Necessary Optimality Conditions for Two Stochastic Control Problems

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Abstract

This thesis consists of two papers concerning necessary conditions in stochastic control problems. In the first paper, we study the problem of controlling a linear stochastic differential equation (SDE) where the coefficients are random and not necessarily bounded. We consider relaxed control processes, i.e. the control is defined as a process taking values in the space of probability measures on the control set. The main motivation is a bond portfolio optimization problem. The relaxed control processes are then interpreted as the portfolio weights corresponding to different maturity times of the bonds. We establish existence of an optimal control and necessary conditons for optimality in the form of a maximum principle, extended to include the family of relaxed controls.

In the second paper we consider the so-called singular control problem where the control consists of two components, one absolutely continuous and one singular. The absolutely continuous part of the control is allowed to enter both the drift and diffusion coefficient. The absolutely continuous part is relaxed in the classical way, i.e. the generator of the corresponding martingale problem is integrated with respect to a probability measure, guaranteeing the existence of an optimal control. This is shown to correspond to an SDE driven by a continuous orthogonal martingale measure. A maximum principle which describes necessary conditions for optimal relaxed singular control is derived.

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Introduction and summary of the papers

In this section we give some background on optimal control theory, stochastic control and in particular the stochastic maximum principle. We also introduce and discuss the notion of relaxed control. We conclude with providing a summary of the two papers.

Optimal control theory

Optimal control theory can be described as the study of strategies to optimally influence a system x with dynamics evolving over time according to a differential equation. The influence on the system is modeled as a vector of parameters, u, called the control. It is allowed to take values in some set U, which is known as the action space. For a control to be optimal, it should minimize a cost functional (or maximize a reward functional), which depends on the whole trajectory of the system $x(t)$ and the control $u(t)$ over some time interval $t \in [0, T]$. The infimum of the cost functional is known as the value function (as a function of the initial time and state). This minimization problem is infinite dimensional, since we are minimizing a functional over the space of functions $u(t), t \in [0, T]$. Optimal control theory essentially consists of different methods of reducing the problem to a less transparent, but more managable problem. The two main methods are dynamic programming and the maximum principle.

As for dynamic programming, it is essentially a mathematical technique for making a sequence of interrelated decisions. By considering a family of control problems with different initial times and states and establishing relationsships between them one obtains a nonlinear first-order partial differential equation known as the Hamilton-Jacobi-Bellman (HJB) equation. Solving this equation gives the value function, after which a finite dimensional maximization problem can be solved.

On the other hand, the maximum principle gives necessary conditions for optimality by perturbing an optimal control on a small time interval of lenght θ. Performing a Taylor expansion with respect to θ and then sending θ to zero one obtains a variational inequality. By duality the maximum principle is obtained. It states that any optimal control must solve the Hamiltonian system associated to the control problem. The Hamiltonian system is a linear differential equation with terminal conditions, called the adjoint equation, and a (finite dimensional) maximization problem.

Optimal control theory and the maximum principle originates from the works of Pontryagin and co-authors [18]. The method of dynamic programming was developed by Bellman [4]. For an introduction to optimal control theory, see e.g. Sontag [20].

Stochastic control

Stochastic control is the extension of optimal control to problems where it is of importance to, in one way or the other, take into account some uncertainty in the system. One possibility is to replace the differential equation by an SDE, e.g. of the Markovian type:

$$x_t = x_0 + \int_0^t b\left(s, x_s, u_s\right) ds + \int_0^t \sigma\left(s, x_s\right) dB_s, \qquad (0.1)$$

where b and σ are deterministic functions and the last term is an Itô integral with respect to the Brownian motion B. More generally, the diffusion coefficient may have an explicit dependence on the control:

$$x_t = x_0 + \int_0^t b\left(s, x_s, u_s\right) ds + \int_0^t \sigma\left(s, x_s, u_s\right) dB_s. \qquad (0.2)$$

When solving these types of stochastic control problems with dynamic programming, the HJB equation obtained is a nonlinear second-order partial differential equation, see e.g. Fleming and Soner [9] or Yong and Zhou [21].

The stochastic maximum principle

The earliest paper on the extension of the maximum principle to stochastic control problems is Kushner and Schweppe [13]. One major difficulty that arises in such an extension is that the adjoint equation becomes an SDE with terminal conditions. In contrast to a deterministic differential equation, one cannot simply reverse the time since the control process, and consequently the solution to the SDE, is required to be adapted to the filtration. Bismut solved this problem by introducing conditional expectations and obtained the solution to the adjoint equation from the Martingale Representation Theorem, see e.g. [6] and also Haussmann [12]. An extensive study of these so-called backward SDEs can be found in e.g. Ma and Yong [15].

