Control and Analysis
of Pulse-Modulated Systems

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The future’s so bright, I gotta’ wear shades

- Timbuk 3
Abstract

The thesis consists of an introduction and four appended papers. In the introduction we give an overview of pulse-modulated systems and provide a few examples of such systems. Furthermore, we introduce the so-called dynamic phasor model which is used as a basis for analysis in two of the appended papers. We also introduce the harmonic transfer function and finally we provide a summary of the appended papers.

The first paper considers stability analysis of a class of pulse-width modulated systems based on a discrete time model. The systems considered typically have periodic solutions. Stability of a periodic solution is equivalent to stability of a fixed point of a discrete time model of the system dynamics. Conditions for global and local exponential stability of the discrete time model are derived using quadratic and piecewise quadratic Lyapunov functions. A griding procedure is used to develop a systematic method to search for the Lyapunov functions.

The second paper considers the dynamic phasor model as a tool for stability analysis of a general class of pulse-modulated systems. The analysis covers both linear time periodic systems and systems where the pulse modulation is controlled by feedback. The dynamic phasor model provides an $L_2$-equivalent description of the system dynamics in terms of an infinite dimensional dynamic system. The infinite dimensional phasor system is approximated via a skew truncation. The truncated system is used to derive a systematic method to compute time periodic quadratic Lyapunov functions.

The third paper considers the dynamic phasor model as a tool for harmonic analysis of a class of pulse-width modulated systems. The analysis covers both linear time periodic systems and non-periodic systems where the switching is controlled by feedback. As in the second paper of the thesis, we represent the switching system using the $L_2$-equivalent infinite dimensional system provided by the phasor model. It is shown that there is a connection between the dynamic phasor model and the harmonic transfer function of a linear time periodic system and this connection is used to extend the notion of harmonic transfer function to describe periodic solutions of non-periodic systems. The infinite dimensional phasor system is approximated via a square truncation. We assume that the response of the truncated system to a periodic disturbance is also periodic and we consider the corresponding harmonic balance equations. An approximate solution of these equations is stated in terms of a harmonic transfer function which is analogous to the harmonic transfer function of a linear time periodic system. The aforementioned assumption is proved to hold for small disturbances by proving the existence of a solution to a fixed point equation. The proof implies that for small disturbances, the approximation is good.

Finally, the fourth paper considers control synthesis for switched mode DC-DC converters. The synthesis is based on a sampled data model of the system dynamics. The sampled data model gives an exact description of the converter state at the switching instances, but also includes a lifted signal which represents the inter-sampling behavior. Within the sampled data framework we consider $H_\infty$ control design to achieve robustness to disturbances and load variations. The suggested controller is applied to two benchmark examples; a step-down and a step-up converter. Performance is verified in both simulations and in experiments.

Keywords: Pulse-width modulation, Periodic systems, Stability analysis, Harmonic analysis, Lyapunov methods, Dynamic phasors, Harmonic transfer function, Switched mode power converters, Sampled data modeling, $H_\infty$ synthesis
Preface

The first steps towards this thesis were taken in my M.S. thesis titled “Control and Analysis of a PWM DC-DC Converter”. This work was done at Bombardier Transportation in Västerås under supervision of Jorge Mari and Ulf Jönsson. After completion of this work in 2003, Ulf Jönsson suggested that I pursue doctoral studies at the division of Optimization and Systems Theory and I gladly accepted.

As the title of this thesis suggests, I returned to the problems I worked on at Bombardier. The problems were embedded into a more general mathematical framework, but the DC-DC converter has remained the main motivation for the work and is, as the reader will see, my example of choice.

In my first work I considered the problem of stability of pulse-width modulated (PWM) systems using a discrete time approach. At this time I had the pleasure of meeting Chung-Yao Kao (a.k.a. Isaac) who did a post-doc at the department. Isaac soon became a much appreciated colleague and has made contributions to paper A of the thesis.

In 2004, Hisaya Fujioka made a longer visit to the department. We made sure to utilize Fujioka’s expertise on sampled data control and considered the problem of control of PWM systems using a sampled data approach. About the same time, I joined the European research network HYCON. In particular, I was involved in a work package on control of power systems and power electronics. Within HYCON, the sampled data control approach was pushed all the way to experimental application and the results have been collected in a number of publications. A summary of this work is found in paper D.

It was during a HYCON meeting that I learned about a modeling tool called the dynamic phasor model. Using this modeling tool I returned to the problem of stability of pulse-modulated systems and also considered harmonic analysis. The dynamic phasor model is the basis for the main part of my thesis. The results are found in paper B and C.
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I thank all my colleagues in the research network HYCON. In particular I thank Manfred Morari and Georgios Papafotiou for defining the benchmark problems which I have frequently used as examples in my papers. I thank Sébastien Mariéthoz for helping me perform the experiments which are included in the thesis and I would also like to thank Sébastien Cliquennois for his input on the applications of harmonic analysis. Georgios Papafotiou, Andrea Becutti and Sébastien Mariéthoz also deserve to be acknowledged for their editorial work on our common papers and I want to thank Turhan Demiray for introducing me to the dynamic phasor model.

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Introduction

The thesis considers control and analysis problems associated with pulse-modulated systems. In this section we give a brief introduction to pulse modulation with a special focus on pulse-width modulation. A few practical examples of pulse-modulated systems are also provided.

We further introduce the so-called dynamic phasor model which is used as a basis for stability and harmonic analysis in two of the appended papers. The dynamic phasor model is a Fourier series expansion of the system state over a moving time window. It has proved to be an efficient tool for exploring the structure of periodic and cyclic systems.

Moreover we give a brief description of the harmonic transfer function (HTF) and comment on some similarities between the HTF and the dynamic phasor model. The HTF generalizes the concept of transfer function to linear time periodic (LTP) systems and is thus an efficient tool for harmonic analysis.

Finally we provide a summary of the appended papers and list the publications upon which the appended papers are based.

1 Pulse modulation

Pulse modulation [14] has been applied in a wide range of applications. An early example is the oven temperature control system described by Gouy [16] in 1897. Other examples where feedback has been implemented using pulse modulation include radar systems [5], electrolytic metal refining plants [17] and switched mode power converters [20, 28]. Many electromechanical actuators are implemented using pulse modulation [37, 39] and thus, pulse-modulated dynamics are included in many mechanical systems.

The advantage of pulse modulation lies primarily in its simplicity. Since the control variable typically only takes two or three values, an actuator can be realized using a switch [18]. A drawback is that the inherent nonlinear and discontinuous nature of pulse modulation makes analysis and design problems difficult.

In the section below we provide a mathematical description of pulse modulated systems which follows the outline of [14]. In [14] a pulse modulator is defined as an operator which maps an input signal \( \sigma(t) : [0, \infty) \to \mathbb{R} \) to a piecewise continuous pulse function \( s(t) : [0, \infty) \to \mathbb{R} \). Pulse modulators may take many different forms, but they all share a common principle of operation; the pulse function \( s(t) \) is defined on a partition of the time axis. The pulse modulator divides the time axis into segments \([t_k, t_{k+1}], t_{k+1} > t_k\) which are called sampling intervals. For some types of
In this section we focus on what is probably the most common type of pulse modulation. We consider modulators which produce pulse functions that are composed of rectangular pulses. For pulse modulators, the output pulse function can be written explicitly on the form

\[ s(t) = \begin{cases} 
  a_k, & t \in [t'_k, t'_k + \tau_k) \\
  0, & t \in [t_k, t'_k) \cup [t'_k + \tau_k, t_{k+1}) 
\end{cases} \]

where \( a_k \) is the pulse-amplitude and where \( t_k, t'_k \) and \( \tau_k \) are non-negative real numbers satisfying \( t_k \leq t'_k \leq t'_k + \tau_k < t_{k+1} \). The parameter \( \tau_k \) is the pulse-width and \( t'_k \) determines the position of the leading edge of the pulse. See Fig. 2 for an illustration.

Some of the parameters mentioned above may be constant while other may be a function of the input signal \( \sigma(t) \). Two common cases \cite{14} are illustrated in Fig. 3 below. They are

1. **Pulse-amplitude modulation (PAM)**

   In pulse-amplitude modulation the sampling intervals are of equal length and \( t_k = kT_s \) where \( T_s > 0 \) is the sampling period. It holds \( t'_k = t_k \) so that
the leading edge of the pulse coincides with the beginning of the sampling interval and \( \tau_k = \tau > 0 \ \forall k \) so that all pulses have the same duration. The pulse-amplitudes \( a_k \) are variable and are determined by the signal \( \sigma(t) \).

2. Pulse-width modulation (PWM)

In pulse-width modulation the sampling intervals are of equal length and \( t_k = kT_s \) where \( T_s > 0 \) is the sampling period. It holds \( t'_k = t_k \) so that the leading edge of the pulse coincides with the beginning of the sampling interval and \( a_k = a > 0 \ \forall k \) so that all pulses have the same amplitude. The pulse-durations \( \tau_k \) are variable and are determined by the signal \( \sigma(t) \).

![PAM and PWM diagrams](image)

Figure 3: Two types of pulse modulation with square pulses; pulse-amplitude modulation (PAM) and pulse-width modulation (PWM).

1.1 Pulse-width modulation

The thesis is primarily focused on pulse-width modulated (PWM) systems, although paper B considers more general types of modulation, including PAM. In the special case of PWM, the pulse function is written

\[
s(t) = \begin{cases} 
a, & t \in [kT_s, (k + d_k)T_s) \\
0, & t \in [(k + d_k)T_s, (k + 1)T_s) 
\end{cases}
\]

where \( a > 0 \) is the amplitude of the pulses and \( d_k \in [0, 1] \) is the so-called duty cycle which determines the duration of the \( k \)th pulse.

In PWM, the pulse-width is variable and it is determined by the input signal \( \sigma(t) \). The functional relationship between \( \sigma(t) \) and the duty cycles \( d_k \) may take many different forms. In analogy with [14], we distinguish the following two cases.
1. Pulse-width modulation of the first kind (PWM-1)

In PWM-1, the duty cycle $d_k$ is determined by sampling the input signal according to

$$d_k = F(\sigma(kT_s))$$

where $F(\cdot)$ is a nondecreasing function defined on $\mathbb{R}$ satisfying $0 \leq F(\sigma) \leq 1, \forall \sigma \in \mathbb{R}$ (and typically satisfying $F(0) = 0$).

2. Pulse-width modulation of the second kind (PWM-2)

In PWM-2 the duty cycle $d_k$ is the solution to an equation of the form

$$d_k = F(\sigma(kT_s + d_kT_s))$$

where $F(\cdot)$ satisfies the same assumptions as in PWM-1.

Applications of PWM are frequent in the field of power electronics. Examples include switched mode converters and inverters, see Section 1.3 below. In such devices, the PWM is often implemented according to

$$d_k = \text{sat}_{[0, 1]}(\sigma(kT_s))$$

where $\text{sat}_{[0, 1]}(\cdot) := \min(\max(\cdot, 0), 1)$ is the saturation between zero and one. In other words, PWM-1 is used with $F(\cdot) = \text{sat}_{[0, 1]}(\cdot)$.

Another technique for implementing PWM in power electronics is to use a comparator ramp function as illustrated in Fig. 4. Here, the input signal $\text{sat}_{[0, 1]}(\sigma(t))$ is compared to the saw-tooth ramp function $r(t) = \frac{1}{T_s}(t \mod T_s)$. At the beginning of each sampling interval the pulse function $s(t)$ takes value $a$ and it remains constant until the signal $\sigma(t)$ intersects the ramp. At this time instance the pulse function takes value zero and it remains zero until the beginning of the next sampling interval where the process is repeated. The $k^{th}$ switching instant is determined by the smallest number $d_k \in [0, 1]$ which satisfies the nonlinear equation

$$d_k = \text{sat}_{[0, 1]}(\sigma(kT_s + d_kT_s)).$$

In other words, this is a case of PWM-2.

In the PWM-2 technique described above we only allowed for one pulse in each sampling interval. In power electronics this restriction is often imposed to avoid so-called sliding solutions [7] where there are several switching instants within one sampling interval. Sliding solutions are associated with nonlinear phenomena such as bifurcations and chaos, see [7, 11] and references therein.

In the context of power electronics, the PWM-1 technique (3) offers advantages such as being less sensitive to noise and aging of components [45]. PWM-1 also makes it easier to model the system dynamics in a discrete time framework which can be useful for both control and analysis. However, PWM-1 introduces additional delay compared to PWM-2.
1.2 Modeling PWM systems

The nonlinear and discontinuous characteristics of a pulse-width modulator makes control and analysis of PWM systems complicated. This has motivated a number of approximate models for the dynamics of PWM systems. In the section below we review some of these modeling approaches.

A common technique for approximating pulse-modulated systems is referred to as averaging. Here, the pulse modulator is replaced by a continuous function of the input signal $\sigma(t)$. One common averaging approach is to replace the pulse modulator with the corresponding static characteristic [14]. For a pulse-width modulator, the static characteristic is the time-average value of the pulse function when the input signal $\sigma(t)$ is constant. In other words, the static characteristic is defined as

$$\text{stat}(\sigma_0) := \frac{1}{T_s} \int_0^{T_s} s(\tau)d\tau$$

where $s(t)$ is the pulse function corresponding to the input $\sigma(t) = \sigma_0 = \text{constant}$. For the case of PWM-1 and PWM-2 it is easy to see that

$$\text{stat}(\sigma_0) = a F(\sigma_0).$$

Thus, in the PWM system one would replace the pulse function $s(t)$ with the function $a F(\sigma(t))$.

The averaging approach is particularly common when modeling switched mode DC-DC converters. In this context, an averaging methodology referred to as state space averaging was outlined by Middlebrook et al. [26]. Rigorous mathematical treatments were provided in [21, 22]. Extensions to resonant converters have also been made, see [32, 36].

Figure 4: Comparator ramp function $r(t)$ and input control signal $\sigma(t)$. The duration of the pulses is determined by the intersection of $r(t)$ and $\sigma(t)$. 
The state space averaged model is often simple and is therefore often used for control design. However, the state space averaged model does not account for the rich variety of nonlinear phenomena that can occur in switched mode converters. This was noted in [19] where the authors show that some of the assumptions made in the averaging theory are essential for the validity of the averaged model.

It has also been noted, see [12] and references therein, that the state space averaged model need not capture the stability properties of the system. For instance, the averaged model can be stable even though the actual system is not. Furthermore, the averaged model need not give an accurate description of harmonic properties such as disturbance propagation. This was discussed in [30]. Finally we note that design specifications typically include bounds on the state/output. The state space averaged model does not represent the state/output ripple which is inherent in switched converters and peak values are therefore underestimated.

Recently there has been a number of papers [3, 4, 8, 12, 27, 30, 36, 38, 40–42], addressing the issues of stability, harmonic analysis and ripple estimation of switched mode power converters. The focus of these papers differ, but they share the view that classical state space averaging is insufficient and they develop more sophisticated models.

For a more accurate system description, some authors have modeled PWM systems in discrete time. This allows for the switched nature of the plant to be accounted for explicitly and the resulting discrete time model gives an exact representation of the system state at the sampling instances. The discrete time approach was used for stability analysis in [4, 42] and in paper A of this thesis.

Discrete time models have also served successfully as a basis for hybrid control design techniques such as hybrid model predictive control and relaxed dynamic programming, see e.g., [2, 6, 15, 23, 44]. In paper D of this thesis we consider control synthesis based on a sampled data model which describes the evolution of the state at the sampling instances, but also provides a representation of the inter-sample behavior.

