape</th>
<th>BEM</th>
<th>Kachanov et al.</th>
<th>Jasiuk et al.</th>
<th>Present analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( N = 2 )</td>
<td>( N = 3 )</td>
<td>( N = 4 )</td>
<td>( N = 2 )</td>
</tr>
<tr>
<td>Square</td>
<td>2.3642</td>
<td>2.3956</td>
<td>2.4035</td>
<td>2.364</td>
</tr>
<tr>
<td>Pentagon</td>
<td>2.1699</td>
<td>2.1838</td>
<td>2.1883</td>
<td>2.168</td>
</tr>
<tr>
<td>Hexagon</td>
<td>2.0914</td>
<td>2.1009</td>
<td>2.1038</td>
<td>2.0910</td>
</tr>
</tbody>
</table>
Table 2.2. Percentage increase in $C_{pc}$ upon addition of an additional term to the mapping function.

<table>
<thead>
<tr>
<th>Shape</th>
<th>BEM 3rd term</th>
<th>BEM 4th term</th>
<th>Jasiuk et al. 3rd term</th>
<th>Jasiuk et al. 4th term</th>
<th>Present work 3rd term</th>
<th>Present work 4th term</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>2.22</td>
<td>0.52</td>
<td>2.22</td>
<td>0.55</td>
<td>2.22</td>
<td>0.51</td>
</tr>
<tr>
<td>Square</td>
<td>1.33</td>
<td>0.33</td>
<td>1.33</td>
<td>0.36</td>
<td>1.32</td>
<td>0.35</td>
</tr>
<tr>
<td>Pentagon</td>
<td>0.64</td>
<td>0.21</td>
<td>0.77</td>
<td>0.22</td>
<td>0.75</td>
<td>0.20</td>
</tr>
<tr>
<td>Hexagon</td>
<td>0.45</td>
<td>0.14</td>
<td>0.43</td>
<td>0.16</td>
<td>0.46</td>
<td>0.10</td>
</tr>
</tbody>
</table>

It would be of interest to have “exact” values for the compressibilities of the equilateral polygons that are represented by an infinite number of terms of the Schwarz-Christoffel mappings. If the calculated compressibilities are plotted against $1/N^2$, where $N$ is the number of terms taken in the mapping function, the results seem to be converging linearly with $1/N^2$ (Fig. 2.2). The values for $N \to \infty$ are calculated by extrapolating from the values for $N = 3$ and $N = 4$, using $1/N^2$ as the independent variable, and letting $1/N^2 \to 0$. These extrapolated values are given in Table 2.3, along with the values that computed using the boundary element method. The extrapolated analytical values generally agree with the boundary element calculations to three digits, thus giving confidence that these values are accurate to at least three digits. It should also be noted that the extrapolated values are in each case much closer to the BEM values than are the values calculated using only four terms in the mapping function, which provides some justification for the extrapolation procedure.

![Figure 2.2](image)

Figure 2.2. Determination of pore compressibilities of some regular equilateral polygons, by extrapolating the analytical results for the $N$-term Schwarz-Christoffel approximations. Fitted lines are intended to illustrate the plausibility of a linear extrapolation; the values shown in Table 2.3 are found by linear extrapolation of the values for $N = 3$ and $N = 4$. 

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Table 2.3. Normalized pore compressibilities, $GC_{pc}/(1-\nu)$, of some regular equilateral polygons, as computed by boundary elements and by extrapolation of the analytical results.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Triangle</th>
<th>Square</th>
<th>Pentagon</th>
<th>Hexagon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extrapolated analytical values</td>
<td>3.250</td>
<td>2.418</td>
<td>2.195</td>
<td>2.108</td>
</tr>
<tr>
<td>BEM</td>
<td>3.245</td>
<td>2.416</td>
<td>2.198</td>
<td>2.112</td>
</tr>
</tbody>
</table>

These results show that the normalized compressibility of an equilateral triangle is 3.25, or about 63% larger than that of a circle. The compressibility of these equilateral polygons decreases rapidly as the number of sides increases: a square is 21% more compressible than a circle, the pentagon 10% more compressible, and the hexagon only 5% more compressible. Regular polygons with more than six sides will therefore have compressibilities that are essentially the same as that of a circle.

Although both the conformal mapping approach and the BEM method can in principle be used to compute the compressibility of a pore of any shape, calculation of the mapping coefficients for complex pore shapes is extremely tedious, and computationally non-trivial (Sisavath et al., 2001; Tsukrov & Novak, 2002). So, it would also be useful if the pore compressibility could be calculated from some simple geometric attributes of the pore shape, without requiring elaborate analytical or numerical calculations. Such a capability would be useful in attempts to estimate elastic moduli from images of heterogeneous media, for example (Tsukrov et al., 2005).

Zimmerman (1986) suggested that the pore compressibility $C_{pc}$ scales (approximately) with $P^2/A$, where $P$ is the perimeter and $A$ is the area. Forcing this scaling law to be exact for a circular hole leads to the approximation

$$C_{pc} \approx \frac{2(1-\nu)}{G} \frac{P^2}{4\pi A}.$$  \hspace{1cm} (2.40)

Zimmerman (1991) showed that this approximation has an error of less than 8% for all hypotrochoids (i.e., two terms in the mapping function), and an error of about 23% for thin, crack-like pores, which he suggested might be the “worst-case” shape. Tsukrov & Novak (2002) verified that eq. (2.40) had an error of only 8% for a single, arbitrarily drawn irregular pore. In order to assess the usefulness of this approximation for other pore shapes, we have tested it against some of the shapes that can be generated using our two-term, three-term and four-term mappings.

Tables 2.4 and 2.5 show the pore compressibility values for some two-term and three-term pores having three-fold symmetry (i.e., $n = 2$). The compressibility increases as the hole becomes “less circular”, in some macroscopic sense. The presence of pointed cusps also causes the compressibility to increase. The scaling law (2.40) accounts in a rough sense for the effect of pore shape on compressibility, but in general should not be expected to have better than about 10% accuracy.
Table 2.4. Normalized pore compressibility \( GC_{pc}/(1−\nu) \) of some pores having a three-fold axis of symmetry, and which can be represented by two terms in the mapping function.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Coefficients</th>
<th>Analytical</th>
<th>BEM</th>
<th>Eq. (2.40)</th>
<th>Error of (2.40)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( n=2, m_1 = 1/9 )</td>
<td>2.1013</td>
<td>2.1012</td>
<td>2.1012</td>
<td>0.0%</td>
</tr>
<tr>
<td></td>
<td>( n=2, m_1 = 1/5 )</td>
<td>2.3478</td>
<td>2.3479</td>
<td>2.3352</td>
<td>0.5%</td>
</tr>
<tr>
<td></td>
<td>( n=2, m_1 = 1/3 )</td>
<td>3.1429</td>
<td>3.1430</td>
<td>3.1916</td>
<td>1.6%</td>
</tr>
<tr>
<td></td>
<td>( n=2, m_1 = 1/2 )</td>
<td>6.0000</td>
<td>5.9780</td>
<td>6.4845</td>
<td>8.1%</td>
</tr>
</tbody>
</table>

Table 2.5. Normalized pore compressibility \( GC_{pc}/(1−\nu) \) of some pores having a three-fold axis of symmetry, and which can be represented by three terms in the mapping function.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Coefficients</th>
<th>Analytical</th>
<th>BEM</th>
<th>Eq. (40)</th>
<th>Error of eq. (40)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( n=2 ) ( m_1 = 1/3, m_2 = 1/15 )</td>
<td>3.475</td>
<td>3.479</td>
<td>3.631</td>
<td>4.5%</td>
</tr>
<tr>
<td></td>
<td>( n=2 ) ( m_1 = 1/6, m_2 = 2/15 )</td>
<td>2.769</td>
<td>2.775</td>
<td>3.140</td>
<td>13.4%</td>
</tr>
<tr>
<td></td>
<td>( n=2 ) ( m_1 = 1/9, m_2 = 7/45 )</td>
<td>2.734</td>
<td>2.740</td>
<td>3.321</td>
<td>21.5%</td>
</tr>
</tbody>
</table>
3. Compressibility of a Rough-Walled Pore

3.1 Introduction

One issue that must be addressed when using geometric information obtained from pore space images is that of surface roughness. It is well known that the apparent roughness of a pore surface, as observed in a micrograph, increases as the magnification increases (Berryman & Blair, 1987). Indeed, some researchers have claimed that pore boundaries in sedimentary rocks may be fractal in nature, in which case the perimeter of a two-dimensional image would increase without bound as the “magnification” increases (Krohn, 1988). With regards to attempts to model the permeability of a porous rock, it is known that roughness of a sufficiently small spatial wavelength and amplitude has no effect on the hydraulic transmissivity of a pore (Phan-Thien, 1995; Bernabe & Olson, 2000). Similarly, it is intuitively expected that small-scale roughness should have no effect on the mechanical compliance of a pore. However, Zimmerman (1991) proposed a scaling law that estimates the compressibility of a pore in terms of its perimeter and area, and this law is used in Chapter 6 to predict the compressibility of some porous materials based on two-dimensional images. Hence, it is necessary to be able to quantify the level of roughness that can (and should) be ignored when interpreting the pore images.

In this chapter, the effect of small-scale roughness on the pore compressibility is studied via a sample problem in which sinusoidal roughness is superimposed onto a nominally circular pore. It seems reasonable to assume that the results will be qualitatively applicable to pores of other shapes. This geometry is not well suited for the conformal mapping method used in Chapter 2. Instead, the boundary perturbation approach will be used to develop an accurate approximate solution to this problem. The basic ideas of the boundary perturbation approach were illustrated by van Dyke (1975) in the context of fluid mechanics. Low & Chang (1967) used this approach to study stresses around nearly circular pores. Wang & Chao (2002) used the boundary perturbation method to solve the problem of a nearly circular inclusion in plane thermoelasticity. Similar studies were carried out by Parnes (1987) and Gao (1990).

Givoli & Elishakoff (1992) used this approach to analyze a corrugated pore in an infinite region, under far-field loading, and found the solution up to terms of second-order in the small parameter \( \varepsilon \) that quantified the amplitude of the corrugations. In the present study, the solution is extended to fourth-order in \( \varepsilon \). For reasons that will be explained in the next section, it is easier to solve this problem for a rough-walled rigid inclusion, and then use some results of Dundurs (1989) and Jasiuk (1995) to extend the rigid inclusion solution to the case of a similarly-shaped vacuous pore. The solution will then be used to investigate the effect of small-scale roughness on the pore compressibility and the effective bulk modulus of a body containing a dilute concentration of such pores.
3.2. Problem formulation for infinite region containing a rigid inclusion

The solution of a two-dimensional isotropic elasticity problem can be represented in terms of the Airy stress function, $\Phi$, which satisfies the bi-harmonic equation $\nabla^2 \nabla^2 \Phi = 0$. The general solution, neglecting those terms that do not correspond to a uniform stress at infinity, can be written in polar co-ordinates as (Little, 1973; Barber, 1992):

$$
\Phi = A_0 r^2 + A_1 \ln r + A_2 \theta + \frac{A_3 \cos \theta}{r} + \frac{A_4 \sin \theta}{r}
+ \sum_{n=2}^{\infty} \left[ A_{5,n} r^{-n+2} \cos n \theta + A_{6,n} r^{-n} \cos n \theta + A_{7,n} r^{-n+2} \sin n \theta + A_{8,n} r^{-n} \sin n \theta \right]. \quad (3.1)
$$

The stresses and displacements associated with this solution are given by

$$
\tau_{rr} = 2A_0 + \frac{A_1}{r^2} - \frac{2A_3 \cos \theta}{r^3} - \frac{2A_4 \sin \theta}{r^3}
- \sum_{n=2}^{\infty} \left[ A_{5,n} (n+2)(n-1)r^{-n} + A_{6,n} n(n+1)r^{-n-2} \right] \cos n \theta
- \sum_{n=2}^{\infty} \left[ A_{7,n} (n+2)(n-1)r^{-n} + A_{8,n} n(n+1)r^{-n-2} \right] \sin n \theta, \quad (3.2)
$$

$$
\tau_{\theta \theta} = 2A_0 - \frac{A_1}{r^2} + \frac{2A_3 \cos \theta}{r^3} + \frac{2A_4 \sin \theta}{r^3}
+ \sum_{n=2}^{\infty} \left[ A_{5,n} (n-1)(n-2)r^{-n} + A_{6,n} n(n+1)r^{-n-2} \right] \cos n \theta
+ \sum_{n=2}^{\infty} \left[ A_{7,n} (n-2)(n-1)r^{-n} + A_{8,n} n(n+1)r^{-n-2} \right] \sin n \theta, \quad (3.3)
$$

$$
\tau_{r \theta} = \frac{A_2}{r^2} - \frac{2A_3 \sin \theta}{r^3} + \frac{2A_4 \cos \theta}{r^3}
+ \sum_{n=2}^{\infty} \left[ A_{5,n} (n-1)(n-2)r^{-n} + A_{6,n} n(n+1)r^{-n-2} \right] \cos n \theta
- \sum_{n=2}^{\infty} \left[ A_{5,n} (n-1)r^{-n} + A_{6,n} n(n+1)r^{-n-2} \right] \sin n \theta, \quad (3.4)
$$

$$
2G\mu_r = A_0 (\kappa - 1) r - \frac{A_1}{r} + \frac{A_3 \cos \theta}{r^2} + \frac{A_4 \sin \theta}{r^2}
+ \sum_{n=2}^{\infty} \left[ A_{5,n} (\kappa + n-1)r^{-n+1} + A_{6,n} n r^{-n-1} \right] \cos n \theta
$$


\[ + \sum_{n=2}^{\infty} \left[ A_{7,n}(\kappa + n - 1)r^{-n+1} + A_{8,n}nr^{-n-1} \right] \sin n\theta, \quad (3.5) \]

\[ 2Gu_0 = \frac{A_2}{r} + \frac{A_3 \sin \theta}{r^2} - \frac{A_4 \cos \theta}{r^2} \]

\[ + \sum_{n=2}^{\infty} \left[ A_{7,n}(\kappa - n + 1)r^{-n+1} - A_{8,n}nr^{-n-1} \right] \cos n\theta \]

\[ + \sum_{n=2}^{\infty} \left[ - A_{5,n}(\kappa - n + 1)r^{-n+1} + A_{6,n}nr^{-n-1} \right] \sin n\theta, \quad (3.6) \]

where \((A_0, A_1, A_2, A_3, A_4, A_5,n, A_6,n, A_7,n, A_8,n)\) are constants, and \((r, \theta)\) are the usual polar co-ordinates.

For an infinite region containing a single inclusion, with hydrostatic stress \(p\) acting at infinity, eq. (3.2) and eq. (3.3) immediately show that \(A_0 = p/2\). The values of the other coefficients will depend on the shape and elastic properties of the inclusion. Jasiuk et al. (1994) and Jasiuk (1995) showed that the effective bulk modulus of a body containing a dilute concentration of these inclusions can be expressed solely in terms of the coefficient \(A_1\), as follows:

\[ \frac{K_{\text{eff}}}{K} = 1 + \frac{\kappa + 1}{\kappa - 1} A_1(\kappa; p = 1) \frac{\pi}{A_{\text{inclusion}}} c, \quad (3.7) \]

where \(K\) is the two-dimensional bulk modulus of the host material, \(\kappa\) is the Muskhelishvili parameter of the host material, which equals \(3 - 4\nu\) for plane strain and \((3 - \nu)/(1 + \nu)\) for plane stress, \(\nu\) is the three-dimensional Poisson ratio of the host material, \(c\) is area fraction of the inclusions, and \(A_1\) is the coefficient of the logarithmic term in the Airy stress function for the case where the magnitude of the applied pressure is \(p = 1\). The essential explanation of this result is as follows. The two-dimensional bulk modulus can be related to the area change of the material, which can be calculated by integrating the radial displacement over a large circle centred on the inclusion. The \(A_0\) term, which is independent of the inclusion, gives the area change that would occur in the absence of an inclusion. Most terms in eq. (3.5) involve \(\sin\) or \(\cos\), and so their contributions to the area change integrate out to zero. Only the \(A_1\) term gives a non-zero contribution to the excess area change due to the presence of the inclusion. A rigorous proof of eq. (3.7) has been given by Jasiuk (1995).

Dundurs (1989) showed that the solution for a body containing a rigid inclusion could be transformed into the solution for the case in which the inclusion is a cavity, by setting \(\kappa = -1\). Jasiuk (1995) then showed that the effective bulk modulus of a body containing a dilute concentration of these cavities would be given by

\[ \frac{K_{\text{eff}}}{K} = 1 + \frac{\kappa + 1}{\kappa - 1} A_1(\kappa = -1; p = 1) \frac{\pi}{A_{\text{pore}}} c, \quad (3.8) \]
where $\kappa$ is set equal to $-1$ in the expression for $A_1$, but the actual value of $\kappa$ of the host material is used when evaluating the term $(\kappa + 1)/(\kappa - 1)$.

This correspondence is exploited in the present work. Although the main interest here is in the case of vacuous pores, when using the perturbation approach the problem of a rigid nearly circular inclusion is much easier to solve. This is because the boundary condition for the rigid inclusion is that the displacement vector must vanish, which directly implies that both components of the displacement, $u_r$ and $u_\theta$, must vanish along the boundary. On the other hand, if the inclusion is a pore, the traction vector must vanish on the boundary. But the normal and tangential components of the traction vector on the actual non-circular boundary are related to the stress components in the polar co-ordinate system in a very complicated manner (Givoli & Elishakoff, 1992). Hence, the solution procedure for a rigid inclusion is substantially simpler than that for a pore. This simplification will allow us to find the solution to fourth-order in the perturbation parameter, as opposed to the second-order solution found by Givoli & Elishakoff (1992).

### 3.3 General solution for rigid inclusion

The method starts by considering a rigid circular disc of radius $a$, in an infinitely large plate subjected to a far-field hydrostatic stress of magnitude $p$. The Airy stress function for this problem is $\Phi = p r^2/2 + (\kappa - 1) p a^2 \ln r/2$ (Barber, 1992) and the associated displacements and stresses are (Dundurs, 1989):

$$u_r = \frac{p(\kappa - 1)r}{4G} - \frac{p(\kappa - 1)a^2}{4Gr}, \quad u_\theta = 0,$$

$$(3.9)$$

$$\tau_{rr} = p + \frac{p(\kappa - 1)a^2}{2r^2}, \quad \tau_{\theta\theta} = p - \frac{p(\kappa - 1)a^2}{2r^2}, \quad \tau_{r\theta} = 0.$$  

$$(3.10)$$

Now consider a nearly circular inclusion that has a corrugated boundary described by

$$r = a(1 + \varepsilon \sin m \theta),$$

$$(3.11)$$

where $\varepsilon$ is a small parameter representing the amplitude of the corrugations, and $m$ is an integer that represents the number of positive (or negative) bumps on the circumference of the circle. As an example, Fig. 3.1 shows the inclusion shape for $m = 8$ and $\varepsilon = 0.25$. 

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Figure 3.1. Nearly-circular corrugated inclusion subjected to far-field hydrostatic stress, shown for the case \(m = 8, \varepsilon = 0.10\).

Following the standard procedure for a regular perturbation problem (van Dyke, 1975; Givoli, & Elishakoff, 1992), it is assumed that the Airy stress function, the displacements and the stresses can each be expressed as power series in the small parameter \(\varepsilon\), i.e.,

\[
\Phi(r, \theta) = \Phi^0(r, \theta) + \varepsilon \Phi^1(r, \theta) + \varepsilon^2 \Phi^2(r, \theta) + \ldots,
\]

\[
u_r(r, \theta) = u^0_r(r, \theta) + \varepsilon u^1_r(r, \theta) + \varepsilon^2 u^2_r(r, \theta) + \ldots,
\]

\[
u_\theta(r, \theta) = u^0_\theta(r, \theta) + \varepsilon u^1_\theta(r, \theta) + \varepsilon^2 u^2_\theta(r, \theta) + \ldots,
\]

where the superscript \(n\) in \(\Phi^n\) denotes the \(n\)-th Airy stress function, and is not a power exponent. When \(\varepsilon = 0\), this solution must reduce to the solution for a circular inclusion, and so:

\[
\Phi^0(r, \theta) = \frac{pr^2}{2} + \frac{(\kappa - 1)pq^2 \ln r}{2},
\]

\[
u^0_r(r, \theta) = \frac{p(\kappa - 1)r}{4G} - \frac{p(\kappa - 1)q^2}{4Gr}, \quad u^0_\theta(r, \theta) = 0.
\]

Each of the subsequent Airy functions will be of the form given by eqs. (3.1)-(3.6). As the far-field boundary conditions are already satisfied by the zeroth-order solution, the \(A_0\) coefficient will vanish in all the higher-order Airy functions. The remaining coefficients in these functions are found by satisfying the boundary conditions at the interface between the matrix and the inclusion. These boundary conditions require that both displacement components vanish at the interface. Hence, for the purposes of deriving the solution, the stresses need not be considered any further.

