# Group Theoretic Classification of Pentaquarks and Numerical Predictions of Their Masses

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# Gruppteoretisk klassificering av pentakvarkar och numeriska förutsägelser av deras massor

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## Abstract

In this report we investigate the exotic hadrons known as pentaquarks. A brief overview of relevant concepts and theory is initially presented in order to aid the reader. Thereafter, the history of this field with regards to theory and experiments is discussed. In particular, a group theoretic classification of these states is studied. A simple mass formula for pentaquark states is examined and predictions are subsequently made about the composition and mass of possible pentaquark states. Furthermore, this mass formula is modified to examine and predict additional pentaquark states. A number of numerical fits concerning the masses of pentaquarks are performd and studied. Future research is explored with regards to the information presented in this thesis.

## Sammanfattning

I denna uppsats undersöker vi de exotiska hadronerna kända som pentakvarkar. En kort överblick av relevanta koncept och teorier är presenterade med syfte att underlätta för läsaren. Historia i detta område med avseende på både teori och experiment är presenterad. Specifikt ingår en undersökning av gruppteoretisk klassificering av dessa tillstånd. En enkel formel för massan hos pentakvarkar undersöks och används för att förutspå massorna och uppbyggnaden av möjliga pentakvarktillstånd. Dessutom modifieras denna formel för att studera och förutspå ytterligare pentakvarktillstånd. Ett antal numeriska anpassningar relaterade till massorna hos pentakvarkarna genomförs och undersöks. Framtida forskning undersöks i koppling med material presenterat i uppsatsen.

# Contents

# 1 Introduction

Physics is one of the most fundamental scientific disciplines, as it tries to answer questions such as what the world is made out of. In ancient times one famous proposal on what the world is made out of was made by Anaximenes of Miletus ($\sim$500 BC) who proposed that air was the primary substance holding the world together and from which earth, fire and water originated from. Around 25 centuries later Dmitri Mendeleev established the periodic table of elements, which went further in the desire to label matter. Anaximenes' proposal was wrong while Mendeleev was not. The periodic system suggested a more fundamental substructure. The discovery of this substructure was one of largest breakthroughs in 20th century physics. The current most successful theory for describing what matter is and how it behaves is the standard model, which is studied in the field of particle physics. It shares the conceptual simplicity of Anaximenes' proposal while at the same time being quantitative and agreeing with experimental results [1].

## 1.1 Standard Model

Particle physics is the study of the fundamental building blocks of matter and their interactions. Currently the most complete theory about these particles and their interactions is called the *standard model*. This theory classifies all known elementary particles, which categorizes these into distinct types: Spin 1/2 fermions, spin 1 gauge bosons and the spin 0 Higgs boson. Spin 1/2-fermions are subsequently divided into the two categories quarks and leptons. Quarks occur in three groups (or *generations*): $(u, d), (c, s), (t, b)$. The different types of quarks are labeled as their *flavor*. In respective order their flavors are up, down, charm, strange, top and bottom. These particles get progressively heavier across the generations. There exists corresponding antiquarks, denoted by: $(\bar{u}, \bar{d}), (\bar{c}, \bar{s}), (\bar{t}, \bar{b})$, which have same mass but opposite charge from their corresponding quarks. The quantum number associated with quarks is the *baryon number B*, which is defined as

$$B = \frac{1}{3}, \text{ for quarks,}$$
$$B = -\frac{1}{3}, \text{ for antiquarks.}$$

(1.1)

This number is zero for all other particles [2].

Leptons also occur in three generations: Electron and electron neutrino, muon and muon neutrino, tau and tau neutrino. The notation is: $(e, \nu_e), (\mu, \nu_\mu), (\tau, \nu_\tau)$. Again the different types are known as flavors. Corresponding to every flavor there exists an antilepton: $(e^+, \bar{\nu}_e), (\mu^+, \bar{\nu}_\mu), (\tau^+, \bar{\nu}_\tau)$. The quantum number associated with leptons are the *lepton numbers* $L_e, L_\mu, L_\tau$ which take the value $+1$ for the corresponding leptons and $-1$ for their respective antileptons. For all other particles these numbers are zero [2].

In quantum mechanics the interactions between particles are mediated by exchange of field quanta. Spin 1 gauge bosons are responsible for the interactions between elementary particles. There are four fundamental forces, of which the standard model describe three of them.[1] *Strong interaction* is the force between quarks and this interaction is responsible for binding the quarks, in for example neutrons. This is mediated by *gluons*, denoted by $g$. The *electromagnetic interaction* which acts between electrically charged particles, it is mediated by the *photon*, denoted by $\gamma$. *Weak interaction* is responsible for the force between leptons.[2] This interaction is mediated by the $W^{\pm}$ and $Z^0$ bosons [2]. The spin 0 Higgs boson is responsible for giving mass to the elementary particles and mediating bosons $W^{\pm}, Z^0$ [2]. The existence of the Higgs boson was only theorized until it was found at the Large Hadron Collider at CERN in 2012 [3].

Quarks and gluons are not observed isolated, but only in composite structures, known as *hadrons*. This is due to the phenomenon known as *color confinement*. Both quarks and gluons carry a property known as *color charge*.[3] There are three kinds of color charges: red, blue and green. Likewise, for the corresponding antiquarks there exists three anticolors: antired, antiblue and antigreen. Red, blue and green add up to a color charge of zero and such a structure is said to be colorless. The same is true when adding antired, antiblue and antigreen. A color added with its corresponding anticolor is also colorless. Quarks may change color charge in an interaction, though color itself is always conserved, hence the process where a quark emits a gluon entails that gluons carry both a unit of color and a unit of anticolor. Quarks and gluons cannot be observed isolated due to the phenomenon known as *color confinement*. Color confinement is the phenomenon that no naturally occurring particles can possess a non-zero color charge, which is why quarks only exist in composite structures. Hadrons are subsequently divided into the categories mesons and baryons. Mesons consist of two quarks, or more explicitly one quark and one antiquark. Baryons consist of three quarks, arranged in a configuration so that the total color charge is zero [4].

The quark model imposes no limitation on the number of quarks composing hadrons other than the fact that they have to be colorless. For example, tetraquarks, which consist of two quarks and two antiquarks, in addition to pentaquarks, which consist of four quarks and one antiquark, are both hadrons with four and five quarks respectively, in contrast to the mroe common hadrons like mesons or baryons. [5]. The main topic of interest in this report is pentaquarks. Below is a short overview of the experimental

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[1]The excluded one is gravitation, which is theorized to be mediated by a spin 2 boson called a *graviton*.

[2]Leptons and quarks also interact through the weak interaction, though with considerably reduced strength.

[3]This is not in any way related to the everyday meaning of color.

history regarding this type of exotic hadrons.

## 1.2  Pentaquarks

For many years physicists where naturally searching for pentaquarks, since the same theory leading to several important discoveries also predicts the existence of pentaquarks. In 2003 a claim of pentaquark discovery was recorded at LEPS in Japan, where a resonance peak was observed at $1.54 \pm 0.01$ GeV. The peaks were significant to 4.6 standard deviations [6]. Naturally, this generated interest among physicists. Several other research groups in the following year subsequently reported negative experimental searches in addition to criticism regarding the reported discovery [7].

In 2006 the Review of Particle Physics from the Particle Data Group discusses the discovery made in 2003, which in particular was an exotic positive-strangeness baryon, $\Theta^+$, which consisted of the quark combination $uudd\bar{s}$. It also mentions that in 2004 the particle physics review assigned this supposed pentaquark a 3-star rating out of 4 regarding the validity of its existence. However, as mentioned above, further experiments cast this discovery into doubt. Specifically the 2006 review said: *"The conclusion that pentaquarks in general, and the $\Theta^+$, in particular, do not exist, appears compelling"*, and cited two high-statistics repeats from the Jefferson lab in USA [8]. In 2008 the review of particle physics from the Particle Data group went further and said:
*"There are two or three recent experiments that find weak evidence for signals near the nominal masses, but there is simply no point in tabulating them in view of the overwhelming evidence that the claimed pentaquarks do not exist."*
This review then lists a table of unsuccessful searches for the $\Theta^+$ particle [9]. Among others, the search at Stanford Linear Accelerator Center as part of the BABAR collaboration [10].

