The Swap Market Model with Local Stochastic Volatility

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Abstract

Modeling volatility is an intricate part of all financial models and the pricing of derivative contracts. And while local volatility has gained popularity in equity and FX models, it remained neglected in interest rates models. In this thesis, using spot starting swaps, the goal is to build a swap market model with non-parametric local volatility functions and stochastic volatility scaling factors. The local stochastic volatility formula is calibrated through a particle algorithm to match the market’s swaption volatility smile. Numerical experiments are conducted for different currencies to compute the local stochastic volatility at different expiry dates, swap tenors and strike values. The results of the simulation show the high quality calibration of the algorithm and the efficiency of local stochastic volatility in interest rate smile building.
Sammanfattning

Acknowledgements

The writing of this thesis was possible through the advice and support of many people. First, I would like to thank my supervisor at the Royal Institute of Technology, Prof. Fredrik Viklund for his invaluable insight and his continuous guidance throughout the different stages of this thesis. I also would like to acknowledge my colleagues at Kidbrooke Advisory. In particular, my two supervisors, Edvard Sjögren and Ludvig Hällman for introducing me to the subject of the thesis, as well as their invaluable inputs, technical support and their insight into the financial industry. As well as Zaliia Gindullina, Mika Lindahl and Filip Mörk for their valuable advice, animated discussions and companionship.

Mohammed Benmakhlouf Andaloussi
Stockholm, April 2019
Nomenclature

Abbreviations

ARR  Alternate Reference Rate
ATM  At-The-Money option, i.e. the strike price is equal to the current spot price of the underlying security.
1 bp  1 Basis Point, equal to 0.01%
IBOR  Inter-bank Offered Rate (LIBOR, EURIBOR ...)
ICE  Inter Continental Exchange
IRS  Interest Rate Swap
ISDA  International Swaps and Derivatives Association
LIBOR  London Inter-bank Offered Rate
LMM  LIBOR Market Model
OIS  Overnight Index Swap
OTC  Over The Counter
RFR  Risk-Free Rate
SMM  Swap Market Model
ZCB  Zero Coupon Bond
xY  x Years (1Y, 2Y, ...)
xm  x months (3m, 6m, ...)

Mathematical notations

1_x  the dirac delta function at point x
δ_i  The time in year fractions between T_{i-1} and T_i, measured with an appropriate day count convention
P_i^t  The discount factor at time t with maturity T_i
A_{i,k}^t  The Annuity factor at time t over the interval [T_j, T_k]
S_{i,j}^{t,k}  The Swap rate at t starting at T_j and ending at T_k
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$B_t$</td>
<td>The value of the Bank account process at time $t$</td>
</tr>
<tr>
<td>$C_{i,i+\alpha_j^*}(T_i, K)$</td>
<td>Price (theoretical) of a payer swaption with strike $K$, on a spot starting swap with expiry date $T_i$ and tenor $\alpha_j^*$.</td>
</tr>
<tr>
<td>$C_{Mkt}^{i,i+\alpha_j^*}(T_i, K)$</td>
<td>Market price of a payer swaption with strike $K$, on a spot starting swap with expiry date $T_i$ and tenor $\alpha_j^*$.</td>
</tr>
<tr>
<td>$E^Q(...)$</td>
<td>The Expected value under the risk neutral measure $Q$</td>
</tr>
<tr>
<td>$E^T(...)$</td>
<td>The Expected value under the forward measure $Q^T$</td>
</tr>
<tr>
<td>$E^{A_{i,j}}(...)$</td>
<td>The Expected value under the measure associated with the annuity factor $A_{i,j}$</td>
</tr>
<tr>
<td>$F_t$</td>
<td>The Filtration at time $t$, which contains all market information up to time $t$</td>
</tr>
<tr>
<td>$\Pi(t, X)$</td>
<td>the value at $t$ of a contingent claim $X$</td>
</tr>
<tr>
<td>$(x)^+$</td>
<td>The positive part function of $x : (x)^+ = \max(x, 0)$</td>
</tr>
<tr>
<td>$g^{i,j}(K)$</td>
<td>The local volatility function at strike $K$</td>
</tr>
<tr>
<td>$c_t$</td>
<td>The stochastic volatility factor at time $t$</td>
</tr>
<tr>
<td>$n_{sim}$</td>
<td>The number of simulation paths in the Monte-Carlo method</td>
</tr>
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Chapter 1

Introduction

In any financial market, volatility is a central concept for all market participants. Yet, it is arguably still one of the most misunderstood concepts in investing. In short, volatility is a measure of the degree of variation in the returns of the traded price of a security or a market index, over a period of time. Generally, a high volatility is equivalent to a risky security, for which the traded price experiences rapid increases and decreases. Volatility is also an important component in pricing derivatives since it can be used to estimate the possible future fluctuations of the underlying asset over short period of time.

There exists multiple methods to measure or parametrize volatility. First is realized or historical volatility, based on the historical return series of the security, and computing the statistical standard deviation or variance. Next is the implied volatility, representing for an option contract, the estimated volatility of the underlying security which returns a theoretical value equal to the current market price of the option, when put into an option pricing model (such as Black-Scholes). Implied volatility is therefore a key feature of option pricing along with the Black-Scholes model as a link between the volatility and the price of the option.

The Black-Scholes model assumes that the volatility is constant. And using historical option prices, it is possible to solve for the implied volatility corresponding to the options prices. In practice however, options with the same underlying asset but with different strikes require different implied volatilities in order to match the market’s prices. But since the implied volatility should not depend on the strike price, the Black-Scholes model falls short. This
inconsistency regarding how the volatility depends on the strike (referred to as a volatility smile or skew), is an example of why it is necessary to explore other methods of defining volatility.

Local volatility models overcome this shortfall by treating volatility as a function of both the current security’s price level and the time $t$ [1]. Stochastic volatility models on the other hand, treats the volatility of the underlying security as a stochastic process [2]. Both models are generalisations of the Black-Scholes model for which volatility is constant, and are calibrated using the options’ market prices in order to fit and reproduce all market prices of options for different strikes and maturities.

In the interest rates markets, modeling volatility hold just as an important position. Aside from using volatility to price interest rates derivatives like swaptions for example, knowing the volatility of an interest rate or a swap rate is used by financial analyst to compute the corresponding forward rates.

In the case of this thesis, we will be interested specifically in swap rates modeling. Generally, in a swap market [3], a borrower with one type of loan exchanges it with another borrower with a different type of loan. The aim of each party is usually to take an advantage that the original loan did not have, such as the currency of the loan, the type of underlying interest rate (fix or float) or the maturity of the swap. Interest rates swaps are also used frequently to hedge against or speculate on changes in interest rates.

The choice of a swap market is not arbitrary since swap rates offer many mathematical properties and practical advantages, and are some of the most liquid benchmark financial products. A lot of interest rates modeling studies focus on the Libor rate instead because it has a more straightforward link to discount factors for example. But aside from that working in a Libor market framework offers no clear advantage over the swap market framework.

1.1 Background

The local volatility model developed by Dupire [1] is regularly used for FX and equity modeling. Yet, the local volatility remained fairly unused for interest rate smile modeling until recently. A recent study [4], attributes this to the fact that derivatives of swaption price with regards to expiry dates
and strikes cannot be inferred from market quoted prices: A swaption price for a underlying swap is available only when the expiry date is the swap fixing date. However, the Dupire formula to compute the local volatility function requires derivatives of an option price with regards to expiry dates and strikes.

Several new studies \[5\][6] utilize rolling maturity swaps in order to model interest rate smiles through local volatility. The difference with rolling maturity swaps is that the prices of swaptions on rolling maturity swaps are available in the market for different expiry dates and strikes. However, a rolling maturity swap process is not a price process of a tradable asset and since it refers to a different underlying swap for a different observation time. Therefore, the Dupire formula can not be applied in this case either but instead a new local volatility function must be computed (\[7\] shows an example of such function in the case of a Libor market model).

Working with local volatility to model the interest rate smile provides high quality calibration in a self-consistent and arbitrage-free framework. However, the dynamic behavior of smiles and skews predicted by local volatility models doesn’t always behave the same way as the market’s dynamic behavior [2]. To resolve this problem, a stochastic component can be added to the volatility model, in which the swap rate process and volatility are correlated. Alone, a stochastic volatility model has its own limitations. In particular, potential arbitrage for low strike options [8].

1.2 Thesis objective

Based on the paper: The swap market model with local stochastic volatility (2018) by Kenjiro Oya [9], the aim of the thesis is to build a swap market model (SMM) with local stochastic volatility.

Using spot starting swaps as the key modeling component of the SMM, the volatility parameterization is split into a non-parametric local volatility function and a stochastic volatility scaling factor.

The calibration of the model is based on market data in the form of implied normal swaption volatility. The calibration algorithm is based on the particle algorithm, and provides high quality calibration in an efficient manner.
Numerical experiments are conducted based on USD and AUD data. They are carried out over multiple expiry dates and for several tenors. The local stochastic volatility surfaces are computed over a range of strikes. Convergence tests and error computations prove the high accuracy of the model.

1.3 Limitations

In the scope of this thesis, we will be working on a swap market model built on spot starting swaps. The correlation structure used in the model is based on historical correlation and parametrized through a low-dimensional functional form (section 4.1). However, the market modeling and the algorithmic calibration can be done using assuming any general correlation structure.

As explained in section 3.1.4, the local stochastic volatility will be computed on certain key tenors only. This is due to the fact that calibrating the volatility smile is done using swaption market data, which is only available for certain tenors like a 5Y or 10Y tenor as opposed to a 7Y tenor. To remedy this interpolation and extrapolation methods are used on the tenors that are not considered key tenors. In this thesis, we will mainly work with linear interpolation but it would be interesting to work with more sophisticated methods that are the subject of recent research such as interpolation using parametrisation [10].

In section 2.4.2, we discuss the new changes in the financial markets related to computing the discount curve. The new market standard involves computing the discounting curve based on the overnight indexed rates rather than Libor or swap rates. The process is a complex one and can be based on different market instruments, bootstrapping methods and optimisation functions. In the scope of the thesis, we chose to compute the discount curve using swap rates of the model itself instead of the new discounting curve method.

1.4 Disposition

The remainder of the thesis is structured as follows. In Chapter 2, we look into the mathematical background of the thesis. We review some fundamentals of the interest rates theory, as well as the important findings regarding
interest rate swaps and the swap market model. Swaptions are also discussed in details through the contract specifications, the payoff and the pricing.

In chapter 3, the focus is on the theory of the swap market model with local stochastic volatility. We will present the problem setting chosen as a framework for the swap market model (SMM), parametrize the volatility component, as well as establish the formula for the local stochastic volatility. We will also look into pricing swaptions under the Black’s model using the market’s implied normal volatility. Finally, a link to the Libor rate is established to show how the thesis work can be translated into a Libor market model instead.

In chapter 4, the aim is to present the practical side in computing the forward dynamics of the rolling maturity swap of the SMM framework and calibrating the local stochastic volatility formula. The model’s correlation structure will be introduced, along with the model chosen for the stochastic volatility scaling factor. The SMM algorithm to calibrate the local stochastic volatility will also be presented.

In chapter 5, we deal with the results of different numerical experiments. Convergence tests will be conducted to insure the convergence of the SMM algorithm output. Then, simulations over two different currencies will be run to plot the local stochastic volatility surface at different expiry dates and for different tenors. The error will be computed to test out the quality of the calibration. Finally, in chapter 6, a summary of the findings of the thesis is presented along side potential leads for further work.
Chapter 2

Mathematical Background

In this chapter, the aim is to present fundamentals of the interest rates theory. These fundamentals are the foundation for the swap market model with local stochastic volatility.

First, the basic concepts for the different interest rates and IBOR rates are presented, and the concepts for discount factors and zero coupon bonds are explained. Second, the interest rate swaps are discussed along side interest rate derivatives, swaptions in particular. Finally, a detailed view of the latest advancements in the interest rates market and the effect of the current changes on the result of this thesis are discussed.

2.1 Interest rates, discount factors and ZCBs

An Interest rate $r(t)$ is the amount charged, expressed as a percentage of principal, by a lender to a borrower for the use of assets. Interest rates are usually noted on an annual basis, and are influenced by various factors: the currency of the principal, the term to maturity of the investment, the perceived default probability of the borrower, government policies to central banks regarding set financial goals like inflation level, etc.

Certain interest rates are considered benchmark rates and are used as basis for interest rates modeling and derivatives pricing. IBOR rates fall into that category. They represent interest rate at which banks lend to and borrow from one another in inter-bank market. IBORs serve as an indicator of levels of demand and supply in all financial markets. The last section of this
chapter offers a detailed discussion of IBOR rates and the current changes of the inter-bank market.

2.1.1 Discount factors, ZCBs

Next, a few key concepts associated with the interest rate market are introduced, the definitions mainly follow Bjork [11].

Consider one unit of currency invested in a bank account at time \( t = 0 \). The value of that unit at \( t \geq 0 \), \( B(t) \) is the price process of a risk-free asset and it has dynamics:

\[
 dB(t) = r(t)B(t)dt, \quad t \geq 0
\]  

(2.1.1)

With \( B(0) = 1 \). The differential equation above has the solution:

\[
 B(t) = B(0) \exp \left( \int_0^t r(s)ds \right) = \exp \left( \int_0^t r(s)ds \right).
\]  

(2.1.2)

An important concept in all financial modeling, is the present value of money. To express this, a discount factor is defined as the value of an asset at time \( T \) brought to its value today \( t \). The notation used here is \( D(t, T) \), which represents the discount factor at time \( t \) with maturity date \( T \). Using the previous notation:

\[
 D(t, T) = \frac{B(t)}{B(T)} = \exp \left( - \int_t^T r(s)ds \right).
\]  

(2.1.3)

In the same sense, another important concept in financial modeling is introduced: The zero coupon bonds (ZCB). A ZCB of a maturity \( T \), is a contract which guarantees the holder 1 currency unit to be paid on the date \( T \). The price at time \( t \) of such a bond is denoted by \( p(t, T) \). The payment at the maturity \( T \) is deterministic and always equals to 1. This translates to \( p(T, T) = 1 \). Under the neutral risk measure \( Q \) (an equivalent measure to the real world probability measure \( P : P \sim Q \)), ZCBs are defined the following expression:

\[
 p(t, T) = E^Q \left[ \exp(- \int_t^T r(s)ds) \right]_{F_t} = E^Q [D(t, T) \right \ F_t] \]  

(2.1.4)

with \( F_t \) the filtration containing all market information at the date \( t \).
2.1.2 Forward rates

For \( t \leq S \leq T \), the forward rate for \([S, T]\) contracted at \( t \), is defined as the interest rate applicable to a financial transaction at time \( S \) over the interval \([S, T]\). Such rate is defined as the solution of the equation (Bjork 2009 [11]):

\[
1 + (T - S)L(t; S, T) = \frac{p(t, S)}{p(t, T)} \tag{2.1.5}
\]

\( L \) is called the simple forward rate or the Libor rate at \( t \) over the interval \([S, T]\). And using the above formula, the Libor rate can be expressed as follow:

\[
L(t; S, T) = \frac{1}{T - S} \left( \frac{p(t, S)}{p(t, T)} - 1 \right) \tag{2.1.6}
\]

The forward rate can also be defined as a continuous rate, by solving the equation:

\[
\exp \left( R(t; S, T)(T - S) \right) = \frac{p(t, S)}{p(t, T)} \tag{2.1.7}
\]

\( R \) is called the compounded forward rate at \( t \) over the interval \([S, T]\).

Several important rates can be defined from the Libor rate formula above. 
\( L(S, T) \) : the simple spot rate over \([S, T]\), using the limit \((t \to S)\). 
\( f(t, T) \) : the instantaneous forward rate with maturity \( T \), using the limit \( S \to T \). And \( r(t) \) : the short rate at time \( t \), using \( r(t) = f(t, t) \).

This results in another way to define ZCBs at \( t \) with maturity \( T \):

\[
p(t, T) = \exp \left( - \int_t^T f(t, u)du \right) = \exp \left( -y(t, T)(T - t) \right) \tag{2.1.8}
\]

With \( y(t, T) = \frac{1}{T - t} \int_t^T f(t, u)du \), the yield to maturity \( T \).

2.1.3 The forward measure

When pricing interest rates derivatives, it is often more practical to use the forward neutral measure \( Q^T \) instead of the common neutral measure \( Q \) (Bjork 2009). \( Q^T \) is a probability measure equivalent to \( Q : Q^T \sim Q \).
For a contingent claim $X$, with expiry $T$, the value of $X$ at time $t$, $\Pi(t, X)$, under the neutral risk measure $Q$, the pricing formula is defined as:

$$\frac{\Pi(t, X)}{B_t} = E^Q \left[ \frac{\Pi(T, X)}{B_T} \mid F_t \right] \quad (2.1.9)$$

Using the fact that $\Pi(T, X) = X$ and the definition of discounts factors gives us:

$$\Pi(t, X) = E^Q [XD(t, T) \mid F_t]. \quad (2.1.10)$$

The expectation in formula (2.1.9) is under the measure $Q$, because the numeraire used is that or the risk free asset $t \to B_t = B(t)$. Changing the numeraire therefore results in a change of the expectation in the pricing formula (cf Appendix A).

Under the forward measure $Q^T$, the numeraire is $t \to p(t, T)$ the price of ZCB with maturity $T$:

$$\frac{\Pi(t, X)}{p(t, T)} = E^T \left[ \frac{\Pi(T, X)}{p(T, T)} \mid F_t \right] = E^T[X \mid F_t], \quad (2.1.11)$$

($p(t, T)$ is known at time $t$). This gives us the relation:

$$\Pi(t, X) = E^Q[XD(t, T) \mid F_t] = p(t, T)E^T[X \mid F_t] \quad (2.1.12)$$

In Bjork (2009) chapter 26 [11], the details of change of numeraire and relations between the different resulting measures and pricing formulas are presented.

