Efficient Monte Carlo Simulation for Counterparty Credit Risk Modeling

SAM JOHANSSON
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Abstract

In this paper, Monte Carlo simulation for CCR (Counterparty Credit Risk) modeling is investigated. A jump-diffusion model, Bates’ model, is used to describe the price process of an asset, and the counterparty default probability is described by a stochastic intensity model with constant intensity. In combination with Monte Carlo simulation, the variance reduction technique importance sampling is used in an attempt to make the simulations more efficient. Importance sampling is used for simulation of both the asset price and, for CVA (Credit Valuation Adjustment) estimation, the default time. CVA is simulated for both European and Bermudan options. It is shown that a significant variance reduction can be achieved by utilizing importance sampling for asset price simulations. It is also shown that a significant variance reduction for CVA simulation can be achieved for counterparties with small default probabilities by employing importance sampling for the default times. This holds for both European and Bermudan options. Furthermore, the regression based method least squares Monte Carlo is used to estimate the price of a Bermudan option, resulting in CVA estimates that lie within an interval of feasible values. Finally, some topics of further research are suggested.

Keywords: CCR, OTC derivatives, European option, Bermudan option, CVA, jump-diffusion model, stochastic intensity model, Monte Carlo, variance reduction, importance sampling, least squares Monte Carlo
Effektiv Monte Carlo-simulering för modellering av motpartskreditrisk

Sammanfattning


Nyckelord: CCR, OTC-derivat, europeisk option, Bermuda-option, CVA, jump-diffusionmodell, stokastisk intensitetsmodell, Monte Carlo, variansreduktion, importance sampling, least squares Monte Carlo
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# List of Abbreviations

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<tr>
<td>CCR</td>
<td>Counterparty Credit Risk</td>
</tr>
<tr>
<td>c.d.f</td>
<td>Cumulative Distribution Function</td>
</tr>
<tr>
<td>CVA</td>
<td>Credit Valuation Adjustment</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>Independent and Identically Distributed</td>
</tr>
<tr>
<td>OTC</td>
<td>Over The Counter</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
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<td>SDE</td>
<td>Stochastic Differential Equation</td>
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## List of Symbols

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<th>Symbol</th>
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<tr>
<td>$r$</td>
<td>Annual risk free interest rate</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Volatility of the price of an asset</td>
</tr>
<tr>
<td>$T$</td>
<td>Time of expiration of a financial contract</td>
</tr>
<tr>
<td>$W_t$</td>
<td>Wiener process at time $t$</td>
</tr>
<tr>
<td>$N_t$</td>
<td>Poisson process at time $t$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Intensity of a Poisson process</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Time of default</td>
</tr>
<tr>
<td>$\mathbb{1}{\cdot}$</td>
<td>Indicator function</td>
</tr>
<tr>
<td>$Q$</td>
<td>Risk neutral measure</td>
</tr>
<tr>
<td>$N(\mu, \sigma)$</td>
<td>Normal distribution with mean $\mu$ and standard deviation $\sigma$</td>
</tr>
<tr>
<td>$\text{Po}(\lambda)$</td>
<td>Poisson distribution with intensity parameter $\lambda$</td>
</tr>
<tr>
<td>$U(a, b)$</td>
<td>Uniform distribution on the interval $[a, b]$</td>
</tr>
<tr>
<td>$\text{Exp}(\lambda)$</td>
<td>Exponential distribution with rate parameter $\lambda$</td>
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Chapter 1

Introduction

1.1 Counterparty Credit Risk

Operating in the financial markets is associated with risk. There are several different kinds of financial risk, and they are connected to each other in different ways. This article focuses on CCR (Counterparty Credit Risk). CCR is the risk of a counterparty defaulting before fulfilling its contractual commitments. CCR is mainly related to both market risk and credit risk. These can be described as follows:

Market risk is the risk associated with price movements that are affecting the performance of the whole market. Market risk often arises from movements in underlying variables affecting the prices of instruments, such as interest rates, foreign exchange rates, credit spreads and market volatility. Credit risk is the risk of losses arising from the possibility that a counterparty might be unable to fulfill its contractual obligations. Associated with credit risk are both the default of a counterparty and deterioration of a counterparty’s credit quality.

To handle CCR is a difficult task. It requires an understanding of both market risk and credit risk, as well as the interactions between the two. Additionally, when dealing with CCR, one must often deal with complex financial instruments. In particular, OTC derivatives are often involved in CCR.

In recent times, CCR has been given increased attention. During the 2007-2009 financial crisis, banks suffered substantial CCR losses on their OTC derivatives portfolios. Thus it was evidenced at the time, that the regulations and practices for mitigating CCR were insufficient. Since then, efforts have been made to improve both regulations governing CCR and the practices of banks related to CCR.

1.2 Credit Valuation Adjustment

Closely related to CCR is CVA (Credit Valuation Adjustment), which is defined as the market value of the CCR losses incurred by a portfolio.

In connection with the larger focus placed on CCR after the 2007-2009 financial crisis, the need for an increased use of CVA also came to light. At the time of the crisis, many firms were...
operating under the Basel II framework. The conditions of this framework required the firms to capitalise for default risk and migration risk. Therefore, the CCR management of firms were mostly designed to protect them against losses incurred from defaults. However, the majority of the losses banks suffered on their OTC derivatives portfolios during the crisis were not from counterparty defaults, but from fair value adjustments on the derivatives. To account for the large proportion of losses being due to CVA changes, regulations governing CVA have since been created. Notably, the Basel III framework released in 2010 included a capital requirement for CVA that must be calculated by banks operating under the framework \[2\].

As the banks themselves are responsible for calculating the CVA capital charge, it is common for them to have desks set up solely for this purpose. While the banks are required to calculate CVA once a month under the Basel III framework, some of them calculate it as frequently as every day. It is common for the CVA calculations to involve large portfolios of complex instruments, and they are therefore often carried out through Monte Carlo simulations. Consequently, these calculations are often very computationally expensive \[4\].

In general, CVA depends on several correlated market factors. These factors are dependent on the portfolio, and could for instance be stock price, interest rate, volatility or default times. A common approach to calculating CVA is to simulate the market factors over a discretized time interval and use the simulated quantities to obtain a Monte Carlo estimate of CVA. In order to conduct these simulations, model assumptions have to be made with regard to the market factors. Usually, a market factor is assigned a probability distribution at each point of calculation or its movement is assumed to be described by a stochastic process \[5\].

1.3 Purpose and Aim

Monte Carlo simulations are of large importance to CCR modeling in general as well as CVA calculations in particular, with the objective being to achieve the desired accuracy while minimizing the computational time or the computational load. In order to achieve this objective, several attempts have been made to improve Monte Carlo simulation techniques within CCR modeling. For example, Glasserman and Li utilized the importance sampling technique for Monte Carlo simulations to decrease the variance when simulating rare events of large losses \[6\], and Antonov et.al. proposed a method of quickly calculating the portfolio exposure in order to speed up CVA calculations in a setting where the portfolio is priced using Monte Carlo simulation \[7\].

For Monte Carlo simulations, the relative error \(\hat{e}_n\) is related to the sample standard deviation \(S_n\) and the number of scenarios (or paths) \(n\) in the following way.

\[
\hat{e}_n \propto \frac{S_n}{\sqrt{n}}. \tag{1.1}
\]

An efficient simulation method will have a small relative error. To decrease the relative error, either the sample standard deviation (or equivalently the sample variance) must be decreased or the number of paths must be increased. As an increase in the number of paths results in both an increased computational time and an increased computational load, it is preferred to reduce the variance. There exist several different variance reduction techniques that can
be used in the Monte Carlo setting. This paper focuses on the technique called importance sampling. The idea behind importance sampling is to change the measure under which the sampling is performed and adjust the resulting Monte Carlo estimate to account for the change of measure. This paper also discusses the implementation of a regression based Monte Carlo method, called least squares Monte Carlo, for the pricing of Bermudan options. The method is later used to calculate CVA for Bermudan options.

The objectives of the paper are to investigate:

- How to decrease the variance of asset price simulation.
- How to estimate the price of a Bermudan option using regression based Monte Carlo simulation.
- How to decrease the variance of CVA simulation for European and Bermudan options.
Chapter 2

Mathematical Background

This section presents the mathematical concepts that lay the foundation of the work performed in the paper. If not stated otherwise, a probability space \((\Omega, \mathcal{F}, \mathbb{Q})\) is assumed, where \(\mathbb{Q}\) is a risk neutral measure. Moreover, for any given stochastic process \(X\), \(X_t\) is used to denote \(X(t)\).

2.1 Preliminaries

2.1.1 Wiener Process

The Wiener process is a continuous time stochastic process that can be seen as a generalization of the random walk to continuous time, as the Wiener process is the limit of a random walk when the size of the time increment approaches zero. In mathematical finance, the Wiener process is often used to model the randomness of price processes. The Wiener process is defined as follows:

Definition 2.1 A stochastic process \(W = \{W_t \mid t \geq 0\}\) is a Wiener process if the following conditions hold.

1) \(W_0 = 0\)

2) The sample paths \(t \mapsto W_t\) are almost surely continuous.

3) \(\{W_t \mid t \geq 0\}\) has stationary and independent increments

4) \(W_t - W_s \in \mathcal{N}(0, t-s)\) for \(t > s\).

2.1.2 Poisson Process

To describe jumps in quantities related to financial mathematics, Poisson processes are often used. A Poisson process is a stochastic point process that only takes non negative values and is increasing. The value of a Poisson process at time \(t\) can be interpreted as the number of occurrences of some event on the time interval \((0, t]\). A Poisson process is defined as follows.

Definition 2.2 A stochastic process \(N = \{N_t \mid t \geq 0\}\) is a Poisson process if the following conditions hold.
CHAPTER 2. MATHEMATICAL BACKGROUND

1) \( N_0 = 0 \).

2) The increments \( N_{t_k} - N_{t_{k-1}} \) are independent stochastic variables for 
   \( 1 \leq k \leq n, 0 \leq t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n \) and all \( n \).

3) \( N_t - N_s \in \text{Po}(\lambda(t-s)) \) for \( t > s \).

Furthermore, the times between jumps in a Poisson process are exponentially distributed with rate parameter \( \lambda \).

From the definition of a Poisson process, it follows that its mean is given by

\[
\mathbb{E}[N_t] = \lambda t. \tag{2.1}
\]

A zero mean process called a compensated Poisson process can be constructed from a Poisson process. A compensated Poisson process is defined as follows.

**Definition 2.3** Let \( N = \{N_t \mid t \geq 0\} \) be a Poisson process. Then the stochastic process \( \tilde{N} = \{\tilde{N}_t \mid t \geq 0\} \), where

\[
\tilde{N}_t = N_t - \lambda t, \quad t \geq 0,
\]

is a compensated Poisson process.

