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Licentiate Thesis

Group Extensions, Gerbes and Twisted K-theory

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Abstract

This thesis reviews the theory of group extensions, gerbes and twisted K-theory. Application to anomalies in gauge theory is briefly discussed. The main results are presented in two appended scientific papers. In the first paper we establish, by construction, a criterion for when an infinite dimensional abelian Lie algebra extension corresponds to a Lie group extension. In the second paper we introduce the fractional loop group $L_q G$, construct highest weight modules for the Lie algebra and discuss an application to twisted K-theory on G.

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Preface

This thesis consists of two scientific papers which are the result of my research at the Department of Theoretical Physics during the period 2006-2007. The thesis is divided into two parts. The first part provides some background needed to understand the scientific results. The second part contains the two research papers listed below.

List of papers

I Pedram Hekmati
Integrability Criterion for Abelian Extensions of Lie Groups
Submitted for publication
`math.DG/0611431`

II Pedram Hekmati and Jouko Mickelsson
Fractional Loop Group and Twisted K-theory
Submitted for publication
`math.DG/0801.2522`

The author's contribution to the papers

I I performed all computations and wrote the paper. The problem was suggested by my supervisor Jouko Mickelsson.

II I performed the computations and wrote the material in Sections 2-5, the Appendix and the last paragraph in the Introduction.

Contents

Part I

Background material

Chapter 1

Introduction

In quantum mechanics the rotation group $SO(3)$ describing angular momentum is an important symmetry. Eigenstates of the hamiltonian are classified by irreducible linear representations of $SO(3)$. By discovering spin in 1924, Wolfgang Pauli realized that projective representations of $SO(3)$ play an equally important role in physics. A projective representation of $SO(3)$ corresponds to a linear representation of its double cover $Spin(3) = SU(2)$. This covering can be written as a short exact sequence

$$\mathbf{1} \to \mathbb{Z}_2 \xrightarrow{i} SU(2) \xrightarrow{p} SO(3) \to \mathbf{1}$$

which means that i is an injective homomorphism, p a surjective homomorphism and $\ker p = \mathrm{im}\ i$. We say that $SU(2)$ is a central extension of $SO(3)$ by $\mathbb{Z}_2 = \pi_1(SO(3))$. This is usually how group extensions occur in physics, that is from the study of projective bundles. In this particular example, there is a topological obstruction for writing $SU(2)$ as a product $SO(3) \times \mathbb{Z}_2$ given by an element in the first Čech cohomology $\check{H}^1(SO(3), \underline{\mathbb{Z}_2})$. Group extensions might also arise from a different source, namely if the group law is twisted algebraically by a 2-cocycle f in the group cohomology. A prominent example is the Virasoro-Bott group $\mathrm{Diff}(S^1) \times_f S^1$ which we discuss in more detail in Section 2.3.

Closely related to the concept of projective bundles is a mathematical object called *gerbe*. Projective principal bundles over a topological space M are classified by the third integral cohomology $H^3(M, \mathbb{Z})$. One can think of gerbes as the third level in a hierarchy of mathematical objects providing geometric realizations for integral cohomology classes. At the first level we have circle-valued functions classified by $H^1(M, \mathbb{Z})$ and at the second level are line bundles classified by their Chern class in $H^2(M, \mathbb{Z})$. Gerbes are in $1-1$ correspondence with elements in $H^3(M, \mathbb{Z})$. Naively one would think that they are vector bundles of rank 2, but this idea fails.

For the 3-sphere for example we have $H^3(S^3,\mathbb{Z})=\mathbb{Z}$, while the set of isomorphism classes of rank 2 vector bundles on S^3 is trivial, $\mathrm{Vect}_2(S^3)=\pi_2(U(2))=0$.

Gerbes were introduced by Jean Giraud in 1971 in the context of non-abelian cohomology. They are roughly speaking fibre bundles where the fibres are groupoids, i.e. categories where every morphism is invertible. More precisely, a gerbe on M is a nonempty transitive stack of groupoids. In this thesis however, we will adopt a more specialized definition suitable in the differential geometric setting. Gerbes are not only classified by their characteristic class in degree 3 cohomology, they share many properties with line bundles. One can define the notion of pullback, dual, product and do geometry on them by introducing connection and curvature.

A physical situation where these structures show up is in canonical quantization of chiral fermions in external fields. Let G be a simple compact Lie group in some fixed unitary matrix representation and $\mathfrak{g}$ its Lie algebra. The chiral Dirac hamiltonian in Minkowski space coupled to a gauge connection $A=A_j dx^j$ is given by

$$D_A=-i\sigma^j\Big(\frac{\partial}{\partial x^j}+A_j\Big)$$

where the Pauli matrices σ^j satisfy $\sigma^j\sigma^k=i\epsilon_{jkl}\sigma^l$ and ϵ_{jkl} is the antisymmetric permutation tensor, $j=1,2,3$. Here the Yang-Mills potential A is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}^3$ satisfying $A_j^*=-A_j$ and we denote by $\mathcal{A}$ the affine space of all such connections. The 1-particle Hilbert space is $\mathcal{H}=L^2(\mathbb{R}^3,\mathbb{C}^2_{spin}\otimes\mathbb{C}^N_{color})$. For simplicity, we consider a one-point compactification of $\mathbb{R}^3$ to S^3 so that the spectrum of D_A will be discrete. This system is flawed from a physical point of view since the requirement that the hamiltonian is a positive operator is not satisfied. To remedy this, one second quantizes the system to obtain a new Hilbert space $\mathcal{F}_{A,\lambda}$ of multiparticle states and a new hamiltonian $\widehat{D_A}$ which is bounded from below and still transforms equivariantly under gauge transformations

$$\widehat{D_{A^g}}=g^{-1}\widehat{D_A}g$$

where $A^g=g^{-1}Ag+g^{-1}dg$. Let λ be a real number *not* in the spectrum of D_A. The Fock space $\mathcal{F}_{A,\lambda}$ is fixed by a spectral decomposition of the 1-particle Hilbert space

$$\mathcal{H}=\mathcal{H}_+(A,\lambda)\oplus\mathcal{H}_-(A,\lambda)$$

where $\mathcal{H}_\pm(A,\lambda)$ are the eigenspaces corresponding to eigenvalues of D_A larger/smaller than λ. We have

$$\mathcal{F}_{A,\lambda}=\bigoplus_{p,q}\mathcal{F}^{p,q}_{A,\lambda}=\bigoplus_{p,q}\bigwedge^p\mathcal{H}_+(A,\lambda)\otimes\bigwedge^q\mathcal{H}_-(A,\lambda)$$

where $\mathcal{F}^{p,q}_{A,\lambda}$ is the Hilbert space corresponding to p particles and q antiparticles. In this way, we obtain a Hilbert bundle of Fock spaces $\mathcal{F}\to\mathcal{A}$ over the space of connections. The splitting of the 1-particle Hilbert space defines the vacuum in each fibre $\mathcal{F}_{A,\lambda}$, but it does not vary continuously due to the spectral flow of D_A. As one varies the potential A, an eigenvalue of D_A will cross λ at some point leading to a discontinuity in the polarization of $\mathcal{H}$. The construction still works locally on open sets $U_\lambda = \{A \in \mathcal{A} \mid \lambda \notin Spec(D_A)\}$ covering $\mathcal{A}$. On overlaps $U_\lambda \cap U_{\lambda'}$ we have

$$\mathcal{F}_{A,\lambda} = \mathcal{F}_{A,\lambda'} \otimes DET_{\lambda\lambda'}(A)$$

where

$$DET_{\lambda\lambda'}(A) = \bigwedge^{top} E_{\lambda\lambda'}(A)$$

is the top exterior product of the finite-dimensional vector space $E_{\lambda\lambda'}(A)$ spanned by eigenvectors of D_A corresponding to eigenvalues in the interval (λ, λ'). This simply means that the "Dirac sea" is filled between vacuum levels λ and λ' when passing from one Fock space to another. Now $\{DET_{\lambda\lambda'}\}$ defines a collection of local line bundles on overlaps and there is a canonical isomorphism

$$DET_{\lambda\lambda'} \otimes DET_{\lambda'\lambda''} \cong DET_{\lambda\lambda''}$$

over triple intersections $U_\lambda \cap U_{\lambda'} \cap U_{\lambda''}$. We will see in Section 3.2 that this is one way to characterize a gerbe over a space, that is as a family of local line bundles satisfying a cocycle condition.

We have thus a Fock bundle on each open set $U_\lambda \subset \mathcal{A}$ with a continuous vacuum section, but on overlaps the fibres are isomorphic only up to a phase defined by $DET_{\lambda\lambda'}(A)$. In order to remove the phase dependence we define a new Fock bundle over U_λ,

$$\tilde{\mathcal{F}}_{A,\lambda} = \mathcal{F}_{A,\lambda} \otimes DET_\lambda(A)$$

by tensoring with a line bundle $DET_\lambda(A)$ satisfying

$$DET_{\lambda'} = DET_{\lambda\lambda'} \otimes DET_\lambda \ .$$

One can prove that such a line bundle exists since $\mathcal{A}$ is contractible. Moreover, on overlaps we have now

$$\tilde{\mathcal{F}}_{A,\lambda} = \tilde{\mathcal{F}}_{A,\lambda'} \ .$$

These patch together to form a global Fock bundle $\tilde{\mathcal{F}} \to \mathcal{A}$ with a global continuous vacuum section.

Next we consider the action of the group of based local gauge transformations $\mathcal{G}_0 = \mathrm{Map}_0(S^3, G)$ on $\tilde{\mathcal{F}}$. It acts freely on the base $\mathcal{A}$ by gauge transformations and the moduli space $\mathcal{A}/\mathcal{G}_0$ is an infinite-dimensional Fréchet manifold. The action on $\tilde{\mathcal{F}}$ however is only projective and the reason is the following. On each open set U_λ the gauge action lifts naturally to $\mathcal{F}_{A,\lambda}$, but the lifting to the complex line bundle $DET_\lambda(A)$ leads to an abelian extension $\widehat{\mathcal{G}_0}$ of $\mathcal{G}_0$ by

$\mathrm{Map}(\mathcal{A}, S^1)$.[22] By modding out by $\mathcal{G}_0$ we obtain a projective bundle $\mathbb{P}\tilde{\mathcal{F}} \to \mathcal{A}/\mathcal{G}_0$, described by its characteristic class $\Omega \in H^3(\mathcal{A}/\mathcal{G}_0, \mathbb{Z})$. This cohomology class can be computed from families index theorem.[9] The 2-cocycle ω of the corresponding Lie algebra extension $Lie\ \mathcal{G}_0 \oplus_\omega \mathrm{Map}\ (\mathcal{A}, i\mathbb{R})$ is determined by transgression $H^3(\mathcal{A}/\mathcal{G}_0, \mathbb{Z}) \to H^2(Lie\ \mathcal{G}_0, \mathrm{Map}\ (\mathcal{A}, i\mathbb{R}))$. This is how gauge anomaly, i.e. breakdown of classical gauge symmetry, manifests itself in hamiltonian quantization.

The interaction between mathematics and theoretical physics has been extremely fruitful the past decades. As the example above demonstrates, the language of group extensions and gerbes lends itself naturally to the description of anomalies in quantum field theory. On the other hand, physical models can in some cases be used to compute topological invariants. An important such example is the use of the supersymmetric Wess-Zumino-Witten model for computing twisted K-theory classes on compact Lie groups. The construction is described in Section 4.3.

The set of isomorphism classes of complex vector bundles $\mathrm{Vect}(M)$ over a compact manifold M forms a semigroup. A semigroup can always be completed to a group via the Grothendieck construction, i.e. by adding formal inverses. This is for example how integers $\mathbb{Z}$ are constructed from natural numbers $\mathbb{N}$. The Grothendieck group of $\mathrm{Vect}(M)$ defines the K-theory group $K^0(M)$.

