Clock synchronization over networks —
Identifiability of the sawtooth model

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Abstract—In this paper, we analyze the two-node joint clock synchronization and ranging problem. We focus on the case of nodes that employ time-to-digital converters to determine the range between them precisely. This specific design leads to a sawtooth model for the captured signal, which has not been studied in detail before from an estimation theory standpoint. In the study of this model, we recover the basic conclusion of a well-known article by Freris, Graham, and Kumar in clock synchronization. Additionally, we discover a surprising identifiability result on the sawtooth signal model: noise improves the theoretical condition of the estimation of the phase and offset parameters. To complete our study, we provide performance references for joint clock synchronization and ranging. In particular, we present the Cramér-Rao lower bounds that correspond to a linearization of our model, as well as a simulation study on the practical performance of basic estimation strategies under realistic parameters. With these performance references, we enable further research in estimation strategies using the sawtooth model and pave the path towards industrial use.

Index Terms—Clock synchronization, ranging, identifiability, sawtooth model, sensor networks, round-trip time (RTT).

I. INTRODUCTION

Clock synchronization across a deployed network is a pervasive and long-standing challenge [1]–[6]. Furthermore, new-generation technologies each require more accurate synchronization. To name a few, i) in cellular communications, synchronization between base stations is fundamental to maintain frame alignment and permit handover among neighboring cells, and has been identified as a crucial requirement for distributed beamforming, interference alignment, and user positioning [7], [8], ii) in radio-imaging technology [9], accurate clock synchronization between the sparse chips that form an array is critical, and, in active-sensing 3-dimensional cases [10], [11], it results in low-cost wide-aperture ultra-short ultra-wideband (UWB) pulses, increasing both the angular and depth resolutions of the captured images, iii) in wireless sensor networks [12], [13], synchronization is critical to data-fusion, channel-sharing, coordinated scheduling [14], [15], and distributed control [2] and iv) in distributed database solutions that provide external consistency, clock synchronization accuracy regulates latency, throughput, and performance [5].

Consequently with this wide range of application, theoretical insights on the fundamental limitations of clock synchronization over networks are likely to incite radical innovations in a number of fields. In [16], Freris, Graham, and Kumar established the fundamental limitations of the clock synchronization problem in an idealized scenario. Particularly, given a network of nodes with noise-less affine clocks and fixed unknown link delays that exchange time-stamped messages, [16] i) showed that clock synchronization was only possible if the link delays were known to be symmetric, and ii) characterized the uncertainty regions of the clock synchronization parameters under different hypotheses. In this paper, we analyze the same problem from a perspective that is closer to real implementation. In short, we analyze the two-node joint clock synchronization and ranging problem [17]–[19] with noisy round-trip time (RTT) measurements without time-stamps [20], for a node design originally proposed in [21] to improve ranging accuracy. The resulting analysis has several advantages. First, considering RTT-based protocols is consistent with applications that require minimizing the communication overhead [6, p. 29]. Second, because we consider hardware specifically tailored to ranging accuracy, we reveal how applications that require this accuracy, such as cooperative localization [22], positioning [23], and control [2], can harness the same hardware and protocols for synchronization. Third, the analysis takes into account the real-world stochasticity of the measurements, resulting in more meaningful conclusions. In particular, in our analysis of the problem we i) unveil the need for symmetric delays in RTT-based protocols, in a direct parallel to the discovery in [16], ii) find novel results on the identifiability of sawtooth signal models under diverse conditions, which are also of interest by their own to chaotic system analysis [24], [25] and control, and iii) provide performance references for practitioners to guide their decisions in the use of this technology.

In summary, in this paper we first derive from basic principles a model for RTT measurements between two nodes equipped with time-to-digital converters (TDC) in a network with fixed, unknown link delays (Theorem [1]). Then, we shift our focus towards an encompassing family of signal models, i.e., sawtooth signal models, when one considers different

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stochastic effects. In this context, we provide results on the identifiability of these models, both negative (Lemma 1) and positive (Theorem 2), under different noise conditions. Here, we obtain the surprising result that the presence of a noise term inside a non-linear model term makes said model identifiable. We then shift the focus again towards the particular case of clock synchronization and ranging, and provide estimation performance bounds (Cramér-Rao lower bounds (CRLB) in (19), (21) and (30)) for a related linearized model. Then, we present simple and intuitive algorithms for clock synchronization and ranging that build on the work in [19], along with public implementations (accessible through [26]) that make our results reproducible. Together with the CRLBs, these estimators and their thorough empirical evaluation (Figs. 9–13) provide performance references for clock synchronization and ranging using sawtooth modeling of RTT measurements in nodes equipped with TDCs. This performance references are of use to both practitioners considering the use of this technology and theoreticians aiming to develop estimation techniques for sawtooth signal models. In particular, we identify new directions of research in frequency and phase estimation using the sawtooth signal model that could hold the key to further improving this technology.

A. Notation

Discrete random processes will be in uppercase letters and square brackets, such as \( Y[n] \), while deterministic sequences, e.g., realizations of said processes, will be lower case with square brackets, i.e., \( y[n] \). For both these sequences, the notation will be simplified by omitting the discrete time index when it can be established by context. Vector random variables will be bold uppercase letters, e.g., \( \mathbf{Y} \), while deterministic vectors will be bold lowercase letters, e.g., \( \mathbf{y} \in \mathbb{R}^N \). Functions, on the other hand, will be non-italics lower-case letters, e.g., the probability density function (PDF) of a vector random variable \( \mathbf{Y} \) on a parametric family with vector parameter \( \theta \) will be \( f_{\mathbf{Y}}(\mathbf{y}; \theta) \). Through the paper, we view the modulus equivalence as a function, i.e., we note the mapping \( x \mapsto y \in [0, a) \mid (y = x) \mod(a) \) as \( \mod_a : \mathbb{R} \to [0, a) \).

II. THE SAWTOOTH MODEL

In applications in which high ranging accuracy with low communication overhead is desired, a low-cost solution in terms of both complexity and power consumption is using node designs that include TDCs to measure RTTs [21]. Indeed, such sensors were successfully incorporated in a prototype system to aid firefighters by providing on-site infrastructure-free indoor positioning [23]. TDCs, however, induce an asymmetry between the rate at which nodes can measure time and the rate at which they can act upon their environment. This asymmetry provokes an unexpected waveform in the sequence of RTT measurements over time. This phenomenon was first reported by [19], where a sawtooth model was proposed and empirically validated, and possible applications to clock synchronization over networks were identified. In this section, we reintroduce the design of [21] and derive the sawtooth model from a few simple assumptions.

Consider now the design of [21], described in Fig. 1. Here, each node or sensor \( N \) has a processing unit, a transceiver and a TDC. With this design, a sensor can measure RTTs at the resolution of the TDC, usually in the order of ps, much finer than the period of the processing unit’s clock, usually in the order of tens of ns. Besides the clear advantage of this design for ranging through RTT measurements, this creates an interesting asymmetric behavior of the node as an agent and as a measuring device. As we will show below, this asymmetry produces a sawtooth waveform in the measured RTTs that depends on the synchronization parameters. A final by-product of this design is that we can consider that each node has perfect knowledge of its own clock period, which it can measure directly with its TDC.

A. Deterministic model

Consider two nodes, \( M \) and \( S \), designed as \( N \) in Fig. 1. These two nodes execute the RTT measurement scheme illustrated in Fig. 2. In this scheme, \( M \) measures the RTT between itself and \( S \) by sending pulses (a.k.a. pings) to \( S \) and using its TDC to accurately record when a response (a.k.a. pong) is received from \( S \). In particular, \( M \) sends a pulse at some of the times at which its clock has upflanks, i.e., at the times \( t_n = KT_Mn \). Here \( T_M > 0 \) [s] is \( M \)’s clock period, the sampling factor \( K \in \mathbb{N} \) is designed to determine the sampling period \( T_s = KT_M \) [s], \( n \in \mathbb{N} \) is a discrete-time index, and we assume without loss of generality that \( M \)’s clock phase offset is zero, i.e., \( \phi_M = 0 \) [rad]. If we...
We now allow for a known delay magnitude finer than that of the nodes’ clocks, the resolution of the TDC, which is usually four orders of magnitude before responding with a pong pulse. The pong is recorded in $M$’s TDC as soon as it arrives.