It was not until 2015 that the Large Hadron Collider experiment at CERN reported the discovery of pentaquark states. Physicists examined the decay of a baryon known as bottom lambda $\Lambda_b$ into three other particles: a $J/\psi$-meson, a kaon $K^-$ and a proton $p$. By studying the mass spectrum of these particles it was revealed that intermediate states were involved in this decay. These states were named $P_c(4450)^+$ and $P_c(4380)^+$. The first of these was observed as a peak in the data and the other one was required to fit the data. Researchers at CERN examined all possibilities for these signals and concluded that they can only be described as pentaquark states. Further analysis could reveal how these pentaquarks are bound together, since this information is still unknown [11]. These two states both had statistical significances over 9 $\sigma$, which is enough to warrant a formal discovery. The results from this experiment can be found in Ref. [12]. In the 2018 review both the history mentioned above and current information are summarized.

This can be found in Ref. [13].

# 2  Preparatory Physics

Recurring mathematical objects in the field of particle physics include, among others, groups. For a short review of these concepts, readers are encouraged to read the Appendix. The most important objects analyzed in this report are the special unitary groups $SU(n)$ consisting of unitary $n \times n$ matrices with determinant 1, which forms a Lie group.

## 2.1  Conservation and Quantum Numbers

In this section important quantum numbers for the discussion concerning pentaquarks will be presented. In the 1930s Werner Heisenberg observed that the neutron and proton had similar mass, leading to a proposition that these particles are to be regarded as two states of the same particle, called a *nucleon*. That the strong interaction does not distinguish between protons and neutrons suggests that this degeneracy in mass is an approximate symmetry. This leads to *isospin* $\vec{I}$ being introduced, which is a vector in isospin space, where the name is due to the similarities to the intrinsic spin. The nucleon carries isospin 1/2 and the third component $I_3$ has eigenvalues $+1/2$ for the proton and $-1/2$ for the neutron. Generally the isospin quantum number $I$ corresponds to a vector $\vec{I}$ and the projection $I_3$ of $\vec{I}$ ranges from $-I$ to $I$ in integer steps, resulting in $2I + 1$ different values.

If rotations in isospin space leave the strong force invariant then it follows that isospin is conserved in all strong interactions. This symmetry is in fact not exact and only approximate. The strong interaction is regarded as invariant under the symmetric Lie group $SU(2)$. Hadrons are grouped in isospin multiplets when they possess nearly degenerate masses.[4] In terms of the three light quark flavors $u, d, s$ the relation between $I_3$ and the charge of the particle $Q$ is given by the *Gell-Mann-Nishijima formula*:

$$Q = I_3 + \frac{1}{2}(B + S), \tag{2.1}$$

where $B$ denotes the baryon number and $S$ denots the *strangeness*, which quantifies the number of strange quarks present. The component $I_3$ for a hadron can generally be calculated as

$$I_3 = \frac{1}{2}(n_u - n_d), \tag{2.2}$$

where $n_u$ denotes the number of up quarks and $n_d$ denotes the number of down quarks. In term of these quarks the quantum number *hypercharge* $Y$ is defined as $Y = B + S$ [2].

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[4]The term multiplet refers to the representation of a Lie group in a certain vector space, usually irreducible.

Inverting spatial coordinates may produce difference in behaviour, which is described by the *parity operator* $P$. Since two inversions in succession is equivalent to the identity operator $I$ it follows that the eigenvalues for $P$ is $\pm 1$. Hadrons can be classified according to their parity eigenstate. Weak interactions are not parity invariant while strong interactions and electromagnetic interactions are.

The operator converting every particle in a system into its antiparticle is known as *charge conjugation operator* $C$, which also satisfies $C^2 = I$. Hence the eigenvalues are $\pm 1$. Particles that are eigenstates of this operator are rather limited, as these particles have to be their own antiparticle, for example the photon. Among hadrons this eigenvalue is only relevant for mesons consisting of a quark and its corresponding antiquark. As with parity, this operator is not invariant under weak interactions but is invariant under both strong and electromagnetic interactions [4].

## 2.2 Group Theory

The set of rotations forms a Lie group, where each rotation is an element in the Lie group. Each rotation can be expressed as the product of infinitesimal rotations generated by the so-called Lie algebras. Rotations may leave systems unchanged by rotations and in that case the rotation group would also be a symmetry group for that system, i.e. the system is invariant under rotations. These groups can be described using the special unitary group $SU(N)$, where the number of generators are given by $N^2 - 1$. The lowest dimensional representation is $SU(2)$, where the generators may be chosen as

$$J_i = \frac{1}{2}\sigma_i, \quad i = 1, 2, 3, \tag{2.3}$$

where $\sigma_i$ denote the *Pauli matrices*. This is the two-dimensional special unitary group, in accordance with isospin for the nucleon described above where the states were either protons or neutrons. The fundamental representation of this Lie group is composed of the Pauli matrices, which means every other representation may be obtained from them. A common example of a decomposition of a system with two spin-1/2 into a triplet and a singlet. Labeling the states by their dimensions this can be symbolically represented as

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}. \tag{2.4}$$

This is related to the so-called Clebsch-Gordan coefficients.

For $SU(3)$ the eight generators are known as the *Gell-Mann matrices.*, which are $3 \times 3$ matrices. A triplet forms the fundamental representation of this Lie group, additionally the three color charges mentioned in section 1.1 form the fundamental representation of an $SU(3)$ symmetry group known as color $SU(3)$. Furthermore, the three light quark flavors $u, d, s$ also form an $SU(3)$ symmetry group known as flavor $SU(3)$. The corre-

sponding antiquarks are represented by the complex conjugate representation, denoted by $\bar{\mathbf{3}}$.

A meson consist of a quark and an antiquark, with decomposition as

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}. \tag{2.5}$$

As we can observe a meson can be decomposed into an octet and a singlet. This asserts that under operations of the $SU(3)$ group the eight states corresponding to the octet transform among themselves, but are unable to transform into the singlet.

For baryons consisting of the light quarks there are a total of 27 combinations due to the three color charges. Decomposition of two of the quarks is given by

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}, \tag{2.6}$$

where $\mathbf{6}$ is symmetric under interchange of two quarks and $\bar{\mathbf{3}}$ is antisymmetric. Adding the third quark yields the decomposition

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{6} \otimes \mathbf{3}) \oplus (\bar{\mathbf{3}} \otimes \mathbf{3}) = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}, \tag{2.7}$$

hence it consists of a decuplet, two octets and a singlet, adding up to a total of 27 combinations. This way of organizing the hadrons is known as the *eightfold way*.

Taking into account the two different spins per flavor allows representation with symmetry group $SU(6)$, known as *spin-flavor* [1]. This will be used when classifying pentaquarks in Section 3.2

A *Young tableau* is a combinatorial object useful in representation theory as it provides an graphical way to describe irreducible representations of symmetry groups, which correspond to tensors. The notation consists of boxes. A Young tableau corresponding to a tensor with $n$ indices is represented by $n$ boxes. Restrictions are that no row is longer than any row above it and that no more than $N$ boxes occur in a column when studying $SU(N)$. A rank $n$ tensor is represented by row of $n$ boxes if it is symmetric and by a column of $n$ boxes if it is antisymmetric. As a tensor may have mixed symmetry a general tableau will also have this property. As an example the relation (2.4) related to $SU(2)$ written in terms of Young tableau is

$$\square \otimes \square = \square\square \oplus \begin{matrix} \square \\ \square \end{matrix} \tag{2.8}$$

Additional information can be found in Ref. [14].

## 2.3   Quantum Chromodynamics

Quantum chromodynamics (QCD) is the theory of the strong interaction and is mathematically based on the color symmetry group $SU(3)$. QCD examines the color charge

defined in Section 1.1 and studies how this color field affects the interactions between quarks, in analogy to how an electromagnetic field affects electrically charged particles. The strong interaction is mediated by gluons, which carry both a unit of color and a unit of anticolor, hence there is a possibility of nine gluon states. In terms of $SU(3)$ symmetry this is divided into a color octet and a color singlet. As previously mentioned color confinement refers to the observation that naturally occurring particles need to be color singlets. Better insight into exotic hadrons would then improve our understanding concerning the strong interaction and hence of QCD [4].