The compensated Poisson process has the desirable property of being a martingale with respect to its own filtration \((\mathcal{F}_t)_{t \geq 0}\).

### 2.1.3 Compound Poisson Process

The Poisson process has limited modeling capabilities in financial mathematics, as its jumps are constant in size. For the purpose of modeling jumps that are random in size, a compound Poisson process can be used. The compound Poisson process is defined as follows.

**Definition 2.4** Let \((Z_k)_{k \geq 1}\) denote an i.i.d. sequence of square-integrable random variables and let \( N = \{N_t \mid t \geq 0\} \) be a Poisson process independent of the sequence \((Z_k)_{k \geq 1}\). Then the process \( M = \{M_t \mid t \geq 0\} \), where

\[
M_t = \sum_{k=1}^{N_t} Z_k, \quad t \geq 0,
\]

is a compound Poisson process.

The compound Poisson process has independent increments \( M_{t_k} - M_{t_{k-1}} \), and its mean is given by

\[
\mathbb{E}[M_t] = \lambda t \mathbb{E}[Z_1]. \tag{2.2}
\]

Similarly to the case with the compensated Poisson process, the compensated compound Poisson process can be constructed using the compound Poisson process. The compensated compound Poisson process is defined by
\[ \hat{M} = \{ \hat{M}_t \mid t \geq 0 \}, \]
where \( \hat{M}_t = M_t - \lambda t \mathbb{E}[Z_1] \).

(2.3)

The compensated compound Poisson process is a martingale with respect to its own filtration.

### 2.1.4 Markov Processes

*Markov Processes* are a family of stochastic processes that are characterized by the future state only depending on the present state. This property is called the *Markov property*. Formally, a Markov process has the following definition.

**Definition 2.5** A continuous time process \( X = \{ X_t \mid t > 0 \} \) is a Markov process on the state space \( S \) if for any \( t \geq 0 \), \( \Delta t \geq 0 \) and \( x \in S \)

\[
P[X_{t+\Delta t} \leq x \mid X_s, s \leq t] = P[X_{t+\Delta t} \leq x \mid X_t].
\]

Markov processes are especially useful for the purpose of simulations, as the Markov property allows for computationally efficient simulations.

### 2.1.5 Jump-Diffusion Processes

*Jump-diffusion processes* are used in mathematical finance to describe both the randomness and the jumps of a price process. In general, a jump-diffusion process utilizes both a Wiener process and a compound Poisson process, in addition to a drift component, to describe the dynamics of the process. In the general case, a one-dimensional jump-diffusion process \( X = \{ X_t \mid t \geq 0 \} \) can be described by the following SDE.

\[
dX_t = a(X_t, t) dt + b(X_t, t) dW_t + c(X_t, Z, t) d\hat{N}_t,
\]

(2.4)

where \( W_t \) is a Wiener process and \( \hat{N}_t \) is a compensated Poisson process.

A process \( Y_t \) on the form

\[
dX_t = a(Y_t, t) dt + b(Y_t, t) dW_t,
\]

(2.5)

is called a diffusion-process.

From equations 2.4 and 2.5, it is apparent that jump-diffusion processes and diffusion processes in the general case are Markov processes. Therefore, these processes are well suited for computationally heavy simulations.

### 2.1.6 Radon-Nikodym Derivative

In finance, it is often desirable to change from one probability measure to another. For example, one may want to change from the risk neutral measure to the historical measure. The *Radon-Nikodym derivative* expresses a relation between two measures. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \( Z \) be a non-negative Borel function. Then, it holds that
\[ \tilde{P}(A) = \int_A Z(\omega) \, dP(\omega), \quad A \in \mathcal{F} \]  

(2.6)
is a measure satisfying

[ \[ P(A) = 0 \Rightarrow \tilde{P}(A) = 0. \] ]  

(2.7)

This is expressed as \( \tilde{P} \) being absolutely continuous with regard to \( P \) and is written as \( \tilde{P} \ll P \).

Furthermore, if \( E_P[Z] = 1 \) then \( \tilde{P}(A) \) is a probability measure. The variable \( Z \) is called the Radon-Nikodym derivative and is defined as follows.

**Definition 2.6** Let \( P \) and \( \tilde{P} \) be two measures on \( (\Omega, \mathcal{F}) \) and \( P \) \( \sigma \)-finite. If \( \tilde{P} \ll P \), then there exists a non-negative Borel function \( Z \) on \( \Omega \) such that

\[ \tilde{P}(A) = \int_A Z(\omega) \, dP(\omega), \quad A \in \mathcal{F}. \]

Furthermore, \( Z \) is almost everywhere unique. The function \( Z \) is called the Radon-Nikodym derivative of \( \tilde{P} \) with regard to \( P \) and is denoted by \( \frac{d\tilde{P}}{dP} \).

### 2.1.7 The Girsanov Theorem

The Girsanov theorem describes the change in behaviour of stochastic processes when there is a change of measure. For diffusion processes, an additional theorem called Novikov’s condition is useful. Novikov’s condition is given by the following.

**Theorem 2.1** Let \( W_t \) be a Wiener process defined on the filtered probability space \( (\Omega, \mathcal{F}_t, P, (\mathcal{F}_t)_{t \in [0,\infty)}) \)
and let \( \theta_t, t \geq 0 \) be an adapted process. Define

\[ Z_t = \exp \left\{ \int_0^t \theta_s \, dW_s - \int_0^t \theta_s^2 \, ds \right\}. \]

Then, if for each \( t \geq 0 \)

\[ E_P \left[ \exp \left\{ \frac{1}{2} \int_0^t \theta_s^2 \, dW_s \right\} \right] < \infty, \]

then for each \( t \geq 0 \)

\[ E_P[Z_t] = 1. \]

If this is the case, then the process \( (Z_t)_{t \geq 0} \) is a positive martingale.

Novikov’s condition can be utilized to state Girsanov’s theorem for diffusion processes, which is given below.

**Theorem 2.2** Let \( W_t \) be a Wiener process defined on the filtered probability space \( (\Omega, \mathcal{F}_t, P, (\mathcal{F}_t)_{t \in [0,T]}) \)
and let \( \theta_t, t \geq 0 \) an adapted process. Define
\[ Z_t = \exp \left\{ \int_0^t \theta_s dW_s - \int_0^t \theta_s^2 ds \right\}, \quad (2.8) \]

\[ \tilde{W}_t = W_t - \int_0^t \theta_s ds. \quad (2.9) \]

If \( \theta_t, \ t \geq 0 \) fulfills Novikov’s condition, then the stochastic process \((\tilde{W}_t)_{0 \leq t \leq T}\) is a Wiener process under the probability measure \(\tilde{P}\) given by

\[ \tilde{P}(A) = \int_A Z(\omega) \, dP(\omega), \quad A \in \mathcal{F}. \]

### 2.1.8 Monte Carlo Estimators

In the Monte Carlo simulation framework, a large number of simulations of a quantity are made in order to obtain an estimate of that quantity. Suppose that \(X\) is a random variable and that \(E[f(X)]\) is to be estimated. The standard Monte Carlo estimator of \(E[f(X)]\) is given by the sample average of \(n\) independently simulated copies of \(f(x)\).

\[ \mu_n = \frac{1}{n} \sum_{i=1}^{n} f(X_i). \quad (2.10) \]

By the law of large numbers, the standard Monte Carlo estimator \(\mu_n\) will converge to \(E[f(X)]\) as \(n \to \infty\). By the central limit theorem, it holds that

\[ \frac{\sqrt{n}(\mu_n - E[f(X)])}{\sigma} \to N(0,1), \quad (2.11) \]

where \(\sigma^2 = \text{Var}[f(X)]\). An unbiased estimate of \(\sigma\) is given by the sample standard deviation \(S_n\),

\[ S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (f(X_i) - \mu_n)^2. \quad (2.12) \]

From equations 2.10, 2.11 and 2.12, an expression of a confidence interval of \(E[f(X)]\) can be obtained. Given significance level \(p\), the confidence interval is given by

\[ \left( \mu_n - \frac{S_n}{\sqrt{n}} \Phi^{-1}(1-p/2), \ \mu_n + \frac{S_n}{\sqrt{n}} \Phi^{-1}(1-p/2) \right). \quad (2.13) \]

One measure of the performance of a Monte Carlo estimator \(\mu_n\) is the relative error \(\hat{e}_n\), which is defined as

\[ \hat{e}_n = \frac{S_n}{\mu_n \sqrt{n}}. \quad (2.14) \]
2.1.9 Importance Sampling

The confidence interval of $E[f(X)]$, as well as the relative error, can be made smaller by increasing the sample size or by reducing the sample standard deviation (or equivalently reducing the sample variance). To reduce the computational load, it is therefore desirable to reduce the sample standard deviation. This can be achieved through several different variance reduction techniques. In this paper, the variance reduction technique called *importance sampling* is considered.

The idea behind importance sampling is to change the probability measure that is sampled under in order to achieve a smaller variance. Consider the continuous random variable $X$, the function $f(X)$ and two different probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$. The following holds for the expected value of $f(X)$.

$$E_{\mathbb{P}}[f(X)] = \int_{\Omega} f(X) \, d\mathbb{P} = \int_{\Omega} f(X) \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \, d\tilde{\mathbb{P}} = E_{\tilde{\mathbb{P}}}[f(X) \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}] .$$

(2.15)

The Monte Carlo estimator when using importance sampling can be formulated as

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} f(X_i) \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}(X_i),$$

(2.16)

where $(X_i)_{1 \leq i \leq n}$ are sampled under $\tilde{\mathbb{P}}$. The variance of the estimator is given by

$$\text{Var}_{\tilde{\mathbb{P}}}[f(X) \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}] = E_{\tilde{\mathbb{P}}}[f(X)^2 \left( \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \right)^2] - \left( E_{\tilde{\mathbb{P}}}[f(X) \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}] \right)^2$$

(2.17)

In order to minimize the variance, the change of measure should be chosen so that

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \frac{E_{\mathbb{P}}[f(X)]}{f(X)} .$$

(2.18)

This is however impossible, as $E_{\mathbb{P}}[f(X)]$ is the quantity that is to be estimated. Thus, alternative methods must be used in order to find changes of measure that are close to optimal. The change of measure is however not always derived with the explicit goal of minimizing the variance. Depending on the application, importance sampling can be used to give the new probability measure desirable properties that in turn lead to a reduced variance. For example, in the context of mathematical finance, importance sampling can be used to increase the probability of a rare event occurring or to drive the value of a process up or down.