Zero-displacement boundary conditions must be applied at the boundary of the inclusion, where \(r = r^* = a(1 + \varepsilon \sin m \theta)\). But each function \(u^0_r(r, \theta)\) is expressed...
naturally in terms of \( r \) and \( \theta \). In order to re-express the boundary conditions at \( r = a \) instead of \( r = a(1 + \varepsilon \sin m\theta) \), the displacements are first expanded in a Taylor series about the value \( r = a \), with \( a\varepsilon \sin m\theta \) as the small variation in \( r \). Following the procedure used in (Givoli & Elishakoff, 1992) for the stresses, one arrives at

\[
\begin{align*}
\varepsilon \theta (r*, \theta) &= u_r(a + a\varepsilon \sin m\theta, \theta) = u_r(a, \theta) + a\varepsilon \sin m\theta \frac{\partial u_r\varepsilon (r, \theta)}{\partial r} |_{r=a} \\
+ \frac{a^2\varepsilon^2 \sin^2 m\theta \ \partial^2 u_r\varepsilon (r, \theta)}{2} |_{r=a} + \frac{a^3\varepsilon^3 \sin^3 m\theta \ \partial^3 u_r\varepsilon (r, \theta)}{6} |_{r=a} + ..., \quad (3.17)
\end{align*}
\]

and similarly for \( u_\theta(r*, \theta) \). Applying this expansion to every term in (3.13), and then grouping terms according to ascending powers of \( \varepsilon \), leads to

\[
\begin{align*}
u_r(r*, \theta) &= 0 = u_r\varepsilon (a, \theta) + \left[ a \sin m\theta \frac{\partial u_r\varepsilon (r, \theta)}{\partial r} |_{r=a} + u_1\varepsilon (a, \theta) \right] \varepsilon \\
+ \left[ \frac{a^2 \sin^2 m\theta \ \partial^2 u_r\varepsilon (r, \theta)}{2} |_{r=a} + a \sin m\theta \frac{\partial u_r\varepsilon (r, \theta)}{\partial r} |_{r=a} + u_2\varepsilon (a, \theta) \right] \varepsilon^2 + ..., \quad (3.18)
\end{align*}
\]

and similarly for the tangential displacement,

\[
\begin{align*}
u_\theta(r*, \theta) &= 0 = u_\theta\varepsilon (a, \theta) + \left[ a \sin m\theta \frac{\partial u_\theta\varepsilon (r, \theta)}{\partial r} |_{r=a} + u_1\varepsilon (a, \theta) \right] \varepsilon \\
+ \left[ \frac{a^2 \sin^2 m\theta \ \partial^2 u_\theta\varepsilon (r, \theta)}{2} |_{r=a} + a \sin m\theta \frac{\partial u_\theta\varepsilon (r, \theta)}{\partial r} |_{r=a} + u_2\varepsilon (a, \theta) \right] \varepsilon^2 + .... \quad (3.19)
\end{align*}
\]

Requiring the no-displacement boundary condition to be satisfied at all orders of \( \varepsilon \), the order-\( \varepsilon \) term in eq. (3.18) yields

\[
\begin{align*}
u_1\varepsilon (a, \theta) &= -a \sin m\theta \frac{\partial u_\varepsilon \varepsilon (r, \theta)}{\partial r} |_{r=a} = -a \sin m\theta \left\{ \frac{p(\kappa - 1)}{4G} + \frac{p(\kappa - 1)a^2}{4Gr^2} \right\} |_{r=a} \\
= -\frac{p}{2G} (\kappa - 1) a \sin m\theta , \quad (3.20)
\end{align*}
\]

which serves as a boundary condition at \( r = a \) for the as-yet unknown function \( u_1\varepsilon (r, \theta) \). Likewise, the order-\( \varepsilon \) term in eq. (3.19) yields

\[
\begin{align*}
u_1\varepsilon (a, \theta) &= -a \sin m\theta \frac{\partial u_\varepsilon \varepsilon (r, \theta)}{\partial r} |_{r=a} = -a \sin m\theta[0] = 0 . \quad (3.21)
\end{align*}
\]
Examination of the general solution for $\Phi^1$, as given by (3.1)-(3.6), shows that the only terms that yield a $\sin m\theta$ variation for $u^1(r, \theta)$ are those involving $A^1_{7,m}$ and $A^1_{8,m}$. Hence, the first-order displacement functions must have the form

$$2Gu^1(r, \theta) = \frac{A^1_{7,m}(\kappa + m - 1)\sin m\theta}{\rho^{m-1}} + \frac{A^1_{8,m}m\sin m\theta}{\rho^{m+1}}, \quad (3.22)$$

$$2Gu^1_0(r, \theta) = \frac{A^1_{7,m}(\kappa - m + 1)\cos m\theta}{\rho^{m-1}} - \frac{A^1_{8,m}m\cos m\theta}{\rho^{m+1}}. \quad (3.23)$$

Forcing these expressions to satisfy the boundary conditions (3.20) and (3.21) yields

$$\frac{A^1_{7,m}(\kappa + m - 1)}{a^{m-1}} + \frac{A^1_{8,m}m}{a^{m+1}} = -p(\kappa - 1)a, \quad (3.24)$$

$$\frac{A^1_{7,m}(\kappa - m + 1)}{a^{m-1}} - \frac{A^1_{8,m}m}{a^{m+1}} = 0, \quad (3.25)$$

the solution to which is

$$A^1_{7,m} = -\frac{a^m p(\kappa - 1)}{2\kappa}, \quad (3.26)$$

$$A^1_{8,m} = -\frac{a^{m+2} p(\kappa - 1)(\kappa - m + 1)}{2m\kappa}. \quad (3.27)$$

Hence, the first-order displacement functions are

$$2Gu^1(r, \theta) = -\frac{p(\kappa - 1)a^m}{2\kappa} \left[ \frac{(\kappa + m - 1)}{\rho^{m-1}} + \frac{(\kappa - m + 1)a^2}{\rho^{m+1}} \right] \sin m\theta, \quad (3.28)$$

$$2Gu^1_0(r, \theta) = -\frac{p(\kappa - 1)a^m(\kappa - m + 1)}{2\kappa} \left[ \frac{1}{\rho^{m-1}} - \frac{a^2}{\rho^{m+1}} \right] \cos m\theta. \quad (3.29)$$

Next, the coefficient of $\varepsilon^2$ is set to zero in expressions (3.18) and (3.19), to find the boundary conditions for $u^2(a, \theta)$ and $u^2_0(a, \theta)$. First, from the coefficient of $\varepsilon^2$ in eq. (3.18), and using eq. (3.16) for $u^0(r, \theta)$ and eq. (3.28) for $u^1(r, \theta)$, it is found that

$$2Gu^2(a, \theta) = \frac{ap(\kappa - 1)[(\kappa - 2) - 2m(\kappa - 1)]}{4\kappa} (1 - \cos 2m\theta), \quad (3.30)$$

where the identity $2\sin^2 m\theta = 1 - \cos 2m\theta$ has been used to express the boundary condition as a trigonometric series. Likewise, eqs. (3.16), (3.19) and (3.29) give
\[
2Gu_0^2(a, \theta) = \frac{ap(\kappa - 1)(\kappa - m + 1)}{2\kappa} \sin 2m\theta. \quad (3.31)
\]

Hence, both \(u_r^2(r, \theta)\) and \(u_\theta^2(r, \theta)\) will consist of a term that is independent of \(\theta\), and a term that varies as \(\cos 2m\theta\). Examination of the general solution (3.1)-(3.6) shows that the second-order displacement functions must therefore have the form

\[
2Gu_r^2(r, \theta) = -\frac{A_r^2}{r} + \left[ A_{5,2m}^2 (\kappa + 2m - 1)r^{-2m+1} + 2A_{6,2m}^2 mr^{-2m-1}\right] \cos 2m\theta, \quad (3.32)
\]

\[
2Gu_\theta^2(r, \theta) = -\left[ A_{5,2m}^2 (\kappa - 2m + 1)r^{-2m+1} + 2A_{6,2m}^2 mr^{-2m-1}\right] \sin 2m\theta. \quad (3.33)
\]

Requiring that the functions given by eqs. (3.32) and (3.33) satisfy the boundary conditions (3.30) and (3.31) leads to the following two equations for the \(A_{i,2m}^2\) coefficients:

\[
\frac{ap(\kappa - 1)(\kappa - 2) - 2m(\kappa - 1)}{4\kappa} \left(1 - \cos 2m\theta\right) = -\frac{A_r^2}{a} + \left[ A_{5,2m}^2 (\kappa + 2m - 1)a^{-2m+1} + 2A_{6,2m}^2 ma^{-2m-1}\right] \cos 2m\theta, \quad (3.34)
\]

\[
\frac{ap(\kappa - 1)(\kappa - m + 1)}{2\kappa} \sin 2m\theta = \left[ A_{5,2m}^2 (\kappa - 2m + 1)a^{-2m+1} + 2A_{6,2m}^2 ma^{-2m-1}\right] \sin 2m\theta. \quad (3.35)
\]

Matching separately the coefficients of the constant term, the cosine term and the sine term yields the required three equations that allows the three coefficients to be found:

\[
A_1^2 = \frac{-a^2 p(\kappa - 1)(\kappa - 2) - 2m(\kappa - 1)}{4\kappa}, \quad (3.36)
\]

\[
A_{5,2m}^2 = \frac{a^{2m} p(\kappa - 1)(2m - 3)}{8\kappa}, \quad (3.37)
\]

\[
A_{6,2m}^2 = \frac{a^{2m+2} p(\kappa - 1) [4(\kappa - m + 1) + (2m - 3)(\kappa - 2m + 1)]}{16m\kappa}. \quad (3.38)
\]

The third-order and fourth-order solutions can be found by following this same methodology. Each of these solutions will have the form given by (3.5) and (3.6), but with only a small number of non-zero coefficients. These non-zero coefficients are given in the Appendix to this chapter. It is noted here the pattern that only the even-order terms in the perturbation solution contain non-zero values of \(A_i\), and so the expression for the effective bulk modulus will contain only even powers of \(\varepsilon\). However, the even-order terms cannot be computed without having already computed all lower-order terms, and so the odd terms must be computed, despite the fact that
they do not contribute to the bulk modulus. The fact that the expression for the bulk modulus contains only even powers of \( \varepsilon \) could have been anticipated by noting that letting \( \varepsilon \rightarrow -\varepsilon \) corresponds to rotating the cavity by one-half of a wavelength. As the far-field stress is isotropic, a rotation of the cavity cannot alter the bulk modulus, and so the expression for \( K_{\text{eff}} \) must be an even function of \( \varepsilon \).

### 3.4 Effective bulk compressibility and pore compressibility

To be specific, consider plane strain, in which case \( (\kappa+1)/(\kappa-1) = 2(1-\nu)/(1-2\nu) \). The expression for the effective bulk modulus given by eq. (3.8) can then be written in terms of compressibility as follows:

\[
\frac{C}{C_{\text{eff}}} = 1 + \frac{2(1-\nu)}{(1-2\nu)} A_1(\kappa = -1; p = 1) - \frac{\pi}{A_{\text{pore}}} \phi, \tag{3.39}
\]

where \( C = 1/K \) is the bulk compressibility of the host material, and the inclusion area fraction \( c \) is replaced by the usual symbol for porosity, \( \phi \). For small values of \( \phi \), eq. (3.39) can be expanded as

\[
C_{\text{eff}} = C - \frac{2(1-\nu)C}{(1-2\nu)} A_1(\kappa = -1; p = 1) - \frac{\pi}{A_{\text{pore}}} \phi. \tag{3.40}
\]

But the effective compressibility can also be expressed in the present notation as (Zimmerman, 1991)

\[
C_{\text{eff}} = C + \phi C_{pc}, \tag{3.41}
\]

where \( C_{pc} \) is the compressibility of the pore with respect to the far-field “confining” pressure. Comparison of eqs. (3.40) and (3.41) shows that the pore compressibility is given by

\[
C_{pc} = -\frac{2(1-\nu)}{G} \frac{A_1(\kappa = -1; p = 1)\pi}{A_{\text{pore}}}. \tag{3.42}
\]

The pore compressibility parameter is of great importance in petroleum engineering, but also provides a convenient means to discuss the effect of pores on the bulk modulus, as shown by eq. (3.41).

An interesting qualitative implication of eq. (3.42) is that, since the term \( A_1(-1,1) \) does not depend on the elastic moduli, the pore compressibility is always proportional to \((1-\nu)/G\), with a dimensionless multiplicative factor that depends only on the shape of the pore. This result is consistent with the exact solutions found in Chapter 2 for a large family of pores having \( n \)-fold rotational symmetry, but it is shown here now that it is completely general. Indeed, this fact can also be obtained from some of the recent results of Vigdergauz (2006), if they are translated into the present terminology of “pore compressibility.”
Returning to the corrugated pore, the pore area can easily be shown to be given by \( A_{\text{pore}} = \pi a^2 [1 + (\varepsilon^2 / 2)] \). Hence, using eqs. (3.36) for \( A_1(\kappa, p) \), it is found that the second-order solution in \( \varepsilon \) is

\[
C_{pc} = \frac{2(1-\nu) [1 + (4m-3)\varepsilon^2 / 2]}{G \left( 1 + (\varepsilon^2 / 2) \right)}. \tag{3.43}
\]

The fourth-order perturbation solution yields

\[
C_{pc} = \frac{2(1-\nu) \left[ 1 + \varepsilon^2(4m-3)/2 - \varepsilon^4 f \right]}{G \left( 1 + (\varepsilon^2 / 2) \right)}, \tag{3.44}
\]

\[
f = \left[ \frac{3}{16} - \frac{1}{32} \frac{m(m+1)(2m+1) - (2m-3)}{8} \right] - \frac{1}{4} \left[ \frac{A_{7,2m}^3 (m-1)(m-2)}{a^m} + \frac{A_{8,2m}^3 m(m+1)}{a^{m+2}} \right]
- \frac{m}{4} \left[ \frac{A_{5,2m}^2 (2m-1)(m-1)}{a^{2m}} + \frac{A_{6,2m}^2 m(2m+1)(m+1)}{a^{2m+1}} \right], \tag{3.45}
\]

where the \( A_{i,2m}^2 \) and \( A_{i,2m}^3 \) coefficients are given in the Appendix.

### 3.5 Stress concentration

The stress concentration at the boundary of a hole in a stressed body is of great engineering interest (Savin, 1961). Givoli & Elishakoff (1992) discussed the stress concentration at the boundary of holes described by eq. (3.11), as given by their second-order perturbation solution. It has been verified that the stresses obtained from the present second-order solution agree with those found by Givoli & Elishakoff (1992). But as the present focus is on the pore compressibility and the effective bulk modulus, the issue of stress concentration will not be pursued any further.

### 3.6 Pore compressibility of corrugated pore

The compressibilities predicted by the two-term and four-term perturbation solutions are now compared with the values obtained by boundary element calculations, and with some upper and lower bounds that can be derived. The boundary element calculations are performed using a code developed by Martel & Muller (2000), which is a simplified version of the more general two-dimensional boundary element method (BEM) code from Crouch & Starfield (1983) that is based on the displacement discontinuity method. This code allows the solution of the problem of a pore of specified shape in an infinite elastic region, with prescribed stresses at infinity, and requires discretization only of the pore boundary. Typically, about 300 boundary elements are needed to obtain a converged solution.
An upper bound on the pore compressibility parameter can be obtained by starting with the fact that if the actual pore is replaced by a pore defined by the smallest possible circumscribed circle, which has radius \( a(1+\varepsilon) \), the overall bulk compressibility of the body cannot decrease, since removal of solid material cannot stiffen the body (Rice & Drucker, 1967; Gol’dshtein & Entov, 1994). Considering a large body of area \( A \) containing only one pore, it can be stated, from eq. (3.41), that

\[
C + \frac{A_{\text{pore}}}{A} C_{pc}(\text{pore}) \leq C + \frac{A_{\text{circumscribed circle}}}{A} C_{pc}(\text{circle}).
\]  

But \( C_{pc}(\text{circle}) = 2(1-\nu)/G \), \( A_{\text{pore}} = \pi a^2[1+(\varepsilon^2/2)] \), and \( A_{\text{circle}} = \pi a^2(1+\varepsilon)^2 \), and so it can be found that

\[
C_{pc}(\text{pore}) \leq \frac{(1+\varepsilon)^2}{[1+(\varepsilon^2/2)]} \frac{2(1-\nu)}{G}.
\]  

The Hashin-Shtrikman upper bound on the effective bulk modulus of a two-dimensional porous body (Hashin, 1983) can be used to show that the pore compressibility of the rough pore cannot be less than \( \frac{GC_{pc}}{(1-\nu)} \). This is equivalent to stating that no pore can be stiffer than a circular pore. A different lower bound on the compressibility of the rough pore could be found by using an argument based on the largest possible inscribed circle, which has radius \( a(1-\varepsilon) \). But this lower bound would be lower than \( \frac{2(1-\nu)}{G} \), and so is not useful. Hence, the normalized compressibility of the corrugated pore is bounded as follows:

\[
1 \leq \frac{G C_{pc}}{2(1-\nu)} \leq \frac{(1+\varepsilon)^2}{1+(\varepsilon^2/2)}.
\]

As, by definition, only small values of \( \varepsilon \) are of interest, one can say roughly that the normalized pore compressibility must lie between 1 and \( 1+2\varepsilon \).

The two-term and four-term perturbation expressions are plotted in Figs. 3.2-3.4 as functions of \( \varepsilon \), for the values \( m = 8, 16, \) and 32. Also plotted are the upper and lower bounds, and the values computed using the boundary element method. As the influence of roughness on compressibility is of second-order, the compressibility lies close to the lower bound for very small values of \( \varepsilon \). But as \( \varepsilon \) increases, the compressibility moves closer to the upper bound. Moreover, the difference between the exact value and the upper bound becomes smaller as the number of corrugations, \( m \), increases. As the upper bound corresponds to the case in which the small bumps of solid material have been removed, this shows that, particularly as \( m \) increases, these small bumps provide no stiffness to the pore. Although one might think that this result could have been anticipated, it is pointed out that the stress concentration increases drastically with \( m \) and \( \varepsilon \) (Givoli & Elishakoff, 1992) and it has been frequently asserted that the pore compressibility correlates with the stress concentration (Warren, 1973; Rice, 1998). But the present results show that as \( m \) increases, the pore compressibility approaches an upper bound that is independent of \( m \). Hence, this supposed correlation between pore compressibility and stress concentration, which had been inferred from the solutions for ellipsoidal and spheroidal pores, has no general validity whatsoever.
Figure 3.2. Pore compressibility as a function of the roughness amplitude, for the case $m = 8$.

Figure 3.3. Pore compressibility as a function of the roughness amplitude, for the case $m = 16$. 

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Figure 3.4. Pore compressibility as a function of the roughness amplitude, for the case \( m = 32 \).

As expected, the two-term perturbation expression agrees closely with the numerical values for a certain range of \( \varepsilon \), and then rapidly becomes unrealistically large, exceeding the upper bound. Using the compressibility predicted by the four-term solution extends the range of accuracy by about a factor of 2 in \( \varepsilon \). For example, for \( m = 8 \), the two-term solution is quite accurate until about \( \varepsilon = 0.07 \), whereas the four-term solution is accurate until \( \varepsilon = 0.14 \). The range of accuracy decreases as \( m \) increases, as was found by Givoli & Elishakoff (1992) for the stress concentrations. Roughly, the four-term solution is accurate until a critical value of about \( \varepsilon^* = 1/m \), beyond which its accuracy deteriorates rapidly.