Assume that the delays involved in the pulse traveling from $M$ to $S$ accumulate to a constant value $\delta_s [s]$, the $n$-th pulse arrives at $S$ and is recorded in its TDC at time $t_n + \delta_s$. Nonetheless, $S$ will not be able to access the TDC’s memory before its next clock upflank, and consequently, any action by $S$ will be further delayed until $t_n + \delta_s + \Delta_n$. Here, $\Delta_n \geq [s]$ is the time remaining until $S$’s next clock upflank. If we consider that $S$ has a clock with period $T_S$ and phase $\phi_S$ [rad], i.e., an offset delay of $\phi_S = T_S \phi_S/(2\pi)$ [s], then $S$ has its clock upflanks at those times that are at an integer number of periods away from $\phi_S$, i.e., at the times $\tau \geq 0$ when $\text{mod}_{T_S}(\tau + \phi_S) = 0$. Furthermore, we know that $\Delta_n \leq T_S$, as $T_S$ is the time between consecutive upflanks. Consequently, to obtain a closed-form expression for $\Delta_n$, we need to find

$$\Delta_n = \min \{ \tau \in (0, T_S) : \text{mod}_{T_S}(t_n + \delta_s + \tau + \phi_S) = 0 \}.$$ 

Because $\text{mod}_a(b + c) = \text{mod}_a(\text{mod}_a(b) + \text{mod}_a(c))$ for any $a \geq 0$ and $b, c \in \mathbb{R}$, and $\tau \in (0, T_S)$, the condition for $t_n + \delta_s + \tau$ to be the time of one of $S$’s clock upflanks can be rewritten as $\tau = QT_S - \text{mod}_{T_S}(t_n + \delta_s + \phi_S)$ for some $Q \in \mathbb{Z}$. Then, because $\text{mod}_{T_S}(t_n + \delta_s + \phi_S) < T_S$, we conclude that

$$\Delta_n = T_S - \text{mod}_{T_S}(t_n + \delta_s + \phi_S). \quad (1)$$

We now allow for a known delay $\delta_0 [s]$ to be introduced by $S$, which can account for any processing required to read the TDC’s state and prepare the new pulse, and will usually be an integer number of $S$’s clock periods, i.e., $\delta_0 = K_0 T_S$. Finally, as we did for the ping pulse, we consider $\delta_s$ to express the fixed delay for a pong pulse from $S$ to reach $M$ and be captured by the TDC. In conclusion, if we disregard the effect of the resolution of the TDC, which is usually four orders of magnitude finer than that of the nodes’ clocks, the $n$-th RTT measurement will amount to

$$y_{\text{det}}[n] = \delta_0 + \delta_s + \Delta_n, \quad (2)$$

where $\delta_{\alpha} = \delta_s + \delta_{\alpha}$. We summarize our result in the following theorem.

**Theorem 1 (Deterministic RTT measurement model):** Consider two nodes $M$ and $S$ designed as specified in Fig. 1. Then, if $M$ and $S$ follow the RTT measurement protocol specified above, $f_0 = 1/T_S - 1/T_M$ and $\delta_{\alpha} = \delta_s + \delta_{\alpha}$, the $n$-th RTT measurement $y_{\text{det}}[n]$ can be expressed as

$$y_{\text{det}}[n] = \delta_0 + \delta_s + T_S h[n], \quad (3)$$

where

$$h[n] = 1 - \text{mod}_1 \left[ T_s f_d n + \frac{\delta_s + \delta_s}{T_S} + \frac{\phi_S}{2\pi} \right].$$

**Proof:** From (1) we have that

$$\Delta_n = T_S - \text{mod}_{T_S}(T_n + \delta_s + \phi_S) \quad (4)$$

$$= T_S \left( 1 - \text{mod}_1 \left[ \frac{K T_M - \delta_s}{T_S} + \frac{\phi_S}{2\pi} \right] \right) \quad (5)$$

$$= T_S \left( 1 - \text{mod}_1 \left[ \frac{K T_M - \delta_s}{T_S} + \frac{\phi_S}{2\pi} \right] \right) \quad (6)$$

$$= T_S \left( 1 - \text{mod}_1 \left[ T_s f_d n + \frac{\delta_s + \phi_S}{2\pi} \right] \right) \quad (7)$$

Here, we have used that $T_s = K T_M$ and $\psi_S = T_S \phi_S/(2\pi)$ in (5), that $\text{mod}_1$ is periodic with period one in (6), and that $f_d = 1/T_S - 1/T_M$ in (7). Finally, (3) follows from substituting (7) in (1).

Incidentally, under the simple assumption that $T_s \geq \delta_0 + T_S$, which can be guaranteed under any reasonable $f_d$ if $K > K_0 + 1$, if we assume that $S$’s TDC starts measuring every time $S$ sends a pong and stops measuring when the next ping is received, the $n$-th measurement $x[n]$ taken by $S$’s TDC can be expressed as

$$x_{\text{det}}[n] = T_s - \delta_0 - T_S h[n]. \quad (8)$$

As we will see, this will imply that even while $M$ is running the RTT measurement protocol, $S$ could still perform frequency synchronization. Nonetheless, we will not consider $S$’s TDC measurements for most of the paper, and focus on determining the conditions under which $M$ can achieve full synchronization and ranging. More details on the derivations of (3) and (8) can be found in the supplementary material to this paper.

In this project’s repository, accessible at [26], we validate (3) by simulating an ideal physical system as described above and verifying the exact correspondence between the model and the obtained measurements. In Fig. 3 we show the fits of (3) and (8) on the TDC measurements of $M$ and $S$ throughout a simulated run with noisy clock periods and noisy transmission delays. Other formulations of the model in terms of the usual synchronization parameters for affine clocks, i.e., the clock skew $\alpha_S = T_S/T_M$ and the offset delay $\varphi_S$, including the general expression for when $\varphi_M \neq 0$, can be found in the supplementary material to this paper. Nonetheless, the expression in (3) remains the most practical, because it expresses the compromise between the sampling period $T_s$ and the frequency difference $f_d$ of the system, which will prove to be relevant to our analysis.
B. Stochastic model

In Fig. 4, we show real RTT data obtained in [19] from the ultra-wide band testbed of [21] using this RTT scheme, accompanied by an example model fit. Given the observed signal and its expected shape, a simple observation is that S’s clock was faster than that of M in the specific experimental set-up, because the ramps in the sawtooth signal have negative slope, which implies that \( f_d > 0 \). Fig. 4 also exemplifies the two distinct effects that random deviations of the physical parameters can produce on the data. On one hand, large jumps of approximately \( T_S \) in the measured RTT are observed (effect i) if a random deviation influences the specific clock period at which \( S \) reads the arrival of a ping pulse from its TDC, i.e., it changes which is the first up-flank in \( S \)’s clock after the ping pulse arrives. On the other hand, if this does not happen, random deviations appear directly in the signal as additive noise (effect ii). From a modeling perspective, these two effects are not easily represented distinctively. Indeed, variations of the transmission time from \( M \) to \( S \), \( \delta_+ \), or jitter in any of the two clock periods, \( T_M \) or \( T_S \), could lead to any of the two described effects, while variations of the transmission time from \( S \) to \( M \), \( \delta_- \), can only ever lead to effect ii. In this paper, we will consider the effect of random variations on the physical parameters, as well as the quantization by the TDC, in the form of two additive white noise processes, one inside and one outside the non-linearity. In short, our stochastic model for the RTT measurements taken by \( M \) is

\[
Y[n] = \delta_+ + \delta_0 + W[n] + T_S H[n], \quad \text{where} \quad H[n] = 1 - \text{mod}(T_s f_d n + \frac{\delta_-}{T_s} + \frac{\phi_S}{2\pi} + V[n], 1) \tag{9}
\]

For simplicity, we will assume that \( W[n] \) and \( V[n] \) are zero-mean Gaussian processes with respective standard deviations \( \sigma_w \) and \( \sigma_v \), and we will consider them independent. Analyzing the effect of the existing dependence between them, or evaluating the magnitude of this dependence, is outside of the scope of this paper. Only adding \( V[n] \), i.e., setting \( \sigma_w = 0 \), could explain the two effects explained above for most sample indices \( n \). Nonetheless, as shown by the indicators of the maximum and minimum of the model fit in Fig. 4, the experimental RTT measurements are not bounded, indicating that the noise term outside the non-linearity \( W[n] \) is necessary. Furthermore, random variations in \( \delta_- \) or \( \delta_0 \) cannot be meaningfully represented by \( V[n] \), since variations of these parameters of any magnitude will never affect which up-flank of \( S \) detects the ping pulse. Fig. 5 exemplifies the effect of each of the noise terms by showing two realizations of our stochastic model (9), one in which \( \sigma_w = 0 \) and \( \sigma_v > 0 \), and one in which \( \sigma_w > 0 \) and \( \sigma_v = 0 \).