Gerbes can be used to twist K-theory. This is most easily explained by using an equivalent definition of $K^0(M)$, namely as the set of homotopy classes of maps $[M, Fred(\mathcal{H})]$ from M to the space of bounded Fredholm operators $Fred(\mathcal{H})$ of a complex separable Hilbert space $\mathcal{H}$. A gerbe corresponding to $\Omega \in H^3(M, \mathbb{Z})$ fixes a principal $PU(\mathcal{H})$-bundle P over M, where $PU(\mathcal{H}) = U(\mathcal{H})/S^1$ is the group of projective unitaries. The twisted K-theory group $K^0(M, \Omega)$ is defined by replacing (homotopy classes of) families of Fredholm operators $M \to Fred(\mathcal{H})$ by (homotopy classes of) sections of the associated bundle $P \times_{PU(\mathcal{H})} Fred(\mathcal{H})$, where $PU(\mathcal{H})$ acts on $Fred(\mathcal{H})$ by conjugation. By restricting the classifying space $Fred(\mathcal{H})$ to the subspace $Fred_*(\mathcal{H})$ of self-adjoint Fredholm operators with both positive and negative essential spectrum, we obtain the odd twisted K-theory group $K^1(M, \Omega)$. This is discussed in more detail in Section 4.2.

K-theory twisted by a torsion class Ω first appeared in a paper by Donovan and Karoubi in 1970. The non-torsion case was introduced by Rosenberg in the context of C^*-algebras some twenty years later. In recent years there has been a renewed surge of interest in twisted K-theory, partly due to its role in classification of D-brane charges in string theory. For compact Lie groups G there is a major result by Freed-Hopkins-Teleman,[11] which identifies the G-equivariant twisted K-theory with the Verlinde ring of projective highest weight representations of the loop group LG,

$$R(LG, k) \cong K_G^{dim\ G}(G, k + \kappa)$$

where κ denotes the dual Coxeter number and $k \in \mathbb{N}$ is some fixed level. The non-equivariant case has also been worked out using techniques from homological algebra[7] and spectral sequences.[10] It is interestingly all torsion, in contrast to the G-equivariant case.

Chapter 2

Group Extensions

2.1 Infinite dimensional Lie groups

This section will be a brief outline of basic concepts in infinite dimensional Lie theory. A Lie group is a smooth manifold endowed with a compatible group structure, i.e. multiplication and inversion are smooth maps. This definition still holds in infinite dimensions, what is different is the notion of a manifold. For finite dimensional manifolds, the model space is fixed once the ground field (e.g. $\mathbb{R}$, $\mathbb{C}$) is chosen. In infinite dimensions there are many alternatives and how far the classical Lie theory is extended depends very much on the choice of model space. This is guided by two conflicting criteria. On one hand, one wants to adopt a general approach that includes most known examples of Lie groups. On the other hand, the powerful theorems of finite dimensional Lie theory should be retained as far as possible. For example, many results in ordinary calculus can be extended to Banach spaces, but the category of Banach manifolds does not contain a single diffeomorphism group. Our aim is to treat Lie groups modeled on arbitrary locally convex spaces. With the possible exception of diffeomorphism groups of noncompact manifolds, this includes all Lie groups usually encountered in physics.

Definition 2.1. A topological vector space E over $\mathbb{R}$ (or $\mathbb{C}$) is a vector space with Hausdorff topology such that addition and scalar multiplication are continuous functions of two variables. It is called locally convex if every zero neighborhood contains a convex zero neighborhood.

Local convexity leads to a Hahn-Banach theorem, ensuring the existence of many continuous functionals $E \to \mathbb{R}$. This property is needed for defining a sensible calculus on these spaces. For this aim, we will also need completeness.

Definition 2.2. A locally convex topological vector space E is said to be sequentially complete if every Cauchy sequence in E converges.

Recall that a sequence $x_1, x_2, \ldots$ in E is a Cauchy sequence, if for every zero neighborhood $U \subset E$, the difference $x_i - x_j \in U$ for large enough indices i, j. Sequential completeness guaranties the existence of Riemann integral of continuous paths in the vector space.

Next we develop differential calculus on these spaces. This is a complicated matter and there are different approaches.[20] The main problem is that there is no topology on the space of continuous linear maps $\mathcal{L}(E, F)$ for which the evaluation map is continuous. The definition of the differential df of a continuous linear map $f : E \to F$ should therefore not depend on the topology on $\mathcal{L}(E, F)$, but instead be considered as a continuous map of two arguments. We adopt the following definition:

Definition 2.3. Let E, F be sequentially complete locally convex spaces over $\mathbb{R}$ (or $\mathbb{C}$) and $f : U \to F$ a continuous map on an open subset $U \subseteq E$. Then f is said to be differentiable at $x \in U$ if the directional derivative

$$df(x)(v) = \lim_{t \to 0} \frac{1}{t}(f(x + tv) - f(x))$$

exists for all $v \in E$. It is of class C^1 if it is differentiable at all points of U and

$$df : U \times E \to F,\ (x, h) \mapsto df(x)(h)$$

is a continuous function on $U \times E$. Inductively, we say that $f \in C^n$ if $df \in C^{n-1}$ and $f \in C^\infty$ is called smooth if $f \in C^n$ for every $n \geq 1$.

Using local convexity and completeness properties, one can prove the chain rule and show that $df = 0$ is equivalent to f being constant on each connected component of U. However, beyond Banach spaces the inverse function theorem and the implicit function theorem cannot be proven in general. This is a major factor behind the subtleties in infinite dimensions, as these theorems underlie many results in classical Lie theory.[15]

Definition 2.4. A smooth manifold M is a Hausdorff topological space together with an open cover $M = \cup_{i \in I} U_i$ and a collection of homeomorphisms $\phi_i : U_i \to E_i$ to open subsets $E_i \subset E$ such that the transition functions $\phi_i \circ \phi_j^{-1}$ on overlaps $U_i \cap U_j$ are smooth.

One can proceed to define tangent vectors as equivalence classes of curves, the tangent bundle as the disjoint union of tangent spaces and other familiar structures in the same way as in finite dimensions. For a detailed discussion consult.[27]

Definition 2.5. A Lie group G is a smooth manifold and a group such that the multiplication $G \times G \to G,\ (g, h) \mapsto gh$ and the inversion map $G \to G,\ g \mapsto g^{-1}$ are smooth.

For every $g \in G$ there is an associated diffeomorphism $L_g : G \to G,\ h \mapsto gh$ defined by left translations. A vector field $X : G \to TG$ is called left-invariant if

$$L_{g*}X = X, \ \ \forall g \in G \ .$$

where L_{g*} denotes the pushforward map. The space of left-invariant vector fields is closed under the Lie bracket of vector fields and defines the Lie algebra $\mathfrak{g}$ of G. Since left-invariant vector fields are completely determined by their value at the identity, $\mathfrak{g}$ can be identified with the tangent space $T_{\mathbf{1}}G$ endowed with a continuous Lie bracket.

The passage from Lie algebra to Lie group goes via the exponential function. Absence of the inverse function theorem makes it extremely hard to prove its existence in general. In fact, without the completeness property of the model space there are counterexamples where it does not exist.[15] An additional requirement is needed:

Definition 2.6. A Lie group G is called *regular* if for each $X \in C^\infty([0,1],\mathfrak{g})$, there exists $\gamma \in C^\infty([0,1],G)$ such that

$$\gamma'(t) = L_{\gamma(t)*}(\mathbf{1}).X(t), \qquad \gamma(0) = \mathbf{1}$$

and the evolution map

$$\mathrm{evol}_G : C^\infty([0,1],\mathfrak{g}) \to G, \qquad X \mapsto \gamma(1)$$

is smooth.

In plain text, this means that every smooth path in the Lie algebra arises in a well-behaved way as the logarithmic derivative of a path in the Lie group. This is actually a stronger condition than requiring that the exponential map exists. Even so, we do not know of any Lie group modeled on a sequentially complete locally convex space which is not regular. Assuming regularity, one can prove that every continuous homomorphism between Lie algebras $\mathfrak{g} \to \mathfrak{h}$ gives rise to a unique smooth homomorphism $G \to H$, if G is simply connected. However regularity is not a sufficient condition for the exponential map to be a local diffeomorphism. A striking counterexample is the group of diffeomorphisms of the circle $\mathrm{Diff}(S^1)$. The diffeomorphism $f : S^1 \to S^1$ defined by

$$f(e^{i\phi}) = e^{i(\phi + \frac{2\pi}{n} + \varepsilon \sin n\phi)}$$

can be made arbitrarily close to the identity $\mathrm{id}(e^{i\phi}) = e^{i\phi}$ by choosing n large and ε small, but it does not belong to any one-parameter subgroup.[27] This is not a peculiarity of the circle. For any manifold M of dimension at least 2, one can construct a smooth path in $\mathrm{Diff}(M)$ based at the identity, such that the points of this curve form a set of generators for a free subgroup of Diff(M) which intersects the image of the exponential mapping only in the identity.[20]

A Lie group where the exponential function exists and is a local diffeomorphism at the identity is called locally exponential. Banach Lie groups and gauge groups Map(M, G) are important examples. For locally exponential Lie groups one can prove the classical result that continuous homomorphism between Lie groups are necessarily smooth.

2.2 Lie group and Lie algebra cohomology

Cohomology of Lie groups and Lie algebras appear in the classification of abelian extensions. In this section we set out the basic definitions.

Lie group cohomology. Let G be a Lie group. A smooth G-module is an abelian Lie group A on which G act smoothly by automorphisms

$$G \times A \to A,\ (g,a) \mapsto g.a \ .$$

For $n \geq 0$ we denote by $C^n(G,A)$ the set of smooth maps $f : G^n \to A$ normalized in the sense that $f(g_1,\ldots,g_n) = 0$ whenever $g_j = \mathbf{1}$ for some j. Under pointwise addition $C^n(G,A)$ forms an abelian group. We introduce a family of homomorphisms $\delta_n : C^n(G,A) \to C^{n+1}(G,A)$,

$$(\delta_n f)(g_1,\ldots,g_{n+1}) = g_1.f(g_2,\ldots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\ldots,g_i g_{i+1},\ldots,g_{n+1}) +$$

$$+(-1)^{n+1} f(g_1,\ldots,g_n) \ ,$$

satisfying $\delta_{n+1}\circ\delta_n = 0$. This defines a cochain complex $\big(C^n(G,A),\delta_n\big)_{n\geq 0}$ and the nth Lie group cohomology is given by

$$H^n(G,A) = \frac{Z^n(G,A)}{B^n(G,A)} = \frac{\ker\,\delta_n}{\mathrm{im}\,\delta_{n-1}} \ .$$

Lie algebra cohomology. Let $\mathfrak{g}$ and $\mathfrak{a}$ be topological Lie algebras. The abelian Lie algebra $\mathfrak{a}$ is a continuous $\mathfrak{g}$-module if there is a continuous bilinear map

$$\mathfrak{g} \times \mathfrak{a} \to \mathfrak{a},\ (X,v) \mapsto X.v$$

such that for every $v \in \mathfrak{a}$ and $X_1, X_2 \in \mathfrak{g}$

$$[X_1,X_2].v = X_1.(X_2.v) - X_2.(X_1.v) \ .$$

For $n \geq 0$ we denote by $C^n(\mathfrak{g}, \mathfrak{a})$ the vector space of continuous antisymmetric multilinear maps $\omega : \mathfrak{g}^n \to \mathfrak{a}$. We define the coboundary operators $d_n : C^n(\mathfrak{g}, \mathfrak{a}) \to C^{n+1}(\mathfrak{g}, \mathfrak{a})$ by Palais' formula

$$(d_n\omega)(X_1, \ldots, X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} X_i.\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1}) +$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{n+1})$$

where $\hat{X}_i$ means that the entry in position i is omitted. The linear maps d_n satisfy $d_{n+1} \circ d_n = 0$ and we have thus a cochain complex $\big(C^n(\mathfrak{g}, \mathfrak{a}), d_n\big)_{n \geq 0}$ with the resulting cohomology

$$H^n(\mathfrak{g}, \mathfrak{a}) = \frac{Z^n(\mathfrak{g}, \mathfrak{a})}{B^n(\mathfrak{g}, \mathfrak{a})} = \frac{\ker\, d_n}{\operatorname{im}\, d_{n-1}} \,.$$

There is a map connecting Lie group cohomology to Lie algebra cohomology.