# 3   Pentaquarks

## 3.1   Experiments

Below particularly important experiments regarding pentaquarks will be expanded upon. One particle of particular interest for studies about pentaquarks is the $J/\psi$ meson, consisting of a charm quark and a charm antiquark ($c\bar{c}$). This was the first particle discovered to contain a charm quark, even though it is heavier than other particles containing charm quarks. It was discovered simultaneously by two different experiments, hence the double name. One of these experiments used collision of electron and positron beams. In these collisions incoming particles annihilate each other which produces a virtual photon. As a consequence of this only particles with the same quantum numbers as photons can be produced, which is $J = 1, P = -1$ and $C = -1$. Following conservation of flavor number charm states can only be produced in pairs, which implies needing a center of mass energy larger than the mass of other known particles containing charm quarks. Additionally, strong interaction decays of $J/\psi$ are suppressed by the Okubo-Zweig-Iizuka rule [15]. This rule explains why certain decay modes are more probable than others. The $J/\psi$ meson has a longer lifetime than expected, which is due to this rule that suppresses certain decays and therefore reduces the number of possible final states. In light of the uncertainty principle $\Delta E \Delta t \leq \frac{\hbar}{2}$, this longer lifetime corresponds to a narrow width in energy distribution or the equivalent mass distribution. In particles physics this is commonly referred to as *resonance width* $\Gamma$ and is related to the *lifetime* $\tau$ as $\Gamma = \frac{\hbar}{\tau}$ [4].

What follows from this rule is that the decays of $J/\psi$ that are most probable are those to leptonic states $e^+e^-$ or $\mu^+\mu^-$, i.e. they have large branching fractions. All this amounts to the $J/\psi$ having a distinctive signature which proved to be central to the study of tetra- and pentaquarks. Resonances show up as peaks in the invariant mass spectrum of the decay products of a particular reaction. This resonant amplitude is in a simple case modeled after the Breit-Wigner distribution and relates the resonant amplitude $A(E)$ to the center of mass energy $E$ of the process, resonance mass $m$ and

resonance width $\Gamma$ as follows

$$A(E) \propto \frac{1}{m^2 - E^2 + im\Gamma}. \tag{3.1}$$

As can be observed this amplitude is a complex number, hence it is suitable to plot this in an Argand diagram[5] [15].

As mentioned previously the successful quark model predicts the existence of exotic hadron structures, such as tetraquarks and pentaquarks [5]. Experimental searches for this had not been successful before the 21st century. There has been continual progress the last 16 years. Numerous experiments were carried out, some of which used B meson decays and some of which used collisions of electron and positron beams. A comprehensive review up until the start of 2019 which also includes a more detailed overview of methods for detection can be found in Ref. [16]. It was not until 2015 that the existence of pentaquarks where confirmed at the LHCb experiment at CERN.

At LHCb large amounts of $\Lambda_b^0 \to J/\psi K^- p$ decays are recorded, i.e., decays of a $\Lambda_b^0$ bottom lambda baryon ($bud$) into a $J/\psi$ meson ($c\bar{c}$), a $K^-$ kaon ($s\bar{u}$) and a proton $p$ ($uud$). However in 2015, this was found to be able to decay not only via the expected dominating $\Lambda^* \to K^- p$ decays[6], but also via intermediate pentaquark states. Feynman diagrams for these decays are found in Figure 1.



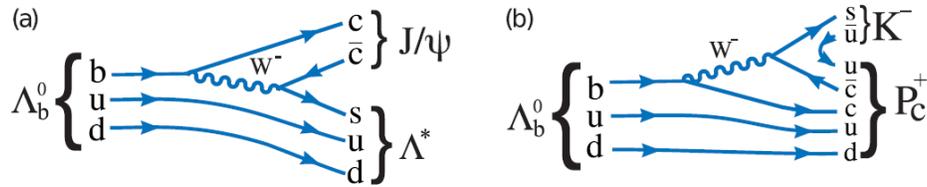Figure 1: Feynman diagrams for (a) $\Lambda_b^0 \to J/\psi \Lambda^*$ and (b) $\Lambda_b^0 \to P_c^+ K^-$. Figure has been taken from Ref. [12].

Strong decays into $J/\psi\, p$ must have a minimal quark content of $c\bar{c}uud$. To confirm that the states observed are in fact pentaquark states and not caused by the lambda states a thorough analysis is needed. This is done by analyzing the resonance amplitudes.

Initially the observed data were fit with an amplitude model corresponding to the known $\Lambda^*$ states. This proved to be unsatisfactory, which lead scientists to add a pentaquark state $P_c(4380)^+$.[7] However, this was also not satisfactory and so a second pentaquark state with larger mass $P_c(4450)^+$ was added, which proved to be a desirable fit. The statistical significances of the lower mass and higher mass states were 9 and 12,

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[5]Perhaps more commonly known as the complex plane.

[6]The * denotes excited states of $\Lambda$.

[7]The number denotes the mass of the hadron in natural units, i.e this structure has mass 4380 MeV.

respectively, which implies a combined significance of 15. Data and corresponding fits are found in Figure 2.
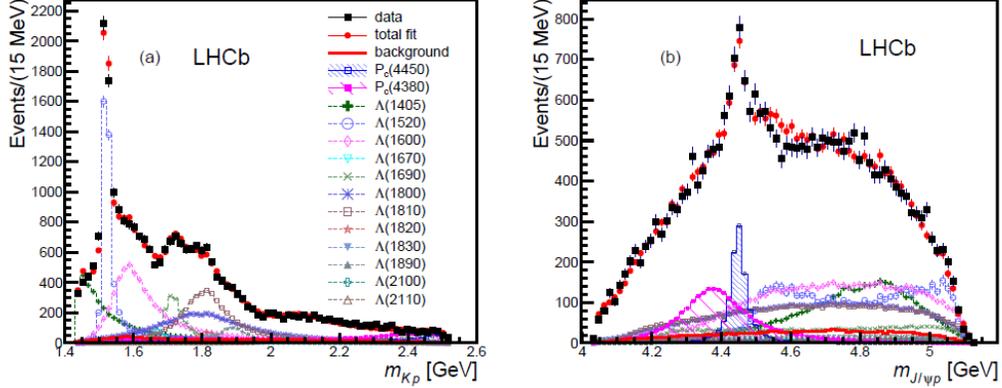


Figure 2: Fits for invariant mass of $m_{K^-p}$ to left and $m_{J/\psi p}$ to right, using the $\Lambda^*$ model with added pentaquark states. Figure has been taken from Ref. [12]. Note how the peaking structure is reproduced by the fit. Fits without the added pentaquark states do not reproduce the peaking behaviour for $m_{J/\psi p}$ and can also be found in Ref. [12].

Knowledge on the quantum numbers of these discovered pentaquarks would provide crucial information about their structure and nature. Despite the statistically significant results acceptable fits are possible with different quantum numbers. The best fit solution has spin-parity $J^P$ $(3/2^-, 5/2^+)$, where the first represents $P_c(4380)^+$ and the second $P_c(4450)^+$. However, acceptable fits are still possible with $(3/2^+, 5/2^-)$ and $(5/2^+, 5/2^-)$.

As mentioned previously Argand diagrams are useful for analyzing resonances. This is found for both pentaquarks states in Figure 3. For the $P_c(4450)^+$ state this is seen to be exhibiting the expected resonant behaviour. For $P_c(4380)^+$ it is not as apparent, which might be a consequence of the amplitude being very sensitive to details in the model. This state also possesses a larger resonance width than $P_c(4450)^+$ [12].