Although this may seem like a severe limitation on the usefulness of the perturbation approach, in practice it is not, as it is now shown. There are \( m \) sine waves of the corrugations along the circumference of the nominal circle, which has a circumference of \( 2\pi a \), and so the half-wavelength of each bump is \( \pi a/m \). The change in radius that occurs over this half-wavelength is \( 2\varepsilon a \). If the reasonable assumption is made that the change in radius does not exceed the half-wavelength (otherwise, the corrugations would look like thin spikes), it is found that \( \varepsilon \) is restricted to be less than \( \pi/2m \), which essentially overlaps the range of \( \varepsilon \) values for which the perturbation solution is accurate. So, it seems that the range of accuracy of the pore compressibility predicted by the fourth-order perturbation solution probably covers most cases of practical interest.
3.7 Appendix: Coefficients in the third-order and fourth-order solutions

The non-zero coefficients required for the third-order solution are

\[
A_{3r,m}^3 = \frac{-a^{m-1}G(T_{r,m}^3 + T_{\theta,m}^3)}{\kappa},
\]

(3.49)

\[
A_{5r,m}^3 = a^{m+1}G \left[ \frac{(\kappa + m - 1)(T_{r,m}^3 - T_{\theta,m}^3)}{\kappa m} - \frac{2T_{r,m}^3}{m} \right],
\]

(3.50)

\[
A_{3\theta,3m}^3 = \frac{-Ga^{3m-1}(T_{r,m}^3 + T_{\theta,m}^3)}{\kappa},
\]

(3.51)

\[
A_{5\theta,3m}^3 = \frac{2a^{3m+1}G}{3m} \left[ \frac{(\kappa + 3m - 1)(T_{r,3m}^3 + T_{\theta,3m}^3)}{2\kappa a^{3m-1}} - T_{r,3m}^3 \right],
\]

(3.52)

\[
T_{r,3m}^3 = \frac{1}{4G} \left[ A_{5,2m}^2 \frac{(\kappa + 2m - 1)(2m - 1)}{a^{2m-1}} + A_{6,2m}^2 \frac{2m(2m + 1)}{a^{2m}} \right]
\]

\[
+ a(\kappa - 1) \frac{3}{16G} \left[ 1 - \frac{1}{2\kappa} \left[ m(m - 1)(\kappa + m - 1) + (m + 1)(m + 2)(\kappa - m + 1) \right] \right],
\]

(3.53)

\[
T_{\theta,3}^3 = \frac{1}{4G} \left[ - A_{5,2m}^2 \frac{(\kappa - 2m + 1)(1 - 2m)}{a^{2m-1}} - A_{6,2m}^2 \frac{2m(2m + 1)}{a^{2m}} \right]
\]

\[
+ \frac{a}{16G\kappa} \left[ (\kappa - 1)(\kappa - m + 1)(2m + 1) \right],
\]

(3.54)

\[
T_{r,3m}^3 = -\frac{1}{4G} \left[ A_{5,2m}^2 \frac{(\kappa + 2m - 1)(2m - 1)}{a^{2m-1}} + A_{6,2m}^2 \frac{2m(2m + 1)}{a^{2m}} \right]
\]

\[
+ a(\kappa - 1) \frac{1}{16G} \left[ 1 - \frac{1}{\kappa} \left[ m(m - 1)(\kappa + m - 1) + (m + 1)(m + 2)(\kappa - m + 1) \right] \right],
\]

(3.55)
Estimation of the Elastic Moduli of Porous Materials

\[
T_{\theta,3m}^3 = \frac{1}{4G} \left[ A_{5,2m}^2 \left( \frac{\kappa - 2m + 1}{a^{2m-1}} \right) + A_{6,2m}^2 \left( \frac{2m(2m+1)}{a^{2m}} \right) \right] - \frac{a}{16G\kappa} \left[ (\kappa - 1)(\kappa - m + 1)(2m + 1) \right]
\]

(3.56)

The non-zero coefficients required for the fourth-order solution are

\[
A_4^4 = \frac{-3a^2}{32G} (\kappa - 1) + \frac{a}{64G} \left[ \frac{p(\kappa - 1)(m + 1)}{\kappa} \left[ (m - 1)m(\kappa + m - 1) + (m + 2)(m + 3)(\kappa - m + 1) \right] \right]
\]

\[
+ \frac{a^2}{16G} \left[ \frac{p(\kappa - 2) - 2m(\kappa - 1)}{\kappa} \right] + \frac{a}{16G} \left[ A_{5,2m}^2 \left( \frac{\kappa + 2m - 1}{a^{2m-1}} \right) + A_{6,2m}^2 \left( \frac{2m(2m+1)}{a^{2m+3}} \right) \right]
\]

\[
- \frac{a}{4G} \left[ A_{5,2m}^3 \left( \frac{\kappa + m - 1}{a^{m-1}} \right) + A_{8,4m}^3 \left( \frac{m(m + 1)}{a^{m+1}} \right) \right]
\]

(3.57)

\[
A_{5,2m}^4 = \frac{Ga^{2m-1}(T_{\theta,2m}^4 - T_{r,2m}^4)}{\kappa}, \quad (3.58)
\]

\[
A_{6,2m}^4 = \frac{Ga^{2m+1}}{2m} \left[ -2T_{r,2m}^4 + (T_{\theta,2m}^4 - T_{r,2m}^4) \frac{(\kappa - 2m + 1)}{\kappa} \right], \quad (3.59)
\]

\[
A_{5,4m}^4 = \frac{Ga^{4m-1}(T_{\theta,4m}^5 - T_{r,4m}^5)}{\kappa}, \quad (3.60)
\]

\[
A_{6,4m}^4 = \frac{Ga^{4m+1}}{4m} \left[ -2T_{r,4m}^4 + \frac{(\kappa - 4m + 1)}{\kappa} (T_{\theta,4m}^4 - T_{r,4m}^4) \right], \quad (3.61)
\]
\[ T_{r,2m}^4 = \frac{ap(k-1)}{16\kappa} [(k-2) - 2m(k-1)] \]
\[ + \frac{p(k-1)a}{8} \left\{ \frac{ap(k-1)(m+1)}{6} \left[ (m-1)m(k+m-1) + (m+2)(m+3)(k-m+1) \right] - 1 \right\] \]
\[ - \frac{1}{4} \left[ A_{1,m}^{3} \frac{(k+m-1)(m+1)}{a^{m-1}} + A_{8,m}^{3} \frac{m(m+1)}{a^{m+1}} \right] \]
\[ + \frac{1}{4} \left[ A_{1,m}^{3} \frac{(k+3m-1)(3m-1)}{a^{3m-1}} + A_{8,m}^{3} \frac{3m(3m+1)}{a^{3m+1}} \right] \]
\[ - \frac{a^2}{8} \left[ A_{5,2m}^{3} \frac{(k+2m-1)(2m-1)2m}{a^{2m+1}} + A_{6,2m}^{3} \frac{2m(2m+1)(2m+2)}{a^{2m+3}} \right] \]
\[ \text{(3.62)} \]
\[ T_{r,4m}^4 = \frac{1}{32} \left\{ \frac{p(k-1)a - ap(k-1)(m+1)}{6\kappa} \left[ (m-1)m(k+m-1) + (m+2)(m+3)(k-m+1) \right] \right\} \]
\[ + \frac{1}{16} \left[ A_{5,2m}^{3} \frac{(k+2m-1)(2m-1)2m}{a^{2m-1}} + A_{6,2m}^{3} \frac{2m(2m+1)(2m+2)}{2a^{2m-3}} \right] \]
\[ - a \left[ A_{1,m}^{3} \frac{(k+3m-1)(3m-1)}{a^{3m-1}} + A_{8,m}^{3} \frac{3m(3m+1)}{a^{3m+1}} \right] \]
\[ \text{(3.63)} \]
\[ T_{\theta,2m}^4 = - \frac{p(k-1)a(k+m-1)}{96G\kappa} \left[ (1-m)m(m+1) + (m+1)(m+2)(m+3) \right] \]
\[ - \frac{a^2}{8G} \left[ A_{5,2m}^{3} \frac{(k-2m-1)(1-2m)2m}{a^{2m+1}} + A_{8,m}^{3} \frac{2m(2m+1)(2m+2)}{a^{2m+3}} \right] \]
\[ - \frac{a}{4G} \left[ A_{1,m}^{3} \frac{(k-m-1)(1-m)}{a^{m}} + A_{8,m}^{3} \frac{m(m+1)}{a^{m+2}} \right] \]
\[ + \frac{a}{4G} \left[ A_{1,m}^{3} \frac{(k-3m-1)(1-3m)}{a^{3m}} + A_{8,m}^{3} \frac{3m(3m+1)}{a^{3m+2}} \right] \]
\[ \text{(3.64)} \]
\[ T_{\theta,4m}^4 = - \frac{p(k-1)a(k+m-1)}{192G\kappa} \left[ (1-m)m(m+1) + (m+1)(m+2)(m+3) \right] \]
\[ + \frac{a^2}{16G} \left[ A_{5,2m}^{3} \frac{(k-2m-1)(1-2m)2m}{a^{2m+1}} + A_{6,m}^{3} \frac{2m(2m+1)(2m+2)}{a^{2m+3}} \right] \]
\[ - \frac{a}{4G} \left[ A_{1,m}^{3} \frac{(k-3m-1)(1-3m)}{a^{3m}} + A_{8,m}^{3} \frac{3m(3m+1)}{a^{3m+2}} \right] \]
\[ \text{(3.65)} \]
4. Shear Compliance of Two-Dimensional Symmetric Pores

4.1. Introduction

In the previous two chapters, the compressibility of a pore under a far-field hydrostatic stress was investigated by various analytical means. Hydrostatic loading is conceptually simple, since in this case issues such as the orientation of the pore with respect to the stress field, and the possibility of macroscopic anisotropy due to these orientation effects, does not arise.

Even though the present thesis is mainly focused on isotropic materials, the issue of pore orientation cannot be ignored when discussing the effect of shear loading on a pore. Consequently, it is necessary to use a more robust formalism when discussing the behavior of a pore under shear loading. In this chapter the formalism used by Kachanov et al. (1994) and Sevostianov & Kachanov (2007) will be used, in which the excess strain, $\Delta\varepsilon$, due to an inhomogeneity in an otherwise homogeneous medium subjected to a far-field stress $\sigma^\infty$, is expressed in terms of the fourth-order $H$ tensor, as follows:

$$\Delta\varepsilon = H : \sigma^\infty,$$

where the colon denotes the tensor inner product. For example, $H_{1212}$ connects the excess shear strain $\Delta\varepsilon_{12}$ to the applied remote shear stress, $\sigma_{12}^\infty$.

For ellipsoidal pores, or special cases thereof, the $H$ tensor can be expressed in terms of the Eshelby tensor, whose components are already known (Eshelby, 1957; Wu, 1966). For non-ellipsoidal pores, however, exact known results are sparse. Three-dimensional non-ellipsoidal shapes remain generally intractable by analytical methods. In two dimensions, the complex variable methods pioneered by Kolosov and Muskhelishvili (1963) can be used, although the calculations are tedious, and essentially must be developed on a case-by-case basis. In Chapter 2, the coefficient of $H$ that correspond to a hydrostatic far-field stress was found, for a large family of pores having an $N$-fold axis of symmetry, and which can be represented by four terms in the mapping function from the unit circle. Jasiuk et al. (1994) and Kachanov et al. (1994) found the components of $H$ that correspond to deviatoric loading, for hypotrochoidal pores and some quasi-polygonal pores.

In this chapter the shear-compliance coefficient $H_{1212}$ will be calculated for the same family of rotationally symmetric pores that were treated in Chapter 2 for the case of hydrostatic loading. Symmetric pores provide simplified models of real pores, but are of interest in their own right, as symmetric inclusions often have interesting properties (Xu & Wang, 2007). Moreover, these analytical solutions will be used to test a new proposed scaling law for the shear compliance (see section 4.6). In the process of deriving the solutions, the fact, proved by Eroshkin & Tsukrov (2005) in a more general but somewhat abstract context, that pores of $N$-fold symmetry are “isotropic” with regards to the effect of far-field stresses, except for the anomalous
cases of $N = 2$ or 4, will be explicitly verified. Furthermore, it will be shown that for the cases $N = 4$, as well as for pores possessing no symmetry, the term $H_{1212}$ varies as $\cos 4\theta$, where $\theta$ measures the orientation of the pore with respect to the far-field shear.

### 4.2 Formulation of the basic problem

Two-dimensional plane stress or plane strain elastostatic states can be represented in terms of two complex potentials, as follows (Sokolnikoff, 1956; Muskhelishvili, 1963):

$$2G(u_x + iu_y) = \kappa \phi(z) - z\phi'(z) - \psi(z). \tag{4.1}$$

where $\phi$ and $\psi$ are complex-valued analytic functions, and the $\kappa = 3 - 4\nu$ for plane strain and $(3 - \nu)/(1 + \nu)$ for plane stress. To formulate the problem of a body containing a cavity, with a state of pure shear stress at infinity, the following superposition argument can be used (Savin, 1961). First, consider a homogeneous body without a cavity, subjected to a pure shear of magnitude $\tau$ at infinity. For now, the shear is assumed to be aligned with the $(x, y)$ axes, so that $\tau_{xy} = \tau_{yx} = \tau$. As pointed out by Tsukrov & Novak (2002), for pores possessing symmetry, the coefficients of $H$ are insensitive to the orientation of the far-field stresses relative to the pore, except for the special cases of two-fold or four-fold symmetry, which will be treated later (section 4.5). The complex potentials corresponding to this state of stress are

$$\phi(z) = 0, \quad \psi(z) = i\tau z. \tag{4.2}$$

This will be referred to as “problem 1”.

Now imagine that the body contains a single cavity defined by some simple closed contour $\Gamma$. The region outside of $\Gamma$ can be mapped (Fig. 4.1) from the interior of the unit circle through a conformal mapping $z = \omega(\zeta)$, about which more will be said below. If there is no stress acting at infinity, and a complex traction vector $t = (t_x, t_y)$ acting along the cavity surface, then the boundary condition for the two potentials can be written along $\Gamma$ as (Sokolnikoff, 1956)

$$\phi(z) + z\phi'(z) + \psi(z) = f_x + i f_y = F, \tag{4.3}$$

where $F$ is equal to $i$ times the integral of the complex traction vector, $t_x + it_y$, along the boundary contour, starting from some arbitrary point $z_0$ on $\Gamma$. 


The stress state of problem 1 will lead to the correct stresses at infinity, but an unwanted, non-zero traction on $\Gamma$. To remove this traction, “problem 2” is defined for the body with the cavity as the one in which the tractions on $\Gamma$ are the negatives of those that would occur along $\Gamma$ in problem 1, and in which the stresses at infinity are zero. Clearly, the solution for a body with a traction-free cavity and shear stress at infinity is given by the sum of the solutions for problems 1 and 2.

The traction vector $t$ on $\Gamma$ in problem 1 is given by $t = Tn$, where $T$ is the stress tensor, and $n$ is the unit normal vector to $\Gamma$, which must point away from the solid body, which is to say, into the cavity. This normal vector is given by $n = [y'(\zeta), -x'(\zeta)]/|z'(\zeta)|$ (Zimmerman, 1986). So, the traction on $\gamma$ in problem 1 is

$$t = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \begin{bmatrix} y'(\zeta) \\ -x'(\zeta) \end{bmatrix} \frac{1}{|z'(\zeta)|} = -\tau \begin{bmatrix} x'(\zeta) \\ -y'(\zeta) \end{bmatrix}. \quad (4.4)$$

The function $F$ that appears on the right side of (4.2) is therefore

$$F = -i \int (tx + ity)dz = i\tau \int \frac{x'(\zeta) - iy'(\zeta)}{|z'(\zeta)|}dz = i\tau \int [x'(\zeta) - iy'(\zeta)]d\zeta$$

$$= i\tau(x - iy) = i\tau \bar{z}. \quad (4.5)$$

The liberty is taken of sometimes writing $t$ as a column vector, sometimes as a row vector, and sometimes in complex form as $t = tx + ity$, depending on whichever is most convenient for the purpose at hand.

The conformal mapping function will be of the form $\omega(\zeta) = \zeta^{-1} + m_1\zeta + m_2\zeta^2 + m_3\zeta^3 \ldots$. If only two non-zero terms in the mapping are taken, i.e., $\omega(\zeta) = \zeta^{-1} + m_n\zeta^n$, the hole is a hypotrochoid, which is a quasi-polygon having $n+1$ equal ‘sides’ (England, 1971; Zimmerman, 1986). In order for the mapping to be single-valued, and for $\Gamma$ not to contain any self-intersections, $m_n$ must satisfy the restriction $0 \leq m_n < 1/n$. The choice $m_n=0$ yields a circle, whereas the limiting value $m_n=1/n$ gives a pore with $n+1$ pointed cusps. For the particular choice $m_n=2/n(n+1)$, the mapping coincides with the first two terms of the Schwarz–
Christoffel mapping for an \((n+1)\)-sided equilateral polygon, and resembles a polygon with slightly rounded corners.

If the pore contour possesses an \((n+1)\)-fold axis of symmetry, only powers that differ by \((n+1)\) will appear in the mapping function, \textit{i.e.},

\[ z = \omega(\zeta) = \zeta^{-1} + m_1\zeta^n + m_2\zeta^{2n+1} + \ldots \] \hspace{1cm} (4.6)

Attention will be focussed on this family of mappings, which contains the Schwarz-Christoffel quasi-polygons as special cases. The solution will be derived in detail for the case of a two-term mapping function. For space considerations, the solution will be presented without detailed derivation for the three-term case, which poses no fundamental additional difficulties aside from increased algebraic complexity. To simplify the notation slightly, in the sequel the mapping function will be written as

\[ z = \omega(\zeta) = \zeta^{-1} + m_1\zeta^n + m_2\zeta^{2n+1} + \ldots \] \hspace{1cm} (4.7)

### 4.3. Derivation of complex potentials

The stress potentials are now found for problem 2, in which there is traction acting on the pore contour, but no traction at infinity. First, the chain rule is used to write eq. (4.3) in terms of \(\zeta\) as follows:

\[ \phi(\sigma) + \omega(\sigma) \frac{\overline{\phi'(\sigma)}}{\omega'(\sigma)} + \psi(\sigma) = F(\sigma), \] \hspace{1cm} (4.8)

where \(\sigma\) is used to represent values of \(z\) on the unit circle \(\gamma\) in the \(\zeta\)-plane. It is noted for later use that for points on \(\gamma\), \(\sigma = e^{i\alpha}\), so \(\overline{\sigma} = e^{-i\alpha} = 1/\sigma\). For problems such as this with no far-field traction, the complex potentials can be expressed (Sokolnikoff, 1956) as power series that converge for \(\zeta < 1\), \textit{i.e.},

\[ \phi(\zeta) = b_0 + b_1\zeta + b_2\zeta^2 + \ldots, \] \hspace{1cm} (4.9)

\[ \psi(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + \ldots. \] \hspace{1cm} (4.10)

The first term of (4.8) takes the form

\[ \phi(\sigma) = b_0 + b_1\sigma + b_2\sigma^2 + \ldots. \] \hspace{1cm} (4.11)

Recalling that the mapping function has the form \(\omega(\zeta) = \zeta^{-1} + m_1\zeta^n\), the second term of eq. (4.8) is expanded in a power series as follows:
\[
\frac{\phi'(\sigma)}{\phi(\sigma)} = \left(\frac{1}{\sigma} + m_1 \sigma^n\right)\left(\frac{b_1 + \frac{2b_2}{\sigma} + \frac{3b_3}{\sigma^2} + \ldots}{\sigma^2 + \frac{m_1 n}{\sigma^{n-1}}}\right)
\]

\[
= -\left(\frac{1 + m_1 \sigma^{n+1}}{\sigma^3}\right)\left(\frac{b_1 + \frac{2b_2}{\sigma} + \frac{3b_3}{\sigma^2} + \ldots}{1 + \frac{m_1 n}{\sigma^{n+1}} + \frac{m_1^2 n^2}{\sigma^{2n+2}} + \ldots}\right). \tag{4.12}
\]

There is no advantage to explicitly expanding out the triple product in eq. (4.12), as there is no convenient way to display all the resulting terms. Instead, when one wants to isolate, say, the \( \sigma^1 \) terms, for example, it is straightforward to see directly from eq. (4.12) that the coefficient of \( \sigma^1 \) will be \(-m_1(n-2)b_{n-2}\).

The third term in eq. (4.8) takes the form

\[
\bar{\psi}(\sigma) = c_0 + \frac{c_1}{\sigma} + \frac{c_2}{\sigma^2} + \ldots. \tag{4.13}
\]

The function \( F \) on the right-hand side of eq. (4.8) is given in (4.5) as \( i\tau \bar{z} \), and \( z \) is given by \( z = \omega(\zeta) = \zeta^{-1} + m_1\zeta^n \). So, recalling that \( \bar{\sigma} = 1/\sigma \) on \( \gamma \), the right side of eq. (4.8) takes the form

\[
F(\sigma) = i\tau(\sigma + m_1 \sigma^{-n}). \tag{4.14}
\]

Inserting eqs. (4.11)-(4.14) into eq. (4.8), and equating the coefficients of each power of \( \sigma \) on both sides of the equation, yields equations involving the \( b_k \) and \( c_k \) coefficients. Assuming that \( n \geq 2 \), the coefficients of the \( \sigma^1 \) term yield the condition

\[
b_1 - (n-2)m_1\bar{b}_{n-2} = i\tau. \tag{4.15}
\]

The coefficients of the \( \sigma^{n-2} \) term yield the condition

\[
b_{n-2} - m_1\bar{b}_1 = 0. \tag{4.16}
\]

Solution of these two equations gives \( b_1 = i\tau/[1 - m_1^2(n-2)] \) and \( b_{n-2} = -im_1\tau/[1 - m_1^2(n-2)] \). The coefficients of all other terms \( \sigma^q \), for \( q \) not equal to 1 or \( n-2 \), yield homogeneous algebraic equations for the \( b_k \), showing that all other \( b_k \) are zero. This procedure also yields equations for the \( c_k \), but these are not as easily solved; instead, the \( c_k \) are found by a different method, as shown below. Hence, the first complex potential is

\[
\phi(\zeta) = \frac{i\tau(\zeta - m_1\zeta^{n-2})}{1 - m_1^2(n-2)}. \tag{4.17}
\]
This result is valid for \( n \geq 2 \). If \( n = 1 \), which corresponds to an elliptical pore, the term \( b_{n-2} = b_{-1} \) does not appear in series (4.9), and so the foregoing calculations do not apply. Elliptical pores require separate, although simpler, calculations. Fortunately, elliptical pores have received extensive treatment already (Savin, 1961; Thorpe & Sen, 1985; Kachanov et al., 1994), and so attention will now be focussed on the cases \( n \geq 2 \).