Theorem 2.7. For $n \geq 1$, there is a homomorphism

$$D_n : H^n(G, A) \to H^n(\mathfrak{g}, \mathfrak{a})$$

given by

$$(D_n f)(X_1, \ldots, X_n) = \frac{\partial^n}{\partial t_1 \ldots \partial t_n} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) f\Big(\gamma_{\sigma(1)}(t_{\sigma(1)}), \ldots, \gamma_{\sigma(n)}(t_{\sigma(n)})\Big)\Big|_{t_i=0}$$

where $\gamma_1(t_1), \ldots, \gamma_n(t_n)$ is any set of smooth curves in G satisfying $\gamma_i(0) = \mathbf{1}$ and $\gamma_i'(0) = X_i \in \mathfrak{g}$.

Proof. See.[30] □

We conclude by stating two important results in the finite dimensional case.

Theorem 2.8. *(Whitehead)* Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and $\mathfrak{a}$ a finite dimensional irreducible, non-trivial $\mathfrak{g}$-module. Then

$$H^n(\mathfrak{g}, \mathfrak{a}) = 0$$

for all $n \geq 0$.

Proof. See[4] □

In the next section we will see that the second cohomology group parametrizes equivalence classes of split abelian extensions. The Whitehead theorem thus implies that there are no non-trivial abelian Lie algebra extensions of finite dimensional semisimple Lie algebras.

Theorem 2.9. Let G be a finite dimensional compact Lie group with Lie algebra $\mathfrak{g}$. Then

$$H^n(\mathfrak{g}, \mathbb{R}) \cong H^n(G)$$

where $H^n(G)$ denotes the de Rham cohomology group of G.

Proof. For every closed form ω on G, the averaged form $\int_G L_g^* \omega dg$ with respect to the Haar measure is left-invariant and represents the same cohomology class. The de Rham cohomology group is therefore determined by the cohomology of left-invariant forms on G, which is the same as $H^n(\mathfrak{g}, \mathbb{R})$. □

2.3 Lie group extensions

Definition 2.10. An extension of Lie groups is a short exact sequence with smooth homomorphisms

$$\mathbf{1} \to A \xrightarrow{i} \widehat{G} \xrightarrow{p} G \to \mathbf{1}$$

such that p admits a smooth local section $\sigma : U \to \widehat{G}$, $p \circ \sigma = \mathrm{id}_U$, where $U \subset G$ is an open identity neighborhood.

The existence of a smooth local section means that $\widehat{G}$ is a principal A-bundle over G. The extension is called *abelian* if A is abelian and *central* if $i(A)$ belongs to the center $Z(\widehat{G})$. There is a natural equivalence between two Lie group extensions $\widehat{G}_1$ and $\widehat{G}_2$. They are said to be equivalent if there exists a smooth isomorphism $\phi : \widehat{G}_1 \to \widehat{G}_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{i_1} & \widehat{G}_1 & \xrightarrow{p_1} & G \\ {\scriptstyle id_A}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle id_G}\downarrow \\ A & \xrightarrow{i_2} & \widehat{G}_2 & \xrightarrow{p_2} & G \end{array}$$

We write $\mathrm{Ext}(G, A)$ for the set of equivalence classes of Lie group extensions of G by A. An abelian extension is called *split* if there exists a smooth global section $\sigma : G \to \widehat{G}$. Such extensions can be constructed explicitly.

Proposition 2.11. Let G be a Lie group, A a smooth G-module and $f \in Z^2(G, A)$ a smooth 2-cocycle. The product manifold $G \times A$ endowed with the multiplication

$$(g_1, a_1)(g_2, a_2) = (g_1 g_2, a_1 + g_1.a_2 + f(g_1, g_2))$$

defines an abelian extension $\widehat{G} = G \times_f A$ of G by A.

Proof. We check the group axioms. Associativity of the group law

$$(g_1, a_1)\{(g_2, a_2)(g_3, a_3)\} =$$

$$= (g_1 g_2 g_3, a_1 + g_1.a_2 + (g_1 g_2).a_3 + g_1.f(g_2, g_3) + f(g_1, g_2 g_3))$$

and

$$\{(g_1, a_1)(g_2, a_2)\}(g_3, a_3) =$$

$$= (g_1 g_2 g_3, a_1 + g_1.a_2 + f(g_1, g_2) + (g_1 g_2).a_3 + f(g_1 g_2, g_3))$$

implies that

$$g_1.f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = (\delta_2 f)(g_1, g_2, g_3) = 0 \ ,$$

which is satisfied since $f \in Z^2(G, A)$. The normalization condition $f(g, \mathbf{1}) = f(\mathbf{1}, g) = 0$ ensures that $(\mathbf{1}, 0)$ is the identity element and the inversion formula is given by

$$(g, a)^{-1} = \big(g^{-1}, -g^{-1}.(a + f(g, g^{-1}))\big) \ .$$

Since the group operations in G and A are smooth and f is a smooth 2-cocycle, it follows that the group law in $\widehat{G}$ is smooth. □

We note that the conjugation action of $\widehat{G}$ on A

$$(g, 0)(\mathbf{1}, a)(g, 0)^{-1} = (g, g.a)(g^{-1}, -g^{-1}.f(g, g^{-1})) = (\mathbf{1}, g.a)$$

induces the original G-action. One can prove that $\widehat{G}$ is regular if and only if G and A are regular. When the cocycle f is trivial, $\widehat{G}$ is just the semidirect product of G and A.

Example 2.12. (Virasoro-Bott group). Let $\mathrm{Diff}^+(S^1)$ denote the Lie group of orientation preserving diffeomorphisms of the circle. The group law is given by composition

$$(\varphi \circ \psi)(z) = \varphi(\psi(z)) \ .$$

One can show that $\mathrm{Diff}^+(S^1)$ is a simple regular Lie group with the "Witt algebra" $\mathrm{Vect}(S^1)$ of smooth vector fields on S^1 as Lie algebra.[27] The map $f : \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1) \to \mathbb{R}$ defined by

$$f(\varphi, \psi) \equiv \frac{1}{2} \int_{S^1} \log(\varphi \circ \psi)' d \log \psi'$$

is a non-trivial smooth 2-cocycle, called the Bott cocycle. By exponentiation $e^{2\pi i f}$, we obtain a cocycle with values in S^1. The corresponding central extension $\mathrm{Diff}^+(S^1) \times_f S^1$ is called the Virasoro-Bott group. The ring $H^*(\mathrm{Diff}^+(S^1), \mathbb{R})$ is actually known to be generated by two elements α, β subject to the relation $\beta^2 = 0$. The connecting map $D_n : H^n(\mathrm{Diff}^+(S^1), \mathbb{R}) \to H^n(\mathrm{Vect}(S^1), \mathbb{R})$ sends β to zero and α to a nonzero element.[13]

Next we discuss classification of split abelian extensions. Let $\widetilde{\mathrm{Ext}}(G,A)$ denote the set of equivalence classes of such extensions.

Theorem 2.13. The assignment

$$H^2(G,A) \to \widetilde{\mathrm{Ext}}(G,A), \quad [f] \mapsto G \times_f A$$

is a bijection.

Proof. By Proposition 2.11 we know that every cocycle f defines a split abelian extension. If two extensions $G \times_{f_1} A$ and $G \times_{f_2} A$ are equivalent, then the isomorphism $\phi : G \times_{f_1} A \to G \times_{f_2} A$ is necessarily of the form

$$(g,a) \mapsto (g, a + c(g))$$

where $c \in C^1(G,A)$. Indeed since any element in $G \times_{f_1} A$ can be written $(g,a) = (g,0)(\mathbf{1},a)$, ϕ is completely fixed once its values on $(g,0)$ and $(\mathbf{1},a)$ are determined. By the commutativity of the diagram, we have $\phi \circ i_1 = i_2$ and $p_2 \circ \phi = p_1$ which imply $\phi(\mathbf{1},a) = (\mathbf{1},a)$ and $\phi(g,0) = (g,c(g))$ respectively. Hence $\phi(g,a) = (g,a+c(g))$ is the generic form for the isomorphism. The condition that ϕ is a homomorphism

$$(gg', a + g.a' + f_1(g,g') + c(gg')) = (g', a + c(g), g)(a' + c(g'))$$

$$= (gg', a + c(g) + g.a' + g.c(g') + f_2(g,g'))$$

implies that the 2-cocycles differ by a coboundary

$$(f_1 - f_2)(g,g') = g.c(g') - c(gg') + c(g) = (\delta_1 c)(g,g') \in B^2(G,A) \ .$$

Thus the assignment factors through to cohomology.

Conversely, if $p : \widehat{G} \to G$ is a split abelian extension, then by global trivialization it is diffeomorphic to $G \times A$. Fixing a smooth global section $\sigma : G \to \widehat{G}$, we define a smooth 2-cocycle f by

$$f(g_1,g_2) = \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1}$$

and construct the Lie group $G \times_f A$. Then the following map defines a Lie group isomorphism

$$G \times_f A \to \widehat{G}, \quad (g,a) \mapsto \sigma(g)a \ .$$

A different choice of trivializing section would give an equivalent extension, amounting to a change in f by a coboundary. □

The general situation when the principal A-bundle $\widehat{G}$ is topologically non-trivial, i.e. when there is no continuous global section, has been investigated independently by Graeme Segal and Dennis Johnson:

Theorem 2.14. The set of equivalence classes $\mathrm{Ext}(G, A)$ is described by the short exact sequence

$$\mathbf{0} \to H^2(G, A) \to \mathrm{Ext}(G, A) \to \check{H}^1(G, \underline{A}) \to \mathbf{0}$$

Proof. See.[33,34] □

Here $\check{H}^1(G, A)$ is the first Čech cohomology group of G with coefficients in the sheaf of continuous functions on G with values in A. This makes clear how the topological and algebraic twists are related.

Example 2.15. (Loop group). The group LG of smooth maps from the unit circle to a finite dimensional Lie group G, has central extensions

$$\mathbf{1} \to S^1 \xrightarrow{i} \widehat{LG} \xrightarrow{p} LG \to \mathbf{1}$$

which are topologically non-trivial. In fact when G is compact, the group cohomology $H^2(LG, S^1) = 0$ is trivial[13] and by Theorem 2.14 all central extensions are characterized by the Čech cohomology group $\check{H}^1(LG, \underline{S^1})$. There are various geometric constructions of $\widehat{LG}$.[22,32] In Paper I we have generalized the path group construction of $\widehat{LG}$ to general abelian extensions.

2.4 Lie algebra extensions

Definition 2.16. An extension of topological Lie algebras is a short exact sequence with continuous homomorphisms

$$\mathbf{0} \to \mathfrak{a} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to \mathbf{0}$$

such that p admits a continuous global section $s : \mathfrak{g} \to \widehat{\mathfrak{g}}$, $p \circ s = \mathrm{id}_{\mathfrak{g}}$.

Two extensions $\hat{\mathfrak{g}}_1$ and $\hat{\mathfrak{g}}_2$ are said to be equivalent if there is an isomorphism of topological Lie algebras $\phi : \hat{\mathfrak{g}}_1 \to \hat{\mathfrak{g}}_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathfrak{a} & \xrightarrow{i_1} & \hat{\mathfrak{g}}_1 & \xrightarrow{p_1} & \mathfrak{g} \\ {\scriptstyle id_{\mathfrak{a}}}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle id_{\mathfrak{g}}}\downarrow \\ \mathfrak{a} & \xrightarrow{i_2} & \hat{\mathfrak{g}}_2 & \xrightarrow{p_2} & \mathfrak{g} \end{array}$$

We write $\mathrm{Ext}(\mathfrak{g}, \mathfrak{a})$ for the set of equivalence classes of Lie algebra extensions of $\mathfrak{g}$ by $\mathfrak{a}$. As in the case of Lie groups, there is a cocycle construction of abelian Lie algebra extensions.

Proposition 2.17. If $\mathfrak{a}$ is a topological $\mathfrak{g}$-module and $\omega \in Z^2(\mathfrak{g}, \mathfrak{a})$ a continuous 2-cocycle, then the topological sum vector space $\mathfrak{g} \oplus \mathfrak{a}$ endowed with the Lie bracket

$$[(X_1, v_1), (X_2, v_2)] = ([X_1, X_2], X_1.v_2 - X_2.v_1 + \omega(X_1, X_2))$$

defines an abelian extension $\hat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{a}$.