In order to simplify the notation for the rest of the paper and abstract some of our theoretical results, we will express the stochastic sawtooth model in terms of four generic parameters, an offset \( \alpha \in \mathbb{R} \), a non-zero amplitude \( \psi \in \mathbb{R} \setminus \{0\} \) with known sign \( \text{sign}(\psi) \), a normalized frequency \( \beta \in [-1/2, 1/2) \), and a normalized phase offset \( \gamma \in [0, 1) \). In other words, we will...
express the sawtooth signal model as

\[ Y[n] = \alpha + W[n] + \psi \bmod_1(\beta n + \gamma + V[n]), \]  

with \( W[n] \) and \( V[n] \) independent additive white Gaussian noise processes. An empirical analysis verifying this model \(^{10}\) on real data from the testbed of \(^{21}\) can be found in \(^{19}\). Here, the restriction of the \( \beta \) and \( \gamma \) parameters simply reflects the maximum ranges that we can expect to distinguish, given the periodicity of \( \bmod_1(\cdot) \) as a function. Indeed, adding any integer factor of \( n \) inside the modulus, or any integer by itself, will not change \( Y[n] \), and establishes an equivalence of period one for both \( \beta \) and \( \gamma \). Here, we have chosen \( \beta \in [-1/2, 1/2] \) and \( \gamma \in [0, 1) \) to preserve their intuitive meanings as a normalized frequency and a phase term, respectively. Several initial insights can be drawn from the parallel between \(^{9}\) and \(^{10}\). First, the condition \( |f_d| < 1/(2T_s) \), resembling the Nyquist sampling condition, arises from the restriction in \( \beta \). Second, if we consider this restriction and examine the relation between the parameters of both models, we observe that

\[ \alpha = \delta_0 + \delta_\psi + T_S, \psi = -T_S, \beta = f_dT_s, \text{ and} \]

and, incorporating that \( T_S = T_M/(T_M f_d + 1) \) and that \( T_M, T_s \) and \( \delta_0 \) are known,

\[ \delta_\psi = \alpha - \delta_0 - \frac{T_M}{T_M f_d + 1}, \]

\[ f_d = \beta T_s = -\left( \frac{1}{T_M} + \frac{1}{\psi} \right), \text{ and} \]

\[ \phi_S = 2\pi \bmod_1(\gamma - \bmod_1\left( \frac{T_M f_d}{T_M} + \frac{1}{T_M} \right)). \]

Clearly, then, unless further constraints relating \( \delta_\psi, \delta_\psi \) and \( f_d \) are given, it is impossible to recover \( \delta_\psi, \delta_\psi, f_d, \) and \( \phi_S \) from \( \alpha, \psi, \beta \) and \( \gamma \). In the context of clock synchronization over networks, this is equivalent to the impossibility result of \(^{16}\), which studied the uncertainty sets where the synchronization parameters are known to lie given time-stamped message exchanges under different conditions. An analysis similar to that in \(^{16}\) under idealized, noise-free conditions could be reproduced for \(^{9}\), but is outside of the scope of this paper. In contrast, we will provide an analysis of identifiability when every physical parameter can be subject to noise. In fact, this analysis will reveal that synchronization with the sawtooth signal model requires a certain level of randomness, i.e., it is impossible without it. Consequently with the discussion above, then, we will assume that \( \delta_\psi \) is given when one knows \( \delta_\psi \) and \( f_d \), as it happens in a number of applications. For example, in wireless sensor networks, one may generally consider that all nodes are equal and the channels between any two of them are symmetric, and thereby one can assume \( \delta_\psi = \delta_\psi/2 = \delta_1 + \rho/c \) where \( \delta_1 > 0 \) is a known delay, \( \rho > 0 \) is the unknown range between \( \mathcal{M} \) and \( \mathcal{S} \) in the communication medium and \( c [\text{m/s}] \) is the speed of light in the medium. Even in this context, line-of-sight communication is not a requirement, as \( \rho \) may simply be the length of the shortest path between \( \mathcal{M} \) and \( \mathcal{S} \). When convenient in the paper, we will use this assumption combined with \( \delta_1 = 0 \), and consider the ranging problem of \(^{21}\), \(^{23}\) jointly with clock synchronization \(^{19}\). In the following section, we will characterize the model \(^{10}\) statistically, providing conditions for its identifiability. Our aims in doing that are 1) to present novel results on the sawtooth signal model, and 2) to provide guarantees for the design of practical synchronization systems using nodes modeled by the design in Fig. 4.

### III. IDENTIFIABILITY OF THE SAWTOOTH MODEL

Identifiability is a basic requirement on any statistical model that relates to the minimal conditions that make parameter estimation a reasonable goal \(^{27}\).

**Def. 1 (Identifiability):** (From \(^{28}\) Definition 11.2.2, p. 523) Let \( \mathcal{Y}_\theta \) be a statistical model with parameter \( \theta \in \Omega \). Assume that if \( Y \sim \mathcal{Y}_\theta \) for some given \( \theta \), \( Y \) has PDF \( f_Y(y; \theta) \). Then, \( \mathcal{Y}_\theta \) is an identifiable model, and \( \theta \) is an identifiable parameter, if and only if

\[ f_Y(y; \theta^{(1)}) = f_Y(y; \theta^{(2)}), \forall Y \iff \theta^{(1)} = \theta^{(2)}. \]

That is, the mapping between the parameter \( \theta \) and the distribution specified by \( \mathcal{Y}_\theta \) is one-to-one. If \(^{14}\) is not met, the data observed when the parameter value is \( \theta^{(1)} \) and the data observed when the parameter value is \( \theta^{(2)} \) have the same distribution, and therefore, distinction between these two parameters from observed data is impossible. Unintuitively, even if \(^{14}\) is not given, one could possibly design good estimators for \( \theta \). Specifically, as long as the selected metric in the space of parameters \( \Omega \) does not assign much importance to the difference between the pairs \( \theta^{(1)} \) and \( \theta^{(2)} \) that do not fulfill \(^{14}\), estimation could remain a sensible objective. In this paper, the data model \( \mathcal{Y}_\theta \) is defined by \(^{10}\).
and the considered parameters are $\theta = [\alpha, \psi, \beta, \gamma]^T$. Hence, this section will be dedicated to establishing under which conditions, in terms of the values of $\sigma_u$ and $\sigma_v$ in (10), a one-to-one relation between $\theta$ and the distribution of the data $Y = [Y[0], Y[1], \ldots, Y[N-1]]^T$ can be ensured.

A. Unidentifiability without inner noise

In order to analyze the relation between $\theta$ and $f_Y(y; \theta)$, we will first consider the simplifying assumption $\sigma_v = 0$. This is an unrealistic assumption under most applicable uses of the sawtooth model (10), including that of clock synchronization, but it will be useful for our analysis. We will show that under this assumption, (10) yields an unidentifiable model $Y_\theta$ in which the effect of $\alpha$ and $\gamma$ cannot be fully distinguished in the observed data $Y \sim Y_\theta$.

Consider first (10) with $\sigma_v = 0$ and observe that then, $Y_\theta = N(\mu_\theta, \sigma^2_\epsilon 1_N)$, where $1_N$ is the $N \times N$ identity matrix and

$$\mu_\theta = \alpha 1_N + \psi \mod_1(\beta n + \gamma 1_N),$$

with the modulus operation $\mod_1(\cdot)$ applied component-wise, $1_N = [1, 1, \ldots, 1]^T \in \mathbb{R}^N$, and $n = [0, 1, \ldots, N-1]^T$. Because the normal distribution is fully characterized by its location and scale parameters, we know that changes in $\theta$ will only affect the distribution in terms of its location, controlled by its mean $\mu_\theta$. Consequently, the condition for identifiability in Def. 1, i.e., (14), can be restated as $\mu_{\theta(1)} = \mu_{\theta(2)} \Leftrightarrow \theta^{(1)} = \theta^{(2)}$. Lem. 1 establishes that, when $\sigma_v = 0$, there are changes in $\alpha$ and $\gamma$ that violate this condition.

Lem. 1 (Unidentifiability of (10) when $\sigma_v = 0$): Let $Y_\theta$ express the model of the data $Y = [Y[0], Y[1], \ldots, Y[N-1]]^T$ given by (10) with $\theta = [\alpha, \psi, \beta, \gamma]^T$, $\alpha \in \mathbb{R}$, $|\psi| > 0$, $\beta \in [-1/2, 1/2]$ and $\gamma \in (0, 1)$, when the inner noise is disregarded, i.e., $\sigma_v = 0$. Then, $Y_\theta$ is unidentifiable. In particular, there are different combinations of $\alpha$ and $\gamma$ that yield the same distribution of $Y$ under $Y_\theta$.

Proof: We will show that $\mu_{\theta(1)} = \mu_{\theta(2)} \Leftrightarrow \theta^{(1)} = \theta^{(2)}$, i.e., the forward implication of (14) in Def. 1 is not fulfilled. In particular, given a parameter vector $\theta = [\alpha, \psi, \beta, \gamma]^T$, we will find $\theta^{(1)}, \theta^{(2)}$ such that $\theta^{(1)} \neq \theta^{(2)}$ and $\mu_{\theta^{(1)}} = \mu_{\theta^{(2)}}$.

Observe that, because $\mod_1 : \mathbb{R} \rightarrow \{0, 1\}$, $\epsilon_+ = 1 - \max_{n \in \{0, 1, \ldots, N-1\}} \{ \mod_1(\beta n + \gamma) \} > 0$.

Then, $\forall n \in \{0, 1, \ldots, N-1\}$ and $\forall \epsilon \in (0, \epsilon_+)$,

$\alpha + \psi \mod_1(\beta n + [\gamma + \epsilon]) = [\alpha + \psi] + \psi \mod_1(\beta n + \gamma)$.

Therefore, for any $\epsilon \in (0, \epsilon_+)$,

$\theta^{(1)} = [\alpha + \psi, \psi, \beta, \gamma]^T$ and $\theta^{(2)} = [\alpha, \psi, \beta, \gamma + \epsilon]^T$,

yield $\mu_{\theta^{(1)}} = \mu_{\theta^{(2)}}$, i.e., $\mu_{\theta^{(1)}} = \mu_{\theta^{(2)}} \Leftrightarrow \theta^{(1)} = \theta^{(2)}$ and $Y_\theta$ is unidentifiable.