### 2.2 Interest Rate Swaps

Interest Rate Swaps (IRS) are the most basic interest rates derivatives. In an IRS, a series of payments at a fixed predetermined rate of interest (called the swap rate), are exchanged for a series of payments at a floating rate (typically a Libor rate), for a fixed tenor. The value of an IRS is then the difference between the value of the floating leg (Fl) and the value of the fixed leg (Fi). By convention, a long IRS contract corresponds to paying the fixed leg and receiving a floating leg. The value of an IRS at a day $t$ is therefore:

$$\Pi_{IRS}(t) = \Pi_{Fl}(t) - \Pi_{Fi}(t). \quad (2.2.1)$$
In the rest of this thesis, the notation used is $S_t^{j,k}$ which refers to a swap rate starting at $T_j$ and with a tenor $T_k - T_j$ (end date of the swap is $T_k$). Payments are done at predetermined times $T_{j+1}, T_{j+2}, \ldots, T_k$, which are usually equally spaced $\delta_i = T_i - T_{i-1} = \delta$, $\forall i$. The floating rate is usually a Libor spot rate with tenor $\delta$.

The most common IRS is the forward swap settled in arrears: For a principal $K$ and a swap rate $R$ (the fixed rate), at a time $T_i$ ($T_{j+1} \leq T_i \leq T_k$), the holder of the contract receives:

$$K \delta_i L(T_{i-1}, T_i),$$

$L(T_{i-1}, T_i)$ is known at $T_{i-1}$, and receives:

$$K \delta_i R.$$

Which results in a net cash flow at $T_i$ equal to:

$$K \delta_i [L(T_{i-1}, T_i) - R] \quad (2.2.2)$$

To get the value of the IRS at time $t < T_j$, all the future cash flows at payment times need to be discounted to their value at $t$.

$$\Pi_{Fl}(t) = \sum_{i=j+1}^{k} K \delta_i L(T_{i-1}, T_i)p(t, T_i)$$

$$= K \sum_{i=j+1}^{k} (1 + \delta_i L(T_{i-1}, T_i))p(t, T_i) - p(t, T_i)$$

$$= K \sum_{i=j+1}^{k} \frac{1}{p(T_{i-1}, T_i)}p(t, T_i) - p(t, T_i)$$

$$= K \sum_{i=j+1}^{k} p(t, T_{i-1}) - p(t, T_i) \quad \text{(a telescopic sum)}$$

$$= K(p(t, T_j) - p(t, T_k))$$

And:

$$\Pi_{Fi}(t) = \sum_{i=j+1}^{k} K \delta_i Rp(t, T_i)$$

$$= KR \sum_{i=j+1}^{k} \delta_i p(t, T_i)$$
The value of an IRS at time $t$ is then:

$$\Pi_{IRS}(t) = K(p(t, T_j) - p(t, T_k)) - KR \sum_{i=j+1}^{k} \delta_i p(t, T_i). \tag{2.2.3}$$

For a contract written today $t$, the value of the IRS should be 0, this gives the following swap rate formula:

$$S_{t}^{j,k} = R = \frac{p(t, T_j) - p(t, T_k)}{\sum_{i=j+1}^{k} \delta_i p(t, T_i)}. \tag{2.2.4}$$

From here forward, the notation used will be:

$$p(t, T_i) := P_i.$$

And we also introduce the Annuity factor at time $t$ over the interval $[T_j, T_k]$, defined as:

$$A_{t}^{j,k} = \sum_{i=j+1}^{k} \delta_i p(t, T_i) = \sum_{i=j+1}^{k} \delta_i P_i \tag{2.2.5}$$

Finally, the swap rate $S_t^{j,k}$ is:

$$S_t^{j,k} = \frac{P_t^j - P_t^k}{A_t^{j,k}} \tag{2.2.6}$$

An important example of very common IRS are the Overnight Index Swaps (OIS). In an OIS, the agreement is to exchange a fixed rate against a pre-determined published index of a daily overnight reference rate such as the overnight LIBOR rate or the SONIA rate (GBP) over an agreed period (tenor). OIS are special cases of interest rate swaps so valuating them is done in a very similar way. The OIS market has grown significantly in importance during few years, specifically the LIBOR-OIS spread, which is now considered a reference indicator of the health of the global credit markets, figure (2.1).
2.3 Swaptions

2.3.1 Definition, characteristics and contract terms

A swaption or a swap option, is an interest rate option where the underlying asset is a swap rate. A swaption give the holder the right but not the obligation to enter at a future date known as the expiry date, into an interest rate swap of a pre-specified tenor. In a payer swaption, the owner of the swaption has the right to enter, at the expiration date, into a swap where they pay the fixed leg and receive the floating leg. In a receiver swaption, if the owner of the swaption chooses to enter into a swap, they will receive the fixed leg, and pay the floating leg. The swaption market is mostly comprised of large corporations, banks and hedge funds. These institutions are in need of managing their interest rate risk and do so through the trading of swaptions.
Swaptions, unlike equity options and future contracts, are Over-the-Counter financial securities, meaning they are not standardized and are traded directly between two parties, without the supervision of an exchange. Thus, the buyer and seller have the liberty to agree on the price of the swaption, its expiration date, the notional amount, tenor and the fixed and floating rates. Expiration dates can span anywhere from 3 months to 20 years and more. By convention, upon execution of the swaption, the IRS starts two business days after the expiration date. The tenor of IRS also depends on the contract and can go anywhere from 1Y to 30Y.

The buyer and seller also agree on the fixed rate (which is the strike of the swaption) and the payment frequency for the fixed leg. The frequency of floating rate payments is also up to the contracting parties to decide. Along with the floating rate choice. For example, in the big majority of USD swaptions the floating rate in equal to the 3 month USD-Libor and is paid quarterly, no matter the tenor of the swap. While for GBP swaptions, it’s more common to choose the 3m Libor paid quarterly as the floating rate for underlying IRS with a maturity under 2 years, and choose the 6m Libor paid semi-annually for the underlying IRS with maturities above 2Y.
The delivery of the swaption contract is also an open choice between the contractors. A physical delivery swaption is such that an actual interest rate swap is entered into if the option is exercised. This type of swaptions follow the original definition of a swaption and is still the most popular contract especially in European markets. A cash settled swaption is a derivative contract that is settled by paying a cash amount computed based on the equivalent value of the IRS future cash flows if the option is exercised. Cash settled swaption are more popular in American markets and the valuation is done by different methods such as the Internal Rate of Return approach IRR or the Cash Collateralized Price approach CCP (Supplement number 58 to the 2006 ISDA Definitions [14]).

Beyond all the choices of contract terms, there exists 3 styles of swaptions. European swaptions which can only be exercised on the expiration date. Bermudan swaptions, in which the holder can choose to exercise the option on any one of a number of predetermined dates. And American swaptions, where purchaser can exercise the option and enter into the swap on any day between the contract signing and the expiration date.

### 2.3.2 Payoff and pricing

For a payer swaption with strike $K$, expiration date $T_i$ and an underlying swap with tenor $T_j$, the payoff is:

$$X^{i,j} = A^{i,j}_{T_i} \max \{S^{i,j}_{T_i} - K, 0\}$$

$$= A^{i,j}_{T_i} (S^{i,j}_{T_i} - K)^+.$$  \hspace{1cm} (2.3.1)

With $A^{i,j}_{T_i}$ being the accrual factor between $T_i$ and $T_j$, and $S^{i,j}_{T_i}$ is the underlying swap rate at the expiry date $T_i$.

$X^{i,j}$ is a contingent $T_i$-claim. The payoff is the same as a call option on $S^{i,j}_{T_i}$ with strike $K$. The arbitrage free pricing formula under the neutral measure $Q$ of the claim $X^{i,j}$, is:

$$\frac{\Pi(t, X^{i,j})}{B_t} = \mathbb{E}^Q \left[ \frac{\Pi(T_i, X^{i,j})}{B_{T_i}} \right| F_t], \quad t < T_i$$

$$= \mathbb{E}^Q \left[ A^{i,j}_{T_i} (S^{i,j}_{T_i} - K)^+ \right| F_t]. $$ \hspace{1cm} (2.3.2)
This formula is quite difficult to implement or compute, that’s why for swaptions, the pricing is usually done under a different probability measure.

Consider the probability measure with numeraire \( t \to A^{i,j}_t \). The expectation under this new measure is expressed as: \( E^{A^{i,j}}(\ldots) \). Under the no arbitrage assumption, the change of numeraire under the Guirsanov theorem applies and the equivalent pricing formula is written:

\[
\Pi(t, X^{i,j}) = E^{A^{i,j}} \left[ \frac{\Pi(T_i, X^{i,j})}{A^{i,j}_T} \right] f_t, \quad t < T_i
\]

\[
= E^{A^{i,j}} \left[ \frac{A^{i,j}_T (S^{i,j}_{T_i} - K)^+}{A^{i,j}_T} \right] f_t
\]

\[
= E^{A^{i,j}} \left[ (S^{i,j}_{T_i} - K)^+ f_t \right].
\]

Which can also be written as:

\[
\Pi(t, X^{i,j}) = A^{i,j}_t E^{A^{i,j}} \left[ (S^{i,j}_{T_i} - K)^+ f_t \right], \quad t < T_i.
\]

(2.3.4)

Once the distribution of \( t \to A^{i,j}_t (\ldots) \) is determined, computing the expectation in the pricing formula is a simple matter of integration.

The market practice is to price the swaptions under the Black’s model by using a similar formula to the Black-76 formula used for pricing options in the equity market (Bjork 2009 [11]):

\[
P_{\text{Payer, LogNormal}}(t) = \Pi(t, X^{i,j}) = A^{i,j}_t [S^{i,j}_t \Phi(d_1) + K \Phi(d_2)],
\]

(2.3.5)

Such that:

\[
\begin{align*}
    d_1 &= \log \left( \frac{S^{i,j}_t}{K} \right) + \frac{1}{2} \sigma_{LN}^2 (T_i - t) / \sigma_{LN} \sqrt{T_i - t} \\
    d_2 &= d_1 - \sigma_{LN} \sqrt{T_i - t}
\end{align*}
\]

\( \sigma_{LN} = \sigma_{LogNormal}^{i,j} \) is the log normal implied Black volatility of a swaption with expiry \( T_i \) and underlying swap with tenor \( T_j \). \( \Phi \) is the cumulative distribution function of a standard normal distribution \( N(0,1) \). In this thesis, normal volatility will be used instead of the usual log normal volatility. The Black’s model formula for pricing a payer swaption under the normal volatility model will be discussed in more detail in the next chapter of the thesis.
In the Black-76 formula, there is a direct relationship between the price of a swaption and the implied volatility (normal or log normal). Therefore, swaption prices are usually quoted in term of the implied volatility, which yields the option’s price when used in Black’s model.

From the at-the-money volatility data found on the CME group website for USD swaptions on 19/11/2017, we are interested in the expiration dates $T_i$: 1m, 3m, 6m, 1Y and 2Y. Next is the tenors $\alpha$: 1Y, 2Y, 5Y, 10Y, 15Y, 20Y and 30Y. The normal implied volatility $\sigma_{i,\alpha}^N$, the log normal implied volatility $\sigma_{i,\alpha}^{LN}$ and the option price $\Pi(0, S_{i,i+\alpha})$.

Figure 2.3: USD ATM Swaption prices
Figure 2.3 shows the market price of USD swaptions according to the CME group. The shape of the surface is such that the price of a swaption increases when the expiration date or the tenor grows larger. This is to be expected since the further the payoff date is, the more uncertainties the buyer faces and thus the higher the price should be.

Figures (2.4) and (2.5) show respectively the implied log normal volatility
and the implied normal volatility for the same USD swaption data. Both surfaces are very similar, and it can easily be checked that the swap rate approximation that links the normal volatility to the log normal volatility through the appropriate swap rate is:

\[ \sigma_{LN}^{i,\alpha} \approx \frac{\sigma_N^{i,\alpha}}{S_{i,i+\alpha}} \]  

(2.3.6)

Given that far forward rates are generally less volatile than near rates, and that long rates are also less volatile than short rates, it makes sense to expect the swaption volatility to decline with both increasing option maturity (Expiration date) and increasing swap maturity (tenor). The shape of both implied volatility figures proves this point.

### 2.3.3 Bermudan swaptions

Bermudan swaptions are a type of swaption contracts that gives the holder the right, but not the obligation, to enter into an interest rate swap on one of several predetermined dates. These derivative contracts differ from European swaptions which can only be exercised on the expiry date. Pricing Bermudan swaptions is more complex since the addition of more potential exercise dates, complicates the calculations.

While European swaptions can be priced directly using the formula under Black’s model, Bermudan swaptions are priced using methods based on simulations such as the Monte-Carlo simulation. Different research papers offer a few methods to price Bermudan (and American) swaptions such as the Mesh and tree methods or the Regression-Based methods [13].

Consider a Bermudan swaption with expiry date \( T_i \), tenor \( \alpha \) and strike \( K \). Let \( \Delta \) a set of early exercise dates : \( \tau \in [0, T_i] \).

If the swaption is not exercised early, the price at \( t \) is:

\[ \Pi(t, X) = A_t^{i,i+\alpha} E^{A^{i,i+\alpha}} [(S_{T_i}^{k,i+\alpha} - K)^+ \setminus F_i], \quad t < T_i. \]  

(2.3.7)

But if the option is exercised early, the value would be:

\[ \Pi(t, X) = A_t^{\tau,\tau+\alpha} E^{A^{\tau,\tau+\alpha}} [(S_{\tau}^{\tau,\tau+\alpha} - K)^+ \setminus F_t], \quad t < \tau. \]  

(2.3.8)

Therefore, the pricing problem at time \( t = 0 \), becomes the calculation of:

\[ \Pi(0, X) = \sup_{\tau \in \Delta} [A_0^{\tau,\tau+\alpha} E^{A^{\tau,\tau+\alpha}} [(S_{\tau}^{\tau,\tau+\alpha} - K)^+ \setminus F_0]]. \]  

(2.3.9)
2.4 Discussion: Current changes in the interest rates market

2.4.1 End of the IBOR era

IBORs, the Inter Bank Offered Rates have been for over 40 years the go to reference rates for variable-rate financial instruments used by the financial services industry.

By definition, an IBOR is the interest rate that banks in a jurisdiction charge one another for short-term, inter bank loans. The major inter bank offered rates, notably LIBOR (London Inter Bank Offered Rate) and USD Libor, are used as reference rates for Sterling and US dollar-denominated forward rate agreements, short-term interest rate futures contracts and interest rate swaps (where the inter bank offered rate is used as the reference rate for the floating payer).

Since 2012, Libor has been subject to many controversies and gained negative public attention when accusations of manipulating IBOR submissions were made against several global banks during the financial crisis [16]. In fact, since the Libor is an average interest rate calculated through submissions of interest rates by major banks across the world, the scandal arose when it was discovered that banks were falsely inflating or deflating their rates so as to profit from trades, or to give the impression that they were more creditworthy than they were. Libor underpins approximately $ 350 trillion in derivatives, so the manipulation of submissions used to calculate it can have significant negative effects on consumers and financial markets worldwide.

Since then, global regulators have taken several steps to strengthen the IBOR, including appointment of a new benchmark administrator, ICE Benchmark Administration. However, IBORs are no longer deemed to be a desirable benchmark due to the recent liquidity decline in the unsecured inter-bank lending market, which is the basis for the Libor.

The reform efforts have then been focused on developing overnight risk-free rates (RFRs) that are based on durable, liquid, underlying markets that conform to the International Organization of Securities Commissions (IOSCO) Principles for Financial Benchmarks. These overnight RFRs are based on
active, liquid underlying markets and that makes them much more robust and dynamic. By 2021, inter bank rates are expected to be replaced by the new RFRs as the global interest rate benchmarks (ISDA 2018 - Consultation on Term Fixings and Spread Adjustment Methodologies [14] & [15]).

<table>
<thead>
<tr>
<th>Jurisdiction</th>
<th>Current Benchmark</th>
<th>Alternative RFR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Europe</td>
<td>EURIBOR, EONIA</td>
<td>Euro Short-Term Rate (ESTER)</td>
</tr>
<tr>
<td>Japan</td>
<td>TIBOR, JPY LIBOR</td>
<td>Tokyo OverNight Average rate (TONA)</td>
</tr>
<tr>
<td>Switzerland</td>
<td>CHF LIBOR</td>
<td>Swiss Average Rate OverNight (SARON)</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>GBP LIBOR</td>
<td>reformed Sterling OverNight Index Average (SONIA)</td>
</tr>
<tr>
<td>United States</td>
<td>USD LIBOR</td>
<td>Secured Overnight Financing Rate (SOFR)</td>
</tr>
<tr>
<td>Sweden</td>
<td>STIBOR</td>
<td>Recommendation for complement/alternative to STIBOR in H2 2019</td>
</tr>
</tbody>
</table>

Table 2.1: New recommended RFRs for major world jurisdictions

Some of the major world jurisdictions have established ARR Working Groups (WGs) to conduct reviews and identify alternative RFRs. Table (2.1) presents some the recommended alternative RFRs [17]. These RFRs are all overnight and transaction based. Some of them are secured (collaterised and/or based on rates from the repo markets) such as the US SOFR, and others are unsecured such as the UK SONIA.

Significant progress has been made in developing market liquidity for the new RFRs in derivatives markets; for example, SOFR and SONIA futures have been launched and the volume of contracts traded continues to increase. Another significant milestone in the adoption of the new RFRs, was the launch of overnight index swap (OIS) on the SOFR rate by CME in the second half of 2018 (Floating index : USD-SOFR-COMPOUND, with a maturity up to 30 years).

The transition from an IBOR to the corresponding RFR is expected to be a significant transformation effort for financial services firms and market participants that have extensive exposure to IBOR-linked products and contracts. The transition will therefore will bring about a different number of challenges that banking and capital markets organizations and other financial market participants will face.
In particular, for a Swap Market Model, IBORs are used as the reference rates for interest rate swaps and are published for various tenors. For example, in a "plain vanilla" interest swap, the floating rate payment is that of the 3-months USD Libor term rate (known 3 months in advance). Since the new RFRs are produced at an overnight maturity only, the way interest rate swaps are defined no longer holds.