Proof. We check the Lie algebra axioms. Linearity and antisymmetry follow from the definition of the 2-cocycle ω. The Jacobi identity

$$[[(X_1, v_1), (X_2, v_2)], (X_3, v_3)] + cycl. = 0$$

implies that

$$\omega([X_1, X_2], X_3) + \omega([X_2, X_3], X_1) + \omega([X_3, X_1], X_2) -$$

$$-X_1.\omega(X_2, X_3) - X_2.\omega(X_3, X_1) - X_3.\omega(X_1, X_2) = (d_2\omega)(X_1, X_2, X_3) = 0 \ ,$$

which is satisfied since $\omega \in Z^2(\mathfrak{g}, \mathfrak{a})$. Since ω and the Lie bracket in $\mathfrak{g}$ are continuous, it follows that the commutator in $\hat{\mathfrak{g}}$ is continuous. □

Example 2.18. (Virasoro algebra). In Example 2.12 we discussed a central extension of $\mathrm{Diff}^+(S^1)$ by the circle. The corresponding Lie algebra extension $\mathrm{Vect}(S^1) \oplus_\omega i\mathbb{R}$ is defined by the non-trivial Gelfand-Fuchs cocycle

$$\omega(X, Y) = \frac{c}{2\pi} \int_{S^1} X' dY'$$

for some constant $c \in \mathbb{R}$. A vector field on S^1 can be written $f(\theta)\frac{d}{d\theta}$ where $f(\theta) \in C^\infty(S^1)$ and $\theta \in [0, 2\pi)$. Since smooth functions on the circle are square-integrable, we can decompose $f(\theta)$ into Fourier modes and construct a basis for $\mathrm{Vect}(S^1)$,

$$L_n = ie^{in\theta}\frac{d}{d\theta}, \quad n \in \mathbb{Z}$$

which satisfy

$$[L_m, L_n] = i(n - m)L_{m+n} \ .$$

In this basis the cocycle is given by

$$\omega(L_m, L_n) = \frac{c}{2\pi} im \cdot n^2 \int_{S^1} e^{i(m+n)\theta} d\theta = icm^3 \delta_{m,-n} \ .$$

There is a general result concerning the Lie algebra cohomology of $\mathrm{Vect}(M)$, which states that for a compact oriented manifold M, $H^n(\mathrm{Vect}(M), \mathbb{R})$ is a finite dimensional vector space for all $n \geq 0$.[4] In particular it is known that $H^2(\mathrm{Vect}(S^1), \mathbb{R}) = \mathbb{R}$. This combined with Theorem 2.20 imply that all central extensions of the Lie algebra $Vect(S^1)$ by $\mathbb{R}$ are isomorphic to the Virasoro algebra.

Example 2.19. (Loop algebra). The Lie algebra of the central extension $\widehat{LG}$ discussed in Example 2.15 is defined by the continuous non-trivial 2-cocycle

$$\omega(X,Y) = \frac{1}{2\pi}\int_{S^1} < X, dY > ,$$

where $< \cdot,\cdot >: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denotes an invariant symmetric bilinear form. When $\mathfrak{g}$ is a compact semisimple Lie algebra, $L\mathfrak{g} \oplus_\omega i\mathbb{R}$ defines a Kac-Moody algebra and is the essentially unique central extension of $L\mathfrak{g}$ by $\mathbb{R}$. Let $\{T^1, T^2, \ldots, T^n\}$ be a basis for $\mathfrak{g}$ satisfying $[T^a, T^b] = \lambda^{abc}T^c$ and set $T^a_n = T^a e^{in\theta}$, $n \in \mathbb{Z}$. These provide an infinite set of generators for $L\mathfrak{g}$ and satisfy

$$[T^a_m, T^b_n] = \lambda^{abc}T^c_{m+n} \; .$$

In this basis the cocycle is given by

$$\omega(T^a_m, T^b_n) = i < T^a, T^b > n\frac{1}{2\pi}\int_{S^1} e^{i(m+n)\theta}d\theta = in < T^a, T^b > \delta_{m,-n} \; .$$

Theorem 2.20. The assignment

$$H^2(\mathfrak{g},\mathfrak{a}) \to \mathrm{Ext}(\mathfrak{g},\mathfrak{a}), \quad [\omega] \mapsto \mathfrak{g} \oplus_\omega \mathfrak{a}$$

is a bijection.

Proof. The proof follows the same line of arguments as in Theorem 2.15 and will not be repeated here. □

The cocycle construction of Lie group and Lie algebra extensions are compatible:

Proposition 2.21. Let A denote a smooth G-module of the form $\mathfrak{a}/\Gamma$ for some discrete subgroup $\Gamma \subset \mathfrak{a}$. If $\widehat{G} = G \times_f A$ is an extension of G by A, then

$$\hat{\mathfrak{g}} \cong \mathfrak{g} \oplus_{D_2 f} \mathfrak{a} \; .$$

Proof. See.[30] □

Chapter 3

Gerbes

In this chapter we review basic concepts concerning line bundles and discuss their generalization to gerbes. The relation between gerbes and group extensions is most clearly seen in Example 2.15 of loop groups. When G is a compact, connected and simply-connected Lie group, the central extensions of the loop group are classified by $\check{H}^1(LG, \underline{S^1}) \cong H^2(LG, \mathbb{Z})$. Now there is a transgression map $H^2(LG, \mathbb{Z}) \to H^3(G, \mathbb{Z})$ given in two steps,

$$H^3(G, \mathbb{Z}) \to H^3(S^1 \times LG, \mathbb{Z}) \to H^2(LG, \mathbb{Z})$$

where the first map is induced by the evaluation map $S^1 \times LG \to G$ and the second is integration over the fibre S^1. This is in fact an isomorphism. First we note that $LG = \Omega G \times G$ as topological spaces, where ΩG is the group of loops in G based at the identity $\mathbf{1} \in G$. Since $\pi_1(G) = 0$, the loop group is also simply connected, $\pi_1(LG) = \pi_1(\Omega G) = \pi_2(G) = 0$ where the last equality holds for any finite dimensional Lie group. The isomorphism then follows by Hurewicz theorem $H^2(LG, \mathbb{Z}) = \pi_2(LG) = \pi_3(G) = H^3(G)$. Thus to every circle bundle over LG there is an associated gerbe on G. In other words, central extensions of the infinite dimensional Lie group LG are completely determined by the third cohomology of the finite dimensional group G.

3.1 Line bundles

Global description.

Definition 3.1. A complex line bundle L over a manifold M is a complex vector bundle of rank 1. More precisely, L is a manifold with a smooth projection $\pi : L \to M$ such that each fibre $\pi^{-1}(x) = L_x$ is a complex one-dimensional vector space. It is also *locally trivial,* i.e. for any $x \in M$ there is an open neighborhood U and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{C}$ such that $\phi(L_x) \subset \{x\} \times \mathbb{C}$ for all $x \in U$ and the induced map $\phi|_{L_x} : L_x \to x \times \mathbb{C}$ is a linear isomorphism.

There is a bijective correspondence between complex line bundles and principal $\mathbb{C}^*$-bundles. Given a line bundle L, the complement L^+ of the zero section in L defines a principal $\mathbb{C}^*$-bundle. Two elements y_1 and y_2 in the fibre are related by the natural right action by $\frac{y_1}{y_2} \in \mathbb{C}^*$. Conversely, a principal $\mathbb{C}^*$-bundle L^+ always gives rise to an associated line bundle $L = L^+ \times_{\mathbb{C}^*} \mathbb{C}$, where $\lambda \in \mathbb{C}^*$ acts on $L^+ \times \mathbb{C}$ by $\lambda(y, \mu) = (y\lambda^{-1}, \mu\lambda)$.

Two bundles $\pi_1 : L_1 \to M$ and $\pi_2 : L_2 \to M$ are isomorphic if there exists a diffeomorphic bundle map, i.e. a diffeomorphism $\phi : L_1 \to L_2$ compatible with the projections $\pi_1 = \pi_2 \circ \phi$ and such that the induced map $\phi|_{L_{1x}} : L_{1x} \to L_{2x}$ is a linear isomorphism. A line bundle is trivial if it is isomorphic to the trivial bundle $M \times \mathbb{C}$. In particular, every line bundle over a contractible space M is trivial. Given a smooth map $f : N \to M$ between manifolds, one can always pull back $\pi : L \to M$ to a line bundle $f^*L = \{(y, x) \in L \times N \mid \pi(y) = f(x)\}$ over N.

The set of isomorphism classes of line bundles over M carries a natural abelian group structure. Tensor product $L_1 \otimes L_2$ defines the group multiplication and for every element L there is an inverse in the form of the dual bundle L^* satisfying $L \otimes L^* \cong M \times \mathbb{C}$. This group is called the Picard group and is denoted by $Pic^\infty(M)$. Passing to a local picture, we can describe it through the first Čech cohomology group.

Local description. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite good cover of M. By a good cover we mean that all nonempty finite intersections of open sets are contractible. Every finite dimensional smooth manifold has a good cover.[6] A line bundle $\pi : L \to M$ can be described locally by a collection of sections $\sigma_i : U_i \to L$, $\pi \circ \sigma_i = id_{U_i}$. The existence of local nowhere vanishing sections is in fact equivalent to the local triviality condition. On overlaps we can introduce transition functions $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ by

$$g_{ij} = \sigma_i \circ \sigma_j^{-1} \ .$$

These capture all the information contained in L and can be used to completely reconstruct the bundle via the so called clutching construction.[18] The transition functions must satisfy certain compatibility relations, namely $g_{ij} = g_{ji}^{-1}$, $g_{ii} = 1$ and $g_{ik} = g_{ij} g_{jk}$ on triple overlaps. If two bundles are isomorphic, then their transition functions are related according to

$$g'_{ij} = h_j^{-1} g_{ij} h_i = h_i h_j^{-1} g_{ij}$$

for some functions $h_i : U_i \to \mathbb{C}^*$. The transition functions of the dual bundle L^* are given by g_{ij}^{-1} and $g_{ij} g'_{ij}$ describe the tensor product $L \otimes L'$. A line bundle is trivial if and only if it has a nowhere vanishing global section. This means that there exists a collection of functions $h_i : U_i \to \mathbb{C}^*$ such that

$$g_{ij} = h_i h_j^{-1}$$

on all overlaps, i.e the bundle can be described by transition functions that take the value 1 everywhere. This data suggests a commutative group structure on the set of equivalence classes of transition functions. Indeed, this defines the first Čech cohomology group. More generally, the Čech groups are defined in terms of "transition functions" on multiple overlaps. Introduce the notation $U_{i_0,\dots,i_k} = U_{i_0} \cap \cdots \cap U_{i_k}$. Denote by $\underline{A}$ the sheaf of continuous A-valued functions and A the sheaf of constant A-valued functions, where A is an abelian group. Recall that a sheaf is an assignment of abelian groups to open sets satisfying certain restriction conditions, see[35] for a precise definition. In our case, we have the natural restriction of a function on $U_{i_0,\dots,i_k}$ to $U_{i_0,\dots,i_{k+1}}$. We write $\underline{A}(U_{i_0,\dots,i_k})$ for the functions on $U_{i_0,\dots,i_k}$ and define the set of n-cochains by

$$C^n(\mathcal{U}, \underline{A}) = \prod_{i_0,i_1,\dots,i_n} \underline{A}(U_{i_0,\dots,i_n})$$

where the product ranges over all $n+1$-tuples in the index set I^{n+1}. Moreover, the n-cochains are assumed to be antisymmetric in the indices $g_{i_0,\dots,i_a,\dots,i_b,\dots,i_k} = g^{-1}_{i_0,\dots,i_b,\dots,i_a,\dots,i_k}$. Define a homomorphism $\partial_n : C^n(\mathcal{U}, \underline{A}) \to C^{n+1}(\mathcal{U}, \underline{A})$ by

$$(\partial_n g)_{i_0,i_1,\dots,i_{n+1}} = \prod_{j=0}^{n+1} g^{(-1)^j}_{i_0,i_1,\dots,\hat{i}_j,\dots,i_{n+1}}|_{U_{i_0,\dots,i_{n+1}}} .$$

One verifies easily that the coboundary operator ∂_n satisfies $\partial_{n+1} \circ \partial_n = 0$. The Čech cohomology groups are then

$$\check{H}^n(\mathcal{U}, \underline{A}) = \frac{\ker\, \partial_\mathrm{n}}{\mathrm{im}\, \partial_{\mathrm{n}-1}} .$$

The definition can be made independent of the open cover $\mathcal{U}$ by introducing an ordering on the set of all open covers (through refinements) and taking the direct limit.[6] Moreover, on smooth finite dimensional manifolds M one can show that Čech and singular cohomology with integer coefficients coincide

$$\check{H}^n(M, \underline{\mathbb{Z}}) = \check{H}^n(M, \mathbb{Z}) = H^n(M, \mathbb{Z}) .$$

We have the following result:

Theorem 3.2. The assignment

$$Pic^\infty(M) \to \check{H}^1(M, \underline{\mathbb{C}^*}), \quad L \to g_{ij}$$

is an isomorphism.