To perform Monte Carlo simulations without any variance reduction techniques is often referred to as *naive Monte Carlo simulation*.

2.1.10 Importance Sampling for Diffusion Processes

One method of finding a close to optimal change of measure for diffusion processes is outlined by Nykvist [8]. The aim of the method is to find a close to optimal parameter $\theta$ for the change of measure given by Girsanov’s theorem,
\[
\frac{d\tilde{p}}{dp} = \exp \left\{ \int_0^t \theta_s dW_s - \int_0^t \theta_s^2 ds \right\}.
\]

(2.19)

Consider a general diffusion process, with dynamics given by

\[
dX_t = a(X_t)dt + b(X_t)dW_t, \\
X_0 = x_0.
\]

(2.20)

The dynamics of the process are obtained by inserting 2.9 into 2.20

\[
dX_t = (a(X_t) + b(X_t)\theta(X_t, t))dt + b(X_t)d\tilde{W}_t, \\
X_0 = x_0.
\]

(2.22)

(2.23)

The method involves constructing classical subsolutions to a Hamilton-Jacobi type of PDE. The solutions are constructed from the Mané potential. With \(c, x, y \in \mathbb{R}\), the Mané potential at level \(c\) is given by

\[
S_c(x, y) = \inf \left\{ \int_t^\tau \left( c + L(\psi(s), \dot{\psi}(s)) \right) ds, \ \psi \in AC([0, \infty); \mathbb{R}^n), \ \psi(0) = x, \ \psi(\tau) = y, \ \tau > 0 \right\},
\]

(2.24)

where \(AC\) is the set of all absolutely continuous functions \(\psi : [0, T] \mapsto \mathbb{R}\) and \(L\) is the local rate function given by

\[
L(x, v) = \frac{1}{2} \left( b^{-1}(x)(v - a(x)) \right)^2.
\]

(2.25)

If the Mané potential is finite, a viscosity sub-solution to the stationary Hamilton-Jacobi equation

\[
H(y, \nabla S^c(x, y)) = c, \quad y \in \mathbb{R}
\]

is given by

\[
y \mapsto S^c(x, y).
\]

(2.26)

(2.27)

An asymptotically optimal change of measure is given by the Radon-Nikodym derivative with parameter

\[
\theta(t, x) = b(x)\nabla S^c(x_0, x).
\]

(2.28)

The Hamiltonian associated with the process 2.20 is given by

\[
H(x, p) = a(x)\nabla S^c(x_0, x) + \frac{1}{2} (\sigma(x)\nabla S^c(x_0, x))^2.
\]

(2.29)

The stationary Hamilton-Jacobi equation 2.26 then becomes

\[
a(x)\nabla S^c(x_0, x) + \frac{1}{2} (\sigma(x)\nabla S^c(x_0, x))^2 = c.
\]

(2.30)

This equation has the solution
\[ \nabla S'(x_0, x) = \frac{a(x)}{b(x)^2} \pm \frac{\sqrt{a(x)^2 + 2cb(x)^2}}{b(x)^2}, \quad x \neq x_0. \] (2.31)

From equation 2.28, it follows that an asymptotically optimal change of measure is given by the Radon-Nikodym derivative with parameter

\[ \theta(t, x) = -\frac{a(x)}{b(x)} \pm \frac{\sqrt{a(x)^2 + 2cb(x)^2}}{b(x)}, \quad x \neq x_0. \] (2.32)

### 2.2 Financial Pricing

#### 2.2.1 Types of Options

The most common types of options are European and American options. A European option gives the buyer the right to exercise the option at the expiration date, while an American option gives the buyer the right to exercise the option at the expiration date as well as any date between acquiring the option and the expiration date. European and American options have the same payoff function, which is given by

\[ \text{Payoff} = \max(S - K, 0), \quad \text{for call options}, \] (2.33)

\[ \text{Payoff} = \max(K - S, 0), \quad \text{for put options}, \] (2.34)

where \( S \) is the spot price of the underlying asset and \( K \) is the strike price. While there exist formulas for pricing European options, it is considerably harder to price American options. This is because the possibility of early exercise has to be taken into account at each date. American options are in general priced higher than European options because the buyer of an American option has the additional right of early exercise.

Options with other exercise rights or payoff structures are called exotic options. There exist many types of exotic options, but in this paper only Bermudan options are considered. Bermudan options have the same payoff function as American and European options, but give the buyer the right to exercise the option only at certain specified dates. The American option can therefore be seen as a specific case of the Bermudan option, where all dates are eligible for early exercise.

#### 2.2.2 Credit Valuation Adjustment

Assume a complete and arbitrage-free market. Let \((\Omega, \mathcal{F}_t, \mathbb{Q})\) be a filtered probability space. \(\mathcal{F}_t\) is the filtration modeling market information up to time \( t \) and \( \mathbb{Q} \) is the risk neutral measure. From here on, \( \mathbb{E}[\cdot | \mathcal{F}_t] \) is denoted by \( \mathbb{E}_t[\cdot] \).

Suppose a portfolio is held at time \( t \) and consists of OTC derivative contracts with a common expiration time \( T \). Also suppose that the counterparty has a risk of defaulting and that \( \tau \) is the stochastic default time of the counterparty. In this paper, only unilateral CVA is considered, meaning that the investor is assumed to have default probability zero. As previously mentioned,
CHAPTER 2. MATHEMATICAL BACKGROUND

CVA is the market value of the CCR losses incurred by a portfolio. Using this definition, CVA can be rewritten as

\[ \text{CVA}_t = \mathbb{E}_t^Q[\mathbb{I}\{\tau \leq T\} L_\tau D(t, \tau)], \]

(2.35)

where \( L_\tau \) is the portfolio exposure at time \( \tau \) and \( D(t, \tau) \) is the discount factor, based on the risk free rate, between times \( t \) and \( \tau \).

To be able to calculate CVA, it is required to calculate the exposure of the portfolio. For OTC derivatives, the exposure depends on the value of the contract. If the value of the contract is positive from the investor’s perspective and the counterparty defaults, the investor will only receive a fraction \( R \) of the agreed upon payments. On the other hand, if the value of the contract is negative from the investor’s perspective and the counterparty defaults, the investor is still obliged to pay the full amount on the remaining payment. Thus, given \( N \) different contracts in the portfolio, no netting agreements and no margin agreements, the exposure of the portfolio at time \( t \) is given by

\[ L_t = (1 - R) \sum_{i=1}^{N} \max \{ V^i_t, 0 \}, \]

(2.36)

where \( V^i_t \) is the value of derivative contract \( i \) at time \( t \). In this paper, it is assumed that \( R = 0 \).

The exposure of the portfolio at time \( t \) is then given by

\[ L_t = \sum_{i=1}^{N} \max \{ V^i_t, 0 \}. \]

(2.37)

Furthermore, assuming that default probability, credit exposure and discount rate are independent of each other, the expression 2.35 can be rewritten in the following way.

\[ \text{CVA}_t = \mathbb{E}_t^Q[\mathbb{I}\{\tau \leq T\} L_\tau D(t, \tau)] = \int_t^T \mathbb{E}_t^Q[L_s D(t, s)] dP(s), \]

(2.38)

where \( P \) is the distribution of \( \tau \).

Henceforth, if nothing else is specified, CVA refers to CVA\(_0\).
Chapter 3

Monte Carlo Pricing

3.1 Options

In the Monte Carlo setting, options are generally priced by discretizing the time interval until expiration and then completing the following steps.

- Simulate a large number of paths for the price of the underlying asset.
- Calculate the option value at each time point for each path.
- Obtain the Monte Carlo estimate of the option value using the simulated option values, i.e.

\[ \hat{V}_t = \frac{1}{n} \sum_{i=1}^{n} V^i_t, \]  

where \( V^i_t \) is the value of the option for path \( i \) at time \( t \).

It is the second step of the procedure that varies when pricing different types of options. For European options, it is straightforward to calculate the option value. At expiration, the payoff is known. The values at remaining times are acquired by discounting the payoff at expiration. Assume a discretized time interval \( t_1, \ldots, t_m \), where \( t_m = T \). For a European call option, the option values of path \( i \) are given by

\[ V^i_T = \max(S^i_T - K, 0) \]
\[ V^i_j = D_{t_j,T} V^i_T, \quad j = 1, \ldots, m - 1, \]  

where \( S^i_T \) is the asset spot price for path \( i \) at time \( T \) and \( D_{t_j,T} \) is the discount factor between times \( t_j \) and \( T \). In order to pathwise obtain the values of a European put option, the payoff function of a European put option is instead used.

When evaluating an American option in the Monte Carlo setting, the time discretization only allows for a finite number of exercise dates. This gives an error when calculating the value of an American option. It also means that an American option is treated as a Bermudan option when performing Monte Carlo simulations. The values of a Bermudan call option for path \( i \) are obtained in the following way.
CHAPTER 3. MONTE CARLO PRICING

- \( V^i_T = \max(S_T^i - K, 0) \)
- for \( j = m - 1, \ldots, 1 \)
  \( V^i_j = \max(S_j^i - K, C^i_j) \), if \( t_j \) is an early exercise date,
  \( V^i_j = D_{t_j, t_{j+1}} V^i_{t_{j+1}} \), else,

where \( C^i_j \) is the value of the option at path \( i \) conditional on not exercising at time \( t_j \). \( C^i_j \) is called the continuation value at time \( t_j \) and path \( i \). A possible way to calculate the continuation value is to for each path set the continuation value at time \( t_j \) and path \( i \) to

\[
C^i_j = D_{t_j, t_{j+1}} V^i_{t_{j+1}}.
\]

This approach, however, uses future market information to assess the continuation value and determine the optimal exercise strategy, and will therefore overestimate the option value. There are several approaches to estimating the continuation value in the Monte Carlo setting. One of these is to use regression based Monte Carlo methods \[9\].

3.2 Least Squares Monte Carlo

Least Squares Monte Carlo is a method for valuation of American and Bermudan options that was first introduced by Longstaff and Schwartz \[10\]. Least squares Monte Carlo estimates by regression the continuation value of an option.