Additional results in 2019 shed further light unto this topic, as LHCb announced new discovered pentaquark states, presented in Ref. [17] This analysis included nine times more $\Lambda_b^0 \to J/\psi K^- p$ decays than the previously discussed result. This additional data were fitted with the same amplitude model as in 2015 and the results were found to be consistent with the previous data regarding $P_c(4380)^+$ and $P_c(4450)^+$. However, analysis of this much larger data samples reveals previously undetected structure. A narrow peak is observed at 4312 MeV. The peak corresponding to $P_c(4450)^+$ can now be distinguished as in fact being two narrow peaks at 4440 MeV and 4457 MeV, respectively. The fit to
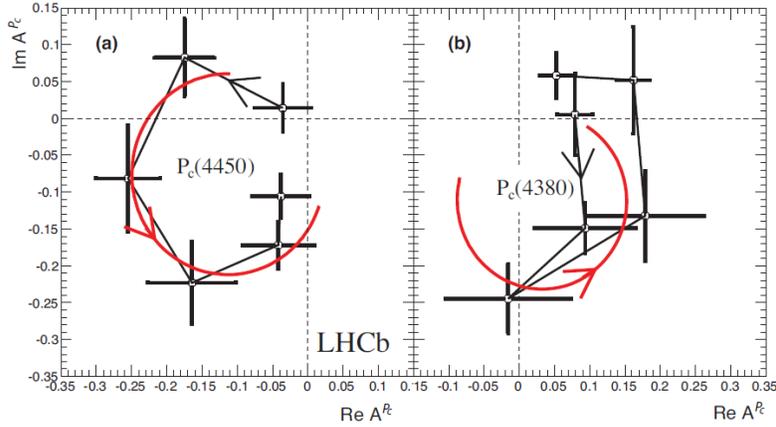
Figure 3: Fits for the real and imaginary parts of the amplitude. Red curves represent the prediction from the Breit-Wigner distribution. The six black bins represent the collected data.

data can be seen in Figure 4. Notice how the peak corresponding to $P_c(4380)^+$ is not found. This is because the fits using this state in the model and the ones not using it both describe the data well, hence this can neither confirm nor disprove the existence of this exotic hadron. Superimposed is the mass threshold for the states $\Sigma_c^+ \bar{D}^0$ and $\Sigma_c^+ \bar{D}^{*0}$, which has quark content $(udcu\bar{c})$ and $(udcu\bar{c})$. Due to the close proximity to the supposed pentaquark these are good candidates for in fact being bound states of these structures. Also different spin-parity quantum numbers give good fits while slightly affecting the binding energy of the structures. This might entail that these pentaquark states might be $\Sigma_c^+ \bar{D}^0$ and $\Sigma_c^+ \bar{D}^{*0}$ with some additional binding energy and it provides the strongest experimental evidence to date for possible existence of bound states of a baryon and a meson [17].

## 3.2   Mathematical Classification

In this section we will here discuss a group theoretic classification of pentaquark states on the form $(qqqc\bar{c})$, which are related to the now validated pentaquarks. Here $q$ denotes light quarks of the flavor up, down or strange. The internal degrees of freedom consists of flavor triplet, flavor anti-triplet, spin doublet and a color triplet. The corresponding algebraic structure for spin-flavor and color triplet is $SU_{sf}(6) \otimes SU_c(3)$. Spin-flavor is categorized by the three light quark flavors and their two possible spins. The spin-flavor can be decomposed into the following, giving a more in depth view of the mathematical
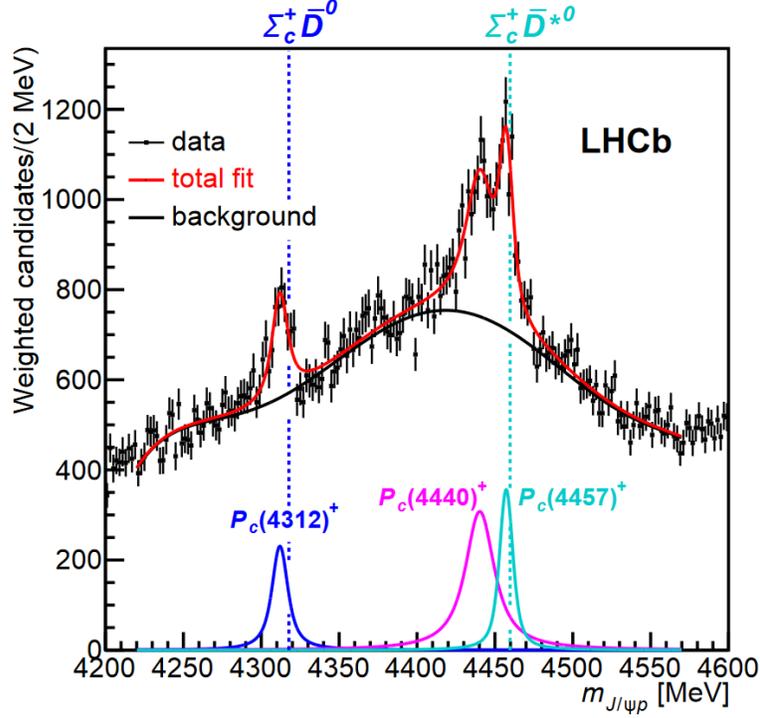
Figure 4: This weighted fit is used to determine the masses and resonance widths of the $P_c^+$ pentaquark states. The mass threshold for $\Sigma_c^+ \bar{D}^0$ and $\Sigma_c^+ \bar{D}^{*0}$ are superimposed, revealing that these hadrons may have a role in the bound states of the pentaquark structures. Figure has been taken from Ref. [17]

structure:

$$SU_{sf}(6) \supset SU_f(3) \otimes SU_s(2) \supset SU_I(2) \otimes U_Y(1) \otimes SU_s(2), \tag{3.2}$$

where $f$ denotes flavor, $s$ spin, $I$ isospin and $Y$ hypercharge [19]. The combination $c\bar{c}$ can be in either a color octet or color singlet, with spin 0 or spin 1. As a consequence the color wave function of the pentaquark should, as all physical states, be an $SU_c(3)$ singlet, i.e. a color singlet. Hence the remaining three quarks have to be in either a color singlet or color octet. As the pentaquark has to be a color singlet, this creates some constraints on the algebraic structure. A consequence of the Pauli exclusion principle implies that the wave function must have a spin-flavor part and a color part that are conjugated. It follows that the spin-flavor state is symmetric if the color part is a singlet or in a mixed symmetric state if the color part is an octet [20]. In the following we will use the notation $[f_1, f_2, ..., f_n]_d$, where $f_i$ denotes the number of boxes in the $i$-th row of the Young tableau and $d$ denotes the dimension of the representation. If one denotes the

fundamental representation of $SU(6)$ by $\mathbf{6}$, then the decomposition of the three quarks is

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56}_S \oplus \mathbf{70}_M \oplus \mathbf{70}_M \oplus \mathbf{20}_A, \qquad (3.3)$$

where $S$ denotes a symmetric state, $M$ a mixed symmetric state and $A$ an anti-symmetric state. Hence the allowed spin-flavor configurations are the 56-plet $\mathbf{56}_S([3]_{56})$ and the 70-plet $\mathbf{70}_M([21]_{70})$. As mentioned above the spin-flavor part can be decomposed according to $SU_{sf}(6) \supset SU_f(3) \otimes SU_s(2)$. The 70-plet contains a flavor singlet $[111]_1$, two octets $[21]_8$ and decuplet $[3]_{10}$, while the 56-plet contains an $SU_f(3)$ flavor octet $[21]_8$ and a decuplet $[3]_{10}$. This shows there are three allowed flavor representations. More details can be found in Ref. [18]. Our analysis is most concerned with the pentaquark composition $uudc\bar{c}$ discovered at LHCb and will as such focus on that specific configuration. Recognizing the relevant quark flavors reduces the hypercharge $Y$ to be defined as $Y = B + S$, which was mentioned in section 2.2. No strange quarks are presented and hence $S = 0$. Baryon number is 1, since we have three quarks, which leads to the hypercharge having the value $Y = 1$. We are therefore looking for flavor states with $Y = 1$. This excludes the singlet $[111]_1$, as it does not have any appropriate submultiplets. As such there are two possible flavor states, $[21]_8$ and $[3]_{10}$, additional discussion can be found in Ref. [20].