To find the second complex potential, the conjugate of the boundary condition (4.8) is taken, to arrive at

\[
\frac{\phi(\sigma) + \omega(\sigma)\phi'(\sigma)}{\omega(\sigma)} + \psi(\sigma) = i\tau z = -i\tau z = -i\tau \left( \frac{1}{\sigma} + m_1\sigma^n \right).
\]  (4.18)

Each term of this equation is now divided by \( 2\pi i(\sigma - \zeta) \), where \( \zeta \) is an arbitrary point inside the unit circle \( \gamma \); and then the terms are integrated around \( \gamma \). With \( \phi \) given by eq. (4.9), the first term integrates out to

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{\phi(\sigma)}{\sigma - \zeta} d\sigma = \frac{1}{2\pi i} \int \frac{i\tau}{1 - m_1^2(n-2)} \left( \frac{1}{\sigma} - \frac{m}{\sigma^{n-2}} \right) d\sigma = 0,
\]  (4.19)

since \( \frac{i}{\sigma^k}(\sigma - \zeta)^{-1}d\sigma = 0 \) for any \( \zeta \) inside the unit circle, and all \( k \geq 1 \) (Godfrey, 1959, p. 279).

The second integral takes the form

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{\omega(\sigma)\phi'(\sigma)}{\omega'(\sigma)(\sigma - \zeta)} d\sigma = \frac{1}{2\pi i} \oint_{\gamma} \frac{(\sigma + m_1\sigma^{-n})i\tau}{(-\sigma^{-2} + m_1n\sigma^{n-1})} \left[ \frac{1 - m_1(n-2)\sigma^{n-3}}{1 - m_1^2(n-2)} \right] d\sigma - \zeta.
\]  (4.20)

The integrand has two poles inside the unit circle, at \( \sigma = \zeta \) and \( \sigma = 0 \). There is also a pole at \( \sigma = (m_1n)^{-1/(n+1)} \), but this lies outside \( \gamma \); due to the constraint \( m_1n < 1 \) that must hold in order for the mapping to be single-valued. The residue at \( \sigma = \zeta \) can be read off directly from the integrand of the left-hand side of eq. (4.20), in the form

\[
\frac{\omega(\zeta)\phi'(\zeta)}{\omega'(\zeta)}.
\]  (4.21)

To find the residue at \( \sigma = 0 \), the terms in the denominator are expanded as a power series, the resulting series are multiplied together, and terms of similar powers are collected, after which the residue is identified as the coefficient of \( \sigma^{-1} \); the result is

\[
\frac{i\tau[m_1\zeta^{2-n} - m_1^2(n-2)\zeta^{-1}]}{1 - m_1^2(n-2)}.
\]  (4.22)
As \( \psi \) is analytic inside \( \gamma \), the integral of the third term in eq. (4.18) is readily found to be

\[
\frac{1}{2\pi i} \oint \frac{\psi(\sigma)}{\sigma - \zeta} d\sigma = \psi(\zeta) .
\] (4.23)

The right-hand side of eq. (4.18) can be integrated by expanding the integrand in partial fractions, and using the residue theorem on each term:

\[
-\frac{1}{2\pi i} \int i\tau \left( \frac{1}{\sigma} + m_1\sigma^n \right) \frac{d\sigma}{\sigma - \zeta} = -i\tau \left[ \frac{-1/\zeta + 1/\zeta + m_1\sigma^n}{\sigma - \zeta} \right] d\sigma = -i\tau m_1\zeta^n .
\] (4.24)

Collecting all the integrals gives the second complex potential in the form

\[
\psi(\zeta) = -im_1\tau\zeta^n - \frac{\omega(\zeta)\phi'(\zeta)}{\omega'(\zeta)} + \frac{im_1\tau}{1 - m_1^2(n - 2)} \left[ m_1(n - 2)\zeta^{-1} - \zeta^{2-n} \right].
\] (4.25)

The displacement is found by inserting eqs. (4.17) and (4.25) into eq. (4.1):

\[
2G(u_x + iu_y) = \kappa \left[ \frac{i\tau(\zeta - m_1\zeta^{n-2})}{1 - m_1^2(n - 2)} \right] - im_1\tau\zeta^n + \frac{im_1\tau[m_1(n - 2)\zeta^{-1} - \zeta^{2-n}]}{1 - m_1^2(n - 2)}. \] (4.26)

Note that the second term on the right side of eq. (4.25) has been cancelled out by an equivalent term that arises from \(-z\phi'(z)\).

For a pore with three terms in its mapping function, \( z = \zeta^{-1} + m_1\zeta^n + m_2\zeta^{2n+1} \), the same procedure eventually yields the following expressions for the two complex potentials:

\[
\phi(\zeta) = B_1\zeta + B_2\zeta^{n-2} + B_3\zeta^{2n-1} + B_4\zeta^{n+2},
\] (4.27)

\[
\psi(\zeta) = -i\tau m_1\zeta^n - i\tau m_2\zeta^{2n+1} - \frac{\omega(\zeta)}{\omega'(\zeta)} \phi'(\zeta) + i\tau \omega(\zeta)
\]

\[
\left( \frac{m_1B_1}{\zeta^{-2}} + \frac{m_1(n - 2)B_2}{\zeta} + \frac{m_2(n - 2)B_2}{\zeta} + \frac{m_2(2n - 1)B_3}{\zeta} \right), \] (4.28)

\[
B_1 = i\tau + m_1(n - 2)[1 + nm_2]\bar{b}_{n-2} + m_2(2n - 1)\bar{b}_{2n-1}, \] (4.29)

\[
B_2 = \bar{b}_1m_1 + \bar{b}_1m_2m_1n + m_2(n + 2)\bar{b}_{n+2}, \] (4.30)
\[ B_3 = \overline{b_1 m_2} , \quad (4.31) \]

\[ B_4 = m_2 (n-2) \overline{b_{n-2}} , \quad (4.32) \]

\[ b_1 = i \tau [1-m_2^2(n+2)(n-2)]/T , \quad (4.33) \]

\[ b_{n-2} = -i \tau m_1 (1+m_2 n)/T , \quad (4.34) \]

\[ b_{2n-1} = -i m_2 \tau [1-m_2^2(n+2)(n-2)]/T , \quad (4.35) \]

\[ b_{n+2} = i \tau m m_2 (n-2)(1+m_2 n)/T , \quad (4.36) \]

\[ T = 1-m_2^2(n-2)-2m_2^2m_2(n-2)-m_2^2(2n-1)-(n+2)(n-2)m_2^2 - n^2(n-2)m_2^2m_2^2 . \quad (4.37) \]

4.4. Calculation of the excess strain energy and the $H_{1212}$ coefficient

The stored strain energy is the sum of the energy calculated in problem 1, and the energy from problem 2. In problem 1, the strain energy per unit volume of material is $\tau^2/2G$, corresponding to the energy of the material without a pore. The excess strain energy due to the pore is found from the solution to problem 2, by calculating the work done by the traction that acts over the pore surface. This traction vector is, from eq. (4.4), given by $|t|/|z'(\alpha)|$, where the angular variable $\alpha$ in the $\zeta$-plane is used to parameterise the pore boundary. Hence,

\[ W = \frac{1}{2} \int \left( t \cdot u \right) |dz| = -\frac{\tau}{2} \int_0^{2\pi} \frac{|x'\alpha - y'(\alpha)|}{|z'(\alpha)|} |u_x(\alpha), u_y(\alpha)| z'(\alpha) |d\alpha \]

\[ = -\frac{\tau}{2} \int_0^{2\pi} [x'(\alpha) - y'(\alpha)] [u_x(\alpha), u_y(\alpha)] d\alpha = -\frac{\tau}{2} \int_0^{2\pi} (x' u_x - y' u_y) d\alpha \]

\[ = -\frac{\tau}{2} \int_0^{2\pi} \text{Re}[u(\alpha) z'(\alpha)] d\alpha = -\frac{\tau}{2} \int_0^{2\pi} \text{Re} u(\alpha) z'(\alpha) d\alpha , \quad (4.38) \]

where in the last integral the displacement $u$ is treated as a complex number, $u = u_x + iu_y$.

With the displacement given by eq. (4.26), and noting that $\zeta = e^{ia}$ on $\gamma$, then $z(\alpha) = e^{-ia} + m_1 e^{ia}$, and the work term for problem 2 becomes

\[ W_2 = \frac{\tau^2}{4G} \text{Re} \int_0^{2\pi} \left[ \frac{\kappa}{D} [e^{i\alpha} - m_1 e^{i(n-2)\alpha}] - m_1 e^{-i\alpha} + \frac{m_1}{D} [m_1 (n-2)e^{i\alpha} - e^{(n-2)i\alpha}] \right] \times (-e^{-i\alpha} + nm_1 e^{i\alpha}) d\alpha \]
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\[ \int \frac{\kappa}{D} \left( -1 + m_1 e^{(n-3)\alpha} + nm_1 e^{-i(\alpha+1)} - nm_1^2 e^{2i(\alpha+1)} + m_1 e^{-i(\alpha+1)} - nm_1^2 \right) d\alpha, \]

where \( D = 1 - m_1^2(n-2) \). Recalling that it is being assumed for now that \( n \neq 1 \) or 3, only three terms in the integrand lead to non-zero contributions to the integral, and it is found that

\[ W_2 = \frac{\tau^2}{2G} \left[ \frac{\pi(\kappa+1)}{1-m_1^2(n-2)} + \pi(1-m_1^2n) \right]. \]

But \( \pi(1-m_1^2n) \) is precisely the area of the pore, \( A \), (Zimmerman, 1986), so

\[ W_2 = \frac{\tau^2}{2G} \left[ \frac{\pi(\kappa+1)}{1-m_1^2(n-2)} + A \right]. \]

To this must now be added the energy due to problem 1, which is \( \tau^2(A_\infty - A)/2G \), where \( A_\infty \) is the total area of the body, including the pore, and \( A \) is the area of the pore. The total energy in the body is therefore given by

\[ W = \frac{\tau^2}{2G} \left[ \frac{\pi(\kappa+1)}{1-m_1^2(n-2)} + A_\infty \right] = \frac{\tau^2 A_\infty}{2G} + \frac{\tau^2}{2G} \left[ \frac{\pi(\kappa+1)}{1-m_1^2(n-2)} \right]. \]

Recall that \( A = \pi(1-m_1^2n) \), and then divide \( W \) by \( A_\infty \), to express the excess energy density, per unit area of the body, as

\[ \frac{\tau^2}{2G} = \frac{\pi(\kappa+1)}{1-m_1^2(n-2)(1-m_1^2n)} \],

where \( c = A/A_\infty \) is the volume fraction of the pores, \( i.e., \) the porosity. The excess energy is proportional to the volume fraction of pores, because interactions between nearby pores are of course being ignored.

The term \( \tau^2/2G \) is the energy density that would exist in the body, under stress, but in the absence of the pore. Hence, the second term on the right in eq. (4.43) is the excess energy due to the pores, and comparison with eq. (4.1) shows that, aside from the \( \tau^2 \) factor, it represents \( H_{1212} \):

\[ H_{1212} = \frac{(\kappa+1)c}{2G[1-m_1^2(n-2)](1-m_1^2n)}. \]

For plane strain, \( \kappa + 1 = 4(1-\nu) \), and so
For a pore with three terms in its mapping function, the result of a lengthy calculation, analogous to that shown above for the two-term case, is

$$H_{1212} = \frac{(\kappa + 1)B|c|}{2G[1 - m_1^2 n - m_2^2 (2n + 1)]},$$  \hspace{0.5cm} \text{(4.46)}$$

where $B_1$ is given by eq. (4.29).

### 4.5. Special case of four-fold symmetry

As discussed in several previous papers (Kachanov et al., 1994; Eroshkin & Tsukrov, 2005), pores with 4-fold rotational symmetry are not “isotropic” with respect to a far-field shear stress. The fact that $n = 3$ is a special case can be anticipated from eq. (4.39), where additional terms are seen to arise in the stored strain energy expression.

To analyse this case, a more general applied shear stress that may be oriented at an arbitrary angle must be considered. If the far-field shear stresses of eq. (4.4) are rotated by an angle $\theta$, then by applying the usual rules of stress transformation, and performing calculations analogous to those given in eqs. (4.4) and (4.5), it is seen that the function $F$ in eq. (4.5) will be multiplied by $e^{-2i\theta}$. This multiplicative factor of $e^{-2i\theta}$ then carries through to the right sides of eqs. (4.15) and (4.16), having the effect of multiplying $b_1$ by $e^{-2i\theta}$, and multiplying $b_{n-2}$ by $e^{2i\theta}$. Hence, the first complex potential is

$$\phi(\zeta) = \frac{i\tau(e^{-2i\theta}\zeta - m_1 e^{2i\theta}\zeta^{-n-2})}{1 - m_1^2 (n - 2)}. \hspace{0.5cm} \text{(4.47)}$$

Similarly, the second complex potential takes the form

$$\psi(\zeta) = -im_1\tau e^{2i\theta}\zeta^n - \frac{\omega(\zeta)\phi'(\zeta)}{\omega'(\zeta)} + \frac{im_1\tau[m_1(n-2)e^{2i\theta}\zeta^{-1} - e^{-2i\theta}\zeta^{2-n}]}{1 - m_1^2(n - 2)}. \hspace{0.5cm} \text{(4.48)}$$

The displacement is found by inserting the potentials (4.47) and (4.48) into eq. (4.1):

$$2G(u_x + iu_y) = \kappa \left[ \frac{i\tau(e^{-2i\theta}\zeta - m_1 e^{2i\theta}\zeta^{-n-2})}{1 - m_1^2 (n - 2)} \right] - im_1\tau e^{-2i\theta}\zeta^{-n}$$

$$+ \frac{im_1\tau[m_1 e^{-2i\theta}(n-2)\zeta^{-1} - e^{2i\theta}\zeta^{2-n}]}{1 - m_1^2(n - 2)}. \hspace{0.5cm} \text{(4.49)}$$
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Calculations similar to those that led to eq. (4.39) show that the work term gains an additional multiplicative factor of $e^{2i\theta}$. Hence, for the case where the far-field stresses are rotated by an angle $\theta$,

$$\int_{0}^{2\pi} \left[ \frac{\kappa}{D} \left[ e^{-2i\theta}e^{i\alpha} - m_1e^{2i\theta}e^{i(n-2)\alpha} \right] - m_1e^{-2i\theta}e^{-i\alpha} \right] \left( -e^{-i\alpha} + nm_1e^{i\alpha} \right) e^{2i\theta} d\alpha$$

This new factor of $e^{4i\theta}$ survives the integration only if $n = 3$ (recalling that the above equations do not apply for $n = 1$). Hence, the result obtained in section 4.4 for the cases $n = 2, 4, 5, \ldots$ but $\theta = 0$, in fact holds for arbitrary $\theta$, thus explicitly verifying that these pores are “isotropic” with respect to shear.

For $n = 3$, the integral in eq. (4.50) gives, aside from the terms already found in section 4.4, an additional term $-\tau^2\pi m_1(n+1)\cos(4\theta)/2DG$. Again recalling that $A = \pi(1 - m_1^2n)$, one finds

$$H_{1212} = \frac{(\kappa + 1)c}{2G[1 - m_1^2(n-2)][1 - m_1^2n]}(1 - m_1\cos4\theta). \quad (4.51)$$

The shear compliance therefore contains a small term that varies as $\cos 4\theta$. Because of the constraint $m_1n \leq 1$, the relative amplitude of this term cannot exceed $1/3$. Equation (4.51) also shows that expression found in section 4.4 for $\theta = 0$ in fact represents the average value of $H_{1212}$ over all angles, as would apply to a collection of such pores whose orientation angles are randomly distributed. Note that eq. (4.51) holds only for $n = 3$, but it has been written with $n$ appearing explicitly so as to allow easy comparison with eq. (4.41), which holds for $n = 2$ or $n \geq 4$.

4.6. Results and discussion

To validate the above results, they can be compared to the results for several simple geometrical shapes that have been obtained previously, and also to the values obtained by boundary element calculations. The boundary element calculations were performed using a code described by Martel & Muller (2000), which is a simplified version of the more general two-dimensional BEM code from Crouch & Starfield (1983) that is based on the displacement discontinuity method.
The boundary element calculations correspond to problem 2, in which the tractions are applied to the boundary of the pore in an infinite region, with no stresses at infinity. The pore boundary is discretised into a number of equal-length elements. The number of elements is always taken to be a multiple of \( n+1 \), the number of “sides” of the pore, to ensure that two boundary elements meet precisely at each corner or cusp, so that the corners are not chopped off. After the stress and displacement fields are calculated, the strain energy is calculated by numerically performing the integral specified in eq. (4.38). It was generally found that about 300 boundary elements are sufficient to achieve convergence of the shear compliance.

The analytical expressions for the shear compliance parameter \( H_{1212} \) showed that for plane strain this parameter is proportional to \( (1-\nu)/G \), and also proportional to the porosity, \( c \). Hence, one can define a dimensionless shear compliance as \( h = H_{1212}G/(1-\nu)c \). For a circular hole, eq. (4.45) recovers the well-known result that that \( h = 2 \).

The normalized shear compliances of some quasi-polygonal holes represented by two or three terms in the Schwarz-Christoffel mapping function are shown in Table 4.1. The present results agree with those cases previously derived by Jasiuk et al. (1994) and Kachanov et al. (1994). By extrapolating the results out to \( N \rightarrow \infty \), where \( N \) here represents the number of terms taken in the mapping function, one can find the shear compliance of regular polygons, which are represented by an infinite number of terms in the Schwarz-Christoffel mapping. As shown by Ekneligoda & Zimmerman (2006) for the case of the pore compressibility, this can be done by linear extrapolation, using \( N^{-2} \) as the independent variable, and extrapolating the results for \( 1/4 \) and \( 1/9 \) down to \( 1/N^2 = 0 \). The results are \( h = 2.667, 2.359, 2.143, \) and \( 2.100 \), for triangles, squares, pentagons and hexagons. (This result for a square represents an average over all angles of the applied shear stress, relative to the pore.) For the case of pore compressibility, these values were \( 3.250, 2.414, 2.192 \) and \( 2.105 \), showing that the shearability of a polygonal pore varies with the number of sides less drastically than does the compressibility.

### Table 4.1. Normalised shear compliance, \( GH_{1212}/(1-\nu)c \), of some quasi-polygonal holes represented by two or three terms in the Schwarz-Christoffel mapping function, as calculated by various methods. The value for a circular hole is 2. (The values for a square are averaged over all angles).

<table>
<thead>
<tr>
<th>Shape</th>
<th>BEM</th>
<th>Jasiuk et al.</th>
<th>Kachanov et al.</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>triangle</td>
<td>2.5966</td>
<td>2.6400</td>
<td>2.5715</td>
<td>2.5834</td>
</tr>
<tr>
<td>Square</td>
<td>2.2550</td>
<td>2.2765</td>
<td>2.2442</td>
<td>2.2605</td>
</tr>
<tr>
<td>pentagon</td>
<td>2.1318</td>
<td>2.1373</td>
<td>2.1126</td>
<td>2.1389</td>
</tr>
<tr>
<td>hexagon</td>
<td>2.0413</td>
<td>2.0805</td>
<td>2.0730</td>
<td>2.0800</td>
</tr>
</tbody>
</table>
The present results are more general than previous available results such as those shown in Table 4.1, as the derived closed-form expressions are valid for all values of the parameters $m_1$ and $m_2$, not necessarily restricted to be those corresponding to the Schwarz-Christoffel quasi-polygons. Explicit results for some additional shapes are shown in Tables 4.2 and 4.3.