Some recent work on PWM systems rely on the so-called dynamic phasor model which was introduced by Verghese et al. [36]. The dynamic phasor model is obtained from a Fourier series expansion of the system state over a moving time window. It can be seen as a generalization of the state space averaged model which includes higher harmonics (than just the constant). An introduction to the dynamic phasor model is given in Section 2 below.

The dynamic phasor model has been applied to a number of different problems. For example, it has been used to develop efficient simulation tools for switched systems [10, 24] and for stability analysis of DC-DC converters. The papers [12, 40, 41] use the dynamic phasor model for stability analysis of a number of different converter topologies. These papers focus on open loop systems, but the closed loop case is also commented.

The theoretical aspects of the dynamic phasor model were addressed in [38]. In this paper, a number of challenging convergence issues was identified and some useful analysis techniques were outlined.
The dynamic phasor model provides information about the harmonics of a dynamic system through time varying Fourier coefficients. An alternative tool for characterizing the harmonics is the harmonic transfer function (HTF) [29,35,43,47]. The HTF is an infinite dimensional input-output map which generalizes the notion of transfer function to linear time periodic (LTP) systems. It is thus an efficient tool for describing the harmonic couplings in time periodic systems. An introduction to the HTF is given in Section 3 below.

The HTF has been applied to switched mode power converters to better understand their harmonic properties. See e.g., [30] which shows that it is essential to understand the harmonic couplings in power converters.

Because the HTF is infinite dimensional, there are some issues on well-posedness. Convergence issues regarding the HTF were cleared in [34,35]. Related work on LTP systems can also be found in [3,47,48].

In this thesis, the dynamics phasor model serves as a basis for analysis in both paper B and C. Paper C is devoted to harmonic analysis of PWM systems. In this paper we consider a square truncated approximation of the phasor dynamics. The truncated system is used to derive a frequency domain input-output map which is analogous to the HTF of an LTP system and which reduces to a truncated HTF for open loop systems. In paper B we consider stability analysis of pulse-modulated systems. We introduce a skew truncated approximation of the phasor dynamics and use it to compute time periodic quadratic Lyapunov functions. We also show how the linearized skew truncated dynamics can be used to to derive a frequency domain input-output map analogous to the one mentioned above. Part of the contribution of the thesis is to show a connection between the dynamic phasor model and the HTF.

### 1.3 Examples

**Example 1: Switched mode DC-DC converter**

![Diagram of a step-down DC-DC converter](image)

Figure 5: Right; step-down DC-DC converter. Left; pulsed voltage.

The first example we consider is the step-down DC-DC converter [20] depicted in Fig. 5. The purpose of the step-down converter is to convert the source voltage $v_s$ into a DC voltage $v_o$ at some given reference level. The source $v_s$ is converted by opening and closing the switch $s$ at a high frequency. The switch produces a
voltage \( v_p \) in the shape of a rectangular wave. The average value of this waveform is filtered out using a low pass LC filter and this gives an approximate DC voltage. The level of the DC output voltage depends on the duration of the pulses of \( v_p \). In other words, the level is controlled by pulse-width modulation.

**Example 2: Voltage source inverter**

![Diagram of voltage source inverter](image)

Figure 6: Right; voltage-source inverter. Left; pulsed voltage

The second example we consider is the voltage source inverter [20] depicted in Fig. 6. The purpose of this device is to convert a given DC source voltage into an approximate AC voltage. The AC waveform is approximated by square pulses as illustrated in Fig. 6.

To produce the rectangular waveform the switches are operated as follows. If \( s_1, s_4 \) are closed and \( s_2, s_3 \) open there is a positive voltage over the load. If \( s_1, s_4 \) are open and \( s_2, s_3 \) closed there is a negative voltage over the load. Finally, if \( s_2, s_4 \) are closed and \( s_1, s_3 \) open there is zero voltage over the load.

## 2 The dynamic phasor model

The papers B and C in the thesis consider analysis based on the so-called dynamic phasor model. This section gives an introduction to the dynamic phasor model and describes two approximation techniques used on the infinite dimensional phasor dynamics.

To our knowledge, the dynamic phasor model was introduced in the field of power electronics as a tool for modeling the transients of switched converters, see e.g., [8, 25, 33, 36]. Furthermore, it has been applied in the analysis of power systems [1, 9] and for developing efficient simulation tools for switched converters, see [10, 24] and references therein. It has also been used for stability analysis of switched power converters, see e.g., [12, 40, 41].

The dynamic phasor model is a Fourier series representation of the state of a system over a moving time window. The principle is illustrated in Fig. 7. In this figure we imagine that \( x(t) \) is a solution to some dynamic system. A time window is placed over the time axis stretching from the current time \( t \) back to time \( t - T_s \) where \( T_s > 0 \). By integrating over the window we compute Fourier coefficients of
$x(t)$ according to

$$
\langle x \rangle_n (t) := \frac{1}{T_s} \int_{t-T_s}^{t} x(\tau) e^{-j n \omega_s \tau} d\tau
$$

where $\omega_s := 2\pi/T_s$. We note that the Fourier coefficients are time dependent. We also note that if $x(t)$ is periodic with period $T_s$, then $\langle x \rangle_n$ is constant.

The time dependent Fourier coefficients, which are also known as dynamic phasors, are used to define a Fourier series according to

$$x(t, \tau) := \sum_{n=-\infty}^{\infty} \langle x \rangle_n (t) e^{j n \omega_s (t+\tau)}, \quad \tau \in [-T_s, 0]. \quad (5)$$

In the Fourier series $x(t, \tau)$, the first time index $t$ denotes the position of the time window on the time axis while the second time index $\tau \in [-T_s, 0]$ gives the position inside the window. We note that $x(t) \neq x(t, \tau)$, but under weak assumptions on $x(t)$, the equality $x(t + \tau) = x(t, \tau)$ holds a.e. on the set $\{ \tau \mid \tau \in [-T_s, 0] \}$. Thus, the dynamic phasor model provides an $L_2$-equivalent representation of the part of the solution which lies inside the time window. As the window moves ahead in time, the time dependent Fourier coefficients change and at each moment in time we have a representation of a section of the solution.

![Figure 7: Dashed line; signal $x(t)$. Solid line; Fourier series representation $x(t, \tau)$.

The dynamic phasor model is conceptually appealing but poses some mathematical difficulties. The main problem is that the Fourier series expansion of the system state over a given time window in general does not converge uniformly. These convergence problems and the corresponding differentiability problems were revealed in [38]. In [38] the author outlines analysis techniques similar to those applied in this thesis. However, we consider different problems in a somewhat different context.

The key idea of the dynamic phasor model is to consider the dynamics of the time dependent Fourier coefficients (or dynamic phasors) rather than the dynamic equations of the system directly. This approach explores some of the structure of
time periodic and cyclic systems. Using partial integration we have

\[ -j n \omega_s \langle x \rangle_n(t) = \frac{1}{T_s} \int_{t-T_s}^{t} x(\tau)(-j n \omega_s)e^{-j n \omega_s \tau} d\tau \]

\[ = \frac{1}{T_s} \left( x(t)e^{-j n \omega_s t} - x(t - T_s)e^{-j n \omega_s t} \right) - \frac{1}{T_s} \int_{t-T_s}^{t} \dot{x}(\tau)e^{-j n \omega_s \tau} d\tau \]

and noting that the first term in the sum above is equal to \( \frac{d}{dt} \langle x \rangle_n(t) \) we conclude that the phasor coefficients \( \langle x \rangle_n(t) \) satisfy the dynamic equation

\[ \frac{d}{dt} \langle x \rangle_n(t) = -j n \omega_s \langle x \rangle_n(t) + \left\langle \frac{d}{dt} x \right\rangle_n(t). \quad (6) \]

For a more compact notation, we will often suppress the explicit time dependence of the phasor coefficients.

To illustrate the possible benefits of the dynamic phasor model we now consider a simple PWM system and derive an explicit expression for the corresponding dynamic phasor system (6). Let \( x(t) \in \mathbb{R}^n \) be a solution of the system

\[ \dot{x}(t) = A(t)x(t) \quad (7) \]

where \( A(t) = s(t)A_0 + (1 - s(t))A_1 \) with \( A_i \) constant matrices and where \( s(t) \) is the PWM function

\[ s(t) = \begin{cases} 
1, & t \in [kT_s, (k + d_k)T_s) \\
0, & t \in [(k + d_k)T_s, (k + 1)T_s)
\end{cases} \]

where the duty cycle is either constant or determined by feedback according to PWM-1 as defined in (1). It holds

\[ \left\langle \frac{d}{dt} x \right\rangle_n = \langle Ax \rangle_n = \langle (sA_0 + (1 - s)A_1)x \rangle_n = \langle A_0 - A_1 \rangle \langle sx \rangle_n + A_1 \langle x \rangle_n \]

\[ = (A_0 - A_1) \left( \sum_{k=-\infty}^{\infty} \langle s \rangle_{n-k} \langle x \rangle_k \right) + A_1 \langle x \rangle_n. \]

Thus, the phasor coefficients \( \langle x \rangle_n \) of \( x(t) \) satisfy

\[ \frac{d}{dt} \langle x \rangle_n = -j n \omega_s \langle x \rangle_n + (A_0 - A_1) \left( \sum_{k=-\infty}^{\infty} \langle s \rangle_{n-k} \langle x \rangle_k \right) + A_1 \langle x \rangle_n \quad (8) \]

where

\[ \langle s \rangle_n(t) := \frac{1}{T_s} \int_{t-T_s}^{t} s(\tau)e^{-j n \omega_s \tau} d\tau \]

is obtained by integrating over the PWM function \( s(t) \) as illustrated in Fig. 8.
Figure 8: The phasors $\langle s \rangle_n(t)$ of $s(t)$ are computed by integrating from time $t - T_s$ to $t$.

By introducing some notation, the phasor dynamics (8) can be stated in a more compact form. Let

$$\hat{x} := \begin{bmatrix} \ldots, \langle x \rangle_2^*, \langle x \rangle_1^*, \langle x \rangle_0^*, \langle x \rangle_{-1}^*, \langle x \rangle_{-2}^*, \ldots \end{bmatrix}^*$$

be an infinite dimensional vector containing the phasor coefficients of $x(t)$. Let

$$T[s] := \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \langle s \rangle_0 I_n & \langle s \rangle_1 I_n & \langle s \rangle_2 I_n & \cdots \\ \cdots & \langle s \rangle_{-1} I_n & \langle s \rangle_0 I_n & \langle s \rangle_1 I_n & \cdots \\ \langle s \rangle_{-2} I_n & \langle s \rangle_{-1} I_n & \langle s \rangle_0 I_n & \cdots & \cdots \end{bmatrix}$$

be a doubly infinite dimensional block Toeplitz matrix with the phasor coefficients $\langle s \rangle_n$ of $s(t)$ along the sub- and super-diagonals. Finally let

$$\mathcal{N} := \text{blkdiag}(..., 2I_n, I_n, 0, -I_n, -2I_n, \ldots)$$

be a doubly infinite dimensional block diagonal matrix with the integers along the diagonal. Using this notation, the dynamics of the phasors $\langle x \rangle_n$ are written

$$\frac{d}{dt} \hat{x} = (-j \omega_s \mathcal{N} + (I \otimes A_1) + (I \otimes (A_0 - A_1)) T[s]) \hat{x}$$

where $\otimes$ denotes the Kronecker product.

As was mentioned above, the phasor coefficients of a $T_s$-periodic signal are constant. If the duty cycle in (7) is constant so that $d_k = d \forall k$, then $s(t)$ is $T_s$-periodic and the matrix $T[s]$ is constant. Thus, via the dynamic phasor model we can represent the periodically switched system (7) with time invariant equations. A drawback
of this approach is that the phasor dynamics (10) are infinite dimensional. Furthermore, the systems matrix in (10) is an unbounded operator on the space $l_2$. This is due to the term $N$.

In [48] the authors consider stability analysis of LTP systems based on harmonic equations analogous to (10). The authors derive their results by viewing the system matrix of the harmonic equations as an operator and the unboundedness of this operator is dealt with by restricting it to a dense subset of $l_2$. In our work we avoid the problem of unboundedness by considering solutions of (10) rather than the system matrix itself.

If the duty cycle in (7) is not constant but determined according to the PWM-1 technique (1) (where $\sigma(t)$ is a function of the state), then the vector field in (10) is not time invariant. In this case, the phasor coefficients $\langle s \rangle_n$ are time varying and depend on sampled values of the state. From Fig. 8 it is clear that if $t \in [k T_s, (k + 1) T_s]$, then $\langle s \rangle_n(t)$ depends on the duration of the pulses that begin at times $k T_s$ and $(k - 1) T_s$. These durations are determined by the duty cycles $d_k$ and $d_{k-1}$ which in the case of PWM-1 will be functions of the state sampled at times $k T_s$ and $(k - 1) T_s$ respectively.

### 2.1 Approximate models

The phasor dynamics (10) are infinite dimensional. In the case of feedback where the duty cycle is determined according to (1), the phasor dynamics are also nonlinear and depend on sampled values of the state with a delay.

These difficulties motivate us to consider approximate models of (10). In the section below we introduce a skew truncated averaged approximation and a square truncated averaged approximation of (10). The skew and square truncation techniques were applied in paper B and C respectively.

The approximate systems are derived in two steps. Firstly, the phasor coefficients $\langle s \rangle_n$ of $s(t)$ are replaced with the averaged approximation

$$s_{av,n}(d) = \begin{cases} d, & n = 0 \\ \frac{j}{n2\pi} (e^{-jn2\pi d} - 1), & n \neq 0 \end{cases}$$

where $d = F(\sigma) \in [0, 1]$. We note that $s_{av,n}(d)$ is a nonlinear function of $d$, but unlike $\langle s \rangle_n$ it is a continuous function of $d$ and does not depend on sampled values. Furthermore we note that in the open loop case where the duty cycle is constant, it holds $s_{av,n}(d) = \langle s \rangle_n$. This means that if the duty cycle varies slowly (compared to the sampling period $T_s$) then $s_{av,n}(d)$ is a good approximation of $\langle s \rangle_n$. This claim is stated formally and proved in paper C.

In the second step of the approximation procedure the infinite dimensional system is truncated. In the case of skew truncation, we pick an integer $N \geq 0$ and replace the phasor coefficients $\langle s \rangle_n$ in $T[s]$ with zero if the order satisfies $|n| > N$. The resulting skew truncated system is still infinite dimensional, but the system matrix is band diagonal. See Fig. 9 for an illustration.
In the case of square truncation, we pick an integer $N \geq 0$ and pick out the central block of dimension $(2N+1)$ from the system matrix. This finite dimensional matrix defines a finite dimensional system which approximates the low order phasor coefficients. See Fig. 10 for an illustration.

The square truncated approximation has practical advantages since it provides a finite dimensional system which is straightforward to implement in simulations. On the other hand, the skew truncated approximation has theoretical advantages since it retains the structure of the original infinite dimensional system and has an immediate time domain interpretation.

3 The harmonic transfer function

This section gives a sketch of derivation of the harmonic transfer function (HTF) [29,35,43,46-48]. The HTF is a generalization of the transfer function to linear time periodic (LTP) systems and it is thus a powerful tool for illustrating the frequency couplings in periodic systems.
From our perspective, the HTF is relevant because the pulse-modulated systems we consider include LTP systems as a special case. Moreover, the HTF is also interesting because of its similarities with the dynamic phasor model. These similarities have inspired us to extend the notion of HTF to describe periodic solutions of closed loop, non-periodic pulse-modulated systems. Results are found in paper B and C of the thesis. We give some further comments on the connection between the HTF and the dynamic phasor model at the end of this section.