A mass formula for baryons related to the $SU(6)$ symmetry is presented by Gürsey and Radicati in Ref. [21]. In order to dicuss the mass of the discovered pentaquarks some modifications are made to the formula presented by Gürsey and Radicati. A simplified model extension is given by [20]:

$$M = M_0 + AS(S + 1) + DY + E\left[I(I + 1) - \frac{1}{4}Y^2\right] + GC_2(SU(3)) + FN_C, \qquad (3.4)$$

where $M_0$ is scale factor, related to the number of particles making up the hadron. Furthermore, $S$ is the hadron's spin, $I$ is the isospin, $Y$ is the hypercharge, $C_2$ is the eigenvalue of the $SU_f(3)$ Casimir operator and $N_C$ counts the number of charm quarks or antiquarks. Due to the lack of experimental results the parameters are determined from known baryon spectrums. Parameters and corresponding uncertainties are presented in Table 1.

|                    | $M_0$ | $A$  | $D$      | $E$  | $G$  | $F$    |
|--------------------|-------|------|----------|------|------|--------|
| Value [MeV]        | 940.0 | 23.0 | $-158.3$ | 32.0 | 52.5 | 1354.6 |
| Uncertainties [MeV]| 1.5   | 1.2  | 1.3      | 1.3  | 1.3  | 18.2   |

Table 1: Parameters for the generalized Gürsey-Radicati mass formula taken from Ref. [20].

As discussed in Section 3.1 different values of spin-parity were satisfactory. We will attempt to analyze the lightest pentaquark state and hypothesize that this state has

$J = 3/2$. The mass splitting between the two allowed multiplets $[21]_8$ and $[3]_{10}$ arises due to the different eigenvalues of the $SU_f(3)$ Casimir operator. For constructing the lightest state we want to choose the multiplets with the lowest eigenvalue, which in this case is the octet $[21]_8$ with eigenvalue 3, as the decuplet $[3]_{10}$ has eigenvalue 6. Using the given parameters and quantum numbers of the observed pentaquark yields the predicted mass $M = 4377 \pm 49$ MeV. Compared to the observed value $M = 4380 \pm 8 \pm 29$ MeV this is in agreement, which is interesting considering the simplistic approach. since this pentaquark, due to the discussion above, is expected to belong to a color octet other predicted states can also be analyzed. These will not necessarily possess the same quantum numbers. See Figure 5 for a graphical overview of these octet states [20].
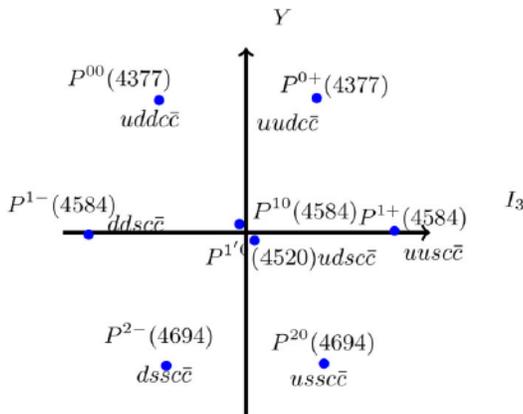


Figure 5: States are labeled as $P^{ij}(M)$, where $i$ denotes number of strange quarks, $j$ denotes the charge of the pentaquark and $M$ denotes the predicted mass. The axes correspond to isospin $I_3$ and hypercharge $Y$. Figure has been taken from Ref. [20]

## 4 Analysis

### 4.1 Method

In this section we will perform a fit of the generalized Gürsey-Radicati mass formula, presented in section 3.2, which reads

$$M = M_0 + AS(S+1) + DY + E\left[I(I+1) - \frac{1}{4}Y^2\right] + GC_2(SU(3)) + FN_C. \quad (4.1)$$

We will first calculate the coefficients with the same data used in Ref. [20] and compare the results. As they do not mention the method used to obtain these parameters it is

challenging to determine how they obtained those values. We can, however, compare our results with theirs.

As of now, as there are only three experimentally known pentaquarks containing $c\bar{c}$ there is not enough information to determine the parameters $M_0, A, D, E, G, F$. Thus we will use the values from the known baryon spectrum and assume the coefficients are the same for the pentaquark states, which is the same approach used in Ref. [20]. Values are listed in Table 2.[8] The parameters fitted in Ref. [20] can be found in Table 1.

| Baryon | Experimental mass [MeV] | Experimental error [MeV] | $SU_f(3)$ multi-plet | $C_2(SU(3))$ | $S$ | $Y$ | $I$ | $N_c$ |
|---|---|---|---|---|---|---|---|---|
| $N(940)$ | 939.565413 | $10^{-6}$ | $[21]_8$ | 3 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 |
| $\Lambda^0(1116)$ | 1115.683 | 0.006 | $[21]_8$ | 3 | $\frac{1}{2}$ | 0 | 0 | 0 |
| $\Sigma^0(1193)$ | 1192.642 | 0.024 | $[21]_8$ | 3 | $\frac{1}{2}$ | 0 | 1 | 0 |
| $\Xi^0(1315)$ | 1314.86 | 0.2 | $[21]_8$ | 3 | $\frac{1}{2}$ | $-1$ | $\frac{1}{2}$ | 0 |
| $\Delta^0(1232)$ | 1232 | 2 | $[3]_{10}$ | 6 | $\frac{3}{2}$ | 1 | $\frac{3}{2}$ | 0 |
| $\Sigma^{*0}(1385)$ | 1383.7 | 1.0 | $[3]_{10}$ | 6 | $\frac{3}{2}$ | 0 | 1 | 0 |
| $\Xi^{*0}(1530)$ | 1531.80 | 0.32 | $[3]_{10}$ | 6 | $\frac{3}{2}$ | $-1$ | $\frac{3}{2}$ | 0 |
| $\Omega^-(1672)$ | 1672.45 | 0.29 | $[3]_{10}$ | 6 | $\frac{3}{2}$ | $-2$ | 0 | 0 |
| $\Lambda_c^+(2286)$ | 2286.46 | 0.14 | $[11]_3$ | $\frac{4}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | 0 | 1 |
| $\Sigma_c^0(2455)$ | 2453.75 | 0.14 | $[2]_6$ | $\frac{10}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | 1 | 1 |
| $\Xi_c^0(2471)$ | 2470.85 | $\begin{smallmatrix}-0.40\\+0.28\end{smallmatrix}$ | $[11]_3$ | $\frac{4}{3}$ | $\frac{1}{2}$ | $-\frac{1}{3}$ | $\frac{1}{2}$ | 1 |
| $\Xi_c^{0'}(2576)$ | 2577.9 | 2.9 | $[2]_6$ | $\frac{10}{3}$ | $\frac{1}{2}$ | $-\frac{1}{3}$ | $\frac{1}{2}$ | 1 |
| $\Omega_c^0(2695)$ | 2695.2 | 1.7 | $[2]_6$ | $\frac{10}{3}$ | $\frac{1}{2}$ | $-\frac{4}{3}$ | 0 | 1 |
| $\Omega_c^{*0}(2770)$ | 2765.9 | 2.0 | $[2]_6$ | $\frac{10}{3}$ | $\frac{3}{2}$ | $-\frac{4}{3}$ | 0 | 1 |
| $\Sigma_c^{*0}(2520)$ | 2518.48 | 0.2 | $[2]_6$ | $\frac{10}{3}$ | $\frac{3}{2}$ | $\frac{2}{3}$ | 1 | 1 |
| $\Xi_c^{*0}(2645)$ | 2649.9 | 0.5 | $[2]_6$ | $\frac{10}{3}$ | $\frac{3}{2}$ | $-\frac{1}{3}$ | $\frac{1}{2}$ | 1 |

Table 2: Mass spectrum of selected baryons used to determine parameters in formula (3.4). All values are taken from Ref. [20], where the data comes from the Particle Data Group Review from 2016.