Let \((\Omega, \mathcal{F}_t, \mathbb{Q})\) be a filtered probability space. \( \mathcal{F}_t \) is the filtration modeling market information up to time \( t \) and \( \mathbb{Q} \) is the risk neutral measure. Let \( F(\omega, s; t, T) \) denote the cash flow of the option conditional on the option not being exercised prior to time \( t \) and the optimal exercise strategy being followed between times \( t \) and \( s \). Here, \( t < s \leq T \). Let \( t_m = T \). The continuation value at time \( t_k \) is given by

\[
C(\omega; t_k) = E^Q_{t_k} \left[ \sum_{j=k+1}^{m} D_{t_k, t_j} F(\omega, t_j; t_k, T) \right].
\]

To implement the least squares Monte Carlo method, \( C(\omega; t_m-1) \) is estimated by least squares regression of the discounted values of \( F(\omega, t_m; t_k, T) \) onto a basis \( \Phi_{t_m-1} \) consisting of functions of the underlying asset’s price at time \( t_m-1 \). The option value at time \( t_m-1 \) can then be estimated by taking the maximum of the immediate exercise value at time \( t_m-1 \) and the estimated continuation value. Then \( C(\omega; t_m-2) \) is estimated by least squares regression of the discounted values of \( F(\omega, t_m-1; t_k, T) \) onto the basis \( \Phi_{t_m-2} \), and the option value at \( t_m-2 \) is estimated by taking the maximum of the immediate exercise value at time \( t_m-2 \) and the estimated continuation value. By recursion, the option value can be estimated at each time point until \( t_1 \). Only in the money paths are used for the regression, as out of the money paths will never be exercised.

In a Monte Carlo setting with \( n \) paths and \( m \) time points, the least squares Monte Carlo method is implemented by the following. Let \( C^i_j \) denote the continuation value for path \( i \) at time \( t_j \) conditional on the estimated option value \( V^i_{t_{j+1}} \) at time \( t_{j+1} \) and path \( i \), given by
\[ C_{t_j}^i = D_{t_j, t_{j+1}} V_{t_{j+1}}^i. \]  
(3.6)

Let \( \hat{C}_{t_j}^i \) denote the estimated continuation value of path \( i \) at time \( t_j \). Furthermore, let \( \{ \phi_1(\cdot), \ldots, \phi_M(\cdot) \} \) denote the basis functions used for regression and \( S_{t_j}^i \) the spot price of the underlying asset at path \( i \) and time \( t_j \). The estimated continuation value for an in the money path \( i \) at time \( t_j \) is given by

\[ \hat{C}_{t_j}^i = \sum_{k=1}^M \hat{\beta}_k \phi_k(S_{t_j}^i), \]  
(3.7)

where \( \hat{\beta} \) is given by

\[ \hat{\beta} = \min_{\beta \in \mathbb{R}^M} \sum_{l=1}^{n'} \left( C_{t_j}^l - \sum_{k=1}^M \beta_k \phi_k(S_{t_j}^l) \right)^2, \]  
(3.8)

and \( l = 1, \ldots, n' \) denote the indices of the in the money paths. The values of a Bermudan call option at path \( i \) are estimated in the following way.

- \( V_T^i = \max(S_T^i - K, 0) \)
- for \( j = m - 1, \ldots, 1 \)
  - \( V_{t_j}^i = \max(S_{t_j}^i - K, \hat{C}_{t_j}^i), \) if \( t_j \) is an early exercise date and \( S_{t_j}^i > K \),
  - \( V_{t_j}^i = D_{t_j, t_{j+1}} V_{t_{j+1}}^i, \) else.

Bermudan put options are valued by using the payoff function of a put option and the criteria \( S_{t_j}^i < K \) for determining whether path \( i \) is in the money at time \( t_j \).

### 3.3 Credit Valuation Adjustment

When constructing a Monte Carlo estimator of CVA it is utilized that CVA can be expressed in the following way

\[ \text{CVA}_t = E^Q_t[f(L, \tau)], \]  
(3.10)

where

\[ f(L, \tau) = \mathbb{1}\{\tau \leq T\} L_\tau D_{t, \tau}. \]  
(3.11)

Thus, a Monte Carlo estimator of CVA can be constructed as follows

\[ \hat{\text{CVA}}_t = \frac{1}{n} \sum_{i=1}^n f(L_i^i, \tau_i) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\tau_i \leq T\} L_i^i D_{t, \tau_i}. \]  
(3.12)

To obtain this Monte Carlo estimator, the default time as well as the portfolio exposure must be pathwise simulated.
Chapter 4

Method and Models

4.1 Asset Price Model

Within mathematical finance, there exist several different formulations of jump-diffusion processes. Most of these are based on Merton’s model \[11\]. Merton’s model is described by the SDE

\[
\frac{dX_t}{X_t} = \mu dt + \sigma dW_t + \hat{M}_t, \tag{4.1}
\]

where \(\hat{M} = \{\hat{M}_t \mid t \geq 0\}\) is a compensated compound Poisson process with the jumps being described by the random variable \((J_t - 1)_{t \geq t_1}\), where \((J_t)_{t \geq t_1}\) are i.i.d. log-normal distributed random variables. \(\mu\) is the instantaneous expected return on \(X_t\) and \(\sigma\) is the volatility of \(X_t\).

In Merton’s model, both \(\mu\) and \(\sigma\) are constant. A common way to present the SDE of Merton’s model is to move the compensation of the Poisson process into the drift. Equation 4.1 then becomes

\[
\frac{dX_t}{X_t} = \left(\mu - \lambda s E[J_t - 1] - \mu s E[J_t - 1]\right) dt + \sigma dW_t + (J_t - 1)dN_t. \tag{4.2}
\]

Here, \(N = \{N_t \mid t \geq 0\}\) is a Poisson process with intensity \(\lambda s\). Extensions of Merton’s model exist, where for example stochastic volatility is used or a different jump size distribution is considered. One of these extensions is Bates’ model \[12\]. Bates’ model has stochastic volatility, but is otherwise equivalent to Merton’s model. The SDE of Bates’ model can be written as follows.

\[
\frac{dX_t}{X_t} = (\mu - \lambda s E[J_{t_1} - 1]) dt + \sqrt{V_t} dW_t + (J_t - 1)dN_t,
\]

\[
dV_t = \kappa(\theta - V_t) dt + \sigma_v \sqrt{V_t} dW'_t,
\]

\[
\text{corr}(W_t, W'_t) = \rho dt, \tag{4.3}
\]

where \(V_t\) is the variance of the diffusion component of \(X_t\) at time \(t\), \(\sigma_v\) is the volatility of \(V_t\) and \(W' = \{W'_t \mid t \geq 0\}\) is a Wiener process. \(V = \{V_t \mid t \geq 0\}\) follows a mean reverting process with drift \(\theta\) and mean reversion rate \(\kappa\). \((J_t)_{t \geq t_1}\) are i.i.d. log-normal distributed random variables.

In this paper, Bates’ model with uncorrelated Wiener processes are used. Thus, the model given in equation 4.3 becomes
\[
\frac{dX_t}{X_t} = (\mu - \lambda^s E[J_t - 1])dt + \sqrt{V_t}dW_t + (J_t - 1)dN_t, \quad (4.4)
\]

\[
dV_t = \kappa(\theta - V_t)dt + \sigma \sqrt{V_t}dW_t', \quad (4.5)
\]

where \(W'_t\) is an independent copy of \(W_t\).

### 4.2 Default Time Model

A common family of models for counterparty default is the family of \textit{stochastic intensity models}. A stochastic intensity model is characterized by the stochastic intensity process

\[ \lambda^\tau_t, \quad \lambda^\tau_t \geq 0 \text{ for all } t \geq 0. \quad (4.5) \]

Under a stochastic intensity model, the probability of default is given by

\[ P(\tau \leq t) = 1 - \exp \left\{ - \int_0^t \lambda^\tau_u du \right\}. \quad (4.6) \]

From equation (4.6), it is evident that the default probability is an increasing function of the default intensity \(\lambda^\tau\).

For further reading on how stochastic intensity models can be used for CVA calculations, see for example the works of Crépey and Rahal \[13\], or Brigo and Vrins \[14\].

In this paper, a constant default intensity \(\lambda^\tau\) is used. The expression of the default probability in equation (4.6) can then be simplified to

\[ P(\tau \leq t) = 1 - e^{-\lambda^\tau t}. \quad (4.7) \]

Thus for a constant default intensity \(\lambda^\tau\), the occurrence of a default event is described by the first jump of a Poisson process with intensity \(\lambda^\tau\). Hence, for a constant intensity \(\lambda^\tau\), the default time \(\tau\) is exponentially distributed with rate parameter \(\lambda^\tau\),

\[ \tau \in \text{Exp}(\lambda^\tau). \quad (4.8) \]

### 4.3 Price Process Importance Sampling

When employing importance sampling on the asset price process, the computations can be simplified by handling the jump component and the diffusion component separately.

The diffusion component of the SDE driving Bates’ model is given by

\[ (\mu - \lambda^s E[J_t - 1])dt + \sqrt{V_t}dW_t. \quad (4.9) \]

With the notation of a general diffusion process given in equation (2.5), the coefficients of Bates’ model are given by
\\( a(x, t) = x(\mu - \lambda s E[J_{t_1} - 1]) , \)  
\\( b(x, t) = x \sqrt{V_t} . \)  

These coefficients can be used in the method described in section 2.1.10 to obtain the optimal change of measure parameter at level \( c \),

\[
\theta(x, t) = -\frac{a(x, t)}{b(x, t)} \pm \frac{\sqrt{a(x, t)^2 + 2cb(x, t)^2}}{b(x, t)} = \frac{\mu - \lambda s E[J_{t_1} - 1]}{\sqrt{V_t}} \pm \frac{\sqrt{\left(\mu - \lambda s E[J_{t_1} - 1]\right)^2 + 2cV_t}}{\sqrt{V_t}} .
\]

It is clear that two asymptotically optimal changes of measure can be chosen from equation 4.12 depending on the sign. In this paper, the choice was made to take the parameter

\[
\theta(x, t) = \frac{\mu - \lambda s E[J_{t_1} - 1]}{\sqrt{V_t}} + \frac{\sqrt{\left(\mu - \lambda s E[J_{t_1} - 1]\right)^2 + 2cV_t}}{\sqrt{V_t}} .
\]

The change of measure for the diffusion process is then given by

\[
\frac{d\tilde{P}}{dP} = \exp \left\{ \int_0^t \theta(X_s, s) dW_s - \int_0^t \theta(X_s, s)^2 ds \right\} .
\]

### 4.4 CVA and Default Time Importance Sampling

The importance sampling of the default time is based on a method described by Joshi and and Leung [15]. The idea behind the method is to sample only from relevant areas. In the context of CVA calculations, it is of interest to sample the default time such that the counterparty defaults before an option expires. This is especially effective if the default probability is very low, as naive Monte Carlo simulation would result in few defaults before option expiration and therefore few non-zero values to be used for estimating CVA. To motivate this sampling method, the Monte Carlo estimator of CVA given by equation 3.12 is rewritten.

