Combinatorics and zeros of multivariate polynomials

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Abstract

This thesis consists of five papers in algebraic and enumerative combinatorics. The objects at the heart of the thesis are combinatorial polynomials in one or more variables. We study their zeros, coefficients and special evaluations.

Hyperbolic polynomials may be viewed as multivariate generalizations of real-rooted polynomials in one variable. To each hyperbolic polynomial one may associate a convex cone from which a matroid can be derived - a so called hyperbolic matroid. In Paper A we prove the existence of an infinite family of non-representable hyperbolic matroids parametrized by hypergraphs. We further use special members of our family to investigate consequences to a central conjecture around hyperbolic polynomials, namely the generalized Lax conjecture. Along the way we strengthen and generalize several symmetric function inequalities in the literature, such as the Laguerre-Turán inequality and an inequality due to Jensen. In Paper B we affirm the generalized Lax conjecture for two related classes of combinatorial polynomials: multivariate matching polynomials over arbitrary graphs and multivariate independence polynomials over simplicial graphs. In Paper C we prove that the multivariate d-matching polynomial is hyperbolic for arbitrary multigraphs, in particular answering a question by Hall, Puder and Sawin. We also provide a hypergraphic generalization of a classical theorem by Heilmann and Lieb regarding the real-rootedness of the matching polynomial of a graph.

In Paper D we establish a number of equidistributions between Mahonian statistics which are given by conic combinations of vincular pattern functions of length at most three, over permutations avoiding a single classical pattern of length three.

In Paper E we find necessary and sufficient conditions for a candidate polynomial to be complemented to a cyclic sieving phenomenon (without regards to combinatorial context). We further take a geometric perspective on the phenomenon by associating a convex rational polyhedral cone which has integer lattice points in correspondence with cyclic sieving phenomena. We find the half-space description of this cone and investigate its properties.

Sammanfattning

Denna avhandling består av fem artiklar i algebraisk och enumerativ kombinatorik. Objekten som ligger till hjärtat av avhandlingen är kombinatoriska polynom i en eller flera variabler. Vi studerar deras nollställen, koefficienter och speciella evalueringar.

Hyperboliska polynom kan ses som multivariata generaliseringar av reellrootade polynom i en variabel. Till varje hyperboliskt polynom kan en konvex kon associeras från vilket en matroid kan härledas - en så kallad hyperbolisk matroid. I Artikel A bevisar vi existensen av en oändlg familj av ickerepresenterbara hyperboliska matroider som parametriseras av hypergrafer. Vidare använder vi speciella medlemmar av vår familj för att undersöka konsekvenser till en central förmodan kring hyperboliska polynom, nämligen den generaliserade Lax förmodan. Längst vägen stärker och generaliserar vi ett flertal symmetriska olikheter i literaturen så som Laguerre-Túran olikheten och en olikhet av Jensen. I Artikel B bekräftar vi den generaliserade Lax förmodan för två relaterade klasser av kombinatoriska polynom: multivariata matchningspolynom över godtyckliga grafer, samt multivariata oberoendepolynom över simpliciala grafer. I Artikel C bevisar vi att det multivariata d-matchningspolynomet är hyperboliskt för godtyckliga multigrafer vilket i synnerhet besvarar en fråga av Hall, Puder och Sawin. Vi tillhandhåller även en hypergrafisk generalisering av en klassisk sats av Heilmann och Lieb angående reell-rotenheten hos matchningspolynomet för en graf.

I Artikel D fastställer vi en rad olika ekvidistributioner mellan Mahoniska statistiker som ges av koniska kombinationer av generaliserade mönsterfunktioner av längd som mest tre, över permutationer som undviker ett enstaka klassiskt mönster av längd tre.

I Artikel E hittar vi nödvändiga och tillräckliga villkor för att ett kandidatpolynom ska kunna komplementeras till ett cykliskt sållfenomen (utan hänsyn till kombinatoriskt kontext). Vi tar dessutom ett geometrisk perspektiv på fenomenet genom att associera en konvex rationell polyhedral kon vars gitterpunkter är i korrespondens med cykliska sållfenomen. Vi finner halvrymdsbeskrivningen av denna kon och undersöker dess egenskaper.

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II Scientific papers
   Paper A
   Non-representable hyperbolic matroids.
   (joint with Petter Brändén)
   Advances in Mathematics 43 (2018) 417–449.
   Paper B
   Spectrahedrality of hyperbolicity cones of multivariate matching polynomials.
   Journal of Algebraic Combinatorics (to appear)
   https://doi.org/10.1007/s10801-018-0848-9
   Paper C
   Stable multivariate generalizations of matching polynomials.
   Journal of Combinatorial Theory, Series A (accepted subject to revisions)
   Paper D
   Equidistributions of Mahonian statistics over pattern avoiding permutations.
   Electronic Journal of Combinatorics 25, No.1 (2018) P7.
   Paper E
   The cone of cyclic sieving phenomena.
   (joint with Per Alexandersson)
   Discrete Mathematics 342, No.6 (2019) 1581–1601.

Part I

Introduction and summary

1   Overview

Polynomials have a long history in mathematics and remain relevant to almost all branches of mathematical science. In combinatorics, polynomials are an indispensable tool for studying quantitative properties associated with discrete structures. In this thesis this manifests itself in at least three different ways:

- The geometry of zeros of combinatorial polynomials

- Generating polynomials of combinatorial statistics

- Counting via evaluation of polynomials

The geometry of zeros of combinatorial polynomials

The problem of locating zeros of polynomials is almost as old as mathematics itself and includes fundamental theoretical contributions by mathematicians such as Cauchy, Fourier, Gauss, Hermite, Laguerre, Newton, Pólya, Schur and Szegö.

In combinatorics there are numerous examples of polynomials which are known to have zero sets confined to a prescribed region in the complex plane. Many of them are polynomials associated with combinatorial objects such as graphs, matroids, posets and lattice polytopes etc. For a combinatorialist the zero set of a univariate polynomial is mainly interesting due its relationship with the polynomial coefficients. This relationship is especially pronounced when the polynomial vanishes only at real points, a property which is known to imply both unimodality and log-concavity of the coefficients. Unimodality and log-concavity are properties exhibited by many important combinatorial sequences and have been the subject of much research. More recently, with breakthroughs by Borcea, Brändén and others, analogues of real-rootedness in multivariate polynomials have attracted a lot of attention. These ideas are captured in the notion of hyperbolic/stable polynomials which is fundamentally the subject of papers A, B and C in this thesis. Although hyperbolic polynomials originated in PDE-theory with the works of Gårding, Hörmander and others, they have recently found applications in diverse areas such as optimization, real algebraic geometry, computer science, probability theory and combinatorics. They were notably used by Marcus, Spielman and Srivastava in 2013 to give an affirmative answer to the longstanding Kadison-Singer problem from 1959 - a problem originally formulated in the area of operator theory but with far-reaching consequences for other areas of mathematics. Linear transformations preserving stability were fully characterized in seminal work of Borcea and Brändén, completing a century old classification program going back to Pólya and Schur. Their characterization have since been applied to a multitude of combinatorial settings as a tool for establishing stability through primarily linear differential operators.

Generating polynomials of combinatorial statistics

A combinatorial statistic may be loosely defined as a function which associates to each object in a combinatorial set a non-negative integer which is derived in some concrete way from the object. Generating polynomials are standard tools in enumerative combinatorics for reasoning about multi-dimensional arrays of combinatorial data. In essence, the coefficients of a generating polynomial represent the number of objects in the combinatorial set grouped by the statistics under consideration. Two tuples of statistics (on possibly different combinatorial objects) are said to be equidistributed if their generating polynomials have the same coefficients. Many interesting and sometimes unexpected equidistributions have been identified in combinatorics through a variety of different techniques, ranging from generating function manipulations to concrete bijective proofs. Perhaps the most well-known equidistribution is that between the inversion statistic and the major index statistic on permutations.

Pattern avoidance is an area of combinatorics which has seen considerable expansion in the last couple of decades, now even boasting a dedicated annual conference. The study of pattern avoidance in permutations was pioneered by Donald Knuth. He showed in his book *The art of computer programming Vol 1*, that a permutation is sortable by a stack if and only if it avoids the pattern 231, and moreover that these permutations are enumerated by the Catalan numbers. Since then, a main objective in the community have been to enumerate pattern classes and finding similar pattern restrictions in sorting procedures with other data structures. However the study has now expanded well beyond this endeavour.

More recently people including Claesson-Kitaev and Sagan-Savage have combined the study of combinatorial statistics with pattern avoidance in order to refine patterns classes and study statistic-preserving bijections between them. This is the context for paper D in this thesis.

Counting via evaluation of polynomials

The chromatic polynomial of a graph and the Ehrhart polynomial of a lattice polytope are examples of combinatorial polynomials which when evaluated at a natural number n count the number of n-colourings of a graph and the number of lattice points inside the nth dilation of a lattice polytope respectively. The evaluation of combinatorial polynomials at non-natural numbers may sometimes count interesting quantities too, despite there being no a priori reason for it to do so. A prime example of this so called *combinatorial reciprocity* is due to Stanley and occurs when the chromatic polynomial is evaluated at -1. By a combinatorial miracle this evaluation amounts to the number of acyclic orientations of G, a quantity which is seemingly unrelated to counting colourings. Other examples of this phenomenon occurs when counting fixed points under a cyclic action. The phenomenon is exhibited when the evaluations of a combinatorial polynomial at roots of unity coincides with the number of fixed points under a cyclic action on a combinatorial

set. This so called *cyclic sieving phenomenon* was introduced by Reiner, Stanton and White, and there are plenty of examples of it in the literature. Again there is no a priori reason why evaluating a combinatorial polynomial at roots of unity should mean anything at all. In paper E we look closer at the nature of the cyclic sieving phenomenon.

2   Background

Stable polynomials

For a subset $\Omega \subseteq \mathbb{C}^n$, a polynomial $P(\mathbf{z}) \in \mathbb{C}[z_1, \ldots, z_n]$ is called Ω-*stable* if $P(z) \neq 0$ for all $z \in \Omega$. Let $H := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, denote the open upper complex half-plane. Conventionally H^n-stable polynomials are simply referred to as *stable*. If P is a stable polynomial with only real coefficients, then P is referred to as a *real stable polynomial*. It is worth noting that real stable polynomials in one variable are precisely the real-rooted polynomials. Indeed if a real univariate polynomial is non-vanishing on H, then it must also be non-vanishing on $-H$ since its complex roots come in conjugate pairs. Therefore all roots must lie on the real line. In this sense real stability is a multivariate generalization of the notion of real-rootedness. Examples of stable polynomials occurring in combinatorics include:

- *Elementary symmetric polynomials*:

$$e_d(\mathbf{z}) := \sum_{\substack{S \subseteq [n] \\ |S|=d}} \prod_{i \in S} z_i.$$

- *Spanning tree polynomials*:

$$P_G(\mathbf{z}) := \sum_T \prod_{e \in T} z_e,$$

where the sum runs over all spanning trees T of a graph G.

- *Matching polynomials*:

$$\mu_G(\mathbf{z}) := \sum_M (-1)^{|M|} \prod_{ij \notin M} z_i z_j,$$

where the sum runs over all matchings M of a graph G.

- *Eulerian polynomials*:

$$A(\mathbf{y}, \mathbf{z}) := \sum_\sigma \prod_{i \in \mathrm{DB}(\sigma)} y_i \prod_{j \in \mathrm{AB}(\sigma)} z_j,$$

where the sum runs over all permutations σ in $\mathfrak{S}_n$ and $\mathrm{DB}(\sigma)$ (resp. $\mathrm{AB}(\sigma)$) denote the set of descent (resp. ascent) bottoms of σ.

Linear transformations preserving stability

A common technique for proving that a polynomial is stable is to realize the polynomial as the image of a known stable polynomial under a stability preserving linear transformation.

Stable polynomials satisfy a number of basic closure properties:

(i) *Permutation*: for any permutation $\sigma \in \mathfrak{S}_n$, $P(\mathbf{z}) \mapsto P(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$.

(ii) *Scaling*: for $\lambda \in \mathbb{C}$ and $\mathbf{a} \in \mathbb{R}_+^n$, $P(\mathbf{z}) \mapsto \lambda P(a_1 x_1, \ldots, a_n z_n)$.

(iii) *Diagonalization*: for $1 \leq i < j \leq n$, $f(\mathbf{z}) \mapsto f(\mathbf{z})|_{z_i = z_j}$.

(iv) *Specialization*: for $1 \leq i \leq n$ and $\zeta \in \mathbb{C}$ with $\text{Im}(\zeta) \geq 0$, $f(\mathbf{z}) \mapsto f(\mathbf{z})|_{z_i = \zeta}$.

(v) *Translation*: $f(\mathbf{z}) \mapsto f(\mathbf{z} + \mathbf{t}) \in \mathbb{C}[\mathbf{z}, \mathbf{t}]$.

(vi) *Inversion*: if $\deg_{z_i}(f) = d$, $f(\mathbf{z}) \mapsto z_i^d f(z_1, \ldots, z_{i-1}, -z_i^{-1}, z_{i+1}, \ldots, z_n)$.