- One way of doing things is to compute term rates for the RFRs at the end of a period or term based on observed rates during the period (‘backward-looking’). This can be done by compounding the actual overnight rate over the length of the period. This "Compounded Setting in Arrears Rate" can translated to the following formula [14], for a RFR at \( t \) over the term \( f \) and between the period \([T, T + f]\):

\[
ARR_f(t) = \frac{1}{\delta_f} \left( \prod_{u=T}^{T+f-1bd} (1 + \delta_u RFR_u) - 1 \right)
\] (2.4.1)

with \( \delta_f \) the cash day count fraction for the accrual period, and \( \delta_u \) the cash day count fraction for the overnight accrual period from \( u \) to \( u + 1bd \) (1 business day).

This approach mirrors the structure of OISs referencing the RFRs which is considered to be a good indicator of the interbank credit markets and less risky than other traditional interest rate spreads. but the information needed to determine the rate is not available at the start of the relevant period.

- A second approach would be to do the same compounding technique but over a period prior to the start of the relevant IBOR tenor and with equal length of tenor. With this approach the rate us available at the beginning of the relevant IBOR tenor and it reflects actual daily interest rate movements over a comparable tenor during a period near the relevant period, but it’s inherently backward-looking and market conditions may have changed since the relevant historical period, which could lead to differences from the current market term structure and may affect hedging [14].

- Some of the working groups on RFRs [15] have also been considering the development of new RFR derived term rates that could be based on transactions or executable quotes. Such RFR-derived term rates would measure a forward expectation of overnight RFRs over a designated period
or term. Robustness of these RFRs term rates will depend on derivatives market liquidity which is not as deep or continuous as in overnight funding markets. Therefore, RFR-derived term rates cannot equal the robustness of the overnight RFRs. So even though in some cases there may be a role for RFR term rates, it will be important that the transition away from IBORs is to the new overnight RFRs rather than the term rates in order to avoid the same weaknesses of the current IBORs.

2.4.2 OIS vs LIBOR discounting

Another big change that has been happening in the financial markets the past few years is the move from using Libor and Libor-swap rates as proxies for risk-free rates in derivatives valuation, to using overnight indexed swap (OIS) as the risk-free rate when collateralized portfolios are valued while leaving the LIBOR usage for portfolios that are not collateralized. Prior to the 2007 financial crisis, market participants used Libor, as a proxy for the risk-free rate. Although Libor is used a the short-term borrowing rate of AA-rated financial institutions, it still is not risk-free. Most notably, in stressed market conditions as the 2007 financial crisis, the spread between three month US Libor and three month US treasury rate-increased dramatically (reaching over 450 basis points (4.5%) in October 2008). Moreover, the 2007 financial crisis triggered high basis spreads for swaps characterized by different underlying rate tenors (floating leg based on 3 months Libor vs 6-months Libor). As a result, the Libor curve couldn’t be regarded as risk free anymore.

The main attraction of using Libor as the risk-free rate was that the valuation of derivatives was straightforward because the reference interest rate was the same as the discount rate. For example, in the current thesis, the swap rate is the basis used to compute forward swap rates, but it’s also used in the recursive formula (3.3.1) to get discount rates, bank process and annuity factors.

Most derivatives dealers now use interest rates based on overnight indexed swap rates rather than Libor when valuing collateralized derivatives. The OIS rate was chosen as the new standard for the discount rate because it’s derived from the fed funds rate which is the interest rate usually paid on
collateral. As such the fed funds rate and OIS rate are the relevant funding rates for collateralized transactions.

There are different ways to build an OIS curve using different market instruments, bootstrapping methods and optimisation functions. The most liquid for building an OIS curve are Fed Fund Futures and OIS swaps that pay at the daily compounded Fed Fund rate. The Fed Fund Futures are currently only liquid up to two years and OIS swaps up to ten years. Therefore, beyond 10 years the most liquid instruments are Fed Fund versus 3M Libor basis swaps, which are liquid up to thirty years.

Working with multiple financial instruments gives rise to the following issue: to price the basis swaps one needs both the OIS curve, to project the Fed Fund rate, and the LIBOR curve, to project the Libor rate. In the past one could have generated the LIBOR curve separately, by using the single curve for both forward projection and discounting. However with the new convention, Libor swaps are quoted using OIS discounting. This means that in order to generate a forward LIBOR curve from Libor swap quotes one must first have the OIS curve already constructed so that one knows how to discount the cash flows. So neither the OIS curve nor the Libor curve can be built without the other. The two curves must be generated simultaneously.

This method proceeds as follows [18]:

- From the underlying instruments, determine which define a point on the OIS curve and which define a point on the LIBOR curve.
- All missing values have to be interpolated using an interpolation method on rates.
- Create these two sets of unknown curve points and make an initial guess for their values.
- Price all of the given instruments using the initial guess of the two curves
- Compare the prices with the market quotes and adjust the initial guess accordingly.
- Repeat the pricing and adjustment until the error reaches acceptable levels.
There are several research reports about the construction of the OIS curve through different bootstrapping and optimisation methods. The algorithms to generate these types of curve are very complex and are the subject of full master theses. So in this paper, the difference between the OIS risk free curve and the LIBOR curve is ignored with the later curve used for discounting. This isn’t a big approximation since the spread between risk free curve and LIBOR curve is only significant when some big event shakes up the financial markets like the case of the financial crisis. Besides, in local volatility formula, the discounting only appears in the swaption price term. This can be easily changed if the OIS curve is available and therefore match the market standards of pricing.

Figure 2.6: Example of a Swap Spread for USD on 19/11/2018

Figure (2.6) shows the swap spread (difference between the swap rate and a corresponding government bond yield with the same maturity) for USD on 19/11/2019 (data source ”theice.com”). The values range around 10 – 20 bps for the different maturities, which shows why the approximation taken in this thesis is justified.
Chapter 3

The Swap Market Model Theory

In this chapter, the focus is on presenting the theoretical background and problem setting chosen as a framework for the swap market model (SMM) as well as establishing the formula for the local volatility function and its calibration algorithm.

First, the problem setting for the swap market model (SMM) is introduced along side the dynamics of its instruments. Second, the computations to find the local stochastic volatility formula are presented. And finally, the algorithm to calibrate the volatility surfaces to the market’s volatility smiles.

3.1 The SMM with local stochastic volatility

The swap market model is an arbitrage free financial model that uses forward swap rates as its main modeling component. Similar to the more widely used Libor market model, the SMM offers many mathematical and practical advantages, such as intuitive key modeling component, a liquid market for its derivative contracts and a flexible volatility structure.

Instead of the general SMM, this thesis will focus on the spot SMM for which the rolling maturity swap is the main modeling component.
3.1.1 The SMM setting

In the complete stochastic basis \((\Omega, F, \mathbb{F}, P)\), where the filtration \(\mathbb{F}\) on \(F\) satisfies the usual conditions of right-continuity and completeness, and \(P\) denotes the physical measure of the market (Huang and Scaillet [12]), consider the following set up:

A discrete time grid \(\{T_i\}_{i \geq 0}\), and the corresponding accrual factors \(\{\delta_k\}_{k=1,2,...}\) such that:

\[
\begin{aligned}
T_i &= \sum_{k=1}^{i} \delta_k, \quad i = 1, 2, ... \\
T_0 &= 0
\end{aligned}
\]  

(3.1.1)

Thus, the accrual factor, which is equal to the time in year fractions measured with an appropriate day count convention satisfies the formula:

\[\delta_k = T_k - T_{k-1}, \quad k = 1, 2, ...\]

From the previous chapter, consider the zero-coupon bond at time \(t\) and maturity \(T_i\):

\[p(t, T_i) = P^i_t\]

The Annuity factor at time \(t\) over the interval \([T_j, T_k]\):

\[A_{j,k}^{t} = \sum_{l=j+1}^{k} \delta_l P(t, T_l) = \sum_{l=j+1}^{k} \delta_l P^l_t\]

(3.1.2)

And the swap rate at \(t\) starting at \(T_j\) and with a tenor \(T_k - T_j\):

\[S_{j,k}^{t} = \frac{P^j_t - P^k_t}{A_{j,k}^{t}}\]

(3.1.3)

Working under the risk neutral probability measure \(Q\), means using the Bank account process \(t \rightarrow B(t) = B_t\) as numeraire. In the previous chapter, the system of equations (2.1.1) defines the dynamics of the process \(B_t\) in its most used form. But since the work here is within the framework of an SMM, the process \(B_t\) can be defined in an equivalent form more convenient to working with forward swap rates:

\[
\begin{aligned}
B_t &= \left(\prod_{k=0}^{i-1} P^{k+1}_t\right)^{-1} P^i_t, \quad T_{i-1} \leq t < T_i \\
B_0 &= 1
\end{aligned}
\]

(3.1.4)
3.1.2 Dynamics of a rolling maturity swap

A rolling maturity swap is the price process of a spot starting swap with a fixed tenor. For a fixing date \( t \), a starting date \( T_i \) and a tenor period \( \alpha \), the spot starting swap is expressed as \( S_i^{i,i+\alpha} \). The IRS is entered into at the date \( T_i \), with no waiting period, making \( T_{i+1} \) the first payment date and \( T_{i+\alpha} \) the last payment date.

Under \( Q \), the dynamics of a rolling maturity swap rate process starting at \( T_i \) for \( T_{i-1} \leq t < T_i \) and with a fixed tenor \( \alpha \), are written:

\[
dS_i^{i,i+\alpha} = \mu_i^{i,\alpha} dt + \sigma_i^{i,\alpha} dW_{t}^{\alpha,Q}, \quad T_{i-1} \leq t < T_i, \quad \alpha = 1, 2, ...
\]  

(3.1.5)

The term \( \mu_i^{i,\alpha} \) is the stochastic drift term of the process \( S_i^{i,i+\alpha} \) and \( \sigma_i^{i,\alpha} \) the diffusion term. The term \( dW_{t}^{\alpha,Q} \) is a Brownian motion defined under \( Q \). The correlation factor between two Brownian motions is defined as the Swap-Swap rates correlation:

\[
<dW_{t}^{\alpha,Q},dW_{t}^{\beta,Q}>_t = \rho_t^{\alpha,\beta} dt
\]  

(3.1.6)

The dynamics chosen for the forward swap rate here are characteristics of the normal model. The log normal model is widely used for equity and interest rates modeling alike. However, in the current financial markets, interest rates values are close to 0% or even negatives in some currencies. The log normal model alone isn’t enough to predict the future dynamics of such Libor or swap rates. The market practice is thus to either apply a ”shift” [19], to elevates the rate values away from 0% or to use the normal dynamics which have a non zero probability for the forward swap rate to be negative. The latter is the main reason why the chosen dynamics for the SMM in this thesis, are that of a normal model (figure 3.1).
3.1.3 The drift term

**Result 1**: The drift term \( \mu_{i,\alpha}^t \), for \( t \geq 0 \):

For an expiry date \( T_i \) and a fixed tenor \( \alpha \), the drift term in the dynamics of \( S_{i,i+\alpha}^t \) at time \( t \), is defined by the formula:

\[
\mu_{i,\alpha}^t = \sum_{j=i+1}^{i+\alpha} \frac{\delta_j \rho_{j-i,\alpha} \delta_{j-i} \sigma_{j-i,\alpha} v_{i,i+\alpha,j}^t}{1 + \delta_j S_{i,j}^t v_{i,i+\alpha}^t} \quad (3.1.7)
\]

Where an auxiliary function \( v_{i,j,k}^t \) is introduced:

\[
\begin{align*}
v_{i,j,k}^t &= \sum_{l=i+1}^{k} \delta_l \prod_{m=l}^{j} (1 + \delta_m S_{i,m}^t)^{-1}, \quad i < k \\
v_{i,j}^t &= v_{i,j,j}^t
\end{align*}
\]

A detailed proof of the drift formula, using the Girsanov theorem is presented in Appendix B.

The formula (3.1.7) looks quite complicated but it only depends on known parameters, mainly the accrual constant, the correlation and volatility factors and the swap rate at different maturities. For \( t \) such that \( T_{i-1} \leq t < T_i \), all those components are available at \( T_{i-1} \) and thus \( \mu_{i,\alpha}^t \) is possible to compute.
3.1.4 Volatility parameterisation

Strategy: Compute the volatility function for keys tenors then use interpolation and extrapolation methods.

For a fixed expiration date \( T_i \), specify volatility for \( N \) key tenors \( \alpha \in \{ \alpha_j^* \}_{j=1,\ldots,N} \). The key tenors are chosen such that the corresponding swap rates and IRS derivatives (mainly swaptions) are liquid market contracts. Then, use interpolation and extrapolation methods to compute the volatility for all other relevant tenors.

For a key tenor \( \alpha_j^* \), the volatility term in the dynamics of \( S_{i,i+\alpha_j^*} \) at time \( t \), is parameterised as:

\[
\sigma_{i,\alpha_j^*}^t = c_j^t g_{i,j}^t (S_{i,i+\alpha_j^*}^t), \quad T_{i-1} \leq t < T_i \tag{3.1.8}
\]

The volatility \( \sigma_{i,\alpha_j^*}^t \) is composed of two terms:

- \( g_{i,j}^t \): a non-parametric local volatility function. A Dupire-like formula that is calibrated using swaption market data. \( g_{i,j}^t \) is a function of the time \( t \) and the forward swap rate \( S_{i,i+\alpha_j^*}^t \).

- \( c_j^t \): stochastic volatility scaling factors. By having a volatility component randomly distributed, new factors are introduced in the scaling process. These stochastic scaling factors are chosen to minimize the residual error between the output volatility surface using only local volatility and the market’s implied volatility data.

Both components contribute to building the volatility smile and skew which are observed in the financial markets (implied volatility varying with respect to strike price and expiry). Thus, resolving the main shortcoming of the traditional Black–Scholes model [20].

The stochastic volatility scaling factors \( c_j^t \) follow a stochastic process that satisfies the following equation:

\[
dc_j^t = \mu_c^{c,j} dt + \nu_c^{c,j} dZ_{i,j}^Q, \quad c_0 = 1, \quad j = 1, \ldots, N \tag{3.1.9}
\]

Where \( \mu_c^{c,j} \) and \( \nu_c^{c,j} \) are the stochastic factor’s drift and volatility terms, and \( dZ_{i,j}^Q \) is a Brownian motion under \( Q \).
The Volatility-Volatility correlation is the correlation factor between two Brownian motions from the stochastic volatility dynamics (3.1.9), and is defined as:

\[
<d Z^j_t, d Z^k_t> = \rho_{VV,t}^{j,k} dt 
\]

Finally, by introducing two different Brownian motions in the same SMM, there is a third correlation factor, the Swap-Volatility correlation factor defined by the equation:

\[
<d W^{\alpha}_t, d Z^k_t> = \rho_{SV,t}^{\alpha,k} dt 
\]

### 3.2 The Local Volatility function

In this section, the goal is to find the local volatility function \( g^{i,j} \) in terms of \( t, S^{i,i+\alpha_j}_t \) and market data in the form of swaption prices.

Consider a payer swaption on the underlying asset \( S^{i,j}_t, T_{i-1} \leq t \leq T_i \). From the previous chapter, the pricing formula of this swaption at today \( T_0 = 0 \) is a function of \( t \):

\[
\Pi(0, X^{i,j}_t) = E^Q \left[ \frac{A^{i,j}_t}{B_t} (S^{i,j}_t - K)^+ \right]_F^Q = C^{i,j}(t, K)
\]

Conditioning under \( F_0 \), the previous equation can be written as follows:

\[
C^{i,j}(t, K) = E^Q \left[ \frac{A^{i,j}_t}{B_t} (S^{i,j}_t - K)^+ \right] = \frac{A^{i,j}_0}{B_0} E^{A^{i,j}} \left[ (S^{i,j}_t - K)^+ \right].
\]

Next, the derivative with respect to the time \( t \) of the price \( C^{i,j}(t, K) \) is computed:

\[
\partial_t C^{i,j}(t, K) = \frac{1}{2} E^Q \left[ \frac{A^{i,j}_t}{B_t} (\sigma^{i,j-i}_t)^2 I_{S^{i,j}_t - K} \right] = \frac{1}{2} \frac{A^{i,j}_0}{B_0} E^{A^{i,j}} \left[ (\sigma^{i,j-i}_t)^2 I_{S^{i,j}_t - K} \right].
\]
Using the approximation (explored in section 4.4):

Thus:

Next, consider the Taylor expansion on the price process $C^{i,i+\alpha_j}$. The use of the Ito-Tanaka formula.

Which means the local volatility function $\frac{\partial^2 C^{i,i+\alpha_j}}{\partial t} = 1$

Thus:

$$
\frac{C^{i,i+\alpha_j}(T_i, K) - C^{i,i+\alpha_j}(T_{i-1}, K)}{T_i - T_{i-1}} = \partial_t C^{i,i+\alpha_j}(T_{i-1}, K) \left[ 1 + \sum_{k=2}^{\infty} \frac{\partial^k C^{i,i+\alpha_j}(T_{i-1}, K)}{\partial t} (T_i - T_{i-1})^{k-1} \right]
$$

$$
= \frac{1}{2} \mathcal{E}_Q \left[ \frac{A^{i,i+\alpha_j}}{B^{i,i+\alpha_j}} \left( \sigma^{i,i+\alpha_j} \right)^2 \mathbb{I}_{i,i+\alpha_j - K} \right] \left[ 1 + \sum_{k=2}^{\infty} \frac{\partial^k C^{i,i+\alpha_j}(T_{i-1}, K)}{\partial t} (T_i - T_{i-1})^{k-1} \right]
$$

Which means the local volatility function $g^{i,j}(K)$ can be written:

$$
g^{i,j}(K) = \sqrt{\frac{2}{T_i - T_{i-1}} \mathcal{E}_Q \left[ \frac{A^{i,i+\alpha_j}}{B^{i,i+\alpha_j}} \left( \sigma^{i,i+\alpha_j} \right)^2 \mathbb{I}_{i,i+\alpha_j - K} \right] \left[ 1 + \sum_{k=2}^{\infty} \frac{\partial^k C^{i,i+\alpha_j}(T_{i-1}, K)}{\partial t} (T_i - T_{i-1})^{k-1} \right]^{-1/2}}
$$

Using the approximation (explored in section 4.4):

$$
\left[ 1 + \sum_{k=2}^{\infty} \frac{\partial^k C^{i,i+\alpha_j}(T_{i-1}, K)}{\partial t} (T_i - T_{i-1})^{k-1} \right]^{-1/2} \sim 1
$$
The approximation formula of the local volatility function $g^{i,j}(K)$, for the expiration date $T_i$ and the key tenor $\alpha_j$:

$$g^{i,j}(K) \sim \sqrt{\frac{2}{T_i - T_{i-1}}} \frac{C^{i,i+\alpha_j^*}(T_i, K) - C^{i,i+\alpha_j^*}(T_{i-1}, K)}{E \left[ \frac{A_{T_{i-1}}}{B_{T_{i-1}}} (c^{j,i+\alpha_j^*}_{T_{i-1}})^2 \mathbf{1}_{i,i+\alpha_j^*}(S_{T_{i-1}}-K) \right]}.$$ (3.2.6)

On one hand, $C^{i,i+\alpha_j^*}(T_i, K)$ refers to the price of a payer swaption with strike $K$, on a spot starting swap with expiry date $T_i$ and tenor $\alpha_j^*$. This is the type of swaptions traded on the market. Therefore, a calibrated local volatility function, is such that the market observed price $C^{i,i+\alpha_j^*}_{Mkt}(T_i, K)$ matches the model’s theoretical price $C^{i,i+\alpha_j^*}(T_i, K)$.