In finding the best fitting parameter values we will be using the Chi-squared test which tests for goodness of fit. The best fitting parameter values can be found by minimizing the Chi-squared sum with respect to the parameters in the mass formula (3.4). More precisely, we want to minimize

$$\chi^2 = \sum_i \left( \frac{M_i - M_i^{\mathrm{GR}}(M_0, A, D, E, G, F)}{\sigma_i} \right)^2,$$

[8]In the Particle review from 2016 it is evident that the experimental error for $N(940)$ is actually $6 \cdot 10^{-6}$ MeV and not $10^{-6}$ MeV. The mass value for $\Xi_c^{*0}(2645)$ is actually $2645.9 \pm 0.5$ MeV

where $M_i$ denotes the masses of the baryons listed in Table 2, $M_i^{\mathrm{GR}}$ denotes the predicted masses from the generalized Gürsey-Radicati mass formula (3.4) and $\sigma_i$ denotes the experimental errors listed in Table 2. Further information on statistics can be found in Ref. [22]. Firstly, a few things can be observed. Since the Chi-squared sum weighs the terms based on their experimental errors we expect the minimizing parameter values to give an accurate value for the mass of $N(940)$, which has the lowest experimental error by far. Furthermore, the mass formula is linear and is therefore computationally easy to fit.

Calculating the Chi-squared sum for the parameters values listed in Table 1 and the data listed in Table 2 we obtain $\chi^2 \approx 10^{15}$, which we acknowledge is a very large value for the Chi-squared sum. As the authors of Ref. [20] do not mention the method used it is difficult to say why this value is so large.

### 4.1.1  Algorithm

In computing these parameters the basin-hopping algorithm will used. Basin hopping is a stochastic algorithm which attempts to find the global minimum of a scalar function of one or several variables. The algorithm performs local minimization at each step and the local minimization method used is the Nelder-Mead method. As this algorithm is stochastic it will not return the exact same result each simulation. [9] The uncertainty is approximated as the standard deviation related to the determined parameters. Because this method is stochastic several simulations will be run and using these results one can calculate both the average values of the parameters and their respective standard deviations.

## 4.2  Numerical Fits

In all fits the resulting parameter values were known to high accuracy. Running several simulations it was evident that the deviation of the parameters values are of the order $10^{-2}$ or smaller. This results in non-significant errors for the pentaquark masses since this is a simple model and we are interested in the approximate mass ranges we might expect to find these structures. For this reason the standard deviations for the parameter values are not presented. Furthermore, note that the value of the Chi-squared sum is sensitive to changes in the parameters since the masses of the baryons are known to great accuracy. The parameter values are returned with eight decimals and these values are stored for calculating the pentaquark masses. However, the values presented in the tables are rounded to three decimals. Each pentaquark state will be assumed to have $S = \frac{3}{2}$.

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[9]More info on the function used can be found at `https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.basinhopping.html`

### 4.2.1 Comparison of Fits

In this section we will perform a numerical fit using the basin-hopping algorithm with the same data used in Ref. [20]. The smallest value of the Chi-squared sum obtained was $\chi^2 = 121993$. The values of the fitted parameters are found in table 3.

| | $M_0$ | $A$ | $D$ | $E$ | $G$ | $F$ |
|---|---|---|---|---|---|---|
| Value [MeV] | 980.805 | 17.085 | $-195.143$ | 38.075 | 40.684 | 1377.914 |

Table 3: Parameter values obtained using the same data as in Ref. [20].

In particular, this fit gives the mass of the $N(940)$ baryon as 939.565413 MeV, while the parameter values listed in Table 1 gives the mass as 972.45 MeV. This partially explains why the Chi-squared sum for that fit is so large. The pentaquark masses obtained from this fit is listed in Table 4.

| Predicted pentaquarks | Masses [MeV] | $S$ | $Y$ | $I$ | Masses Ref. [20] [MeV] |
|---|---|---|---|---|---|
| $uudc\bar{c}, uddc\bar{c}$ | 4400.519 | $\frac{3}{2}$ | 1 | $\frac{1}{2}$ | 4377 |
| $dssc\bar{c}, ussc\bar{c}$ | 4790.805 | $\frac{3}{2}$ | $-1$ | $\frac{1}{2}$ | 4694 |
| $ddsc\bar{c}, uusc\bar{c}$ | 4652.774 | $\frac{3}{2}$ | 0 | 1 | 4584 |
| $udsc\bar{c}$ | 4576.624 | $\frac{3}{2}$ | 0 | 0 | 4520 |
| $udsc\bar{c}$ (Excited state) | 4652.774 | $\frac{3}{2}$ | 0 | 1 | 4584 |

Table 4: Predicted pentaquark states with the data from Table 2. The excited state of $udsc\bar{c}$ has isospin $I = 1$ instead of $I = 0$.

We can observe that the predicted masses from this method are slightly larger than the ones presented by Santopinto and Giachino in Ref. [20].

### 4.2.2 Updated Data

In this section updated data for the baryons will be used. As the previous fit used the ground-state baryons mass spectrum from the 2016 Review of Particle Physics we will use the updated 2018 review, found in Ref. [23]. The masses and their experimental errors for the baryons are mostly the same, expect for $\Xi_c^0(2471) = (2470.87^{+0.28}_{-0.31})$ MeV, $\Xi_c^{'0}(2576) = (2578.8 \pm 0.5)$ MeV and $\Xi_c^{*0}(2645) = (2646.32 \pm 0.31)$ MeV and the mass of for $N(940)$, which is $939.5654133 \pm 5.8 \cdot 10^{-6}$ MeV. The Chi-squared sum obtained has the value $\chi^2 = 129831$. The values of the fitted parameters are found in table 5.

In particular, this fit gives the mass of the $N(940)$ baryon as 939.5654133 MeV. The pentaquark masses obtained from this fit is listed in Table 6

| | $M_0$ | $A$ | $D$ | $E$ | $G$ | $F$ |
|---|---|---|---|---|---|---|
| Value [MeV] | 980.303 | 17.085 | $-195.133$ | 38.069 | 40.845 | 1376.311 |

Table 5: Parameter values obtained using updated data from the 2018 Review of Particle Physics, found in Ref. [23].

| Predicted pentaquarks | Masses [MeV] | $S$ | $Y$ | $I$ |
|---|---|---|---|---|
| $uudc\bar{c}, uddc\bar{c}$ | 4396.978 | $\frac{3}{2}$ | 1 | $\frac{1}{2}$ |
| $dssc\bar{c}, ussc\bar{c}$ | 4787.243 | $\frac{3}{2}$ | $-1$ | $\frac{1}{2}$ |
| $ddsc\bar{c}, uusc\bar{c}$ | 4649.214 | $\frac{3}{2}$ | 0 | 1 |
| $udsc\bar{c}$ | 4573.076 | $\frac{3}{2}$ | 0 | 0 |
| $udsc\bar{c}$ (Excited state) | 4649.214 | $\frac{3}{2}$ | 0 | 1 |

Table 6: Predicted pentaquark states using updated data from the 2018 Review of Particle Physics.

### 4.2.3 Fit without $N(940)$

As previously mentioned the term in the Chi-squared sum corresponding to the $N(940)$ baryon might dominate due to its low experimental error. For this reason we examine the results obtained if we exclude this baryon from the data set. Naturally the updated data from 2018 was used for this fit. The Chi-squared sum obtained has the value $\chi^2 = 111109$, which we see is smaller than the previous values. The values of the fitted parameters are found in table 7.

| | $M_0$ | $A$ | $D$ | $E$ | $G$ | $F$ |
|---|---|---|---|---|---|---|
| Value [MeV] | 971.161 | 19.495 | $-179.184$ | 38.035 | 43.315 | 1367.970 |

Table 7: Parameter values obtained using updated data from 2018, while excluding the $N(940)$ baryon.