(vii) *Differentiation*: for $1 \leq i \leq n$, $f(\mathbf{z}) \mapsto (\partial/\partial z_i)f(\mathbf{z})$.

Despite the elementary nature of the above facts they accomplish a fair amount. For instance, both the Newton inequalities and the Gauss-Lucas theorem are straightforward consequences of the last two facts.

It is natural to ask more generally, which linear transformations preserve stability? For real univariate polynomials this question was already considered by Pólya and Schur in [57] where they characterized diagonal operators preserving real-rootedness. However it was not until nearly a century later that Borcea and Brändén gave a complete answer to this question. They later generalized their results to the multivariate setting [11, 12], in the most general case characterizing stability preservers on Cartesian products of open circular domains (i.e. images of H under Möbius transformations). We state one version of the characterization below. The key to the characterization is an associated $2n$-variate polynomial which characterizes the stability-preserving properties of the linear transformation.

Let $\boldsymbol{\kappa} \in \mathbb{N}^n$ and let $\mathbb{C}_{\boldsymbol{\kappa}}[z_1, \ldots, z_n]$ be the space of polynomials $P \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\deg_{z_i}(P) \leq \kappa_i$ for each $1 \leq i \leq n$. Given a linear transformation $T : \mathbb{C}_{\boldsymbol{\kappa}}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$, define its *algebraic symbol* G_T by

$$G_T(\mathbf{z}, \mathbf{w}) := T\left(\prod_{j \in [n]} (x_j + w_j)^{\kappa_j}\right) \in \mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n].$$

Theorem 2.1 (Borcea-Brändén [11]). *A linear transformation* $T : \mathbb{C}_{\boldsymbol{\kappa}}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ *preserves stability if and only if either*

(i) T has range of dimension at most one and is of the form

$$T(f) = \alpha(f)P,$$

> where α is a linear functional on $\mathbb{C}_{\boldsymbol{\kappa}}[z_1, \ldots, z_n]$ and P is a stable polynomial, or

(ii) $G_T(\mathbf{z}, \mathbf{w})$ is stable.

Stable multiaffine polynomials

A polynomial $P(\mathbf{z}) \in \mathbb{C}[z_1, \ldots, z_n]$ is said to be *multiaffine* if each variable occurs to at most the first power in P, that is, $\deg_{z_i}(P) \leq 1$ for all $i = 1, \ldots, n$. Stable multiaffine polynomials play a special role in the theory and applications of stable polynomials, primarily due to important results by Grace-Walsh-Szegö, Borcea-Brändén-Liggett and Choe-Oxley-Sokal-Wagner.

The Grace-Walsh-Szegö theorem is a cornerstone which is often relied upon when proving results on stability. The theorem is in essence a polarization procedure which proclaims the equivalence between stability and multiaffine stability.

Theorem 2.2 (Grace-Walsh-Szegö [31, 68, 66]). *Suppose $P(\mathbf{z}) \in \mathbb{C}[z_1, \ldots, z_n]$ is a polynomial of degree at most d in the variable z_n. Write*

$$P(\mathbf{z}) = \sum_{k=0}^{d} P_k(z_1, \ldots, z_{n-1}) z_n^k.$$

Let Q be the polynomial in variables $z_1, \ldots, z_{n-1}, w_1, \ldots, w_{n-1}$ given by

$$Q = \sum_{k=0}^{d} P_k(z_1, \ldots, z_{n-1}) \frac{e_k(w_1, \ldots, w_d)}{\binom{d}{k}}.$$

Then P is stable if and only if Q is stable.

The following corollary is nearly a restatement of Theorem 2.2, often quoted in practise to depolarize symmetries in a multiaffine polynomial for achieving a reduction in the number of variables.

Corollary 2.3. *If $P(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$ is a multiaffine and symmetric polynomial, then $P(z_1, \ldots, z_n)$ is stable if and only if $P(z, \ldots, z) \in \mathbb{C}[z]$ is stable.*

Example 2.4. The elementary symmetric polynomial $e_d(z_1, \ldots, z_n)$ is a multiaffine and symmetric polynomial of degree d. By Corollary 2.3 we have that $e_d(z_1, \ldots, z_n)$ is stable if and only if $e_d(z, \ldots, z) = \binom{n}{d} z^d$ is stable, the latter of which is clear since $\binom{n}{d} z^d$ is trivially a real-rooted univariate polynomial.

Brändén [14] proved that real stability in multiaffine polynomials is equivalent to certain polynomial inequalities being satisfied.

Theorem 2.5. *Let $P(\mathbf{z}) \in \mathbb{R}[z_1, \ldots, z_n]$ be a multiaffine polynomial. Then P is stable if and only if*

$$\frac{\partial P}{\partial z_i}(\mathbf{z})\frac{\partial P}{\partial z_j}(\mathbf{z}) \geq \frac{\partial^2 P}{\partial z_i \partial z_j}(\mathbf{z})P(\mathbf{z})$$

for any $\mathbf{z} \in \mathbb{R}^n$ and $i, j \in [n]$.

The inequalities in Theorem 2.5 are similar, but stronger than those satisfied by the partition function of a Rayleigh measure, leading to an interesting connection between stable polynomials and probability theory. This topic was investigated closer in a paper by Borcea, Brändén and Liggett [13].

The significance of stable multivariate polynomials in combinatorics first became apparent in a long paper by Choe, Oxley, Sokal and Wagner [22]. The authors discovered a highly fascinating connection between matroids and stable homogeneous multiaffine polynomials. Matroids are structures which try to capture the combinatorial essence of independence. They admit several cryptomorphic axiomatizations which is an important reason why they serve as useful abstractions. The definition we give here is the most relevant for our current purposes. We refer to [54] for further background on matroid theory.

A *matroid* is a pair $(\mathcal{M}, E)$, where $\mathcal{M}$ is a collection of subsets of a finite ground set E satisfying,

(1) If $B \in \mathcal{M}$ and $A \subseteq B$, then $A \in \mathcal{M}$,

(2) The collection $\mathcal{B}(\mathcal{M})$ of maximal (with respect to inclusion) elements of $\mathcal{M}$ satisfies the *basis exchange axiom*:

$A, B \in \mathcal{B}(\mathcal{M})$ and $x \in A\backslash B$ implies $y \in B\backslash A$ such that $A\backslash\{x\}\cup\{y\} \in \mathcal{B}(\mathcal{M})$.

The elements of $\mathcal{M}$ are called *independent sets* and the elements of $\mathcal{B}(\mathcal{M})$ are called *bases* of $\mathcal{M}$. The *support*, $\mathrm{supp}(P)$, of a polynomial $P(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a(\boldsymbol{\alpha}) \prod_{i=1}^n z_i^{\alpha_i}$ is defined by

$$\mathrm{supp}(P) := \{\boldsymbol{\alpha} \in \mathbb{N}^n : a(\boldsymbol{\alpha}) \neq 0\}.$$

Theorem 2.6 (Choe-Oxley-Sokal-Wagner)**.** *The support of a stable homogeneous multiaffine polynomial is the set of bases of a matroid.*

In fact Brändén later proved that the support of an arbitrary stable polynomial posesses the structure of a so called *jump system*, see [14] for further details. The converse to Theorem 2.6 is false however, the *weighted bases generating polynomial*

$$P_\mathcal{M}(\mathbf{z}) := \sum_{B \in \mathcal{B}(\mathcal{M})} a(B) \prod_{i \in B} z_i$$

of every matroid is not necessarily a stable polynomial for some weighting $a(B) \in \mathbb{R}$, $B \in \mathcal{B}(\mathcal{M})$. One such example is given by the Fano matroid. A matroid is said

to have the *weak half-plane property* (WHPP) if $P_{\mathcal{M}}$ is a stable polynomial and is said to have the *half-plane property* (HPP) if $P_{\mathcal{M}}$ is stable with $a(B) = 1$ for all $B \in \mathcal{B}(\mathcal{M})$. Despite the Fano matroid there are many important matroid classes which have HPP and WHPP, e.g., the class of uniform matroids and the class of $\mathbb{C}$-representable matroids respectively. There are also matroids, e.g. the Pappus matroid, which have WHPP but not HPP. A natural question is thus to properly characterize these two matroid classes, but the problem remains elusive.

Hyperbolic polynomials

A polynomial $h(\mathbf{z}) \in \mathbb{R}[z_1, \ldots, z_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if

(1) $h(\mathbf{z})$ is a homogeneous polynomial (i.e., $h(t\mathbf{z}) = t^d h(\mathbf{z})$),

(2) $h(\mathbf{e}) \neq 0$,

(3) for all $\mathbf{x} \in \mathbb{R}^n$, the univariate polynomial

$$t \mapsto h(t\mathbf{e} - \mathbf{x})$$

has real zeros only.

Geometrically speaking hyperbolicity means that any line parallel to the direction $\mathbf{e}$ of hyperbolicity must intersect the real algebraic variety cut out by $h(\mathbf{z})$ in exactly d points (counting multiplicity), where d is the degree of $h(\mathbf{z})$. Thus the notion of hyperbolicity may, in addition to the notion of stability, be viewed as a multivariate generalization of real-rootedness. As we will point out in the next section, hyperbolicity is essentially a more general notion than real stability.

It is worth giving a brief explanation regarding the origins of this definition. Hyperbolic polynomials first appeared in the theory of partial differential equations with the works of Petrowsky, Gårding, Hörmander, Atiyah and Bott [6, 38, 42, 56]. Let $h(z_1, \ldots, z_n)$ be a polynomial and consider the Cauchy problem,

$$h(\partial/\partial z_1, \ldots, \partial/\partial z_n)u(\mathbf{z}) = f(\mathbf{z}),$$

where $f \in C_0^\infty(H)$ and $H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{e} \geq 0\}$. The analytical significance of hyperbolicity is that the PDE above has a unique solution $u(\mathbf{z})$ supported on H for every $f \in C_0^\infty(H)$ if and only if h is a hyperbolic polynomial with respect to $\mathbf{e}$. Whenever $h(\mathbf{z})$ is a hyperbolic polynomial with respect to $\mathbf{e} \in \mathbb{R}^n$, such equations are therefore naturally referred to as *hyperbolic partial differential equations*. A classical example is the second order wave equation $(\partial^2/\partial z_1^2 - c^2 \partial^2/\partial z_2^2)f = 0$ in two variables.

Example 2.7. Below we list a few examples of hyperbolic polynomials:

- Any product $h(\mathbf{z}) = \prod_{i=1}^{d} \ell_i(\mathbf{z})$ of linear forms $\ell_i(\mathbf{z})$ is a hyperbolic polynomial with respect to any direction $\mathbf{e} \in \mathbb{R}^n$ without a zero coordinate.

- The determinant polynomial $\det(Z)$, where $Z = (z_{ij})$ is a symmetric matrix with $\binom{n+1}{2}$ indeterminate entries, may be regarded as a quintessential example of a hyperbolic polynomial due to its prominent role in the theory. If X is a real symmetric $n \times n$ matrix and I is the identity matrix, then $t \mapsto \det(tI - X)$ is the characteristic polynomial of a symmetric matrix and is thus real-rooted. Hence $\det(Z)$ is a hyperbolic polynomial with respect to I.

- Let $h(\mathbf{z}) = z_1^2 - z_2^2 - \cdots - z_n^2$. Then $h(\mathbf{z})$ is hyperbolic with respect to $\mathbf{e} = (1, 0, \ldots, 0)^T$.

Hyperbolicity cones

Let h be a hyperbolic polynomial with respect to $\mathbf{e}$ of degree d. We may write

$$h(t\mathbf{e} - \mathbf{x}) = h(\mathbf{e}) \prod_{j=1}^{d} (t - \lambda_j(\mathbf{x})),$$

where

$$\lambda_{\max}(\mathbf{x}) := \lambda_1(\mathbf{x}) \geq \cdots \geq \lambda_d(\mathbf{x}) =: \lambda_{\min}(\mathbf{x})$$

are called the *eigenvalues* of $\mathbf{x}$ (with respect to $\mathbf{e}$). By homogeneity of h one sees that

$$\lambda_j(s\mathbf{x}) = s\lambda_j(\mathbf{x}) \text{ and } \lambda_j(\mathbf{x} + s\mathbf{e}) = \lambda_j(\mathbf{x}) + s,$$

for all $j = 1, \ldots, d$, $\mathbf{x} \in \mathbb{R}^n$ and $s \in \mathbb{C}$. The *hyperbolicity cone* of h with respect to $\mathbf{e}$ is the set

$$\Lambda_+(h, \mathbf{e}) := \{\mathbf{x} \in \mathbb{R}^n : \lambda_{\min}(\mathbf{x}) \geq 0\}.$$

The interior of $\Lambda_+(h, \mathbf{e})$ is denoted $\Lambda_{++}(h, \mathbf{e})$. Note that $\mathbf{e} \in \Lambda_{++}(h, \mathbf{e})$ since $h(t\mathbf{e} - \mathbf{e}) = h(\mathbf{e})(t - 1)^d$. We usually abbreviate and write $\Lambda_+(\mathbf{e})$, or even Λ_+, if there is no risk for confusion.