On the other, the term $C^{i,i+\alpha_j^*}(T_{i-1}, K)$ is a mathematical quantity that can be computed, knowing the distributions of $\{S_{T_{i-1}}\}$ and $\{B_{T_{i-1}}\}$:

$$C^{i,i+\alpha_j^*}(T_{i-1}, K) = E \left[ \frac{A_{T_{i-1}}}{B_{T_{i-1}}} (c^{j,i+\alpha_j^*}_{T_{i-1}})^2 \mathbf{1}_{i,i+\alpha_j^*}(S_{T_{i-1}}-K) \right]$$ (3.2.7)

**Result 2**: The local volatility function $g^{i,j}(K)$, for $T_{i-1} \leq t < T_i$:

For an expiry date $T_i$ and a fixed key tenor $\alpha_j^*$, the calibration formula for local volatility function $g^{i,j}(K) = g^{i,j}(K; T_{i-1}, T_i)$ is:

$$g^{i,j}(K) = \sqrt{\frac{2}{T_i - T_{i-1}}} \frac{C^{i,i+\alpha_j^*}_{Mkt}(T_i, K) - E \left[ \frac{A_{T_{i-1}}}{B_{T_{i-1}}} (c^{j,i+\alpha_j^*}_{T_{i-1}})^2 \mathbf{1}_{i,i+\alpha_j^*}(S_{T_{i-1}}-K) \right]}{E \left[ \frac{A_{T_{i-1}}}{B_{T_{i-1}}} (c^{j,i+\alpha_j^*}_{T_{i-1}})^2 \mathbf{1}_{i,i+\alpha_j^*}(S_{T_{i-1}}-K) \right]}.$$ (3.2.8)

**Control Variate for variance reduction**

The purpose of the control variate method is to reduce the variance by exploiting information about the errors in estimates of known quantities in order to reduce the error of an estimate of an unknown quantity [21].
In the Monte-Carlo simulation of the expectation (3.2.7):

\[ C_{i,i}^{i,i+\alpha_j^*}(T_{i-1}, K) = \mathbb{E}^Q \left[ \frac{A_{T_{i-1}}^{i,i+\alpha_j^*}}{B_{T_{i-1}}}(S_{T_{i-1}}^{i,i+\alpha_j^*} - K)^+ \right], \]

we can use the swaption price at \( T_{i-1} \) as a control variate for variance reduction. The formula becomes:

\[ C_{i,i}^{i,i+\alpha_j^*}(T_{i-1}, K) = \mathbb{E}^Q \left[ \frac{A_{T_{i-1}}^{i,i+\alpha_j^*}}{B_{T_{i-1}}}(S_{T_{i-1}}^{i,i+\alpha_j^*} - K)^+ \right] \\
+ C_{Mkt}^{i,i+\alpha_j^*}(T_{i-1}, K). \tag{3.2.9} \]

The control variate here is based on the difference between the model price \( C_{i-1,i-1+\alpha_j^*}(T_{i-1}, K) \) (the estimate of the unknown quantity) and the market observed price \( C_{Mkt}^{i,i+\alpha_j^*}(T_{i-1}, K) \) (the known quantity) at \( T_{i-1} \). Using this control variate technique in the calibration algorithm of the local stochastic volatility, can help significantly reduce the variance of the expectation term (3.2.7) simulated through the Monte-Carlo method.

3.3 The swap rate discount curve

In the local volatility formula (3.2.8), two terms still need to be specified: \( B_{T_{i-1}} \) and \( A_{T_{i-1}}^{i,i+\alpha_j^*} \). Both terms depend on the ZCB discounting factors:

\[
\begin{cases}
B_t = \left( \prod_{k=0}^{i-1} P_{T_k}^{k+1} \right)^{-1} P_t^i, & T_{i-1} \leq t < T_i \\
B_0 = 1
\end{cases}
\]

And

\[ A_t^{j,k} = \sum_{i=j+1}^k \delta_i p(t, T_i) = \sum_{i=j+1}^k \delta_i P_t^i \]

Thus \( P_t^T \) needs to be computed at \( t = \{T_i\}_{i=0,1,...} \) and for all necessary maturities.
There are different methods to compute the discount curve. But in the framework of an SMM, it is conventional to use the IRS data as basis to build the curve (detailed discussion of discount curve building in chapter 2 section 2.4.2).

The first step is to start with a swap curve, which represents a plot of swap rates across different maturities. Figure (3.2) show the swap curve plot for USD data on 19/11/2018 and the source of the data is the Inter-Continental Exchange ("theice.com"). Using the current thesis notation, the curve represents the swap rates $S_{T_0}^{0,0+\alpha}$, $\alpha \in [1Y : 30Y]$ and $T_0 = 0$.

![Example of a Swap Curve for USD on 19/11/2018](image)

For an expiration date $T_i$ and a tenor $\alpha$, the swap rate changes from $\{S_t^{i,i+\alpha}\}$ to $\{S_t^{i+1,i+1+\alpha}\}$ when $t$ goes across from one time interval $[T_{i-1}, T_i]$ to the next $[T_i, T_{i+1}]$. Using the formula for the annuity factor (2.2.5) and the swap rate (2.2.6), a recursive formula is established to compute the ZCBs at time $T_i$:

\[
\begin{align*}
P_{T_i}^{i,i+1} &= \frac{1}{1+\delta_{i+1}S_{T_i}^{i,i+1}} \quad \text{(initial step)} \\
P_{T_i}^{i,i+1+\alpha} &= \frac{1-S_{T_i}^{i,i+1+\alpha}A_{T_i}^{i,i+\alpha}}{1+\delta_{i+1+\alpha}S_{T_i}^{i,i+1+\alpha}}; \quad \alpha \geq 1.
\end{align*}
\]
The above formula shows that using the swap curve at \( t = T_0 \), it is possible to compute the discount rates at \( T_0 \) for all available maturities. While for the discount rates at \( T_i, i \geq 1 \), it’s necessary to first compute the swap rates \( S_{T_i}^{i, i+\alpha} \), \( \alpha \geq 1 \), using the swap rates dynamics.

Figure 3.3: ZCB discount curve for USD data on 19/11/2018, at \( T_0 = 0 \) plot (a) and at \( T_1 = 3m \) plot (b)

Figure 3.3 shows an example of ZCB discount curves for ICE data in USD on 19/11/2018. The curve at \( T_1 = 3m \) is a result of a simulation that will be explained in detail later on in the thesis. The shape of both curves is logical, both start at the value 1 since \( P(T, T) = 1 \), \( \forall T \), and then it’s a decreasing function as the maturity increases.

### 3.4 Swaption pricing under the Normal model

The only term left in the local volatility formula (3.2.8) that is yet to be discussed is the market price of a swaption denoted : \( C_{Mkt}^{i, i+\alpha_j}(T_i, K) \).

As explained previously, it is common to price swaption in the market in term of the Black implied volatility instead of the actual price of the swaption. Therefore, under the framework of the normal swap dynamics (3.1.5), a general formula needs to be established to price swaptions from normal implied volatility data coming from the market.

Under a no-arbitrage assumption, and under the martingale measure associated with the annuity factor \( A^{i,j} \), consider the following normal dynamics of
a swap rate \(dS_t^{i,i+\alpha} :\)
\[
dS_t^{i,i+\alpha} = \sigma_t^{i,\alpha} dW_t^{\alpha,A^{i,j}}
\] (3.4.1)

With : \(\sigma_t^{i,\alpha} = \sigma_N\) being a normal volatility.

In this case, the forward swap rate is assumed to be normally distributed according to a standard Brownian motion. And under this Normal model, there is a non zero probability for the forward swap rate to be negative.

Under this model, the price at \(t=0\) of payer swaption starting at \(T_i\), ending at \(T_i+\alpha\) (tenor \(\alpha\)) and with strike \(K\) is:
\[
P_{\text{Payer,Normal}}(0) = \sigma_N \sqrt{T_i}(\hat{d}_1 \Phi(\hat{d}_1) + \varphi(\hat{d}_1)) \sum_{n=1}^{\alpha} \delta_n p(0, T_{i+n}), \quad (3.4.2)
\]

\[
\hat{d}_1 = \frac{S_t^{i,i+\alpha}(0) - K}{\sigma_N \sqrt{T_i}},
\]

where \(\varphi\) is the probability distribution function of a standard normal distribution, \(N(0, 1)\), and \(\Phi\) is the cumulative distribution function of a standard normal distribution, \(N(0, 1)\).

**Proof :**

The Payoff of the swaption at the expiration date \(T_i\) is:
\[
X = (S_t^{i,i+\alpha}(T_i) - K)^+ \sum_{n=1}^{\alpha} \delta_n p(T_i, T_{i+n}). \quad (3.4.3)
\]

Thus, the arbitrage free pricing formula is written:
\[
P_{\text{Payer,Normal}}(0) = p(0, T_i) E^{A^{i,j}}[X]
\]
\[
= p(0, T_i) E^{A^{i,j}} [(S_t^{i,i+\alpha}(T_i) - K)^+ \sum_{n=1}^{\alpha} \delta_n p(T_i, T_{i+n})]
\]
\[
= \sum_{n=1}^{\alpha} \delta_n p(0, T_{i+n}) E^{A^{i,j}} [(S_t^{i,i+\alpha}(T_i) - K)^+]
\]

Using the dynamics of the forward swap presented at the beginning of this section, the following equation can be written:
\[
S_t^{i,i+\alpha}(T_i) = S_t^{i,i+\alpha}(0) + \sigma_N \sqrt{T_i} Z, \quad Z \sim N(0, 1)
\]

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Therefore:

\[
E[(S^{i,i+\alpha}(T_i) - K)^+] = E^{A^{i,j}}\left[(S^{i,i+\alpha}(0) + \sigma_N \sqrt{T_i} Z - K)^+\right]
\]

\[
= \sigma_N \sqrt{T_i} E^{A^{i,j}}\left[(S^{i,i+\alpha}(0) - K \sigma_N \sqrt{T_i} + Z)^+\right]
\]

\[
= \sigma_N \sqrt{T_i} E^{A^{i,j}}\left[(\hat{d}_1 + Z)^+\right]
\]

Then:

\[
E^{A^{i,j}}\left[(\hat{d}_1 + Z)^+\right] = \int_{-\hat{d}_1}^{+\infty} (\hat{d}_1 + z) \varphi(z) dz
\]

\[
= \int_{-\hat{d}_1}^{+\infty} \hat{d}_1 \varphi(z) dz + \int_{-\hat{d}_1}^{+\infty} z \varphi(z) dz
\]

\[
= \hat{d}_1 \int_{-\hat{d}_1}^{+\infty} \varphi(z) dz + \int_{-\hat{d}_1}^{+\infty} \frac{1}{\sqrt{2\pi}} ze^{-z^2/2} dz
\]

\[
= \hat{d}_1 P_S(Z > -\hat{d}_1) + \frac{1}{\sqrt{2\pi}} \left[-e^{-z^2/2}\right]_{-\hat{d}_1}^{+\infty}
\]

\[
= \hat{d}_1 \Phi(\hat{d}_1) + \frac{1}{\sqrt{2\pi}} e^{-(-\hat{d}_1)^2/2}
\]

\[
= \hat{d}_1 \Phi(\hat{d}_1) + \varphi(\hat{d}_1).
\]

Finally:

\[
P_{Payer,Normal}(0) = \sigma_N \sqrt{T_i} (\hat{d}_1 \Phi(\hat{d}_1) + \varphi(\hat{d}_1)) \sum_{n=1}^{\alpha} \delta_n p(0, T_{i+n}),
\]

\[
\hat{d}_1 = \frac{S^{i,i+\alpha}(0) - K}{\sigma_N \sqrt{T_i}}
\]

From the above formula and using the market’s normal implied volatility, we can write:

\[
C_{Mkt}^{i,i+\alpha}(T_i, K) = \sigma_{Mkt}^{i,i+\alpha} \sqrt{T_i} (\hat{d}_1 \Phi(\hat{d}_1) + \varphi(\hat{d}_1)) \sum_{n=1}^{\alpha} \delta_n p(0, T_{i+n})
\]

\[
, \quad \hat{d}_1 = \frac{S^{i,i+\alpha}(0) - K}{\sigma_{Mkt}^{i,i+\alpha} \sqrt{T_i}}.
\]

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3.5 Relation to LMM and Libor rates

As mentioned earlier, it is much more common to work within an LMM framework while modeling interest rates instead of the SMM framework which is the subject of this thesis. Thus, the purpose of this section is to present a few indicative points of how to transfer the results found in this thesis to their equivalent if one chooses to work with Libor rates and an LMM instead.

Consider a rolling maturity swap \( S^{i,i+\alpha}_t \), using the definition of a swap rate and the formulas for ZCBs in terms of Libor rates, we have:

\[
S^{i,i+\alpha}_t = \frac{P^{i}_t - P^{i+\alpha}_t}{A^{i,i+\alpha}_t} = \frac{P^{i}_t - P^{i+\alpha}_t}{\sum_{k=1}^{\alpha} \delta_{i+k} P^{i+k}_t} = 1 - \frac{P^{i+\alpha}_t}{P^{i}_t} \sum_{k=1}^{\alpha} \delta_{i+k} \frac{P^{i+k}_t}{P^{i}_t} = 1 - \frac{1}{\sum_{k=1}^{\alpha} \delta_{i+k} \prod_{l=1}^{k} \frac{1}{1 + \delta_{i+l} L^{i+l}_t}}
\]

Therefore it can be established that:

\[
L^{i+\alpha}_t = S^{i+\alpha-1,i+\alpha}_t \tag{3.5.1}
\]

Using the dynamics of \( S^{i,i+\alpha}_t \), it can be proven that for \( T_{i-1} \leq t < T_i \), and \( \alpha \geq 1 \), \( L^{i+\alpha}_t \) has dynamics:

\[
dL^{i+\alpha}_t = \sum_{j=1}^{\alpha} \frac{\delta_{i+j} \rho^{L,i+j,i+\alpha}_t \sigma^{L,i,j}_t \sigma^{L,i+\alpha}_t}{1 + \delta_{i+j} L^{i+j}_t} + \sigma^{L,i+\alpha}_t dW^{L,i+\alpha}_t \tag{3.5.2}
\]

With:

\[
\begin{align*}
\rho^{L,i+\alpha,i+\beta}_t &= <dW^{L,i+\alpha}_t, dW^{L,i+\beta}_t> _t \\
\rho^{L,i+\alpha,k}_t &= <dW^{L,i+\alpha}_t, dZ^{k}_t> _t 
\end{align*}
\]

Where \( \sigma^{L,i+\alpha}_t \) is the Libor volatility, \( dW^{L,i+\alpha}_t \) is a Brownian motion defined under \( Q \), the \( \rho^{L,i+\alpha,i+\beta}_t \) the forward Libor-Libor correlation and \( \rho^{L,i+\alpha,k}_t \) the forward Libor-Volatility correlation factor.
The Libor rate volatility and the two correlation factors can be deduced from their equivalent swap rate factors using the dynamics of $S_{t}^{i,i+\alpha}$ and the Ito formula (Bjork 11).

Using the relation between $L_{t}^{i+\alpha}$ and $S_{t}^{i,i+\alpha}$, the Ito formula is written as:

$$dS_{t}^{i,i+\alpha} = \sum_{j=1}^{\alpha} \frac{\partial S_{t}^{i,i+\alpha}}{\partial L_{t}^{i+j}} dL_{t}^{i+j} + (...)dt$$  \hspace{1cm} (3.5.3)

The rest is a matter of replacing the formulas of the Libor dynamics, and identifying the results with the already established formulas for the swap dynamics.

Using all this information, it’s easy to see how all the work here on the SMM model can be translated into an LMM model and use the non-parametric local volatility calibration method. This falls outside the setting of the thesis, and is left under the umbrella of further work.
Chapter 4

Methodology

In this chapter, the aim is to present the practical methods used to compute the forward dynamics of the rolling maturity swap of the SMM framework and build the local stochastic volatility surface for different expiration dates and different tenors.

First, practical methods such as computing the historical correlation and the volatility interpolation and extrapolation are presented. Then, important results about the SABR model will be presented in order to build a 'Target' market. Finally, the algorithm to compute the local stochastic volatility formula will be presented in details and discussed.

4.1 Swap-Swap rates correlation

In the definition of the dynamics of a rolling maturity swap in equation (3.1.5), the correlation between two forward swap rates at time $t$ and maturities $\alpha$ and $\beta$ is defined as the correlation between the two corresponding Brownian motions.