Table 4.2. Normalized shear compliance, $GH_{1212}/(1-\nu)c$, of some pores having a three-fold axis of symmetry, and which can be represented by two terms in the mapping function, i.e., $\phi(\zeta) = \zeta^{-1} + m_1\zeta^2$.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Coefficients</th>
<th>Analytic</th>
<th>Eq. (4.52)</th>
<th>Error of (4.52)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n = 2, m_1 = 1/9$</td>
<td>2.0507</td>
<td>2.0506</td>
<td>0.00%</td>
</tr>
<tr>
<td></td>
<td>$n = 2, m_1 = 1/5$</td>
<td>2.1739</td>
<td>2.1763</td>
<td>0.11%</td>
</tr>
<tr>
<td></td>
<td>$n = 2, m_1 = 1/3$</td>
<td>2.5715</td>
<td>2.5968</td>
<td>0.98%</td>
</tr>
<tr>
<td></td>
<td>$n = 2, m_1 = 1/2$</td>
<td>4.0000</td>
<td>4.2418</td>
<td>6.05%</td>
</tr>
</tbody>
</table>

Table 4.3. Normalized shear compliance, $GH_{1212}/(1-\nu)c$, of some pores having a three-fold axis of symmetry, and which can be represented by three terms in the mapping function, i.e., $\phi(\zeta) = \zeta^{-1} + m_1\zeta^2 + m_2\zeta^5$.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Coefficients</th>
<th>Analytic</th>
<th>Eq. (4.52)</th>
<th>Error of (4.52)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m_1 = 1/3, m_2 = 1/15$</td>
<td>2.6471</td>
<td>2.8163</td>
<td>6.4%</td>
</tr>
<tr>
<td></td>
<td>$m_1 = 1/6, m_2 = 2/15$</td>
<td>2.3377</td>
<td>2.5733</td>
<td>10.1%</td>
</tr>
<tr>
<td></td>
<td>$m_1 = 1/9, m_2 = 7/45$</td>
<td>2.5243</td>
<td>2.6347</td>
<td>4.4%</td>
</tr>
</tbody>
</table>

The complex variable method can in principle be used to compute the shear compliance of a pore of any shape. However, calculation of the mapping coefficients for the sort of complex pore shapes that are observed in rocks or ceramics is extremely tedious, and computationally non-trivial (Sisavath et al., 2001; Tsukrov & Novak, 2002). Moreover, the additional complexity exhibited by the analytic solution for the three-term mapping as opposed to the two-term mapping indicates that analytical treatment of pores whose mappings contain more than three terms is not feasible. Hence, it would be useful if the pore shear compliance could be calculated from simple geometric attributes of the pore shape. Such a capability would be useful in attempts to estimate elastic moduli from images of heterogeneous media, as
attempted, for example, by Tsukrov et al. (2005), and in Chapter 6 of the present thesis.

Zimmerman (1991) proposed a scaling law in which the normalized pore compressibility is proportional to the dimensionless geometric parameter $P^2 / 2\pi A$, where $P$ is the perimeter and $A$ is the area of the pore. This parameter measures deviations from circularity, and in fact $4\pi A / P^2$ is often referred to as the “roundness” in particle technology literature. The multiplicative factor $1/2\pi$ in the scaling law was chosen so that the correlation is exact for a circular pore. This law was tested on various pore shapes by Zimmerman (1991), Tsukrov & Novak (2002), and Ekneligoda & Zimmerman (2006), with errors that were always less than 22%, and usually less than 10%.

The shear compliance also seems to scale linearly with the geometrical parameter $P^2 / 2\pi A$, according to the following expression:

$$h = \frac{GH_{1212}}{(1-v)c} = 1 + \frac{1}{2} \left( \frac{P^2}{2\pi A} \right). \quad (4.52)$$

Results for a few symmetric pores are shown in Tables 4.2 and 4.3, where it is seen that the scaling law (4.52) is very accurate in these cases. These results are also plotted in Fig. 4.2, along with the results for a few other symmetric pores not shown in the tables. Figure 4.2 shows graphically that the correlation between $h$ and $P^2 / 2\pi A$ is strong and non-trivial. Equation (4.52) is not the best-fitting line through the data, but represents a compromise between accuracy and elegance, in the sense that it is exact for a circle, and the coefficients of the constant term and the $P^2 / 2\pi A$ term are round numbers. It is interesting that the slope of the normalised shearability, with respect to the geometric parameter $P^2 / 2\pi A$, is half of the corresponding slope in the scaling law for the normalised hydrostatic compressibility.

Validation of the scaling law for symmetric pores does not guarantee that it will be useful for “realistic” pore shapes, so it has been also tested for pores observed in scanning electron micrograph (SEM) images of a SiC ceramic (Reynaud et al., 2005; Fig. 4.3). For the present purposes, issues related to the fact that these are two-dimensional slices of three-dimensional pores (Lock et al., 2002) are ignored, and treat the pores as two-dimensional. In these cases the exact values were obtained from BEM calculations, with the mean value obtained by averaging over all values of $\theta$, where $\theta$ is the angle between the pore orientation and the far-field shear. This averaging would be appropriate if one assumed that the ceramic contains a collection of pores whose orientations are spatially random. It is encouraging that the shear compliances of the ceramic pores tend to fall close to the line of the scaling law, eq. (4.52).
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Fig. 4.2. Normalised shear compliance, \( \frac{GH_{1212}}{(1-\nu)c} \), plotted against \( \frac{P^2}{2\pi A} \). The dotted line shows the proposed scaling law, eq. (4.52).

Fig. 4.3. Shapes of pores of a SiC ceramic (Reynaud et al., 2005), as extracted by image analysis software from a scanning electron micrograph. The shear compliances of these pores, averaged over all orientation angles, are plotted in Fig. 4.2.

The boundary element calculations showed that, for irregular pores having no axis of symmetry, \( H_{1212} \) always varies with \( \cos^4 \theta \), aside from some phase shift in the angle. One particular example is shown in Fig. 4.4, for a pore in a Berea sandstone. This result can be explained as follows. Anti-clockwise rotation of the far-field shear stresses by 90º is equivalent to changing the sign of the stresses, since, for example, the component \( \tau_{xy} = \tau \) that points in the positive \( y \)-direction gets rotated into a component \( \tau_{yx} = -\tau \) that points in the negative \( x \)-direction. But the strain energy is a quadratic form in the stresses, so multiplying the stresses by \(-1\) cannot change the energy, and so cannot change \( H_{1212} \). Hence, \( H_{1212} \) can only vary with angle according to \( \cos 4\theta \), \( \cos 8\theta \), etc. Allowing for a possible phase shift, the term that varies trigonometrically with \( 4\theta \) would also yield a \( \sin 4\theta \) term, etc. But the transformation law for fourth-order tensors contains only trigonometric terms such as \( \sin k\theta \) or \( \cos k\theta \), where \( k \) cannot exceed 4 (Eroshkin & Tsukrov, 2005). Hence, the only angular-dependent terms that can appear in \( H_{1212} \) are those containing \( \cos 4\theta \) and \( \sin 4\theta \), or,
equivalently, \( \cos(4\theta + \delta) \), where the constant \( \delta \) represents the phase shift. Hence, a \( \cos(4\theta + \delta) \) term should be expected to appear, but vanishes when the pore has \( N \)-fold rotational symmetry of order \( N = 3, 5, 6, \text{ etc.} \). Alternatively, by retracing the steps that led to eq. (4.50), one can show that in the general case of an arbitrary number of terms in the mapping function, the angle \( \theta \) enters the integral that expresses the excess energy only in the form of \( e^{4i\theta} \).

Fig. 4.4. Variation of normalised shear compliance, \( h = GH_{1212}/(1-\nu)c \), for a pore from Berea sandstone, from Lock et al. (2002). The angle \( \theta \) is the anti-clockwise angle by which the stress state \( \tau_{xy} = \tau_{yx} = \tau \) is rotated.

### 4.7. Conclusions

A closed-form solution for the shear compliance of some symmetric pores has been obtained, as quantified by the coefficient \( H_{1212} \) defined by Kachanov et al. (1994). This parameter \( H_{1212} \) was found to be proportional to \( (1-\nu)/G \), where \( G \) and \( \nu \) are the elastic parameters of the host material, and to the porosity, \( c \). Hence, a normalised shear compliance can be defined as \( h = GH_{1212}/(1-\nu)c \). The analytical results were validated against some previously computed special cases, and against BEM calculations. It was found that the shear compliance tends to increase with macroscopic “non-circularity” of the pore, and also increases if the pore has cusps. Perhaps unexpectedly, it was found that the shear compliance of a non-circular pore differs from that of a circular pore less drastically than does the analogous variation in hydrostatic pore compressibility.

The analytical expressions derived above for the shear compliance explicitly verify the general theorem of Eroshkin & Tsukrov (2005), which specified that for pores having \( N \)-fold rotational symmetry, \( H_{1212} \) is independent of the orientation of the pore with respect to the shear direction, except for the case of \( N = 4 \). For \( N = 4 \), it was found explicitly that \( H_{1212} \) varies with \( \cos 4\theta \), where \( \theta \) quantifies the orientation...
of the applied shear with respect to the pore, \( \theta \), i.e., \( H_{1212} = H_0 + H_1 \cos 4\theta \), for two constants \( H_0 \) and \( H_1 \). More generally, it was showed that this type of variation, involving \( \theta \) only through terms such as \( \cos 4\theta \) or \( \sin 4\theta \), also occurs for pores possessing no rotational symmetry.

Lastly, it is mentioned as an aside, both for completeness and to avoid any possible confusion, that the case of \( N = 2 \) is anomalous yet again. For such pores containing only two terms in the mapping function (4.7), \( \theta \), ellipses, calculations such as those described above eventually lead to the result
\[
H_{1212} = (\kappa + 1)c/2G(1 - m^2),
\]
which contains no dependence on \( \theta \). At first, this seems to contradict the assertion by Eroshkin & Tsukrov (2005) that pores with 180° rotational symmetry will not be elastically “isotropic”. This paradox is removed by noting that the coefficient \( H_{1111} \) of an elliptical pore, for example, \textit{will} contain angular-dependent terms (Tsukrov & Novak, 2002), and so the overall elastic moduli tensor of a body containing aligned elliptical pores will indeed be anisotropic.

Finally, a scaling law, \( h = GH_{1212}/(1 - \nu)c = 1 + (P^2/4\pi A) \), was proposed, where \( P \) is the perimeter of the pore and \( A \) is its area. This law was tested against the analytical solutions, and on some pores observed in SEM images of a porous ceramic. The agreement was quite good, with an average error, over seventeen ceramic pores, of only -1.9%. In Chapter 6 this law will be used, along with an earlier scaling law for the hydrostatic pore compressibility (Zimmerman, 1991), to estimate the elastic moduli of several porous materials based on information obtained from SEM images.
5 Shear Compliance of a Rough-Walled Pore

5.1 Introduction

In Chapter 3, the boundary perturbation approach was used to study the effect of small-scale roughness on the compressibility of a nominally circular pore. In the present chapter, a similar analysis is carried out to study the effect of roughness on the shear compliance. This is done by solving the problem of a rough-walled pore in an infinite body subjected to a far-field shear stress. As in Chapter 3, the analysis is carried out for a rigid inclusion, and the results of Dundurs (1989) and Jasiuk (1995) are used to convert the solution to one that is applicable to pores. The solution is then used to estimate the effective shear modulus of a body containing a dilute concentration of such pores. In contrast to the analysis in Chapter 3, here the solution is taken only to second-order in the small parameter $\varepsilon$.

5.2 Problem formulation for infinite region containing a rigid inclusion

The solution of a two-dimensional isotropic elasticity problem can be represented in terms of the Airy stress function, $\Phi$, which satisfies the bi-harmonic equation $\nabla^2 \nabla^2 \Phi = 0$. The general solution, neglecting those terms that do not correspond to a uniform stress at infinity, can be written in polar co-ordinates as (Little, 1973; Barber, 1992)

$$\Phi = A_0 r^2 \sin 2\theta + \sum_{n=2}^{\infty} [A_{5,n} r^{-n+2} \cos n\theta + A_{6,n} r^{-n} \cos n\theta + A_{7,n} r^{-n+2} \sin n\theta + A_{8,n} r^{-n} \sin n\theta]. \quad (5.1)$$

The stresses and displacements associated with this solution are given by

$$\tau_{rr} = -2A_0 \sin 2\theta - \sum_{n=2}^{\infty} [A_{5,n} (n+2)(n-1)r^{-n} + A_{6,n} n(n+1)r^{-n-2}] \cos n\theta$$

$$- \sum_{n=2}^{\infty} [A_{7,n} (n+2)(n-1)r^{-n} + A_{8,n} n(n+1)r^{-n-2}] \sin n\theta, \quad (5.2)$$

$$\tau_{\theta\theta} = 2A_0 \sin 2\theta + \sum_{n=2}^{\infty} [A_{5,n} (n-1)(n-2)r^{-n} + A_{6,n} n(n+1)r^{-n-2}] \cos n\theta$$

$$+ \sum_{n=2}^{\infty} [A_{7,n} (n-2)(n-1)r^{-n} + A_{8,n} n(n+1)r^{-n-2}] \sin n\theta, \quad (5.3)$$

$$\tau_{r\theta} = -2A_0 \cos 2\theta + \sum_{n=2}^{\infty} [A_{5,n} (n)(n-1)r^{-n} + A_{6,n} n(n+1)r^{-n-2}] \cos n\theta$$
\[- \sum_{n=2}^{\infty} \left[ A_{5,n} (n)(n-1)r^{-n} + A_{6,n} n(n+1)r^{-n-2} \right] \sin n\theta, \quad (5.4)\]

\[2G\mu_r = -2A_0 r \sin 2\theta + \sum_{n=2}^{\infty} \left[ A_{7,n}(\kappa + n - 1)r^{-n+1} + A_{8,n} nr^{-n-1} \right] \cos n\theta, \quad (5.5)\]

\[2G\mu_\theta = -2A_0 r \cos 2\theta + \sum_{n=2}^{\infty} \left[ -A_{7,n}(\kappa - n + 1)r^{-n+1} + A_{8,n} nr^{-n-1} \right] \sin n\theta, \quad (5.6)\]

where \((A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9)\) are constants, and \((r, \theta)\) are the usual polar co-ordinates.

For an infinite region containing a single inclusion, with a shear stress \(\tau\) acting at infinity, Jasiuk et al. (1994) and Jasiuk (1995) showed that the effective shear modulus of a body containing these inclusions can be expressed solely in terms of the coefficient \(A_{7,2}\), as follows:

\[\frac{G_{\text{eff}}}{G} = 1 + (\kappa + 1)A_{7,2}(\kappa; \tau = 1)\frac{\pi}{A_{\text{inclusion}}} c, \quad (5.7)\]

where \(G\) is the two-dimensional shear modulus of the host material, \(\kappa\) is the Muskhelishvili parameter of the host material, which equals \(3 - 4\nu\) for plain strain and \((3 - \nu)/(1 + \nu)\) for plain stress, \(\nu\) is the three-dimensional Poisson ratio of the host material, \(c\) is area fraction of the inclusions, and \(A_{7,2}(\kappa; \tau = 1)\) is the one of the coefficients in the Airy stress function, evaluated for the case where the magnitude of the applied shear is \(\tau = 1\).

Dundurs (1989) showed that the solution for a body containing a rigid inclusion could be transformed into the solution for the case in which the inclusion is a cavity, by setting \(\kappa = -1\). Jasiuk (1995) then showed that the effective shear modulus of a body containing a dilute concentration of these cavities would be given by,

\[\frac{G_{\text{eff}}}{G} = 1 + (\kappa + 1)A_{7,2}(\kappa; \tau = 1)\frac{\pi}{A_{\text{pore}}} c, \quad (5.8)\]

where \(\kappa\) is set equal to \(-1\) in the expression for \(A_1\), but the actual value of \(\kappa\) of the host material is used when evaluating the term \((\kappa + 1)\).

This correspondence is exploited in the present study. Although the present interest lies in the case of vacuous pores, when using the perturbation approach the case of a rigid nearly circular inclusion is much easier to solve. This is because the boundary condition for the rigid inclusion is that the displacement vector must vanish, which directly implies that both components of the displacement, \(u_r\) and \(u_\theta\), must
vanish along the boundary. On the other hand, if the inclusion is a pore, the traction vector must vanish on the boundary. But the normal and tangential components of the traction vector on the actual non-circular boundary are related to the stress components in the polar co-ordinate system in a very complicated manner (Givoli & Elishakoff, 1992). Hence, the solution procedure for a rigid inclusion is substantially simpler than that for a pore.

### 5.3 Solution for rigid inclusion

The solution procedure starts by considering a rigid circular disc of radius \(a\), in an infinitely large plate subjected to a far-field pure shear stress of magnitude \(\tau\). The Airy stress function for this problem is given by (Barber, 1992)

\[
\Phi = \left[ (-a^2/\kappa) + (a^4/2\kappa r^2) + (-r^2/2) \right] \tau \sin 2\theta,
\]

and the associated displacements and stresses are

\[
\begin{align*}
u_r &= \frac{\tau \sin 2\theta}{2G} \left[ r + \frac{a^4}{\kappa r^3} - \frac{(\kappa + 1)a^2}{\kappa r} \right], \\
u_\theta &= \frac{\tau \cos 2\theta}{2G} \left[ r - \frac{a^4}{\kappa r^3} - \frac{(\kappa - 1)a^2}{\kappa r} \right], \\
\tau_{rr} &= \tau \sin 2\theta \left[ \frac{4a^2}{\kappa r^2} + \frac{3a^4}{\kappa r^4} + 1 \right], \\
\tau_{\theta\theta} &= \tau \sin 2\theta \left[ \frac{3a^4}{\kappa r^4} - 1 \right], \\
\tau_{r\theta} &= \tau \cos 2\theta \left[ -\frac{2a^2}{\kappa r^2} - 1 + \frac{3a^4}{\kappa r^4} \right].
\end{align*}
\]

Now consider a nearly circular inclusion that has a corrugated boundary described by

\[
r = a(1 + \varepsilon \sin m\theta),
\]

where \(\varepsilon\) is a small parameter representing the amplitude of the corrugations, and \(m\) is an integer that represents the number of bumps on the circumference of the circle. As an example, Fig. 5.1 shows the inclusion shape for \(m = 8\) and \(\varepsilon = 0.25\).
Following the standard procedure for a regular perturbation problem, it is now assumed that the Airy stress function, the displacements, and the stresses can each be expressed as perturbation series in the small parameter $\varepsilon$ (van Dyke, 1975; Givoli & Elishakoff, 1992), i.e.,

$$
\Phi(r,\theta) = \Phi^0(r,\theta) + \varepsilon \Phi^1(r,\theta) + \varepsilon^2 \Phi^2(r,\theta) + \ldots, \tag{5.12}
$$

$$
u_r(r,\theta) = u_r^0(r,\theta) + \varepsilon u_r^1(r,\theta) + \varepsilon^2 u_r^2(r,\theta) + \ldots, \tag{5.13}
$$

$$u_\theta(r,\theta) = u_\theta^0(r,\theta) + \varepsilon u_\theta^1(r,\theta) + \varepsilon^2 u_\theta^2(r,\theta) + \ldots, \tag{5.14}
$$

where the superscript $n$ in $\Phi^n$ denotes the $n$-th Airy function, and is not a power exponent. When $\varepsilon = 0$, this solution must reduce to the solution for a circular inclusion, in which case

$$
\Phi^0(r,\theta) = \tau \left[ -\frac{a^2}{\kappa} + \frac{a^4}{2\kappa r^2} - \frac{r^2}{2} \right] \sin 2\theta, \tag{5.15}
$$

$$nu_r^0 = \frac{\tau \sin 2\theta}{2G} \left[ r + \frac{a^4}{\kappa r^3} - \frac{(\kappa + 1)a^2}{\kappa r} \right], \tag{5.16a}
$$

$$nu_\theta^0 = \frac{\tau \cos 2\theta}{2G} \left[ r - \frac{a^4}{\kappa r^3} - \frac{(\kappa - 1)a^2}{\kappa r} \right]. \tag{5.16b}
$$