Fig. 6 illustrates the idea of our proof of Lem. 1 when $\psi > 0$, and shows, in Fig. 6(b), changes in $\alpha$ and $\gamma$ that cannot be distinguished in the mean of $Y$ for a simple example with $N = 3$. Consequently, Fig. 6(b) serves as a straightforward counter-example to the identifiability of $Y_\theta$ when $\sigma_v = 0$.

In our proof of Lem. 1 we exploit the formal definition of $\mod_1(\cdot)$ to claim that its value will always be strictly less than one, and therefore, we obtain the margin $\epsilon_+$ under which changes in $\alpha$ of the same sign as $\psi$ and positive changes in $\gamma$ are not distinguishable. However, the real limitation on identifiability is given by the points closest to the discontinuity from both sides, and, in most cases (i.e., $\gamma \neq 0$ and for most $\beta$s), a similar margin $\epsilon_-$ can be obtained under which changes in $\alpha$ of the sign opposite to $\psi$ and negative changes in $\gamma$ are not distinguishable.

While our analysis is concerned with a fixed value of $N$, the lack of identifiability stated in Lem. 1 may be less problematic in an asymptotic regime. In particular, if increasing the sample size tends to reduce the segment at the left of the non-linearity without any sample, i.e., $\epsilon_+ \rightarrow 0$ when $N \rightarrow +\infty$, the size of the changes in $\alpha$ and $\gamma$ that cannot be distinguished in the data would decrease with $N$, making the model identifiable in an asymptotic regime, or at least invalidating our proof of non-identifiability. In particular, if we consider elements of the vector (15), $\mu_{\theta_n}$, as a sequence, we obtain what is known in dynamical systems as the orbit of a rotation of the circle. Then, if $\beta \in \mathbb{R} \setminus \mathbb{Q}$ we have that the orbit is minimal [29, ch. 1.3., proposition 1.3.3.], i.e., that the set $\{\mu_{\theta_n}\}_{\mathbb{N}}^\infty$ when $N \rightarrow +\infty$ is dense in $[0, 1)$, and thus, $\epsilon_+ \rightarrow 0$. This contradicts the intuitive notion of finite-sample identifiability as a necessary
In the next section, we analyze the full model in the presence of noise \( Y \) and the stable transmission delay. This would invalidate the identifiability result obtained from the global grid search (GGS) on (23) (see Section IV-B) for estimating an offset in white noise, which here serves as a reference for the estimation of \( \alpha, \psi, \beta, \gamma \) (see the supplementary material of this paper for details). Here, we observe that while the performance references for estimation problems influence the development of new technology at several critical stages. At an early stage, they guide the efforts in the design of new estimators by identifying the pitfalls of current methods. At a later stage, they guide industrial application by providing an expectation of what benefits can be expected from the use of a specific technology. Here, we provide two performance references. First, we derive error lower bounds for unbiased estimators working on a phase-unwrapped version of our stochastic RTT measurement model (9). Second, we present and thoroughly evaluate simple estimators of the sawtooth parameters on simulated synchronization and ranging problems.

### IV. Performance References

Performance references for estimation problems influence the development of new technology at several critical stages. At an early stage, they guide the efforts in the design of new estimators by identifying the pitfalls of current methods. At a later stage, they guide industrial application by providing an expectation of what benefits can be expected from the use of a specific technology. Here, we provide two performance references. First, we derive error lower bounds for unbiased estimators working on a phase-unwrapped version of our stochastic RTT measurement model (9). Second, we present and thoroughly evaluate simple estimators of the sawtooth parameters on simulated synchronization and ranging problems.

#### A. Cramér-Rao lower bounds for an unwrapped model

Under model (9), the likelihood function is not differentiable at some parameter values, because \( \text{mod}_1(x) \) is not differentiable when \( x \) is not co-prime with \( Q \). This violates the assumptions of the well-known Cramér-Rao lower bound (CRLB) for the mean squared error (MSE) of unbiased estimators. Here, we propose to analyze the linear, unwrapped model that results in condition for consistent estimators to exist, seen, for example, in [30, p. 62]. In contrast, when \( \beta \in \mathbb{Q} \), [15] is periodic and hence \( \epsilon_+ \to \epsilon^*_+ > 0 \). In particular, if \( \beta = \pm M/Q \) with \( M \) and \( Q \) two co-prime naturals, then [15] is periodic with minimal period \( Q \), and increasing the sample size beyond \( Q \) will not result in any further reduction of \( \epsilon_+ \), i.e., any further improvement from an identifiability perspective. In the case of clock synchronization, this specific case corresponds to coherent sampling, in which \( T_{g, d} = \pm M/Q \). The effect of this specific case in the estimation error of a global grid search (GGS) strategy (detailed in Section IV-B) when \( \psi \) and \( \beta \) are known is illustrated in Fig. 7.

Our analysis has assumed that \( \alpha \) was part of the parameter vector \( \theta \), and that one wanted to recover it. Although this can be the prominent case in many applications of the sawtooth model, e.g., synchronization in wireless sensor networks or networks of autonomous vehicles, other applications may consider \( \alpha \) to be known. Within synchronization, this would be the case of base-station synchronization in cellular networks, in which the backhaul links will most likely have a known and stable transmission delay. This would invalidate the identifiability analysis in Lem. [1] and under some additional conditions, \( \gamma \) could be shown to be identifiable. Regardless, in the next section we analyze the full model in the presence of noise inside the \( \text{mod}_1(\cdot) \) non-linearity, i.e., with \( \theta = [\alpha, \psi, \beta, \gamma]^T \) when \( \sigma_v > 0 \), and show its identifiability.

### B. Identifiability with inner noise

**Theorem 2** (Identifiability of (10) when \( \sigma_v > 0 \)): Let \( \mathcal{Y}_\theta \) express the model of the data \( Y = [Y[0], Y[1], \ldots, Y[N-1]]^T \) given by (10) when \( \theta = [\alpha, \psi, \beta, \gamma]^T \), the parameters fulfill \( \alpha \in \mathbb{R}, |\psi| > 0 \) with \( \text{sign}(\psi) \) known, \( \beta \in [-1/2, 1/2] \) and \( \gamma \in [0, 1] \), there is noise inside the \( \text{mod}_1(\cdot) \) non-linearity, i.e., \( \sigma_v > 0 \) and \( \sigma_w > 0 \), and at least two RTT measurements have been taken, i.e., \( N \geq 2 \). Then, \( \mathcal{Y}_\theta \) is an identifiable model and \( \theta \) is an identifiable parameter.

Because the proof of Theorem 2 is rather technical, we place it in the Appendix A. However, it is worthwhile to note here that it is not limited to the case in which \( W[n] \) and \( V[n] \) are Gaussian processes. Indeed, the statements in there apply mutatis mutandis under a wide variety of distributions for \( W[n] \) and \( V[n] \). In particular, any \( W[n] \) consisting of independent and identically distributed (IID) samples from any location-scale family with some reference PDF \( \varphi(w) \) with unbounded support will allow for the conclusion in (27). Furthermore, such a \( W[n] \) together with any \( V[n] \) consisting of IID samples from a location family that accepts a PDF and leads to a monomodal distribution after wrapping with mode equal to the location parameter, e.g., IID Cauchy distributed samples [31, p. 51], will also preserve all the statements therein. Nonetheless, to our knowledge, the literature mostly considers timing errors to be Gaussian (see, among others, [5], [7], [8], [17], [18], [32], [53]), with little empirical incentive to consider other models.

The contrast between Lem. [1] and Theorem 2 is highly non-intuitive. Indeed, it implies that the presence of noise inside the non-linearity improves the theoretical condition of the estimation problem. This result recalls the popular theories of stochastic resonance for testing and estimation [34, 38] and of dithering for improving the signal-independence of quantization noise [39], but is, in fact, less expected. In summary, both these theories delve into using noise to improve the performance of knowingly suboptimal strategies. In contrast, our identifiability result reveals how the inclusion of noise makes the data more informative with respect to the underlying parameters.
from assuming 1) that an oracle has removed the effect of the nonlinearity, and 2) that transmission delays are symmetric, i.e., \( \delta_\omega = 2 \delta_v \), which, as discussed at the end of Section III is the dominant assumption in ranging applications. The model then becomes

\[
Z[n] = \delta_0 + \frac{\delta_\omega}{2} + TS \left( 1 - \frac{\phi_S}{2\pi} \right) - TSf_\delta n + U[n], \quad (16)
\]

with \( U[n] \) composed by the sum of independent samples of \( W[n] \) and \(-T_S V[n], \) i.e., a white Gaussian process such that \( U[n] \sim N(0, \sigma^2) \) with \( \sigma^2 = \sigma_v^2 + T_S^2 \sigma_\theta^2. \) The resulting model, i.e., (16), is not without complications. First, \( \phi_S \) and \( \delta_\omega \) are not jointly identifiable from (16), because only their weighted sum affects the mean of the resulting distribution. This is not jointly identifiable from (16), because only their weighted sum affects the mean of the resulting distribution. This is a much more extreme case than the non-identifiability of \( \alpha \) and \( \gamma \) in Section III-A, as the uncertainty in what each value may be is much larger. Second, the variance of the noise now depends on \( T_S, \) i.e., on \( f_\delta, \) one of the parameters to estimate. In order to address these complications in a structured manner, we propose to analyze a linear model with slope-dependent noise power, i.e., the model

\[
Z = [1_N, n] \left[ \begin{array}{c} \tilde{\alpha} \\ \tilde{\beta} \end{array} \right] + U, \text{ with } U \sim N(0, \sigma^2 I_N) \quad (17)
\]

and \( \sigma^2 = \sigma_\omega^2 + (\sigma_1 + \beta \sigma_2)^2, \)

where \( Z = [Z[0], Z[1], \ldots, Z[N-1]]^T \) and \( \sigma_0 \geq 0, \sigma_1 \geq 0 \) and \( \sigma_2 \geq 0 \) are known. This model is equivalent to (16) with

\[
\tilde{\alpha} = \delta_0 + \frac{\delta_\omega}{2} + TS \left( 1 - \frac{\phi_S}{2\pi} \right) \quad \text{and} \quad \tilde{\beta} = -TSf_\delta, \quad (18)
\]