LTP systems do not have the property of frequency separation which is characteristic for LTI systems. If the input to an LTP system is a sinusoid with frequency \( \omega \), then the steady state output will be a sum of sinusoids with frequencies \( \omega + n\omega_s \) where \( n \in \mathbb{Z} \) and \( \omega_s \) is the frequency of the system.

In [43], an infinite dimensional transfer function (the HTF) was derived which describes the coupling between the frequencies of input and output signals of LTP systems. The authors consider stable LTP systems of the form

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]  

(11)

where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \) and \( A(t), B(t), C(t) \) and \( D(t) \) are \( T_s \)-periodic matrices. The input signal \( u(t) \) is assumed to be an exponentially modulated periodic (EMP) signal which is defined as

\[
u(t) := \sum_{n=-\infty}^{\infty} u_n e^{s_n t}\]

(12)

where \( s_n = s + jn\omega_s \) where \( \omega_s = 2\pi/T_s \) and \( s \in \mathbb{C} \). In [43], the authors note that “EMP signals are to LTP systems, what complex exponentials (sinusoids) are to LTI systems”, meaning that if the input to an LTP system is EMP, then the steady state solution and output will also be EMP. This observation is used to derive the HTF which maps the coefficients \( u_n \) of the input signal to the corresponding coefficients of the EMP output signal.

To derive the HTF, the system matrices in (11) are expanded in Fourier series according to

\[
\begin{align*}
A(t) &= \sum_{n=-\infty}^{\infty} A_n e^{j\omega_s t}, & B(t) &= \sum_{n=-\infty}^{\infty} B_n e^{j\omega_s t} \\
C(t) &= \sum_{n=-\infty}^{\infty} C_n e^{j\omega_s t}, & D(t) &= \sum_{n=-\infty}^{\infty} D_n e^{j\omega_s t}.
\end{align*}
\]

It is assumed that the steady state response of (11) to the EMP signal (12) is also
EMP so that

\[ x(t) = \sum_{n=-\infty}^{\infty} x_n e^{s_n t} \]
\[ \dot{x}(t) = \sum_{n=-\infty}^{\infty} s_n x_n e^{s_n t} \]
\[ y(t) = \sum_{n=-\infty}^{\infty} y_n e^{s_n t}. \]

These expressions for \( x, \dot{x} \) and \( y \) are plugged into the system equations (11) where the system matrices have been replaced by their Fourier series expansions. The principle of harmonic balance yields the following equations for the coefficients \( u_n, x_n \) and \( y_n \)

\[ s_n x_n = \sum_{k=-\infty}^{\infty} A_{n-k} x_k + \sum_{k=-\infty}^{\infty} B_{n-k} u_k \]
\[ y_n = \sum_{k=-\infty}^{\infty} C_{n-k} x_k + \sum_{k=-\infty}^{\infty} D_{n-k} u_k. \]

The equations above are a concise description of the relationship between the input and output signals. However, this relationship can be expressed more clearly by introducing some notation. Let

\[ x = \left[ \ldots, x_2^*, x_1^*, x_0^*, x_{-1}^*, x_{-2}^*, \ldots \right]^* \]

be an infinite dimensional vector containing the coefficients \( x_n \) and let \( u \) and \( y \) be defined similarly. Let

\[
A := \begin{bmatrix}
\ddots & \cdots & \cdots \\
A_0 & A_1 & A_2 \\
\cdots & A_{-1} & A_0 & A_1 & \cdots \\
A_{-2} & A_{-1} & A_0 & \cdots & \cdots \\
\ddots & \cdots & \cdots & \ddots & \ddots
\end{bmatrix}
\]

be a doubly infinite dimensional block Toeplitz matrix containing the Fourier coefficients of \( A(t) \) and let \( B, C \) and \( D \) be defined similarly. The harmonic balance equations can be stated as

\[ s x = (-j \omega_c N + A) x + B u \]
\[ y = C x + D u \] (13)
where

\[ \mathcal{N} := \text{blkdiag}(\ldots, 2I_n, I_n, 0, -I_n, -2I_n, \ldots) \]

is a doubly infinite dimensional block diagonal matrix with the integers on the diagonal. By formally inverting the matrix \( s\mathcal{I} - (-j \omega_p \mathcal{N} + \mathcal{A}) \) (where \( \mathcal{I} \) is the identity) we can express the relationship between \( u \) and \( y \) according to

\[ y = \mathcal{H}(s)u \]

where

\[ \mathcal{H}(s) := \mathcal{C}(s\mathcal{I} - (-j \omega_p \mathcal{N} + \mathcal{A}))^{-1}\mathcal{B} + \mathcal{D} \]

is the harmonic transfer function.

In [29] it is shown that the HTF has the block structure

\[
\mathcal{H}(s) = \begin{bmatrix}
\ddots & & \\
& H_0(s + \omega_s) & H_1(s) & H_2(s - \omega_s) \\
& \vdots & & \\
& H_{-1}(s + \omega_s) & H_0(s) & H_1(s - \omega_s) & \vdots \\
& H_{-2}(s + \omega_s) & H_{-1}(s) & H_0(s - \omega_s) & \ddots \\
& & & & \ddots \\
\end{bmatrix}
\]

where the blocks \( H_n \) are obtained by Laplace-transforming the terms \( h_n \) of a Fourier series expansion of the impulse response \( h \) corresponding to the LTP system. The structure of the HTF can be used to show that the steady state response of an LTP system to a sinusoidal input \( u(t) = \sin(\omega t) \) is given by the formula

\[
y(t) = \sum_{n=-\infty}^{\infty} |H_n(\omega)| \sin((\omega + n\omega_s)t + \phi_n)
\]

where \( \phi_n = \arg H_n(\omega) \). This expression clearly illustrates the frequency coupling property of LTP systems mentioned above. It also shows that the HTF contains sufficient information to describe the frequency coupling.

Because the HTF is infinite dimensional, there are issues on well-posedness. Convergence of the HTF is addressed in [35,47]. In our work, the focus is on the dynamic phasor model and the HTF is introduced mainly to show analogies between the dynamic phasor model and the HTF. We therefore neglect the convergence issues of the HTF and our calculations involving the HTF should be viewed as formal.

Finally we remark on some similarities between the HTF and the dynamic phasor model. The derivation of the HTF was based on the observation that the steady state response of an LTP system to an EMP input signal is also EMP. An EMP signal can be written

\[
u(t) = \sum_{n=-\infty}^{\infty} u_n(t)e^{jn\omega_s t}
\]
where $u_n(t) = u_ne^{st}$ are time varying coefficients. Thus, an EMP signal has the same structure as the Fourier series (5) of the dynamic phasor model. The difference is that in the EMP signal, the time varying coefficients are defined as exponentials while in the dynamic phasor model they are obtained by integrating over the solution of the system.

When the dynamic phasor model is applied to an LTP system, one can (formally) describe a relationship between the phasor coefficients of the dynamic phasor model and the coefficients of the corresponding EMP signals. To see this, let $x$ be a solution of the LTP system (11). Let $\hat{x}$ be the vector of phasor coefficients of $x$ defined according to (9) and let $\hat{u}$ and $\hat{y}$ be defined similarly. The phasors $\hat{x}$ satisfy the dynamic equations

$$\dot{\hat{x}} = (-j\omega_N + A)\hat{x} + B\hat{u}$$
$$\hat{y} = C\hat{x} + D\hat{u}$$

where $A, B, C, D$ and $N$ are defined above. Let $\hat{y}(s)$ and $\hat{u}(s)$ denote the Laplace transform of $\hat{y}$ and $\hat{u}$ respectively. By formally applying the Laplace transform to the system above we obtain the relation

$$\hat{y}(s) = \mathcal{H}(s)\hat{u}(s).$$

From this we conclude that the Laplace transform $(\mathcal{L}(y)_n)(s)$ of the phasor coefficients $(y)_n$ can be identified with the coefficient $y_n$ of the EMP signal.

In this thesis, the connection between the dynamic phasor model and the HTF described above is used to extend the concept of HTF to non-periodic pulse modulated systems controlled by feedback. In paper B we consider a skew truncated approximation of the dynamic phasor model. The approximate model is linearized. The linearized dynamics yield a small-signal input-output map which is analogous to the HTF and which takes the effect of feedback into account. In paper C we obtain similar results by considering a square truncated approximation of the dynamic phasor model.

4 Main contribution of the thesis

In this thesis we consider control and analysis problems associated with pulse-modulated systems and we focus especially on PWM systems.

As was noted in Section 1.2 above, control and analysis of PWM systems is often based on the state space averaged model. This model is often simple and therefore suitable for control design, but it has a number of drawbacks because it only provides an approximation of the slow-scale system dynamics. See Section 1.2 above for further detail.

In this thesis we consider a number of different modeling techniques for pulse-modulated systems which are more accurate than the state space averaged model. These modeling techniques are used to derive methods for analysis and control synthesis which overcome some of the limitations of the state space averaged model. In
papers A and D we derive discrete time models of PWM systems which are used for stability analysis and control synthesis. In papers B and C we consider the dynamic phasor model, which can be seen as a generalization of state space averaging, and use it for the purpose of stability and harmonic analysis. In these papers we also discuss the harmonic transfer function and remark on some similarities with the dynamic phasor model.

5 Summary of papers

This section offers a brief summary of the four papers included in the thesis. We also list the publications upon which the appended papers are based and comment on the contribution of different authors.


Paper A considers stability analysis of a class of pulse-width modulated systems which is particularly suited to model switched mode DC-DC converters. The systems are typically designed to have periodic solutions at stationarity. The switched dynamics are modeled by a discrete time system which represents the state at the sampling instances. Stability of a periodic solution is equivalent to stability of a corresponding fixed point of the discrete time system.

Sufficient conditions for global and local stability of the discrete time system are derived using quadratic and piecewise quadratic Lyapunov functions. The stability conditions are stated in terms of parameterized matrix inequalities. To find a solution to the parameterized matrix inequalities we apply a griding procedure. This yields a systematic method for computing quadratic and piecewise quadratic Lyapunov functions.

Paper A is based on the following publications.


**B: Dynamic Phasor Analysis of Pulse-Modulated Systems**, coauthored with Ulf Jönsson
Submitted to journal.

Paper B considers stability analysis of a general class of pulse-modulated systems which includes both pulse-amplitude and pulse-width modulation as special cases. The analysis is based on the dynamic phasor model which is obtained from a Fourier series expansion of the system state over a moving time window. The dynamic phasor model provides an $L_2$-equivalent representation of the system dynamics in terms of an infinite dimensional system which governs the dynamics of the time varying Fourier coefficients (dynamic phasors). The infinite dimensional system is approximated through averaging and a skew truncation. The truncated approximate system is used to derive a systematic method for computing time periodic quadratic Lyapunov functions.

In the special case of a linear time periodic system, existence of a time periodic quadratic Lyapunov function of a certain form is equivalent to the existence of a solution to a set of LMI. In the case of a non-periodic system where the pulse modulation is controlled by feedback, existence of a time periodic quadratic Lyapunov function is guaranteed by the existence of a solution to a parameterized matrix inequality. A gridding procedure is applied and the stability conditions are stated in terms of LMI.

Paper B is based on the following publications.


**C: Harmonic Analysis of Pulse-Width Modulated Systems**, coauthored with Ulf Jönsson
Submitted to journal.

Paper C considers harmonic analysis of a class of pulse-width modulated systems. We introduce an external disturbance in the system and consider the question; for a given periodic disturbance, what is the spectral content of a certain output signal?

To answer this question we consider the dynamic phasor model which gives an $L_2$-equivalent representation of the switching system in terms of an infinite dimensional system governing the dynamic phasors. In the special case of a linear time periodic system, the infinite dimensional phasor dynamics are time invariant and it is shown that the time invariant phasor dynamics provide an explicit formula for the harmonic transfer function. In the case of a non-periodic system where the switching is controlled by feedback, the infinite
dimensional phasor system is nonlinear and non-autonomous. In this case, the dynamics are approximated through averaging and a square truncation. To determine the impact of the external disturbance we assume that a periodic disturbance results in a periodic solution at stationarity and we consider the corresponding harmonic balance equations. The nonlinear equations are linearized and the solution is stated in terms of an input-output map which is analogous to the harmonic transfer function of a linear time periodic system. It is proved that the aforementioned assumption is valid for small disturbances. The proof provides conditions on the size of the disturbance under which the approximation is accurate.

The results which are summarized in paper C inspired the formulation of some benchmark problems regarding harmonic analysis of switched mode DC-DC converters. These problems are part of a benchmark defined in the HYCON project [31].

Paper C is based on the following publications.


**D: Sampled Data Control of DC-DC Converters**

Paper D summarizes theoretical and experimental work on the problem of control of switched mode DC-DC converters. In particular, we consider sampled data $\mathcal{H}_\infty$ control synthesis for DC-DC converters and test the resulting controller in experimental setups.

Firstly, a brief introduction to DC-DC converters is given. We then discuss the most common modeling techniques used for control design and we note that one of these modeling techniques, the discrete time model, may lead to subharmonic oscillations. This observation motivates the introduction of the sampled data model which is used for $\mathcal{H}_\infty$ control synthesis.

The $\mathcal{H}_\infty$ control synthesis yields a linear controller. To improve performance and to satisfy state constraints, the linear sampled data controller is augmented. We add additional control structure which provides nonlinear compensation when the system is far from steady state.

The control methodology is applied to two benchmark examples and the control performance is verified in both simulations and in experimental setups.

Paper D is based on the following publications.


The papers D5-D9 develop the sampled data modeling technique which is applied in the appended paper D. These results were obtained in cooperation with Hisaya Fujioka and Chung-Yao Kao. The design tools that were developed and applied in paper D rely on the toolbox [13] and its underlying theory.
developed by Fujioka et al. In the appended paper we explain the concept of the sampled data model but we omit most of the technical detail.

The suggested controller was applied to two benchmark examples, a step-down and a step-up DC-DC converter, which were defined by the HYCON network. In the papers D1-D4 a number of different control approaches, including the $\mathcal{H}_\infty$ sampled data technique, are applied to the benchmarks. In papers D3 and D4, performance is compared in simulations and in the papers D1 and D2 the controllers are applied in experimental setups.

The experimental setups were provided by ETH, Zürich, Switzerland and the experiments were performed with help from Sébastien Mariéthoz.

6 References


6. REFERENCES


Stability Analysis of a Class of PWM Systems

Stefan Almér, Ulf Jönsson, Chung-Yao Kao and Jorge Mari

Abstract

The paper considers stability analysis of a class of pulse-width modulated (PWM) systems that incorporates several different switched mode DC-DC converters. The systems of the class typically have periodic solutions. A discrete time model is developed and used to prove stability of these solutions. Conditions for global and local exponential stability are derived using quadratic and piecewise quadratic Lyapunov functions. The state space is partitioned and the stability conditions are verified by checking a set of coupled LMIs.

Keywords: Pulse-width modulated systems, Stability analysis, Lyapunov methods, Sampled data modeling, DC-DC converters

1 Introduction

The paper presents a method for stability analysis of a class of pulse-width modulated (PWM) systems. The systems switch periodically between two affine vector fields to create a periodic solution at stationarity. The only control variable is the so-called duty cycle which determines the fraction of time each vector field is active.