Each of the subsequent Airy functions will be of the form given by (5.1)-(5.6). As the far-field boundary conditions are already satisfied by the zeroth-order solution, the $A_0$ coefficient will vanish in all the higher-order Airy functions. The remaining coefficients in these functions are found by satisfying the boundary conditions at the interface between the matrix and the inclusion. These boundary conditions require that both displacement components vanish at the interface. Hence, for the purposes of deriving the solution, the stresses need not be considered any further.
Zero-displacement boundary conditions must be applied at the boundary of the inclusion, where \( r = r^* = a(1 + \varepsilon \sin m\theta) \). But each function \( u^\beta(r, \theta) \) is expressed naturally in terms of \( r \) and \( \theta \). In order to re-express the boundary conditions at \( r = a \) instead of \( r = a(1 + \varepsilon \sin m\theta) \), the displacements are first expanded in a Taylor series about the value \( r = a \), with \( a\varepsilon \sin m\theta \) as the small variation in \( r \). Following the procedure used in Chapter 3 for the analogous problem involving hydrostatic loading, it can be stated that

\[
u^\beta(r^*, \theta) = u^\beta_r(a + a\varepsilon \sin m\theta, \theta) = u^\beta_r(a, \theta) + a\varepsilon \sin m\theta \left( \frac{\partial u^\beta_r(r, \theta)}{\partial r} \right)_{r=a}
+ \frac{a^2\varepsilon^2 \sin^2 m\theta}{2} \left( \frac{\partial^2 u^\beta_r(r, \theta)}{\partial^2 r} \right)_{r=a}
+ \frac{a^3\varepsilon^3 \sin^3 m\theta}{6} \left( \frac{\partial^3 u^\beta_r(r, \theta)}{\partial^3 r} \right)_{r=a} + \ldots, \tag{5.17}\]

and similarly for \( u^\beta_\theta(r^*, \theta) \). Applying this expansion to every term in eq. (5.13), and then grouping terms according to ascending powers of \( \varepsilon \), leads to

\[
u_r(r^*, \theta) = 0 = u^0_r(a, \theta) + \left[ a \sin m\theta \left( \frac{\partial u^0_r(r, \theta)}{\partial r} \right)_{r=a} + u^1_r(a, \theta) \right] \varepsilon
+ \left[ \frac{a^2 \sin^2 m\theta}{2} \left( \frac{\partial^2 u^0_r(r, \theta)}{\partial^2 r} \right)_{r=a} \right] \varepsilon^2 + \ldots, \tag{5.18}\]

and similarly for the tangential displacement,

\[
u_\theta(r^*, \theta) = 0 = u^0_\theta(a, \theta) + \left[ a \sin m\theta \left( \frac{\partial u^0_\theta(r, \theta)}{\partial r} \right)_{r=a} + u^1_\theta(a, \theta) \right] \varepsilon
+ \left[ \frac{a^2 \sin^2 m\theta}{2} \left( \frac{\partial^2 u^0_\theta(r, \theta)}{\partial^2 r} \right)_{r=a} \right] \varepsilon^2 + \ldots \tag{5.19}\]

Requiring the no-displacement boundary condition to be satisfied at all orders of \( \varepsilon \), the order-\( \varepsilon \) term in eq. (5.18) yields

\[
u^1_r(a, \theta) = -a \sin m\theta \left( \frac{\partial u^0_r(r, \theta)}{\partial r} \right)_{r=a}
= -\frac{\tau a \sin m\theta \sin 2\theta}{2G} \left[ \frac{(\kappa + 1)a^2}{\kappa^2} + 1 - \frac{3a^4}{\kappa^4} \right]_{r=a}
= -\frac{\tau a \sin m\theta \sin 2\theta}{G} \left[ \frac{\kappa - 1}{\kappa} \right]
= -\frac{\tau a (\kappa - 1)}{2G \kappa} [\cos(m - 2)\theta - \cos(m + 2)\theta], \tag{5.20}\]
which serves as a boundary condition at \( r = a \) for the as-yet unknown function \( u_1^I(r, \theta) \). Likewise, the order-\( \varepsilon \) term in eq. (5.19) yields

\[
u_1^g(a, \theta) = -a \sin m\theta \frac{\partial u_0^g(r, \theta)}{\partial r} \bigg|_{r=a} = -\frac{\tau a \sin m\theta \cos 2\theta}{2G} \left[ \frac{(\kappa - 1)a^2}{\kappa a^2} + 1 + \frac{3a^4}{\kappa a^4} \right].
\]

\[
u_1^g(a, \theta) = -\frac{\tau a (\kappa + 1)}{2G} \left[ \sin(m + 2)\theta + \sin(m - 2)\theta \right] \tag{5.21}
\]

Examination of the general solution for \( \Phi^1 \), as given by eqs. (5.1)-(5.6), shows that the only terms that yield \( \cos(m+2)\theta \) or \( \cos(m-2)\theta \) variations for \( u^I_1(r, \theta) \) are those involving \( A_{5,m}^1 \) and \( A_{6,m}^1 \). Hence, the first-order displacement functions must have the form

\[
2G u_1^I(r, \theta) = \frac{A_{5,m-2}^1 (\kappa + m - 3) \cos(m - 2)\theta}{r^{m-3}} + \frac{A_{6,m-2}^1 (m - 2) \cos(m - 2)\theta}{r^{m-1}}
\]

\[
+ \frac{A_{5,m+2}^1 (\kappa + m + 1) \cos(m + 2)\theta}{r^{m+1}} + \frac{A_{6,m+2}^1 (m + 2) \cos(m + 2)\theta}{r^{m+3}}, \tag{5.22}
\]

\[
2G u_1^g(r, \theta) = -\frac{A_{5,m-2}^1 (\kappa - m + 3) \sin(m - 2)\theta}{r^{m-3}} + \frac{A_{6,m-2}^1 (m - 2) \sin(m - 2)\theta}{r^{m-1}}
\]

\[
- \frac{A_{5,m+2}^1 (\kappa - m - 1) \sin(m + 2)\theta}{r^{m+1}} + \frac{A_{6,m+2}^1 (m + 2) \sin(m + 2)\theta}{r^{m+3}} \tag{5.23}
\]

Forcing these expressions to satisfy the boundary conditions (5.20) and (5.21) yields

\[
\frac{A_{5,m-2}^1 (\kappa + m - 3)}{a^{m-3}} + \frac{A_{6,m-2}^1 (m - 2)}{a^{m-1}} = -\frac{\tau a (\kappa - 1)}{\kappa}, \tag{5.24}
\]

\[
-\frac{A_{5,m-2}^1 (\kappa - m + 3)}{a^{m-3}} + \frac{A_{6,m-2}^1 (m - 2)}{a^{m-1}} = -\frac{\tau a (\kappa + 1)}{\kappa}, \tag{5.25}
\]

\[
\frac{A_{5,m+2}^1 (\kappa + m + 1)}{a^{m+1}} + \frac{A_{6,m+2}^1 (m + 2)}{a^{m+3}} = \frac{\tau a (\kappa - 1)}{\kappa}, \tag{5.26}
\]

\[
-\frac{A_{5,m+2}^1 (\kappa - m - 1)}{a^{m+1}} + \frac{A_{6,m+2}^1 (m + 2)}{a^{m+3}} = -\frac{\tau a (\kappa + 1)}{\kappa}. \tag{5.27}
\]

The solutions to these equations are
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\[ A_{5,m-2}^1 = \frac{\tau a^{m-2}}{\kappa^2}, \]  
\[ A_{6,m-2}^1 = -\tau a^m \frac{(\kappa^2 + m - 3)}{(m - 2)\kappa^2}, \]  
\[ A_{5,m+2}^1 = \frac{\tau a^{m+2}}{\kappa}, \]  
\[ A_{6,m+2}^1 = -\tau a^{m+4} \frac{1}{\kappa}. \]  

Hence, the first-order displacement functions are

\[ u_h^1(r, \theta) = \frac{\tau a^{m-2}(\kappa + m - 3)\cos(m - 2)\theta}{2G\kappa^2 r^{m-3}} - \frac{\tau a^m(\kappa^2 + m - 3)\cos(m - 2)\theta}{2G\kappa^2 r^{m-1}} \]
\[ + \frac{\tau a^{m+2}(\kappa + m + 1)\cos(m + 2)\theta}{2G\kappa r^{m+1}} - \frac{\tau a^{m+4}(m + 2)\cos(m + 2)\theta}{2G\kappa r^{m+3}}, \]  
\[ u_r^1(r, \theta) = -\frac{\tau a^{m-2}(\kappa - m + 3)\sin(m - 2)\theta}{2G\kappa^2 r^{m-3}} - \frac{\tau a^m(\kappa^2 + m - 3)\sin(m - 2)\theta}{2G\kappa^2 r^{m-1}} \]
\[ - \frac{\tau a^{m+2}(\kappa - m - 1)\sin(m + 2)\theta}{2G\kappa r^{m+1}} - \frac{\tau a^{m+4}(m + 2)\sin(m + 2)\theta}{2G\kappa r^{m+3}}. \]  

Next, the coefficient of \( \varepsilon^2 \) is set to zero in expressions (5.18) and (5.19), to find the boundary conditions for \( u_h^2(a, \theta) \) and \( u_r^2(a, \theta) \). First, from the coefficient of \( \varepsilon^2 \) in eq. (5.18), and using eq. (5.16) for \( u_h^0(r, \theta) \) and eq. (5.28) for \( u_h^1(r, \theta) \), one obtains

\[ u_h^2(a, \theta) = -\frac{1}{2} \left[ L_{r1} + a(Q_{r1} + Q_{r2}) - a(Q_{r3} + Q_{r4}) \right] \sin 2\theta \]
\[ + \frac{1}{4} \left[ L_{r1} - 2a(Q_{r3} + Q_{r4}) \right] \sin 2(m + 1)\theta - \frac{1}{4} \left[ L_{r1} + 2a(Q_{r1} + Q_{r2}) \right] \sin 2(m - 1)\theta, \]

where

\[ L_{r1} = \frac{a \tau (5 - \kappa)}{2G\kappa}, \]  
\[ Q_{r1} = \frac{\tau (\kappa + m - 3)(3 - m)}{2G\kappa^2}, \]  
\[ Q_{r2} = \frac{\tau (\kappa^2 + m - 3)(m - 1)}{2G\kappa^2}, \]  

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\[ Q_{r3} = -\frac{\tau(k + m + 1)(m + 1)}{2G\kappa}, \quad (5.34d) \]
\[ Q_{r4} = \frac{\tau(m + 3)(m + 2)}{2G\kappa}. \quad (5.34e) \]

The identity \(2\sin^2 m\theta = 1 - \cos 2m\theta\) has been used, together with trigonometric manipulations, to express the boundary condition as a trigonometric series. Likewise, eqs. (5.16), (5.19) and (5.29) give

\[u_2^2(a, \theta) = \frac{1}{2} \left[ L_{\theta 1} + a(Q_{\theta 1,1} + Q_{\theta 1,2}) + a(Q_{\theta 1,3} + Q_{\theta 1,4})\right] \cos 2\theta\]
\[+ \frac{1}{4} \left[ L_{\theta 1} + 2a(Q_{\theta 1,1} + Q_{\theta 1,2})\right] \cos 2(m - 1)\theta + \frac{1}{4} \left[ L_{\theta 1} + 2a(Q_{\theta 1,3} + Q_{\theta 1,4})\cos 2(m + 1)\theta \right], \quad (5.35)\]

where

\[ L_{\theta 1} = -\frac{\tau a(k + 5)}{2G\kappa}, \quad (5.35a) \]
\[ Q_{\theta 1} = \frac{\tau(k - m + 3)(m - 3)}{2G\kappa^2}, \quad (5.35b) \]
\[ Q_{\theta 2} = \frac{\tau(k^2 + m - 3)(m - 1)}{2G\kappa^2}, \quad (5.35c) \]
\[ Q_{\theta 3} = -\frac{\tau(k - m - 1)(m + 1)}{2G\kappa}, \quad (5.35d) \]
\[ Q_{\theta 4} = \frac{\tau(m + 3)(m + 2)}{2G\kappa}. \quad (5.35e) \]

Hence, both \(u_2^2(r, \theta)\) and \(u_2^2(r, \theta)\) will consist of a term that is varies with \(\sin 2\theta\) and \(\cos 2\theta\). Examination of the general solution (5.1)-(5.6) shows that the second-order displacement functions must have the form

\[u_2^2(r, \theta) = \frac{A_{r,2}^2(k + 1)}{2Gr} \sin 2\theta + \frac{2A_{r,2}^2}{2Gr^3} \sin 2\theta \]
\[+ \frac{A_{r,2m+2}^2(k + 2m + 1)}{2Gr^{2m+1}} \sin 2(m + 1)\theta + \frac{A_{r,2m+2}^2(2m + 2)}{2Gr^{2m+3}} \sin 2(m + 1)\theta \]
\[+ \frac{A_{r,2m-2}^2(k + 2m - 3)}{2Gr^{2m-3}} \sin 2(m - 1)\theta + \frac{A_{r,2m-2}^2(2m - 2)}{2Gr^{2m-1}} \sin 2(m - 1)\theta, \quad (5.36)\]
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\[ u_\theta(r, \theta) = \frac{A_{7,2}^2(\kappa - 1)}{2Gr} \cos \theta - \frac{2A_{8,2}^2}{2Gr^3} \cos \theta \]

\[ + \frac{A_{7,2m+2}^2(\kappa - 2m - 1)}{2Gr^{2m+1}} \cos 2(m + 1)\theta - \frac{A_{8,2m+2}^2(2m + 2)}{2Gr^{2m+3}} \cos 2(m + 1)\theta \]

\[ + \frac{A_{7,2m-2}^2(\kappa - 2m + 3)}{2Gr^{2m-3}} \cos 2(m - 1)\theta - \frac{A_{8,2m-2}^2(2m - 2)}{2Gr^{2m-1}} \cos 2(m - 1)\theta . \quad (5.37) \]

Requiring that the functions given by eqs. (5.36) and (5.37) satisfy the boundary conditions (5.34) and (5.35) leads to the following two equations for the \( A_{7,2m}^2 \) coefficients:

\[ \frac{A_{7,2}^2(\kappa + 1)}{Ga} + \frac{2A_{8,2}^2}{Ga^3} = -\left[ L_{r1} + a(Q_{r1} + Q_{r2}) - a(Q_{r3} + Q_{r4}) \right] , \quad (5.38) \]

\[ \frac{A_{7,2m-2}^2(\kappa + 2m - 3)}{Ga^{2m-3}} + \frac{A_{8,2m-2}^2(2m - 2)}{Ga^{2m-1}} = -\frac{1}{2} \left[ L_{r1} + 2a(Q_{r1} + Q_{r2}) \right] , \quad (5.39) \]

\[ \frac{A_{7,2m+2}^2(\kappa + 2m + 1)}{Ga^{2m+1}} + \frac{A_{8,2m+2}^2(2m + 2)}{Ga^{2m+3}} = -\frac{1}{2} \left[ -L_{r1} + 2a(Q_{r3} + Q_{r4}) \right] , \quad (5.40) \]

\[ \frac{A_{7,2}^2(\kappa - 1)}{Ga} - \frac{2A_{8,2}^2}{Ga^3} = -\frac{1}{2} \left[ L_{\theta1} + a(Q_{\theta1} + Q_{\theta2}) + a(Q_{\theta3} + Q_{\theta4}) \right] , \quad (5.41) \]

\[ \frac{A_{7,2m-2}^2(\kappa - 2m + 3)}{Ga^{2m-3}} - \frac{A_{8,2m-2}^2(2m - 2)}{Ga^{2m-1}} = -\frac{1}{4} \left[ -L_{\theta1} - 2a(Q_{\theta3} + Q_{\theta4}) \right] , \quad (5.42) \]

\[ \frac{A_{7,2m+2}^2(\kappa - 2m - 1)}{Ga^{2m+1}} - \frac{A_{8,2m+2}^2(2m + 2)}{Ga^{2m+3}} = -\frac{1}{4} \left[ -L_{\theta1} - 2a(Q_{\theta1} + Q_{\theta2}) \right] . \quad (5.43) \]

Solving the each pair of simultaneous equations yields all six coefficients:

\[ A_{7,2}^2 = \frac{G(F_{1r}^2 + F_{1\theta}^2)a}{\kappa} , \quad (5.44) \]

\[ A_{8,2}^2 = Ga^3 \left[ F_{1r}^2 - \frac{(F_{1r}^2 + F_{1\theta}^2)(\kappa + 1)}{2\kappa} \right] , \quad (5.45) \]

\[ F_{1r}^2 = -\frac{1}{2} \left[ L_{r1} + a(Q_{r1,1} + Q_{r1,2}) - a(Q_{r1,3} + Q_{r1,4}) \right] , \quad (5.46) \]
\[ F_{1,\theta}^2 = -\frac{1}{2} \left[ L_{\theta 1} + a(Q_{\theta 1,1} + Q_{\theta 1,2}) + a(Q_{\theta 1,3} + Q_{\theta 1,4}) \right]; \]  
\[ A_{7,2m-2}^{2} = \frac{Ga^{2m-3}(F_{4,r}^2 + F_{4,\theta}^2)}{\kappa}, \]  
\[ A_{8,2m-2}^{2} = \frac{Ga^{2m-3}}{2\kappa(m-1)} \left[ 2\kappa F_{4,r}^2 - (F_{4,r}^2 + F_{4,\theta}^2)(\kappa + 2m - 3) \right], \]  
\[ F_{4,r}^2 = -\frac{1}{4} \left[ L_{r 1} + 2a(Q_{r 1,1} + Q_{r 1,2}) \right]; \]  
\[ F_{4,\theta}^2 = \frac{1}{4} \left[ L_{\theta 1} + 2a(Q_{\theta 1,1} + Q_{\theta 1,2}) \right]; \]  
\[ A_{7,2m+2}^{2} = \frac{Ga^{2m+1}(F_{3,r}^2 + F_{3,\theta}^2)}{\kappa}, \]  
\[ A_{8,2m+2}^{2} = \frac{Ga^{2m+2}}{2\kappa(m+1)} \left[ 2\kappa F_{3,r}^2 - (F_{3,r}^2 + F_{3,\theta}^2)(\kappa + 2m + 1) \right], \]  
\[ F_{3,r}^2 = -\frac{1}{4} \left[ L_{r 1} + 2a(Q_{r 1,3} + Q_{r 1,4}) \right]; \]  
\[ F_{3,\theta}^2 = \frac{1}{4} \left[ L_{\theta 1} + 2a(Q_{\theta 1,3} + Q_{\theta 1,4}) \right]. \]

5.4 Effective shear modulus

It should be first noted that the pores considered in this chapter all possess rotational symmetry. The special case (recall Chapter 4) of \( m = 4 \) can be ignored, since only large values of \( m \) represent small-scale roughness, it is clear that the effective shear modulus will be insensitive to any angular rotation of the pore. Hence, with regard to the effective shear modulus, the above solution is perfectly general, and there is no need to consider a phase shift of the sinusoidal bumps with respect to the far-field shear stress.

To be specific, plane strain will be considered, in which case \( (\kappa + 1)/(\kappa - 1) = 2(1-v)/(1-2v) \). The expression for the effective shear modulus given by eq. (5.8) can then be written as follows:

\[ \frac{1}{G_{eff}} = \frac{1}{G} \cdot \frac{4(1-v)A_{7,2}(\kappa = -1; \tau = 1)}{A_{pore}} \cdot \frac{\pi}{\phi}, \]

where \( G \) is the shear modulus of the host material, and the inclusion area fraction \( c \) is replaced by the porosity, \( \phi \). In terms of the \( H_{1212} \) parameter defined and used in Chapter 4,
Comparison of the two previous equations shows that the shear compliance parameter $H_{1212}$ of the corrugated pore is given by

$$H_{1212} = -\frac{2(1-\nu)}{G} A_{1/2}(\kappa = -1; \tau = 1)\pi \phi. \tag{5.58}$$

As was found for the family of symmetric pores studied in Chapter 4, and also for the pore compressibility problems analyzed in Chapters 2 and 3, the shear compliance is proportional to $(1-\nu)/G$, and also to a dimensionless multiplicative factor that depends only on the pore shape. As in Chapter 4, a normalized shear compliance can be defined as

$$h = \frac{G H_{1212}}{(1-\nu)\phi} = \frac{A_{1/2}(\kappa = -1; \tau = 1)\pi}{A_{\text{pore}}} \tag{5.59}$$

Recalling that $A_{\text{pore}} = \pi a^2 \left[1 + (\varepsilon^2/2)\right]$, and using eq. (5.44) for $A_{1/2}(\kappa, \tau)$, it follows that, to second-order in $\varepsilon$, the normalized shear compliance of the corrugated pore is given by

$$h = \frac{2[1 + (4m - 1)\varepsilon^2/2]}{1 + (\varepsilon^2/2). \tag{5.60}$$

Setting $\varepsilon = 0$ recovers the value $h = 2$ for the smooth-walled circular pore. It is worth noting that this expression agrees with result (3.43) for the pore compressibility, except that $(4m - 3)$ has been replaced by $(4m - 1)$. Since $m$ must be a large integer in order to represent small-wavelength roughness, for practical purposes the relative effect of roughness on the pore shear compliance is the same as for the pore compressibility.