\[
\frac{1}{n} \sum_{i=1}^{n} 1\{ \tau^i \leq T \} L^i_t D_{t, \tau^i} = \frac{1}{n} \sum_{j}^{n'} L^j_t D_{t, \tau^j} ,
\]

where \( j = 1, \ldots, n' \) denote the indices of the paths where \( \tau^j \leq T \). The expression 4.15 can be further rewritten as

\[
\frac{1}{n} \sum_{j}^{n'} L^j_t D_{t, \tau^j} = \left( \frac{1}{n'} \sum_{j}^{n'} L^j_t D_{t, \tau^j} \right) \frac{n'}{n} .
\]

It is noted that \( \frac{n'}{n} \) is a Monte Carlo estimator of the probability \( P(\tau \leq T) \) of a counterparty default occurring before expiration. Thus, the following Monte Carlo estimator of CVA is obtained.
\[
\hat{\text{CVA}}_t = \left( \frac{1}{n'} \sum_{i}^{n'} L_{i,t,\tau^i} \right) P(\tau \leq T). \tag{4.17}
\]

The algorithm for estimating CVA becomes:

- Simulate \( n \) paths of exposures \( (L_{i,t})_{i=1,...,n} \)
- Simulate \( n \) default times, \( \bar{\tau}^i \), that occur before time of expiration.
- Calculate the Monte Carlo estimate \( \hat{\text{CVA}}_t = \left( \frac{1}{n} \sum_{i}^{n} L_{i,t,\bar{\tau}^i} \right) P(\tau \leq T) \) \tag{4.18}

In order to perform the steps in the above algorithm, the probability \( P(\tau \leq T) \) must be calculated and the default times before expiration must be sampled. Let the investor hold a derivative asset at time \( t \) with no counterparty default having occurred before time \( t \). For a constant default intensity \( \lambda \), the probability of a counterparty default event before expiration is given by

\[
P(\tau \leq T) = 1 - e^{-\lambda^\tau (T-t)}. \tag{4.19}
\]

Given a constant default intensity \( \lambda^\tau \), the default time is described by

\[
\tau = t + \Delta t, \quad \Delta t \in \text{Exp}(\lambda^\tau). \tag{4.20}
\]

To sample a default time, inverse transform sampling can be used. A random sample of \( \tau \) is given by

\[
\tau^i = t - \log \left( \frac{1 - u}{\lambda^\tau} \right), \quad u \in U(0,1). \tag{4.21}
\]

To exclude default times that are after expiration, let \( u \in U(0, P(\tau \leq T)) \) when performing inverse transform sampling. Thus, a sampled default time before expiration is acquired by

\[
\bar{\tau}^i = t - \log \left( \frac{1 - u}{\lambda^\tau} \right), \quad u \in U(0, P(\tau \leq T)). \tag{4.22}
\]

For the purpose of calculating CVA, it is difficult to use importance sampling of asset prices. The difficulty lies in finding the factor \( \frac{dP}{d\tilde{P}} \), which is needed to adjust the Monte Carlo estimate of CVA under the new measure in order to obtain the estimate under the original measure. Finding this factor is outside the scope of this paper. However, CVA will be estimated under the new measure given by the optimal parameter \( \theta(x,t) \) for asset price simulation. This estimate is expected to differ from the estimate of CVA under the original measure.

### 4.5 Parameter Selection

The choice of parameters in both the asset price model and the default time model is highly dependent on the counterparties and the economy involved. It is outside the scope of this paper to calibrate the parameters to specific counterparties and economies, and the focus is instead
on the simulation methods.

The parameters chosen for Bates’ model, which was used for the asset price simulation, were

\[ \kappa = 2, \quad \theta = 0.04, \quad \rho = 0, \quad V_0 = 0.04 \]
\[ \lambda^s = 2, \quad \mu = 0.02, \quad J_0 \in \text{lognorm}(-0.2, 0.2). \] (4.23)

The portfolio for CVA calculation was chosen to consist of one call option with parameters

\[ S_0 = 100, \quad K = 100, \quad T = 1. \] (4.24)

The Bermudan option was chosen to have 9 equidistant early exercise dates,

\[ t' \in \{\Delta t, \ldots, T - \Delta t\}, \quad \Delta t = \frac{T}{10}. \] (4.25)

The discount factor used for calculations was obtained by using continuous compounding with the risk free interest rate \( r = 0.02, \)

\[ D_{s,t} = e^{r(t-s)} = e^{0.02(t-s)}. \] (4.26)

To minimize the variance of the simulated asset price, it is essential to determine a close to optimal level \( c \) for the change of measure. This was done by performing a grid search over a range of levels \( c \) and choosing the value that minimized the relative error.

Similarly, a suitable basis should be chosen for the least squares Monte Carlo method used for Bermudan option valuation. This was done by simulating a sample of several Monte Carlo estimates of CVA for the Bermudan option for different sets of basis functions. The most suitable basis was then chosen based on the sample average Monte Carlo estimate, the sample relative error of the Monte Carlo estimate and simplicity of the basis. The considered bases were the following.

\[ \{1, P_1(S), P_2(S), P_3(S)\}, \quad \{1, P_1(S), P_2(S), P_3(S)\}, \quad \{1, P_1(S), P_2(S), P_3(S)\}, \] (4.27)

where \( P_i(\cdot) \) is the \( i \)th Legendre polynomial. In particular

\[ P_1(x) = x, \quad P_2(x) = \frac{3x^2 - 1}{2}, \quad P_3(x) = \frac{5x^3 - 3x}{2}. \] (4.28)

### 4.6 Performance Evaluation

The performance evaluation of the importance sampling method for asset prices takes the following into account: the estimated asset price at all time points should be similar to that of naive Monte Carlo and the variance should be as low as possible.

For CVA calculations, the estimated CVA of naive Monte Carlo and of the method using importance sampling should be similar. For CVA, as opposed to the asset price, the sample
variance of one run of simulations does not always give a good reflection of the performance of the method. Because CVA for a path is zero if the simulated default time is after expiration, the CVA sample variance of one run of simulations is bound to be very large if there is a probability of default after expiration. Therefore when dealing with CVA simulations without importance sampling for the default times, the sample variance of Monte Carlo estimates of CVA for different runs of simulations is used to evaluate the performance of the method. This variance should be as low as possible. Furthermore, the estimated CVA of a Bermudan option should be higher than the estimated CVA of a European option with the same parameters where applicable, and lower than the estimated CVA using the optimal exercise strategy conditional on all market information up to expiration.

To determine whether there is a significant difference in variance between different runs of Monte Carlo simulations, Levene’s test can be used. Levene’s test is a statistical test that assesses the null hypothesis that the variances of two or more populations are equal. Thus, a low p-value implies that the variances of the populations are not equal. Levene’s test is well suited to non-normal distributed populations and approximately normal distributed populations. Thus, Levene’s test can be used on two populations of simulated quantities from two different Monte Carlo methods in order to assess differences in variance of the two methods. It can also be used on two samples of Monte Carlo estimates obtained from two different Monte Carlo methods in order to assess differences in the variance of the Monte Carlo estimates of the two methods. For details about Levene’s test as well as discussions of its usage, the reader is referred to the work of Gastwirth et.al. [16]. Unless stated otherwise, in this paper, a significant difference in variance between two populations refers to a difference in variance between the populations at the significance level 0.05.

4.7 Algorithms

The simulations are performed by computations at each time step of a discretized time interval. The time interval $[0, T]$ is discretized into $m$ equidistant time steps, i.e.

$$\Delta t = \frac{T}{m-1}, \quad t_j = (j - 1)\Delta t, \quad j = 1, \ldots, m. \quad (4.29)$$

The asset price simulation without importance sampling is performed according to the following algorithm.

---

**Algorithm 1:** simulateBatesModel

**Result:** Returns asset spot price and variance at all $m$ time points for all $n$ paths.

1. *for each* $i \in \{1, \ldots, n\}$ *
do*
2. $S_i^1 = s_0$
3. $V_i^1 = v_0$
4. *for each* $j \in \{1, \ldots, m - 1\}$ *
do*
5. Use $S_i^j$ and $V_i^j$ to calculate $S_i^{j+1}$ and $V_i^{j+1}$ according to Bates’ model.
6. *end*
7. *end*
8. *return*

---
With importance sampling, the asset price simulation algorithm becomes.

**Algorithm 2: simulateBatesModel_importanceSampling**

**Result:** Returns asset spot price and variance at all \( m \) time points for all \( n \) paths.

1. **for each** \( i \in \{1, \ldots, n\} \) **do**
2. \( S_1^i = s_0 \)
3. \( V_1^i = v_0 \)
4. **for each** \( j \in \{1, \ldots, m - 1\} \) **do**
5. Calculate the optimal change of measure parameter \( \theta_j^i \)
6. Calculate the change of measure \( \left( \frac{dP}{d\tilde{P}} \right)^i_j \)
7. Use \( S_j^i \) and \( V_j^i \) to calculate \( S_{j+1}^i \) and \( V_{j+1}^i \) under the new measure.
8. Multiply the asset price with the importance sampling weight:
   \[ S_{j+1}^i = S_{j+1}^i \left( \frac{dP}{d\tilde{P}} \right)^i_j. \]
9. **end**
10. **end**
11. **return**

To calculate CVA for an option, the default time needs to be simulated and the option exposure must be estimated. In the case of the Bermudan option, the option exposure is estimated using least squares Monte Carlo. The following algorithm describes how to evaluate a Bermudan option and calculate its exposure using least squares Monte Carlo.
Algorithm 3: exposureBermudan

**Result:** Returns exposure $L^i_j$ at all $m$ time points for all $n$ paths.

**Data:** Asset spot price $S^i_j$ at all $m$ time points for all $n$ paths.

1. Choose a set of basis functions $\{\phi_1(\cdot), \ldots, \phi_M(\cdot)\}$

2. for each $i \in \{1, \ldots, n\}$ do
   
   // Calculate exposure at expiration
   
   3. $V^i_m = \text{Payoff}(S^i_m, K)$
   
   4. $L^i_m = \max(V^i_m, 0)$

   // Compute option values using regression
   
   5. for each $j \in \{m-1, \ldots, 1\}$ do
      
      if $j$ is an early exercise index then
         
         Obtain in the money indices $\{k_1, \ldots, k_{m'}\}$
      
      Obtain $(\hat{C}^k_j)_{k \in \{k_1, \ldots, k_{m'}\}}$ by regressing $(D_{t_j,t_{j+1}}V^{k_1}_{j+1}, \ldots, D_{t_j,t_{j+1}}V^{k_{m'}}_{j+1})$ on $\left( \begin{array}{c} \phi_1(S^{k_1}_{j+1}) \\ \vdots \\ \phi_1(S^{k_{m'}}_{j+1}) \\ \phi_M(S^{k_1}_{j+1}) \\ \vdots \\ \phi_M(S^{k_{m'}}_{j+1}) \end{array} \right)$

      // Compute option value for in the money paths
      
      9. for each $k \in \{k_1, \ldots, k_{m'}\}$ do
         
         $\hat{V}^k_j = \max(\text{Payoff}(S^k_j, K), \hat{C}^k_j)$
      
      end
   
   // Compute option value for out of the money paths
   
   12. for each $l \in \{1, \ldots, n\} \setminus \{k_1, \ldots, k_{m'}\}$ do
         
         $\hat{V}^l_j = D_{t_j,t_{j+1}}\hat{V}^l_{j+1}$
      
      end
   
   else
      
      15. for each $i \in \{1, \ldots, n\}$ do
         
         $\hat{V}^i_j = D_{t_j,t_{j+1}}\hat{V}^i_{j+1}$
      
      end
   
   end

   19. $L^i_j = \max(\hat{V}^i_j, 0)$

22. end

The following algorithm describes how to calculate the exposure of a European option.
Algorithm 4: exposureEuropean

Result: Returns exposure $L^i_j$ at all $m$ time points for all $n$ paths.
Data: Asset spot price $S^i_j$ at all $m$ time points for all $n$ paths.