The short exact sequence

$$\mathbf{0} \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to \mathbf{1}$$

induces a short exact sequence of the corresponding sheaves

$$\mathbf{0} \to \underline{\mathbb{Z}} \to \underline{\mathbb{C}} \xrightarrow{\exp} \underline{\mathbb{C}^*} \to \mathbf{1} \ .$$

This sequence of coefficient groups induces a long exact sequence in Čech cohomology

$$\begin{aligned} \cdots \to \check{H}^n(M,\underline{\mathbb{Z}}) \to \check{H}^n(M,\underline{\mathbb{C}}) \to \check{H}^n(M,\underline{\mathbb{C}^*}) \to \\ \to \check{H}^{n+1}(M,\underline{\mathbb{Z}}) \to \check{H}^{n+1}(M,\underline{\mathbb{C}}) \to \check{H}^{n+1}(M,\underline{\mathbb{C}^*}) \to \dots \end{aligned}$$

Using the fact that Čech cohomology with coefficients in *fine* sheaves (i.e. with partition of unity) such as $\underline{\mathbb{R}}$ or $\underline{\mathbb{C}}$ is trivial,

$$\check{H}^n(M,\underline{\mathbb{R}}) = \check{H}^n(M,\underline{\mathbb{C}}) = 0, \quad \forall\ n \geq 1 \ ,$$

we get the following isomorphisms

$$\mathbf{0} \to \check{H}^n(M,\underline{\mathbb{C}^*}) \xrightarrow{\sim} \check{H}^{n+1}(M,\underline{\mathbb{Z}}) \to \mathbf{0}$$

for $n \geq 1$. This isomorphism can be made explicit using the logarithm function (with a fixed branch),

$$g_{i_0,\dots,i_n} \mapsto h_{i_0,\dots,i_{n+1}} = \frac{i}{2\pi} \sum_{j=0}^{n+1} (-1)^j \log g_{i_0,\dots,\hat{i}_j,\dots,i_{n+1}}$$

In particular, for $\check{H}^1(M,\underline{\mathbb{C}^*}) \cong \check{H}^2(M,\underline{\mathbb{Z}})$ we have

$$2\pi i \ h_{ijk} = \log g_{jk} - \log g_{ik} + \log g_{ij}$$

which is a $\underline{\mathbb{Z}}$-valued Čech 2-cocycle. This cocycle defines a characteristic class $c(L) \in H^2(M,\mathbb{Z})$ of the line bundle L called the first Chern class. The Chern class is natural with respect to pullback, dual, and tensor product,

$$f^*(c(L)) = c(f^*L), \quad c(L^*) = -c(L), \quad c(L_1 \otimes L_2) = c(L_1) + c(L_2) \ .$$

Two isomorphic line bundles have the same Chern class and $c(L) = 0$ if and only if L is trivial. The de Rham representative F of the Chern class on $U_i \subset M$ can be expressed in terms of the transition functions,

$$F_i = \frac{i}{2\pi} d \sum_{j \in I} \rho_j g_{ij}^{-1} dg_{ij}$$

where $\{\rho_j\}_{j\in I}$ is a partition of unity subordinate to the cover $\mathcal{U}$, i.e. a set of functions $\rho_j : M \to [0,1]$ such that $\sum_{j\in I} \rho_j(x) = 1$ for all $x \in M$ and the support of ρ_j is contained in U_j. This map can actually be generalized to $\check{H}^n(M, \underline{\mathbb{C}^*}) \to H^{n+1}(M, \mathbb{C})$ by

$$F_i = \frac{i}{2\pi} d \sum_{i_0,\dots,i_{n-1} \in I} \rho_{i_0} d\rho_{i_1} \wedge \dots \wedge d\rho_{i_{n-1}} \wedge g^{-1}_{i,i_0,\dots,i_{n-1}} dg_{i,i_0,\dots,i_{n-1}}$$

3.2 Gerbes

Global description. Our definition of gerbes is restricted to smooth manifolds and follows closely,[28, 29] where they are referred to as *bundle gerbes*. Every bundle gerbe corresponds to a gerbe in the original sense as a sheaf of groupoids.[8] In fact, every isomorphism class of gerbes has a bundle gerbe representative.

Definition 3.3. A map $f : M \to N$ between two manifolds is called a submersion if the rank of the derivative $T_x f : T_x M \to T_{f(x)} N$ equals the dimension of N for all $x \in M$.

Let $\pi : Y \to M$ be a surjective submersion and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of M. We denote by $Y^{[p]}$ the p-fold fibre product

$$Y^{[p]} = \{(y_1, ..., y_p) \mid \pi(y_1) = ... = \pi(y_p)\}$$

and define projections $\pi_i : Y^{[p]} \to Y^{[p-1]}$ by deleting the entry at position i. By the submersion property of π it follows that $Y^{[p]} \subset Y^p$ is a submanifold and π_i are smooth for all $i = 1, \dots, p$.

Definition 3.4. A bundle gerbe over M is a pair (P, Y) consisting of a surjective submersion $\pi : Y \to M$ and a circle bundle $S^1 \to P \to Y^{[2]}$. There is a multiplication defined by a smooth isomorphism of circle bundles over $Y^{[3]}$,

$$m : \pi_3^* P \otimes \pi_1^* P \to \pi_2^* P$$

which is associative in the sense that the following diagram

$$\begin{array}{ccc} P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes P_{(y_3,y_4)} & \longrightarrow & P_{(y_1,y_3)} \otimes P_{(y_3,y_4)} \\ \downarrow & & \downarrow \\ P_{(y_1,y_2)} \otimes P_{(y_2,y_4)} & \longrightarrow & P_{(y_1,y_4)} \end{array}$$

commutes for all $(y_1, y_2, y_3, y_4) \in Y^{[4]}$, where $P_{(y_1,y_2)}$ denotes the fibre over $(y_1, y_2) \in Y^{[2]}$.

The bundle gerbe multiplication defines an isomorphism

$$P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \cong P_{(y_1,y_3)}$$

for all $(y_1, y_2, y_3) \in Y^{[3]}$ and implies that

$$P_{(y_1,y_2)} \cong P^*_{(y_2,y_1)}, \quad P_{(y,y)} \cong \{(y,y)\} \times S^1 \ .$$

Two bundle gerbes (P, Y) and (Q, Y) are said to be isomorphic when the circle bundles P and Q over $Y^{[2]}$ are isomorphic. Similar to line bundles, one can define the notion of pullback, dual and product. For any smooth map $f : N \to M$, the pullback

$$\begin{array}{ccc} f^*Y & \xrightarrow{\hat{f}} & Y \\ \Big\downarrow & & \Big\downarrow \pi \\ N & \xrightarrow{f} & M \end{array}$$

induces a map $\hat{f}^{[2]} : f^*(Y)^{[2]} \to Y^{[2]}$. The pullback of the bundle gerbe over M is defined by $f^*(P, Y) = (f^*(Y)^{[2]}, f^*Y)$. The dual $(P, Y)^* = (P^*, Y)$ is given by the dual of the circle bundle P. We can introduce a product between bundle gerbes (P, Y) and (Q, X) by

$$(P, Y) \otimes (Q, X) = (P \otimes Q, Y \times_M X)$$

where $Y \times_M X \to M$ is the surjective submersion defined by the fibre product and $P \otimes Q$ is the circle bundle

$$S^1 \to P \otimes Q \to (Y \times_M X)^{[2]} = Y^{[2]} \times_M X^{[2]}$$

given by

$$(P \otimes Q)_{((y_1,x_1),(y_2,x_2))} = P_{(y_1,y_2)} \otimes Q_{(x_1,x_2)} \ .$$

A bundle gerbe (P, π) is trivial if it can be trivialized by a circle bundle $S^1 \to R \to Y$, meaning that the induced circle bundle

$$\delta(R) = \pi_1^* R \otimes (\pi_2^* R)^*$$

over $Y^{[2]}$ is isomorphic to P. One can prove that a bundle gerbe is trivial if $\pi : Y \to M$ admits a global section, but the converse statement is not true.

It so happens, the definition of isomorphism given above is not appropriate for classification of bundle gerbes. For that purpose we need a weaker notion:

Definition 3.5. Two bundle gerbes (P, Y) and (Q, Y) are called *stably isomorphic* if $(P, Y)^* \otimes (Q, X)$ is trivial. A stable isomorphism from (P, Y) to (Q, Y) corresponds to fixing a trivialization by a circle bundle $S^1 \to R \to Y$.

Stable isomorphism defines an equivalence relation on the set of all bundle gerbes. By passing to a local picture, we will see that the set of equivalence classes is in $1-1$ correspondence with the second Čech cohomology group.

For any open cover $\mathcal{U}$ of M, there is a natural surjective submersion $\pi : Y_{\mathcal{U}} \to M$ defined by the disjoint union

$$Y_{\mathcal{U}} = \{(x,i) \in M \times I \mid x \in U_i\}$$

with the projection $\pi(x,i) = x$.

Definition 3.6. A bundle gerbe (P,Y) over M is called *local* if $Y = Y_{\mathcal{U}}$ with respect to some open cover $\mathcal{U}$ of M.

One can prove that every bundle gerbe is stably isomorphic to a local one.[29]

Local description. Every surjective submersion $\pi : Y \to M$ admits local sections $s_i : U_i \to Y$ on some open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M. Assuming that $\mathcal{U}$ is a good cover, these define local sections

$$(s_i, s_j) : U_i \cap U_j \to Y^{[2]}$$

on overlaps in M which can be used to pull back the circle bundle P over $Y^{[2]}$

$$P_{ij} = (s_i, s_j)^* P$$

to circle bundles P_{ij} over $U_i \cap U_j$. For any section $\sigma_{ij} : U_i \cap U_j \to P_{ij}$, the bundle gerbe multiplication

$$m : P_{(s_i(x),s_j(x))} \otimes P_{(s_j(x),s_k(x))} \to P_{(s_i(x),s_k(x))}$$

implies that

$$m(\sigma_{ij}(x), \sigma_{jk}(x)) = g_{ijk}(x)\sigma_{ik}(x)$$

over triple overlaps, for a circle-valued function $g_{ijk} : U_i \cap U_j \cap U_k \to S^1$ satisfying

$$g_{ijk} = g_{jik}^{-1} = g_{jki} = g_{kji}^{-1} .$$

The associativity condition for the multiplication translates to

$$g_{ijk} g_{ijl}^{-1} g_{ikl} g_{jkl}^{-1} = 1$$

on quadruple intersections $U_i \cap U_j \cap U_k \cap U_l$. The "transition functions" of the dual bundle gerbe $(P,Y)^*$ are given by g_{ijk}^{-1} and $g_{ijk} g'_{ijk}$ describe the product $(P,Y) \otimes (Q,X)$. If two bundle gerbes are stably isomorphic, then their transition functions are related according to

$$g'_{ijk} = g_{ijk} h_{ij} h_{jk} h_{ik}^{-1}$$

for some function $h_{ij} : U_i \cap U_j \to S^1$. In terms of Čech cocycles, this simply means that they are cohomologous, $g'_{ijk} = g_{ijk}(\partial_1 h)_{ijk}$. A bundle gerbe is trivial if and only if $g_{ijk} = h_{ij} h_{jk} h_{ik}^{-1}$, i.e. the transition functions can be chosen to be 1 everywhere.

Theorem 3.7. The assignment

$$(P, Y) \mapsto g_{ijk}$$

defines a bijection from the set of stable isomorphism classes of bundle gerbes to the second Čech cohomology $\check{H}^2(M, \underline{S^1})$.