This fit gives the mass of the $N(940)$ baryon as 955.562 MeV. As expected this value is not very close to the actual mass. The pentaquark masses obtained from this fit is listed in Table 8

### 4.2.4 Data with 1% Error

In this section we will perform a fit using the same data as in Section 4.2.2. However, instead of using the listed experimental error we will consider the errors being 1% of the mass value and examine how this affects the parameter values. The Chi-squared sum obtained has the value $\chi^2 = 25.4361$, which as expected is significantly smaller than the

| Predicted pentaquarks | Masses [MeV] | $S$ | $Y$ | $I$ |
|---|---|---|---|---|
| $uudc\bar{c}, uddc\bar{c}$ | 4397.427 | $\frac{3}{2}$ | 1 | $\frac{1}{2}$ |
| $dssc\bar{c}, ussc\bar{c}$ | 4755.794 | $\frac{3}{2}$ | $-1$ | $\frac{1}{2}$ |
| $ddsc\bar{c}, uusc\bar{c}$ | 4633.664 | $\frac{3}{2}$ | 0 | 1 |
| $udsc\bar{c}$ | 4557.593 | $\frac{3}{2}$ | 0 | 0 |
| $udsc\bar{c}$ (Excited state) | 4633.664 | $\frac{3}{2}$ | 0 | 1 |

Table 8: Predicted pentaquark states using the updated data from 2018 while excluding the $N(940)$ baryon.

previous values. The values of the fitted parameters are found in table 9.

| | $M_0$ | $A$ | $D$ | $E$ | $G$ | $F$ |
|---|---|---|---|---|---|---|
| Value [MeV] | 965.349 | 21.778 | $-175.319$ | 22.545 | 46.145 | 1365.536 |

Table 9: Parameter values obtained using updated data from 2018 with the experimental error defined as 1% of the mass for every baryon.

This fit gives the mass of the $N(940)$ baryon as 956.070 MeV. The pentaquark masses obtained from this fit is listed in Table 10. These pentaquark masses varies in regard to the previous fits. However, there is not a very large difference.

| Predicted pentaquarks | Masses [MeV] | $S$ | $Y$ | $I$ |
|---|---|---|---|---|
| $uudc\bar{c}, uddc\bar{c}$ | 4396.043 | $\frac{3}{2}$ | 1 | $\frac{1}{2}$ |
| $dssc\bar{c}, ussc\bar{c}$ | 4746.681 | $\frac{3}{2}$ | $-1$ | $\frac{1}{2}$ |
| $ddsc\bar{c}, uusc\bar{c}$ | 4605.179 | $\frac{3}{2}$ | 0 | 1 |
| $udsc\bar{c}$ | 4560.089 | $\frac{3}{2}$ | 0 | 0 |
| $udsc\bar{c}$ (Excited state) | 4605.179 | $\frac{3}{2}$ | 0 | 1 |

Table 10: Predicted pentaquark states using the updated data from 2018 with the experimental error being 1% of the mass for every baryon.

## 4.3   Pentaquarks Containing Bottom Quarks

In this section the generalized mass formula (3.4) will be modified to account for the possibility of bottom quarks included in pentaquark states. As this formula has a variable $N_C$ corresponding to the number of charm and anticharm quarks one can replace this

with a variable $N_B$ counting the number of bottom and antibottom quarks, yielding

$$M = M_0 + AS(S+1) + DY + E\left[I(I+1) - \frac{1}{4}Y^2\right] + GC_2(SU(3)) + FN_B. \quad (4.2)$$

Apart from the variable $N_C$ the discussion leading up to the mass formula was general enough for this to also apply for other similar pentaquark states, and hence the modification performed above is reasonable. We will now, in analogy to the previous section, fit these parameters. Again we will be looking for pentaquarks belonging to the $[21]_8$ color octet. In calculating these parameters the eight baryons not containing charm quarks in Table 2 will be used and additionally five baryons containing bottom quarks. The reason for taking only five baryons is because there is an uncertainty in the properties of these baryons and hence a few of the most researched are included. These additional baryons are presented in Table 11.[10] In calculating the hypercharge $Y$ a more general formula than the ones presented previously is used, which is found in Ref. [23]:

$$Y = B + S - \frac{C - B' + T}{3}, \quad (4.3)$$

where $B$ is the baryon number, $S$ strangeness, $C$ charmness, $B'$ bottomness and $T$ topness. These values reveal how many of the corresponding quark flavors are present in a specific hadron. The Chi-squared sum obtained has the value $\chi^2 = 99254$. The resulting parameter values can be found in Table 12.

| Baryon | Experimental mass [MeV] | Experimental error [MeV] | $SU_f(3)$ multi- plet | $C_2(SU(3))$ | $S$ | $Y$ | $I$ | $N_B$ |
|--------|-------------------------|--------------------------|-----------------------|--------------|-----|-----|-----|-------|
| $\Lambda_b^0$ | 5619.60 | $\pm 0.17$ | $[11]_3$ | $\frac{4}{3}$ | $\frac{1}{2}$ | $\frac{4}{3}$ | $0$ | $1$ |
| $\Xi_b^0$ | 5791.9 | $\pm 0.5$ | $[11]_3$ | $\frac{4}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $1$ |
| $\Sigma_b^+$ | 5811.3 | $^{+0.9}_{-0.8} \pm 1.7$ | $[2]_6$ | $\frac{10}{3}$ | $\frac{1}{2}$ | $\frac{4}{3}$ | $1$ | $1$ |
| $\Sigma_b^{*+}$ | 5832.1 | $\pm 0.7^{+1.7}_{-1.8}$ | $[2]_6$ | $\frac{10}{3}$ | $\frac{3}{2}$ | $\frac{4}{3}$ | $1$ | $1$ |
| $\Omega_b^-$ | 6046.1 | $\pm 1.7$ | $[2]_6$ | $\frac{10}{3}$ | $\frac{1}{2}$ | $-\frac{2}{3}$ | $0$ | $1$ |

Table 11: Mass spectrum of bottom baryons used to determine parameter values in formula (4.2). All values are taken from Ref. [23]. For the sigma baryons the experimental errors in the optimization were taken as the square root of the sum of the variances.

Using these parameters we can calculate the predicted masses of the pentaquark octet states containing $b\bar{b}$. Each state will be assumed to have spin $S = J = \frac{3}{2}$. The results are listed in Table 13.

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[10]To determine the $SU_f(3)$ multiplet we used the fact that baryons containing bottom baryons essentially act the same as charm baryons with just the $c$ and $b$ interchanged, which is stated in the 2018 Review of Particle Physics, found in Ref. [23].

| | $M_0$ | $A$ | $D$ | $E$ | $G$ | $F$ |
|---|---|---|---|---|---|---|
| Value [MeV] | 1002.703 | 18.305 | $-195.154$ | 37.992 | 33.098 | 4829.411 |

Table 12: Parameter values obtained for the modified mass formula (4.2) related to bottom quarks.

| Predicted pentaquarks | Masses [MeV] | $S$ | $Y$ | $I$ |
|---|---|---|---|---|
| $uudb\bar{b}, uddb\bar{b}$ | 11321.769 | $\frac{3}{2}$ | 1 | $\frac{1}{2}$ |
| $dssb\bar{b}, ussb\bar{b}$ | 11712.078 | $\frac{3}{2}$ | $-1$ | $\frac{1}{2}$ |
| $ddsb\bar{b}, uusc\bar{c}$ | 11573.911 | $\frac{3}{2}$ | 0 | 1 |
| $udsb\bar{b}$ | 11497.928 | $\frac{3}{2}$ | 0 | 0 |
| $udsb\bar{b}$ (Excited state) | 11573.911 | $\frac{3}{2}$ | 0 | 1 |

Table 13: Predicted pentaquark states containing bottom quarks. The excited state of $udsb\bar{b}$ has isospin $I = 1$ instead of $I = 0$.

# 5    Discussion

One significant difficulty regarding research of pentaquark states is the fact that relevant experiments other than LHCb at CERN are limited. As with all discoveries it is desirable to have several independent experiments to confirm the discovery. Since the progress in the study of these states has been very rapid the last decade we can reasonably expect more results from experiments other than LHCb [15].

The discussion in Section 4 predicts both pentaquarks containing $c\bar{c}$ or $b\bar{b}$. As LHCb has already discovered pentaquarks containing $c\bar{c}$ there will likely be more states with this composition to be discovered in the future. The pentaquarks containing $b\bar{b}$ has a larger mass and might be more difficult to experimentally detect. There are less discovered baryons containing bottom quarks than containing charm quarks, which is 9 and 24, respectively. An overview of this can be found in Ref. [23]. The generalized mass formula used for predicting the masses is a simplistic model that was extended to account for mass splitting between different $SU_f(3)$ multiplets. Additional research from different perspectives on the topic of the general spectroscopy of pentaquarks, their mathematical classification and other properties can be found in Ref. [24], [25], [26], [27], [28] and [29].