1 for each $i \in \{1, \ldots n\}$ do
  // Calculate exposure at expiration
  2 $V^i_m = \text{Payoff}(S^i_m, K)$
  3 $L^i_m = \max(V^i_m, 0)$
  4 for each $j \in \{m - 1, \ldots, 1\}$ do
    5 $V^i_j = D_{t_j, t_{j+1}} V^i_{j+1}$
  6 end
  7 $L^i_j = \max(V^i_j, 0)$
8 end

The CVA calculations without importance sampling of the default time are carried out in the following way.

Algorithm 5: calculateCVA

Result: Returns CVA estimates for all $n$ paths.
Data: Asset spot price $S^i_j$ at all $m$ time points for all $n$ paths.

1 Simulate $n$ default times $(\tau^i)_{i=1,\ldots,n}$
2 Estimate the exposures $\{L^i_j\}$ by exposureBermudan or exposureEuropean
3 for each $i \in \{1, \ldots, n\}$ do
  4 CVA$^i$ = $L^i_{\tau^i} D_{0,\tau^i} 1\{\tau^i \leq T\}$
5 end
6 return

With importance sampling of the default time, the CVA calculation algorithm becomes

Algorithm 6: calculateCVA_importanceSampling

Result: Returns CVA estimates for all $n$ paths.
Data: Asset spot price $S^i_j$ at all $m$ time points for all $n$ paths.

1 Calculate probability of default before exposure, $P(\tau \leq T)$
2 Simulate $n$ default times $(\tau^i)_{i=1,\ldots,n}$ before expiration
3 Estimate the exposures $\{L^i_j\}$ by exposureBermudan or exposureEuropean
4 for each $i \in \{1, \ldots, n\}$ do
  5 CVA$^i$ = $L^i_{\tau^i} D_{0,\tau^i}$
    Multiply with the importance sampling weight: CVA$^i$ = CVA$^i$ $P(\tau \leq T)$
6 end
7 return
4.8 Simulations

This section describes the simulations performed in order to evaluate the performance of the employed methods. All simulations were performed on a time interval discretized into \( m = 101 \) equidistant time points and the parameter values specified in section 4.5 were used.

4.8.1 Asset Price

To evaluate the effectiveness of the importance sampling method used for the asset price, the optimal energy level was first found. This was done through a grid search where the mean relative error taken over all time points was plotted against the energy level \( c \). The relative error was based on \( n = 20000 \) paths.

Simulations of the asset price for \( n = 100000 \) paths were performed in order to investigate the performance of the importance sampling method. The simulation was performed both with importance sampling and with naive Monte Carlo. For the importance sampling method, the optimal energy level \( c \) was used. The asset prices obtained from these simulations were used to compute the Monte Carlo estimate of the asset price at each time point for both methods. Through these quantities, a two sided confidence interval of the Monte Carlo estimate could be calculated at a 95% confidence level.

4.8.2 CVA of a European Option

For estimation of CVA for the European option, the price paths were simulated without importance sampling. 10 different runs of simulations were performed with \( n = 100000 \) paths for each of the default intensities \( \lambda^\tau \in \{0.01, 1, 10\} \), resulting in 10 different sets of asset price paths and default time paths. The default time paths were calculated both with naive Monte Carlo and with importance sampling. With these values, a sample of 10 CVA Monte Carlo estimates could be computed for each default intensity, both for naive Monte Carlo and for the method utilizing importance sampling of the default times. The mean, variance and relative error of these samples could then be calculated.

Simulations of \( n = 100000 \) paths were performed in order to investigate the variance of CVA under the optimal measure for the asset price. For the new measure given by equation 4.13, the optimal energy level \( c \) found as described above was used. Under the new measure, a Monte Carlo estimate of CVA, an estimate of the variance of CVA and a two sided confidence interval of the CVA estimate at a 95% confidence level were calculated using importance sampling of the default time. The default intensity \( \lambda^\tau = 1 \) was used. Simulations of \( n = 100000 \) paths were also performed under the original measure for the asset price. Importance sampling was used for the default times. Using the simulated asset prices and default times, an estimate of CVA, an estimate of the variance of CVA and a 95% confidence interval of the CVA estimate were computed.
4.8.3 CVA of a Bermudan Option

In order to determine an appropriate basis for CVA calculations, a grid search was performed
over different sets of basis functions, for which 10 runs of Monte Carlo simulations were per-
formed, yielding 10 different Monte Carlo estimates of CVA of the Bermudan option. The
number of paths per run of simulations was $n = 10000$. The most suitable basis was chosen
based on sample average Monte Carlo estimate, sample relative error of the Monte Carlo esti-
mate and simplicity of the basis.

Similarly to the case with the European option, the asset price paths were calculated without
importance sampling. 10 different sets of $n = 100000$ asset price paths were simulated for
default intensities $\lambda^\tau \in \{0.01, 1, 10\}$. 10 different sets of $n = 100000$ default times were sim-
ulated both with naive Monte Carlo and importance sampling. With these values, a sample
of 10 CVA Monte Carlo estimates could be computed for each default intensity and for both
methods. For these calculations, the most suitable basis obtained as described above was used.
The mean, variance and relative errors of these samples were calculated.

Simulations with $n = 100000$ paths were also performed under the optimal measure for the
asset price given by equation 4.13. For these simulations, the optimal energy level $c$ found as
described above was used, and the default intensity was $\lambda^\tau = 1$. Under the new measure, the
Monte Carlo estimate of CVA, the estimate of the variance of CVA and a two sided confidence
interval of the CVA estimate at a 95% confidence level were calculated using importance sam-
ping of the default time. Additionally, simulations of $n = 100000$ paths were performed under
the original measure for the asset price, with importance sampling being used for the default
times. Using the simulated asset prices and default times, an estimate of CVA, an estimate of
the variance of CVA and a 95% confidence interval of the CVA estimate were computed.
Chapter 5

Results

The simulations from which the results in this chapter were obtained were all performed on a time interval discretized into \( m = 101 \) equidistant time points. All simulations used the parameter values specified in section 4.5.

5.1 Asset Price

The mean relative error taken over all time points is plotted against energy level \( c \) in figure 5.1. The relative errors were calculated using \( n = 20000 \) asset price paths.

![Figure 5.1: Relative error plotted against energy level \( c \).](image)

The energy level minimizing the relative error is \( c = 0.37 \). Hence, \( c = 0.37 \) was chosen as the optimal energy level.
In figure 5.2, the first 20 paths of the asset price as well as the volatility are plotted. These values were obtained by simulation with naive Monte Carlo.

Figure 5.2: Plots of first 20 paths of asset price (left) and volatility of asset price (right) against the indices of the time points.

In figure 5.3, the Monte Carlo estimate of the asset price with naive Monte Carlo, as well as the difference between the Monte Carlo estimate of the asset price with importance sampling and with naive Monte Carlo, $\hat{S}_n^{IS} - \hat{S}_n$, are plotted against the indices of the time points. The Monte Carlo estimates were computed from $n = 100000$ paths.

Figure 5.3: Plot of the asset price Monte Carlo estimate with naive Monte Carlo against the indices of the time points (left). Plot of the difference between the asset price Monte Carlo estimate with importance sampling and with naive Monte Carlo against the indices of the time points (right).

Figure 5.3 shows an increasing trend of the asset price Monte Carlo estimate with time. This is consistent with the fact that the drift of the asset price model was set to $r = 0.02$. The plot
of the difference in Monte Carlo estimate between the case with importance sampling and the case with naive Monte Carlo indicates that the two Monte Carlo estimates are similar. There is also no clear sign of overestimation or underestimation of the asset price by the importance sampling method compared to the naive Monte Carlo method.

In figure 5.4 the Monte Carlo asset price variance estimates are plotted at all time points. The variance was calculated over \( n = 100000 \) simulated paths.

![Figure 5.4: Plots of the Monte Carlo estimate of asset price variance against the indices of the time points. To the left is the variance for naive Monte Carlo and to the right is the variance for the method using importance sampling.](image)

It is seen in figure 5.4 that the Monte Carlo estimate of the variance of the asset price for both naive Monte Carlo and importance sampling are approximately linear functions of time. It is also seen that the variance is consistently smaller for the estimate obtained using importance sampling. The average relative decrease in variance calculated over all time points was 0.2761. This is a substantial decrease in variance. Levene’s test on the populations of asset prices at expiration of all paths, for important sampling and naive Monte Carlo, returned a p-value of 0. This indicates a significant difference in variance between the two populations.

In figure 5.5 the asset price Monte Carlo estimate at expiration together with a two sided 95% confidence interval is plotted against the number of paths, both for the case with naive Monte Carlo and the case with importance sampling. The results were based on \( n = 100000 \) paths.
CHAPTER 5. RESULTS

Figure 5.5: Plots of asset price Monte Carlo estimate at expiration together with a two sided 95% confidence interval against the number of paths. For the case with naive Monte Carlo (left) and for the case with importance sampling (right).

Notably, by inspection of figure 5.5, the variance reduction of the method using importance sampling is not discernible. Moreover, for this run of simulations, the importance sampling method underestimates both the upper and the lower confidence interval limits of the asset price at expiration, compared to naive Monte Carlo. However, the asset price estimate at expiration is similar for both methods, which is consistent with the differences in the estimate plotted in figure 5.3.