Example 2.8. Below we list the hyperbolicity cones associated with the hyperbolic polynomials in Example 2.7.

- $\Lambda_+(\mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : \ell_i(\mathbf{x})e_i \geq 0 \text{ for all } i\}$.

- $\Lambda_+(I)$ is the cone of positive semidefinite matrices.

- $\lambda_+(1, 0, \ldots, 0) = \left\{\mathbf{x} \in \mathbb{R}^n : x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2}\right\}$ is the *Lorentz light cone*.

The following facts are due to Gårding.

Theorem 2.9 (Gårding)**.** *Let h be a hyperbolic polynomial with respect to $\mathbf{e}$. Then*

   (i) $\Lambda_+(h, \mathbf{e})$ is a convex cone.

   (ii) $\Lambda_+(h, \mathbf{e})$ is the connected component of

$$\{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \neq 0\}$$

   which contains $\mathbf{e}$.

   (iii) If $\mathbf{v} \in \Lambda_{++}(h, \mathbf{e})$, then h is hyperbolic with respect to $\mathbf{v}$, and $\Lambda_{++}(h, \mathbf{v}) = \Lambda_{++}(h, \mathbf{e})$.

   (iv) $\lambda_{min} : \mathbb{R}^n \to \mathbb{R}$ is a concave function.

Another natural property of the hyperbolicity cone is its facial exposure, that is, the property that all its faces are intersections between the cone itself and one of its supporting hyperplanes (see [59]). The following elementary lemma is a consequence of Rolle's theorem from real analysis and states that taking directional derivatives of a hyperbolic polynomial relaxes the hyperbolicity cone.

Lemma 2.10. *If h is a hyperbolic polynomial and $\mathbf{v} \in \Lambda_+$ such that $D_{\mathbf{v}}h \not\equiv 0$, then $D_{\mathbf{v}}h$ is hyperbolic with respect to $\mathbf{v}$ and $\Lambda_+(h, \mathbf{v}) \subseteq \Lambda_+(D_{\mathbf{v}}h, \mathbf{v})$.*

Finally we remark on the connection between hyperbolic polynomials and homogeneous real stable polynomials.

Proposition 2.11. *Let $P \in \mathbb{R}[z_1, \dots, z_n]$ be a homogeneous polynomial. Then P is stable if and only if P is hyperbolic with $\mathbb{R}_+^n \subseteq \Lambda_+(P)$.*

It is also worth noting that the homogenization of a real stable polynomial is a polynomial hyperbolic with respect to any vector with non-negative coordinates. Therefore the real stable polynomials essentially form a subclass of hyperbolic polynomials with hyperbolicity cone containing the positive orthant.

Hyperbolic polymatroids

Let E be a finite set. A *polymatroid* is a function $r : 2^E \to \mathbb{N}$ satisfying

   1. $r(\emptyset) = 0$,

   2. $r(S) \leq r(T)$ whenever $S \subseteq T \subseteq E$,

   3. r is *semimodular*, i.e.,

$$r(S) + r(T) \geq r(S \cap T) + r(S \cup T),$$

   for all $S, T \subseteq E$.

Rank functions of matroids on E coincide with polymatroids r on E with $r(\{i\}) \leq 1$ for all $i \in E$. The connection between hyperbolic polynomials and polymatroids was noted by Gurvits in [35].

In analogy with the rank of a matrix, the *hyperbolic rank*, $\mathrm{rk}(\mathbf{x})$, of $\mathbf{x} \in \mathbb{R}^n$ is defined as the number of non-zero eigenvalues of $\mathbf{x}$, i.e., $\mathrm{rk}(\mathbf{x}) := \deg h(\mathbf{e} + t\mathbf{x})$. Note that the rank is independent of the direction $\mathbf{e}$ of hyperbolicity.

Theorem 2.12 (Gurvits). *Let $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be a tuple of vectors in $\Lambda_+(h, \mathbf{e})$. Define a function $r_{\mathcal{V}} : 2^{[m]} \to \mathbb{N}$, where $[m] := \{1, 2, \ldots, m\}$, by*

$$r_{\mathcal{V}}(S) = \mathrm{rk}\left(\sum_{i \in S} \mathbf{v}_i\right).$$

Then r is the rank function of a polymatroid.

The polymatroid constructed in Theorem 2.12 is called a *hyperbolic polymatroid*. If the vectors in $\mathcal{V}$ have rank at most one, then we obtain the hyperbolic rank function of a *hyperbolic matroid*.

Example 2.13. Let $A_1, \ldots, A_n$ be positive semidefinite matrices over $\mathbb{C}$. Define $r : 2^{[n]} \to \mathbb{N}$ by $r(S) = \dim\left(\sum_{i \in S} A_i\right)$ for all $S \subseteq [n]$. Then $r : 2^{[n]} \to \mathbb{N}$ is a hyperbolic polymatroid on $[n]$. In particular, if $A_1, \ldots, A_n$ are positive semidefinite matrices of rank at most one, then we obtain the rank function of a hyperbolic matroid on $[n]$. These are the matroids representable over $\mathbb{C}$.

Hyperbolic matroids are in fact equivalent to WHPP matroids, see [5].

The generalized Lax conjecture

The generalized Lax conjecture is one of the major outstanding problems in the theory of hyperbolic polynomials. Interest in it is largely driven by the connection between hyperbolic polynomials and convex optimization. The field of hyperbolic programming was introduced by Güler [36] for studying efficient optimization of linear functionals over hyperbolicity cones. A *hyperbolic program* is an optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \text{ and} \\ & \mathbf{x} \in \Lambda_+, \end{aligned}$$

where $\mathbf{c} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ is a system of linear equations and Λ_+ is a hyperbolicity cone. Notable subfields of hyperbolic programming are linear programming (LP) and semidefinite programming (SDP). Linear programming arises by taking Λ_+ to be the positive orthant in $\mathbb{R}^n$ and semidefinite programming arises by taking Λ_+ to be the cone of positive semidefinite matrices. Recall that these cones are associated with the hyperbolic polynomials $h(\mathbf{z}) = z_1 \cdots z_n$ and $h(Z) = \det(Z)$ respectively.

The generalized Lax conjecture roughly asserts that hyperbolic programming is in fact not a generalization of semidefinite programming at all, but that the two fields are equivalent.

A convex cone in $\mathbb{R}^n$ is said to be *spectrahedral* if it is of the form

$$\left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i A_i \text{ is positive semidefinite} \right\}$$

where $A_1, \ldots, A_n$ are symmetric matrices such that there exists a vector $(y_1, \ldots, y_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n y_i A_i$ positive definite.

Remark 2.14. It is not difficult to see that spectrahedral cones are the hyperbolicity cones associated with the hyperbolic polynomials

$$h(\mathbf{z}) = \det\left(\sum_{i=1}^n z_i A_i \right).$$

The generalized Lax conjecture asserts more precisely that every hyperbolicity cone is conversely an affine section of the cone of positive semidefinite matrices.

Conjecture 2.15 (Generalized Lax conjecture (geometric version))**.** *All hyperbolicity cones are spectrahedral.*

Remark 2.16. Note that h_1 and h_2 are hyperbolic polynomials with respect to $\mathbf{e}$ if and only if $h_1 h_2$ is hyperbolic with respect to $\mathbf{e}$. In that case we also have

$$\Lambda_+(h_1 h_2, \mathbf{e}) = \Lambda_+(h_1, \mathbf{e}) \cap \Lambda_+(h_2, \mathbf{e}).$$

Moreover if C_1 and C_2 are two spectrahedral cones with respect to symmetric matrices $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ respectively, then their intersection

$$C_1 \cap C_2 = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i \begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix} \text{ is positive semidefinite} \right\},$$

is again spectrahedral. Hence it suffices to prove the generalized Lax conjecture for hyperbolicity cones associated with irreducible hyperbolic polynomials.

The generalized Lax conjecture can also be formulated algebraically as follows, see [41].

Conjecture 2.17 (Generalized Lax conjecture (algebraic version))**.** *If $h(\mathbf{z}) \in \mathbb{R}[\mathbf{z}]$ is hyperbolic with respect to $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$, then there exists a polynomial $q(\mathbf{z}) \in \mathbb{R}[\mathbf{z}]$, hyperbolic with respect to $\mathbf{e}$, such that $\Lambda_+(h, \mathbf{e}) \subseteq \Lambda_+(q, \mathbf{e})$ and*

$$q(\mathbf{x})h(\mathbf{z}) = \det\left(\sum_{i=1}^n z_i A_i \right) \tag{2.1}$$

for some real symmetric matrices $A_1, \ldots, A_n$ of the same size such that $\sum_{i=1}^n e_i A_i$ is positive definite.

Indeed if the conditions in Conjecture 2.17 are satisfied, then $\Lambda_+(qh, \mathbf{e})$ is a spectrahedral cone by Remark 2.14, and by Remark 2.16 we have that

$$\Lambda_+(qh, \mathbf{e}) = \Lambda_+(q, \mathbf{e}) \cap \Lambda_+(h, \mathbf{e}) = \Lambda_+(h, \mathbf{e}).$$

Conversely if $\Lambda_+(h, \mathbf{e})$ is a spectrahedral cone, then by Remark 2.14 there exists symmetric matrices $A_1, \ldots, A_n$ such that $\Lambda_+(h, \mathbf{e}) = \Lambda_+(f, \mathbf{e})$ where $f(\mathbf{z}) := \det(z_1 A_1 + \cdots + z_n A_n)$. By Remark 2.16 we may assume that h is irreducible. Furthermore h and f both vanish on the boundary $\partial \Lambda_+(h, \mathbf{e})$ of $\Lambda_+(h, \mathbf{e})$. Therefore h must divide f i.e. $f(\mathbf{z}) = q(\mathbf{z})h(\mathbf{z})$ for some hyperbolic polynomial $q(\mathbf{z})$ with respect to $\mathbf{e}$. Hence

$$\Lambda_+(q, \mathbf{e}) \cap \Lambda_+(h, \mathbf{e}) = \Lambda_+(f, \mathbf{e}) = \Lambda_+(h, \mathbf{e}),$$

implying that $\Lambda_+(h, \mathbf{e}) \subseteq \Lambda_+(q, \mathbf{e})$. This establishes the equivalence between Conjecture 2.15 and Conjecture 2.17.

For hyperbolic polynomials $h(z_1, z_2, z_3)$ in three variables more is true, namely there exists symmetric matrices A_1, A_2, A_3 satisfying Conjecture 2.17 with $q(\mathbf{z}) \equiv 1$, i.e., h has a *definite determinantal representation*. This property was initially conjectured by Peter Lax [46] (originally known as the *Lax conjecture*), and was proved by Helton and Vinnikov [41] as pointed out in [48]. However the former conjecture cannot extend to more than three variable. This may be seen by comparing dimensions. The set of polynomials on $\mathbb{R}^n$ of the form $\det(x_1 A_1 + \cdots x_n A_n)$ with A_i a $d \times d$ symmetric matrix for $1 \leq i \leq n$, has dimension at most $n\binom{d+1}{2}$ (as an algebraic image $(A_1, \ldots, A_n) \mapsto \det(x_1 A_1 + \cdots x_n A_n)$ of a vector space of the same dimension) whereas the set of hyperbolic polynomials of degree d on $\mathbb{R}^n$ has non-empty interior in the space of homogeneous polynomials of degree d in n variables (see [53]) and therefore has the same dimension $\binom{n+d-1}{d}$.

Apart from the theorem by Helton and Vinnikov for $n = 3$, the generalized Lax conjecture, as it currently stands (Conjecture 2.17), is known to be true only in a few special cases, see [5] for an up to date summary at the time of writing.

Permutation patterns

There are many different notions of "patterns" in combinatorics involving objects such as graphs, matrices, partitions, words and permutations etc. In this section we shall give a brief (and by no means comprehensive) background on permutation patterns. For a more extensive introduction we refer to books by Kitaev [44] and Bona [10].

Let $\mathcal{S}_n$ denote the set of permutations on $[n]$. A permutation $\sigma \in \mathcal{S}_n$ is said contain an *occurrence* of the *classical pattern* $\pi \in \mathcal{S}_m$, $m \leq n$ if there exists a subsequence in σ whose letters are in the same relative order as those in π i.e. there exists $i_{\pi(1)} < i_{\pi(2)} < \cdots < i_{\pi(m)}$ such that $\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_m)$.

Example 2.18. The permutation $\sigma = 241563 \in \mathcal{S}_6$ has four occurrences of the pattern $\pi = 231 \in \mathcal{S}_3$ given by the subsequences 241, 453, 463 and 563 in σ. On the other hand σ avoids the pattern 321.

Remark 2.19. It is also possible to visualize the definition using permutation matrices. Let M_σ denote the permutation matrix of $\sigma \in \mathcal{S}_n$. Then a permutation $\sigma \in \mathcal{S}_n$ contains an occurrence of the pattern $\pi \in \mathcal{S}_m$ if and only if M_π is a submatrix of M_σ.