Recall that line bundles are described by functions g_{ij} on double overlaps satisfying $g_{ij} = g_{ji}^{-1}$ and $g_{ik} = g_{ij} g_{jk}$. This raises the question whether bundle gerbes have an analogous description on $U_i \cap U_j$. The answer is affirmative and in fact leads to an alternative definition of gerbes due to Hitchin[17] and Chatterjee. They define a gerbe as a collection of line bundles $\{L_{ij}\}$ on two-fold intersections satisfying $L_{ij} = L_{ji}^*$ and the cocycle identity

$$L_{ij} \otimes L_{jk} = L_{ik}$$

on triple overlaps. This condition means that there is a trivialization

$$L_{ij} \otimes L_{jk} \otimes L_{ki} = g_{ijk} \mathbf{1}$$

where $g_{ijk} : U_i \cap U_j \cap U_k \to S^1$ is a Čech cocycle of degree two. A gerbe defined in this way is the same thing as a local bundle gerbe. Two systems of line bundles $\{L_{ij}\}$ and $\{L'_{ij}\}$ define the same gerbe if and only if there is a family of line bundles $\{L_i\}$ such that

$$L'_{ij} = L_{ij} \otimes L_i^* \otimes L_j \ .$$

In particular, a gerbe defined by $\{L_{ij}\}$ is trivial if

$$L_{ij} = L_i^* \otimes L_j \ .$$

From this definition it is evident why a gerbe is not a finite dimensional manifold, unlike the total space of a line bundle. While transition functions g_{ij} can be used to sew together fibres of line bundles over double overlaps, there is no such construction for a collection of line bundles or for g_{ijk} over triple intersections.

This idea of defining gerbes using line bundles can be generalized in an inductive manner to higher dimensional objects providing geometric realizations of higher cohomology groups $H^n(M, \mathbb{Z})$. Thus a n-gerbe corresponding to Čech cocycle $g_{i_0,\ldots,i_n}$ can be defined as a collection of $n-1$-gerbes on double overlaps satisfying a list of compatibility conditions. However, this list grows with the degree n and makes this definition rather awkward for large values of n.

The cocycle g_{ijk} defines a characteristic class $DD(P,Y)$ of the bundle gerbe, called the Dixmier-Douady class. Using the isomorphism between Čech and singular cohomology groups discussed in the previous section,

$$\check{H}^2(M,\underline{S^1}) \cong \check{H}^3(M,\underline{\mathbb{Z}}) \cong H^3(M,\mathbb{Z})$$

there is an explicit formula for the image of $DD(P,Y)$ in $\check{H}^3(M,\underline{\mathbb{Z}})$,

$$2\pi i\ h_{ijkl} = \log g_{ikl} - \log g_{jkl} + \log g_{ijk} - \log g_{ijl}$$

with a fixed branch for the logarithm. The Dixmier-Douady class is natural with respect to pullback, dual and product,

$$f^*DD(P,Y) = DDf^*(P,Y), \quad DD(P,Y)^* = -DDf^*(P,Y)\ ,$$

$$DD(P,Y)\otimes(Q,X) = DD(P,Y) + DD(Q,X)\ .$$

Stably isomorphic bundle gerbes have the same Dixmier-Douady class and a bundle gerbe is trivial if and only if $DD(P,Y) = 0$. Notice that the weaker notion of stable isomorphism is crucial here, since one can easily construct two trivial bundle gerbes which are not isomorphic. The de Rham representative Ω_i of the Dixmier-Douady class on $U_i \subset M$ expressed in terms of the transition functions is given by

$$\Omega_i = \frac{i}{2\pi} d \sum_{j,k\in I} \rho_j d\rho_k \wedge g_{ijk}^{-1} dg_{ijk}$$

where $\{\rho_j\}_{j\in I}$ is a partition of unity subordinate to the cover $\mathcal{U}$, i.e. a set of functions $\rho_j : M \to [0,1]$ such that $\sum_{j\in I}\rho_j(x) = 1$ for all $x \in M$ and the support of ρ_j is contained in U_j.

Projective bundle. There is yet another side to the theory of gerbes which has to do with projective bundles. Let $\mathcal{H}$ denote a complex separable Hilbert space and $PU(\mathcal{H}) = U(\mathcal{H})/S^1$ the projective unitary group with the norm topology. Gerbes arise as an obstruction to lifting bundles of projective Hilbert spaces to a global Hilbert bundle, or equivalently to lifting a principal $PU(\mathcal{H})$-bundle to $U(\mathcal{H})$. Principal $PU(\mathcal{H})$-bundles are classified by third integral cohomology

$$\check{H}^1(M,\underline{PU(\mathcal{H})}) \cong \check{H}^2(M,\underline{S^1}) \cong H^3(M,\mathbb{Z})\ .$$

Notice that two cocycles $g, g' \in \check{H}^1(M,\underline{PU(\mathcal{H})})$ are equivalent if $g'_{ij} = h_j^{-1} g_{ij} h_i$, but this can in general not be interpreted as a Čech cocycle since $PU(\mathcal{H})$ is non-abelian. Thus by $\check{H}^1(M,\underline{PU(\mathcal{H})})$ we mean the set of isomorphism classes. The reason behind their simple classification is that $PU(\mathcal{H})$ is a model for the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, i.e. $\pi_q(PU(\mathcal{H})) = \mathbb{Z}$ for $q = 2$ and zero otherwise. Indeed, the short exact sequence

$$\mathbf{1} \to S^1 \to U(\mathcal{H}) \to PU(\mathcal{H}) \to \mathbf{1}$$

induces a long exact sequence in homotopy

$$\cdots \to \pi_2(S^1) \to \pi_2(U(\mathcal{H})) \to \pi_2(PU(\mathcal{H})) \to \pi_1(S^1) \to \pi_1(U(\mathcal{H})) \to \pi_1(PU(\mathcal{H}))$$

and the claim follows by the fact that $U(\mathcal{H})$ is contractible in norm topology (Kuiper's theorem). Now the classifying space of $K(\mathbb{Z}, n)$ is an Eilenberg-MacLane space of one degree higher $BK(\mathbb{Z}, n) = K(\mathbb{Z}, n+1)$ and such bundles are parametrized by singular cohomology $[M, K(\mathbb{Z}, n)] = H^n(M, \mathbb{Z})$. This also explains the classification of line bundles by $H^2(M, \mathbb{Z})$, since S^1 has the homotopy type of $K(\mathbb{Z}, 1)$.

Let E denote a $PU(\mathcal{H})$-bundle over M and choose a good cover $\mathcal{U}$ of M. Since all double overlaps are contractible, the transition functions $g_{ij} : U_i \cap U_j \to PU(\mathcal{H})$ lift to the unitary group $\hat{g}_{ij} : U_i \cap U_j \to U(\mathcal{H})$ satisfying

$$\hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ik}^{-1} = f_{ijk}\mathbf{1} \ .$$

One can show that $f_{ijk} : U_i \cap U_j \cap U_k \to S^1$ defines a Čech cocycle and vanishes if and only if P is the projectivization of a global Hilbert bundle. This 2-cocycle defines the Dixmier-Douady class of the bundle E and provides the link to other descriptions of the gerbe. More precisely, if $L = U(\mathcal{H}) \times_{S^1} \mathbb{C}$ is the line bundle associated to the central extension $S^1 \to U(\mathcal{H}) \to PU(\mathcal{H})$, then the transition functions $g_{ij} : U_i \cap U_j \to PU(\mathcal{H})$ can be used to pull back L to a system of local line bundles L_{ij} over $U_i \cap U_j$. Over triple overlaps we have a canonical isomorphism

$$L_{ij} \otimes L_{jk} = L_{ik}$$

induced by the group structure on $U(\mathcal{H})$. In the language of bundle gerbes, we have a map $f : E^{[2]} \to PU(\mathcal{H})$ defined by $e_1 f(e_1, e_2) = e_2$. Pullback of the central extension of $PU(\mathcal{H})$ by this map yields a circle bundle $S^1 \to P \to E^{[2]}$ and the bundle gerbe multiplication $P_{(e_1,e_2)} \otimes P_{(e_2,e_3)} = P_{(e_1,e_3)}$ follows by $f(e_1, e_2)f(e_2, e_3) = f(e_1, e_3)$. The bundle gerbe (P, E) is called the lifting bundle gerbe of E.

Gerbes can thus be viewed as principal bundles, with infinite dimensional fibre $PU(\mathcal{H})$. This point of view is useful when considering twistings in K-theory. When the Hilbert space $\mathcal{H}$ is finite dimensional, the circle bundle $U(\mathcal{H})$ over $PU(\mathcal{H})$ reduces to

$$\mathbf{0} \to \mathbb{Z}_N \to SU(N) \to PU(N) \to \mathbf{1}$$

since $PU(N)/S^1 = SU(N)/\mathbb{Z}_N$, where $N = \dim\, \mathcal{H}$. In this case, the Dixmier-Douady class is necessarily a torsion class and all information is contained in integral cohomology. A familiar example of a finite dimensional projective bundle is the Clifford bundle $Cliff(M)$ over an oriented Riemannian manifold M of dimension $dim(M) = n$. This bundle carries a projective representation of $SO(n)$ in dimension $2^{[\frac{n}{2}]}$, corresponding to a linear representation of the double cover $Spin(n)$. Over contractible two-fold intersections we may lift transition functions $g_{ij} : U_i \cap U_j \to SO(n)$ to the spin group up to a phase $f_{ijk} : U_i \cap U_j \cap U_k \to \mathbb{Z}_2$. The lifting gerbe here is simply the obstruction to the existence of spin structure on M and $[f_{ijk}] = w_2(M) \in \check{H}^2(M, \underline{\mathbb{Z}_2})$ is known as the second Stiefel-Whitney class. The Dixmier-Douady class in the image of the so called Bockstein map $\check{H}^2(M, \underline{\mathbb{Z}_2}) \to H^3(M, \mathbb{Z})$ is a 2-torsion class, $2\Omega = 0$.

3.3 Basic gerbe on $SU(n)$

In this section we discuss the construction of gerbes on $SU(n)$.[23, 25] Since $H^3(G, \mathbb{Z}) = H^3_G(G, \mathbb{Z}) = \mathbb{Z}$ for any compact simple Lie group G, we expect all gerbes on $SU(n)$ to be generated by a single element corresponding to $1 \in \mathbb{Z}$. Gerbes with Dixmier-Douady class $k \in \mathbb{Z}$ are obtained by k-fold products of this basic gerbe and its dual. The de Rham representative of the Dixmier-Douady class is given by the third power of the Maurer-Cartan form,

$$\Omega = k\Omega_0 = \frac{k}{24\pi^2} \mathrm{Tr}(g^{-1}dg)^3 \ .$$

Set $G = SU(n)$ and define ΩG as the group of smooth loops $f : S^1 \to G$ based at the identity, $f(0) = f(1) = \mathbf{1} \in G$, where the circle is parametrized by $[0, 1]$. We denote by $\mathcal{P}G$ the space of smooth paths $\hat{g}$ in G, originating at the identity $\hat{g}(0) = \mathbf{1}$ and with smooth periodic logarithmic derivative $\hat{g}^{-1}\hat{g}(0) = \hat{g}^{-1}\hat{g}(1)$. The group ΩG acts freely on $\mathcal{P}G$ by pointwise multiplication and we have

$$\Omega G \to \mathcal{P}G \to G$$

with projection $\hat{g} \mapsto \hat{g}(1) = g$. This fibration is in fact a universal bundle for ΩG, since the total space is contractible. A smooth contraction of $\mathcal{P}G$ is defined by the homotopy

$$F(t, s) = \hat{g}(ts), \quad F(t, 1) = \hat{g}(t), \quad F(t, 0) = \hat{g}(0) = \mathbf{1} \ .$$

Next we consider a projective representation $\phi : \Omega G \to PU(\mathcal{H})$ through the basic representation of the affine Kac-Moody group based on G. The principal $PU(\mathcal{H})$-bundle over G corresponding to the basic gerbe is then given by the associated bundle $P = \mathcal{P}G \times_\phi PU(\mathcal{H})$.