\( \sigma_0 = \sigma_\omega, \sigma_1 = T_M \sigma_v, \) and \( \sigma_2 = \sigma_v / K. \) Here, recall that \( K = T_S / T_M. \) The advantages of (17) are that 1) it is an identifiable model, and 2) it can be analyzed using standard results for the Fisher information matrix of Gaussian models [40] ch. 3.9, p. 47. Furthermore, given the Fisher information matrix \( I_\omega \) for (17), one can obtain CRLBs for \( f_\delta, \) for \( \phi_S \) when \( \delta_\omega \) is known, and for \( \delta_v \) when \( \phi_S \) is known by using the CRLB on functions of vector parameters [41] corollary 5.23, p. 306]. In short, the CRLB on functions of vector parameters states that if \( \omega = [\tilde{\alpha}, \tilde{\beta}]^T, \) any unbiased estimator of a differentiable bounded function of the parameters \( g(\omega), \) i.e., \( \hat{g}(Z), \) fulfills that

\[
\text{MSE}(\hat{g}(Z)) \geq \text{CRLB}_\omega(g(\omega)) = (\nabla_\omega g)^T I_\omega^{-1} (\nabla_\omega g), \quad (19)
\]

where \( \nabla_\omega g \) is the gradient of \( g. \) The derivation and statement of the inverse Fisher information matrix necessary for (19) can be found in Appendix B. From the relations in (18), one can obtain the parameters of interest, i.e., \( f_\delta, \delta_\omega, \) and \( \phi_S \) in (16), as functions of the vector parameter \( \omega = [\tilde{\alpha}, \tilde{\beta}]^T, \) i.e.,

\[
f_\delta = g_{f_\delta}(\omega) = -\frac{\tilde{\beta}}{T_M \left( KT_M + \tilde{\beta} \right)}, \quad (20)
\]

\[
\phi_S = g_{\phi_S}(\omega) = 2\pi + \frac{2\pi}{T_M + \frac{\tilde{\beta}}{K}} \left( \frac{\delta_\omega}{2} + \delta_0 - \tilde{\alpha} \right),
\]

\[
\delta_\omega = g_{\delta_\omega}(\omega) = 2 \left( \tilde{\alpha} - \delta_0 - \left( T_M + \frac{\tilde{\beta}}{K} \right) \left( 1 - \frac{\phi_S}{2\pi} \right) \right).
\]

Clearly, in obtaining the expressions for \( \phi_S \) or \( \delta_\omega, \) we had to assume that \( \delta_v \) or \( \phi_S \) were known, respectively. Consequently, the resulting bounds will not take into account that both need to be estimated simultaneously. This is, in fact, intended, because these two parameters can not be jointly estimated from (16), and our only purpose in deriving this bounds is to use them as a plausible reference for the performance one can obtain using (9). Because the functions of \( \omega \) in (20) are bounded and differentiable, we can use their gradients to obtain the CRLBs for (16) using (19). In particular, we have that

\[
\nabla_\omega g_{f_\delta}(\omega) = -\frac{1}{T_S K} [0, 1]^T, \quad (21)
\]

\[
\nabla_\omega g_{\phi_S}(\omega) = 2\left[ 1, \frac{\tilde{\beta}}{K} \left( \frac{\phi_S}{2\pi} - 1 \right) \right]^T,
\]

\[
\nabla_\omega g_{\delta_\omega}(\omega) = -2\pi \left[ 1, \frac{\tilde{\beta}}{K} \left( \frac{\phi_S}{2\pi} - 1 \right) \right]^T.
\]

A more detailed derivation of (20) and (21) can be found in the supplementary material of this paper.

The obtained CRLBs are valid for unbiased estimators from data \( Z \) generated according to (16), but they are not guaranteed to hold for unbiased estimators from data \( Y, \) generated from (9). Furthermore, they are definitely not valid bounds on the MSE of biased estimators from either model. Nonetheless, we believe they provide a linear intuition that, as our experimental results confirm, is still practically relevant.

B. Basic estimation strategies

In this section, we present simple techniques to estimate the parameters of a sawtooth signal model (10) based on those proposed in [19]. In their simplicity, they show remarkable robustness for the ranges of parameters \( \alpha, \beta, \gamma \) and \( \psi \) that arise in practical clock synchronization and ranging scenarios. Consequently, we consider them to be a good reference on the minimum expected performance that can be obtained from systems that use the proposed technology. For the sake of reproducibility and direct impact onto the interested communities, we provide thoroughly documented notebooks that contain the implementation of all the techniques presented in this section in this project’s repository [26].

For the sake of generality and clear exposition, we will present these estimators on the parameters of the generic sawtooth signal model (10). Nonetheless, we will consider that the frequency and amplitude parameters \( \beta \) and \( \psi \) are intimately related, i.e., that given one the other is fully determined. We do this to parallel the case of clock synchronization and ranging, in which \( \beta = T_M f_\delta \) and \( \psi = -T_S = T_M / (T_M f_\delta + 1). \) In practice, we will aim to find good estimators \( \tilde{\beta} \) of the frequency parameter \( \beta, \) and consider that they result in good derived estimators of the amplitude \( \psi, \) which we will note \( \tilde{\psi} \). For practical application of the techniques we present to clock synchronization and ranging, it suffices to transform the estimators of \( \alpha, \beta, \gamma \) and \( \psi \) to suitable estimators of \( \rho, f_\delta \) and \( \phi_S \) through the expressions in [11].
1) Periodogram and correlation peaks (PCP), a fast and simple solution: The first technique we present is deliberately developed to be computationally cheap, and uses only very simple and efficient operations such as discrete Fourier transforms (DFTs), sorting algorithms, and sample means. Basically, the estimator is divided in three fundamental steps, and relies on the assumption that the sign of the amplitude \( \hat{\gamma} \) is known. First, one uses a periodogram of the \( L-1 \) times zero-padded centered data as an estimate of the absolute value of the frequency parameter \( \beta \), i.e.,

\[
|\hat{\beta}| = \max_{n \in K} \left\{ |\text{DFT}_N (\hat{y}[n])|^2 \right\} / (NL)
\]

where \( \hat{y}[n] \) is a length \( NL \) signal such that

\[
\hat{y}[n] = \begin{cases} 
  \hat{y}[n] - \frac{1}{N} \sum_{m=0}^{N-1} y[m] & \text{if } n < N, \\
  0 & \text{if } N \leq n \leq NL - 1,
\end{cases}
\]

and \( K = \{0, 1, \ldots, [NL/2]\} \).

Second, one uses this unsigned frequency estimate to build two length \( [1/|\hat{\beta}|] \) signals \( p_+[n] \) and \( p_-[n] \) such that

\[
p_{\pm}[n] = \text{sign} (\hat{\psi}) \mod (\pm|\hat{\beta}|) \quad \text{for } 0 \leq n < [1/|\hat{\beta}|].
\]

These two reference signals and the first estimated period of the data, i.e., the length \( [1/|\hat{\beta}|] \) signal \( \hat{y}[n] \) such that \( \hat{y}[n] = y[n] \) for \( 0 \leq n < 1/|\hat{\beta}| \), are centered, max-normalized, and circularly correlated using DFTs to estimate the sign of \( \beta \) and the value of \( \gamma \). In particular, if \( \hat{y}[n], p_{+}[n], \) and \( p_{-}[n] \) are the centered and successively max-normalized versions of \( y[n] \), \( p_{+}[n] \) and \( p_{-}[n] \), respectively, one computes two numbers \( l_{+} \) and \( l_{-} \) given by

\[
l_{\pm} = \max_{0 \leq n < 1/|\hat{\beta}|} \left\{ 1 \text{DFT} \left[ \text{DFT} (p_{\pm}[n]) \text{DFT} (\hat{y}[n])^* \right] \right\},
\]

along with the indices \( n_{\pm}^{\text{opt}} \in \{0, 1, \ldots, [1/|\hat{\beta}|] - 1\} \) at which the maxima \( l_{\pm} \) are achieved, respectively. Here, the DFT and inverse DFT operations were considered to work on the length \( [1/|\hat{\beta}|] \) signals, without any zero padding, and \( \ast \) represents complex conjugation. Then, if \( l_{+} > l_{-} \), one estimates \( \hat{\beta} = +|\hat{\beta}| \) and sets \( n_{\text{opt}} = n_{+}^{\text{opt}} \), while if \( l_{-} > l_{+} \), one estimates \( \hat{\beta} = -|\hat{\beta}| \) and sets \( n_{\text{opt}} = n_{-}^{\text{opt}} \). At this point, then, the amplitude of the signal is also considered estimated as \( \hat{\psi} \). Furthermore, the phase parameter \( \gamma \) is estimated as \( \hat{\gamma} = \text{mod}|\hat{\beta}| \).