For a set Π of patterns, let $\mathcal{S}_n(\Pi)$ denote the set of permutations in $\mathcal{S}_n$ avoiding all of the patterns in Π simultaneously. Two pattern classes Π_1 and Π_2 are called *Wilf-equivalent* if $|\mathcal{S}_n(\Pi_1)| = |\mathcal{S}_n(\Pi_2)|$. Unfortunately the problem of enumerating $\mathcal{S}_n(\Pi)$ is very difficult in general, even for small patterns. However one of the earliest results in the area relates to the enumeration of permutations avoiding patterns of length three, a result that goes back to MacMahon [49] and Knuth [45].

Theorem 2.20 (MacMahon, Knuth). *If $\pi \in \mathcal{S}_3$, then $|\mathcal{S}_n(\pi)| = C_n$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the nth Catalan number.*

In other words the theorem says that all classical patterns of length three are Wilf-equivalent. This no longer remains true for classical patterns of length greater than three. Already for patterns of length four we have three different Wilf-equivalence classes, one of which has not yet been enumerated.

Another early result (famous from Ramsey theory) is due to Erdős and Szekeres [28] which in the language of permutation patterns states the following.

Theorem 2.21 (Erdős-Szekeres [28]). *Let a, b be positive integers and $n = (a - 1)(b-1) + 1$. Then any permutation $\sigma \in \mathcal{S}_n$ contains an occurrence of the pattern $123 \cdots a$ or an occurrence of the pattern $b \cdots 321$.*

A milestone was reached when Marcus and Tardos [52] proved the Stanley-Wilf conjecture which asserts that for each pattern $\pi \in \mathcal{S}_m$ there exists a constant C such that $|\mathcal{S}_n(\pi)| \leq C^n$. The conjecture is equivalent to the following statement.

Theorem 2.22 (Marcus-Tardos [52]). *For any pattern $\pi \in \mathcal{S}_m$, the limit $\lim\limits_{n \to \infty} \sqrt[n]{|\mathcal{S}_n(\pi)|}$ exists and is finite.*

There are several different generalizations of classical patterns. One such generalization is the notion of a vincular pattern introduced by Babson and Steingrímsson. A *vincular pattern* is a permutation $\pi \in \mathcal{S}_m$ some of whose consecutive letters are underlined. If π contains $\underline{\pi(i)\pi(i+1)\cdots\pi(j)}$, then the letters corresponding to $\pi(i), \pi(i+1), \ldots, \pi(j)$ in an occurrence of π in $\sigma \in \mathcal{S}_n$ must be adjacent, whereas there is no adjacency condition for non-underlined consecutive letters. Moreover if π begins with $[\pi(1)$, then any occurrence of π in σ must begin with the leftmost letter of σ. Similarly if π ends with $\pi(m)]$, then any occurrence of π in σ must end with the rightmost letter of σ.

Example 2.23. Let $\sigma = 241563$.

Pattern π	Occurrences in σ
231	$241, 453, 463, 563$
<u>23</u>1	$241, 563$
23<u>1</u>	$241, 463, 563$
[2<u>3</u>1	241
231]	$453, 463, 563$

More recently vincular patterns have been generalized a step further to so called *mesh patterns* introduced by Brändén and Claesson in [17].

Permutation patterns and statistics

A *statistic* on a combinatorial set S is a function stat $: S \to \mathbb{N}$ that keeps track of a particular quantity associated with S. A plethora of statistics have been studied on a number of different combinatorial objects in the literature. Many of them are currently being collected in the findstat database [61]. The *generating polynomial* of a statistic stat $: S \to \mathbb{N}$ is given by

$$f^{\mathrm{stat}}(q) := \sum_{\sigma \in S} q^{\mathrm{stat}(\sigma)}$$

The polynomials $f^{\mathrm{stat}}(q)$ provide natural q-analogues to the enumeration sequence of the combinatorial family. Furthermore $f^{\mathrm{stat}}(q)$ may have other natural properties of interest such as real-rootedness and coefficient unimodality etc. Generating polynomials of statistics defined on two different combinatorial objects may occasionally coincide leading to new and sometimes unexpected connections in combinatorics and beyond.

Example 2.24. The inversion statistic is a particularly well-studied statistic on permutations. The *inversion set* of $\sigma \in \mathcal{S}_n$ is defined by $\mathrm{Inv}(\sigma) := \{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$. The *inversion statistic* inv $: \mathcal{S}_n \to \mathbb{N}$ is given by $\mathrm{inv}(\sigma) := |\mathrm{Inv}(\sigma)|$. Rodrigues [60] showed in 1839 that

$$\sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!,$$

where $[n]_q! := [1]_q[2]_q \cdots [n]_q$ and $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$. It is not difficult to show that $[n]_q!$ is a polynomial with unimodal coefficients.

Example 2.25. The *descent set* of σ is defined by $\mathrm{Des}(\sigma) := \{i : \sigma(i) > \sigma(i+1)\}$ and the *descent statistic* by $\mathrm{des}(\sigma) := |\mathrm{Des}(\sigma)|$. The coefficients of the polynomial $f^{\mathrm{des}}(q)$ are given by the Eulerian numbers and the Eulerian polynomial $f^{\mathrm{des}}(q)$ is well-known to be real-rooted (see e.g. [55]).

Example 2.26. The *major index statistic* is defined by $\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i$. MacMahon [49] showed that the maj and inv statistics are equidistributed, i.e., $f^{\mathrm{maj}}(q) = f^{\mathrm{inv}}(q)$. Permutation statistics which are equidistributed with inv are called *Mahonian*.

Patterns give rise to statistics as well. A *pattern function* $(\pi) : \mathcal{S}_n \to \mathbb{N}$ is a statistic that is induced by a permutation pattern π, counting the number of occurrences of π in a permutation $\sigma \in \mathcal{S}_n$. The *length* of a pattern function is the length of its underlying pattern. Babson and Steingrímsson[7] classified (up to trivial bijections) all Mahonian statistics that are conic combinations of pattern functions of length at most 3. Among them are inv and maj.

Sagan and Savage [63] introduced a q-analogue of Wilf-equivalence in order to refine Wilf-classes by statistic equidistribution. Formally two sets of patterns Π_1 and Π_2 are said to be *st-Wilf equivalent* with respect to the statistic $\mathrm{st} : \mathcal{S}_n \to \mathbb{N}$ if

$$\sum_{\sigma \in \mathcal{S}_n(\Pi_1)} q^{\mathrm{st}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\Pi_2)} q^{\mathrm{st}(\sigma)}.$$

Clearly st-Wilf equivalence implies Wilf-equivalence but not conversely. Dokos et.al. [25] completed the inv-Wilf and maj-Wilf classifications over $\mathcal{S}_n(\pi)$ where π is a classical pattern of length three. The st-Wilf classification of other permutation statistics such as fixed points, exceedances, peak and valley have also been investigated in detail, see [9, 26].

The cyclic sieving phenomenon

Let C_n be a cyclic group of order n generated by σ_n, X a finite set on which C_n acts and $f(q) \in \mathbb{N}[q]$. Let $X^g := \{x \in X : g \cdot x = x\}$ denote the fixed point set of X under $g \in C_n$. A triple $(X, C_n, f(q))$ is said to exhibit the *cyclic sieving phenomenon (CSP)* if

$$f(\omega_n^k) = |X^{\sigma_n^k}|, \text{ for all } k \in \mathbb{Z}, \tag{2.2}$$

where ω_n is any fixed primitive n^{th} root of unity. The cyclic sieving phenomenon was introduced by Reiner, Stanton and White in [58]. Although it is always possible to find a (generally uninteresting) polynomial satisfying the equations in (2.2) when provided with a cyclic action, namely,

$$f(q) = \sum_{\mathcal{O} \in \mathrm{Orb}_{C_n}(X)} \frac{q^n - 1}{q^{n/|\mathcal{O}|} - 1}, \tag{2.3}$$

it sometimes happens that a polynomial $f(q) \in \mathbb{N}[q]$ can be found which satisfies (2.2) and is intrinsically related to the set X on which C_n acts. Generally we would consider a CSP "interesting" if for example

- $f(q) = \sum_{x \in X} q^{\mathrm{stat}(x)}$ where $\mathrm{stat} : X \to \mathbb{N}$ is a natural statistic on X.

- $f(q)$ is the formal character of some representation $\rho : C_n \to GL(V)$.

- $f(q)$ is the Hilbert series $\mathrm{Hilb}(R, q) := \sum_i \dim(R_i) q^i$ of some graded ring $R = \bigoplus_i R_i$.

- $f(q)$ at $q = p^d$ counts the number of points of a variety over a finite field $\mathbb{F}_q$.

There is no a priori reason why one would expect the existence of polynomials with any of the above properties. Nevertheless such situations occur quite ubiquitously in combinatorics, as witnessed by the growing literature on the phenomenon. See [62] for an extensive survey on CSP.

Example 2.27. The prototypical example of CSP is given by $X = \binom{[n]}{k}$ and

$$f(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where $[m]_q! := [m]_q[m-1]_q \cdots [2]_q[1]_q$ and $[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$. Here the generator σ_n of C_n acts on $S = \{i_1, \ldots, i_k\} \in X$ via

$$\sigma_n \cdot S := \{i_1 \ (\mathrm{mod} \ n) + 1, \ldots, i_k \ (\mathrm{mod} \ n) + 1\}.$$

By [58] the triple $(X, C_n, f(q))$ exhibits CSP. The following facts are also proved in [58]:

- If $\mathrm{sum} : X \to \mathbb{N}$ is the statistic defined by $\mathrm{sum}(S) := \sum_{i \in S} i$, then

$$f(q) = q^{-\binom{k+1}{2}} \sum_{S \in X} q^{\mathrm{sum}(S)}.$$

- Let $V = \bigwedge^k(\mathbb{C}^n)$ denote the kth exterior power of the vector space $\mathbb{C}^n$. The action of C_n on X induces an action of C_n on V, giving rise to a representation $\rho : C_n \to GL(V)$. Denote the character of ρ by $\chi_\rho(x_1, \ldots, x_n) : C_n \to \mathbb{C}[x_1, \ldots, x_n]$, defined for $\sigma \in C_n$ as the trace of the matrix $\rho(\sigma)$ with eigenvalues $x_1, \ldots, x_n$. Then

$$f(q) = q^{-\binom{k}{2}} \chi_\rho(1, q, q^2, \ldots, q^{n-1}).$$

- Let $\mathbb{Z}[\mathbf{x}]^G$ denote the ring of polynomials in variables $\mathbf{x} = (x_1, \ldots, x_n)$ invariant under the action of the group G. Then

$$f(q) = \mathrm{Hilb}(\mathbb{Z}[\mathbf{x}]^{\mathcal{S}_k \times \mathcal{S}_{n-k}} / \mathbb{Z}[\mathbf{x}]_+^{\mathcal{S}_n}, q),$$

  where $\mathcal{S}_k \times \mathcal{S}_{n-k}$ and $\mathcal{S}_n$ act as usual on $\mathbb{Z}[\mathbf{x}]$, and $\mathbb{Z}[\mathbf{x}]_+$ denotes the ring of polynomials with positive degree.

- $f(q)$ counts the number of k-subspaces of a vector space of dimension n over a finite field $\mathbb{F}_q$ with q elements i.e. the number points in the Grassmanian variety $\mathrm{Gr}_{\mathbb{F}_q}(k, n)$.

3 Summary of results

Paper A [5]

In the wake of Helton and Vinnikov's celebrated proof of the Lax conjecture [41] the follow up question was how the theorem should be generalized to more than three variables. Stronger versions of Conjecture 2.17 were initially believed to be true. For instance it was conjectured in [41] that if $h(\mathbf{z})$ is a hyperbolic polynomial, then $h(\mathbf{z})^N$ has a definite determinantal representation for some positive integer N. This belief is not totally unreasonable given that for $p(\mathbf{z})$ homogeneous and irreducible, it is well-known that $p(\mathbf{z})^N$ has a (not necessarily definite) determinantal representation for some N, see [8]. The claim was however disproved by Brändén in [15] via the bases generating polynomial of a certain non-representable hyperbolic matroid.

The *Vámos matroid* V_8 is the matroid with ground set $E = \{1, \ldots, 8\}$ and bases

$$\mathcal{B}(V_8) = \binom{[8]}{4} \setminus \{\{1,2,3,4\}, \{3,4,5,6\}, \{1,2,5,6\}, \{1,2,7,8\}, \{5,6,7,8\}\}.$$

Theorem 3.1 (Wagner-Wei [67])**.** V_8 *is a HPP matroid (and therefore hyperbolic).*

In 1969 Ingleton [43] proved a necessary condition for a matroid to be representable.

Theorem 3.2 (Ingleton)**.** *Suppose* $r : 2^E \to \mathbb{N}$ *is the rank function of a representable matroid and* $A, B, C, D \subseteq E$. *Then*

$$\begin{aligned} r(A \cup B) + r(A \cup C \cup D) + r(C) + r(D) + r(B \cup C \cup D) \leq \\ r(A \cup C) + r(A \cup D) + r(B \cup C) + r(B \cup D) + r(C \cup D) \end{aligned} \tag{3.1}$$

Considering V_8 and setting

$$A = \{1,2\}, \quad B = \{3,4\}, \quad C = \{5,6\}, \quad D = \{7,8\},$$

the Ingleton inequality (3.1) reads, $4 + 4 + 2 + 2 + 4 \leq 3 + 3 + 3 + 3 + 3$ which is a contradiction. Hence V_8 cannot be representable.