### 5.5 Comparison with BEM calculations

The shear compliance given by the second-order perturbation solution will now be compared with the values obtained by boundary element calculations, and to some upper and lower bounds that can be derived. The boundary element calculations were performed using a code developed by Martel & Muller (2000), and which is described in Chapter 3. The number of elements needed to achieve convergence will increase with $m$, since, as the spatial wavelength of the roughness decreases, the spacing between boundary nodes must decrease. Similarly, the mesh must also be finer if the roughness amplitude increases. For the calculations shown in Figs. 5.2-5.4 below, it was found that 360 elements were always sufficient.

An upper bound on the pore compliance parameter $H_{1212}$ can be obtained by starting with the fact that if the actual pore is replaced by a pore defined by the smallest possible circumscribed circle, which has radius $a(1+\varepsilon)$, the overall shear compliance of the macroscopic body cannot decrease, since removal of solid material...
cannot stiffen the body (Rice & Drucker, 1967; Goldshtein & Entov, 1994). Considering a large body of area $A_\infty$, containing only one pore, it follows from eq. (5.57) that

$$\frac{1}{G} + 2H_{1212}(\text{pore}) \leq \frac{1}{G} + 2H_{1212}(\text{circumscribed circle}).$$ (5.61)

In terms of $h$, as defined in eq. (5.59),

$$\frac{h_{\text{pore}}(1-\nu)A_{\text{pore}}}{GA_\infty} \leq \frac{h_{\text{circle}}(1-\nu)A_{\text{circle}}}{GA_\infty}.$$ (5.62)

But $h(\text{circle}) = 2$, $A_{\text{pore}} = \pi a^2[1 + (\varepsilon^2/2)]$, and $A_{\text{circle}} = \pi a^2(1 + \varepsilon)^2$, and so

$$h(\text{corrugated pore}) \leq \frac{(1 + \varepsilon)^2}{1 + (\varepsilon^2/2)}.$$ (5.63)

In analogy with the case of pore compressibility, the Hashin-Shtrikman bounds on the effective shear modulus can be used to show that the normalized shear compliance $h$ for a pore of any shape, averaged over all angular orientations if necessary, can never be less than 2, the value obtained for a smooth circle. Hence, the normalized shear compliance of the corrugated pore is bounded as follows:

$$2 \leq h = \frac{GH_{1212}}{(1-\nu)\phi} \leq \frac{2(1 + \varepsilon)^2}{1 + (\varepsilon^2/2)}.$$ (5.64)

Again, recalling that only small values of roughness are of present interest, one can say that the normalized pore shearability must lie between 2 and $2(1 + 2\varepsilon)$, which are the same bounds as apply to the pore compressibility.

The second-order perturbation expression for the shear compliance is plotted in Figs. 5.2-5.4 as a functions of $\varepsilon$, for the values $m = 8, 16, \text{and } 32$. Also plotted are the upper and lower bounds, eq. (5.64), and the values computed using the boundary element method. As the influence of roughness on shearability is of second-order, the normalized shear compliances lie close to the lower bound for very small values of $\varepsilon$. But as $\varepsilon$ increases, the compliances move closer to the upper bound. Moreover, the difference between the exact value and the upper bound becomes smaller as the number of corrugations, $m$, increases. As the upper bound corresponds to the case in which the small bumps of solid material have been removed, this shows that, particularly as $m$ increases, these small bumps provide very little shear stiffness to the pore. The second-order perturbation solution is reasonably accurate for $\varepsilon < 1/2m$, and this range of accuracy deteriorates as $m$ increases. These results are in almost all respects similar to those found in Chapter 3 for the effect of small roughness on the hydrostatic pore compressibility.
Figure 5.2. Normalized shear compliance of a corrugated pore (see eq. 5.11, Fig. 5.1) as a function of the roughness amplitude, for the case $m = 8$.

Figure 5.3. Normalized shear compliance of a corrugated pore (see eq. 5.11, Fig. 5.1) as a function of the roughness amplitude, for the case $m = 16$. 
Figure 5.4. Normalized shear compliance of a corrugated pore (see eq. 5.11, Fig. 5.1) as a function of the roughness amplitude, for the case $m = 32$. 
6. **Image Analysis and Prediction of the Elastic Moduli**

6.1 **Introduction**

Image analysis is a powerful tool, which is now widely used in several fields such as in engineering and medicine (Chermant, 2001; Chermant et al., 2001; Mouret et al., 2001). In the recent past this technique has been successfully used in modelling the mechanical and transport properties of porous and inhomogeneous materials. Yue et al. (2003) used the image analysis technique together with the finite element method analysis to model asphalt concrete under different loading conditions. Chen et al. (2004) used image analysis to aid in the prediction of inhomogeneous rock failure. Coster and Chermant (2001) discussed in detail how the image analysis technique can be incorporated into the modeling of civil engineering materials. Lock et al. (2002) used information about rock pores obtained from image analysis of electron micrographs to predict permeability of sandstones.

With specific reference to the topic of this thesis, few works have been carried out that attempt to use data obtained from image analysis to model the mechanical behaviour of porous materials. Most previous modelling work has proceeded on the basis of assuming idealized pore geometries, such as circular cylinders, oblate spheroids, or penny-shaped cracks (Zimmerman, 1991; Mavko et al., 1998). Any connection with the actual pore geometry is made, if at all, in an inverse manner, by fitting measured moduli data to the models to infer the pore aspect ratios, for example.

Recently, some researchers have begun to use pore space images in the modelling process. Arns et al. (2002) reconstructed the three-dimensional pore structure of a small (20 mm$^3$) volume of Fontainebleau sandstone from X-ray tomographic images, and then computed the effective elastic moduli by performing three-dimensional finite element calculations. The agreement between the predicted and measured moduli was good, but their method is very time-consuming with regards to both data collection and simulation, and requires expensive X-ray tomography equipment. Prokopiev & Sevostianov (2007) took the same data set, and computed a representative aspect ratio of an “equivalent” spheroidal pore that had the same surface/volume ratio as the actual pores. Several classical effective medium theories (Mori-Tanaka, self-consistent, and differential) were then used to predict the moduli as a function of porosity. Rögen et al. (2005) and Fabricius et al. (2007) used image analysis, along with appropriate combinations of the Reuss average and the Hashin-Shtrikman bounds, to model the elastic properties of some North Sea chalks.

In this study, actual pore space images are used to study the relationship between the effective elastic properties and pore geometry. Commercially available image analysis software is used to extract the pore geometry information, such as the perimeter and the area, from the pore images. The elastic compliances of the pores are estimated either by performing boundary element calculations, or using the scaling laws, (2.40) and (4.52). The differential effective medium theory is used to...
(approximately) account for the elastic interactions between nearby pores. The moduli predictions are then compared to previously-published experimental data. This methodology is applied to Berea sandstone, Fontainebleau sandstone, and an SiC ceramic.

6.2 Image analysis

The three key steps followed in image analysis are image acquisition, enhancement, and segmentation (Fig. 6.1). According to Coster & Cherment (2001), image acquisition can be made either with scanning electron microscopy (SEM) or optical systems using visible light.

Images are saved in computers in the form of a rectangular array of pixels (Glasbey & Horgan, 1995). The pixel value carries information of the image. In the image enhancement procedure, pixel values are altered to achieve the desirable features. The image enhancement procedure is normally carried out using filtering techniques. Mode, mean, and Laplacian are few of the commonly used filtering methods. The above mention three types of filtering techniques are now briefly described.

6.2.1 Mode and mean filters

Both of these filters may be called linear filters. In this method, each pixel is replaced by the average of pixel values or mode values in the central pixel value of the selected square. The process can be explained further, as follows.

Let \( f_{ij} \), for \( i,j = 1,2,\ldots,n \), denote the pixel values in the image. If we represent output pixel values with \( g_{ij} \), a linear filter kernel of size \((2m+1) \times (2m+1)\), with specified weights \( w_{kl} \), gives

\[
g_{ij} = \sum_{k=-m}^{m} \sum_{l=-m}^{m} w_{kl} f_{i+k,j+l} \quad \text{for} \quad i,j = (m+1)\ldots,(n-m). \quad (6.1)
\]

Now, \( g_{ij} \) can be determined depending on whether one uses mode or mean filters. In the present analysis, in order to remove random noise values, both the mode and mean filters were used.

6.2.2 Laplacian filter

This filter type may be called a “second derivative filter”. The new pixel value is derived considering the second derivative

\[
g_{ij} = \sum_{k=-m}^{m} \sum_{l=-m}^{m} w_{kl} f_{i+k,j+l} \approx \frac{\partial^2 f_g}{\partial x^2} + \frac{\partial^2 f_g}{\partial y^2}. \quad (6.2)
\]

After the image enhancement, image segmentation is carried out. Image segmentation is the division of an image into different regions or several categories.
All the pixels in the image have its own unique class. According to Glasbey & Horgan (1995), a good segmentation is typically one in which (a) pixels in the same category have similar greyscale or multivariate values, and (b) neighbouring pixels that are in different categories have dissimilar values.

There are three ways to carry out image segmentation. These methods are thresholding, edge-based, and region-based. Among these three methods, threshold segmentation is the simplest one. In this method all pixels located at \((i,j)\) with greyscale value \(f_{ij}\) will be categorized into two classes, above and below the given threshold value \(t\). The threshold value \(t\) can be determined either manually or by a histogram. In the present analysis, the histogram was used to determine the threshold value. Figure 6.2 represents a typical histogram obtained for the ceramic image. The \(y\)-axis of the histograms represents the number of pixels, and the \(x\)-axis gives the value of pixels.
Figure 6.1. Schematic flowchart of the image analysis process.
6.3 Effective elastic moduli

Image analysis procedures are used to identify individual pores, and convert the images of the pore boundaries into some digital format. The individual pore compressibilities and pore shear compliances can then be estimated by either using boundary elements, or using the two scaling laws, eqs. (2.40) and (4.52), which are based on the area and perimeter of the pore. From these data, an areally-weighted mean value of \( C_{pc} \) and \( S_{pc} \) can be calculated, where \( 2H_{1212}/\phi \) is now defined as \( S_{pc} \), in order to have a similar notation for both of the two isotropic elastic moduli, \( K \) and \( G \). But these values strictly apply only in the hypothetical case in which the pores are all “infinitely” far from each other, so that the stress fields around each pore do not overlap. In reality, each pore is surrounded by porous material that is less stiff than the “non-porous” rock, and so \( C_{pc} \) and \( S_{pc} \) will each be higher than the values calculated for isolated pores. Numerous methods have been proposed to relate these compliances in the case of finite porosity to the values that would occur in the “isolated pore, small porosity” limit (Christensen, 1990; Zimmerman, 1991; Grimvall, 1999). However, no single method has gained universal acceptance, and a review of them would not be appropriate to the present purposes. Instead, a version of the “differential” scheme will be used. The differential method is known to be fairly accurate for materials containing spherical pores (Zimmerman, 1991).

Consider the expressions \( C_{bc} = C_o + \phi C_{pc} \) and \( S_{bc} = S_o + \phi S_{pc} \), where an obvious notation is introduced for the shear modulus, in which \( S_{bc} = 1/G_{eff} \). Hereafter “\( R \)” with the relevant subscript will be used to represent either the compressibility or the shearability parameters, as the following derivation is common for both. If one considers the thought experiment of inserting a small differential amount of pores into the otherwise non-porous rock, then one can say that

\[
R_{eff}(\delta \phi) = R_o + R_{pc}\delta \phi. \tag{6.3}
\]
Noting that $R_o$ is the effective compliance when the porosity is zero, one can convert eq. (6.3) into the following differential equation:

$$\frac{1}{R_{\text{eff}}} \frac{dR_{\text{eff}}}{d\phi} = \frac{R_{pc}}{R_{\text{eff}}}.$$  \hspace{1cm} (6.4)

This differential equation can be used to model the evolution of the compliance as additional pores are added into the rock, with the initial condition that $R_{\text{eff}} = R_o$ when $\phi = 0$.

The plausible assumption is now made that, as pores are added to the rock, the ratio of the compliance of the new pore, $R_{pc}$, to the compliance of the host material, $R_{\text{eff}}$, is constant. Such an assumption is entirely consistent with the spirit of the differential scheme. If this is the case, this ratio must be equal to the value it has in the “isolated pore, zero-porosity” limit. This value is called $R_{pc}^0 / R_o$, with the superscript “$0$” used to denote the zero-porosity limit. The main point to note for now is that this parameter does not vary with porosity, in which case eq. (6.4) can be integrated to find

$$\frac{R_{\text{eff}}}{R_o} = \phi(R_{pc}^0 / R_o)^\phi.$$  \hspace{1cm} (6.5)

In the above scheme it was implicitly assumed that, as new pores with incremental porosity $\delta\phi$ are placed into the material, the total porosity is increased by $\delta\phi$. But Norris (1985) and others have pointed out that if these additional pores are imagined to be randomly placed in the rock, they will replace solid material with probability $1-\phi$, and replace existing pores with probability $\phi$. Hence, the increment in total porosity will be $\delta\phi(1-\phi)$. If $\delta\phi$ in eq. (6.4) is replaced with $\delta\phi(1-\phi)$, this has the effect of transforming $\phi$ into $-\ln(1-\phi)$, and so eq. (6.5) is transformed into (Zimmerman, 1991)

$$\frac{R_{\text{eff}}}{R_o} = (1 - \phi)^{-R_{pc}^0 / R_o}.$$  \hspace{1cm} (6.6)

In terms of the effective moduli, eq. (6.6) takes the forms

$$\frac{K_{\text{eff}}}{K_o} = (1 - \phi)^{C_{pc}^0 / C_o},$$  \hspace{1cm} (6.7)

$$\frac{G_{\text{eff}}}{G_o} = (1 - \phi)^{S_{pc}^0 / S_o}.$$  \hspace{1cm} (6.8)

These expressions have the same form, $(1 - \phi)^m$, that has often been proposed empirically to model the elastic moduli of porous ceramics (Rice, 1998; Reynaud et al., 2005), but the present derivation assigns a physical interpretation to the exponent $m$. All that remains is to estimate the value of the parameters $C_{pc}^0 / C_o$ and $S_{pc}^0 / S_o$, which depend on pore shape.
6.4 Pore compressibility and shearability estimations

The process starts with a scanning electron micrograph (SEM) image of a region of rock surface. Figure 6.3 (Schlueter et al., 1997) shows such an image for Berea sandstone, Figs. 6.6 (Colón et al., 2004) and 6.7 (Holmes & Packer, 2004) are from Fontainebleau sandstone, and Fig. 6.10 (Reynaud et al., 2005) is taken from an SiC ceramic. In Fig. 6.3a the dark grey regions are quartz grains, the black regions are epoxy-filled pores, the silver regions are pores filled with Wood’s metal, and the light grey regions are clays and other minerals. This image was initially taken for a study of two-phase flow properties, but for the present purposes the only important distinction to be made is between pores and minerals. The size of the original image in Fig. 6.3a was 375×250 μm. The properties of all four images are summarized in Table 6.1.

The Image Analysis software packages Idrisi and CartaLinx are used to extract the pores from the images, and to compute their relevant properties, area and perimeter. As the first step, the original image is enhanced using 4×4 kernel with mode filtering, to arrive at Fig. 6.3b. Then, based on the histogram of color distribution, a binary image was created to separate the pores from the minerals (Fig. 6.3c). As an intermediate step, smaller features of size less than 8 μm² were eliminated, using a histogram of pore areas. These features may simply be artifacts of the image analysis procedure, or may indeed be small pores. With regard to the latter possibility, it should be noted that these smaller pores contribute only 2.5% of the total porosity of the analyzed section. (This does not mean 2.5 “porosity points”. For example, if \( \phi = 10\% \) for an image, eliminating these smaller pores will decrease \( \phi \) to say, 9.75%.) Finally, a vector image is created in which the boundary of each pore is associated with a numerical \((x,y)\) co-ordinate pair (Fig. 6.3d).

To remove artificially introduced waviness in the pore boundary, the vector image was exported to another image analysis software package, Carta-Linx. Smoothing was carried out using the option called “generalization”, based on the individual perimeter values of the pores. This smoothing is permissible, as waviness with small amplitudes does not contribute any appreciable stiffness to the pore, as shown in Chapters 3 and 5.

Table 6.1. Properties of the images.

<table>
<thead>
<tr>
<th>Image</th>
<th>Original size (μm)</th>
<th>Image enhancement</th>
<th>Fractional reduction in porosity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ceramic</td>
<td>200×125</td>
<td>4×4</td>
<td>1.5%</td>
</tr>
<tr>
<td>Berea</td>
<td>375×250</td>
<td>4×4</td>
<td>2.5%</td>
</tr>
<tr>
<td>Fontainebleau (F1)</td>
<td>400×300</td>
<td>4×4</td>
<td>&lt; 1%</td>
</tr>
<tr>
<td>Fontainebleau (F2)</td>
<td>850×620</td>
<td>4×4</td>
<td>&lt; 1%</td>
</tr>
</tbody>
</table>

The boundary element method is then used to calculate the parameters \( C_{phC} / C_o \) and \( S_{phC} / S_o \), assuming each pore to be isolated in an infinite rock matrix. The calculations are performed using a code developed by Martel & Muller (2000), which
is a simplified version of the more general two-dimensional BEM code from Crouch & Starfield (1983) that is based on the displacement discontinuity method. This code is optimized for the problem of a single void or crack in an infinite elastic body, with many of the options included in the original code removed, thus rendering it easier to use for the problem at hand.

In the calculations, all far-field stresses and body forces are set to zero. A uniform normal traction of unit magnitude is prescribed over the surface of the hole to determine the area change of the single pore. The cavity boundary is discretized into a number of equal-length elements. It is generally found that about 300 boundary elements are sufficient to achieve convergence of the computed compressibilities. After calculating the $C_{pc}$ values for all the pores, the area-weighted average value of $C_{pc}$ is computed. The pore compressibility is also estimated from the area and perimeter, using the scaling law, eq. (2.40). Finally, the macroscopic bulk modulus is predicted from eq. (6.7).

However for the calculation of shearability parameters, a slightly different approach is adopted. A shear traction is applied to the pore surface, corresponding to “problem 2” of the superposition procedure used in Chapter 4. The stored strain energy is calculated by numerically performing the integrations described in Chapter 4, from which the shear compliance is found. But, as shown in Chapter 4, the shear compliance of a pore, such as those observed in the images, which possesses no axes of rotational symmetry, will be of the form $h = h_o + h_1 \cos 4\theta$ (whether written in terms of $h$, $H_{1212}$, or $S_{pc}$, the same argument applies). If it is assumed that these pores are randomly oriented, the desired value of $h$ is the mean value over all angles $\theta$, which is $h_o$. This mean value can be found by taking the mean of two values of $h$ for two orientation angles that differ by $45^\circ$, since in this case

$$h(\theta) + h(\theta + 45^\circ) = h_o + h_1 \cos 4\theta + h_o + h_1 \cos[4(\theta + 45^\circ)] \over 2$$

$$= h_o + {h_1 \cos 4\theta + h_1 \cos(4\theta + 180^\circ) \over 2} = h_o + {h_1 \cos 4\theta - h_1 \cos 4\theta \over 2} = h_o.$$  \hspace{1cm} (6.9)

After calculating $C_{pc}$ and $S_{pc}$ for each pore, the areally-weighted mean values are determined. Finally, the macroscopic moduli are predicted from eqs. (6.7) and (6.8).

6.5 Elastic moduli predictions

6.5.1 Berea sandstone

The area and perimeter of each individual pore from the Berea sandstone image in Fig. 6.3a are shown in Table 1, along with the pore compressibility that is calculated from boundary elements (BEM), and estimated from the scaling law, eq. (2.49). The same information is also plotted in Fig. 6.4. The scaling law usually overestimates the compressibility, and indeed it is found empirically that the scaling law is more accurate if a multiplicative factor of 0.9 is introduced. However, the idea
of optimizing this coefficient will not be pursued in any detail here, and the proposed scaling law is used without modification.

The area-weighted mean value of the normalized pore compressibility, \( G_oC_{pc}/(1-\nu_o) \), as calculated by the BEM, is 3.96. The elastic moduli of the mineral phase of Berea, as reported in Zimmerman (1991), are \( G_o = 31.34 \) GPa, \( K_o = 39.75 \) GPa, and \( \nu_o = 0.188 \). Hence, it can be seen that \( C_{pc}^o/C_o = 4.58 \). The porosity of Berea is 0.22, so eq. (6.7) gives an effective bulk modulus of 13.0 GPa. The mean value of \( G_oC_{pc}/(1-\nu_o) \), as calculated by the scaling law, is 4.44, in which case the analogous calculation procedure yields \( K_{eff} = 11.1 \) GPa. Both of these values are reasonably close to the experimental value, measured at high stresses when all microcracks are closed, which is 9.6 GPa (Zimmerman, 1991).