The motivation for the analysis comes mainly from switched mode DC-DC converters [8] which are used extensively in power supplies of various electronic circuits. However, PWM systems are found in a wide range of applications, ranging from power conversion to hydraulic systems, see e.g., [4, 6, 15, 17].

Conventionally, DC-DC converters are controlled using analog PWM techniques that rely on a comparator ramp function. In this paper we consider a switching technique referred to as digital PWM where the switching is based on the sampled state. Digital PWM offers advantages such as being less sensitive to noise and aging of components and has received much attention recently, see e.g., [17]. It should be
noted that analog PWM can also be treated in our framework. See [2] for a detailed description.

Much of the reported analysis on PWM systems is based on the averaging approach [6, 13]. However, averaging is only an approximation of the low frequency system dynamics and it requires sufficiently high switching frequency to be adequate. The contribution of this paper is to provide a systematic method for stability analysis which does not resort to averaging or linearization.

Our starting point is a stationary periodic solution and we proceed to derive criteria for stability and uniqueness of such a solution. The first step is to introduce an equivalent discrete time model of the PWM system, similar to what is done in [11, 12, 14]. Our contribution compared to these works is to provide a systematic procedure to search for Lyapunov functions. We exploit the fact that the discrete time model is affine when the state is restricted to certain hyperplanes and state necessary and sufficient conditions for the existence of a quadratic Lyapunov function. These conditions can be verified by checking a set of coupled linear matrix inequalities (LMIs). The structure of the feedback control is used to partition the state space and each region of the partition implies a different LMI. In some of the regions it is necessary to sweep the duty cycle over its domain of definition ([0, 1]). This is analogous to the time axis sweep used to compute quadratic surface Lyapunov functions in [7, 16], which has proved to be an effective approach for analysis of piecewise linear systems. Note that we consider periodically switched systems and are therefore distinguished from [7, 16].

Conditions are presented for both local and global stability. In the local case we provide an estimate of the region of attraction. To make the conditions less conservative we also consider an extension of the quadratic Lyapunov function to the piecewise quadratic Lyapunov function by using results in [5, 10].

The paper is organized as follows: Section 2 introduces the class of systems considered and describes the PWM technique used for control. In Section 3 the discrete time model is developed and in Section 4 conditions for global exponential stability are derived. Section 5 derives further stability conditions using piecewise quadratic Lyapunov functions and Section 6 presents conditions for local stability. Section 7 provides an example and Section 8 deals with computational issues in verifying the stability conditions. Section 9 discusses modeling aspects and how the class of systems considered can be generalized. Finally, Section 10 contains our conclusions.

Notation

For a given continuous time state vector $x(t)$ a discrete time signal $x_k$ is defined according to $x_k := x(kT_s)$ where $T_s > 0$ is the sampling period. $\bar{x}_k$ denotes an extension of the signal $x_k$ according to $\bar{x}_k := [x'_k 1]'$. Let $\Omega$ be a subset of $\mathbb{R}^n$ containing the origin. We say that a dynamic system $y_{k+1} = f(y_k)$ is exponentially stable on $\Omega$ if for each $y_0 \in \Omega$ there exists $c > 0$ and $|\alpha| < 1$ such that $\|y_k\| \leq c \alpha^k \|y_0\|$ where $\|y\| = \sqrt{y'y}$. 
2 A class of PWM systems

This paper presents a method for stability analysis of a class of PWM systems which incorporates many different DC-DC converters. The systems of the class switch periodically between two modes and are of the form

\[ \dot{x}(t) = (A_0 + s(t)A_1)x(t) + B_0 + s(t)B_1 \]  \hspace{1cm} (A.1)

where \( x(t) \in \mathbb{R}^n, A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n}, i = 0, 1 \) and the switching signal \( s(t) \) has the form

\[ s(t) = \begin{cases} 1, & t \in [kT_s, (k + d_k)T_s) \\ 0, & t \in [(k + d_k)T_s, (k + 1)T_s). \end{cases} \]  \hspace{1cm} (A.2)

Here, \( T_s > 0 \) is a fixed time period and the duty cycle \( d_k \in [0, 1], k \in \mathbb{Z}_{+} \), determines the fraction of each time period that is spent in each mode. The system dynamics are controlled by varying the duty cycle. This control technique is referred to as pulse-width modulation (PWM) Note that the state vector \( x \) may also contain the state of a dynamic controller.

DC-DC converters are typically designed to have a periodic solution when the duty cycle is constant. This justifies the following assumption on (A.1).

**Assumption 2.1.** There exists at least one point \( (x^0, d^0) \) where \( x^0 \in \mathbb{R}^n, d^0 \in [0, 1] \) such that (A.1) attains a periodic solution \( x^0(t) = x^0(t + T_s) \) when the initial state is \( x(0) = x^0 \) and the duty cycle is fixed so that \( d_k = d^0 \) \( \forall k \).

Different types of PWM may be used in DC-DC converters. The paper considers a technique referred to as digital PWM [17] where the state is sampled at time instants \( kT_s \) to determine a duty cycle \( d_k \). The \( k^{th} \) duty cycle is determined by linear feedback according to

\[ d_k = \psi \left( d^0 + F(x(kT_s) - x^0) \right) \]  \hspace{1cm} (A.3)

where \( (x^0, d^0) \) satisfy the conditions in Assumption 2.1, \( F \) is a vector and \( \psi \) is the saturation

\[ \psi(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1. \end{cases} \]  \hspace{1cm} (A.4)

**Remark 2.1.** The method of analysis proposed in this paper can be extended to systems with \( n \) switching signals \( s_i(t) \) that switch among \( 2^n \) subsystems. Here we consider the case where there is only one switching signal because it serves to present our techniques in a more transparent way. The extension has been carried out. Details can be found in [1].
3 Sampled data modeling

Consider the system (A.1) with the switching function \( s(t) \) defined in (A.2)-(A.3). Over a time period \([kT_s, (k+1)T_s]\) with corresponding duty cycle \( d_k \) the state evolves according to

\[
\dot{x} = \begin{cases} 
(A_0 + A_1) x + B_0 + B_1, & t \in [kT_s, (k + d_k)T_s) \\
A_0 x + B_0, & t \in [(k + d_k)T_s, (k + 1)T_s). 
\end{cases}
\]

The solutions on the two intervals are stacked to yield a discrete time dynamic system describing the state at switching instants \( kT_s \). Defining \( x_k := x(kT_s) \) the system is written as

\[
x_{k+1} = \Phi(d_k) x_k + \Gamma(d_k)
\]

where the matrices \( \Phi(d) \) and \( \Gamma(d) \) are nonlinear functions of \( d \). Explicit expressions are found in the appendix.

By assumption 2.1 there is at least one fixed point of the discrete time system, i.e., a point \( (x^0, d^0) \in \mathbb{R}^n \times [0, 1] \) that satisfies

\[
x^0 = \Phi(d^0)x^0 + \Gamma(d^0).
\]  

(A.5)

Because the dynamics between switching instants are linear, the periodic solution \( x^0(t) \) corresponding to a fixed point \( (x^0, d^0) \) is unique. Furthermore, if for a fixed \( d^0 \) the state transition matrix \( \hat{\Phi}(d^0) \) and feedback matrix \( F \) satisfy

\[
\ker \left( \begin{bmatrix} \hat{\Phi}(d^0) - I \\ F \end{bmatrix} \right) = \{0\}
\]

then there is at most one solution \( x^0(t) \) with duty cycle \( d^0 \). Sufficient conditions for a periodic solution \( x^0(t) \) to be unique will follow from the stability analysis.

To prove stability of a periodic solution \( x^0(t) \) corresponding to \( (x^0, d^0) \) we consider the discrete time system and define the deviation \( y_k := x_k - x^0 \) from the fixed point. The dynamics of the error are

\[
y_{k+1} = \Phi(d_k)y_k + \Gamma(d_k) \\
d_k = \psi(d^0 + F y_k)
\]

(A.6a)  

(A.6b)

where \( \psi \) is the saturation in (A.4) and

\[
\Phi(d) = \hat{\Phi}(d), \quad \Gamma(d) = (\hat{\Phi}(d) - \Phi(d^0)) x^0 + \hat{\Gamma}(d) - \Gamma(d^0).
\]

The discrete time error model (A.6) describes how a perturbation \( x(t) - x^0(t) \) of a periodic solution of (A.1)-(A.3) evolves at the switching instants \( kT_s \). If the fixed point \( (0, d^0) \) of (A.6) is globally exponentially stable, then the periodic solution \( x^0(t) \) is globally exponentially stable. Furthermore, \( x^0(t) \) is unique since the fixed point is unique.
4 Stability analysis

To obtain sufficient conditions for the error model to be exponentially stable we use discrete time Lyapunov theory and quadratic Lyapunov functions. The following well-known result is fundamental to our analysis.

Lemma 4.1. Consider a sequence \( \{y_k\} \in \mathbb{R}^n \). If there is a matrix \( P = P' \) and real numbers \( \alpha_2 > \alpha_1 > 0, \beta > 0 \) such that \( \alpha_1 \|y_k\|^2 \leq y'_k P y_k \leq \alpha_2 \|y_k\|^2 \) and

\[
y'_{k+1} P y_{k+1} - y'_k P y_k \leq -\beta \|y_k\|^2
\]

for all \( k \) then the sequence is exponentially stable (about the origin) with rate of convergence

\[
\|y_k\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \sqrt{\left( 1 - \frac{\beta}{\alpha_2} \right)^k} \|y_0\|.
\]

Consider a sequence \( \{y_k\} \) satisfying the error dynamics (A.6). The inequality of Lemma 4.1 is written in the quasi-quadratic form.

\[
y'_{k} \Pi(d_k, \beta, P) y_{k} \leq 0 \quad (A.7)
\]

where \( \bar{y}_k = [y'_k \; 1]' \) and

\[
\Pi(d_k, \beta, P) = \begin{bmatrix}
\Phi(d_k)' P \Phi(d_k) - P + \beta I & \Phi(d_k)' P \Gamma(d_k) \\
\Gamma(d_k)' P \Phi(d_k) & \Gamma(d_k)' P \Gamma(d_k)
\end{bmatrix}.
\]

To verify inequality (A.7) the state space is partitioned into three (unbounded) polyhedral sets with pairwise disjoint interior. The sets are denoted

\[
S_i = \left\{ y \mid \bar{E}_i \bar{y} = \begin{bmatrix} E_i & e_i \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix} \geq 0 \right\}
\]

where

\[
\begin{bmatrix} E_0 & e_0 \end{bmatrix} = \begin{bmatrix} -F & -d^0 \end{bmatrix}, \quad \begin{bmatrix} E_1 & e_1 \end{bmatrix} = \begin{bmatrix} F & d^0 - 1 \end{bmatrix}
\]

\[
\begin{bmatrix} E_2 & e_2 \end{bmatrix} = \begin{bmatrix} -F & 1 - d^0 \\ F & d^0 \end{bmatrix}
\]

and the vector inequality \( z \geq 0 \) is component-wise. With the matrices \( \bar{E}_i \) defined above, \( S_0 \) and \( S_1 \) are the regions where the feedback \( d_k = \psi(d^0 + F y_k) \) saturates (at \( d = 0 \) and \( d = 1 \) respectively) and \( S_2 \) is the non-saturated region.

On \( S_i \), \( i = 0, 1 \) the matrix \( \Pi(d, \beta, P) \) is independent of \( y \). There the inequality (A.7) can be stated in terms of LMIs and the constraints \( \bar{E}_i \bar{y} \geq 0 \) can be relaxed using the S-procedure, see e.g., [3]. The relaxation is lossless, i.e., the relaxed formulation is equivalent to the constrained LMI condition.

On \( S_2 \), the inequality (A.7) is formulated as a parameterized matrix inequality. To this end we use the structure of the feedback matrix \( F \) and introduce a
parameterization of $S_2$. Let $F^\perp$ be the orthogonal complement of $F$ which satisfies $FF^\perp = 0$ and $(F^\perp)'F^\perp = I$ and denote $Z := F^\perp$. The set $S_2$ can be expressed as

$$S_2 = \{ y \mid y = Z \xi + \gamma(d), \xi \in \mathbb{R}^{n-1}, d \in [0,1] \}$$

where

$$\gamma(d) = \frac{d - d^0}{F F^\prime - F'}.$$

Replacing $y$ with $Z \xi + \gamma(d)$, condition (A.7) is formulated as a matrix inequality parameterized in the duty cycle $d$. Lemma 4.1 and the subsequent discussion imply the following result.

**Proposition 4.1.** Consider the error model (A.6). If there are real numbers $\beta > 0$, $\tau_i \geq 0$ and a matrix $P = P' > 0$ such that

$$\Pi(i, \beta, P) + \tau_i \begin{bmatrix} 0 & E_i' \\ E_i & 2e_i \end{bmatrix} \leq 0 \quad (A.8)$$

for $i = 0, 1$ and

$$\begin{bmatrix} Z & \gamma(d)' \\ 0 & 1 \end{bmatrix} \Pi(d, \beta, P) \begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix} \leq 0 \quad (A.9)$$

for $d \in [0,1]$, then (A.6) is globally exponentially stable and the periodic solution $x^0(t)$ of (A.1)-(A.3) corresponding to $(x^0, d^0)$ is globally exponentially stable.

**Remark 4.1.** Since the S-procedure is lossless, conditions (A.8) and (A.9) are also necessary for the existence of a quadratic Lyapunov function proving global stability of (A.1)-(A.3).

**Remark 4.2.** In order to check the last condition of Proposition 4.1 we partition the interval $[0,1]$ into points $d_i$. For a fixed $d_i$ inequality (A.9) is an LMI and we thus search for a solution $P$ by solving a set of coupled LMIs. In Section 8 it is shown that one can indeed find a matrix $P$ satisfying (A.9) by solving a finite set of LMIs.

**Remark 4.3.** The stability conditions presented in Proposition 4.1 rely on two key features of the error dynamics (A.6). Firstly, equation (A.6a) is affine in the state $y$ although nonlinear in the feedback control $d$ (the duty cycle). Secondly, the feedback control $d$, defined in (A.6b), depends on the state $y$ in such a way that any constant $d$ corresponds to a hyperplane or a half-space in the state space. The necessary and sufficient conditions for existence of a quadratic Lyapunov function presented in Proposition 1 apply not only to systems of the form (A.1), but to any system having the two features described above. A similar idea was used in [7].
5 Extended stability analysis

Proposition 4.1 was derived using quadratic Lyapunov functions and only provides sufficient conditions for stability. Should a system fail to satisfy the conditions of Proposition 4.1, a natural way to proceed is to consider piecewise quadratic Lyapunov functions. In this section we consider functions of the form

\[ V(y) = \begin{cases} 
\dot{y}'P_i\dot{y}, & y \in \text{int}(\mathcal{S}_i), \ i = 0, 1 \\
\dot{y}'P_2\dot{y}, & y \in \mathcal{S}_2
\end{cases} \]  \hspace{1cm} (A.10)

where \( \dot{y} = [y' \ 1]' \) and \( \text{int}(\mathcal{S}_i) \) denotes the interior of \( \mathcal{S}_i \). The matrices \( P_i = P_i' \) have the structure

\[ P_2 = \begin{bmatrix} P_{211} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_i = \begin{bmatrix} P_{i11} & P_{i12} \\ P_{i12}' & P_{i22} \end{bmatrix}, \ i = 0, 1 \]  \hspace{1cm} (A.11)

and are required to satisfy

\[ \dot{y}'P_i\dot{y} \geq \alpha_1||y||^2 \quad \forall y \in \mathcal{S}_i. \]

where \( \alpha_1 > 0 \). These conditions can be relaxed without loss using the S-procedure [3] and stated equivalently in terms of LMIs. For \( i = 0, 1 \), \( P_i \) need not be positive definite in general. However, the blocks \( P_{111} \) must be positive definite for all \( i \).