Theorem 3.3 (Brändén)**.** *There exists no positive integer* N *such that* $P_{V_8}(\mathbf{z})^N$ *has a definite determinantal representation where* $P_{V_8}(\mathbf{z})$ *denotes the bases generating polynomial of* V_8.

Proof sketch. Suppose

$$P_{V_8}(\mathbf{z}) = \det(\sum_{i=1}^{n} z_i A_i),$$

for some positive integer N and symmetric matrices $A_1, \ldots, A_n$. The bases generating polynomial $P_{V_8}(\mathbf{z})$ is stable by Theorem 3.1, so it is hyperbolic with respect to $\mathbf{1}$. The rank function of the hyperbolic matroid associated with the hyperbolic

polynomial $P_{V_8}(\mathbf{z})^N$ with respect to $\mathcal{V} = \{\delta_1, \ldots, \delta_8\} \subseteq \mathbb{R}_+^8 \subseteq \Lambda_+$ can be expressed as

$$r_{\mathcal{V}}(S) = \deg\left(P_{V_8}\left(\mathbf{1} + t\sum_{i \in S} \delta_i\right)^N\right) = N r_{V_8}(S)$$

where $\delta_1, \ldots, \delta_8$ denote the standard basis vectors of $\mathbb{R}^8$ and r_{V_8} denotes the rank function of the matroid V_8. Now consider the representable matroid given by

$$r(S) := \mathrm{rk}\left(\sum_{i \in S} A_i\right).$$

By initial assumption we have

$$r(S) = r_{\mathcal{V}}(S) = N r_{V_8}(S).$$

However we know that r_{V_8} violates the Ingleton inequalities (3.1) which contradicts the fact that r is the rank function of a representable matroid. $\qquad\square$

Remark 3.4. Brändén [15] in fact proved a slightly stronger statement: There exists no positive integers M, N and no linear form $\ell(\mathbf{z})$ such that $\ell(\mathbf{z})^M P_{V_8}(\mathbf{z})^N$ has a definite determinantal representation.

It is not known whether $P_{V_8}(\mathbf{z})$ satisfies the generalized Lax conjecture (Conjecture 2.17). In order to find potential obstructions to the generalized Lax conjecture it is worthwhile understanding the role of non-representable hyperbolic matroids in the context of the conjecture and finding additional instances of them. Prior to Paper A, only the Vámos matroid V_8 and a certain generalization of it were known to be both non-representable and hyperbolic.

A *paving matroid* of rank r is a matroid such that all its circuits (minimal dependent sets) have size at least r. A paving matroid of rank r is called *sparse* if all its hyperplanes (flats of rank $r - 1$) have size $r - 1$ or r.

Further instances of non-representable hyperbolic matroids come from finite projective geometry. Sparse paving matroids of rank three can be obtained from finite point-line configurations in which every line contains three points. Such matroids are obtained by letting a subset of three points define a circuit hyperplane if and only if there is a line containing them. The Pappus and Desargues configurations are geometrical configurations with 9 and 10 points respectively such that every line contains three points and every point is incident to three lines (note that such configurations need not be unique). The *Non-Pappus* and *Non-Desargues* matroids are obtained from the Pappus and Desargues configurations by deleting one line. Both of these matroids are not representable over any field. However the Non-Pappus matroid can be shown to be representable over every skew-field e.g. the quaternions $\mathbb{H}$, see [43]. The Non-Desargues matroid on the other hand is not even representable over any skew-field [43], but is known to be representable over the octonions $\mathbb{O}$, see [37]. The algebras $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$ of Hermitian 3×3 matrices

over $\mathbb{H}$ and $\mathbb{O}$ respectively, are examples of real Euclidean Jordan algebras. All real Euclidean Jordan algebras A come equipped with a hyperbolic determinant polynomial det : $A \to \mathbb{R}$, in particular realizing the cone of positive semidefinite matrices in $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$ as hyperbolicity cones. Hence we obtain:

Theorem 3.5. *The Non-Pappus and Non-Desargues matroids are hyperbolic matroids not representable over any field.*

Burton et.al. [19] defined a class of matroids V_{2n} for $n \geq 4$ with base set $\mathcal{B}(V_{2n}) := \binom{[2n]}{4} \setminus \mathcal{H}_{2n}$ where

$$\mathcal{H}_{2n} := \{1, 2, 2k-1, 2k\} \cup \{2k-1, 2k, 2k+1, 2k+1\} \ \text{ for } 2 \leq k \leq n,$$

extending the Vámos matroid. They made the following conjecture regarding the family V_{2n} for $n \geq 4$.

Conjecture 3.6 (Burton-Vinzant-Youm). *For each $n \geq 4$, V_{2n} is a HPP matroid.*

Burton et.al. confirmed Conjecture 3.6 for $n = 5$. In Paper A we prove a sweeping generalization of Conjecture 3.6, in particular proving Conjecture 3.6 in the affirmative for all $n \geq 4$.

Theorem 3.7. *Let H be a d-uniform hypergraph on $[n]$, and let $E = \{1, 1', \ldots, n, n'\}$. Let*

$$\mathcal{B}(V_H) = \binom{E}{2d} \setminus \{e \cup e' : e \in E(H)\},$$

in which $e' := \{i' : i \in e\}$ for each $e \in E(H)$. Then $\mathcal{B}(V_H)$ is the set of bases of a sparse paving matroid V_H of rank $2d$.

Theorem 3.8. *If G is a simple graph, then V_G is a HPP matroid.*

Theorem 3.8 unfortunately does not admit a full generalization to matroids V_H parametrized by hypergraphs H. An obstruction is e.g. given by the complete 3-uniform hypergraph on [6]. Nevertheless we can prove the following.

Theorem 3.9. *If H is a d-uniform hypergraph, then V_H is a WHPP matroid.*

Remark 3.10. Since the class of hyperbolic matroids is equivalent to the class of WHPP matroids [5], all matroids V_H are hyperbolic by Theorem 3.9.

Remark 3.11. The family $\{V_{2n}\}_{n \geq 4}$ studied by Burton et al. [19] corresponds to V_{G_n} where G_n is an n-cycle with edges $\{1, i\}$, $i = 2, \ldots, n$, adjoined. Thus Theorem 3.8 implies Conjecture 3.6.

Remark 3.12. Since representability is closed under taking minors, any matroid V_H containing the Vámos V_8 as a minor is necessarily non-representable (and fails to satisfy Ingleton's inequality (3.1)).

The proof of Theorem 3.9 depends on certain symmetric function inequalities. These inequalities are also of independent interest.

Recall that a *partition* of a natural number d is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of natural numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_1 + \lambda_2 + \cdots = d$. We write $\lambda \vdash d$ to denote that λ is a partition of d. The *length*, $\ell(\lambda)$, of λ is the number of nonzero entries of λ. If λ is a partition and $\ell(\lambda) \leq n$, then the *monomial symmetric polynomial*, m_α, is defined as

$$m_\lambda(z_1, \ldots, z_n) := \sum z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n},$$

where the sum is over all distinct permutations $(\beta_1, \beta_2, \ldots, \beta_n)$ of $(\lambda_1, \ldots, \lambda_n)$. If $\ell(\lambda) > n$, we set $m_\lambda(\mathbf{z}) = 0$. The dth *elementary symmetric polynomial* is $e_d(\mathbf{z}) := m_{1^d}(\mathbf{z})$. Lemma 3.13 below is a refinement of the Laguerre-Turán inequalities

$$0 \leq re_r(\mathbf{z})^2 - (r+1)e_{r-1}(\mathbf{z})e_{r+1}(\mathbf{z}),$$

and is used in the proof of Theorem 3.14.

Lemma 3.13. *If $r \geq 1$, then*

$$m_{2^r}(\mathbf{z}) \leq re_r(\mathbf{z})^2 - (r+1)e_{r-1}(\mathbf{z})e_{r+1}(\mathbf{z}).$$

The theorem below is a central ingredient to the proof of Theorem 3.9.

Theorem 3.14. *Let $r \geq 2$ be an integer, and let*

$$M(\mathbf{z}) = \sum_{|S|=r} a(S) \prod_{i \in S} z_i^2 \in \mathbb{R}[z_1, \ldots, z_n],$$

where $0 \leq a(S) \leq 1$ for all $S \subseteq [n]$, where $|S| = r$. Then the polynomial

$$4e_{r+1}(\mathbf{z})e_{r-1}(\mathbf{z}) + \frac{3}{r+1}M(\mathbf{z})$$

is stable.

In light of Remark 3.4 it is natural to question whether it is possible to put any kind of restrictions on the factor $q(\mathbf{z})$ in Conjecture 2.17 when it comes to a prescribed bound on its degree and its number of irreducible factors. The answer turns out to be *no*. We construct a family of hyperbolic polynomials obtained from the bases generating polynomials of specific members of the family V_H, such that for sufficiently many variables $\mathbf{z} = (z_1, \ldots, z_n)$, the factor $q(\mathbf{z})$ in Conjecture 2.17 must either have an irreducible factor of large degree or have a large number of irreducible factors of low degree.

Given positive integers n and k, consider the k-uniform hypergraph $H_{n,k}$ on $[n+2]$ containing all hyperedges $e \in \binom{[n+2]}{k}$ except those for which $\{n+1, n+2\} \subseteq e$. By Theorem 3.9 the matroid $V_{H_{n,k}}$ is hyperbolic and therefore has a hyperbolic

bases generating polynomial $h_{V_{H_{n,k}}}(\mathbf{z})$ with respect to $\mathbf{1}$. The polynomial $h_{n,k}(\mathbf{z}) \in \mathbb{R}[z_1, \ldots, z_{n+2}]$, obtained from the multiaffine polynomial $h_{V_{H_{n,k}}}(\mathbf{z})$ by identifying the variables z_i and $z_{i'}$ pairwise for all $i \in [n+2]$ is therefore hyperbolic with respect to $\mathbf{1}$.

Theorem 3.15. *Let n and k be a positive integers. Suppose there exists a positive integer N and a hyperbolic polynomial $q(\mathbf{z})$ such that*

$$q(\mathbf{z})h_{n,k}(\mathbf{z})^N = \det\left(\sum_{i=1}^{n+2} z_i A_i\right) \tag{3.2}$$

with $\Lambda_+(h_{n,k}) \subseteq \Lambda_+(q)$ for some symmetric matrices $A_1, \ldots, A_{n+2}$ such that $A_1 + \cdots + A_{n+2}$ is positive definite and

$$q(\mathbf{z}) = \prod_{i=1}^{s} p_j(\mathbf{z})^{\alpha_i}$$

for some irreducible hyperbolic polynomials $p_1, \ldots, p_s \in \mathbb{R}[z_1, \ldots, z_{n+2}]$ of degree at most $k-1$ where $\alpha_1, \ldots, \alpha_s$ are positive integers. Then

$$n < (2s+1)k - 1.$$

Paper B [2]

Although there is not an extensive amount of evidence for the generalized Lax conjecture (Conjecture 2.17), the conjecture is known to hold for some specific classes of hyperbolic polynomials (see [5]). In particular Brändén [16] confirmed the conjecture for elementary symmetric polynomials, extending work of Zinchenko [69] and Sanyal [64]. Brändén applied the matrix-tree theorem, which implies that every spanning tree polynomial has a definite determinantal representation, and realized the spanning tree polynomial of a certain series-parallel graph as a product of elementary symmetric polynomials. A consequence of Brändén's result is that hyperbolic polynomials which are iterated derivatives of products of linear forms have spectrahedral hyperbolicity cones. Moreover the hyperbolicity cone of the spanning tree polynomial of a complete graph is linearly isomorphic to the cone of positive semidefinite matrices. Hence the generalized Lax conjecture is equivalent to the assertion that each hyperbolicity cone is an affine slice of the hyperbolicity cone of a spanning tree polynomial.

In Paper B we consider hyperbolicity cones of multivariate matching polynomials in context of the generalized Lax conjecture. Two main reasons for considering matching polynomials are the well-known facts that the univariate matching polynomial of a tree coincide with its characteristic polynomial and that every univariate matching polynomial divides the matching polynomial of a tree. Multivariate versions of the above two facts are important inputs for proving the generalized Lax

conjecture for the class of multivariate matching polynomials. As an application
we reprove Brändén's result by realizing the elementary symmetric polynomials of
degree k as a factor in the matching polynomial of the length-k truncated path tree
of the complete graph.