For the dynamics of $S_t^{i,i+\alpha}$ and $S_t^{i,i+\beta}$:

\[
dS_t^{i,i+\alpha} = \mu_t^{i,\alpha} dt + \sigma_t^{i,\alpha} dW_t^{\alpha,Q}
\]

\[
dS_t^{i,i+\beta} = \mu_t^{i,\beta} dt + \sigma_t^{i,\beta} dW_t^{\beta,Q}
\]

The Brownian motions $dW^{\alpha,Q}$ and $dW^{\beta,Q}$ are correlated according to :

\[
< dW^{\alpha,Q}, dW^{\beta,Q} >_t = \rho_t^{\alpha,\beta} dt
\]
This correlation appears in the drift term of the dynamics of the process $S_t^{i,i+\alpha}$ in the formula (3.1.7):

$$\mu_t^{i,\alpha} = \sum_{j=i+1}^{i+\alpha} \delta_j \rho_t^{j-i,\alpha} \sigma_t^{j-i,\alpha} \delta_t v_t^{i,i+\alpha,j} \frac{1 + \delta_j S_t^{i,j-1}}{v_t^{i,i+\alpha}}$$

This makes the Swap-Swap rate correlation, $\rho_t^{\alpha,\beta}$, a fundamental component of the SMM model presented in this thesis.

In this thesis (and traditionally in most LMMs and SMMs), we will be working with time invariant correlation: $\rho_t^{\alpha,\beta} = \rho^{\alpha,\beta}$, $\forall \alpha, \beta$. The correlation will therefore not dependent on the time $t$ but only on the forward rate maturity ($\alpha$ and $\beta$).

### 4.1.1 Historical Correlation

The estimation of the correlation term can be a difficult task. But in the particular cases where the payoff is only slightly correlation dependent which is the case for SMMs [24], we can extract the correlation from a time series of historical swap rates.

For a set of length $N$ of historical dates $t_n$, we consider the swap rates $S^{0,0+\alpha}(t_n) = S^\alpha(t_n)$ for different maturities $\alpha$. This results in a time series of swap rates for each maturity $\alpha$. Then since we work in a normal distribution for the swap rates, we can define the returns directly as:

$$\Delta S^\alpha(t_n) = S^\alpha(t_n) - S^\alpha(t_{n-1})$$ (4.1.1)

From the historical returns series, we can compute the average historical change in each rate for a maturity $\alpha$:

$$\overline{\Delta S^\alpha} = \frac{1}{N} \sum_{n=1}^{N} \Delta S^\alpha(t_n)$$ (4.1.2)

From this, we obtain the standard deviation:

$$\text{Std}(\Delta S^\alpha) = \sqrt{\frac{1}{N-1} \sum_{n=1}^{N} (\Delta S^\alpha(t_n) - \overline{\Delta S^\alpha})^2}$$ (4.1.3)
and the covariance between two swap rates with different maturities $\alpha$ and $\beta$:

$$\text{Cov}(\Delta S^\alpha, \Delta S^\beta) = \frac{1}{N-1} \sum_{n=1}^{N} (\Delta S^\alpha(t_n) - \bar{\Delta S^\alpha})(\Delta S^\beta(t_n) - \bar{\Delta S^\beta}).$$

(4.1.4)

Finally, the historical correlation follows as:

$$\rho^{\alpha,\beta} = \frac{\text{Cov}(\Delta S^\alpha, \Delta S^\beta)}{\text{Std}(\Delta S^\alpha)\text{Std}(\Delta S^\beta)}.$$  

(4.1.5)

Using data retrieved from The Federal Reserve Bank of St. Louis, we can use swap rate historical data for almost 10 years in the past and the different maturities (1Y, 2Y, 5Y, 10Y, 30Y). Figure (4.1) shows a plot of the historical swap rates for 3 different maturities 2Y, 5Y and 10Y. Visually, it can be seen that for most periods of time, the swap rate evolves in a similar manner across the different maturities, which indicates a positive and strong correlation factor.

Figure 4.1: Historical swap rates (in %) for maturity 2Y, 5Y and 10Y over the period 2010-2018

The correlation matrix retrieved is:
Table 4.1: Historical correlation matrix for swap rates data

<table>
<thead>
<tr>
<th>Correlation</th>
<th>1Y</th>
<th>2Y</th>
<th>5Y</th>
<th>10Y</th>
<th>30Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>1</td>
<td>0.84</td>
<td>0.47</td>
<td>0.28</td>
<td>0.17</td>
</tr>
<tr>
<td>2Y</td>
<td>0.84</td>
<td>1</td>
<td>0.83</td>
<td>0.63</td>
<td>0.48</td>
</tr>
<tr>
<td>5Y</td>
<td>0.47</td>
<td>0.83</td>
<td>1</td>
<td>0.94</td>
<td>0.84</td>
</tr>
<tr>
<td>10Y</td>
<td>0.28</td>
<td>0.63</td>
<td>0.94</td>
<td>1</td>
<td>0.98</td>
</tr>
<tr>
<td>30Y</td>
<td>0.17</td>
<td>0.48</td>
<td>0.84</td>
<td>0.98</td>
<td>1</td>
</tr>
</tbody>
</table>

The values obtained using this procedure are only spurious entries for a limited number of maturities. In order to get a full correlation matrix, we use a correlation parametrization method.

4.1.2 Smooth Correlation Matrix

In order to generate a smooth and well-behaved correlation matrix, we can use a parametrization method through a low-dimensional functional form.

In theory, we look into minimizing the sum of squared errors:

$$\sum_{\alpha,\beta \in \Omega} (\rho_{hist}^{\alpha,\beta} - \rho^{\alpha,\beta})^2$$  \hspace{1cm} (4.1.6)

Where \(\Omega\) is the set of available maturities.

In literature, there are different parametric forms that provide smooth and valid correlation matrices. We choose the following form [24] :

$$\rho^{\alpha,\beta} = \rho_\infty + (1 - \rho_\infty)e^{-c|\alpha-\beta|}$$  \hspace{1cm} (4.1.7)

This is an 'Exponential parametrization with decay control' (others are presented and discussed in more details in Rebonato 2004 [24]). This form offers a good calibration of the correlation for LMMs and SMMs, with the added benefit of consisting of very few parameters.
In the formula (4.1.7), $\rho_\infty$ is a fixed parameter that is equal to the correlation between the farthest apart swap rates observed in the historical matrix. In other words, $\rho_\infty = \rho^{1Y:30Y}$ in our case. $c$ is also a constant that can be deduced using the same data. For $\alpha$ and $\beta$ in the historical data:

$$c = -\frac{1}{|\alpha - \beta|} \log\left(\frac{\rho^{\alpha,\beta} - \rho_\infty}{1 - \rho_\infty}\right)$$

Figure 4.2 presents the swap rate correlation matrix $\rho^{\alpha,\beta}$, generated through equation (4.1.7) with $c = 0.22$ and $\rho_\infty = 0.17$. The diagonal of the matrix is constant and equals 1. The values are symmetric and match the one from the historical matrix.

We also notice that $\rho^{\alpha,\beta}$ is decreasing for $\alpha \geq \beta$. This makes sense since movements of far away rates are less correlated than movements of the rates with close maturities.

Finally, we see that when moving along the yield curve, the larger the maturity, the more correlated the adjacent forward rates are.

Figure 4.2: Swap rate correlation matrix generated through equation (4.1.7) with $c = 0.22$ and $\rho_\infty = 0.17$
4.2 Stochastic Volatility Scaling

The volatility parametrization for the dynamics of a swap rate starting at $T_i$ and for a key tenor $\alpha_j^*, j = 1, ..., N$, was introduced in section 3.1.4 as follow:

$$\sigma_{i,\alpha_j^*}^t = c_j^i g^{i,j}(S_{i,i}^t + \alpha_j^*)^t, \quad (4.2.1)$$

with $g^{i,j}$ being the non-parametric local volatility function.

The stochastic volatility scaling factor $c_j^i$ follows the general dynamics formula written as:

$$dc_j^i = \mu_c^j dt + v_c^j dZ_Q^j, \quad c_0^j = 1, \quad j = 1, ..., N \quad (4.2.2)$$

In this thesis, we chose to define the stochastic scaling factor as a log normal process under $Q$, which is the most common way of defining such factors (the same is done in the standard SABR models [2]):

$$dc_j^i = dc_t = v_t dZ_Q^j, \quad c_0^j = 1, \quad j = 1, ..., N \quad (4.2.3)$$

With Swap-Volatility correlation:

$$<dW^{\alpha,Q},dZ^Q>_t = \rho_{SV}^t dt \quad (4.2.4)$$

The process and the correlation factor are independent from the tenor of the swap rate, this approximation is common for stochastic scaling factor and can be easily verified numerically.

The constant $v$ is the volatility term in the stochastic volatility factor and can therefore be seen as a volatility of the volatility of the swap rate.

To determine the value of $\rho_{SV}^t$, the same data from the Swap-Swap correlation is used again (section 4.1).

for a fixed maturity $\alpha$, we define the Standard deviation of the swap rate process in the same way as before:

$$\text{Std}(\Delta S^\alpha) = \sqrt{\frac{1}{N-1} \sum_{n=1}^N (\Delta S^\alpha(t_n) - \overline{\Delta S^\alpha})^2}. \quad (4.2.5)$$
Then we define a subset of length $\tilde{N}$ of historical dates $t_n$ from the larger set $N$ of available historical dates. We can then compute the standard deviation over the subset $\tilde{N}$:

$$\text{Std}(\Delta S^\alpha) = \sqrt{\frac{1}{N-1} \sum_{n=1}^{\tilde{N}} (\Delta S^\alpha(t_n) - \bar{\Delta S^\alpha})^2} \quad (4.2.6)$$

Then we add a new historical date to $\tilde{N}$ from the set $N\backslash \tilde{N}$, to get a set $\tilde{N}_1$ larger than $\tilde{N}$ by one element. We compute the standard deviation over the new set and retain the new value. We repeat the same method for all the element in $N\backslash \tilde{N}$ every time adding a new one to the growing subset and computing the standard deviation.

The result is a time series $\{\Sigma_t\}$ of historical standard deviations of length $N - \tilde{N} + 1$. Then all is left to express this standard deviation in term of annualized volatility. We simply multiply the standard deviation term of the series by the square root of 252, which represents the number of trading days in a given year.

The other constant left to discuss is $v$. Instead of fixing its value based on historical data, we made the choice of determining the optimal value of $v$ that accurately matches our own model’s output to the implied volatility curves observed in the marketplace, using a minimisation algorithm based on the least squares method. In practice, at an expiry $T_i$ and for a key tenor $\alpha^*$, we search to minimize the sum of squared errors:

$$\sum_{K \in \mathbb{K}} (g^{i,\alpha^*}(K) - \sigma^{i,\alpha^*}_{\text{Target}}(K))^2 \quad (4.2.7)$$

Where $\mathbb{K}$ is the set of available strikes, $g^{i,\alpha^*}$ the local stochastic volatility output and $\sigma^{i,\alpha^*}_{\text{Target}}$ the target volatility.

Once the constants $v$ and $\rho^{SV}$ are determined, the expression for the stochastic scaling factor $c_t$ is expressed by the following formula:

$$c_t = c_0 \exp \left\{ -\frac{v^2}{2} t + v Z^Q(t) \right\}$$

$$= c_0 \exp \left\{ -\frac{v^2}{2} t + v \left( \rho^{SV} X_1^Q + \sqrt{1 - (\rho^{SV})^2} X_2^Q \right) \right\} \quad (4.2.8)$$
where $X_1^Q(t)$ and $X_2^Q(t)$ are two uncorrelated standard Brownian motions corresponding to the Brownian motions in the swap rate dynamics and the stochastic volatility dynamics. The details of how to find the above formula are presented in Appendix A.4.

4.3 Volatility interpolation/extrapolation

In the volatility parametrisation of section 3.1.4, the strategy chosen was to compute the volatility function for keys tenors then use interpolation and extrapolation methods for the other tenors. The reason behind this is the market data needed for the calibration algorithm. Since the calibration is based on market implied normal volatility, it is only available for certain tenors that are usually bought and sold by market participants. Generally, swaptions with tenors of 5 and 10 years are the most common on the market, while it’s rare to see a swaption on a underlying swap with a tenor of 7 years.

In this thesis, we chose to carry out the interpolation and extrapolation in swap rate volatility space. It is also possible to perform the interpolation and extrapolation in Libor volatility space, to ensure the volatility dynamics transparent in the framework of a Libor market model [9].

For a set of $N$ key tenors $\alpha \in \{\alpha_j^*\}_{j=1,\ldots,N}$ of ascending order ($\alpha_1^* < \alpha_2^* < \ldots < \alpha_N^*$), the interpolation is carried out for tenors $\alpha$ between $\alpha_1^*$ and $\alpha_N^*$, while extrapolation is performed on tenors $\alpha$ such that: $\alpha < \alpha_1^*$ or $\alpha > \alpha_N^*$.

For extrapolation, we chose a flat extrapolation method for this thesis. This translates into: $\sigma_i^{\alpha,t} = \sigma_i^{\alpha_1^*,t}$, $\forall \alpha < \alpha_1^*$ and $\sigma_i^{\alpha,t} = \sigma_i^{\alpha_N^*,t}$, $\forall \alpha > \alpha_N^*$.

For interpolation, several methods can be performed to achieve a volatility surface that is arbitrage consistent. We consider the interpolation of the variable $f(T_{i+\alpha}) = \sigma_i^{i,\alpha}\sigma$ with regards to $T_{i+\alpha}$. Then for every $\alpha$ between $\alpha_1^*$ and $\alpha_N^*$, the interpolated $\sigma_i^{i,\alpha}$ is given as $\hat{f}$: the interpolated variable at $T_{i+\alpha}$.

In this thesis, we consider two different interpolation methods: a linear interpolation and a quadratic interpolation. More sophisticated methods such as the parametric interpolation can be carried out to test out their effect on the quality of the calibration [10]. Figure (4.3) shows the result of the
interpolation and extrapolation methods on USD implied normal volatility data with the key tenors: 1Y, 2Y, 5Y, 10Y and 15Y.

Figure 4.3: Example of volatility interpolation and extrapolation results based on key tenor \{1Y, 2Y, 5Y, 10Y, 15Y\}

4.4 A time series approximation

While computing the formula for the local volatility function $g^{i,j}$ (3.2.8), the following approximation have been made:

$$[1 + \sum_{k=2}^{\infty} \frac{\partial_t^k C^{i,i+j}(T_{i-1}, K)}{\partial_t C^{i,i+j}(T_{i-1}, K)}(T_i - T_{i-1})^{k-1}]^{-\frac{1}{2}} \sim 1 \quad (4.4.1)$$

With, for $k \in \{1, 2, \ldots\}$:

$$\partial_t^k C^{i,i+j}(T_{i-1}, K) = \frac{\partial^k C^{i,i+j}(T_{i-1}, K)}{\partial t^k}$$

is the derivative of order $k$ with respect to time $t$ of $C^{i,i+j}(T_{i-1}, K)$: the price process of a payer swaption with strike K, on the underlying asset $S^{i,j}_{T_{i-1}}$.

The closer the time $t$ is to the expiration date $T_i$, the lower the price of the swaption is, since there is more information about the underlying asset on
the market and less uncertainty for its participants. In order to prove that
the approximation formula holds, it’s necessary to start by proving that the
first term of the sum is negligible compared to 1.

The first term of the sum \((k=2)\) is:

\[
\frac{\partial^2 C_{i,i+\alpha^j}(T_{i-1}, K)}{\partial t C_{i,i+\alpha^j}(T_{i-1}, K)} (T_i - T_{i-1})
\]

It has already been proven that:

\[
C_{i,j}(t, K) = \mathbb{E}^Q\left[\frac{A_{i,j}^t}{B_t} (S_{i,j}^t - K)^+ \right] = \frac{A_{0,i,j}^t}{B_0} \mathbb{E}^{A_{i,j}^t}\left[(S_{i,j}^t - K)^+ \right]
\]

And:

\[
\partial_t C_{i,j}(t, K) = \frac{1}{2} \frac{A_{0,i,j}}{B_0} E^{A_{i,j}^t}\left[(\sigma_{i,j}^t)^2 \mathbb{1}_{S_{i,j}^t - K} \right]
\]

The first term of the sum can be approximated using the finite difference
method: at a fixed strike \(K\), the time step \(h_i = T_i - T_{i-1}\), the notation
\(C_{i,i+\alpha^j}(T_{i-1}, K) = X_{i-1}(K)\) and using a second order finite difference ap-
proximation:

\[
\frac{\partial^2 C_{i,i+\alpha^j}(T_{i-1}, K)}{\partial t C_{i,i+\alpha^j}(T_{i-1}, K)} (T_i - T_{i-1}) = \frac{\partial^2 X_{i-1}(K)}{\partial t X_{i-1}(K)} h_i
\]

\[
= \frac{X_i(K) - 2X_{i-1}(K) + X_{i-2}(K)}{h_i^2} \frac{X_i(K) - X_{i-2}(K)}{2h_i}
\]

\[
= 2 \frac{X_i(K) - 2X_{i-1}(K) + X_{i-2}(K)}{X_i(K) - X_{i-2}(K)}.
\]

What’s left to do is for an expiration date \(T_i\) and a tenor \(\alpha\), compute the
quantity \(C_{i,i+\alpha^j}(t, K)\) for \(t = T_{i-2}, T_{i-1}\) and \(T_i\), then plot the finite difference approximation formula as a function of the strike \(K\).
Figure 4.4: Finite difference approximation of the first term of the time series sum for $T_i = 9m$ and $\alpha = 10Y$.

Figure (4.4) shows the numerical approximation of the first term of the time series sum for the expiration date $T_i = 9m$ and the tenor $\alpha = 10Y$. From the plot, we can see that even though the value of the term varies depending on the strike $K$, it always stays at a level that is negligible compared to 1.

The other terms in the series corresponding to $\partial_t \partial_{\alpha}^i C^{i,i+\alpha}(T,K)/\partial_{\alpha}^i C^{i,i+\alpha}(T,K)$ for $k > 2$, decrease rapidly as $T$ increases for an option $C^{i,i+\alpha}(T,K)$ computed with wide range of models. In the Bachelier model for example, $\partial_t \partial_{\alpha}^i C^{i,i+\alpha}(T,K)/\partial_{\alpha}^i C^{i,i+\alpha}(T,K)$ is proportional to $T^{1-k}$ [28]. This proves that the approximation error is smaller for large expiry dates, and that the approximation formula (4.4) stands.