Studying the forward difference \( V(y_{k+1}) - V(y_k) \) of functions of the form (A.10) leads to the following result. A proof can be found in e.g., [5].

**Lemma 5.1.** Consider a sequence \( \{y_k\} \in \mathbb{R}^n \) and let \( \mathcal{S}_{ij} = \{y_k \mid y_k \in \mathcal{S}_i, \ y_{k+1} \in \mathcal{S}_j\} \). If there are matrices \( P_i = P_i' \) as in (A.11) and real numbers \( \alpha_2 > \alpha_1 > 0, \ \beta > 0 \) such that \( \alpha_1||y_k||^2 \leq \dot{y}'_k P_i \dot{y}_k \leq \alpha_2||y_k||^2 \) for all \( y_k \in \mathcal{S}_i, \ i = 0, 1, 2 \) and for all \( (i, j) \) it holds

\[ \dot{y}'_{k+1} P_j \dot{y}_{k+1} - \dot{y}'_k P_i \dot{y}_k \leq -\beta||y_k||^2 \]

for all \( y_k \in \mathcal{S}_{ij} \), then the sequence is exponentially stable (about the origin).

Consider a sequence \( \{y_k\} \) satisfying the error dynamics (A.6). The extended state \( \tilde{y} \) satisfies

\[ \dot{\tilde{y}}_{k+1} = \tilde{\Phi}(d_k)\tilde{y}_k \]

where

\[ \tilde{\Phi}(d) = \begin{bmatrix} \Phi(d) & \Gamma(d) \\ 0 & 1 \end{bmatrix}. \]

Using this notation the inequality of Lemma 5.1 is written

\[ \dot{\tilde{y}}'_k \Pi_{ij}(d_k)\tilde{y}_k \leq 0, \quad \forall y_k \in \mathcal{S}_{ij} \]  \hspace{1cm} (A.12)
where
\[
\Pi_{ij}(d) = \Phi(d)'P_j\Phi(d) - P_i + \beta \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.
\]

To formulate (A.12) in terms of LMI we follow the rationale of [10]: The sets $S_{ij}$ are described according to $S_{ij} = \{ y \mid G_{ij}(d)y \geq 0 \}$ where
\[
G_{ij}(d) = \begin{bmatrix} E_i \\ \bar{E}_j\Phi(d) \end{bmatrix}
\]
and the matrices $G_{ij}$ are used to define matrices $\bar{G}_{ij}$ and $H_{ij}$ according to
\[
\bar{G}_{ij}(d) = \begin{bmatrix} G_{ij}(d) \\ 0 \\ 1 \end{bmatrix}, \quad H_{ij}(d) = G_{ij}(d)'U_{ij}\bar{G}_{ij}(d)
\]
where $U_{ij}$ is a matrix of appropriate dimensions with non-negative entries. Notice that with such $U_{ij}$, the matrix $H_{ij}$ satisfies $y'H_{ij}y \geq 0 \forall y \in S_{ij}$, and inequality (A.12) is therefore implied by
\[
y'y\Pi_{ij}(d)y \leq 0 \quad \forall y \text{ s.t. } y'H_{ij}(d)y \geq 0.
\]

(A.13)

For $(i, j) \in \{0, 1\} \times \{0, 1, 2\}$ the matrices $\Pi_{ij}$ and $H_{ij}$ are constant. Inequality (A.13) is relaxed and stated in terms of LMI. For $(i, j) \in \{2\} \times \{0, 1, 2\}$ we use the parameterization $y = Z\xi + \gamma(d)$ and proceed with a sweeping procedure analogous to the one in the previous section. For fixed $d \in [0, 1]$ inequality (A.13) is equivalently expressed as
\[
\begin{bmatrix} \xi' \\ 1 \end{bmatrix}' \begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix}' \Pi(d, \beta, P) \begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \leq 0
\]
\forall \xi \in \mathbb{R}^{n-1} \text{ s.t.}
\begin{bmatrix} \xi' \\ 1 \end{bmatrix}' \begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix}' \bar{G}_{ij}(d)'U_{ij}\bar{G}_{ij}(d) \begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \geq 0.
\]

For each $d \in [0, 1]$ we choose a matrix $U_{ij}$ and relax the inequality. Lemma 5.1 and the subsequent discussion imply the following result.

**Proposition 5.1.** Consider the error model (A.6). If there are matrices $P_i = P_i'$ as in (A.11) satisfying
\[
P_{211} > 0, \quad P_i - \tau_i \begin{bmatrix} 0 & E_i' \\ E_i & 2\epsilon_i \end{bmatrix} > 0, \quad i = 0, 1
\]
for some $\tau_i \geq 0$ and if there are matrices $U_{ij}, U_{ij}(d)$ with non-negative components such that
\[
\Pi_{ij}(i) + \bar{G}_{ij}(i)'U_{ij}\bar{G}_{ij}(i) \leq 0
\]
for \((i, j) \in \{0, 1\} \times \{0, 1, 2\}\) and
\[
\begin{bmatrix}
Z \gamma(d)
\end{bmatrix}' \left( \Pi_{ij}(d) + \bar{G}_{ij}(d)'U_{ij}(d)\bar{G}_{ij}(d) \right) \begin{bmatrix}
Z \gamma(d)
\end{bmatrix} \leq 0
\]

for \((i, j) \in \{2\} \times \{0, 1, 2\}\) and \(d \in [0, 1]\), then (A.6) is globally exponentially stable
and the periodic solution \(x^0(t)\) of (A.1)-(A.3) corresponding to \((x^0, d^0)\) is globally
exponentially stable.

**Remark 5.1.** If there is no step transition \(i \rightarrow j\) (i.e., no point \(y_k\) such that \(y_k \in S_i\)
and \(y_{k+1} \in S_j\)) then the corresponding inequality of Proposition 5.1 is redundant and
can be removed. A systematic way to determine if there is a step transition \(i \rightarrow j\) is
described in [2].

6 Local results

If the error model (A.6) does not satisfy the conditions for global stability, we may
proceed to examine local stability. In the section below the region of attraction is
estimated by using quadratic Lyapunov functions. We consider a fixed ellipsoid
\(E_Q = \{y \mid y'Qy \leq 1\}\) and search for a contractive set in \(E_Q\). The idea, which
straightforwardly follows Lemma 4.1, is stated as follows.

**Lemma 6.1.** Consider a sequence \(\{y_k\} \in \mathbb{R}^n\) and let the ellipsoid \(E_Q = \{y \mid y'Qy \leq 1\}\) be given. If there is a matrix \(P = P'\) and real numbers \(\alpha_2 > \alpha_1 > 0, \beta > 0\) such that
\[
\alpha_1 ||y_k||^2 \leq y_k'P_y y_k \leq \alpha_2 ||y_k||^2 \quad \forall y_k
\]

for all \(y_k \in E_Q\) and
\[
E_P := \{y \mid y'P_y \leq 1\} \subseteq E_Q
\]

then, \(y_0 \in E_P\) implies that the sequence \(\{y_k\}_{k=0}^{\infty} \subseteq E_P\) will converge exponentially
to the origin.

When applied to a sequence satisfying the error dynamics (A.6) the inequalities of Lemma 6.1 are written

\[
y'\Pi(d, \beta, P)y \leq 0 \quad \forall y \in E_Q \quad (A.14a)
\]
\[
y'Qy \leq 1 \quad \forall y \text{ s.t } y'Py \leq 1 \quad (A.14b)
\]

where \(\Pi(d, \beta, P)\) is the matrix in (A.7). For \(y \in S_i, \ i = 0, 1\), the matrix \(\Pi(d, \beta, P)\)
reduces to a constant matrix and (A.14a) can be relaxed and stated in terms of
LMIs. For \(y \in S_2\) we proceed with a sweeping procedure analogous to what is
done in the previous section. Lemma 6.1 and the subsequent discussion imply the
following result.
Proposition 6.1. Consider the error model (A.6) and let the ellipsoid 
\( E_Q = \{ y \mid y' Q y \leq 1 \} \) be given. If there are real numbers \( \beta > 0, \lambda > 0, \tau_{k_i} \geq 0 \), a non-negative function \( \tau(d) : [0, 1] \to \mathbb{R} \) and a matrix \( P = P' > 0 \) such that

\[
\lambda \begin{bmatrix} Q & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & -1 \end{bmatrix} \leq 0
\]

\[
\Pi(i, \beta, P) + \tau_{1i} \begin{bmatrix} 0 & E'_i \\ E_i & 2e_i \end{bmatrix} + \tau_{2i} \begin{bmatrix} -Q & 0 \\ 0 & 1 \end{bmatrix} \leq 0
\]

for \( i = 0, 1 \) and

\[
\begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix}' \left( \Pi(d, \beta, P) + \tau(d) \begin{bmatrix} -Q & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix} \leq 0
\]

for \( d \in [0, 1] \), then (A.6) is exponentially stable on \( E_P = \{ y \mid y' P y \leq 1 \} \). The periodic solution \( x_0(t) \) of (A.1)-(A.3) corresponding to \( (x_0, d^0) \) converges exponentially whenever the initial point \( x_0(0) \) is in \( E_P \).

**Remark 6.1.** When solving for a matrix \( P \) (and other parameters) satisfying the conditions of Proposition 6.1 we do not only want a feasible point but also want to maximize the volume of \( E_P \). One way to obtain a “large” \( E_P \) is to minimize the largest eigenvalue of \( P \). Thus we consider the problem

\[
\min \mu
\]

\[
s.t \quad \mu > 0, \quad P \preceq \mu I
\]

while observing the conditions of Proposition 6.1.

**Remark 6.2.** As an alternative to the first inequality of Lemma 6.1 we may impose the condition

\[
y_{k+1}P_{y_{k+1}} - y_kP_{y_k} \leq -\beta \| y_k \|^2
\]

for all \( y_k \) s.t \( y_k \in E_Q \) and \( y_{k+1} \in E_Q \). This could lead to a less conservative result since there is an additional term to relax in the S-procedure. However, we omit this extra term for clarity of the presentation.

**Remark 6.3.** Local stability results have also been derived using the piecewise quadratic Lyapunov function presented in Section 5. Such results can be found in [2].

### 7 Example

To illustrate the theory presented in this paper we consider a simple numerical example: The synchronous step-up converter [8] depicted in Fig. A.1 is a switched
linear system of the form (A.1). The state is chosen as $x = [x_1 \ x_2]'$ where $x_1$ is the inductor current and $x_2$ is the capacitor voltage. The system matrices are

$$A_0 = \begin{bmatrix} -r_l/x_l & 0 \\ 0 & -1/(r_o x_c) \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1/x_l \\ 1/x_c & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} v_s/x_l \\ 0 \end{bmatrix}$$

and $B_1$ is the zero vector. The parameter values are taken from [9]. They are $x_l = 10$ μH, $x_c = 50$ μF, $r_o = 30$ Ω, $r_l = 1$ μΩ, $v_s = 1$ V and the switch period is $T_s = 20$ μs. Finally, the reference output voltage is $v_{\text{ref}} = 5$ V. In the section below we prove that the system is globally exponentially stable with a control of the form (A.2)-(A.3).

The stationary duty cycle is chosen to make the average output voltage equal $v_{\text{ref}}$ at stationarity. This yields $d^0 = 0.8003$ and the corresponding fixed point satisfying (A.5) is $x^0 = [0.0323 \ 5.0235]'$. The feedback vector is chosen as

$$F = [-0.0356 \ -0.0291].$$

The corresponding error model does not satisfy the conditions of Proposition 4.1 and we therefore proceed to search for a piecewise quadratic Lyapunov function. The inequalities of Proposition 5.1 are solved with $\beta = 1 \cdot 10^{-6}$. The domain $[0,1]$ of the duty cycle is partitioned with grid size $h = 1 \cdot 10^{-6}$ and the solution satisfies the LMIs obtained. We conclude that the system is globally exponentially stable. Fig. A.2 shows the partition of the state space and level sets of the Lyapunov function obtained. Three trajectories are also plotted. The initial states are marked by circles. Note that the Lyapunov function is not continuous.

8 **Computational aspects**

The stability criteria in this paper are all stated in terms of parameterized matrix inequalities. To find a solution satisfying these inequalities we grid the domain of the parameter and for each grid point we obtain an LMI. In this section we consider the inequality of Proposition 4.1. Assuming there is a matrix $P = P' > 0$ satisfying inequality (A.9) we show that such a matrix can be found by solving a finite number
of LMIs. This is done using the Lipschitz continuity of the matrix functions. The inequalities of Proposition 5.1 and 6.1 are dealt with analogously, see [2].

Let

$$\Psi(d, \beta, P) := \begin{bmatrix} Q(d, \beta, P) & S(d, P) \\ S(d, P)' & R(d, \beta, P) \end{bmatrix}, \quad \Pi(d, \beta, P) := \begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix},$$

be the matrix operator in Proposition 4.1 and assume there is a matrix $P = P' > 0$ satisfying

$$\Psi(d, \beta, P) \leq 0 \quad \forall d \in [0, 1] \tag{A.15}$$

for some $\beta > 0$. Note that $R(d, P)$ and $S(d, P)$ vanish at $d^0$ (since $\gamma(d^0) = 0$ and $\Gamma(d^0) = 0$). Hence, $\Psi(d)$ has a zero eigenvalue at $d^0$ and one cannot satisfy $\Psi(d) < 0 \quad \forall d \in [0, 1]$. Also note that if (A.15) holds for some $\beta > 0$, then by decreasing $\beta$ (A.15) can be satisfied with strict inequality $\forall d \in [0, 1] \setminus d^0$.