Third, one employs the minimum mean squared error estimator for the offset parameter \( \alpha \) assuming that \( \hat{\beta}, \hat{\gamma} \) and \( \hat{\psi} \) are correct. In other words, one solves

\[
\min_{\alpha \in \mathbb{R}} \left\{ \sum_{n=0}^{N-1} \left( y[n] - \hat{\psi} \mod |\hat{\beta}| \right)^2 \right\},
\]

through its closed-form solution, fully determined by \( \hat{\beta} \) and \( \hat{\gamma} \)

\[
\hat{\alpha}_{\hat{\beta}, \hat{\gamma}} = \sum_{n=0}^{N-1} y[n] - \sum_{m=0}^{N-1} \hat{\psi} \mod |\hat{\beta}| m + \hat{\gamma}. \tag{22}
\]

Although this three-step estimator is clearly heuristic and will not have good statistical properties, its computational cost is very low, and it can be implemented in lightweight hardware, such as sensors in a wireless sensor network. Furthermore, while some of its steps are rather counter-intuitive, they show remarkable robustness. For example, using only the first estimated period of the data \( \hat{y}[n] \) to estimate the phase parameter \( \gamma \) clearly is not an optimal strategy, but shows unparalleled robustness to errors in the estimation of the unsigned frequency parameter \( |\beta| \), while steeply reducing the computational burden.

2) Local or global grid search (LGS or GGS), an exhaustive and costly solution: As a contrast to PCP, the second technique we propose is computationally heavy. Nonetheless, our simulation study in Section V will suggest that it exhibits desirable statistical properties. In particular, we propose to minimize the prediction MSE (PMSE), i.e.,

\[
\min_{(\beta, \gamma) \in \mathbb{R} \times \mathbb{R}} \left\{ \sum_{n=0}^{N-1} \left( y[n] - \hat{\alpha}_{\beta, \gamma} \hat{\psi}_{\beta, \gamma} \mod |\beta n + \gamma| \right)^2 \right\}. \tag{23}
\]

In (23), \( \hat{\psi}_{\beta, \gamma} \) is the implied estimator of \( \psi \) for a given \( \beta \) we mentioned at the start of Section IV-B and \( \hat{\alpha}_{\beta, \gamma} \) is the \( \alpha \) that minimizes the cost function in (23), parametrized by \( \beta \) and \( \gamma \) and given mutatis mutandis by the expression in (22). Regrettably, not only does (23) not have a closed-form solution, but the PMSE over \( \beta \) and \( \gamma \) is highly non-convex, which implies that most current iterative solvers are unable to find the global minimum. Axis-aligned cuts of an example profile of the PMSE over \( \beta \) and \( \gamma \) are reported in Fig. 8. We propose to build a grid over some given ranges \( \mathcal{G} \subset [0, 1] \) for \( \gamma \) and \( \mathcal{B} \subset [-1/2, 1/2] \) for \( \beta \) and approximately solve (23) by picking the parameters \( (\beta, \gamma) \) in the grid that yield the smallest value of the cost function. We refer to this technique as global grid search (GGS) if \( \mathcal{B} \) and \( \mathcal{G} \) are defined simply on the region of plausible values of \( \beta \) and

![Fig. 8. Example of the prediction mean squared error (PMSE, (23)) of the model (10) obtained in a global grid search procedure (as described in Section IV-B2) with a \( 10^3 \times 10^3 \) grid with \( B = [0, 10^{-2}] \) and \( G = [0, 1] \), when either \( \beta \) or \( \gamma \) are fixed to their minimizing values \( \beta^{\text{opt}} \) and \( \gamma^{\text{opt}} \). In this example, \( N = 500, T_M = 10 \text{ ns}, T_S = 100 \mu\text{s}, \delta_0 = 5 \mu\text{s}, f_s = 73 \text{ Hz}, \phi_S = \pi \text{ rad}, \) and \( \rho = 2 \text{ m.} \) For more details about the example and our implementation, as well as the image representation of the prediction MSE jointly over \( \beta \) and \( \gamma \), see this project’s repository [29].](image-url)
TABLE I

VALUES FOR THE PARAMETERS OF PCP, LGS AND GGS THROUGHOUT THE PAPER, UNLESS OTHERWISE STATED.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Interpretation</th>
<th>Default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>zero-padding factor in PCP</td>
<td>5</td>
</tr>
<tr>
<td>$\beta_{\text{LGS}}$</td>
<td>range for $\beta$ in LGS</td>
<td>$\beta_{\text{PCP}} + [5, 5] \cdot 10^{-4}$</td>
</tr>
<tr>
<td>$\gamma_{\text{LGS}}$</td>
<td>gridpoints for LGS</td>
<td>$\gamma_{\text{PCP}} + [28, 28] \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$(N_B, N_G)$</td>
<td>range for $\beta$ in GGS</td>
<td>$10^2$, $10^3$</td>
</tr>
<tr>
<td>$(N_B, N_G)$</td>
<td>gridpoints for GGS</td>
<td>$[10^{-4}, 10^{-2}]$</td>
</tr>
<tr>
<td>$(N_B, N_G)$</td>
<td>gridpoints for GGS</td>
<td>$(0, 1)$</td>
</tr>
</tbody>
</table>

TABLE II

VALUES FOR THE PHYSICAL AND SIMULATION PARAMETERS IN OUR NUMERICAL RESULTS, UNLESS OTHERWISE STATED.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Interpretation</th>
<th>Value</th>
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</thead>
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<td>$N$</td>
<td>sample size</td>
<td>2000</td>
</tr>
<tr>
<td>$T_M$</td>
<td>$\mathcal{M}$’s period [s]</td>
<td>$5 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$T_s$</td>
<td>sampling period [s]</td>
<td>$= 10^{-4}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>range [m]</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$f_1$</td>
<td>frequency difference [Hz]</td>
<td>$U([-200, 200])$</td>
</tr>
<tr>
<td>$\phi_S$</td>
<td>$S$’s phase [rad]</td>
<td>$U(0, 2\pi)$</td>
</tr>
<tr>
<td>$\text{SNR}_{\text{in}}$</td>
<td>SNR for $V_s[n], 1/\sigma^2_S$ [dB]</td>
<td>40</td>
</tr>
<tr>
<td>$\text{SNR}_{\text{out}}$</td>
<td>SNR for $W_{\text{s}}[n], 1/\sigma^2_S$ [dB]</td>
<td>20</td>
</tr>
</tbody>
</table>

$\gamma$, respectively, and as local grid search (LGS) if $B$ and $G$ are defined as small neighborhoods around the PCP estimates $(\hat{\beta}_{\text{PCP}}, \hat{\gamma}_{\text{PCP}})$. Clearly, the performance of LGS and GGS will critically depend on the number and location of the grid points in $G \times B$, which are design parameters that set the compromise between accuracy and computational complexity. The simplest and most naive distribution of these points is uniformly across $G \times B$, with $N_G$ possible values for $\gamma$ and $N_B$ possible values for $\beta$.

V. NUMERICAL RESULTS

The performance references we present in Section [IV] are illustrated here in a series of realistic simulation studies on the problem of joint clock synchronization and ranging. In the exposition of these results we intend to aid 1) practitioners that aim to apply this technology, by providing reasonable expectations on its current possibilities, and 2) theoreticians that aim to develop estimators for the parameters of the sawtooth signal model (10), by revealing the strengths and pitfalls of the techniques that are currently available. With respect to 1), we include in all our results references that make it easier to identify different standard performance measures, regardless of the scale and aspect of each figure. In particular, a) in figures reporting ranging performance, i.e., $\text{MSE}(\rho)$, horizontal lines corresponding to standard errors of 1 cm or 0.1 cm are shown, b) in figures reporting frequency-difference estimation performance, i.e., $\text{MSE}(f_1)$, horizontal lines corresponding to standard errors of 10 ppb (1 Hz) and 1 ppb (0.1 Hz) of $\mathcal{M}$’s frequency, $1/T_M = 100$ MHz (see Table [I]), are shown, and c) in figures reporting $S$’s phase estimation performance, i.e., $\text{MSE}(\phi_S)$, horizontal lines corresponding to standard errors in $\phi_S$ that are one or two orders of magnitude below $T_S$ are shown. In reference to c), we intentionally report performance on the estimation of the phase parameter $\phi_S$ instead of the absolute time delay $\varphi_S = T_S \phi_S/(2\pi)$. In our opinion, this is a better and fairer measure of how useful a specific clock synchronization technique is, because the scale of the errors in $\varphi_S$ will always be dominated by the order of magnitude of $T_S$. In other words, if $T_S \approx 10$ ns, even guessing $\phi_S$ at random between 0 and $2\pi$ achieves errors in $\varphi_S$ that are on the order of $\mu$s. In order to streamline the exposition of this section and avoid unnecessary repetitions of the experimental conditions, we detail the default values for the parameters of our algorithms in Table [I] and the default values for the physical and simulation parameters in Table [II].