4.5 The SABR Model for normal volatility

The SABR volatility model is a stochastic model used to capture the volatility smile in the derivatives markets. SABR stands for "stochastic alpha, beta, rho" model, which refers to the parameters of the model. Developed by Patrick S. Hagan, Deep Kumar, Andrew Lesniewski, and Diana Woodward [2], The SABR model is widely used by practitioners in the financial industry, especially in the interest rate derivative markets. Since it is considered the
industry’s standard for interest rate smile, we will use the SABR model as basis for comparison with the output of the local stochastic volatility algorithm.

For a single forward $F$, (a swap forward rate in our case but it could also be Libor forward rate or a forward stock price), the volatility of the forward, $F$, is described by a parameter $\sigma$ following a system of stochastic differential equations:

\[
\begin{align*}
    dF_t &= \sigma_tF^\beta dW_{1,t} \\
    d\sigma_t &= v\sigma_t dW_{2,t}
\end{align*}
\] (4.5.1)

The forward and the volatility are correlated:

\[<dW_{1,t},dW_{2,t}> = \rho dt\] (4.5.2)

$\alpha = \alpha_0$, $v$ and $\beta$ are constant parameters, satisfying : $v \geq 0$ and $0 \leq \beta \leq 1$. The original paper by Hagan [2], describes in details the significance of all the SABR parameters and how to find them.

For an option expiring at $T_i$ with initial value $F(0) = f$, the SABR model provides the following formula for normal implied volatility $[24][25]$: 

\[
\sigma_{N,Imp}(K,f) = \alpha(Kf)^\beta \frac{1 + \frac{1}{24} \log^2(f/K) + \frac{1}{1920} \log^4(f/K) + \ldots}{1 + \frac{(1-\beta)^2}{24} \log^2(f/K) + \frac{(1-\beta)^4}{1920} \log^4(f/K) + \ldots} \cdot \left(\frac{z}{\epsilon(z)}\right) \cdot \left\{ 1 + \left[ -\beta(2-\beta) \cdot \frac{\alpha^2}{24} + \frac{1}{4} \cdot \frac{\alpha\beta\rho v}{(Kf)^{1-\beta}} + \frac{2 - 3\rho^2 v^2}{24} \right]T_i + \ldots \right\}
\] (4.5.3)

with:

\[
z = \frac{v}{\alpha} (fK)^{(1-\beta)} \log(f/K)
\]

\[
\epsilon(z) = \log(\sqrt{1 - 2\rho z + z^2} + z - \rho)
\]

In the special case of an ATM option, the formula becomes:

\[
\sigma_{N,ATM}(f) = \sigma_{N,Imp}(f,f)
\]

\[
= \alpha f^\beta \left\{ 1 + \left[ -\beta(2-\beta) \cdot \frac{\alpha^2}{24} f^{2(1-\beta)} + \frac{1}{4} \frac{\alpha\beta\rho v}{f^{1-\beta}} + \frac{2 - 3\rho^2 v^2}{24} \right]T_i + \ldots \right\}
\] (4.5.4)
The implied normal volatility formula is quite complex and hard to implement, but there exist an approximation formula given also by Hagan [2] :

\[
\sigma_{N,Imp}(K, f) = \alpha f^\beta [1 - \frac{1}{2}(1 - \beta - \rho\lambda) \log(\frac{f}{K}) + \frac{1}{12}[(1 - \beta^2) + (2 - 3\rho^2)\lambda^2] \log^2(\frac{f}{K}) + ...]
\]

with :

\[
\lambda = \frac{v}{\alpha} f^{1-\beta}
\]

With the correct values of the SABR parameters, \( f, \alpha, v, \beta \) and \( \rho \), one can construct an implied volatility surface for the SMM that captures the market view.

### 4.6 The SMM Algorithm

Finally, after determining all the components appearing in the volatility function (3.2.8), we can move on to presenting the algorithm to calibrate the formula for different expiration dates and key tenors. The idea for the SMM algorithm is based on the particle algorithms [26].

**Step I** - Set \( k = 1 \):
Starting at date \( t = 0 \), the swap rate values can be written as \( S_{0,0+\alpha}^0 \) for \( \alpha = 1, 2, ... \). Using this data and the recursive formula (3.3.1), we can compute \( S_{0,1+\alpha}^1 \) for \( \alpha = 1, 2, ... \) : the value of a swap rate starting at \( T_1 \) and a tenor of \( \alpha \). Also, initialise \( \sigma_{0,\alpha_j}^1 \) for key tenors \( \{\alpha_j\} \) with a market at-the-money implied normal volatility of expiry \( T_1 \). Then using an interpolation and an extrapolation methods, get \( \sigma_{0,\alpha}^1 \) for all other tenors.

**Step II** - Simulating paths \( S_{k,k+\alpha}^k \) from \( T_{k-1} \) to \( T_k \):
Using the dynamics equation of \( S_{k,k+\alpha}^k \):

\[
dS_{t}^{k,\alpha} = \mu_{t}^{k,\alpha} dt + \sigma_{t}^{k,\alpha} dW_{t}^{\alpha,Q}, \quad T_{k-1} \leq t < T_k, \quad \alpha = 1, 2, ...
\]

We can compute \( S_{T_k}^{k,k+\alpha} \) using a discretization scheme, like the Euler method, which works in the following way :

Example for \( k = 1 \) : We choose a constant time step \( dt = \frac{T_1 - T_0}{N} \) with \( N \) the number of steps and then use the Euler approximation, for every tenor \( \alpha \) :
- Set initial point: $S_0^{1,1+\alpha}$.
- Recursively define $S_n^{1,1+\alpha}$ for $1 \leq n \leq N$ by:

$$S_{n+1}^{1,1+\alpha} = S_n^{1,1+\alpha} + \mu_n^{1+\alpha} dt + \sigma_n^{1+\alpha} dW_n^{\alpha,Q}$$

The random variables $dW_n^{\alpha,Q}$ are independent and identically distributed normal random variables with expected value zero and variance $dt$.

**Step III** - Compute local volatility functions $g^{k+1,j}(K)$.

Once $S_{T_k}^{k,k+\alpha}$ for $\alpha = 1, 2, \ldots$ is determined, we then use the recursive formula (3.3.1) again to obtain $S_{T_k}^{k+1,k+1+\alpha}$, $\alpha = 1, 2, \ldots$. Then for the key tenors $\{\alpha_j^\ast\}$, we can compute the local volatility function $g^{k+1,j}(K)$. Followed by interpolation and extrapolation for the rest of the tenors.

In the term $E^Q\left[\frac{\Delta T_k}{B_{T_k}} (c_j^T)^2 \mathbb{I}_{S_{T_k}^{k+1,k+1+\alpha_j^\ast} - K}\right]$, a dirac mass is too singular for numerics, so we approximate it by a smooth function that integrates to 1 and is localized where the Dirac has its mass. The function used is:

$$\mathbb{I}_a(x) := \frac{1}{|a|\sqrt{\pi}} e^{-\frac{(x-a)^2}{2}}, \quad a \to 0$$

The function gives good results and isn’t very sensitive of the values of $a$.

Once we have computed the function $g^{k+1,j}(K)$, we can draw the local volatility surface at $t = T_k$ for a range of strike values $K$.

**Step IV** - set $k = k+1$ and repeat steps II and III:

The rest of the algorithm is recursive:

at step $k+1$, for the key tenors, we compute the at-the-money normal volatility using the formula:

$$\sigma_t^{k+1,\alpha_j^\ast} = c_j^T g^{k+1,j}(S_{T_k}^{k+1,k+1+\alpha_j^\ast}), \quad T_k \leq t < T_{k+1}$$

Then repeat the Euler method used in steps II and III to get the local volatility function at the next expiry $T_{k+1}$ until the maturity date.
Chapter 5

Results and Analysis

In this Chapter, the results are presented and analyzed. The aim is to look into the different outputs of the algorithm and put it to the test using real market data for different currencies, expiration dates and tenors. First, we look into details of the SMM algorithm, the Monte-carlo simulation, and a study of convergence. Next, we move on to simulations on two sets of the data, corresponding to the currencies USD and AUD over a range of strike prices K. For different expiration dates and tenor, we compute the error between the model output and the target values. Finally, we explore an application of the local stochastic volatility as we price Bermudan swaptions using the same algorithm.

5.1 Algorithm Convergence

5.1.1 Monte-Carlo simulation

In the expression of the volatility function $g^{i,j}$ (3.2.8), the terms:

$$E^Q \left[ \frac{A_{T_{i-1}}^{i,i+\alpha_j^*}}{B_{T_{i-1}}} (S_{T_{i-1}}^{i,i+\alpha_j^*} - K)^+ \right] \quad \text{and} \quad E^Q \left[ \frac{A_{T_{i-1}}^{i,i+\alpha_j^*}}{B_{T_{i-1}}} (c_{T_{i-1}}^{j})^2 \mathbb{1}_{S_{T_{i-1}}^{i,i+\alpha_j^*} - K} \right]$$

are computed using the simulated values of $S_{T_{i-1}}^{i,i+\alpha_j^*}$. To compute both expectation the Monte-Carlo simulation is needed. The Euler scheme explained in the previous chapter is repeated for a number of times ($2^{11}$ times is our
simulations) which leaves us with a series of values for $S_{T_{i-1}}^{i,i+\alpha_j^*}$. Then we can compute 2 series of values corresponding to:

$$\frac{A_{T_{i-1}}}{B_{T_{i-1}}} (S_{T_{i-1}}^{i,i+\alpha_j^*} - K)^+ \quad \text{and} \quad \frac{A_{T_{i-1}}}{B_{T_{i-1}}} (c_{T_{i-1}}^j)^2 I_{S_{T_{i-1}}^{i,i+\alpha_j^*} - K}.$$ 

The 2 expectations are then, the mean of each value series.

Using the following USD swap curve data from the 'theice.com' on January 5, 2017 - figure (5.1), a series of expiration date: 3m, 6m, ... 10Y and a time step of $N = 90$ (number of days between the three months that separates each expiration dates) and a 211 simulation paths, we can simulate the swap rate $S_{3m}^{3m,3m+\alpha}$ for $\alpha = 1, 2, \ldots$ which we use to compute the 2 expectations at $T_1 = 3m$.

![Figure 5.1: USD Swap Curve on 05/01/2017](image)

Figure 5.1: USD Swap Curve on 05/01/2017
Figure 5.2: Monte Carlo simulation for USD data of $S_{t}^{3m,3m+5Y}$ for $t \in [0, 3m]$ (2\textsuperscript{11} paths simulated)

Figure (5.2) show the result of the Monte-Carlo simulation from $t = 0$ to $t = T_1 = 3m$ where we used the swap rate dynamics equation (3.1.5). The results of this simulation are then implemented to the volatility function $g^{i,j}$ as part of the SMM algorithm.

5.1.2 Convergence and Confidence Intervals

As seen in the previous section, the Euler scheme is repeated for a different number of times in order to compute the expectation terms in the volatility function. The question that arises then is whether the number of simulations impacts the end result or not.

Logically, a bigger number of simulations would eliminate any statistical variants and provide a stable outcome. But the number of simulations has a big influence on the time of execution (complexity of the algorithm) and the data space used in the algorithm. So the question here is of compromise between number of simulations and the algorithm’s complexity.

5.1.2.1 The mean swap rate

To test out the convergence of the SMM algorithm, we can first start by studying the convergence of the swap rate simulated through the Monte-Carlo simulation.
According to the Central Limit Theorem, the convergence rate of a Monte-Carlo method is \( \frac{1}{\sqrt{n_{sim}}} \), \( n_{sim} \) being the number of simulated paths, a convergence that is considered slow [26]. Moreover, the approximation error is random and may take large values even if \( n_{sim} \) is large. Therefore it is important to set a certain accuracy level for the calibration algorithm.

To do so, we simulate for example the USD forward swap rate \( S_{t}^{1Y^{10Y}} \) from \( t = 0 \) to \( t = 1Y \) based on the Monte-Carlo method with \( n_{sim} \) paths raging from 100 to 10000. We then compute the mean value of \( S_{t}^{1Y^{10Y}} \) corresponding to a number of simulations \( n_{sim} \) and plot figure (5.3).

![Figure 5.3: Convergence test for the mean of \( S_{t}^{1Y^{10Y}} \) as a function of the number of simulation paths](image)

We can see that the mean value of \( S_{t}^{1Y^{10Y}} \) converges to a value around 1.89\%, which can be seen as the mean value of the different mean values computed based on a different number of simulation paths.

Next, we plot the absolute error corresponding to the same simulation, figure (5.4). The absolute error has an order of magnitude of \( 10^{-4} \). Using statistical test like the p-value and the Anderson–Darling test, we prove that the error series is normally distributed. And by computing the expectation of the error term we can prove that the Monte-carlo estimation of mean \( S_{t}^{1Y^{10Y}} \) is unbiased [26].
Finally, using the fact that for a number of simulated paths $n_{sim}$, the mean $\mu_{sim}$ of $S_{1Y}^{1Y_10Y}$ is distributed as follows:

$$\mu_{sim} + \frac{\sigma}{\sqrt{n_{sim}}} Z,$$

where $\sigma$ is the standard deviation of the series of mean values and $Z$ is a standard normal distribution. By estimating $\sigma$ from the series of mean values, we can compute the 95% confidence intervals with represents a 95% confidence that the true mean of $S_{1Y}^{1Y_10Y}$ will fall inside the corresponding interval. The resulting plot is figure (5.5).
5.1.2.2 The local stochastic volatility

The formula of the local stochastic volatility in (3.2.8) is a complex function of the simulated swap rate. This means that it isn’t enough to show the convergence of the mean swap rate alone but the convergence of the local stochastic volatility itself is necessary. Especially with the presence of terms like the dirac mass in the formula, which can be unstable numerically.

Using the same USD data, we compute for example the volatility values $g^{6m5Y}(K = 3\%)$. The number of simulation paths is the only variant from one test to the other.

Figure 5.5: Convergence test, $S_{1Y}^{10Y}$, with the 95% confidence interval
Figure 5.6: Convergence test for the model output $g_{6m5Y}(K = 3\%)$ as a function of the number of simulation paths.

Figure (5.6) shows the result of the simulation for a different number of simulation paths varying from 50 to 10000. We can see that the value of the model output $g_{6m5Y}(K = 3\%)$ converges to a value around 0.01778 after the first few values of simulation paths. In order to prove this, we first plot the absolute error plot which we define as the absolute difference between $g_{6m5Y}(K = 3\%)$ for a certain simulation path minus the average value of $g_{6m5Y}(K = 3\%)$.

Figure 5.7: The absolute error of the convergence test for $g_{6m5Y}(K = 3\%)$.

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From the figure (5.7), we can notice that the absolute error has an order of magnitude of $10^{-4}$. In order to prove that the error follows a normal distribution, we perform a statistical test. Using the p-value and the Anderson–Darling test we prove that the error series is normally distributed.

![Figure 5.8: Convergence test, $g^{6m5Y} (K = 3\%)$, with the 95% confidence interval](image)

Finally, we can draw the 95% confidence intervals (CI) on the same series. The results are presented in figure (5.8). We also added to the same plot the mean value of the model output $g^{6m5Y} (K = 3\%)$.

## 5.2 USD Simulation

### 5.2.1 Model parameters

The first numerical experiment to test out the SMM algorithm is based on the USD currency. We set out to compute the local stochastic volatility surfaces for at different expiry dates and different tenors.

As explained before, the main initial inputs in the SMM algorithm are the swap rate curve at $t=0$ and the implied normal swaption volatility at $t=0$ for the different expiry dates and at the key tenors.

The first data is easy to access and is available for for an array of starting dates and maturities (in our thesis, the Inter Continental Exchange is the source of our swap curves).
The swaption volatility however is difficult to access, since the swaption market’s predominant participants are large corporations, banks, financial institutions and hedge funds. It was therefore difficult to acquire all the swaption data we need even with access to financial data terminals such as Bloomberg or Reuters. Specific subscriptions are instead needed to access swaption data. The CME-group offers implied normal swaption volatility date for at-the-money strikes but not values for other strikes.

For the USD simulation, instead of using market swaption data, we used a volatility smile based on the SABR parameters which matches the real market smile [9]. The parameters are as follow:

<table>
<thead>
<tr>
<th>Expiry</th>
<th>Tenor</th>
<th>$f$</th>
<th>$\alpha$</th>
<th>$v$</th>
<th>$\rho$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Y</td>
<td>5Y</td>
<td>0.0217</td>
<td>0.0281</td>
<td>0.045</td>
<td>-0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>1Y</td>
<td>10Y</td>
<td>0.0242</td>
<td>0.0243</td>
<td>0.035</td>
<td>-0.15</td>
<td>0.3</td>
</tr>
<tr>
<td>5Y</td>
<td>5Y</td>
<td>0.0266</td>
<td>0.0252</td>
<td>0.045</td>
<td>-0.15</td>
<td>0.3</td>
</tr>
<tr>
<td>5Y</td>
<td>10Y</td>
<td>0.0275</td>
<td>0.0229</td>
<td>0.035</td>
<td>-0.1</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Table 5.1: SABR parameters for the USD simulation

Since the SABR smile parameters are available for the tenors 5Y and 10Y, they will be the key tenors in the SMM algorithm. The normal volatility values for the other tenors are computed using interpolation and extrapolation. The plot for USD SABR volatility surface at the 1Y expiry and the key tenors 5Y and 10Y is presented in figure (5.9).
The swap rate curve for the USD simulation is the same as the previous section, figure (5.1). The aim is to compute the local stochastic volatility surface at the expiry 1Y and key tenors 5Y and 10Y. We chose a constant accrual factor $\delta = 3m$. We therefore compute the volatility function at times $T_i = 3m, 6m, 9m$ and 1Y. The time step $N$ between the $T_i$s is 90 and the number of simulation paths for the Monte Carlo simulation is $2^{11}$. The volatility interpolation was carried out using the $f_1$ linear method (section 4.3). The Swap-Swap rate correlation matrix is the one computed in section 4.1, based on historical swap rates data and the exponential parametrization with decay control. The stochastic scaling factors parameters chosen for the simulation are: A Swap-Volatility correlation:

$$
\rho^{SV} = \langle dW^{\alpha,Q}, dZ^{Q} \rangle_t / \langle dt = 0.2. 
$$

The value is based on historical swap rate data (section 4.2). To compute the value of $v$, we use the least squares method to match the algorithm’s output with the target smiles. The results for a simulation based on the USD data are presented in figure (5.10). In the figure, we plot the value of the local stochastic volatility at expiry 1Y and tenor 5Y for different values of the volvol $v$. We then compute the least squares error corresponding to that $v$ based on the target value from the SABR smile. The figure shows that the optimal value of $v$ is 0.2.
5.2.2 Local stochastic volatility surfaces

After fixing all the simulation parameters, we can launch the calibration algorithm and plot the local stochastic volatility surface at different expiry dates, figure (5.11).