To find a matrix satisfying (A.15), let $\Delta > 0$ and let $\{0 = d_1 \leq d_2 \leq \cdots \leq d_N = 1\}$ be a partition of $[0, 1]$ such that $d^0 - d_j = d_{j+1} - d^0 = \Delta$ for some $j$ and $d_{i+1} - d_i \leq \Delta$ for $i \neq j$. Also let $B_\Delta := [d^0 - \Delta, d^0 + \Delta]$ be a neighborhood of $d^0$. By assumption, there is a matrix $P_\Delta = P'_\Delta > 0$ satisfying

$$\Psi(d_i, \beta, P_\Delta) \leq -\alpha_i I \quad i = 1, \ldots, N \tag{A.16}$$

where $\alpha_i > 0$ and such a matrix can be found by solving a set of LMI. In the sequel we use Lipschitz arguments to bound the eigenvalues of $\Psi$ in between grid points. However, because of the zero eigenvalue of $\Psi$, Lipschitz bounds will not apply directly near $d^0$. The region $B_\Delta$ requires some extra care and it is treated separately from the remaining set $[0, 1] \setminus B_\Delta$. The idea is illustrated in Fig. A.3.
First consider the set \([0, 1] \setminus B_\Delta\). For fixed \(P\) and \(\beta\) we define

\[
L_i[\Psi](\beta, P) := \operatorname{Lip}_{[d_i, d_{i+1}]}[\Psi(d_i, \beta, P)]
\]

for \(i = 1, \ldots, N\), \(i \neq j\) where

\[
\operatorname{Lip}_{[a, b]}[\Psi] = \inf\{L \mid \|\Psi(\delta_1, \beta, P) - \Psi(\delta_2, \beta, P)\| \leq L|\delta_1 - \delta_2| \forall \delta_1, \delta_2 \in [a, b]\}
\]

is the Lipschitz constant on \([a, b]\) and where the norm \(\|\cdot\|\) is the maximum singular value. For a fixed \(d\) the matrix \(\Psi(d, \beta, P)\) is affine in \(P\) and \(\beta\) and there are constants \(c_i > 0\) such that

\[
L_i[\Psi](\beta, P) \leq c_i\|P\| + \beta\eta_i \tag{A.17}
\]

where

\[
\eta_i := \operatorname{Lip}_{[d_i, d_{i+1}]}[\|\gamma(d)\|^2] = \max_{d \in [d_i, d_{i+1}]} \frac{2|d - d^0|}{FF^T}.
\]

We now bound the maximum eigenvalue of \(\Psi\) as follows: Suppose \(\delta \in [d_i, d_{i+1}]\), \(i \neq j\) and let \(\vec{v}\) be an eigenvector corresponding to the maximum eigenvalue of \(\Psi(\delta)\) with norm \(\|\vec{v}\| = 1\); it holds

\[
\lambda_{\text{max}}(\Psi(\delta)) = \vec{v}'\Psi(\delta)\vec{v}
\]

\[
= \vec{v}'\Psi(d_i)\vec{v} + \vec{v}'(\Psi(\delta) - \Psi(d_i))\vec{v}
\]

\[
\leq -\alpha_i + \sup_{\|v\|=1} \vec{v}'(\Psi(\delta) - \Psi(d_i))v
\]

\[
\leq -\alpha_i + \|\Psi(\delta) - \Psi(d_i)\|
\]

\[
\leq -\alpha_i + L_i\Delta
\]

where we have used the Lipschitz continuity of \(\Psi\) and the inequality \(\Psi(d_i) \leq -\alpha_i I\).

We now use (A.17) and conclude

\[
\lambda_{\text{max}}(\Psi(\delta, \beta, P\Delta)) \leq -\alpha_i + (c_i\|P\Delta\| + \beta\eta_i)\Delta
\]
for all \( \delta \in [d_i, d_{i+1}] \), \( i \neq j \). Thus, if the grid is dense enough to satisfy
\[
\Delta (c_i\|P_\Delta\| + \beta \eta_i) \leq \alpha_i
\]
then \( \Psi(d, \beta, P_\Delta) \leq 0 \) holds on \([0, 1] \setminus B_\Delta\).

Now consider the neighborhood \( B_\Delta \). The matrix inequality (A.15) is rearranged according to
\[
\Psi(d, 0, P) \leq -\beta \begin{bmatrix} I & 0 \\ 0 & \gamma(d)'(d) \end{bmatrix} \quad \forall d \in B_\Delta
\] (A.18)
where we have used that \( Z'Z = I \) and \( Z'(d) = 0 \). Let
\[
\bar{R}(d, P) = \begin{cases} (d - d^0)^{-2} R(d, P), & d \neq d^0 \\ \frac{1}{2} \frac{\partial^2 R}{\partial d^2}, & d = d^0 \end{cases}
\]
\[
\bar{S}(d, P) = \begin{cases} (d - d^0)^{-1} S(d, P), & d \neq d^0 \\ \frac{\partial S}{\partial d}, & d = d^0 \end{cases}
\]
and let
\[
\bar{\Psi}(d, \beta, P) = \begin{bmatrix} Q(d, P) & \bar{S}(d, P) \\ \bar{S}(d, P)' & \bar{R}(d, P) \end{bmatrix}.
\]
By pre and post-multiplying both sides of inequality (A.18) with the matrix
\[
\begin{bmatrix} I & 0 \\ 0 & (d - d^0)^{-1} \end{bmatrix}
\]
and noting that \( \bar{R} \) and \( \bar{S} \) are continuous at \( d^0 \) one can show that (A.18) is equivalent to
\[
\bar{\Psi}(d, 0, P) \leq -\bar{\beta} I \quad \forall d \in B_\Delta
\]
where \( \bar{\beta} = \beta \cdot \min\{1, 1/FF'\} \). Let
\[
L[\bar{\Psi}](P) := \text{Lip}_{B_\Delta} [\bar{\Psi}(d, 0, P)].
\]
For a fixed \( d \) the matrix \( \bar{\Psi}(d, 0, P) \) is linear in \( P \) and there is a number \( \bar{c} > 0 \) such that
\[
L[\bar{\Psi}](P) \leq \bar{c}\|P\|.
\]
Inequality (A.16) implies that \( \bar{\Psi}(d_j, \beta, P_\Delta) \leq -\bar{\beta} I \). Applying arguments analogous to the one above one can show that if the grid is dense enough to satisfy \( \Delta \bar{c}\|P_\Delta\| \leq \bar{\beta}/2 \) then \( \bar{\Psi}(d, \beta, P_\Delta) \leq 0 \) holds on \( B_\Delta \). The above discussion is summarized in the following proposition.
Proposition 8.1. Fix a grid size $\Delta > 0$ and let \( 0 = d_1 \leq d_2 \leq \ldots \leq d_N = 1 \) be a partition of \([0, 1]\) such that \( d^0 - d_j = d_{j+1} - d^0 = \Delta \) for some \( j \) and \( d_{i+1} - d_i \leq \Delta \) for \( i \neq j \). Let \( \eta_i = \max_{d[d_i,d_{i+1}]} \frac{2(d-d^0)}{FF^2} \) and let \( c_i, \bar{c} > 0 \) be such that \( L_i[\Psi](\beta, P) \leq c_i \|P\| + \beta \eta_i \) and \( L[\Psi](P) \leq \bar{c} \|P\| \). Suppose there are numbers \( \alpha_i, \beta, \beta_i, \beta_u > 0 \) and a matrix \( P_\Delta = P_\Delta^* \) such that \( \beta_i I \leq P_\Delta \leq \beta_u I \) and

\[
\Psi(d_i, \beta, P_\Delta) \leq -\alpha_i I, \quad i = 1, \ldots, N
\]

\[
\Delta(c_i \beta_u + \beta \eta_i) \leq \alpha_i, \quad i = 1, \ldots, N, \quad i \neq j
\]

\[
\Delta \bar{c} \beta_u \leq \bar{\beta}/2
\]

where \( \bar{\beta} = \beta \cdot \min\{1, 1/FF^2\} \). Then \( P_\Delta \) satisfies (A.15).

Remark 8.1. The proposition suggests an iterative procedure where the grid size is chosen ever smaller until the corresponding LMIs are feasible. If the system is quadratically stable the procedure will terminate.

9 Modeling aspects

The system (A.1) is not an exact description of a DC-DC converter. A simplifying assumption is made on the sampling where the samples \( x(kT_s) \) are taken at the switching instants. Because of the switching these measurements would be noisy. In practice one would sample “far away” from the switching instants at the center of the “on” or “off” intervals. The model used in this paper is easily adapted to this case by extending the state space. This is very similar to what is done in [1] when modeling systems with multiple switches.

Another limitation of the model is that it only describes bidirectional converters. A more general model would have to take into account the possibility of discontinuous conduction mode. The sampled data model would become state dependent which would require further partitioning of the state space and cause increased complexity. However, as long as the periodic solution is not in discontinuous conduction mode, the local results in the paper are applicable to converters with discontinuous conduction mode.

The paper only considers digital PWM. However, the approach used here can also be applied to systems controlled by analog PWM. See [2] for a detailed description.

As a final note we comment on the controller. We consider linear state feedback, but it is easily replaced by a dynamic controller in either continuous or discrete time. The controller dynamics are simply incorporated in the system matrices.

10 Conclusions

The paper provides a systematic method for stability analysis which does not resort to averaging or linearization. Sufficient conditions for global and local stability are stated in terms of matrix inequalities which can be checked numerically. The
conditions of Proposition 4.1 are both necessary and sufficient for the existence of a quadratic Lyapunov function. The method was successfully applied to a non-trivial example which was proved to be globally stable.

A topic for further research may be to find new classes of Lyapunov functions. By exploiting more of the structure of the system one might obtain less conservative results.

Appendix: Derivation of the discrete time dynamic system

Using the notation \( x_k = x(kT_s) \) and \( x_k^1 = x((k + d_k)T_s) \) we have

\[
\begin{align*}
    x_k^1 &= \Phi_1(d_k)x_k + \Gamma_1(d_k) \\
    x_{k+1} &= \Phi_2(d_k)x_k^1 + \Gamma_2(d_k)
\end{align*}
\]

where

\[
\begin{align*}
    \Phi_1(d_k) &= e^{(A_0 + A_1)d_k T_s} \\
    \Phi_2(d_k) &= e^{A_0(1-d_k)T_s} \\
    \Gamma_1(d_k) &= (A_0 + A_1)^{-1}(e^{(A_0 + A_1)d_k T_s} - I)(B_0 + B_1) \\
    \Gamma_2(d_k) &= A_0^{-1}(e^{A_0(1-d_k)T_s} - I)B_0.
\end{align*}
\]

Note that in the case where \( A \) is singular, the matrix operator \( A^{-1}(e^{AdT_s} - I) \) should be interpreted as the integral \( \int_0^{dT_s} e^{At}dt \). The solutions on the two intervals are stacked to yield the discrete-time dynamic system

\[
x_{k+1} = \Phi(d_k)x_k + \Gamma(d_k)
\]

where

\[
\begin{align*}
    \Phi(d_k) &= \Phi_2(d_k)\Phi_1(d_k), \\
    \Gamma(d_k) &= \Phi_2(d_k)\Gamma_1(d_k) + \Gamma_2(d_k).
\end{align*}
\]

11 References


11. REFERENCES


Dynamic Phasor Analysis of Pulse-Modulated Systems

Stefan Almér and Ulf Jönsson

Abstract

The paper considers stability analysis of a general class of pulse-modulated systems in a phasor dynamic framework. The dynamic phasor model exploits the cyclic nature of the modulation functions by representing the system dynamics in terms of a Fourier series expansion defined over a moving time window. The contribution of the paper is to show that a special type of periodic Lyapunov function can be used to analyze the system and that the analysis conditions become tractable for computation after truncation. The approach provides a trade-off between complexity and accuracy that includes standard state space averaged models as a special case. We also show how the dynamic phasor model can be used to derive a frequency domain input-to-state map which is analogous to the harmonic transfer function.

Keywords: Pulse-modulated systems, Stability analysis, Lyapunov methods, Dynamic phasors, Periodic systems, Harmonic transfer function

1 Introduction

We consider stability analysis of a class of pulse-modulated control systems. The system class includes pulse-width modulation (PWM) and pulse-amplitude modulation (PAM) as special cases and therefore includes many different applications. Our interest is primarily in power electronics applications but the results also apply to analysis of vibrational control systems and to oscillatory control of under-actuated systems [12].

Control design and analysis of pulse-modulated systems typically rely on state space averaged models [6, 11]. The widespread use of the state space averaged model is a testimony to its many good properties. However, control performance must be evaluated using the actual system dynamics. This claim is supported by
a number of observations. Firstly, it has been noted, see [5] and references therein, that the state space averaged model need not capture the stability properties of the system since the averaged model can be stable even though the actual system is not. Secondly, design specifications typically include bounds on the state/output. The classical averaged model does not represent the state/output ripple which is inherent in switched converters, and hence peak values are underestimated. Thirdly, the averaged model need not give an accurate description of harmonic properties such as disturbance propagation. This was discussed in [14].

The dynamic phasor model was introduced to overcome the limitations of the standard state space averaged model [3,17]. The dynamic phasor model is obtained from a Fourier series expansion of the system state over a moving time window. By truncating the Fourier series we obtain a tractable model that provides a trade-off between complexity and accuracy. In the paper we consider stability analysis of the dynamic phasor model using periodic Lyapunov functions. In the special case of a time periodic systems we obtain a new way of deriving and numerically computing skew truncated harmonic Lyapunov equations and we thus complement [24]. The analysis is also generalized to the case of sampled feedback, which is the main result of the paper. Furthermore, we use the dynamic phasor model to derive an approximate harmonic transfer function [13,16,20,23] which describes the response of the non-periodic systems to periodic disturbances.

The main difficulty is that the Fourier series expansion of the system state over a given time window in general does not converge uniformly and this makes the analysis fairly complicated. The convergence problems were revealed in [18] where the author considers stability analysis based on quadratic Lyapunov functions. In this paper we also use a Lyapunov approach, but we consider different systems and unlike [18], the Lyapunov candidates are time varying.

The paper is outlined as follows. In Section 2 we introduce the class of systems considered. In Section 3 the dynamic phasor model is presented and in Section 4 we introduce the periodic functions which serve as Lyapunov candidates. In Section 5 we consider stability analysis for the special case where the systems are linear time periodic and in Section 6 we consider stability analysis for the general case where the systems are controlled by feedback. Section 7 outlines how the truncated dynamic phasor model can be used for harmonic analysis and in Section 8 we provide two examples to illustrate the theory developed in the paper. Finally, in Section 9 we comment on the form of the feedback which has to be chosen in a special way to conform with the dynamic phasor model. Proofs are collected in Sections 11-15.

**Notation**

In this paper $L_2[-T_s, 0]$ denotes the set of square integrable functions $x : [-T_s, 0] \to \mathbb{R}^n$ with inner product

$$\langle x, y \rangle_{L_2} := \frac{1}{T_s} \int_{-T_s}^{0} x(\tau)'y(\tau)d\tau$$
and corresponding norm \( \|x\|_{L_2} := \langle x, x \rangle_{L_2}^{1/2} \). The set \( L_2[t - T_s, t] \) is defined analogously. \( L_2 \) denotes the set of square summable sequences \( x = \{x_k\}_{k=-\infty}^{\infty} \) where \( x_k \in \mathbb{C}^n \) satisfies \( \bar{x}_k = x_{-k} \) where \( \bar{x}_k \) is the complex conjugate of \( x_k \). The set is equipped with inner product

\[
\langle x, y \rangle_{L_2} := \sum_{k=-\infty}^{\infty} x_k^* y_k
\]

and corresponding norm \( \|x\|_{L_2} := \langle x, x \rangle_{L_2}^{1/2} \). Note that the condition \( \bar{x}_k = x_{-k} \) implies that \( L_2 \) denotes a smaller set than usual. As a final remark on the norms we note that the subindex will often be omitted when the norm is clear from context.

\( \pi_N \) is used to denote projections on both \( L_2 \) and \( L_2 \) as follows: Firstly, the projection \( \pi_N : L_2[-T_s, 0] \rightarrow L_2[-T_s, 0] \) projects onto the subspace spanned by the Fourier basis \( \{e^{jk\omega_s t}\}_{k=-N}^{N} \). Secondly, \( \pi_N : L_2 \rightarrow L_2 \) is defined by the relation

\[
(\pi_N x)_k = \begin{cases} x_k, & |k| \leq N \\ 0, & |k| > N. \end{cases}
\]

The interpretation of \( \pi_N \) will be clear from context.