A. Example of consistency

In Fig. 9 we illustrate the convergence of the MSE of the PCP and LGS estimators proposed in Section [IV-B] with the sample size $N$ and compare it with the CRLBs for the unwrapped model derived in [IV-A] The results we report
were obtained from 300 Monte Carlo repetitions for specific physical parameters, i.e., $f_d = 73 \text{ Hz}$, $\phi_S = 3\pi/4$, and $\rho = 2$ with the remaining parameters set as in Tables I and II. The results suggest that both estimators are consistent for these specific values of the parameters, in the sense that their overall error tends to decrease with increasing sample size, i.e., MSE $\rightarrow 0$ with $N \rightarrow +\infty$. Consistency will also be suggested by the results in Fig. 12 where randomized parameters will be used.

For PCP, the convergence is clearly inefficient, and one observes it only by the decay of the envelope of the error. The regular bumps observed in the graphs of $\text{MSE}(\hat{f}_d)_{PCP}$ and $\text{MSE}(\phi_S)_{PCP}$ are due to the resolution of the underlying periodogram estimate. On one hand, if $\beta = k/NL$ for some $k \in \{0, 1, \ldots, [NL/2]\}$, $\beta$ will be included in the periodogram’s grid and the PCP will be biased towards it and achieve very low MSE. On the other hand, if $\beta$ is between two such points, the PCP’s bias will increase the MSE instead.

For LGS, the error seems to follow the decay of the CRLB in the estimation of $\rho$ and $f_d$. However, the convergence of $\text{MSE}(\phi_S)_{LGS}$ is much slower than that predicted by the CRLB of the unwrapped model. This is to be expected, since the bounds in (19) do not take into account the non-linearity of the model, and therefore, the wrapping effect of the phase term. Although this non-linear behavior is what makes the joint estimation of $\phi_S$ and $\rho$ possible, it also makes $\phi_S$ much harder to estimate than a simple offset. Furthermore, one must consider that the MSE in the estimation of $\phi_S$ only plays a role when one aims to obtain time synchronization. If only phase synchronization is desired, however, consistence and efficiency may be defined using more appropriate evaluation metrics [31, p. 84]. The evaluation with respect to these metrics are outside the scope of this paper.

For both PCP and LGS, the error in the estimation of the range $\rho$ is well below the CRLB, and for $N \geq 500$, it is mostly below 0.1 cm. Similarly, for PCP $N \geq 500$ leads to average frequency estimation errors below 1/T_M and average phase estimation errors well below 2π/10. For LGS, $N \geq 500$ leads to average phase estimation errors below 2π/100, and $N \geq 1500$ to frequency estimation errors of less than 1 ppb of 1/T_M. These very attractive results, however, are obtained for these fixed values of the parameters and under the rather optimistic noise conditions in Table II, i.e., SNR_in = 40 dB and SNR_out = 20 dB, cf. with the noise conditions obtained in real data reported in [19, p. 2386]. In the following, we will analyze how the estimation techniques we present in Section IV-B fare when these conditions are changed.

### B. Sensitivity to the parameter values

One of the weaknesses of the estimation approaches we present in Section IV-B is low performance when $f_d$ is small. In particular, because the PCP includes a mean-removal step before the zero-padded DFT, the low frequencies are suppressed. If $f_d$ is small, then, the peak we aim to detect in the DFT is most likely dampened and we detect noise instead. This is visualized in Fig. 10 which shows the average performance in the estimation of $f_d$ by both PCP and LGS as a function of $f_d$, when all other parameters are randomized according to Table II. Here, we see that the estimators fail when $f_d \in [-10, 10]$. Furthermore, we also observe the bumps corresponding to the bias due to the PCP’s frequency grid, as described in the second paragraph of Section V-A. Remarkably, the performance obtained with LGS remains below 1 ppb of 1/T_M for most frequencies $f_d$ outside the $[-10, 10]$ area even when the other parameters are randomized.

A weakness of the measurement protocol described in Section II-A is that, due to the lack of time-stamps in the exchanged packets, the time origin shift between $M$ and $S$’s clock appears only in terms of a phase term inside the mod(·) function, e.g., $\phi_S$ if we assume that $\phi_M = 0$. If one desires absolute time synchronization, this can have dire consequences. While $\phi_S = 2\pi(1 - \xi/2)$ and $\phi_S = \pi\xi$ for some small $\xi > 0$ are only 2πξ away under the non-linear wrapping, their difference implies errors in $\phi_S$ of $T_S(1 - \xi)$. In order to take this effect into account, we use the conventional Euclidean distance to quantify the error for the phase term $\phi_S$, without taking into account the wrapping effect. The ensuing increase of error around the extremes can be seen plainly in Fig. 11. This weakness is characteristic of RTT protocols for clock synchronization, in which there is no time-stamping. In turn, however, RTT protocols minimize the communication overhead and are more robust to malicious nodes [6, p. 29].
C. Average consistency

In order to characterize consistency beyond the example given in Section V-A in Fig. 12 we report the average performance for randomized parameter values for PCP and LGS in the estimation of the clock synchronization and ranging parameters as a function of the sample size. However, we sample frequency values in a reduced range, i.e., \( f_d \in [-200, -10) \cup [10, 200) \), in order to avoid the instability of our estimators when \( f_d \in [-10, 10) \). This provides an impression of the estimators’ performance within their range of probable use, and significantly reduces the amount of Monte Carlo repetitions necessary to obtain intelligible results.

Fig. 12 strengthens the conclusions from the analysis of Fig. 9. Indeed, 1) the results suggest that both PCP and LGS are consistent for the estimation of \( \rho \), \( f_d \) and \( \phi_g \). Furthermore, average estimation errors under 0.1 cm in the range parameter can be expected for \( N \geq 1000 \), and average estimation errors under 1 ppb in the frequency parameter can be expected for \( N \geq 1500 \). For the phase parameter \( \phi_g \), estimation errors below 2\( \pi /10 \) can only be expected with LGS and for \( N \geq 1000 \).

D. Sensitivity to the inner and outer noises

All the results in Figs. 9, 10, 11 were obtained under the optimistic noise conditions \( \text{SNR}_{\text{in}} = 40 \text{ dB} \) and \( \text{SNR}_{\text{out}} = 20 \text{ dB} \). In Fig. 13 we report the average performance for randomized parameter values for PCP and LGS in the estimation of the clock synchronization and ranging parameters as a function of either \( \text{SNR}_{\text{in}} \) and \( \text{SNR}_{\text{out}} \), when \( \text{SNR}_{\text{out}} = 30 \text{ dB} \) and \( \text{SNR}_{\text{in}} = 40 \text{ dB} \), respectively. Similarly as in Section V-C we sample \( f_d \) in the range \( f_d \in [-200, -10) \cup [10, 200) \), in order to avoid the instability of our estimators when \( f_d \in [-10, 10) \).

Fig. 13 reveals that range estimation performance decays progressively with the decrease of \( \text{SNR}_{\text{out}} \) throughout the investigated range for both PCP and LGS. In contrast, the decrease of \( \text{SNR}_{\text{in}} \) creates a progressive decay of performance only up to a breaking point around \( \text{SNR}_{\text{in}} = 10 \text{ dB} \). In this regime (very low \( \text{SNR}_{\text{in}} \)), one could consider modeling the non-linear term in (10) as a uniform noise term and employ techniques tailored to the estimation of offsets in Gaussian plus uniform noise [42]. Practically, both PCP and LGS achieve estimation accuracies below 0.1 cm only when \( \text{SNR}_{\text{out}} \geq 10 \text{ dB} \) for \( \text{SNR}_{\text{in}} = 40 \text{ dB} \), and only when \( \text{SNR}_{\text{in}} \geq 15 \text{ dB} \) for \( \text{SNR}_{\text{out}} = 30 \text{ dB} \).

For frequency estimation, we see that both PCP and LGS are very robust to a decrease of \( \text{SNR}_{\text{out}} \) up to a breaking point around \( \text{SNR}_{\text{out}} = 10 \text{ dB} \), and LGS maintains an improvement of 15 dB over PCP for any \( \text{SNR}_{\text{out}} \) above the breaking point. In contrast, when \( \text{SNR}_{\text{in}} \) degrades, the breaking point for both PCP and LGS is \( \text{SNR}_{\text{in}} = 20 \text{ dB} \), and the improvement of LGS over PCP increases gradually as \( \text{SNR}_{\text{in}} \) increases, only reaching 15 dB when \( \text{SNR}_{\text{in}} = 40 \text{ dB} \). Practically, on one hand, PCP achieves estimation accuracies below 10 ppb only when \( \text{SNR}_{\text{out}} \geq 10 \text{ dB} \) for \( \text{SNR}_{\text{in}} = 40 \text{ dB} \), and only when \( \text{SNR}_{\text{in}} \geq 20 \text{ dB} \) for \( \text{SNR}_{\text{out}} = 30 \text{ dB} \). On the other hand, LGS consistently improves on it, reaching estimation accuracies below 1 ppb only when \( \text{SNR}_{\text{out}} \geq 10 \text{ dB} \) for \( \text{SNR}_{\text{in}} = 40 \text{ dB} \), and only when \( \text{SNR}_{\text{in}} \geq 30 \text{ dB} \) for \( \text{SNR}_{\text{out}} = 30 \text{ dB} \).