For each plot, the normal volatility is plotted as a certain expiry date $T_i$ and for the tenors $1Y, 2Y, ..., 10Y$. The plot is also done for a range of strikes: $K$ (bps) = $-100$, $-50$, $-25$, 0 (ATM), 25, 50, 100, 150.

The first thing we notice in the four different plots is the shape of the volatility smile. The case here is a volatility skew instead of a smile which is reminiscent of the initial data of SABR swaption volatility which also was a skew. The skew in volatility represents the difference in volatility between out-of-the-money options, at-the-money options, and in-the-money options. It is inherent to the sentiment of the market participants and their preference to call or put options. The SMM algorithm therefore passes an important test by having a volatility output that isn’t constant for the range of different strikes.
For the different expiry dates, we note that the value of the normal local stochastic volatility decreases when the tenor increases. The decreases is more pronounced for higher expiry dates (1Y for examples). This something that can also be related to the market views since the far forward rates are generally less volatile than near rates, we would expect the swaption volatility to decline with increasing tenor. This is the case for all expiry dates.

The values of the normal volatility plotted, range between 0.002 and 0.02. Since we are dealing with normal volatility, the values are in an acceptable range. Roughly, they are equivalent to a log normal volatility spread between
5% and 40%, which corresponds to a normal level of volatility observed in the interest rates markets. In the next section, we will be able to look in details at the accuracy of the local stochastic volatility values obtained.

5.2.3 Results

To test out the results of the SMM algorithm, the idea was to use real market swaption volatility data as input, compute the local stochastic volatility with the SMM algorithm and then compare the results with the SABR volatility data which represents the market’s standard of computing swaption volatility surfaces.

We chose to test out two different outputs of the SMM algorithm: the local stochastic volatility output which corresponds to the volatility surfaces presented in the previous section; and the local volatility output, where the stochastic scaling factor was not incorporated in the computations of the algorithm. The aim is to see whether adding the stochastic component of the algorithm has any big affect on the calibration quality of the algorithm.

Figure (5.12) shows the plots of the local stochastic volatility surface and the local volatility surface based on USD data at the expiry date 1Y and the key tenors 5Y and 10Y, plotted for the same range of strikes $K$ and Tenors 1Y, ..., 10Y.

(a) Local stochastic Volatility  
(b) Local Volatility

Figure 5.12: Local stochastic volatility and Local volatility surfaces comparison for USD data at the 1Y Expiry, key tenors 5Y and 10Y
The two plots have roughly the same skew shape for the different tenors. The values of the normal volatility are also in the same range for both the local stochastic volatility surface and the local volatility surface. The main difference is that the local volatility surface plot (b) seems to be smoother than the plot (a). This can be attributed to the component \( v \) in the stochastic scaling factor, which as explained before can control the shape of the surface to match that of the target model.

In both simulation the key tenors chosen are 5Y and 10Y. This means that the volatility function is actually computed on those tenors and calibrated using market data. The rest of the tenors are deduced using interpolation and extrapolation methods. Therefore, to test out the SMM algorithm output against the SABR model, the comparison will be done solely on the key tenors since that’s where we compute the volatility function and that’s where our SABR data is defined.

Figure (5.13) show the plots of the local stochastic volatility skew, the local volatility skew, and the SABR model skew for USD data at the 1Y expiry date and the tenors 5Y and 10Y. We can see from the figure that the three plots have the same shape and value range across the different strikes and for both tenors. This means that the SMM algorithm with stochastic scaling or without manages not only to capture the shape of the volatility skew but also the values.
We also notice that the simulated values of volatility are sometimes above the market value and sometimes below it. This kinda of behavior is noticed with other local or stochastic volatility methods [20].

Table (5.2) shows the results of our USD simulation where at expiry date 1Y and the tenors 5Y and 10Y and for different strikes, we compute:

- The LSV error: the difference between the local stochastic volatility representing the calibrated output of the SMM algorithm, and the target volatility based on the SABR model.

- The LV error: the difference between the local volatility representing the calibrated output of the SMM algorithm without the stochastic scaling factor, and the target volatility based on the SABR model.

<table>
<thead>
<tr>
<th>Currency</th>
<th>Expiry</th>
<th>Tenor</th>
<th>Strike (%)</th>
<th>LSV error (bps)</th>
<th>LV error (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>USD</td>
<td>1Y</td>
<td>5Y</td>
<td>0.61</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>5Y</td>
<td>1.44</td>
<td>0.78</td>
<td>0.82</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>5Y</td>
<td>1.63</td>
<td>0.3</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>5Y</td>
<td>1.83</td>
<td>0.2</td>
<td>-0.1</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>5Y</td>
<td>2.19</td>
<td>-0.91</td>
<td>-0.95</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>10Y</td>
<td>0.77</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>10Y</td>
<td>1.33</td>
<td>0.18</td>
<td>-0.1</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>10Y</td>
<td>1.48</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>10Y</td>
<td>1.67</td>
<td>-0.1</td>
<td>1.12</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>10Y</td>
<td>2.14</td>
<td>0.14</td>
<td>1.45</td>
</tr>
</tbody>
</table>

Table 5.2: USD simulation results: LSV and LV errors at expiry date 1Y, tenors 5Y and 10Y, over a range of strikes
From the table, we can see that the error term is generally small. From the different strikes the LSV and LV errors stay around a value of 1 bp. We can also notice that the stochastic volatility component does not affect the overall calibration quality in a big way, although the calibration at a few low and high strike values was slightly more affected in some cases.

This numerical experiments therefore illustrates the high accuracy of the swaption smile calibration using the SMM algorithm.

5.3 AUD simulation

5.3.1 Model parameters

Aside from the USD swaption data that we used in the USD simulation in the last section, we also had access to implied volatility data for a second currency: the AUD. The data is courtesy of the quandl website [27]. For this new set of data, we again set out to compute the local stochastic volatility surfaces for at different expiry dates and different tenors.

The data we have for the implied normal swaption volatility is market data and is available for multiple expiry dates especially on the short term. Therefore in this simulation we will test out the LSV and LV error not only for different tenors and strikes but also at different expiry dates.

Figure (5.14), shows the market implied volatility surfaces for AUD data at different Expiry dates, key tenors 5Y and 10Y. We notice that the implied volatility surface is less flat than the USD case. The shape also varies a lot from an expiry to the next. The 6m implied volatility surface has a skew shape while the 1Y and 2Y implied volatility surface have more of a smile shape.

The other input data to change for this simulation is the swap curve. The volatility function is now computed at expiry date $T_i = 3m, 6m, ..., 2Y$. The computations are done over a range of strikes: $K (bps) = -150, -100, -50, -25, 0$ (ATM), $25, 50, 100, 150$. The rest of the parameters of the SMM algorithm are left the same as the USD simulation.
Figure 5.14: Market implied volatility surfaces for AUD data at different Expiry dates, key tenors 5Y and 10Y
5.3.2 Local stochastic volatility surfaces

Figure (5.15), shows the result of the SMM algorithm applied to the AUD data for three different expiration dates: 6m, 1Y and 2Y. The plot are done over a tenor range 1Y, ..., 10Y with 5Y and 10Y being the key tenors.

First, we notice that the shape of the local stochastic volatility surfaces vary with expiry dates. As we noticed for the market implied volatility surface, the 6m local stochastic volatility surface has the shape of a skew where the volatility at the lower strikes is higher than the volatility at higher strikes. This volatility shape reflects that that the in-the-money calls and out-of-the-money puts are more attractive product to financial participants compared to out-of-the-money calls and in-the-money puts. The 1Y and 2Y surfaces however have a smile shape. the volatility smiles tell us that demand is greater for options that are in-the-money or out-of-the-money.

From the plots, we also notice that for the different expiry dates, the value of the normal local stochastic volatility decreases when the tenor increases. This time we also notice that the range of values of the local stochastic volatility at expiry 2Y is lower than that of the local stochastic volatility at expiry 1Y. This doesn’t come as a surprise since given that far forward rates are generally less volatile than near rates, and that long rates are also less volatile than short rates, the swaption volatility is expected to decline with both increasing tenor and increasing expiry date.

Finally we can also notice that the values of the normal volatility plotted, range between 0.002 and 0.025. As explained for the USD simulation, these values represents acceptable normal volatility values for the swaption market. In the next section, we will be able to look in details at the accuracy of the local stochastic volatility values obtained at different expiry date and tenors.
Figure 5.15: Local stochastic volatility surface for AUD data at different Expiry dates, key tenors 5Y and 10Y
5.3.3 Results

In the same manner as the USD simulation, we will test out the results of the SMM algorithm, by computing the local stochastic volatility with the SMM algorithm and then comparing the results with the market implied volatility.

Here again, we chose to test out two different outputs of the SMM calibration algorithm: the local stochastic volatility output and the local volatility output. The aim is to see whether adding the stochastic component of the algorithm has any big affect on the calibration quality of the algorithm.

Figure (5.16), show the plots of the local stochastic volatility surface and the local volatility surface based on AUD data at the expiry date 2Y and the key tenors 5Y and 10Y, plotted for the same range of strikes K and Tenors 1Y, ..., 10Y.

![Figure 5.16: Local stochastic volatility and Local volatility surfaces comparison for AUD data at the 2Y Expiry, key tenors 5Y and 10Y](image)

(a) Local stochastic Volatility

(b) Local Volatility

The two plots have a similar smile shape for the different tenors. The values of the normal volatility are also in the same range for both the local stochastic volatility surface and the local volatility surface. The main difference again is that the local volatility surface plot (b) seems to be smoother than the plot (a). This can be attributed to the component $v$ in the stochastic scaling factor, which as explained before can control the shape of the surface to match that of the target model.
Next, for the key tenors 5Y and 10Y, we draw the plots of the local stochastic volatility smile, the local volatility smile, and the market’s implied volatility for the AUD data at the expiry date 2Y, figure (5.17).

![Graphs](image)

(a) Tenor 5Y  
(b) Tenor 10Y

Figure 5.17: Comparison between the market smile and the SMM output for AUD data at the 2Y Expiry

We notice that both the shapes of the local stochastic volatility and the local volatility smiles mimic that of the market’s implied volatility. This shows that means that the SMM algorithm with stochastic scaling and without manages to capture the shape of the volatility smile and its values.

In table (5.3), we compute the LSV and LV errors where the target model this time is the market’s implied volatility. The computations are done at expiry dates 6m, 1Y and 2Y, at the tenors 5Y and 10Y and for different strikes.

Once again we notice that the different error terms in the table are all generally small (still around 1 bp). We also notice that also notice the low affect of adding the stochastic volatility component to the overall calibration quality of the SMM algorithm (minus a few low and high strike values that were slightly more affected).

Consequently, the second simulation using the AUD data confirms the high accuracy of the calibration of the swaption smile by the SMM algorithm.
<table>
<thead>
<tr>
<th>Currency</th>
<th>Expiry</th>
<th>Tenor</th>
<th>Strike (%)</th>
<th>LSV error (bps)</th>
<th>LV error (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AUD</td>
<td>6m</td>
<td>5Y</td>
<td>1.50</td>
<td>0.05</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>6m</td>
<td>5Y</td>
<td>2</td>
<td>1</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>6m</td>
<td>5Y</td>
<td>3.25</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>6m</td>
<td>10Y</td>
<td>2.50</td>
<td>-0.1</td>
<td>-0.2</td>
</tr>
<tr>
<td></td>
<td>6m</td>
<td>10Y</td>
<td>2.84</td>
<td>1.1</td>
<td>1.1</td>
</tr>
<tr>
<td></td>
<td>6m</td>
<td>10Y</td>
<td>3.30</td>
<td>-0.4</td>
<td>-0.5</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>5Y</td>
<td>2.65</td>
<td>1</td>
<td>1.1</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>5Y</td>
<td>3.60</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>5Y</td>
<td>4.25</td>
<td>0.3</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>10Y</td>
<td>2.75</td>
<td>0.8</td>
<td>0.85</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>10Y</td>
<td>3.40</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>1Y</td>
<td>10Y</td>
<td>4.35</td>
<td>0</td>
<td>-0.1</td>
</tr>
<tr>
<td></td>
<td>2Y</td>
<td>5Y</td>
<td>2.45</td>
<td>1.2</td>
<td>1.23</td>
</tr>
<tr>
<td></td>
<td>2Y</td>
<td>5Y</td>
<td>3.30</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>2Y</td>
<td>5Y</td>
<td>4.15</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>2Y</td>
<td>10Y</td>
<td>2.50</td>
<td>0.6</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>2Y</td>
<td>10Y</td>
<td>3.85</td>
<td>-0.1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2Y</td>
<td>10Y</td>
<td>4.38</td>
<td>0.2</td>
<td>0.32</td>
</tr>
</tbody>
</table>

Table 5.3: AUD simulation results: LSV and LV errors at expiry dates 6m, 1Y and 2Y, tenors 5Y and 10Y, over a range of strikes
5.4 Bermudan Swaptions pricing

Since the SMM model with local stochastic volatility is based on the Monte-Carlo simulation for its calibration, we can use the simulate forward swap rates and volatility surfaces to price exotic options where the underlying is the swap rate.

As discussed in section (2.3.3), for a Bermudan swaption with expiry date $T_i$, a tenor $\alpha$ and strike $K$. If $\Delta$ a set of early exercise dates : $\tau \in [0, T_i]$, then the pricing problem of the Bermudan swaption at time $t = 0$, becomes the calculation of:

$$\Pi(0, X) = \sup_{\tau \in \Delta} \left[ A_0^{\tau, \tau+\alpha} E^{A^\tau, \tau+\alpha} \left[ (S_{\tau}^{\tau, \tau+\alpha} - K)^+ \right] \right].$$  

(5.4.1)

For a numerical experiment, we will try to price the following swaptions, table (5.4), using the SMM algorithm to simulate the forward swap rate based on the USD data.

<table>
<thead>
<tr>
<th>Option Style</th>
<th>Strike</th>
<th>Expiry date</th>
<th>Tenor</th>
<th>Exercise dates</th>
</tr>
</thead>
<tbody>
<tr>
<td>European Swaption</td>
<td>2%</td>
<td>2Y</td>
<td>5Y</td>
<td>2Y</td>
</tr>
<tr>
<td>Bermudan Swaption I</td>
<td>2%</td>
<td>2Y</td>
<td>5Y</td>
<td>1Y, 2Y</td>
</tr>
<tr>
<td>Bermudan Swaption II</td>
<td>2%</td>
<td>2Y</td>
<td>5Y</td>
<td>6m, 1Y, 1Y6m, 2Y</td>
</tr>
</tbody>
</table>

Table 5.4: European and Bermudan Swaption contracts to price, currency USD

Table (5.5) shows the discounted payoffs at different exercise dates based on the SMM algorithm simulation with USD data and the use of the Monte-Carlo simulation to compute the local stochastic volatility and the forward swap rates:
<table>
<thead>
<tr>
<th>Exercise date</th>
<th>6m</th>
<th>1Y</th>
<th>1Y6m</th>
<th>2Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discounted payoff</td>
<td>0.02321</td>
<td>0.02324</td>
<td>0.03572</td>
<td>0.03275</td>
</tr>
</tbody>
</table>

Table 5.5: Discounted payoffs at different exercise date for the USD simulation

It is therefore possible to compute the prices of the swaption contracts defined in Table 5.4, based on the simulated values as well as price the European swaption based on the Black model:

<table>
<thead>
<tr>
<th>Swaption Contract</th>
<th>European Black’s model pricing</th>
<th>European Monte-Carlo pricing</th>
<th>Bermudan I</th>
<th>Bermudan II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>0.03272</td>
<td>0.03275</td>
<td>0.03275</td>
<td>0.03572</td>
</tr>
</tbody>
</table>

Table 5.6: Pricing of European and Bermudan swaptions for USD data

In Table (5.6), we first compute the price of the European swaption 2Y5Y based on the formula under Black’s model and the method based on the Monte-Carlo simulation. We notice that the price for both methods is very close. For the Bermudan swaption I, the discounted payoff at the early exercise date 1Y was lower than the payoff if the option is kept until its expiry and therefore has the same price at $t = 0$. The Bermudan swaption II price however, is higher than the other swaptions and has an optimal early exercise date at the 1Y6m mark.
Chapter 6
Discussion and Conclusion

In this chapter, we present a summary of the thesis and a compilation of its findings. First, we discuss the different results presented in the previous chapter through the different simulations and the overall performance of the swap market model and the calibration of the local stochastic volatility. The next section of the chapter is a summary of the findings of the thesis. And finally, we present different possible extensions to the model and calibration algorithm, as well as possible future work to expand the thesis.

6.1 Discussion

Looking at the results presented in chapter 5, the calibration SMM algorithm based on the swap market model with local stochastic volatility, provides a high accuracy for volatility smile building.

The convergence of the algorithm has been tested on two levels. First, since the forward swap rates are simulated using the Monte-Carlo method, the convergence of mean of the simulated swap rates is studied and the 95% confidence intervals can be interfered based on the distribution of the mean swap rates series for high number of simulation paths. Second, the local stochastic volatility formula being a complex function of the simulated forward swap rates, the convergence of the SMM algorithm is also proven for the volatility function as well. The 95% confidence intervals are also computed in this case.