For a local description as a family of line bundles, we can proceed by pulling back the central extension of $PU(\mathcal{H})$ by transition functions $h_{jk} : U_j \cap U_k \to PU(\mathcal{H})$ as described in the previous section. A finite cover of $SU(n)$ can be constructed in the following way. Pick n different points λ_k on the unit circle not equal to the identity and which satisfy $\lambda_1\lambda_2\ldots\lambda_n \neq 1$. We assume also that they are ordered counter-clockwise on the circle. A possible choice is $\lambda_k = e^{2\pi i k/(n+1)}$ for $k = 1, 2, \ldots, n$. We define n open sets $U_k = \{g \in SU(n) \mid \det(g - \lambda_k \mathbf{1}) \neq 0\}$ consisting of matrices g with eigenvalues different from λ_k. This defines an open cover since if g lies outside of the sets U_k, then all the λ_k's are eigenvalues of g and $\det(g) = \lambda_1\lambda_2\ldots\lambda_n \neq 1$ leads to a contradiction. On each U_k there is a smooth section $\sigma_k : U_k \to \mathcal{P}G$ sending $g \in U_k$ to the path $\hat{g}$ joining $\mathbf{1}$ and g by a smooth contraction. More precisely, if $g = VDV^{-1}$ is a diagonalization by $D = diag(d_1, \ldots, d_n)$, then

$$\hat{g}(t) = \sigma_k(g)(t) = V \begin{pmatrix} h_k(t)(d_1) & & \\ & \ddots & \\ & & h_k(t)(d_n) \end{pmatrix} V^{-1}$$

where $h_k(t)$ is a fixed smooth contraction of the interval $S^1 \backslash \{\lambda_k\}$ with $h_k(0) = 1$ and $h_k(1) = id$. The transition functions $h_{jk} : U_j \cap U_k \to PU(\mathcal{H})$ are then given by

$$h_{jk}(g) = \phi\big(\sigma_j(g)\sigma_k(g)^{-1}\big) \ .$$

There is however a more direct approach to the local line bundle construction using a family of Dirac operators on the unit interval parametrized by points on $SU(n)$. For every $g \in SU(n)$ there is an associated Dirac operator $D_g = -i\frac{d}{dx}$ on $I = [0, 1]$. The domain of D_g is the Sobolev space $H^1(I, \mathbb{C}^n)$ of square-integrable functions satisfying the boundary condition $\psi(1) = g\psi(0)$. The spectrum is given by

$$\mathrm{Spec}(D_g) = \{2\pi m + \mu_j(g) \mid m \in \mathbb{Z},\ j = 1, \ldots, n\}$$

where $\mu_j(g)$ are the eigenvalues of $-i\log(g)$, with a fixed branch of the logarithm function $\log : SU(n) \to \mathbb{C}$. Now the point is that the open sets $U_k = \{g \in SU(n) \mid \det(g - \lambda_k \mathbf{1}) \neq 0\}$ of the covering can be defined equivalently as $U_k = \{g \in SU(n) \mid -i\log(\lambda_k) \notin \mathrm{Spec}(D_g)\}$. On double overlaps $U_j \cap U_k$ the spectral subspaces $E_{jk}(g)$ of D_g, corresponding to eigenvalues in the interval $(-i\log(\lambda_j), -i\log(\lambda_k))$, are finite dimensional and vary continuously as function of g. A system of line bundles can be defined by the top exterior power

$$DET_{ij} = \bigwedge^{top} E_{ij} \ .$$

The cocycle property over triple overlaps

$$DET_{ij} \otimes DET_{jk} \cong DET_{ik}$$

follows by the "exponential" property

$$\bigwedge^{top}(V \oplus W) = \bigwedge^{top} V \otimes \bigwedge^{top} W$$

for any finite dimensional vector spaces V, W. One can prove that DET_{ij} defines the same gerbe as earlier. Notice that we might just as well have worked with the Dirac operator coupled to smooth $\mathfrak{su}(n)$-valued vector potentials on the circle,

$$D_A = -i\frac{d}{dx} + A$$

which is closer to the discussion in the Introduction. The spectrum of D_A with holonomy $g = hol(A)$ coincides however with the spectrum of D_g, so these two families of self-adjoint Fredholm operators are really different sides of the same coin.

For a different approach to the basic gerbe over a compact Lie group G, see[21] in the simply connected case and[14] for the non-simply connected case.

Chapter 4

Twisted K-theory

4.1 K-theory

K-theory is the study of vector bundles by means of abelian groups. In words of Sir Michael Atiyah,[2]

> K-theory may roughly be described as the study of additive (or abelian) invariants of large matrices.

It first appeared in the work of Grothendieck on the Riemann-Roch theorem in algebraic geometry. This was subsequently adopted and developed as part of algebraic topology by Atiyah and Hirzebruch in the late 1950's.

Throughout this chapter we will assume that the base space M is a compact topological space and we restrict our attention to complex vector bundles. The set of isomorphism classes of vector bundles over a fixed base $\mathrm{Vect}(M)$ forms a semigroup under the operation of direct (Whitney) sums. One may extend $\mathrm{Vect}(M)$ to an abelian group by introducing formal inverses. In general for any abelian semigroup H, there is a universal enveloping abelian group $\mathfrak{G}(H)$ called the Grothendieck group of H. $\mathfrak{G}(H)$ can be defined to be the quotient of $H \times H$, under the equivalence relation $(x, y) \sim (x', y')$ if and only if there is an element $z \in H$ such that $x+y'+z = x' + y + z$.

We define the even K-theory group by $K^0(M) = \mathfrak{G}(\mathrm{Vect}(M))$ and think of its elements (E, F) as (equivalence classes) of formal differences $E-F$ of vector bundles over M. The odd counterpart $K^1(M)$ is defined as $K^0(SM)$ of the suspension of M. Recall that the suspension $SM = M \wedge S^1$ is the smash product with the circle, where

$$X \wedge Y = \frac{X \times Y}{X \vee Y} = \frac{X \times Y}{(X \coprod Y)/(\{x_0\} \cup \{y_0\})}$$

for two basepoints $x_0 \in X$ and $y_0 \in Y$. More concretely, it is the quotient of $M \times [0,1]$ by collapsing $M \times \{0\}$ and $M \times \{1\}$ to a single point. By the Bott periodicity theorem,[16] it follows that complex K-theory is periodic with period 2,

$$K(M) = K^0(M) \oplus K^1(M) \ .$$

It defines a generalized cohomology, in the sense that all Eilenberg-Steenrod axioms for a cohomology theory are satisfied except the dimension axiom. The dimension axiom states that the cohomology of a point is trivial. However, if M is a point or more generally a contractible space, then all vector bundles are trivial and fully characterized by their rank. Thus $Vect(M) = \mathbb{N}$ implying that $K^0(M) = \mathbb{Z}$ and $K^1(M) = 0$. Tensor product induces a ring structure in $K(M)$ and the link to ordinary cohomology is provided by the Chern character:

Theorem 4.1. There are isomorphisms

$$ch : K^0(M) \otimes \mathbb{Q} \to \bigoplus_{n\ even} H^n(M, \mathbb{Q})$$

$$ch : K^1(M) \otimes \mathbb{Q} \to \bigoplus_{n\ odd} H^n(M, \mathbb{Q})$$

Thus over rationals, $K^0(M)$ is just the direct sum of the even cohomology groups and $K^1(X)$ the sum with odd ones. There is an important relation between K-theory and the theory of Fredholm operators, established by Atiyah and Jänich in the context of index theory.[1] Let $\mathcal{H}$ denote a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators endowed with the norm topology. The space of Fredholm operators

$$Fred(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \mid dim \text{ Ker } T < \infty, \ dim \text{ Coker } T < \infty \}$$

forms a semigroup and so does the set of homotopy classes $[M, Fred(\mathcal{H})]$ under pointwise multiplication. The Atiyah-Jänich theorem states that the index map

$$\text{ind} : [M, Fred(\mathcal{H})] \to K^0(M)$$

is a group isomorphism. When $M = \{pt\}$ is a point, the index map is given by

$$\text{ind} : [pt, Fred(\mathcal{H})] \to \mathbb{Z}, \quad T \mapsto \text{ind}(T) = dim \text{ Ker } T - dim \text{ Coker } T$$

and the theorem implies that $[pt, Fred(\mathcal{H})] = \pi_0(Fred(\mathcal{H})) \cong \mathbb{Z}$, i.e. the connected components of $Fred(\mathcal{H})$ are parametrized by integers.

The index map provides an alternative definition of $K^0(M)$. The odd K-group has a similar representation but with a slightly different classifying space,

$$K^1(M) \cong [M, Fred_*(\mathcal{H})]$$

where $Fred_*(\mathcal{H})$ is the space of bounded self-adjoint Fredholm operators with both positive and negative essential spectrum. $Fred_*(\mathcal{H})$ has the same homotopy type as $\mathcal{U}_1 = \{1 + u \in U(\mathcal{H}) \mid u \text{ trace class}\}$.

4.2 Twisted K-theory

K-theory on a compact space M can be twisted by a gerbe or equivalently by an element $\Omega \in H^3(M,\mathbb{Z})$. Let P_Ω denote a projective principal bundle with Dixmier-Douady class Ω. The projective unitary group $PU(\mathcal{H})$ acts continuously on $Fred(\mathcal{H})$ via the conjugation action of $U(\mathcal{H})$. We write

$$Fred(P_\Omega) = P_\Omega \times_{PU(\mathcal{H})} Fred(\mathcal{H})$$

for the associated bundle of Fredholm operators over M.

Definition 4.2. Twisted K-theory groups are defined to be

$$K^0(M,\Omega) = \pi_0(\Gamma(M, Fred(P_\Omega)))$$

$$K^1(M,\Omega) = \pi_0(\Gamma(M, Fred_*(P_\Omega)))$$

i.e. spaces of homotopy classes of continuous sections of the associated bundles.

This means that a twisted K^0-class is represented by a projectively twisted family of Fredholm operators. More concretely, it is a family of locally defined Fredholm operators $T_i : U_i \to Fred(\mathcal{H})$ satisfying

$$T_j(x) = Ad_{\hat{g}_{ij}}(T_i)(x) = \hat{g}_{ij}^{-1}(x)T_i(x)\hat{g}_{ij}(x)$$

on contractible double overlaps, where $\hat{g}_{ij}$ are lifts of the transition functions $g_{ij} : U_i \cap U_j \to PU(\mathcal{H})$ to the unitary group $U(\mathcal{H})$. In other words, twisted K-theory on M is the same as ordinary $PU(\mathcal{H})$-equivariant K-theory on P_Ω,

$$K^0(M,\Omega) = [P_\Omega, Fred(\mathcal{H})]_{PU(\mathcal{H})}, \quad K^1(M,\Omega) = [P_\Omega, Fred_*(\mathcal{H})]_{PU(\mathcal{H})} ,$$

that is $S : P_\Omega \to Fred(\mathcal{H})$ satisfying $S(pg) = g^{-1}S(p)g$.

Twisted K-theory is a $\mathbb{Z}_2$-graded generalized cohomology theory. It is in particular a homotopy invariant and satisfies the (six-term) exact Mayer-Vietoris sequence,

$$\begin{array}{ccccc} K^0(M,\Omega) & \longrightarrow & K^0(U_1,\Omega_1)\oplus K^0(U_2,\Omega_2) & \longrightarrow & K^0(U_1\cap U_2,\Omega_{12}) \\ \uparrow & & & & \downarrow \\ K^1(U_1\cap U_2,\Omega_{12}) & \longleftarrow & K^1(U_1,\Omega_1)\oplus K^1(U_2,\Omega_2) & \longleftarrow & K^1(M,\Omega) \end{array}$$

where U_1 and U_2 are closed subsets covering M and Ω_1, Ω_2 and Ω_{12} are the restrictions of Ω to U_1, U_2 and $U_1 \cap U_2$ respectively.