The transformation \( T \) maps an infinite complex vector \( \xi = [\ldots, \xi_1^*, \xi_0^*, \xi_{-1}^* \ldots]^* \) where \( \xi_k \in \mathbb{C} \) to a doubly infinite dimensional block Toeplitz matrix according to

\[
T[\xi] := \begin{bmatrix}
\ddots & \cdots & \cdots \\
\cdot & \xi_0 I_n & \xi_1 I_n & \xi_2 I_n \\
\cdot & \cdot & \xi_0 I_n & \xi_1 I_n & \xi_2 I_n & \cdots \\
\cdot & \cdot & \cdot & \xi_0 I_n & \xi_1 I_n & \xi_2 I_n & \cdots \\
\ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

For a periodic function \( s \), \( T[s] \) is defined as above with the \( k \)th Fourier coefficient of \( s \) in place of \( \xi_k \). \( I \) is used for the identity operator on both finite and infinite dimensional spaces and \( \sigma \) denotes the maximum singular value of a matrix and finally \( \otimes \) is the Kronecker product.

## 2 A class of systems

Consider the class of systems

\[
\dot{x}(t) = A(t, x)x(t) + B(t, x) \tag{B.1}
\]

where \( x \in \mathbb{R}^n \) and

\[
A(t, x) := \sum_{i=1}^{m} s_i(t, x) A_i \tag{B.2}
\]

\[
B(t, x) := \sum_{i=1}^{m} \left( s_i(t, x) - s_i(a_i^0, x^0)(t) \right) B_i
\]
where $A_i$, $B_i$ are constant matrices and the functions $s_i$ are composed of a periodic function $f_i$ and a pulse modulation function according to

$$s_i(t, x) := \begin{cases} 
  a_{i,k} f_i(t), & t \in [kT_s, (k + d_{i,k})T_s) \\
  0, & t \in [(k + d_{i,k})T_s, (k + 1)T_s).
\end{cases} \quad (B.3)$$

Here, $f_i$ are periodic $C^1$ functions of period $T_s > 0$ assumed to satisfy $|f_i| \leq 1$ and the pulse amplitudes $a_{i,k} \in [-1, 1]$ and duty cycles $d_{i,k} \in [0, 1]$ are determined by sampling the state at time instants $kT_s$. The exact form of the feedback is defined in Section 6. Finally we define $s_{i,(a_i,d_i)}(t)$ according to (B.3) but with constant pulse amplitudes and duty cycles so that $a_{i,k} = a_i \forall k$ and $d_{i,k} = d_i \forall k$. It is assumed that $s_i(t, 0) = s_{i,(a_i^0,d_i^0)}(t)$ and thus, $x = 0$ is an equilibrium of (B.1).

The modulation functions $s_i$ generate a sequence of pulses. For each sampling interval $[kT_s, (k + 1)T_s]$ parameters $a_{i,k}$ and $d_{i,k}$ are determined to shape the amplitude and duration of the pulse. Two cases of special interest are pure pulse-width modulation (PWM) and pure pulse-amplitude modulation (PAM). In the case of PWM, the pulse amplitude is fixed ($a_{i,k} = a_i \forall k$) and the duty cycles $d_{i,k}$ are the control variables. In the case of PAM, the duty cycle is fixed ($d_{i,k} = d_i \forall k$) and the pulse amplitudes $a_{i,k}$ are the control variables. Note that both $a_{i,k}$ and $d_{i,k}$ may be constant and equal to $a_i^0$ and $d_i^0$ respectively. In this case the affine term $B(t, x)$ vanishes and the system (B.1) is linear time periodic (LTP).

Note that the inequalities $|f_i| \leq 1$ and $|a_{i,k}| \leq 1$ imply that $|s_i| \leq 1$. Since $a_{i,k}$ and $d_{i,k}$ are constant in each sampling interval and $f_i \in C^1$ it also holds that the functions $s_i$ are piece-wise continuous. It follows that the dynamics can be embedded into the system class $\dot{x}(t) = A(t)x(t) + B(t)$, where $A(t)$, $B(t)$ are piece-wise continuous and bounded. It thus follows that there exists a unique absolutely continuous solution on any finite time interval, see [9]. We will in particular use that for any $t \geq 0$, $x(t) \in L_\infty[t - T_s, t]$ and thus $x(t) \in L_2[t - T_s, t]$ and moreover that $\dot{x}(t) \in L_2[t - T_s, t]$. We will later explore the cyclic structure of (B.1) to derive conditions for (practical) stability of the equilibrium at $x = 0$ of (B.1).

The equations (B.1) represent a large class of pulse-modulated systems and are particularly suited to model switched mode power converters. The example below features a simple converter topology and illustrates how it is represented on the form (B.1).

### 2.1 Example 1: Step-down converter

Consider the synchronous step-down DC-DC converter in Fig. B.1. If we define the state as $z = [i_L \ v_C]^T$ where $i_L$ is the inductor current and $v_C$ is the capacitor voltage, then the dynamics are described by

$$\dot{z} = \begin{cases} 
  Az + B, & t \in [kT_s, (k + d_k)T_s) \\
  Az, & t \in [(k + d_k)T_s, (k + 1)T_s)
\end{cases}$$
where

\[
A = \begin{bmatrix}
    -\frac{r_l}{x_l} & -\frac{1}{x_l} \\
    \frac{1}{x_c} & -\frac{1}{(r_o x_c)}
\end{bmatrix}, \quad B = \begin{bmatrix}
    v_s/x_l \\
    0
\end{bmatrix}.
\]

Let \(d^0\) be the nominal duty cycle and let \(z^0(t) = z^0(t + T_s)\) be the corresponding nominal periodic solution. The dynamics of the error \(x := z - z^0\) are

\[
\dot{x}(t) = Ax(t) + (s(t, x) - s_{\phi}(t))B
\]

where the pulse modulation function is defined as in (B.3) with \(a_k = 1 \ \forall k\) and \(f(t) = 1\).

![Figure B.1: Synchronous step-down DC-DC converter.](image)

## 3 The dynamic phasor model

We use the idea of [3] to represent the solution of (B.1) in the frequency domain where we can distinguish how the various harmonics develop over time. The \(n^{th}\) phasor (Fourier coefficient) of \(x\) is defined as

\[
\langle x \rangle_n(t) := \frac{1}{T_s} \int_{t-T_s}^{t} x(\tau)e^{-jn\omega_s \tau}d\tau
\]

where \(\omega_s = 2\pi/T_s\). Note that the phasors are defined over a moving time window and are thus time dependent. Note also that if \(x\) is periodic with period \(T_s\), then \(\langle x \rangle_n(t)\) is constant.

The time domain signal \(x\) is reconstructed on the interval \([t - T_s, t]\) according to

\[
x(t, \tau) := \sum_{n=-\infty}^{\infty} \langle x \rangle_n(t)e^{jn\omega_s(t+\tau)}, \quad \tau \in [-T_s, 0].
\]

(B.4)

Note that \(x(t) \neq x(t, \tau)\), but the equality \(x(t + \tau) = x(t, \tau)\) holds a.e. on the set \(\{\tau \mid \tau \in [-T_s, 0]\}\). The fact that \(x\) is absolutely continuous implies that the sequence \(\{\langle x \rangle_n(t)\}_{n=-\infty}^{\infty}\) is in \(l_2\) for all \(t\). However, \(\{\langle x \rangle_n(t)\}_{n=-\infty}^{\infty}\) need not be absolutely summable since \(x(t, \tau)\) may be discontinuous in \(\tau\).
The key idea of the subsequent analysis is to consider the dynamics of the phasor coefficients \( \langle x \rangle_n \) rather than the dynamic equations (B.1) directly. It can be shown [3] that the phasor coefficients satisfy

\[
\frac{d}{dt} \langle x \rangle_n = -j n \omega_s \langle x \rangle_n + \left( \frac{d}{dt} x \right)_n \tag{B.5}
\]

where the explicit time dependence of the phasors has been suppressed for a more compact notation.

In the subsections below we introduce two different \( L_2 \)-equivalent systems representing the equation (B.6): The first system is an infinite dimensional state space representation and the second is a Fourier series representation. To justify these descriptions we use the fact that the state \( x \) and the pulse modulation functions \( s_i \) are bounded continuous respectively bounded piece-wise continuous functions over any finite time interval and thus belong to \( L_2[t-T_s,t] \). This implies that the phasor coefficients of the product \( s_i x \) can be computed as

\[
\langle s_i x \rangle_n = \sum_{k=-\infty}^{\infty} \langle s_i \rangle_{n-k} \langle x \rangle_k
\]

where the sum converges absolutely, see e.g., [4]. It follows that the equations (B.5) are equivalent to

\[
\frac{d}{dt} \langle x \rangle_n = -j n \omega_s \langle x \rangle_n + \sum_{i=1}^{m} \sum_{k=-\infty}^{\infty} A_i \langle s_i \rangle_{n-k} \langle x \rangle_k + \sum_{i=1}^{m} B_i \langle s_i - s_i(a_i^0,\theta_i^0) \rangle_n \tag{B.6}
\]

where we have substituted \( \frac{d}{dt} x \) in (B.5) for the right hand side in (B.1).

### 3.1 State space representation

Introducing the notation

\[
\begin{bmatrix}
\vdots \\
\langle x \rangle_1 \\
\langle x \rangle_0 \\
\langle x \rangle_{-1} \\
\vdots
\end{bmatrix}, \quad
\begin{bmatrix}
\vdots \\
\langle s_i \rangle_1 \\
\langle s_i \rangle_0 \\
\langle s_i \rangle_{-1} \\
\vdots
\end{bmatrix}, \quad
\begin{bmatrix}
\langle s_i(a_i^0,\theta_i^0) \rangle_1 \\
\langle s_i(a_i^0,\theta_i^0) \rangle_0 \\
\langle s_i(a_i^0,\theta_i^0) \rangle_{-1} \\
\vdots
\end{bmatrix}
\]

the phasor dynamics (B.6) can be written on the form

\[
\frac{d}{dt} \hat{x} = (-j \omega_s \mathbf{N} + \hat{A}) \hat{x} + \hat{B} \tag{B.8}
\]
where $\mathcal{N} := \text{blkdiag}(\ldots, 2I_n, I_n, 0, -I_n, -2I_n, \ldots)$ is a doubly infinite dimensional block diagonal matrix and

\begin{equation}
\dot{A} := \sum_{i=1}^{m} (I \otimes A_i) T[\hat{s}_i] \tag{B.9}
\end{equation}

\begin{equation}
\dot{B} := \sum_{i=1}^{m} (I \otimes B_i) \left( \hat{s}_i - \hat{s}_{i,(a_i^0, d_i^0)} \right) .
\end{equation}

If the duty cycles and pulse amplitudes are constant, then the modulation functions $s_i$ are $T_s$-periodic and the corresponding phasors are constant. In this case the periodically switched system (B.1) is represented by a linear time invariant system in the frequency domain. This special case is considered in Section 5. In the general case however, the modulation functions $s_i$ are determined by feedback. This means that the equations (B.8) are not time invariant and $\hat{s}_i$ depend on sampled values of the state through feedback.

### 3.2 Fourier series representation

In our analysis of the system (B.1) we will make frequent use of the Fourier series representation (B.4). Using the relation (B.6) one can proceed formally to show that

\begin{equation}
\dot{x}(t, \tau) := \frac{d}{dt} x(t, \tau) = A(t, \tau) x(t, \tau) + B(t, \tau) \tag{B.10}
\end{equation}

where (here, $\tau \in [-T_s, 0]$)

\begin{align*}
A(t, \tau) & := \sum_{i=1}^{m} s_i(t, \tau) A_i \\
B(t, \tau) & := \sum_{i=1}^{m} \left( s_i(t, \tau) - s_{i,(a_i^0, d_i^0)}(t, \tau) \right) B_i \\
s_i(t, \tau) & := \sum_{n=-\infty}^{\infty} \left\langle s_i \right\rangle_n \left( t \right) e^{jn \omega_s (t+\tau)} \\
s_{i,(a_i^0, d_i^0)}(t, \tau) & := \sum_{n=-\infty}^{\infty} \left\langle s_{i,(a_i^0, d_i^0)} \right\rangle_n e^{jn \omega_s (t+\tau)}.
\end{align*}

The derivative in (B.10) is well defined in an $L_2$-sense and this allows us to use it in certain quadratic Lyapunov functions. We refer to Lemma 4.2 in the next section for further detail.

### 3.3 Norm relations and bounds

As mentioned above, the state space and Fourier series representations are equivalent in the $L_2$-sense. We will make frequent use of the Parseval relations

\begin{equation}
\|\dot{x}(t)\|_{L_2} = \|x(t, \cdot)\|_{L_2[-T_s, 0]} = \|x(\cdot)\|_{L_2[0, T_s]}.
\end{equation}
and similarly for the norms of \( \hat{s}_i, s_i \) and \( s_i \). The Parseval identity also implies \( L_2 \)-equivalence between the representations of the system matrices as stated in the following lemma (we note that similar results can be found in e.g., [22]). A proof is given in Appendix 15 for completeness.

**Lemma 3.1.** For any solution \( x(t) \) of (B.1), the corresponding operator \( \hat{A}(t) \) defined in (B.9) is a bounded linear operator on \( l_2 \) with induced norm

\[
\|\hat{A}(t)\|_{l_2-l_2} = \|A(t, \cdot)\|_{L_2[-T_s,0] \to L_2[-T_s,0]} = \|A(\cdot, x(\cdot))\|_{L_2[0,-T_s] \to L_2[0,-T_s]}
\]

\[
\leq \sum_{i=1}^{m} \sigma(A_i) =: |A|.
\]

Analogously, it holds

\[
\|\hat{B}(t)\|_{l_2} = \|B(t, \cdot)\|_{L_2[-T_s,0]}
\]

\[
= \|B(\cdot, x(\cdot))\|_{L_2[t-T_s,t]} \leq \sum_{i=1}^{m} \sigma(B_i) =: |B|.
\]

In the sequel we will make frequent use of these inequalities and of the definitions of \( |A| \) and \( |B| \).

### 4 Quadratic Lyapunov analysis

To derive stability conditions for the system (B.1) we consider quadratic time periodic Lyapunov functions. In the section below we introduce a time periodic matrix function which is used to define a Lyapunov candidate for the system (B.1). The matrix function also defines \( L_2 \)-equivalent Lyapunov functionals for the state space representation (B.8) and Fourier series representation (B.10).