Next, we moved on to building the local stochastic volatility surfaces at
different expiry dates and for the key tenors 5Y and 10Y. For the USD simulation, the swaption implied normal volatility was substituted by a volatility surface generated through the SABR model due to hard access to USD implied normal volatility. The results of the simulation show that the model generated surfaces match the shape of the SABR surface and well and its values. A closer look at the volatility skew at expiry 1Y and for the tenors 5Y and 10Y, compares the SABR model values to the local stochastic volatility values generated by the SMM algorithm as well as local volatility values generated by the same algorithm without the stochastic volatility scaling factor. The results show a negligible difference between the two model outputs and the target SABR model. The error is roughly around 1 bp, and takes both negative and positive values. It is also worth noting that the local volatility output performed with a level of accuracy almost as high as the local stochastic volatility output, except for a few a few low or high strike values.

The simulation of AUD data proceeds in a similar way. The calibration quality of the SMM algorithm is tested at different expiry dates this time around: at 6m, 1Y and 2Y, with key tenors 5Y and 10Y and compared to the market’s implied normal volatility data. Once again the shape of the local stochastic volatility smile generated by the SMM algorithm matches the initial market data as well as its values. By plotting the volatility smiles for a range of strike values at the different expiry dates and for the key tenors, we can once again test the model’s outputs of the local stochastic volatility values and local volatility (with no stochastic scaling factor) against the market’s implied volatility smiles. The error in this case is also very small and stays around a level of magnitude of 1 bp. We also notice that the LSV and LV errors (definition of the errors in section 5.2.3) are highest for deep out-of-money strikes. It is hard to tell if this is inherent to the calibration algorithm itself or due the fact that deep out-of-money swaptions are a less liquid derivative than at-the-money options, but it can be clarified by running the SMM algorithm on swaption market data spanning multiple days and at different expiry dates. Another important observation from the AUD data simulation is that once again adding the stochastic scaling factor to the SMM algorithm had little influence on the overall calibration quality of the model except in a few cases at very high or low strike values, in particular the deep out-of-money strikes.
6.2 Conclusion

In this thesis, the goal was to build a swap market model with non-parametric local volatility functions and stochastic volatility scaling factors. To overcome the difficulty of using local volatility parametrisation in interest rate modeling, the spot starting swaps were chosen as key modeling component of the swap market model. Using the Girsanov theorem and by computing the time derivative of the pricing formula of a swaption on a spot starting swap, a local volatility function can be derived from the dynamics of the SMM model. The volatility function can be computed at different expiry dates, different tenors and a different strikes. The resulting volatility surfaces can be used in pricing exotics interest rates derivatives such as Bermudan swaptions, and to simulate forward swap rates or even Libor rates with the appropriate transformation.

The SMM model is calibrated through a particle algorithm which makes use of the swaption market smile in an efficient way. Through simulations based on swaption implied normal volatility of different currencies, the stochastic local volatility surfaces are computed at different expiry dates. A study of the error between the model output and a target model shows the high accuracy of the calibration. Results are noticed to be of better quality near the at-the-money strike value and the effect of the stochastic scaling factor is proven more significant for a few cases of deep in-the-money and deep out-of-money strikes.

6.3 Further Work

In this final section, we present a few different ways to consider moving forward in the hope of obtaining a higher quality calibration and better capture of the market volatility smile. We will also discuss possible extensions of the swap market model framework outside of smile building.

To improve the quality of the calibration in the SMM model several leads can be explored. First, we can add an intermediate point in the time grid of expiry dates: \[ T_{i-\frac{1}{2}} = \frac{1}{2}(T_{i-1} - T_i) \] for every time evolution from \( T_{i-1} \) to \( T_i \), as well as increase the number of simulation paths in the Monte-Carlo method. It is also interesting to carry out the calibration with the control variate of equation (3.2.9) and test out how the error behaves against the
model without the control variate. The interpolation methods for volatility between key tenors can also be a point of improvement. There are several new and sophisticated interpolation methods than can be carried out and tested to improve the overall quality of calibration.

A big limitation of this thesis was the access to market implied normal volatility. With more market data, we can carry out more simulations and test out the calibration algorithm over different periods where swap rates are more volatile or where interest rates are at a higher or lower level. These tests can prove the resilience of the calibration algorithm and how it adapts to changes in the financial markets. Moreover, in section 2.4.2, we made the decision of using the swap rates to compute the discounting factors as opposed to the OIS discounting. Since the swaptions on the market are priced using the new market standard of OIS discounting, implementing it to the SMM algorithm should improve the calibration quality.

As seen briefly in the last section of chapter 5, the local stochastic volatility computed with the SMM algorithm can be used to price exotic options like the Bermudan swaptions. In the interest rates market, American swaptions are also a liquid traded derivatives. Their pricing can be quite complex but feasible using the local stochastic volatility; a point that can be explored as an extension of the work done in this thesis. Lastly, risk management in the framework of a swap market model can also constitute the next step in expanding this thesis. As mentioned earlier, interest rate swaps and swaptions are frequently used instruments to hedge against interest rate risk. Using the local stochastic volatility, explicit formulas for the Greeks (also known as risk sensitivities or hedge parameters) can be computed and hedging strategies based on their dynamics can be established.
Bibliography


[16] https://www.cfr.org/backgrounder/understanding-libor-scandal


[27] https://www.quandl.com/databases/CSWO/documentation


Appendices

Appendix A

Stochastic calculus

A.1 The Itô formula

The Itô formula is an identity used in the Itô calculus to find the differential of a time-dependent function of a stochastic process.

For a process $X$ with a stochastic differential given by:

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

with $\mu$ and $\sigma$ as two adapted processes.

If $f$ is a $C^{1,2}$ function and $Z$ is a process defined as: $Z(t) = f(t, X(t))$, then $Z$ has the stochastic differential equation defined by:

$$df(t, X(t)) = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dW(t).$$

The above formula is derived from the Taylor expansion.

Using the dynamics of the process $X$ and the following set of formal multiplication table:

$$\begin{cases} (dt)^2 = 0 \\ dt.dW = 0 \\ (dW)^2 = dt. \end{cases}$$

Then $df$ is given by:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2$$
A.2 Equivalent Martingale Measures

The concept of equivalent martingale measures (EMM) is key notion in the pricing of financial derivatives. An EMM represents a probability distribution of the possible expected payouts of an investment adjusted to the investor’s degree of risk aversion. In an efficient market, the computed present value is equal to the current traded price of the security.

According to the Radon-Nikodym theorem, if an EMM $Q \sim P$ exists, then the two probability measures $P$ and $Q$ are linked by a random variable $D$ such that $E^P[D] < \infty$ and :

$$Q(A) = E^P[\mathbb{1}_AD] \quad \text{or} \quad D = \frac{dQ}{dP}$$

The variable $D$ is referred to as the Radon-Nikodym derivative of $Q$ with respect to $P$.

From a risk measure perspective, the derivative $D$ is also defined by the following equation :

$$\frac{dQ}{dP} \bigg|_{F(t)} = \mathcal{E}(-\gamma(t).W(t)).$$

$\mathcal{E}$ is the Doléans-Dade exponential or the stochastic exponential defined as :

$$\mathcal{E}(X(t)) = \exp \left\{ X(t) - \frac{1}{2} < X(t) > \right\}$$

and $\gamma$ is the market price of risk and serves as a link between the two probability spaces.

A.3 The Girsanov theorem

Let $W^P(t)$ a $P$-Brownian motion. For an adapted stochastic process $X(t)$, set :

$$D(t) = \exp \left\{ \int_0^t X(s)dW^P(s) - \frac{1}{2} \int_0^t X(s)^2 ds \right\}$$

Under the assumption that $D(t)$ is a martingale, we have :

$$E^P[D(t)] = 1 \quad \text{and} \quad dD(t) = D(t)X(t)dW^P(t)$$
We define a new measure: \( dQ = D(t)dP \). Then, the Girsanov theorem states that under \( F(t) \), the equation for \( dW^P(t) \) changes under \( Q \) to:

\[
dW^P(t) = dW^Q(t) + X(t)dt
\]

The Girsanov theorem therefore describes how the dynamics of a stochastic process changes when moving from one probability measure to another equivalent probability measure.

**A.4 Geometric Brownian motions**

Geometric Brownian motions (GBM) are one of the fundamental blocks of modeling the price process of financial assets. A GBM is a continuous-time stochastic process with a drift and a stochastic term with a Brownian motion. A Geometric Brownian motion \( X(t) \) satisfies the following stochastic differential equation (SDE):

\[
dX(t) = \alpha X(t)dt + \sigma X(t)dW(t),
\]

\[
X(0) = x_0.
\]

Where \( \mu \) and \( \sigma \) are constants. And \( W(t) \) is a standard Brownian motion. Based on the Itô formula, we can find an analytical solution to the SDE above.

Consider the stochastic process \( Y(t) = \ln(X(t)) \), The Itô formula applied on the process \( Y(t) \) gives:

\[
Y(t) = \frac{dX(t)}{X(t)} - \frac{1}{2} \frac{d< X(t) >}{X^2(t)}
\]

\[
= \frac{1}{X(t)}(\alpha X(t)dt + \sigma X(t)dW(t)) - \frac{1}{2X^2(t)}\sigma^2 X^2(t)dt
\]

\[
= (\alpha - \frac{\sigma^2}{2})dt + \sigma dW(t).
\]

With \( Y(0) = \ln(x_0) \).

This an easy equation to integrate:

\[
Y(t) = \ln(x_0) + (\alpha - \frac{\sigma^2}{2})t + \sigma W(t)
\]
Which means that $X(t)$ is given by:

$$X(t) = x_0 \exp \left\{ (\alpha - \frac{\sigma^2}{2})t + \sigma W(t) \right\}$$

For the log normal dynamics of the stochastic scaling factor introduced in section (4.2):

$$\begin{cases} dc_t = v_c c_t dZ^Q_t \\ c_0 = 1 \end{cases}$$

And the Swap-Volatility correlation:

$$<dW^\alpha, dZ^Q>_t = \rho^{SV} dt.$$  

With $\rho^{SV}$ and $v$ are constant.

The expression for $c_t$ is:

$$c_t = c_0 \exp \left\{ -\frac{v^2}{2} t + v Z^Q(t) \right\}$$

The correlated process $Z^Q(t)$ can be rewritten as:

$$Z^Q(t) = \rho^{SV} X^Q_1(t) + \sqrt{1 - (\rho^{SV})^2} X^Q_2(t),$$

where $X^Q_1$ and $X^Q_2$ are two uncorrelated standard Brownian motions corresponding to the Brownian motions in the swap rate dynamics and the stochastic volatility dynamics.
Appendix B

The drift term in the swap market model

First, we prove that:

\[ v_{i,j}^t = \frac{A_{i,j}^t}{P_t^i}, \quad i < j \]

By induction:

For \( j = i+1 \):

\[ v_{i,i+1}^t = v_{i,i+1}^{i,i+1} = \frac{\delta_{i+1}}{1 + \delta_{t+1}S_t^{i+1}} = \frac{\delta_{i+1}P_t^{i+1}}{P_t^i} = \frac{A_{i,i+1}^t}{P_t^i} \]

Next, we assume the equation holds for the step \( k : i < j \leq k \), then we compute \( v_{i,k+1}^t \):

\[
v_{i,k+1}^t = \sum_{l=i+1}^{k+1} \delta_l \prod_{m=l}^{k+1} (1 + \delta_m S_t^{i,m})^{-1}
\]

\[
= \sum_{l=i+1}^{k} \delta_l \prod_{m=l}^{k+1} (1 + \delta_m S_t^{i,m})^{-1} + \frac{\delta_{k+1}}{1 + \delta_{k+1}S_t^{i,k+1}}
\]

\[
= \frac{v_{i,k}^t + \delta_{k+1}}{1 + \delta_{k+1}S_t^{i,k+1}}
\]

We can compute \( \frac{A_{i,k+1}^t}{P_t^i} \):

\[
\frac{A_{i,k+1}^t}{P_t^i} = \frac{\sum_{l=i}^{k+1} \delta_l P_t^l}{P_t^i} = \frac{A_{i,k}^t}{P_t^i} + \frac{\delta_{k+1}}{P_t^i} \frac{P_t^{k+1}}{P_t^i}
\]

From the definitions of \( A_{i,k}^t \) and \( S_t^{i,k} \), we can write:

\[ P_t^{k+1} = \frac{P_t^i - S_t^{i,k+1} A_t^{i,k}}{1 + \delta_{k+1}S_t^{i,k+1}} \]
Which means:

\[
\frac{A_{t}^{i,k+1}}{P_{t}^{i}} = \frac{A_{t}^{i,k}}{P_{t}^{i}} + \frac{\delta_{k+1}}{1 + \delta_{k+1} S_{t}^{i,k+1}} \left( \frac{P_{t}^{i} - S_{t}^{i,k+1} A_{t}^{i,k}}{1 + \delta_{k+1} S_{t}^{i,k+1}} \right)
\]

\[
= \frac{A_{t}^{i,k}}{P_{t}^{i}} \left( 1 - \frac{\delta_{k+1} S_{t}^{i,k+1}}{1 + \delta_{k+1} S_{t}^{i,k+1}} \right) + \frac{\delta_{k+1}}{1 + \delta_{k+1} S_{t}^{i,k+1}}
\]

This means we proved that the equation hold for \( k+1 \). By induction we can conclude that:

\[
v_{t}^{i,j} = \frac{A_{t}^{i,j}}{P_{t}^{i}}, \quad i < j
\]

We now compute the drift term : \( \mu_{t}^{i,\alpha} \). Since we work under a no-arbitrage condition, \( dS_{t}^{i,i+\alpha} \) has a null drift under the equivalent measure associated with numeraire \( A_{t}^{i,i+\alpha} \). However, using the Girsanov theorem, the drift term under the equivalent Q measure is:

\[
\mu_{t}^{i,\alpha} = -\frac{<d\ln\left(\frac{A_{t}^{i,i+\alpha}}{P_{t}^{i}}\right), dS_{t}^{i,i+\alpha}>_{t}}{dt}
\]

\[
= -\frac{<d\ln\left(v_{t}^{i,i+\alpha}\right), dS_{t}^{i,i+\alpha}>_{t}}{dt}
\]

\[
= -\frac{1}{v_{t}^{i,i+\alpha}} \frac{<dv_{t}^{i,i+\alpha}, dS_{t}^{i,i+\alpha}>_{t}}{dt}
\]

Next, we apply the Ito formula on \( v_{t}^{i,i+\alpha} \):

\[
dv_{t}^{i,i+\alpha} = d\left( \sum_{j=i+1}^{i+\alpha} \delta_{j} \prod_{m=j}^{i+\alpha} \left( 1 + \delta_{m} S_{t}^{i,m} \right)^{-1} \right)
\]

\[
= (...)dt - \sum_{j=i+1}^{i+\alpha} \frac{\delta_{j}}{1 + \delta_{j} S_{t}^{i,j}} \sum_{j=i+1}^{i+\alpha} \delta_{j} \prod_{m=j}^{i+\alpha} \left( 1 + \delta_{m} S_{t}^{i,m} \right)^{-1} dS_{t}^{i,j}
\]

\[
= (...)dt - \sum_{j=i+1}^{i+\alpha} \frac{\delta_{j} v_{t}^{i,i+\alpha,j}}{1 + \delta_{j} S_{t}^{i,j}} dS_{t}^{i,j}
\]
Replacing this in the drift formula gives us:

\[
\mu_t^{i,\alpha} = 0 - \frac{1}{v_t^{i,i+\alpha}} \left< - \sum_{j=i+1}^{i+\alpha} \frac{\delta_j v_t^{i,i+\alpha,j}}{1 + \delta_j S_t^{i,j}} dS_t^{i,j}, dS_t^{i,i+\alpha} \right>_t \\
= \sum_{j=i+1}^{i+\alpha} \frac{\delta_j}{1 + \delta_j S_t^{i,j}} \left< \frac{v_t^{i,i+\alpha,j}}{v_t^{i,i+\alpha}}, dS_t^{i,j}, dS_t^{i,i+\alpha} \right>_t \\
= \sum_{j=i+1}^{i+\alpha} \frac{\delta_j}{1 + \delta_j S_t^{i,j}} \frac{v_t^{i,i+\alpha,j}}{v_t^{i,i+\alpha}} \rho_t^{i-i,\alpha} \sigma_t^{i,j-i} \sigma_t^{i,\alpha} \\
= \sum_{j=i+1}^{i+\alpha} \frac{\delta_j \rho_t^{i-i,\alpha} \sigma_t^{i,j-i} \sigma_t^{i,\alpha}}{1 + \delta_j S_t^{i,j}} \frac{v_t^{i,i+\alpha,j}}{v_t^{i,i+\alpha}} 
\]
Appendix C

Time derivative of the swaption pricing formula

We apply the Ito-Tanaka formula on \((S_t^{i,j} - K)^+\):

\[
(S_t^{i,j} - K)^+ - (S_0^{i,j} - K)^+ = \int_0^t \mathbb{1}_{S_u^{i,j} - K > 0} dS_u^{i,j} + \frac{1}{2} L^K_t(S_t^{i,j})
\]

\(L^K_t(S_t^{i,j})\) is the local time of \(S_t^{i,j}\) defined as:

\[
L^K_t(S_t^{i,j}) = \int_0^t \mathbb{1}_{S_u^{i,j} < K} dS_u^{i,j} = \int_0^t \mathbb{1}_{S_u^{i,j} - K} (\sigma_u^{i,j})^2 du
\]

We apply the expectation under the \(A^{i,j}\) measure on both sides of the first equation, then multiply by \(\frac{\Delta t^{i,j}}{B_0}\) to obtain:

\[
C^{i,j}(t, K) - C^{i,j}(0, K) = \frac{1}{2} A_{0}^{i,j} \frac{E^{A^{i,j}}[L^K_t(S_t^{i,j})]}{B_0} = \frac{1}{2} A_{0}^{i,j} \int_0^t E^{A^{i,j}}[\mathbb{1}_{S_u^{i,j} - K} (\sigma_u^{i,j})^2] du
\]

Then, we differentiate both side with regards to \(t\) to get:

\[
\partial_t C^{i,j}(t, K) = \frac{1}{2} A_{0}^{i,j} E^{A^{i,j}}[(\sigma_t^{i,j})^2 \mathbb{1}_{S_t^{i,j} - K}] = \frac{1}{2} E^Q\left[\frac{A_{0}^{i,j}}{B_t} (\sigma_t^{i,j})^2 \mathbb{1}_{S_t^{i,j} - K}\right]
\]