The analogous prediction of the effective shear modulus proceeds as follows. Table 6.3 shows the pore shear compliances as calculated from boundary elements (BEM), and as estimated from the scaling law. The same information is plotted in Fig. 6.4. The mean value of the numerically computed pore shear compliances is \( S_{pc}^o/S_o = 3.09 \). The porosity of Berea is 0.22, so eq. (6.8) gives an effective shear modulus of 14.5 GPa. The mean value of \( G_oS_{pc}/(1-\nu_o) \), as calculated using the scaling law, is 3.53, in which case the calculation procedure yields \( G_{eff} = 13.0 \) GPa. These compare very well with the high-stress effective shear modulus of 14.1 GPa measured by King (1966).
Fig. 6.3. Image analysis procedure for Berea sandstone; see text for explanation of steps.
Table 6.2. Normalized pore compressibilities of the 14 largest pores from Fig. 6.3.

<table>
<thead>
<tr>
<th>Perimeter (μm)</th>
<th>Area (μm²)</th>
<th>$GC_{pc}/(1-\nu)$ (BEM)</th>
<th>$GC_{pc}/(1-\nu)$ eq. (2.40)</th>
<th>Error of eq. (2.40)</th>
</tr>
</thead>
<tbody>
<tr>
<td>293</td>
<td>4369</td>
<td>2.58</td>
<td>3.12</td>
<td>21%</td>
</tr>
<tr>
<td>475</td>
<td>9080</td>
<td>3.39</td>
<td>3.96</td>
<td>17%</td>
</tr>
<tr>
<td>251</td>
<td>3709</td>
<td>2.50</td>
<td>2.71</td>
<td>8%</td>
</tr>
<tr>
<td>621</td>
<td>12509</td>
<td>3.99</td>
<td>4.90</td>
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</tr>
<tr>
<td>331</td>
<td>3024</td>
<td>5.93</td>
<td>5.76</td>
<td>-3%</td>
</tr>
<tr>
<td>259</td>
<td>3733</td>
<td>2.63</td>
<td>2.87</td>
<td>9%</td>
</tr>
<tr>
<td>728</td>
<td>12475</td>
<td>6.03</td>
<td>6.76</td>
<td>12%</td>
</tr>
<tr>
<td>240</td>
<td>2802</td>
<td>3.33</td>
<td>3.27</td>
<td>-2%</td>
</tr>
<tr>
<td>173</td>
<td>1879</td>
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<tr>
<td>309</td>
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<td>4.56</td>
<td>5.15</td>
<td>13%</td>
</tr>
<tr>
<td>157</td>
<td>1281</td>
<td>3.10</td>
<td>3.04</td>
<td>-2%</td>
</tr>
</tbody>
</table>

Table 6.3. Normalized pore shear compliances of the 14 largest pores from Fig. 6.3.

<table>
<thead>
<tr>
<th>Perimeter (μm)</th>
<th>Area (μm²)</th>
<th>$GS_{pc}/(1-\nu)$ (BEM)</th>
<th>$GS_{pc}/(1-\nu)$ eq. (4.52)</th>
<th>Error of eq. (4.52)</th>
</tr>
</thead>
<tbody>
<tr>
<td>293</td>
<td>4369</td>
<td>2.51</td>
<td>2.56</td>
<td>-2%</td>
</tr>
<tr>
<td>433</td>
<td>6568</td>
<td>3.04</td>
<td>3.27</td>
<td>-7%</td>
</tr>
<tr>
<td>475</td>
<td>9080</td>
<td>3.02</td>
<td>2.98</td>
<td>1%</td>
</tr>
<tr>
<td>251</td>
<td>3709</td>
<td>2.39</td>
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<td>2.73</td>
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<tr>
<td>157</td>
<td>1282</td>
<td>2.66</td>
<td>2.52</td>
<td>5%</td>
</tr>
</tbody>
</table>
Fig. 6.4. Test of the compressibility scaling law (2.40) for the Berea sandstone pores shown in Fig. 6.3a. Values computed with BEM are plotted on the x-axis, predictions of the scaling law are plotted on the y-axis.

Fig. 6.5. Test of shear compliance scaling law (4.52) for the Berea sandstone pores shown in Fig. 6.3a. Values computed with BEM are plotted on the x-axis, predictions of the scaling law are plotted on the y-axis.

6.5.2 Fontainebleau sandstone

Fontainebleau sandstone is often used for testing petrophysical models, due to the facts that its mineral phase consists of 99.8% quartz, it has a very narrow grain size distribution, and quartz cementation has created a continuous range of porosities from 3-30% (Bourbié et al., 1987).
Images of two Fontainebleau sandstone samples, having porosities of 12% and 25%, are shown in Figs. 6.6 and 6.7. These images illustrate the fact that essentially any 2-D pore image can be used as a starting point for our procedure, as long as the pores can be distinguished from the minerals.

![Image of a region of Fontainebleau sandstone, from Colón et al. (2004). Darker regions are pores; total porosity of this sample is 12%.](image1)

Fig. 6.6. Image of a region of Fontainebleau sandstone, from Colón et al. (2004). Darker regions are pores; total porosity of this sample is 12%.

![Image of a region of Fontainebleau sandstone, from Holmes & Packer (2004). Horizontal bar has length of 500 μm. Darker regions are pores, lighter regions are grains. Porosity of this sample is 25%.](image2)

Fig. 6.7. Image of a region of Fontainebleau sandstone, from Holmes & Packer (2004). Horizontal bar has length of 500 μm. Darker regions are pores, lighter regions are grains. Porosity of this sample is 25%.

**Bulk modulus predictions**

The same image analysis procedure as was described above for Berea is now applied to the Fontainebleau images. The area-weighted pore compressibility of the sample with \( \phi = 12\% \) is found to be \( G_o C_{pc}/(1-\nu_o) = 3.81 \) according to the BEM calculations, and 5.02 according to the scaling law. For the \( \phi = 25\% \) sample, these values were 3.84 and 4.97, respectively. Interestingly, the pore compressibility values
do not seem to vary much with porosity; the values computed for the 12% porosity sample are used in the following calculations. The elastic moduli of the mineral phase of Fontainebleau, which is essentially pure quartz, are (Bourbié et al., 1987) \( G_0 = 44 \) GPa, \( K_0 = 37 \) GPa, and \( \nu_0 = 0.074 \). Hence, it is found that \( C_{pc}/C_o = 2.97 \) according to the BEM calculations, and \( C_{pc}/C_o = 3.84 \) using the scaling law. Finally, eq. (6.7) yields

\[
K_{\text{eff}} (\text{GPa}) = 37.0(1 - \phi)^{2.97} \tag{6.10}
\]

when the pore compressibilities are calculated from the BEM, and

\[
K_{\text{eff}} (\text{GPa}) = 37.0(1 - \phi)^{3.84} \tag{6.11}
\]

when the pore compressibilities are calculated from the scaling law, eq. (2.40).

The two effective bulk moduli predictions are shown in Fig. 6.8, compared with the experimental values reported by Arns et al. (1987). The data tend to fall between the two predictions, which could be interpreted as implying that the BEM calculations underestimate the pore compressibility, whereas the scaling law slightly overestimates it. It is noted also that “correcting” the scaling law (2.40) by the aforementioned factor of 0.9 yields a curve that would lay between the two curves shown in Fig. 6.8, and which would fit the data quite well.

![Figure 6.8](image_url)

Fig. 6.8. Bulk modulus of Fontainebleau sandstone, showing values reported by Arns et al. (2002) and values predicted using both methods of estimating the mean pore compressibility, starting with the images shown in Figs. 6.6 and 6.7. Also shown are the predictions made by Prokopiev & Sevostianov (2007).

Also shown in Fig. 6.8 are the three predictions made by Prokopiev & Seviostanov (2007), using three different effective medium theories: Mori-Tanaka
Estimation of the Elastic Moduli of Porous Materials

(MT), the differential scheme (DS), and the self-consistent scheme (SC). Details of these effective medium schemes can be found in Mura (1987) and Nemat-Nasser & Hori (1999). Although it is not the present intention to debate the relative merits of the present approach compared to that used by Prokopiev & Sevostianov (2007), it is clear that the present method compares well with these other recent approaches.

Shear modulus predictions

The area-weighted pore shearability of the sample with $\phi = 12\%$ is found to be $G_0 S_{pc}/(1-\nu_0) = 2.58$ according to the BEM calculations, and 3.50 according to the scaling law. For the $\phi = 25\%$ sample, these values were 3.33 and 3.48 respectively. Interestingly, the pore shearability values also do not seem to vary very much with porosity. The value computed for the 25% porosity sample is used in the following calculations. The elastic moduli of the mineral phase of Fontainebleau, which is essentially pure quartz are (Arns et al., 2002) $G_o = 44$ GPa, $K_o = 37$ GPa, and $\nu_o = 0.074$. Hence, it can be found $S_{pc}^o / S_o = 3.33$ according to the BEM calculations, and $S_{pc}^o / S_o = 3.48$ using the scaling law. Finally, eq. (6.08) yields the following prediction for the effective shear modulus, when the pore shear compliance is calculated by BEM:

$$G_{eff} \text{(GPa)} = 44.0(1-\phi)^{3.33},$$

and yields the following prediction if the pore shear compliance is estimated from the scaling law, eq. (4.52):

$$G_{eff} \text{(GPa)} = 44.0(1-\phi)^{3.48}.$$  

These two effective shear modulus predictions are shown in Fig. 6.9, where they are compared to the experimental values reported by Arns et al. (2002), and to the three predictions made by Prokopiev & Sevostianov (2007). As was the case for the bulk modulus, the present method gives very good predictions, over the entire range of data. Note that the values predicted by the finite element calculations (FEM) of Arns et al. (2002) are indistinguishable from the predictions of eq. (6.12), in which the individual pore compliances were found from BEM calculations, and the differential effective medium method was used to extrapolate from the low-porosity limit out to higher porosities.
Fig. 6.9. Shear modulus of Fontainebleau sandstone, showing values measured by Arns et al. (2002), along with those predicted by the present methods, by the Finite Element calculations of Arns et al. (2002), and by Prokopiev & Sevostianov (2007).

6.5.3 Ceramic

Pore images from an SiC ceramic taken from Reynaud et al. (2005) are shown in Fig. 6.10. The porosity of this image is 36.8%. As this is the only image available, the pore geometry data obtained from it will be assumed to be representative of the ceramic over the entire range of porosities. The compressibilities of the seventeen largest pores in this image, as estimated by BEM calculations, and from the scaling law, eq. (2.40), are shown in Table 6.4 and Fig. 6.11. The pore shear compliances are shown in Table 6.5 and Fig. 6.12. For this material, both scaling laws work very well.

The elastic moduli of the mineral phase of this ceramic, as reported in Reynaud et al. (2004), are $G_o = 183.0$ GPa, $K_o = 256$ GPa, and $\nu_o = 0.17$. Using the same procedure as described above for the sandstones, the following equations are found for the effective moduli when the BEM is used to calculate the individual pore compliances:

$$K_{\text{eff}} \, (\text{GPa}) = 256(1 - \phi)^{2.53}, \quad (6.14)$$

$$G_{\text{eff}} \, (\text{GPa}) = 183(1 - \phi)^{2.52}, \quad (6.15)$$

and the following pair are obtained when the scaling laws are used to calculate the individual pore compliances:

$$K_{\text{eff}} \, (\text{GPa}) = 256(1 - \phi)^{2.71}, \quad (6.16)$$

$$G_{\text{eff}} \, (\text{GPa}) = 183(1 - \phi)^{2.48}. \quad (6.17)$$

Both pairs of equations fit the data very well, over the entire range of measured data.
Fig. 6.10. Image analysis procedure for ceramic, from Reynaud et al. (2005); see text for explanation of steps.
Table 6.4. Normalized pore compressibilities of the 17 largest pores from Fig. 6.10.

<table>
<thead>
<tr>
<th>Perimeter (μm)</th>
<th>Area (μm²)</th>
<th>(GC_{pc}/(1-\nu)) (BEM)</th>
<th>(GC_{pc}/(1-\nu)) eq. (2.40)</th>
<th>Error of eq. (2.40)</th>
</tr>
</thead>
<tbody>
<tr>
<td>186</td>
<td>2259</td>
<td>2.29</td>
<td>2.43</td>
<td>6%</td>
</tr>
<tr>
<td>111</td>
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<td>2%</td>
</tr>
<tr>
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<td>2.46</td>
<td>9%</td>
</tr>
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<td>218</td>
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<td>12%</td>
</tr>
<tr>
<td>83</td>
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<td>3.21</td>
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</tr>
<tr>
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<td>12%</td>
</tr>
<tr>
<td>121</td>
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<td>2.22</td>
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</tr>
<tr>
<td>108</td>
<td>859</td>
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<tr>
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</tr>
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<td>2.11</td>
<td>2.18</td>
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Fig. 6.11. Test of the compressibility scaling law (2.40) for the ceramic pores shown in Fig. 6.10. Values computed with BEM are plotted on the x-axis, predictions of the scaling law are plotted on the y-axis.
Table 6.5. Normalized pore shear shearabilities of the 17 largest pores from Fig. 6.10.

<table>
<thead>
<tr>
<th>Perimeter (μm)</th>
<th>Area (μm²)</th>
<th>( \frac{GS_{pc}/(1-\nu)}{(BEM)} )</th>
<th>( \frac{GS_{pc}/(1-\nu)}{eq. \ (4.52)} )</th>
<th>Error of eq. (4.52)</th>
</tr>
</thead>
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<td>2.25</td>
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<td>2.21</td>
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<tr>
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<td>2.16</td>
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<td>2.29</td>
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<td>413</td>
<td>2.25</td>
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</table>

Fig. 6.12. Test of the pore shearability scaling law (4.52) for the ceramic pores shown in Fig. 6.10. Values computed with BEM are plotted on the x-axis, predictions of the scaling law are plotted on the y-axis.
Fig. 6.13. Bulk modulus of ceramic, showing values measured by Reynaud et al. (2005), and values predicted using both methods of estimating the mean pore compressibility, starting with the image in Fig. 6.10.

Fig. 6.14. Shear modulus of ceramic, showing values reported by (Reynaud et al., 2005) and values predicted using both methods of estimating the mean pore compressibility, starting with the image in Fig. 6.10.
7 Summary, Conclusions and Further Work

7.1 Summary and conclusions

In this work, several closed-form solutions were derived for the problem of an isolated pore in an isotropic elastic medium. In Chapters 2 and 4, conformal mapping and the complex potential method was used to find exact solutions for a rotationally symmetric pore in a body that is subjected to either far-field hydrostatic loading or shear loading. The results were validated against some special cases that have previously been derived by Zimmerman (1986), Jasiuk et al. (1994) and Kachanov et al. (1994), and also against boundary element calculations. By extrapolation of the results for pores obtained from three and four terms of the Schwarz-Christoffel mapping function for regular polygons, the compressibility and shear compliance of a triangle, square, pentagon and hexagon were found to three digits of accuracy. Specific results for some other pore shapes, more general than the quasi-polygons obtained from the Schwarz-Christoffel mapping, were also presented.

An interesting consequence of these solutions was that both the compressibility and shear compliance of the pore depend on the elastic moduli of the matrix only through the combination \( \frac{G}{1 - \nu^2} \). Hence, the results can conveniently be discussed in terms of normalized compliances (normalized with respect to that of a circle) that depend only on the pore geometry.

Although the results for symmetric pores are of some intrinsic scientific interest, their most immediate use in the context of predicting the elastic moduli of real porous materials is that they provide exact results against which to test scaling laws. An approximate scaling law proposed by Zimmerman (1986) for the pore compressibility, in terms of the ratio of perimeter-squared to area, was therefore extensively tested against these analytical solutions. This expression was generally found to overestimate the compressibility, by an amount that varied from about 0-21%. A similar scaling law was then proposed for the shear compliance. Perhaps surprisingly, the scaling law for a pore under shear was more accurate than that for hydrostatic compression. With the exception of thin, crack-like pores, most pores seem to be no more than twice as compliant as a circular pore.

Rotationally symmetric pores with any symmetry other than 4-fold were explicitly found to be “isotropic” with regards to their orientation angle \( \theta \) with respect to the direction of shear, in agreement with the results of Eroshkin & Tsukrov (2005). It was also shown that pores with 4-fold symmetry have a shear compliance that varies with \( \cos 4\theta \), as do arbitrarily-shaped pores that possess no symmetry. Hence, the average shear compliance, which is appropriate for a collection of randomly oriented pores of a given shape, can be found from the average of the compliances calculated for two loading angles that differ by 45°.
In Chapters 3 and 5, the boundary perturbation method was used to study the effect of small-scale roughness on the bulk and shear compliance of a pore. The pore was taken to be nominally circular, with a small sinusoidal roughness superimposed, so that the pore boundary is defined by \( r = a(1 + \epsilon \sin m\theta) \). Actually, the problem was solved for a rigid inclusion, after which some results of Dundurs (1989) and Jasiuk (1995) were used to convert the results from applying to rigid inclusions, to being applicable to pores. The solutions were obtained to 4\(^{th}\)-order in \( \epsilon \) for hydrostatic loading, but only to 2\(^{nd}\)-order in \( \epsilon \) for shear loading. From these solutions, expressions were obtained for the effective bulk modulus and shear modulus of a body that contains a dilute distribution of these pores.

The pore compressibility and pore shear compliance of the corrugated circular pore exceeds that of a smooth circle by a term of order \( \epsilon^2 \). This is in contrast to the stress concentration factor, which differs from those of a circle by a term of order \( \epsilon \). The relative effect of small-scale roughness was found to be nearly the same for the pore compressibility as for the pore shear compliance. The main implication for the use of pore images to estimate the elastic moduli of real porous materials is that small-scale roughness can safely be neglected when estimating the areas and perimeters of the pores. It follows that very high magnification is not necessarily needed or desirable for this purpose. This is similar to what has been found when using image of pores to estimate the permeability of porous rocks, as small-scale roughness has little effect on the hydraulic resistance in the laminar range (Sisavath et al., 2001).

The analytical solutions and boundary element calculations of Chapters 2-5 established the approximate validity of the scaling laws for bulk and shear compliances of the pores. This gave some confidence that the compliance of the pore space of a real porous material could be estimated from knowledge of the areas and perimeters of the pores. In Chapter 6, two-dimensional electron micrographs of pores in Berea sandstone, Fontainebleau sandstone, and a silicon carbide ceramic, were analysed using commercial image analysis software to extract the required areas and perimeters. The pore compliances were calculated using boundary elements, and using the scaling laws, with generally good agreement. The area-weighted mean pore compressibility and pore shear compliance were then used, in conjunction with the differential effective medium theory, to predict the macroscopic bulk modulus and shear modulus. The resulting predictions were in all cases quite close to previously–reported values measured in the laboratory. The methodology used in this thesis therefore offers a promising and simple approach to relate the mechanical properties of porous geological or man-made materials to their pore structure, using only two-dimensional images.

7.2 Recommendations for future work

1. The analysis presented in this thesis assumed that the pores were filled with an “infinitely-compressible” fluid, such as would be appropriate for air. Although this assumption is relevant to ceramics, it is not appropriate to rocks, which are commonly filled either with liquid or with a gas at high pressure, which also has a finite compressibility. However, the Gassmann equation (Mavko et al., 1998) can be used to account for the presence of a pore fluid in an exact manner, so no further analysis is required in this regard.
2. The perturbation solution was used to derive approximate solutions for the effect of small-scale roughness superimposed on a circular pore. Such solutions could probably also be obtained for three-dimensional nominally spherical pores. In fact, the analyses in Chapters 3 and 5 show that the boundary-perturbation method can successfully be used for otherwise intractable problems, and it seems that this method probably has not been used to its full potential in the solid mechanics literature.

3. The differential effective medium theory was used to extend the results for isolated pores (i.e., low porosities) to the finite porosities that are relevant to sandstones and ceramics. The validity of this approximation could be examined by conducting boundary element calculations using, not isolated pores, but an entire collection of nearby pores, as observed in the images, with their actual relative locations. This type of test of the differential effective medium theory is feasible, and should be attempted, but would require a more robust BEM code and a more powerful computer than were available in this project.

4. Other important properties of rocks, such as permeability, also depend on the pore structure. Much more work has been done on predicting permeability than on predicting the elastic moduli, and it would be interesting to try to make some connections between the methods used in these two problems. For example, in permeability modeling, much effort is expended in trying to convert two-dimensional images into three-dimensional pores structures, whereas this aspect was totally ignored in the present thesis, which focused on predicting the elastic moduli. The extent to which the “three-dimensionality” of the pore space can indeed be ignored when modeling elastic properties remains to be fully seen.
References


Estimation of the Elastic Moduli of Porous Materials


Estimation of the Elastic Moduli of Porous Materials


Sevostianov, I., and Kachanov, M. Relations between compliances of inhomogeneities having the same shape but different elastic constants. *Int. J. Eng. Sci.* 2007;45(10):797-806.