5.2 CVA for a European Option

The Monte Carlo estimates of the exposure of one European call option are plotted against the indices of the time points in figure 5.6. Note that the option value at all times is positive and the exposure is therefore equal to the option value. The Monte Carlo estimates were based on 100000 asset price paths.
Figure 5.6: Monte Carlo estimate of the exposure of one European call option plotted against the indices of the time points.

As expected, the estimated exposure of the European option plotted in figure 5.6 is an increasing function of the time indices. The positive slope is caused by the discounting of the payoff at expiration.

In table 5.1 Monte Carlo estimates of CVA, both with naive Monte Carlo and with importance sampling are shown for one European call option. The estimates were based on 10 different runs of simulations with $n = 100000$ paths and were computed for the default intensities $\lambda^\tau \in \{0.01, 0.1, 10\}$.

<table>
<thead>
<tr>
<th>CVA Monte Carlo Estimates Naive</th>
<th>$\lambda^\tau = 0.01$</th>
<th>0.1671</th>
<th>0.1845</th>
<th>0.1623</th>
<th>0.1573</th>
<th>0.1843</th>
<th>0.1562</th>
<th>0.1727</th>
<th>0.1630</th>
<th>0.1705</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda^\tau = 1$</td>
<td>11.0005</td>
<td>10.9300</td>
<td>10.8820</td>
<td>10.9226</td>
<td>10.9583</td>
<td>10.9182</td>
<td>10.9906</td>
<td>10.8831</td>
<td>11.0059</td>
</tr>
<tr>
<td></td>
<td>$\lambda^\tau = 10$</td>
<td>17.3317</td>
<td>17.4479</td>
<td>17.4628</td>
<td>17.3168</td>
<td>17.2975</td>
<td>17.3049</td>
<td>17.3571</td>
<td>17.1947</td>
<td>17.3371</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CVA Monte Carlo Estimates Importance Sampling</th>
<th>$\lambda^\tau = 0.01$</th>
<th>0.1716</th>
<th>0.1727</th>
<th>0.1725</th>
<th>0.1717</th>
<th>0.1730</th>
<th>0.1719</th>
<th>0.1722</th>
<th>0.1734</th>
<th>0.1725</th>
<th>0.1717</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda^\tau = 10$</td>
<td>17.3309</td>
<td>17.4472</td>
<td>17.4624</td>
<td>17.3165</td>
<td>17.2973</td>
<td>17.3041</td>
<td>17.3570</td>
<td>17.1940</td>
<td>17.3369</td>
<td>17.4103</td>
</tr>
</tbody>
</table>

Table 5.1: Monte Carlo estimates of CVA for a European call option for 10 different runs of simulations, for both naive Monte Carlo and Monte Carlo with importance sampling.

The mean, variance and relative error for the samples of Monte Carlo estimates shown in table 5.1 are displayed in table 5.2.
CHAPTER 5. RESULTS

Table 5.2: Mean, variance and relative error for the samples of Monte Carlo estimates of CVA. The estimates were computed for 10 different runs of simulations, both with naive Monte Carlo and Monte Carlo with importance sampling.

<table>
<thead>
<tr>
<th>( \lambda^\tau = 0.01 )</th>
<th>Mean</th>
<th>Mean IS</th>
<th>Var</th>
<th>Var IS</th>
<th>Relative Error</th>
<th>Relative Error IS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^\tau = 1 )</td>
<td>10.9414</td>
<td>10.9313</td>
<td>0.0058</td>
<td>0.0028</td>
<td>0.0058</td>
<td>0.0028</td>
</tr>
<tr>
<td>( \lambda^\tau = 10 )</td>
<td>17.3460</td>
<td>17.3457</td>
<td>0.0063</td>
<td>0.0063</td>
<td>0.0014</td>
<td>0.0014</td>
</tr>
</tbody>
</table>

In table 5.3, the p-values of Levene’s test on the populations of pathwise simulated CVA values with naive Monte Carlo and with importance sampling are displayed.

<table>
<thead>
<tr>
<th>p-value</th>
<th>( \lambda^\tau = 0.01 )</th>
<th>( \lambda^\tau = 1 )</th>
<th>( \lambda^\tau = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2749 ( \cdot 10^{-4} )</td>
<td>0.0373</td>
<td>0.9939</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.3: p-values for Levene’s test on the populations of CVA estimates for a European option, with importance sampling and with naive Monte Carlo.

Tables 5.1 and 5.2 show that the Monte Carlo estimates of CVA are similar for both methods. The CVA estimate increases with the default intensity. This is in line with expectations, as a high default intensity implies a high default probability, which in turn means that the counterparty has a high probability of being unable to fulfil its contractual obligations as a result of a default.

The variance is significantly lower when using importance sampling compared to naive Monte Carlo for \( \lambda^\tau = 0.01 \), which is confirmed by the small p-value of Levene’s test of 1.2749 \( \cdot 10^{-4} \) (see table 5.3). This is expected, as few paths with a counterparty default will be simulated using naive Monte Carlo for such a small default probability. The variance reduction is smaller for \( \lambda^\tau = 1 \), and the p-value of Levene’s test of 0.0373 indicates a significant variance reduction. For \( \lambda^\tau = 10 \), the obtained variances of the two methods are identical, and the high p-value of Levene’s test of 0.9939 indicates no significant difference in variance between the methods. Because the default probability is so high, a large portion of the paths is expected to have a counterparty default before expiration. The number of paths with nonzero CVA contribution are therefore expected to be similar for naive Monte Carlo and importance sampling. The observed relative error follows a similar pattern as the variance. The relative error is much smaller for the method with importance sampling when \( \lambda^\tau = 0.01 \). The difference is smaller for \( \lambda^\tau = 1 \) and the relative errors of the two methods are equal for \( \lambda^\tau = 10 \).

5.3 CVA for a European Option under a new Measure

The Monte Carlo estimate of CVA, the estimated variance and the relative error of the estimate for one European call option, under the original measure and the optimal measure for asset price simulation, are displayed in table 5.4. The estimates were both calculated using importance sampling of the default time. The default intensity was set to \( \lambda^\tau = 1 \) and \( n = 100000 \) paths were simulated.
Table 5.4: CVA Monte Carlo estimate, estimated variance and relative error for a European call option, under the original measure and the new measure.

From table 5.4, it can be seen that the CVA estimate is larger under the new measure. The variance is also larger under the new measure, while the relative error is slightly smaller. Levene’s test on the populations of simulated CVA values under the old and the new measure resulted in a p-value of $1.4050 \cdot 10^{-168}$, which implies a significant difference in variance between the populations.

In figure 5.7, the CVA Monte Carlo estimate together with a two sided 95% confidence interval is plotted against the number of paths, under both the original measure and the new measure. The results were based on $n = 100000$ paths.

Figure 5.7: Plots of CVA Monte Carlo estimate for a European call option together with a two sided 95% confidence interval against the number of paths, under the original measure (left) and the new measure (right).

Figure 5.7 shows that both the CVA Monte Carlo estimate and the estimated variance are consistently larger under the new measure. This is consistent with the previous observations in this section.

5.4 CVA for a Bermudan Option

In table 5.5, the Monte Carlo estimates of CVA of a Bermudan call option for 10 different runs of simulations of $n = 10000$ paths are displayed. The option has 9 equidistant exercise dates. The default intensity was set to $\lambda^v = 1$. Table 5.6 shows the mean, variance and relative error of the sample of Monte Carlo estimates. The estimates were computed for the bases
where $P_i(\cdot)$ is the $i$th Legendre polynomial. The estimates were also computed for the optimal exercise strategy conditional on all market information up to time $T$.

\begin{equation}
\{1\}, \{1, S\}, \{1, S, S^2\}, \{1, S, S^2, S^3\}, \{1, P_1(S)\}, \{1, P_1(S), P_2(S)\}, \{1, P_1(S), P_2(S), P_3(S)\},
\end{equation}

Table 5.5: Monte Carlo estimates of the value of a Bermudan call option for 10 different runs of simulations and for different sets of regression basis functions.

<table>
<thead>
<tr>
<th>Basis</th>
<th>CVA Monte Carlo Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1}</td>
<td>17.2195 17.6302 17.6523 17.3946 17.4047 17.0113 17.1123 17.1144 17.3903 17.0226</td>
</tr>
<tr>
<td>{1, P_1(S), P_2(S)}</td>
<td>11.0121 11.3059 11.1213 10.9113 11.1336 10.8789 10.8406 10.7969 10.8248</td>
</tr>
<tr>
<td>{1, P_1(S), P_2(S), P_3(S)}</td>
<td>11.0121 11.3059 11.1213 10.9113 11.1336 10.8789 10.8406 10.7969 10.8248</td>
</tr>
</tbody>
</table>

The results in tables 5.5 and 5.6 indicate that the basis \{1\} considerably overestimates the option value. The estimated values of CVA for this basis are even larger than the estimates of CVA obtained when following the optimal exercise strategy conditional on all market information up to expiration, which is known to overestimate the option value. However, for the rest of the bases, both the CVA estimates and the relative error are very similar. Therefore, the most simple basis, \{1, S\} was chosen as the most suitable basis.

The Monte Carlo estimate of the exposure of one Bermudan call option with 9 equidistant early exercise dates is plotted against the indices of the time points in figure 5.8. Note that the option value at all times is positive and the exposure is therefore equal to the option value. The regression basis functions \{1, S\} were used and the estimates were based on $n = 100000$ asset price paths.
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Figure 5.8: Monte Carlo estimate of the exposure of one Bermudan call option plotted against the indices of the time points.

Compared to the plot of the estimated value of a European call option (see figure 5.6), the estimated value of the Bermudan option has a similar increasing trend. However, there are jumps in the estimated option value. These are explained by the possibility of early exercise at the time indices of the jumps.

In table 5.7 Monte Carlo estimates of CVA, both with naive Monte Carlo and with importance sampling, for one Bermudan call option and for 10 different runs of simulations with \( n = 100000 \) paths are shown. The simulations were performed for the default intensities \( \lambda \in \{0.01, 1, 10\} \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>CVA Monte Carlo Estimates Naive</th>
<th>CVA Monte Carlo Estimates Importance Sampling</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda = 0.01 )</td>
<td>0.1777 0.1679 0.1593 0.1794 0.1689 0.1629 0.1689 0.1749 0.1689 0.1704</td>
<td>0.1715 0.1716 0.1735 0.1721 0.1715 0.1740 0.1724 0.1728 0.1730</td>
</tr>
<tr>
<td>( \lambda = 10 )</td>
<td>17.4514 17.3587 17.4619 17.4885 17.2837 17.3064 17.4660 17.3850 17.2395 17.3584</td>
<td>17.4435 17.3523 17.4244 17.4959 17.2852 17.3175 17.4422 17.3739 17.2398 17.3252</td>
</tr>
</tbody>
</table>

Table 5.7: Monte Carlo estimates of CVA for a Bermudan call option for 10 different runs of simulations, for both naive Monte Carlo and Monte Carlo with importance sampling.