For the maximum principle, there is a major difference between the cases (0.1) and (0.2). As for (0.2), when performing the expansion with respect to the perturbation θ, the fact that the perturbed Itô integral turns out to be of order $\sqrt{\theta}$ (rather than θ as with the ordinary Lebesgue integral) poses a problem. In fact, one needs to take into account both the first-order and second-order terms in the expansion. This ultimately leads to a maximum principle containing two adjoint equations, both in the form of linear backward SDEs. The first proof of this general stochastic maximum principle is given in Peng [17]. It should be noted that under the assumption that the action space U is convex, it is possible to derive the maximum principle in its original form, by using a convex perturbation of the control instead of a spike variation, see Bensoussan [5]. For an extensive survey of the developments in stochastic control theory see e.g. Fleming and Soner [9] and Yong and Zhou [21].

Relaxed control

Since the maximum principle describes a necessary condition for optimality we know that if there exists an optimal control, there also exists at least one solution to the maximum principle. However, the control problem may fail to have an optimal solution even for quite simple situations. At least if by a solution we mean a process $u(t)$ taking values in U. The idea is then to compactify the space of controls by extending the definition of controls to include the space of probability measures on U. The set of relaxed controls $\mu_t(du)dt$, where μ is a probability measure, is the closure under weak* topology of the measures $\delta_{u(t)}(du)dt$ corresponding to usual, or strict, controls.

Here is a simple example from deterministic control. Consider the cost functional

$$J(u) = \int_0^T \left(x^2(t) + \left(1 - u^2(t) \right)^2 \right) dt,$$

to be minimized over the set of controls $u : [0, T] \mapsto U = [-1, 1]$. The state of the system is given by

$$\begin{cases} dx(t) = u(t)dt \\ x(0) = 0 \end{cases}$$

Now consider the following sequence of controls

$$u^{(n)}(t) = (-1)^k \ \text{ if } \ t \in \left[\frac{k}{n}, \frac{k+1}{n} \right), \ 0 \le k \le n - 1.$$

Then we have $|x^{(n)}(t)| \le 1/n$ and $|J(u^{(n)})| \le T/n^2$, which implies that $\inf_u J(u) = 0$. The limit of $u^{(n)}$ is however not in the space of strict controls. Instead the sequence $\delta_{u^{(n)}(t)}(du)dt$ converges weakly to $1/2 \left(\delta_{-1} + \delta_1 \right)(du)dt$. Thus, there does not exist an optimal strict control in this case but only a relaxed one. But since we can construct a sequence of strict controls such that the cost functional is arbitrarily close to its infimum, it is clear that there does exists an optimal solution, albeit in a wider sense.

This notion of relaxed control is introduced for deterministic optimal control problems in Young [22], and generalized to the stochastic control problem (0.1) in Fleming [10]. A maximum principle for this relaxed control problem is established in Bahlali and Mezerdi [16]. As for the case (0.2), i.e. with controlled diffusion coefficient, the existence of an optimal relaxed control is proved in El Karoui et al. [8], and a maximum principle for this problem is established in Bahlali et al. [2]. In these papers, control is defined as the solution to the appropriate martingale problem, and the relaxation is performed by integrating the generator with respect to a probability measure on U. In Ma and Yong [14] another type of relaxation of the problem (0.2) is studied. The relaxed control problem is then defined by integrating the coefficients in the SDE with respect

to a probability measure. It should be noted that this is not equivalent to the relaxation in El Karoui et al. [8]. Roughly, the difference is that in the former one integrates the diffusion coefficient $\sigma(\cdot, \cdot, u)$, and in the latter $\sigma(\cdot, \cdot, u)\sigma(\cdot, \cdot, u)^*$.

In this thesis we extend the maximum principle to include relaxed controls for two stochastic control problems. In the first paper we study a relaxed control problem of the form in Ma and Yong [14], where the more direct form of relaxation is used to formulate an optimal bond portfolio problem. In the second paper we obtain a maximum principle for a relaxed singular control problem, extending a result in Bahlali et. al [1] to the case of a controlled diffusion coefficient. Here is a summary of the results.

A maximum principle for relaxed stochastic control of linear SDEs with application to bond portfolio optimization

Consider an investor with initial capital x_0 who invests in bonds, i.e. financial contracts that are bought today and pay a fixed amount at some future time, called the maturity time. If the interest rate is stochastic, so are the prices of the bonds since the interest rate of outstanding bonds can be higher or lower than its current market value.