Recall that a *k-matching* in a graph $G = (E, V)$ is a subset $M \subseteq E(G)$ of k
edges, no two of which have a vertex in common. Let $\mathcal{M}(G)$ denote the set of all
matchings in G and for $M \in \mathcal{M}(G)$, let $V(M)$ denote the set of vertices contained in
M. Let $\mathbf{z} = (z_v)_{v \in V}$ and $\mathbf{w} = (w_e)_{e \in E}$ be indeterminates. Define the *homogeneous*
multivariate matching polynomial $\mu(G, \mathbf{z} \oplus \mathbf{w}) \in \mathbb{R}[\mathbf{z}, \mathbf{w}]$ by

$$\mu(G, \mathbf{z} \oplus \mathbf{w}) := \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} \prod_{v \notin V(M)} z_v \prod_{e \in M} w_e^2.$$

As a direct consequence of a theorem by Heilmann and Lieb [40], the polynomial
$\mu(G, \mathbf{z} \oplus \mathbf{w})$ is hyperbolic with respect to $\mathbf{e} = \mathbf{1} \oplus \mathbf{0}$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^V$
and $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^E$. Note that $\mu(G, \mathbf{z} \oplus \mathbf{w})$ specializes to the conventional
univariate matching polynomial $\mu(G, t)$ by putting $\mathbf{z} \oplus \mathbf{w} = t\mathbf{1} \oplus \mathbf{1}$. The following
recursion is immediate from the definition,

$$\mu(G, \mathbf{x} \oplus \mathbf{w}) = z_u \mu(G \setminus u, \mathbf{z} \oplus \mathbf{w}) - \sum_{v \in N(u)} w_{uv}^2 \mu((G \setminus u) \setminus v, \mathbf{z} \oplus \mathbf{w}).$$

Let G be a graph and $u \in V(G)$. The *path tree* $T(G, u)$ is the tree with vertices
labelled by simple paths in G (i.e. paths with no repeated vertices) starting at u
and where two vertices are joined by an edge if one vertex is labelled by a maximal
subpath of the other. Godsil [32] proved the following divisibility relation for the
univariate matching polynomial,

$$\frac{\mu(G \setminus u, t)}{\mu(G, t)} = \frac{\mu(T(G, u) \setminus u, t)}{\mu(T(G, u), t)}.$$

The above identity implies that $\mu(G, t)$ divides $\mu(T(G, u), t)$. To establish a mul-
tivariate version of the above relationship we must consider a natural change of
variables. The technique used to prove the multivariate divisibility relation is very
similar to its univariate counterpart. Let $\phi : \mathbb{R}^{T(G,u)} \to \mathbb{R}^G$ denote the linear
change of variables defined by

$$z_p \mapsto z_{i_k},$$
$$w_{pp'} \mapsto w_{i_k i_{k+1}},$$

where $p = i_1 \cdots i_k$ and $p' = i_1 \cdots i_k i_{k+1}$ are adjacent vertices in $T(G, u)$. For every
subforest $T \subseteq T(G, u)$, define the polynomial

$$\eta(T, \mathbf{z} \oplus \mathbf{w}) := \mu(T, \phi(\mathbf{z}' \oplus \mathbf{w}'))$$

where $\mathbf{z}' = (z_p)_{p \in V(T)}$ and $\mathbf{w}' = (w_e)_{e \in E(T)}$.

Lemma 3.16. *Let* $u \in V(G)$. *Then*

$$\frac{\mu(G \setminus u, \mathbf{z} \oplus \mathbf{w})}{\mu(G, \mathbf{z} \oplus \mathbf{w})} = \frac{\eta(T(G,u) \setminus u, \mathbf{z} \oplus \mathbf{w})}{\eta(T(G,u), \mathbf{z} \oplus \mathbf{w})}.$$

In particular $\mu(G, \mathbf{z} \oplus \mathbf{w})$ *divides* $\eta(T(G,u), \mathbf{z} \oplus \mathbf{w})$.

The next lemma arises as a multivariate analogue to the fact that the matching polynomial of a tree T is equal to the characteristic polynomial of the adjacency matrix of T.

Lemma 3.17. *Let* $T = (V, E)$ *be a tree. Then* $\mu(T, \mathbf{z} \oplus \mathbf{w})$ *has a definite determinantal representation.*

Note that

$$\frac{\partial}{\partial z_u} \mu(G, \mathbf{z} \oplus \mathbf{w}) = \mu(G \setminus u, \mathbf{z} \oplus \mathbf{w}),$$

and therefore

$$\Lambda_+(\mu(G, \mathbf{z} \oplus \mathbf{w})) \subseteq \Lambda(\mu(G \setminus u, \mathbf{z} \oplus \mathbf{w})).$$

Using the above fact, Lemma 3.16 and Lemma 3.17 it follows, using an inductive argument, that multivariate matching polynomials $\mu(G, \mathbf{z} \oplus \mathbf{w})$ satisfy the generalized Lax conjecture for any graph G.

Theorem 3.18. *The hyperbolicity cone of* $\mu(G, \mathbf{z} \oplus \mathbf{w})$ *is spectrahedral.*

By considering the matching polynomial of the partial path tree of the complete graph K_n up to paths of length at most k, along with a suitable linear change of variables, we recover Brändén's result regarding the spectrahedrality of hyperbolicity cones of elementary symmetric polynomials. Hence Theorem 3.18 can be viewed as a generalization of this fact.

A subset $I \subseteq V(G)$ is called *independent* if no two vertices of I are adjacent in G. Let $\mathcal{I}(G)$ denote the set of all independent sets in G. Define the *homogeneous multivariate independence polynomial* $I(G, \mathbf{z} \oplus t) \in \mathbb{R}[\mathbf{z}, t]$ by

$$I(G, \mathbf{z} \oplus t) = \sum_{I \in \mathcal{I}(G)} (-1)^{|I|} \left(\prod_{v \in I} z_v^2 \right) t^{2|V(G)| - 2|I|}.$$

A graph is said to be *claw-free* if it has no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. If G is a claw-free graph, then $I(G, \mathbf{z} \oplus t)$ is hyperbolic with respect to $\mathbf{e} = (0, \dots, 0, 1)$. This fact is a simple consequence of the real-rootedness of the weighted univariate independence polynomial of a claw-free graph, due to Engström [27]. We prove that when G satisfies an additional technical condition (stronger than claw-freeness), then $I(G, \mathbf{z} \oplus t)$ satisfies Conjecture 2.17.

Matching polynomials and independence polynomials are intimately related. The *line graph* $L(G)$ of G is the graph having vertex set $E(G)$ and where two

vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident. The univariate matching polynomial of a graph G can be realized as the univariate independence polynomial of its line graph $L(G)$. With that said, the multivariate polynomial $I(G, \mathbf{z} \oplus t)$ does not strictly generalize $\mu(G, \mathbf{z} \oplus \mathbf{w})$ due to the dummy homogenization in the variable t. Unfortunately we were unsuccessful in constructing a hyperbolic refinement of $I(G, \mathbf{z} \oplus t)$ with respect to the variable t which reduces to $\mu(G, \mathbf{z} \oplus \mathbf{w})$ (after relabelling) when G is a line graph.

The key to proving the generalized Lax conjecture for $I(G, \mathbf{z} \oplus t)$ is to find a tree that plays a role similar to that of the path tree for the matching polynomial. Such a tree was constructed by Leake and Ryder in [47]. We outline its construction below.

An induced clique K in G is called a *simplicial clique* if for all $u \in K$ the induced subgraph $N[u] \cap (G \setminus K)$ of $G \setminus K$ is a clique. In other words the neighbourhood of each $u \in K$ is a disjoint union of two induced cliques in G. Furthermore, a graph G is said to be *simplicial* if G is claw-free and contains a simplicial clique. A connected graph G is a *block graph* if each 2-connected component is a clique.

Given a simplicial graph G with a simplicial clique K we recursively define a block graph $T^{\boxtimes}(G, K)$ called the *clique tree* associated to G and rooted at K.

We begin by adding K to $T^{\boxtimes}(G, K)$. Let $K_u = N[u] \setminus K$ for each $u \in K$. Attach the disjoint union $\bigsqcup_{u \in K} K_u$ of cliques to $T^{\boxtimes}(G, K)$ by connecting $u \in K$ to every $v \in K_u$. Finally recursively attach $T^{\boxtimes}(G \setminus K, K_u)$ to the clique K_u in $T^{\boxtimes}(G, K)$ for every $u \in K$.

Theorem 3.19 (Leake-Ryder)**.** *Let K be a simplicial clique of a simplicial graph G. Then*

$$\frac{I(G, \mathbf{z} \oplus t)}{I(G \setminus K, \mathbf{z} \oplus t)} = \frac{I(T^{\boxtimes}(G, K), \mathbf{z} \oplus t)}{I(T^{\boxtimes}(G, K) \setminus K, \mathbf{z} \oplus t)},$$

where $T^{\boxtimes}(G, K)$ is relabelled according to the natural graph homomorphism $\phi_K : T^{\boxtimes}(G, K) \to G$. Moreover $I(G, \mathbf{z} \oplus t)$ divides $I(T^{\boxtimes}(G, K), \mathbf{z} \oplus t)$.

The following lemma asserts that vertex deletion relaxes the hyperbolicity cone, providing the necessary setup for an inductive argument of spectrahedrality.

Lemma 3.20. *Let $v \in V(G)$. Then $\Lambda_+(I(G, \mathbf{z} \oplus t)) \subseteq \Lambda_+(I(G \setminus v, \mathbf{z} \oplus t))$.*

Using Theorem 3.19, Lemma 3.20 and the fact that the clique tree $T^{\boxtimes}(G, K)$ can be realized as the line graph of an actual tree, one proves the theorem below using an inductive argument which unfolds in an analogous manner to the proof of Theorem 3.18.

Theorem 3.21. *If G is a simplicial graph, then the hyperbolicity cone of $I(G, \mathbf{z} \oplus t)$ is spectrahedral.*

Paper C [3]

A graph G is called *Ramanujan* if the absolute value of its largest non-trivial eigenvalue is bounded above by the spectral radius $\rho(G)$ of its universal covering tree. We refer to [33] for undefined terminology. *Expanders* are graphs which can be informally characterized by being sparse and yet well-connected. Expanders are of importance in e.g. computer science where they serve as basic building blocks for robust network designs (among other things). Due to their spectral properties, Ramanujan graphs are considered optimal expanders in the sense that a random walk on a Ramanujan graph converges to the uniform distribution in the fastest possible way. The existence of Ramanujan graphs is a highly non-trivial issue. A longstanding open question asks about the existence of infinitely many k-regular Ramanujan graphs for every $k \geq 3$. Marcus, Spielman and Srivastava proved that every finite graph G has a 2-sheeted covering (or 2-covering for short) with maximum non-trivial eigenvalue (not induced by G) bounded above by $\rho(G)$, a so called *one-sided Ramanujan covering*. Since coverings of bipartite graphs are bipartite, and the spectrum of a bipartite graph is symmetric around zero, they were able to point to the existence of infinitely many k-regular bipartite Ramanujan graphs.

Subsequently Hall, Puder and Sawin [39] generalized the techniques in [50, 51] and proved that every loopless connected graph has a one-sided Ramanujan d-covering for every $d \geq 1$. An essential polynomial to the proof is the average matching polynomial of all d-coverings of G. For $d \geq 1$, the *d-matching polynomial* of G is defined by

$$\mu_{d,G}(z) := \frac{1}{|\mathcal{C}_{d,G}|} \sum_{H \in \mathcal{C}_{d,G}} \mu_H(z),$$

where $\mathcal{C}_{d,G}$ denotes the set of all d-coverings of G and

$$\mu_G(z) := \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i m_i z^{n-2i} \in \mathbb{Z}[z]$$

denotes the univariate matching polynomial of G. In particular if $d = 1$, then $\mu_{d,G}(z) = \mu_G(z)$.

Using the celebrated technique of interlacing families, developed by Marcus, Spielman and Srivastava, the authors prove that the maximum root of the expected characteristic polynomial over all d-coverings of G is bounded above by their uniform average, which in turn is proved to equal $\mu_{d,G}(z)$. The real roots of $\mu_{d,G}(z)$ on the other hand can easily be deduced to lie in the interval $[-\rho(G), \rho(G)]$ using a well-known theorem of Heilmann and Lieb [40]. Hence there is at least one covering in the family which has its maximal non-trivial eigenvalue less than the maximum root of the average $\mu_{d,G}(z)$, that is, less than $\rho(G)$ as desired.

As implied by the paragraph above we have in particular the following theorem.