The mean, variance and relative error for the sample of Monte Carlo estimates shown in table 5.7 are displayed in table 5.8.
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<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Mean IS</th>
<th>Var</th>
<th>Var IS</th>
<th>Relative Error</th>
<th>Relative Error IS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda^\tau = 0.01)</td>
<td>0.1708</td>
<td>0.1724</td>
<td>4.3660 \cdot 10^{-5}</td>
<td>7.6179 \cdot 10^{-4}</td>
<td>0.0122</td>
<td>0.0016</td>
</tr>
<tr>
<td>(\lambda^\tau = 1)</td>
<td>10.9704</td>
<td>10.9766</td>
<td>0.0095</td>
<td>0.0047</td>
<td>0.0028</td>
<td>0.0020</td>
</tr>
<tr>
<td>(\lambda^\tau = 10)</td>
<td>17.3800</td>
<td>17.3700</td>
<td>0.0074</td>
<td>0.0065</td>
<td>0.0016</td>
<td>0.0015</td>
</tr>
</tbody>
</table>

Table 5.8: Mean, variance and relative error for the sample of Monte Carlo estimates of CVA for 10 different runs of simulations, for both naive Monte Carlo and Monte Carlo with importance sampling.

In table 5.9, the p-values of Levene’s test on the populations of simulated CVA values for all paths, with naive Monte Carlo and with importance sampling, are displayed.

\[
\begin{aligned}
\text{p-value} & \quad \lambda^\tau = 0.01 & \quad \lambda^\tau = 1 & \quad \lambda^\tau = 10 \\
9.2877 \cdot 10^{-4} & 0.5821 & 0.8080 \\
\end{aligned}
\]

Table 5.9: p-values of Levene’s test on the populations of simulated CVA values for a Bermudan option, with importance sampling and with naive Monte Carlo.

The displayed values in tables 5.8 and 5.9 (the corresponding table for the European option) show similar patterns for the CVA estimates. The sample means of the estimates are similar for both methods across the three different default intensities. The relative error and the variance are much smaller for the method with importance sampling for \(\lambda^\tau = 0.01\). This variance reduction is confirmed by the p-value of Levene’s test of 9.2877 \cdot 10^{-4} (see table 5.9). The method using importance sampling has smaller variance and relative error for \(\lambda^\tau = 1\) compared to naive Monte Carlo as well. However, the difference is smaller than for \(\lambda^\tau = 0.01\). The p-value of Levene’s test of 0.5821 indicates that no significant variance reduction is achieved by using importance sampling for \(\lambda^\tau = 1\), as opposed to the case with the European option where a significant difference in variance was observed. The variances and relative errors for naive Monte Carlo and importance sampling are similar for \(\lambda^\tau = 10\). However, both the relative error and the variance are slightly smaller for the method utilizing importance sampling, which is not the case for the European option. Levene’s test indicates no significant difference in variance between the two methods, with a p-value of 0.8080. The relative error for the Bermudan option is similar to that for the European option for all considered default intensities. Notably, the relative error of naive Monte Carlo for \(\lambda^\tau = 1\) is considerably lower for the Bermudan option than for the European option, 0.0028 compared to 0.0058. It is also notable that the CVA estimate is consistently slightly higher for the Bermudan option compared to the European option. This is expected, as a Bermudan option has a higher value than a European option with the same parameters where applicable.

5.5 CVA for a Bermudan Option under a new Measure

The Monte Carlo estimate of CVA, the estimated variance and the relative error of the estimate of one Bermudan call option, under the original measure and the optimal measure for asset price simulation, are displayed in table 5.10. Both of the estimates were calculated by using importance sampling of the default time. The option has 9 equidistant exercise dates. The default intensity was set to \(\lambda^\tau = 1\) and \(n = 100000\) paths were simulated.
CHAPTER 5. RESULTS

<table>
<thead>
<tr>
<th></th>
<th>CVA Estimate</th>
<th>Variance</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original Measure</td>
<td>11.0364</td>
<td>117.9449</td>
<td>0.9842</td>
</tr>
<tr>
<td>New Measure</td>
<td>12.7293</td>
<td>140.7910</td>
<td>0.9321</td>
</tr>
</tbody>
</table>

Table 5.10: CVA Monte Carlo estimate, estimated variance and relative error for a Bermudan call option, under the original measure and the new measure.

From table 5.10 it can be seen that the CVA estimate is larger under the new measure. The estimated variance is also larger under the new measure, while the relative error is slightly lower under the new measure. These results are similar to those of the European option. Levene’s test on the populations of simulated CVA values under the original measure and the new measure resulted in a p-value of $4.6027 \cdot 10^{-110}$, which implies a significant difference in variance between the populations. Notably, the estimated variance and the relative error are considerably smaller for the Bermudan option compared to the European option, both under the original measure and the new measure.

In figure 5.9 the CVA Monte Carlo estimate together with a two sided 95% confidence interval is plotted against the number of paths, under both the original measure and the new measure. The results were based on $n = 100000$ paths.

![Figure 5.9](image)

Figure 5.9: Plots of CVA Monte Carlo estimates for a Bermudan call option together with a two sided 95% confidence interval against the number of paths, under the original measure (left) and the new measure (right).

Figure 5.9 shows that both the CVA Monte Carlo estimate and the estimated variance are consistently larger under the new measure. This is consistent with the previous observations in this section.
Chapter 6

Discussion and Conclusions

The first objective of this paper was to investigate how to decrease the variance of asset price simulation. By using a Monte Carlo importance sampling method with an optimal change of measure based on the diffusion component of the price process, a significant variance reduction was achieved compared to naive Monte Carlo. Because the importance sampling employed in the simulation only takes the diffusion part of the price process into account, it might be possible to reduce the variance further by using an importance sampling algorithm that also takes the jumps into account when changing the measure. An area of further research is to investigate if the variance of the simulation of an asset price following a jump-diffusion process can be further reduced by employing an importance sampling algorithm that takes both the jump and the diffusion components into account.

The second objective of the paper was to investigate how the price of a Bermudan option can be estimated using regression based Monte Carlo simulation. By using least squares Monte Carlo, the asset price of a Bermudan option could be estimated and later used for CVA calculations. It is however hard to evaluate how good the estimate is, as it is difficult to find a baseline Monte Carlo estimate to compare the obtained estimate with. The estimated prices were deemed reasonable, as the estimated CVA of the Bermudan option was larger than the estimated CVA of a European call option with the same parameters where applicable and smaller than the CVA estimate obtained by using the optimal exercise strategy conditional on all market information up to expiration. This is a large interval, and it is of interest to find a more accurate way of evaluating the estimated price of a Bermudan option. A suggestion for further work is to investigate methods of improving the evaluation of the estimated option price. Additional research could also be made on other regression based methods for Bermudan option pricing.

The third objective of the paper was to investigate how to decrease the variance of CVA simulation for European and Bermudan options. For a European option, it was shown that a variance reduction compared to naive Monte Carlo could be achieved by using a Monte Carlo method based on importance sampling of the default time. The variance reduction was significant for the default intensities $\lambda^* = 0.01$ and $\lambda^* = 1$, while the variances of the two methods were not significantly different for $\lambda^* = 10$. This was consistent with the expectation that the variance reduction should be large for small default probabilities and small for large default probabilities. For a Bermudan option, similar results were obtained when comparing simulations of
CVA using naive Monte Carlo and least squares Monte Carlo to simulations of CVA using importance sampling of the default time and least squares Monte Carlo. However, for Bermudan options, the variance reduction was only significant for \( \lambda^\tau = 0.01 \), while the variances were not significantly different for \( \lambda^\tau = 1 \) or \( \lambda^\tau = 10 \). This was, again, consistent with the results expected from theory. As previously discussed, there was no baseline naive Monte Carlo estimate to compare the simulated CVA values to. The comparison was instead made between two methods using least squares Monte Carlo, one of them using importance sampling of the default time and one of them not using importance sampling. Thus for the Bermudan option, it was shown that a significant variance reduction was achieved in the least squares Monte Carlo setting by using importance sampling of the default time. In future work, one could investigate whether the employed importance sampling method achieves variance reduction for alternative ways of pricing Bermudan options. It is also noted that the used importance sampling method only gives a significant variance reduction for sufficiently small default probabilities and is therefore particularly useful for counterparties with high credit ratings. The method is therefore not as useful for counterparties with large default probabilities. A topic of further research is to devise importance sampling methods that also significantly reduce the variance for large counterparty default probabilities.

The asset price importance sampling technique could not be applied on CVA calculations, as the factor associated with the change of measure could not be found. The Monte Carlo estimate under the new measure could therefore not be adjusted in order to obtain the estimate under the original measure. CVA was instead simulated under the optimal measure for asset price importance sampling, without multiplication of this factor. As was expected from theory, the CVA estimate under the new measure was different from the CVA estimate under the original measure. The CVA estimate as well as the estimated variance of CVA was larger under the new measure, for both the European option and the Bermudan option. However, the relative error was smaller under the new measure for both options. It is therefore possible that the larger variance under the new measure is in part explained by the larger estimate of CVA under the new measure. This motivates attempts to find ways of employing asset price importance sampling for CVA simulations. To build upon the findings of this paper, one could try to find the factor associated with the change of measure, that is necessary in order to obtain an estimate of CVA under the original measure by using asset price importance sampling.

Furthermore, the importance sampling method employed for the asset price process was a general one, designed to reduce the variance of the simulation as much as possible. For CVA simulation, or other applications, however, the optimal change of measure might depend on other factors as well. For instance, the CVA contribution will be zero if the path is out of the money at the time of default, which might motivate a change of measure that drives the asset price to the region that puts the option in the money. By further investigating changes of measure based on specific applications, convergence results might be improved. It is also noted that the model used for the default time was quite simple, and the c.d.f of the default time could be computed analytically. To further build upon the findings of this paper, similar importance sampling methods could be investigated for other stochastic intensity models. In particular, it would be of interest to find a method of performing importance sampling when the c.d.f of the default time cannot be computed analytically.
Moreover, the findings of this paper are for asset prices following a jump-diffusion process. It would be interesting to further investigate how similar methods could be used for different types of asset price processes. Finally, in this paper, unilateral CVA was considered and the exposure was assumed to be independent of the default probability. A further topic of research is to adapt the methods in this paper to bilateral CVA, where both the investor and the counterparty run the risk of defaulting. Furthermore, one could also consider the case when the default probabilities of both the investor and the counterparty are correlated to the exposure.
Bibliography