Assume that there exists a market of bonds with time to maturity in some set $U \subset \mathbb{R}_+$. There is one major difference between such a bond market and a stock market. In the bond market there is naturally a continuum of assets, while in the standard model of a stock market there is normally only a finite number of assets. Thus, an appropriate definition of a portfolio strategy should include such portfolios that may contain a continuum of assets. Here we define a portfolio strategy as a measure valued process $\rho_t(du)$, reflecting the "number" of bonds in the portfolio at time t, which have time to maturity in the interval $[u, u+du]$. In the Heath-Jarrow-Morton-Musiela framework, the portfolio value x_t can be derived as

$$x_t = x_0 + \int_0^t \int_U p_s(u) \left(r_s^0 - v_s(u)\Theta_s \right) \rho_s(du)ds + \int_0^t \int_U p_s(u)v_s(u)\rho_s(du)dB_s,$$

where $p_t(u)$ is the price of the bond with time to maturity equal to u, $v_t(u)$ is its volatility, r_t^0 the short rate and Θ_t is the so-called market price of risk. If the investor only takes long positions (positive portfolio weights) we can introduce the relative portfolio μ_t as a process taking values in the space of probability measures, by

$$\rho_t(du) = \frac{x_t}{p_t(u)}\mu_t(du).$$

The relative portfolio, which is the proportion of the portfolio value invested in different bonds, has the following dynamics.

$$x_t = x_0 + \int_0^t x_s \int_U \left(r_s^0 - v_s(u)\Theta_s \right) \mu_s(du)ds + \int_0^t x_s \int_U v_s(u)\mu_s(du)dB_s.$$

We consider this as a controlled SDE with action space U and relaxed control μ. Given a cost functional, we derive existence of an optimal relaxed control as well as necessary conditions for optimality, when the state equation is given by this type of linear SDEs with random and unbounded coefficients.

In the example above motivating the use of relaxed controls, we find an approximating strict control sequence which converges to the optimal relaxed control. This is actually true in general and the result is known as the Chattering Lemma. That is, we can always find a sequence of strict controls which are near optimal. We derive a maximum principle for these near optimal controls. Passing to the limit and using the Chattering Lemma as well as some stability properties of the state and adjoint processes, we obtain our main result: if $\hat{\mu}$ is an optimal relaxed control, it fulfills the usual maximization condition, however with the function appearing in the maximum principle extended to a functional of probability measures on U.

The relaxed stochastic maximum principle in singular optimal control of diffusions with controlled diffusion coefficient

In this paper we study a slightly more general stochastic control problem by adding a singular control component. The state equation is given by

$$x_t = x_0 + \int_0^t b\left(s, x_s, u_s\right) ds + \int_0^t \sigma\left(s, x_s, u_s\right) dB_s + \int_0^t G_s d\xi_s,$$

where the control is the vector (u, ξ). The singular control component is the process ξ, which is increasing and càglàd (french acronym for "left continuos with right limits"). I.e., as opposed to the control u, its influence on the state is not continuous in time. We refer to the process u_t taking values in U, as the absolutely continuous part of the control.

This model appears in applications in mathematical finance. The absolutely continuous control then corresponds to the continuous rebalancing of a portfolio, which becomes too expensive when one takes into account the transaction costs. The singular control ξ_t is in this context the total transactions up to time t, cf. Shreve and Soner [19].

The question of existence of an optimal control is basically the same as in the non-singular case. Indeed, by relaxing the absolutely continuous control an existence result is obtained in Haussmann and Suo [11]. A control is defined as a solution to the martingale problem corresponding to the state equation. The relaxation is performed by integrating the generator with respect to a probability measure. This however leads to problems of interpretation; what kind of process induces a measure which is a solution to this relaxed martingale problem? It turns out that the Itô integral should be replaced by a stochastic integral with respect to a continuous orthogonal martingale measure, i.e. a process which for fixed t is a measure on sets $A \subset U$, and for a fixed set $A \subset U$ is a continuous martingale, see e.g. El Karoui and Méléard [7] .

By the Chattering Lemma one can construct a sequence of strict controls approximating the relaxed optimal control. Therefore, using methods from the derivation of the maximum principle for strict singular control in Bahlali and Mezerdi [16], we are able to derive the corresponding maximum principle for near optimal controls. Passing to the limit gives the relaxed maximum principle, which is the same maximum condition as in the case of only an absolutely continuous control, plus a second condition characterizing the singular part of the optimal control. This is a generalization of a result in Bahlali et. al [1], where the diffusion coefficient σ is independent of the control.

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