For phase estimation, our results are not conclusive, because the randomization of both the frequency and phase parameters result in wide variability that would require further Monte Carlo averaging. Furthermore, while the unfavorable region...
PCP and LGS when either SNR seems to suggest a progressive decay of performance for both \( \phi \) networks in applications that require high ranging accuracy on the fundamental limitations of clock synchronization over estimates. Type techniques complimented with better frequency parameter improvements of phase estimation could be expected using LGS-SNR to a degradation of \( \text{thermore} \), we see that phase estimation is much more sensitive of the estimators when \( \rho \) estimation of the clock synchronization and ranging parameters \( \rho \) and \( \phi \), and \( d \) out SNR was fixed to its maximum when the other was varied. The range and \( \phi \) were randomized as specified in Table II, while \( f \) and the frequency difference \( f \) and the frequency difference \( f \) ∼ \( \mathcal{U}([-200, -10) \cup (10, 200]) \), in order to avoid the wildly irregular behavior of the estimators when \( f > 10 \) dB in Fig. [13], or the use of techniques similar to [43] to improve frequency synchronization. In this sense, we also provide a reference on the potential for improvement in the form of CRLBs for a related linear model. Additionally, we pinpoint the weaknesses of both the model and our estimation strategies. First, we acknowledge that more research is needed to consistently obtain accurate phase synchronization (phase estimation accuracies below \( 2\pi/100 \)). Second, we identify that schemes that rely on this mathematical model are not best suited for absolute time synchronization because the offset delay only appears as part of a phase term, which leads to wrapping errors.

From a theoretical standpoint, we establish that RTT-based schemes for clock synchronization are characterized by similar fundamental limitations as those relying on time-stamped message exchanges, previously discovered by [2]. Namely, synchronization is impossible with unknown path delays \( \delta \rightarrow \) and \( \delta \rightarrow \) if one cannot assume some relation between \( \delta \rightarrow \), \( \delta \rightarrow \), and the frequency difference \( f, \) e.g., that they are symmetric, i.e., \( \delta \rightarrow = \delta \rightarrow \). Furthermore, we have discovered a surprising property of sawtooth signal models, i.e., that adding noise inside the non-linearity allows for the joint identifiability of its offset and phase terms (cf. Lemma 1 and Theorem 2). This result challenges the convention of random variations as a negative component of a model, and is of use beyond our application domain.

**APPENDIX A**

**PROOF OF IDENTIFIABILITY**

**Proof of Theorem 2** Identifiability of (10) when \( \sigma_y > 0 \): The backward implication of identifiability, i.e., that the same parameters lead to the same data distribution (see Def. 1), is immediately clear from (10).

In the following, we will show that under the conditions above, the forward implication is also true, i.e., that

\[
 f_Y \left( y; \theta^{(1)} \right) - f_Y \left( y; \theta^{(2)} \right) = 0, \forall y \in \mathbb{R}^N \tag{24}
\]

implies that \( \theta^{(1)} = \theta^{(2)} \), where \( \theta^{(1)} \) and \( \theta^{(2)} \) are two vectors of parameters, and we denote their respective components by the same super-index, i.e., \( \alpha^{(i)} \), \( \psi^{(i)} \), \( \beta^{(i)} \), and \( \gamma^{(i)} \) for \( i \in \{1, 2\} \). Consequently, we assume (24), and will reach the conclusion that \( \theta^{(1)} = \theta^{(2)} \). We start by defining the random process

\[
 P[n] = \alpha + \psi \mod 1 (\beta n + \gamma + V[n]) \tag{25}
\]
and observe that (10) implies that \( Y[n] = W[n] + P[n] \). Because \( W[n] \) and \( V[n] \) are assumed independent (see (10) and the discussion in Section (II) and \( W[n] \sim N(0, \sigma_w^2) \), we have that the distribution of \( Y[n] \) is, for any given \( n \in \{0, 1, \ldots, N-1\} \),

\[
f_Y(y; \theta) = \left[ \frac{\varphi}{\sigma_w} \ast f_P(\tau; \theta) \right](y),
\]

where \( \ast \) denotes a convolution and \( \varphi(\cdot) \) is the PDF of the standard normal distribution \( N(0, 1) \). Then, consider the difference between the PDFs of \( Y[n] \) that correspond to the parameter vectors \( \theta^{(1)} \) and \( \theta^{(2)} \), i.e.,

\[
\Delta Y(y) = f_Y(y; \theta^{(1)}) - f_Y(y; \theta^{(2)}) = \left[ \frac{\varphi}{\sigma_w} \ast \left( f_P(\tau; \theta^{(1)}) - f_P(\tau; \theta^{(2)}) \right) \right](y),
\]

and observe that (24) implies that \( \Delta Y(y) = 0, \forall y \in \mathbb{R} \) and \( \forall n \in \{0, 1, \ldots, N-1\} \). Because \( \varphi(\tau) > 0 \) for any \( \tau \in \mathbb{R} \), we have that \( \Delta Y(y) = 0, \forall y \in \mathbb{R} \) if and only if \( \Delta P(p) = 0, \forall p \in \mathbb{R} \). Because \( P[n] \in [\alpha, \alpha + \psi, \alpha] \) if \( \psi > 0 \) and \( P[n] \in (\alpha + \psi, \alpha] \) if \( \psi < 0 \), this can only happen if \( \alpha^{(1)} = \alpha^{(2)} \) and \( \psi^{(1)} = \psi^{(2)} \), i.e.,

\[
\Delta Y(y) = 0, \forall y \in \mathbb{R} \Rightarrow \alpha^{(1)} = \alpha^{(2)} \text{ and } \psi^{(1)} = \psi^{(2)}. \quad (27)
\]

Therefore, the ambiguity between \( \alpha \) and \( \gamma \) that led to Lem. 1 has been resolved by the presence of the noise term \( V[n] \) inside the non-linearity. This effect is exemplified by the simple model in which \( Z \sim N(\mu, C) \) as

\[
I_\omega = \left[ \frac{\partial}{\partial \omega} C_{\omega} \right] \left[ \frac{\partial}{\partial \omega} C_{\omega} \right]^{T} + \frac{1}{2} \text{Tr} \left[ C_{\omega}^{-1} \frac{\partial C(\omega)}{\partial \omega} \left( C_{\omega}^{-1} \frac{\partial C(\omega)}{\partial \omega} \right) \right] \quad i,j \in \{1,2\}
\]

For (17), \( \partial \mu/\partial \alpha = 1_N, \partial \mu/\partial \beta = \mathbf{n}, \partial C/\partial \alpha = 01_N \text{ and } \partial C/\partial \beta = 2\sigma_2 \left( \sigma_1 + \hat{\beta} \sigma_2 \right) \mathbf{1}_N \). Considering that \( 1_N^{T}1_N = N, \mathbf{n}^{T}1_N = (N-1)/2 \text{ and } \mathbf{n}^{T} \mathbf{n} = (N-1)(2N-1)/6 \), we obtain the Fisher information matrix for (17), i.e.,

\[
I_\omega = \frac{N}{\sigma^2} \left[ \frac{1}{2} \left( N-1 \right) \left( N-1 \right) + \frac{\left( \sigma^2 \right)}{2N^2} \right] \left( \sigma^2 \right)^2. \quad (29)
\]

Simply inverting (29) leads to

\[
\text{I}_{\omega}^{-1} = \left[ \frac{\sigma^2/N}{N+1} + \frac{2\sigma^2 \left( \sigma_1 + \hat{\beta} \sigma_2 \right)^2}{\sigma^2 \left( N-1 \right)^2} \right] \left( \frac{2N^2}{\sigma^2} \right) \left( \frac{2N^2}{\sigma^2} \right)^2 \quad \text{and } \left( \sigma^2 \right). \quad (30)
\]

which allows for the computation of the CRLBs for the estimation of \( \alpha \) and \( \beta \), and, through the relations (20) in Section IV and their gradients (21), the CRLBs for the estimation of \( f_{\delta}, \delta_{\leftrightarrow} \), \( \psi \) when \( \varphi_S \) is known, and \( \varphi_S \) when \( \delta_{\leftrightarrow} \) is known. In terms of
the rates of convergence for the variance of efficient estimators, one can see that
\[
\mathbf{I}_\omega^{-1} = \left( \frac{\sigma^2(1 + \beta)^2}{N} + \frac{\sigma^2}{N} + O(1) \right) \left( \frac{1}{N} + O(1) \right),
\]
i.e., the efficient estimators of the offset \( \alpha \) and the slope \( \beta \) still have the same rates of convergence as in a standard linear model, with additions of only non-dominating terms. A detailed derivation of (29) and (30) can be found in the supplementary material of this paper.

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