## 5.1    Future Research

Naturally there is large potential for further research into this topic, as we are only starting to explore pentaquark states. An important question to study is how these

pentaquarks are bound together. As the recent paper Ref. [17] shows, the contemporarily known pentaquark states might be a bound state of a baryon with a meson, particularly the states $\Sigma_c^+ \bar{D}^0$ and $\Sigma_c^+ \bar{D}^{*0}$ appear to play an important role in the dynamics of these states.

Several hundreds of conventional mesons and baryons are known, as opposed to tens of multiquark candidates. Much research is being performed in particle physics to better understand these exotic states, since they allow for deeper understanding into the strong force while also providing insight into quantum chromodynamics. Theoretical predictions using, among others, quantum chromodynamics and symmetries are being made in order to examine which bound multiquark states might be possible. However, this field has been expanded by experimental results and will likely continue on that path. Hence the experimental prospects are of specific interest [16].

Up until this point all established multiquark states have been found to contain either $(c\bar{c})$ or $(b\bar{b})$. So far it is not apparent if this is a necessary condition for exotic hadrons [15].

If the discussion in Section 3.2 is correct then we might expect to find other states presented in the pentaquark octet. For the lightest discovered pentaquark the generalized mass formula gave a satisfactory answer that was in agreement with the experiment. As such, looking for the other pentaquarks in the octet with their predicted compositions and masses would be a reasonable starting point for future experiments. This applies to pentaquarks containing $c\bar{c}$ or $b\bar{b}$. It is worth noting that if the pentaquark is indeed a molecular state not all of these octet states may be allowed. In addition the classification can also be done for the decuplet $[3]_{10}$ and singlet $[111]_1$ for predictions of more pentaquark structures as well as using different predicted spin for the structures. Here the ground state was of primary interest, but naturally one can look for excited states of pentaquarks.

Perhaps the most interesting fit is the one which contains all the data. The question is how well this formula succeeds in predicting the pentaquark masses. The mass formula (3.4) is as mentioned a simple formula for predicting the pentaquark masses. Interestingly, the fit not using the $N(940)$ baryon did not have a much lower Chi-squared sum value than the fit using the updated data, which at first glance seemed unexpected. This is most likely due to the fact that the updated fit gives a value for the mass of $N(940)$ that is very close to actual value and, as such, the corresponding term in the Chi-squared sum is not very large. The predicted values of the mass values vary slightly in the different fits. It is worth noting that the masses of many baryons are not within the range of their experimental values. This is unsurprisingly most visible in the fit where the error was defined to be 1%, even if it is evident in all of these fits. Interestingly enough, the mass value of the confirmed pentaquark $uudc\bar{c}$ seems to have similar values in all of these fits, which is approximately 4400 MeV. As for the pentaquarks containing

bottom quarks there is not much to be said about these since there is no experimental verification of these structures existing. However, if these types of structures do exist, we might expect to find them in the mass ranges listed, provided that they belong to the color octet examined in this report. Due to their masses being considerably large it might be difficult to experimentally detect them.

Going forward it is important to link the theory with experiments to better understand the underlying physics. This will aid in more insight into the strong force, quantum chromodynamics and the Standard Model.

# Acknowledgements

# A    Appendix

A short overview of relevant mathematical concepts is presented below.

**Definition 1.** *A group $G$ is a set equipped with a binary operation $*$ that satisfies the axioms:*

1. *If $a, b \in G$, then $a * b \in G$.*

2. *Binary operation $*$ is associative.*

3. *There exists an identity element $e$ in $G$, such that $a * e = e * a = a$ for every $a \in G$.*

4. *For every element $a \in G$, there is a element $a^{-1}$, such that $a * a^{-1} = a^{-1} * a = e$.*

Groups are frequently used to describe algebraic structures which occur in nature, and specifically, physics. These structures include various geometric transformations and notions of symmetry. Another relevant object is the *subgroup*.

**Definition 2.** *Let $G$ be a group with identity element $e$ and binary operation $*$ If $H$ is a subset of the elements of the group $G$ and satisfies the axioms:*

1. *If $a, b \in H$, then $a * b \in H$.*

2. *For every element $a \in H$, there is a element $a^{-1}$, such that $a * a^{-1} = a^{-1} * a = e$,*

*then $H$ is called a subgroup of $G$ and is denoted as $H \subset G$*

Further information can be found in Ref. [14].

For the purpose of making group elements and operations more manageable, representing each element with something more concrete and familiar would be beneficial.

**Definition 3.** *If each element $X$ of a group $G$ can be assigned a non-singular $n \times n$ matrix $\Gamma(X)$ contained in a group of matrices with matrix multiplication as the group operation in such a way that*

$$\Gamma(X_1 X_2) = \Gamma(X_1)\Gamma(X_2), \quad \forall X_1, X_2 \in G, \tag{A.1}$$

*then this set of matrices are said to provide a n-dimensional representation of $G$.*[11]

A representation need not necessarily be unique. Given one representation a equivalent representation can be obtained by a similarity transformation. Given a constant invertible matrix $A$ and $X \in G, \Gamma(X)$ and $\Gamma'(X)$ are equivalent representations if they are related as

$$\Gamma'(X) = A\Gamma(X)A^{-1} \tag{A.2}$$

By using these similarity transformation it is possible to find more useful representations. If a group representation can be converted to a block diagonal form by a similarity transformation it is said to be a reducible representation. If such a transformation is not possible the representation is known as *irreducible*.

Consider two groups $G_1$ and $G_2$ with the set of pairs $(X, Y)$ where $X \in G_1$ and $Y \in G_2$. Define the product of two such pairs $(X, Y)$ and $(X', Y')$ by

$$(X, Y)(X', Y') = (XX', YY'), \tag{A.3}$$

for all $X, X' \in G_1$ and all $Y, Y' \in G_2$. This set of pairs form a group with the equation above defined as the binary operation. This group is denoted $G_1 \otimes G_2$ and is called the *direct product* of $G_1$ and $G_2$ [30].

The next object introduced are *Lie groups*, which are groups where the elements are labelled by a set of continuous parameters with an operation that depends smoothly on the parameters. Formally it is defined as:

**Definition 4.** *A Lie group is a smooth manifold $G$ which is also a group, such that the group product $G \times G \to G$ and the inverse map $G \to G$ are smooth*

A *manifold* is a topological space $M$ that locally looks like a part of $\mathbb{R}^n$. More precisely, an $n$-dimensional manifold is a second-countable, Hausdorff topological space with the property that each $m \in M$ has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

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[11]This can be rephrased as there existing a *homomorphic* mapping of $G$ onto a group of non-singular $n \times n$-matrices with matrix multiplication as group operation.

The fact that the manifold is smooth essentially means that the manifold is equipped with a differential structure that allows one to perform differential calculus and that derivatives of all orders exist on it, hence the name smooth.

In general, simple Lie Groups can be defined as an exponentiation of infinitesimal generators. These generators are known as *Lie algebras*. The number of independent Lie algebras determines the dimension of the Lie group. In this report we will be most concerned with Lie groups consisting of matrices. Lie groups of particular interest will be presented below.

A $n \times n$ complex matrix $A$ is said to be *unitary* if its columns are orthonormal with respect to the standard inner product on $\mathbb{C}^n$, which is equivalent to the matrix preserving the inner product. That is, it satisfies $\langle Ax|Ay \rangle = \langle x|y \rangle$ for all $x, y \in \mathbb{C}^n$. The set of unitary $n \times n$ matrices form a group that is called the *unitary group* and is denoted by $U(n)$. The set of unitary $n \times n$ matrices with determinant 1 is a subgroup of $U(n)$ called the *special unitary group* and is denoted by $SU(n)$. Both of these form Lie groups consisting of matrices and has found wide application in particle physics [31].

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