Example 4.3. Let $M = S^3$ be covered by the upper and lower hemispheres $U_\pm$. Since these are contractible open sets and $U_+ \cap U_- \sim S^2$, the Mayer-Vietoris property yields an exact sequence

$$\mathbf{0} \to K^0(S^3,\Omega) \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z}\oplus\mathbb{Z} \to K^1(S^3,\Omega) \to \mathbf{0}$$

where we have used that $K^0(S^2,\Omega_{+-}) = \mathbb{Z}\oplus\mathbb{Z} = K^0(U_+,\Omega_+) \oplus K^0(U_-,\Omega_-)$ and $K^1(S^2,\Omega_{+-}) = 0 = K^1(U_\pm,\Omega_\pm)$. In the untwisted case, the map $\pi_* : (m,n) \to$

$(m-n,0)$ implies that $K^0(S^3) = K^1(S^3) = \mathbb{Z}$. When $\Omega = k\Omega_0$ is the kth multiple of the generator Ω_0 of $H^3(S^3,\mathbb{Z}) = \mathbb{Z}$, then the map is given by $\pi_* : (m,n) \to (m-n,-kn)$ and it follows that $K^0(S^3,\Omega) = 0$ and $K^1(S^3,\Omega) = \mathbb{Z}/k\mathbb{Z}$.[5]

The group structure in $K^i(M,\Omega)$ is defined by direct sums of Fredholm operators. Under tensor product however

$$K^i(M,\Omega) \times K^i(M,\Omega') \xrightarrow{\otimes} K^i(M,\Omega+\Omega'), \quad (T,T') \mapsto (T \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes T')$$

the twistings add up. Thus tensor product no longer defines a ring structure on the twisted K-theory groups, but they still form modules for ordinary K-theory groups. A proper continuous map $f : M \to N$ induces a natural pullback by

$$f^* : K^i(N,\Omega) \to K^i(M, f^*\Omega) \ .$$

If a group G acts on $Fred(P_\Omega)$ and $\Omega \in H^3_G(M,\mathbb{Z})$, then we may define the G-equivariant twisted K-theory $K^i_G(M,\Omega)$ as homotopy classes of G-equivariant continuous sections. There is a subtlety here if the G-action is not continuous. In that case additional conditions must be fulfilled.[3]

4.3 Twisted K-theory class on $SU(n)$

When M is a compact Lie group, one may use representation theory to make powerful statements about the twisted K-theory. In this section we construct G-equivariant twisted K-theory classes on $G = SU(n)$ using families of cubic Dirac operators for $N = 1$ supersymmetric Wess-Zumino-Witten model.[24,26] Supersymmetry is required since ordinary second quantized Dirac hamiltonians are positive (by construction), whereas in K-theory we need operators of a more general kind.

Recall the universal bundle $\Omega G \to \mathcal{P}G \to G$ from Section 3.3. The total space $\mathcal{P}G$ may be identified with the affine space $\mathcal{A}$ of smooth $\mathfrak{su}(n)$-valued connection 1-forms on the circle. The action of ΩG is now by gauge transformations

$$\Omega G \times \mathcal{A} \to \mathcal{A}, \quad (g,A) \mapsto A^g = g^{-1}Ag + g^{-1}dg$$

and the projection $\mathcal{A} \to G$ is given by the holonomy around the circle.

Our aim is to construct a family of self-adjoint Fredholm operators $Q(A)$, acting on sections of a Hilbert bundle over $\mathcal{A}$ and transforming equivariantly under (projective) gauge transformations. By the embedding $\Omega G \subset PU(\mathcal{H})$, the space $\mathcal{A}$ may be viewed as a reduction of a principal $PU(\mathcal{H})$-bundle over G and $Q(A)$ as a section of the associated bundle of Fredholm operators. We consider Hilbert spaces of the form $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ where $\mathcal{H}_b$ carries a level k representation and $\mathcal{H}_f$ a spin representation of $\widehat{LG}$.[32] The Dixmier-Douady class of the corresponding gerbe is $k+\kappa \in \mathbb{Z}$, where the degree shift $\kappa = n$ is the dual Coxeter number of $SU(n)$. More precisely, let $\mathcal{H}_b$ denote the bosonic Hilbert space carrying an irreducible unitary highest weight representation of the loop algebra $\widehat{L\mathfrak{g}}$, fixed by the level k and a dominant integral weight λ of $\mathfrak{g}$.[12, 19] The level is a non-negative integer multiple of $\frac{2}{\theta^2}$, where θ is the length of the longest root of G. The generators of the loop algebra in Fourier basis T^a_m satisfy

$$[T^a_m, T^b_p] = \lambda^{abc} T^c_{m+p} + \frac{k}{4} m \delta^{ab} \delta_{m,-p}$$

where $m \in \mathbb{Z}$ and $a = 1, 2, \ldots, N = dim(G)$. We fix an orthonormal basis T^a in $\mathfrak{su}(n)$, with respect to the Killing form $< X, Y >= -\mathrm{Tr}(ad_X \cdot ad_Y)$, so that the structure constant λ^{abc} is completely antisymmetric and the Casimir invariant $C_2 = \lambda^{abc}\lambda^{acb}$ equals $-N$. Moreover, we have the hermiticity relation

$$(T^a_m)^* = -T^a_{-m} \ .$$

The fermionic Hilbert space $\mathcal{H}_f$ carries an irreducible representation of the canonical anticommutation relations (CAR)[31]

$$\{\psi^a_m, \psi^b_p\} = 2\delta^{ab}\delta_{m,-p}$$

where $(\psi^a_m)^* = \psi^a_{-m}$. The Fock vacuum is a subspace of $\mathcal{H}_f$ of dimension $2^{[\frac{N}{2}]}$. It carries an irreducible representation of the Clifford algebra spanned by the zero modes ψ^a_0 and is annihilated by all ψ^a_m with $m < 0$. The loop algebra $\widehat{L\mathfrak{g}}$ acts in $\mathcal{H}_f$ through the spin representation of level κ. The operators are realized explicitly as bilinears in the Clifford generators

$$K^a_m = -\frac{1}{4}\lambda^{abc} : \psi^b_{m-p}\psi^c_p :$$

and satisfy

$$[K^a_m, K^b_p] = \lambda^{abc} K^c_{m+p} + \frac{\kappa}{4} m \delta^{ab}\delta_{m,-p} \ .$$

It follows that $(K^a_m)^* = -K^a_{-m}$. The normal ordering $: :$ indicates that operators with positive Fourier index are placed to the left of those with negative index. In case of fermions there is a change of sign, $: \psi^a_{-m}\psi^b_m := -\psi^b_m\psi^a_{-m}$ if $m > 0$. Since λ^{abc} is totally antisymmetric, the normal ordering in K^a_m is actually redundant. The full Hilbert space $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ carries a tensor product representation of $\widehat{L\mathfrak{g}}$ of level $k + \kappa$, with generators $S^a_m = T^a_m \otimes \mathbf{1} + \mathbf{1} \otimes K^a_m$.

The free supercharge operator is defined by

$$Q =: i\Big(\psi^a_m T^a_{-m} + \frac{1}{3}\psi^a_m K^a_{-m}\Big) :=: i\Big(\psi^a_m T^a_{-m} - \frac{1}{12}\lambda^{abc}\psi^a_m\psi^b_{-m-p}\psi^c_p\Big) :$$

and squares to the Hamilton operator $h = Q^2$ of the supersymmetric Wess-Zumino-Witten model

$$\begin{aligned}
h &= -:\Big(\psi^a_m T^a_{-m}\psi^b_p T^b_{-p} + \frac{1}{3}\{\psi^a_m T^a_{-m}, \psi^b_p K^b_{-p}\} + \frac{1}{9}\psi^a_m K^a_{-m}\psi^b_p K^b_{-p}\Big): \\
&= -:\Big(T^a_m T^a_{-m} - \frac{1}{8}km\psi^a_m\psi^a_{-m} + \underbrace{\frac{1}{4}\lambda^{abc}[\psi^a_m,\psi^b_p]T^c_{-m-p} + 2K^a_m T^a_{-m}}_{=0} + \frac{1}{9}\psi^a_m K^a_{-m}\psi^b_p K^b_{-p}\Big): \\
&= -:\Big(T^a_m T^a_{-m} - \frac{1}{8}km\psi^a_m\psi^a_{-m} - \frac{1}{8}\kappa m\psi^a_m\psi^a_{-m} - \frac{N}{24}\Big): \\
&= -:T^a_m T^a_{-m}: + \frac{\bar{k}}{2}:m\psi^a_m\psi^a_{-m}: + \frac{N}{24} = h_b + 2\bar{k}h_f + \frac{N}{24}
\end{aligned}$$

where $\bar{k} = \frac{k+\kappa}{4}$. Next we perturbe Q by coupling to the Clifford action of connection 1-forms A on the circle

$$Q(A) = Q + i\bar{k}\psi^a_m A^a_{-m}$$

where A^a_m are the Fourier coefficients of A in the basis T^a_m, satisfying $(A^a_m)^* = -A^a_{-m}$. This provides us with a family of self-adjoint Fredholm operators $Q(A)$ that is equivariant with respect to the action of $\widehat{LG}$,

$$S(g)^{-1}Q(A)S(g) = Q(A^g)$$

where $S : LG \to PU(\mathcal{H})$ is the (level $k+\kappa$) embedding of the loop group. Infinitesimally this translates to

$$\begin{aligned}
[S^a_m, Q(A)] &= i\Big([T^a_m, \psi^b_p T^b_{-p}] + [K^a_m, \psi^b_p T^b_{-p}] + \frac{1}{3}[K^a_m, \psi^b_p K^b_{-p}] + i\bar{k}[S^a_m, \psi^b_p A^b_{-p}]\Big) \\
&= i\Big(\frac{1}{4}km\psi^a_m + 0 + \frac{1}{4}\kappa m\psi^a_m + i\bar{k}\lambda^{abc}\psi^c_{m+p}A^b_{-p}\Big) \\
&= i\bar{k}\Big(m\psi^a_m + \lambda^{abc}\psi^c_{m+p}A^b_{-p}\Big) = -\mathcal{L}^a_m Q(A)
\end{aligned}$$

where $\mathcal{L}^a_m$ denotes the Lie derivative in the direction $X = S^a_m$. The interacting Hamiltonian $h(A) = Q(A)^2$ is given by

$$h(A) = h - 2\bar{k}S^a_m A^a_{-m} + \bar{k}^2 A^a_m A^a_{-m} = h + h_{int}\ .$$

The supercharge operator $Q(A)$ is unbounded and has a dense domain in $\mathcal{H}$ consisting of states which are finite linear combinations of eigenvectors of h. We can define a family of bounded Fredholm operators by setting

$$F(A) = \frac{Q(A)}{(1+Q(A)^2)^{\frac{1}{2}}}$$

which is the approximate sign operator. This new family is still Fredholm, $\widehat{LG}$-equivariant and varies continuously as function of $A \in \mathcal{A}$. Due to the residual gauge symmetry corresponding to constant (global) gauge transformations, $F(A)$ defines an element in G-equivariant twisted K-theory. One has to draw a distinction between the cases when N is even and odd. In the odd case, $F(A)$ is a Fredholm operator with both positive and negative essential spectrum and hence belongs to $K^1_G(G, k+\kappa)$. When N is even, $F(A)$ gives a trivial element in odd K-theory, but there exists a chirality operator Γ which anticommutes with $F(A)$. Introducing the chiral operators $F(A)^{\pm} = F(A)\frac{1}{2}(\Gamma \pm 1)$ with $(F(A)^+)^* = F(A)^-$, both these families define a class in $K^0_G(G, k+\kappa)$. If ψ_0^{N+1} denotes the chirality operator in the Clifford algebra spanned by the zero modes ψ_0^a and f the fermion number operator satisfying $\{\psi_m^a, f\} = \frac{m}{|m|}\psi_m^a$ for $m \neq 0$, then $\Gamma = (-1)^f \psi_0^{N+1}$. For $n = 2$ we have

$$\begin{aligned} K^0_{SU(2)}(SU(2), k+2) &= K^0(SU(2), k+2) = 0 \\ K^1_{SU(2)}(SU(2), k+2) &= K^1(SU(2), k+2) = \mathbb{Z}/k\mathbb{Z} \end{aligned}$$

and for $n = 3$,

$$K^0(SU(3), k+3) = K^1(SU(3), k+3) = \begin{cases} \mathbb{Z}/\frac{k}{2}\mathbb{Z} & k \text{ even} \\ \mathbb{Z}/k\mathbb{Z} & k \text{ odd} \end{cases}$$

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Part II

Scientific papers

1

Paper 1

Pedram Hekmati

Integrability Criterion for Abelian Extensions of Lie Groups

Submitted for publication

`math.DG/0611431`

2

Paper 2

Pedram Hekmati and Jouko Mickelsson

Fractional Loop Group and Twisted K-theory

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