Theorem 3.22 (Hall-Puder-Sawin)**.** *If G is a finite loopless graph, then $\mu_{d,G}(z)$ is real-rooted.*

The authors gave a rather long and indirect proof of Theorem 3.22. They further asked for a direct proof that includes graphs with loops. In Paper C we answer their question by proving that a multivariate version of the d-matching polynomial is stable, a statement which is more general than their original question. Define the *multivariate d-matching polynomial* of G by

$$\mu_{d,G}(\mathbf{z}) := \mathbb{E}_{H \in \mathcal{C}_{d,G}} \mu_H(\mathbf{z}),$$

where

$$\mu_G(\mathbf{z}) := \sum_M (-1)^{|M|} \prod_{v \in [n] \setminus V(M)} z_v,$$

and the sum runs over all matchings in G. By analysing the algebraic symbol it follows that the *multi-affine part* operator

$$\mathrm{MAP} : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$$
$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a(\boldsymbol{\alpha}) \mathbf{z}^{\boldsymbol{\alpha}} \mapsto \sum_{\boldsymbol{\alpha} : \alpha_i \leq 1, i \in [n]} a(\boldsymbol{\alpha}) \mathbf{z}^{\boldsymbol{\alpha}}$$

is a stability-preserving linear operator. Moreover one sees that

$$\mathrm{MAP} \left(\prod_{uv \in E(G)} (1 - z_u z_v) \right) = \mu_G(\mathbf{z}),$$

proving that $\mu_G(\mathbf{z})$ is stable. By using MAP and the Grace-Walsh-Szegö theorem we prove:

Theorem 3.23. *Let G be a finite graph and $d \geq 1$. Then $\mu_{d,G}(\mathbf{z})$ is stable.*

Corollary 3.24. *Let G be a finite graph and $d \geq 1$. Then $\mu_{d,G}(z)$ is real-rooted.*

Proof. Follows by putting $\mathbf{z} = (z, \ldots, z)$ in Theorem 3.23 $\qquad \square$

In [40] Heilmann and Lieb proved that the matching polynomial $\mu_G(z)$ of any graph G is real-rooted. In analogy with graph matchings, a matching in a hypergraph consists of a subset of (hyper)edges with empty pairwise intersection. However the analogous matching polynomial for hypergraphs is not real-rooted in general, see e.g. [34]. A natural question is thus how to generalize the Heilmann-Lieb theorem to hypergraphs. We consider a relaxation of matchings in general hypergraphs that leads to an associated real-rooted polynomial which reduces to the conventional matching polynomial for graphs.

Consider the problem of assigning a subset of n people with prescribed competencies into teams of no less than two people, working on a subset of m different projects in such a way that no person is assigned to more than one project and each person has the competency to work on the project they are assigned to. We shall call such team assignments "relaxed matchings". More formally define a *relaxed*

matching in a hypergraph $H = (V(H), E(H))$ to be a collection $M = (S_e)_{e \in E}$ of edge subsets such that $E \subseteq E(H)$, $S_e \subseteq e$, $|S_e| > 1$ and $S_e \cap S_{e'} = \emptyset$ for all pairwise distinct $e, e' \in E$.

Remark 3.25. If H is a graph then the concept of relaxed matching coincides with the conventional notion of graph matching. Note also that a conventional hypergraph matching is a relaxed matching $M = (S_e)_{e \in E}$ for which $S_e = e$ for all $e \in E$.

Remark 3.26. The subsets S_e in the relaxed matching are labeled by the edge they are chosen from in order to avoid ambiguity. However if H is a *linear hypergraph*, that is, the edges pairwise intersect in at most one vertex, then the subsets uniquely determine the edges they belong to and therefore no labeling is necessary. Graphs and finite projective geometries (viewed as hypergraphs) are examples of linear hypergraphs.

Let $V(M) := \bigcup_{S_e \in M} S_e$ denote the set of vertices in the relaxed matching. Moreover let $m_k(M) := |\{S_e \in M : |S_e| = k\}|$ denote the number of subsets in the relaxed matching of size k. Define the *multivariate relaxed matching polynomial* of H by

$$\eta_H(\mathbf{z}) := \sum_M (-1)^{|M|} W(M) \prod_{i \in [n] \setminus V(M)} z_i,$$

where the sum runs over all relaxed matchings of H and

$$W(M) := \prod_{k=1}^{n-1} k^{m_{k+1}(M)}.$$

Let $\eta_H(z) := \eta_H(z\mathbf{1})$ denote the *univariate relaxed matching polynomial*.

Remark 3.27. If H is a graph, then $\eta_H(z) = \mu_H(z)$.

Theorem 3.28. *The polynomial $\eta_H(\mathbf{z})$ is stable. In particular*

$$\eta_H(z) = \sum_M (-1)^{|M|} W(M) z^{n-|V(M)|},$$

is a real-rooted polynomial for any hypergraph H.

Paper D [4]

Combining the study of pattern avoidance with combinatorial statistics is a paradigm which has been advocated in papers by Claesson-Kitaev [23] and Sagan-Savage [63] among others. Typically one is interested in the generating polynomial

$$f(q) = \sum_{\sigma \in \mathcal{S}_n(\Pi)} q^{\mathrm{stat}(\sigma)},$$

for some pattern set Π and combinatorial statistic stat $: \mathcal{S}_n(\Pi) \to \mathbb{N}$. Examples of questions one may ask about $f(q)$ have to do with equidistribution, recursion and unimodality/log-concavity/real-rootedness etc. In Paper D we focus on equidistributions of the form

$$\sum_{\sigma \in \mathcal{S}_n(\Pi_1)} q^{\mathrm{stat}_1(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\Pi_2)} q^{\mathrm{stat}_2(\sigma)},$$

where Π_1, Π_2 consist of a single classical pattern of length three and stat_1, stat_2 are Mahonian permutation statistics.

Let Π denote the set of vincular patterns of length at most d. A *d-function* is a statistic of the form

$$\mathrm{stat} = \sum_{\pi \in \Pi} \alpha_\pi \cdot (\pi),$$

where $\alpha_\pi \in \mathbb{N}$ and (π) is the statistic counting the number of occurrences of the pattern π. Babson and Steingrímsson classified all Mahonian 3-functions up to trivial symmetries. Several previously studied Mahonian statistics fall under the classification, including maj and inv. The complete table of Mahonian 3-functions may be found below along with their original references.

Name	Vincular pattern statistic	Reference
maj	$(1\underline{32}) + (2\underline{31}) + (3\underline{21}) + (\underline{21})$	MacMahon [49]
inv	$(2\underline{31}) + (\underline{31}2) + (3\underline{21}) + (\underline{21})$	MacMahon [49]
mak	$(1\underline{32}) + (\underline{31}2) + (3\underline{21}) + (\underline{21})$	Foata-Zeilberger [30]
makl	$(1\underline{32}) + (2\underline{31}) + (\underline{32}1) + (\underline{21})$	Clarke-Steingrímsson-Zeng [24]
mad	$(2\underline{31}) + (2\underline{31}) + (\underline{31}2) + (\underline{21})$	Clarke-Steingrímsson-Zeng [24]
bast	$(1\underline{32}) + (2\underline{13}) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[7]
bast$'$	$(1\underline{32}) + (\underline{31}2) + (3\underline{21}) + (\underline{21})$	Babson-Steingrímsson[7]
bast$''$	$(1\underline{32}) + (3\underline{12}) + (3\underline{21}) + (\underline{21})$	Babson-Steingrímsson[7]
foze	$(\underline{21}3) + (3\underline{21}) + (\underline{13}2) + (\underline{21})$	Foata-Zeilberger [29]
foze$'$	$(1\underline{32}) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Foata-Zeilberger [29]
foze$''$	$(2\underline{31}) + (\underline{31}2) + (\underline{31}2) + (\underline{21})$	Foata-Zeilberger [29]
sist	$(1\underline{32}) + (1\underline{32}) + (2\underline{13}) + (\underline{21})$	Simion-Stanton [65]
sist$'$	$(1\underline{32}) + (1\underline{32}) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [65]
sist$''$	$(1\underline{32}) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [65]

Since all statistics in the table above are Mahonian, they are by definition equidistributed over $\mathcal{S}_n$. In Paper D we ask what equidistributions hold between the statistics if we restrict ourselves to permutations avoiding a classical pattern of

length three. Existing bijections $\phi : S_n \to S_n$ in the literature for proving the Mahonian nature of these statistics do not restrict to bijections over pattern classes. Therefore there is no a priori reason to expect that such equidistributions should continue to hold over $S_n(\pi)$. Another motivation for studying equidistributions over $S_n(\pi)$ where $\pi \in S_3$, is that these pattern classes are enumerated by the Catalan numbers. Thus under appropriate bijections we may get induced equidistributions between combinatorial statistics on other Catalan structures (and vice versa). Below we give an example of such an induced equidistributions from Paper D.

Theorem 3.29. *For any $n \geq 1$,*

$$\sum_{\sigma \in S_n(321)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)} = \sum_{\sigma \in S_n(321)} q^{\mathrm{mak}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)},$$

where $\mathrm{DB}(\sigma) := \{\sigma(i+1) : \sigma(i) > \sigma(i+1)\}$ *and* $\mathrm{DT}(\sigma) := \{\sigma(i) : \sigma(i) > \sigma(i+1)\}$.

The equidistribution in Theorem 3.29 is proved via an explicit involution $\phi : S_n(321) \to S_n(321)$ mapping maj to mak and preserving descent bottoms and descent tops in the process. The involution ϕ induces an equidistribution on shortened polyominoes (another Catalan structure) as we shall now describe.

A *shortened polyomino* is a pair (P, Q) of N (north), E (east) lattice paths $P = (P_i)_{i=1}^n$ and $Q = (Q_i)_{i=1}^n$ satisfying

1. P and Q begin at the same vertex and end at the same vertex.

2. P stays weakly above Q and the two paths can share E-steps but not N-steps.

Denote the set of shortened polyominoes with $|P| = |Q| = n$ by $\mathcal{H}_n$. Let $\mathrm{Valley}(Q) = \{i : Q_i Q_{i+1} = EN\}$ denote the set of indices of the *valleys* in Q and let $\mathrm{nval}(Q) = |\mathrm{Valley}(Q)|$. Define the statistics *valley-column area*, $\mathrm{vcarea}(P, Q)$, and *valley-row area*, $\mathrm{vrarea}(P, Q)$, as illustrated below.
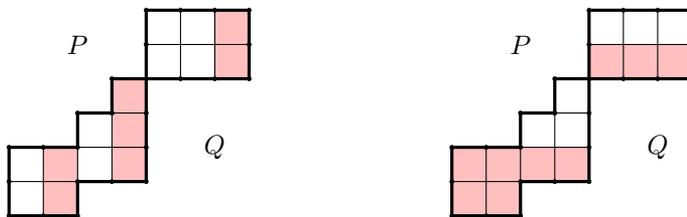
(a) $\mathrm{vcarea}(P, Q) = 2 + 3 + 2 = 7$ (b) $\mathrm{vrarea}(P, Q) = 2 + 4 + 3 = 9$

Cheng, Eu and Fu [21] gave a creative bijection $\Psi : \mathcal{H}_n \to S_n(321)$. In Paper D we show that

- $\mathrm{vcarea}(P, Q) = [(\underline{21}) + (\underline{31}\underline{2})]\Psi(P, Q)$.

- vrarea$(P, Q) = [(\underline{21}) + (23\underline{1})]\Psi(P, Q)$.

From the involution ϕ in Theorem 3.29 one gets

$$[(\underline{21}) + (3\underline{12})]\phi(\sigma) = [(\underline{21}) + (23\underline{1})]\sigma.$$

Hence by considering the composition $\Psi^{-1} \circ \phi \circ \Psi$ we get the following induced equidistribution.

Theorem 3.30. *For any $n \geq 1$,*

$$\sum_{(P,Q)\in\mathcal{H}_n} q^{\text{vcarea}(P,Q)} t^{\text{nval}(Q)} = \sum_{(P,Q)\in\mathcal{H}_n} q^{\text{vrarea}(P,Q)} t^{\text{nval}(Q)}.$$

Conversely we may prove equidistributions between Mahonian 3-functions via equidistributions over an intermediate Catalan structure. Below we give an example of this technique from Paper D.

Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0, 0)$ and $(2n, 0)$ consisting of up-steps $(1, 1)$ and down-steps $(1, -1)$ which never go below the x-axis. For convenience we denote the up-steps by U and the down-steps by D. Let $\mathcal{D}_n$ denote the set of Dyck paths of semi-length n. Under Krattenthaler's well-known bijection $\Gamma : \mathcal{S}_n(321) \to \mathcal{D}_n$, the statistic inv is mapped to the statistic *sumpeaks*, defined for Dyck paths $P = s_1 \cdots s_{2n} \in \mathcal{D}_n$ by

$$\text{spea}(P) := \sum_{p\in\text{Peak}(P)} (\text{ht}_P(p) - 1),$$

where $\text{Peak}(P) := \{p : s_p s_{p+1} = UD\}$ and $\text{ht}_P(p)$ is the y-coordinate of the pth step in P. The figure below illustrates the Dyck path corresponding to $\sigma = 341625978 \in \mathcal{S}_9(321)$ under Krattenthaler's bijection, mapping inv to spea.
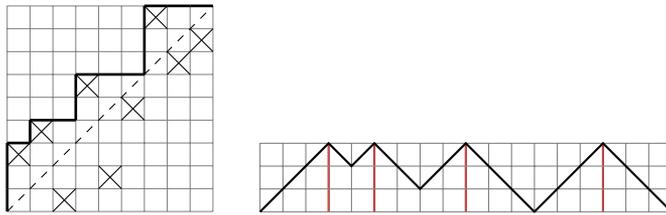
Let $\text{Valley}(P) := \{v : s_v s_{v+1} = DU\}$ denote the set of indices of the valleys in P. For each $v \in \text{Valley}(P)$, there is a corresponding *tunnel* which is the subword $s_i \cdots s_v$ of P where i is the step after the first intersection of P with the line $y = \text{ht}_P(v)$ to the left of step v (see figure below). The length, $v - i$, of a tunnel is always an even number. Let $\text{Tunnel}(P) := \{(i, j) : s_i \cdots s_j \text{ tunnel in } P\}$ denote the set of pairs of beginning and end indices of the tunnels in P. Define the statistic *sumtunnels* by

$$\text{stun}(P) := \sum_{(i,j)\in\text{Tunnel}(P)} (j - i)/2.$$

The tunnel lengths of the Dyck path below are highlighted by dashes.
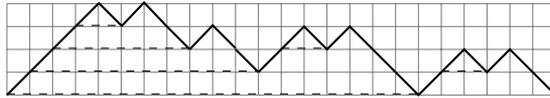
Cheng, Elizalde, Kasraoui and Sagan [20] gave a bijection $\Psi : \mathcal{D}_n \to \mathcal{D}_n$ mapping spea to stun. The *mass* corresponding to two consecutive U-steps, is half the number of steps between their matching D-steps (i.e. if $P = UUP'DP''D$, then the mass of the pair UU is $|P''|/2$). Define the statistics

$$\text{mass}(P) := \text{ sum of masses over all occurrences of } UU$$
$$\text{dr}(P) := \text{ number of double rises } UU \text{ in } P.$$

The part of the Dyck path below contributing to the mass associated with the first double rise is highlighted in red.
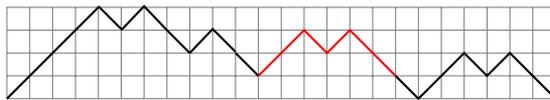
In Paper D we give a bijection $\Phi : \mathcal{D}_n \to \mathcal{D}_n$, mapping stun to mass + dr. Finally via Knuth's standard bijection $\Delta : \mathcal{S}_n(231) \to \mathcal{D}_n$ defined recursively by $k\sigma_1\sigma_2 \mapsto U\Delta(\sigma_1)D\Delta(\sigma_2)$ where $\sigma_1 < k < \sigma_2$, we map the 3-Mahonian statistic mad to mass + dr. Combining all mentioned bijections we obtain the following theorem.

Theorem 3.31. *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}(321)} q^{\text{inv}(\sigma)} = \sum_{P \in \mathcal{D}_n} q^{\text{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\text{stun}(P)} = \sum_{P \in \mathcal{D}_n} q^{\text{mass}(P)+\text{dr}(P)}$$
$$= \sum_{\sigma \in \mathcal{S}_n(231)} q^{\text{mad}(\sigma)}$$

As an aside we find several other related equidistributions with inv and mad over $\mathcal{S}_n(321)$ and $\mathcal{S}_n(231)$ respectively.

Consider the statistic

$$\text{inc} := \iota_1 + \sum_{k=2}^{\infty} (-1)^{k-1} 2^{k-2} \iota_k$$

where $\iota_{k-1} = (12 \ldots k)$ is the statistic that counts the number of increasing subsequences of length k in a permutation. Using the Catalan continued fraction framework of Brändén, Claesson and Steingrímsson[18] we prove the following equidistribution.

Theorem 3.32. *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(132)} q^{\mathrm{inc}(\sigma)}.$$

Let $\mathrm{Up}(P) := \{i : s_i = U\}$ denote the indices of the up-steps in $P = s_1 \cdots s_{2n}$. Define
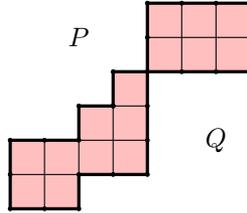
$$\mathrm{sups}(P) := \sum_{i \in \mathrm{Up}(P)} \lceil \mathrm{ht}_P(i)/2 \rceil.$$

By constructing a bijection $\Theta : \mathcal{D}_n \to \mathcal{D}_n$, mapping sups to $\mathrm{mass} + \mathrm{dr}$, we deduce via Theorem 3.31 the following equidistribution.

Proposition 3.33. *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{sups}(P)}.$$

If $(P, Q) \in \mathcal{H}_n$ is a shortened polyomino, then the *area statistic*, $\mathrm{area}(P, Q)$ is defined as the number of boxes enclosed by (P, Q).

It is finally worth mentioning the following equidistribution.

Theorem 3.34 (Cheng-Eu-Fu)**.** *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{(P,Q) \in \mathcal{H}_n} q^{\mathrm{area}(P,Q)}.$$

See Paper D for the full table of established and conjectured Mahonian 3-function equidistributions.

Paper E [1]

Given a cyclic action of C_n on the set X, Reiner, Stanton and White [58] showed that the polynomial $f(q)$ in (2.3) always makes $(X, C_n, f(q))$ into a CSP triple. Many natural CSP triples occurring in the literature have the additional property that $f(q) = \sum_{x \in X} q^{\mathrm{stat}(x)}$ for some combinatorial statistic $\mathrm{stat} : X \to \mathbb{N}$. Conversely it is natural to ask under what circumstances a combinatorial polynomial

$f(q) = \sum_{x \in X} q^{\text{stat}(x)}$ can be complemented with a cyclic action to a CSP? In Paper E we give a necessary and sufficient criterion for this to be the case. In particular the converse is not trivial in the sense that if $f(q) \in \mathbb{N}[q]$ is a polynomial such that $f(\omega_n^j) \in \mathbb{N}$ for all $1 \leq j \leq n$, then one cannot always find a cyclic action complementing $f(q)$ to a CSP. Our main theorem is the following.

Theorem 3.35. *Let $f(q) \in \mathbb{N}[q]$ and suppose $f(\omega_n^j) \in \mathbb{N}$ for each $j = 1, \ldots, n$. Let X be any set of size $f(1)$. Then there exists an action of C_n on X such that $(X, C_n, f(q))$ exhibits CSP if and only if for each $k | n$,*

$$\sum_{j|k} \mu(k/j) f(\omega_n^j) \geq 0. \tag{3.3}$$

The action complementing $f(q)$ to a CSP in Theorem 3.35 is given by the following generic construction.

Construction 3.36. *Let $X = \mathcal{O}_1 \sqcup \mathcal{O}_2 \sqcup \cdots \sqcup \mathcal{O}_m$ be a partition of a finite set X into m parts such that $|\mathcal{O}_i|$ divides n for $i = 1, \ldots, m$. Fix a total ordering on the elements of $\mathcal{O}_i$ for $i = 1, \ldots, m$. Let C_n act on X by permuting each element $x \in \mathcal{O}_i$ cyclically with respect to the total ordering on $\mathcal{O}_i$ for $i = 1, \ldots, m$.*

We call the action in Construction 3.36 an *ad-hoc* cyclic action. The action lacks combinatorial context and merely depends on the choice of partition and total order. By ordinary Möbius inversion, the sums $S_k = \sum_{j|k} \mu(k/j) f(\omega_n^j)$ represent the number of elements of order k under the action of C_n. Thus the only non-trivial issue in the proof of Theorem 3.35 is whether k divides S_k for all k. This is required for the elements to be evenly partitioned into orbits. Rather surprisingly it turns out that the divisibility property always hold as long as $f(\omega_n^j) \in \mathbb{Z}$ for all $1 \leq j \leq n$.

Although we would generally not consider a CSP "interesting" unless both the action and the polynomial are combinatorially meaningful, we think that our criteria serves a useful purpose in the way that a candidate polynomial can be quickly tested for CSP without having a combinatorial cyclic action at hand. A combinatorial polynomial passing the test may be a likely indication that a combinatorially meaningful cyclic action is present explaining the CSP.

Example 3.37. Let $f(q) = q^5 + 3q^3 + q + 9$. Then $f(\omega_6^j)$ takes values $7, 11, 4, 11, 7, 14$ for $j = 1, \ldots, 6$. On the other hand $S_k = \sum_{j|k} \mu(k/j) f(\omega_6^j)$ takes values $7, 4, -3, 0, 0, 6$ for $k = 1, \ldots, 6$. Since we cannot have a negative number of elements of order 3, there is no action of C_6 on a set X of size $f(1) = 14$ such that $(X, C_6, f(q))$ is a CSP-triple.

Thus even if $f(q) \in \mathbb{N}[q]$ satisfies $f(\omega_n^j) \in \mathbb{N}$ for all $j = 1, \ldots, n$, we may not have an associated cyclic action complementing $f(q)$ to a CSP.

In the second part of Paper E we consider CSP from a more geometric perspective. Let stat : $X \to \mathbb{N}$ be a statistic and denote $\text{stat}_n(x) = \text{stat}(x) \pmod{n}$. Consider

the joint distribution

$$\sum_{x \in X} q^{\text{stat}_n(x)} t^{o(x)} = \sum_{i=0}^{n-1} \sum_{j=1}^{n} a_{ij} q^i t^j,$$

where $o(x)$ denotes the order of $x \in X$ under C_n. We can now restate CSP as follows.

Proposition 3.38. *Suppose X is a finite set on which C_n acts and let $f(q) = \sum_{x \in X} q^{stat(x)}$ where stat $: X \to \mathbb{N}$ is a statistic. Then the triple $(X, C_n, f(q))$ exhibits CSP if and only if $A_{(X,C_n,\text{stat})} = (a_{ij})$ satisfies the condition that for each $1 \le k \le n$,*

$$\sum_{\substack{0 \le i < n \\ 1 \le j \le n}} a_{ij} \omega_n^{ki} = \sum_{0 \le i < n} \sum_{j|k} a_{ij}. \tag{3.4}$$

where ω_n is a primitive nth root of unity.

We call a matrix $A = (a_{ij}) \in \mathbb{R}_{\ge 0}^{n \times n}$ a CSP matrix if it satisfies the linear equations in (3.4). Let $\text{CSP}(n)$ denote the set of $n \times n$ CSP matrices.

Example 3.39. Consider all binary words of length 6, with group action being cyclic right-shift by one position and stat being the the major index statistic (sum of all descent indices). Then

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 11 \\ 0 & 0 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 11 \\ 0 & 1 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 11 \\ 0 & 0 & 2 & 0 & 0 & 7 \end{pmatrix}$$

is the corresponding CSP matrix. The above matrix can be checked to satisfy (3.4) with $n = 6$. The entry in the upper left hand corner correspond to the two binary words 000000 and 111111. These have major index 0 and are fixed under a single shift, so they have order one. The words corresponding to the second column are 010101 and 101010. These have major index $6 \equiv 0 \pmod 6$ and $9 \equiv 3 \pmod 6$ respectively and are fixed under a minimum of two consecutive shifts, so they have order two etc.

Define the hyperplanes

$$H_k(\mathbf{x}) := \sum_{i=0}^{n-1} \sum_{\substack{j|n \\ j>1}} \alpha_{ijk} x_{ij} \in \mathbb{Z}[\mathbf{x}],$$

where

$$\alpha_{ijk} := \begin{cases} -n + \frac{n}{j}, & \text{if } i = k \text{ and } k \equiv 0 \ (\text{mod } \frac{n}{j}), \\ -n, & \text{if } i = k \text{ and } k \not\equiv 0 \ (\text{mod } \frac{n}{j}), \\ \frac{n}{j}, & \text{if } i \neq k \text{ and } k \equiv 0 \ (\text{mod } \frac{n}{j}), \\ 0, & \text{if } i \neq k \text{ and } k \not\equiv 0 \ (\text{mod } \frac{n}{j}). \end{cases}$$

Theorem 3.40. *We have*

$$CSP(n) \cong \{\mathbf{x} \in \mathbb{R}_{\geq 0}^{n(d-1)+1} : H_k(\mathbf{x}) \geq 0\},$$

where d denotes the number of divisors of n.

Thus we see that $\mathrm{CSP}(n)$ forms a convex rational polyhedral cone of dimension $n(d-1)+1$. The cone $\mathrm{CSP}(n)$ has several notable properties as summarized below.

- The integer lattice points $\mathrm{CSP}(n) \cap \mathbb{Z}^{n \times n}$ correspond to distributions that are realizable by a CSP triple $(X, C_n, f(q))$.

- Suppose that i and i' are indices such that $\gcd(n, i) = \gcd(n, i')$. Then the operation of swapping rows i and i' preserves the property of being a CSP matrix.

- Adding a matrix B with zero row and column-sum to a CSP matrix A preserves the property of being a CSP matrix provided $A + B \in \mathbb{R}_{\geq 0}^{n \times n}$.

About the joint paper contributions of the author

Papers A and E in this thesis are a result of joint collaboration with two different coauthors. The contribution of the author in each of these papers is described below.

Paper A was written together with the author's advisor Petter Brändén. While the author participated in all aspects of the project, many of the key breakthroughs regarding the symmetric function inequalities were made by the advisor. Initially the hyperbolicity of the matroids in our family was proved only for graphs. The main contribution of the author pertains to the generalization of the inequalities in the graphical case to strengthen the main result to matroids derived from hypergraphs. This later turned out to have consequences for the generalized Lax conjecture and produce instances of non-representable hyperbolic matroids without a Vámos minor. Some smaller results regarding the minor closure of the matroid family and facts regarding representability of matroids derived from tree-like hypergraphs was also contributed by the author. Paper E was written jointly with Per Alexandersson where both authors contributed approximately equal amounts to all aspects of the work.

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Part II

Scientific papers

Paper A

Paper B

Paper C

Paper D

Paper E