For a fixed integer \( N \geq 0 \), let

\[
P_N(t) := \sum_{n=-N}^{N} \hat{P}_N[n] e^{j\omega_s t}, \quad t \in [-T_s,0] \tag{B.11}
\]

where \( \hat{P}_N[n] = \hat{P}_N[n]^* \) and \( \hat{P}_N[-n] = \overline{\hat{P}_N[n]} \) and we assume that \( P_N(t) \geq \beta^2 I \forall t \) for some parameter \( \beta > 0 \). We let \( \|P_N\| \) denote the induced norm of \( P_N \) on \( L_2[-T_s,0] \) and we note that \( \|P_N\| = \sup_{t} \sigma(P_N(t)) \). The function \( P_N \) is used to define a Lyapunov candidate for the system (B.1) according to

\[
V(x(t)) := \langle x, P_Nx \rangle_{L_2[-T_s,t]} = \frac{1}{T_s} \int_{t-T_s}^{t} x(\tau)^* P_N(\tau) x(\tau) d\tau. \tag{B.12}
\]
In the proofs of this paper we make frequent use of two \( L_2 \)-equivalent representations of (B.12). Let \( x \) be a solution of (B.1) and let \( \hat{x} \) and \( x \) be the corresponding phasor coefficients and Fourier series representation. Firstly we define

\[
V(\hat{x}(t)) := \langle \hat{x}(t), T[P_N]\hat{x}(t) \rangle_{L_2} = \hat{x}(t)^* T[P_N]\hat{x}(t)
\]  
\hfill (B.13)

where \( T[P_N] \) is the doubly infinite dimensional band diagonal block Toeplitz matrix determined by the coefficients of \( P_N \);

\[
T[P_N] := \begin{bmatrix}
\vdots & \vdots & \vdots \\
\hat{P}_N[0] & \hat{P}_N[1] & \hat{P}_N[2] \\
\vdots & \hat{P}_N[-1] & \hat{P}_N[0] \\
\hat{P}_N[-2] & \hat{P}_N[-1] & \hat{P}_N[0] \\
\vdots & \vdots & \vdots \\
\end{bmatrix}
\]

Secondly we define

\[
V(x(t, \cdot)) := \langle x(t, \cdot), P_N(t, \cdot) x(t, \cdot) \rangle_{L_2[-T_s, 0]} = \frac{1}{T_s} \int_{-T_s}^{0} x(t, \tau)^* P_N(t, \tau) x(t, \tau) d\tau
\]

where \( P_N(t, \tau) := P_N(t + \tau) \). It is easily shown that \( V(x) = V(\hat{x}) = V(x) \) and the assumption that \( P_N(t) \geq \beta^2 I \forall t \in [-T_s, 0] \) ensures that \( V \) is positive definite. It holds

\[
V(x(t)) = V(\hat{x}(t)) = V(x(t, \cdot)) \geq \beta^2 \| x(t, \cdot) \|^2_{L_2[-T_s, 0]} = \beta^2 \| x(t) \|^2_{L_2[t-T_s, t]} = \beta^2 \| \hat{x}(t) \|^2_{L_2}.
\]

The condition that \( P_N \) is positive definite can be stated in terms of LMI as described in the following lemma.

**Lemma 4.1.** Let \( \{p_k\}_{k=-N}^{N} \) be a (finite) sequence of matrices in \( \mathbb{C}^{n \times n} \) satisfying \( p_k = p^*_k \) and \( p_{-k} = p_k^* \). Define \( P_N(t) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n} \) as

\[
P_N(t) := \sum_{k=-N}^{N} p_k e^{j\omega_s t}
\]

and let

\[
K(p_k, \beta) := \begin{bmatrix}
p_0 - \beta^2 I & p_1 & \cdots & p_N \\
p_1 & \vdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
p_{N} & & & \hat{p}_N
\end{bmatrix}, \quad H(z) := \frac{1}{(z + a)^N} \begin{bmatrix}
I \\
zI \\
\vdots \\
z^N I
\end{bmatrix}
\]

where \( a \neq 1 \) is a real number. The following conditions are equivalent
(i) $P_N(t) \geq \beta^2 I, \forall t \in [-T_s, 0]$

(ii) $\exists Q = Q' \in \mathbb{C}^{N \times N}$ such that

\[
\begin{bmatrix}
A'QA - Q \\
B'QA \\
A'QB \\
B'QB
\end{bmatrix} + \begin{bmatrix}
C' \\
D'
\end{bmatrix} K(p_k, \beta) \begin{bmatrix}
C \\
D
\end{bmatrix} \geq 0
\]

where $(A, B, C, D)$ is a minimal realization of $H(z)$.

**Proof:** A proof is given in Appendix 11.

The following lemma shows that the time-derivative of the suggested Lyapunov candidate (B.12) can be expressed using the Fourier series representation $x$.

**Lemma 4.2.** The time derivative of $V(x(t))$ satisfies

\[
\dot{V}(x(t)) = \left. x, (\dot{P}_N + 2P_N A)x + 2P_N B \right|_{L_2[-T_s, t]}
\]

\[
= \left. x, (\dot{P}_N + 2P_N A)x + 2P_N B \right|_{L_2[-T_s, 0]}
\]

where $\dot{P}_N(t, \tau) := \frac{d}{dt}P_N(t, \tau)$.

**Proof:** A proof is given in Appendix 15.

5 The linear time periodic case

In this section we consider the special case where the pulse amplitudes and duty cycles are constant and equal to $a_i^0$ and $d_i^0$ respectively. In this case the affine term in (B.1) disappears and the system is LTP. For clarity the system equations are repeated below. We consider the system

\[
\dot{x}(t) = \sum_{i=1}^{m} s_i(t) A_i x(t) =: A(t)x(t) \quad (B.14)
\]

where

\[
s_i(t) := \begin{cases} 
a_i^0 f_i(t), & t \in [kT_s, (k + d_i^0)T_s) \\
0, & t \in [(k + d_i^0)T_s, (k + 1)T_s)
\end{cases}
\]

To derive stability conditions for the LTP system (B.14) we introduce an approximate system where the system matrix contains a finite number of harmonics. As an approximation of (B.14) we consider the solution $x_{ap}$ of

\[
\dot{x}_{ap}(t) = \sum_{i=1}^{m} s_{i,M}(t) A_i x_{ap}(t) =: A_M(t)x_{ap}(t) \quad (B.15)
\]
where (recall that $\pi_M$ denotes the projection on the Fourier basis)

$$s_{i,M}(t) := \pi_M s_i(t) = \sum_{n=-M}^{M} \langle s_i \rangle_n e^{jn\omega_s t}$$

where $\langle s_i \rangle_n$ are the Fourier coefficients of $s_i$. Existence of a time periodic quadratic Lyapunov function of the form (B.12) for the approximate system (B.15) can be checked by solving a set of LMI (this follows from Lemma 4.1). If such a function exists it can be used as a Lyapunov candidate for the original system (B.14) and provides sufficient conditions for stability. This claim is formalized in the following proposition.

**Proposition 5.1.** Consider the approximate system (B.15) for a fixed integer $M \geq 0$. Let $N \leq M$ and $L = \lfloor (M - N)/2 \rfloor$ and assume there exists a periodic function $P_N$ as defined in (B.11) such that

(i) there exists $\beta > 0$ such that $\forall t \in [-T_s, 0]$

$$P_N(t) = \sum_{n=-N}^{N} \hat{P}_N[n] e^{jn\omega_s t} \geq \beta^2 I$$

(ii) there exists $\alpha_N > 0$ such that $\forall t \in [-T_s, 0]$

$$\dot{P}_N(t) + A_M(t)' P_N(t) + P_N(t) A_M(t) \leq -\alpha_N P_N(t)$$

(iii) there is a number $\alpha \in (0, \alpha_N)$ such that

$$C(T_s, L)/\sqrt{M + 1} \leq \alpha_N - \alpha$$

where

$$C(T_s, L) := \frac{4T_s |A|^2 \|P_N\|}{\pi^2 \beta^2 \sqrt{L + 1}} \sum_{i=1}^{m} \hat{\sigma}(A_i)(2 + T_s \|\hat{f}_i\|)$$

$$\times \left(2 + T_s |A| \left(1 + \frac{2}{\pi} \ln(L + 1)\right)\right).$$

Then $V(x) := \langle x, P_N x \rangle_{L_2[-T_s, t]}$ is a Lyapunov function for the system (B.14) and satisfies

$$V(x(t)) \leq e^{-\alpha(t-t_0)} V(x(t_0))$$

along the trajectories of (B.14), which is thus Lyapunov stable.

**Proof:** A proof is given in Appendix 12.
Remark 5.1. We note that if $N$ is fixed and $M \to \infty$, then $C(T_s, L) \to 0$ and we obtain the standard periodic Lyapunov inequality. The contribution of the proposition is to show how the periodic Lyapunov inequality can be approximated using a set of LMI. Condition (i) can be stated in terms of LMI by directly applying Lemma 4.1. To state (ii) in terms of LMI we define $Q(t) := \dot{P}_N(t) + A_M(t)P_N(t) + P_N(t)A_M(t) + \alpha_N P_N(t) = \sum_{n=-\infty}^{M+N} \dot{Q}[n]e^{j\omega_n t}$ and apply Lemma 4.1.

Remark 5.2. The approximate system (B.15) can also be represented using the state space representation (B.8). In this case we consider the Lyapunov inequality $\dot{V}(\hat{x}) \leq -\alpha_N V(\hat{x})$ where $V$ is defined in (B.13). This inequality can be formally stated as the (skew truncated) infinite dimensional harmonic Lyapunov inequality

$$(-j\omega_N + \hat{A}_M)^*T[P_N] + T[P_N](-j\omega_N + \hat{A}_M) \leq -\alpha_N T[P_N]$$

where $\hat{A}_M = T[A_M(t)]$. Infinite dimensional harmonic Lyapunov inequalities have been considered in [24] as a means for stability analysis of LTP systems. In [24] it is pointed out that the inequality above should be viewed as an operator valued inequality defined on a certain dense subspace of $l_2$. We circumvent these technicalities by considering solutions of the system rather than the system per se.

6 The closed loop case

In this section we consider the system (B.1) in closed loop where the pulse modulation is determined by feedback. The pulse amplitudes $a_{i,k}$ and duty cycles $d_{i,k}$ are determined by sampling according to

$$a_{i,k} = a_i(x(kT_s)), \quad d_{i,k} = d_i(x(kT_s))$$

where

$$a_i(x(t)) := \text{sat}_{[-1,1]} \left( a_i^0 + \langle F_i^1(\cdot), x(\cdot) \rangle_{L_2[t-T_s,t]} \right)$$
$$d_i(x(t)) := \text{sat}_{[0,1]} \left( d_i^0 + \langle F_i^2(\cdot), x(\cdot) \rangle_{L_2[t-T_s,t]} \right)$$

(B.16)

where for $a < b$, $\text{sat}_{[a,b]}(\cdot) := \min(\max(\cdot, a), b)$ denotes saturation between $a$ and $b$ and where for $\kappa = 1, 2$ we define

$$F_i^\kappa(t) := \sum_{k=-N}^{N} e^{j\omega_N t} F_{i,k}^\kappa$$

(B.17)

where $F_{i,k}^\kappa$ are column vectors in $\mathbb{C}^n$ satisfying $\overline{F_{i,k}^\kappa} = F_{i,-k}^\kappa$. The form of the feedback (B.16) is motivated because it is easily represented in the dynamic phaser model. A discussion on the interpretation and implications of the feedback (B.16) can be found in Section 9.
In order to derive tractable stability conditions we consider the phasor model corresponding to (B.1) and introduce an averaged approximation. We consider the phasor model

$$\dot{x}(t, \tau) = A(t, \tau)x(t, \tau) + B(t, \tau)$$  \hspace{1cm} (B.18)

with feedback

$$a_i(t) := \text{sat}_{[-1,1]} \left( a_i^0 + \langle F_i^1(\cdot), x(t, \cdot) \rangle_{L_2[-T_s,0]} \right)$$

$$d_i(t) := \text{sat}_{[0,1]} \left( d_i^0 + \langle F_i^2(\cdot), x(t, \cdot) \rangle_{L_2[-T_s,0]} \right).$$  \hspace{1cm} (B.19)

The approximate system is derived in two steps. In the first step we replace the phasor coefficients \( \langle s_i \rangle_n \) with the nonlinear approximation

$$s_{av,i,n}(a_i, d_i) = \frac{a_i}{T_s} \int_0^{T_s} f_i(\tau)e^{-j\omega_s \tau} \, d\tau$$  \hspace{1cm} (B.20)

where \( a_i = a_i(t), \) \( d_i = d_i(t) \) are defined in (B.19). Note that if the duty cycle and pulse amplitude are fixed so that \( d_{i,k} = d_i \) \( \forall k \) and \( a_{i,k} = a_i \) \( \forall k \), then \( \langle s_i \rangle_n = s_{av,i,n}(a_i, d_i) \). This implies that if the duty cycle and pulse amplitude vary slowly (compared to the switch period \( T_s \)), then \( s_{av,i,n}(a_i, d_i) \) will be a good approximation of \( \langle s_i \rangle_n \). Note also that the approximation \( s_{av,i,n} \) is a continuous function of the state \( x \) unlike \( \langle s_i \rangle_n \) which depends on sampled values with a delay.

Replacing the phasor coefficients \( \langle s \rangle_n \) with the averaged approximations \( s_{av,i,n} \) in (B.18) yields the following averaged system matrices

$$A_{av}(t, \tau) := \sum_{i=1}^{m} s_{av,i}(t, \tau)A_i$$

$$B_{av}(t, \tau) := \sum_{i=1}^{m} \left( s_{av,i}(t, \tau) - s_{i,(a_i^0,d_i^0)}(t, \tau) \right)B_i$$  \hspace{1cm} (B.21)

$$s_{av,i}(t, \tau) := \sum_{n=-\infty}^{\infty} s_{av,i,n}(a_i(t), d_i(t))e^{j\omega_s(t+\tau)}.$$

Note that \( s_{i,(a_i^0,d_i^0)}(t, \tau) \) remains unchanged from the definition of \( B(t, \tau) \) and that \( A_{av} \) and \( B_{av} \) depend on \( a_i(t) \) and \( d_i(t) \) which in turn depend on \( x(t, \tau) \) as defined in (B.19).

In the second step of the approximation, the averaged system matrices \( A_{av}, B_{av} \) are replaced by system matrices \( A_{av,M}, B_{av,M} \) containing a finite number of
harmonics. We define (recall that $\pi_M$ denotes the projection on the Fourier basis)

\[
A_{av,M}(t, \tau) := \pi_M A_{av}(t, \tau) \\
= \sum_{i=1}^{m} \sum_{n=-M}^{M} s_{av,i,n}(a_i(t), d_i(t)) e^{jn\omega_0(t+\tau)} A_i
\]

\[
B_{av,M}(t, \tau) := \pi_M B_{av}(t, \tau) \\
= \sum_{i=1}^{m} \sum_{n=-M}^{M} \left(s_{av,i,n}(a_i(t), d_i(t)) - s_{av,i,n}(a_i^0, d_i^0)\right) e^{jn\omega_0(t+\tau)} B_i.
\] (B.22)

As an approximation of (B.1) we consider the solution $x_{ap}$ of

\[
\dot{x}_{ap}(t, \tau) = A_{av,M}(t, \tau)x_{ap}(t, \tau) + B_{av,M}(t, \tau). \tag{B.23}
\]

We check stability of the system (B.23) by searching for a Lyapunov function $V(x)$ as defined in Section 4. The matrix function $P_N$ that defines $V(x)$ can be computed using the systematic procedure outlined in Section 6.1 below. If a Lyapunov function $V(x)$ exists for the system (B.23), then it guarantees practical stability of (B.1) under certain conditions. This claim is formalized in the following proposition.

**Proposition 6.1.** Consider the approximate system (B.23) for a fixed integer $M \geq 0$. Let $N \leq M$ and $L = \lfloor (M - N)/2 \rfloor$ and assume there exists a periodic function $P_N$ as defined in (B.11) such that

(i) there exists $\beta > 0$ such that $\forall t \in [-T_x, 0]$

\[
P_N(t) = \sum_{n=-N}^{N} \hat{P}_N[n] e^{jn\omega_0 t} \geq \beta^2 I
\]

(ii) there exists $\alpha_N > 0$ such that $V(x_{ap}) := \langle x_{ap}, P_N x_{ap} \rangle_{L^2}$ satisfies

\[
\dot{V}(x_{ap}) \leq -\alpha_N V(x_{ap})
\]

along the averaged dynamics (B.23).

Let $V(x(t)) := \langle x, P_N x \rangle_{L^2[t-T_x,t]}$ and assume $V(x(t_0)) \leq R^2$ for some positive number $R$. If

(iii) $\beta_1(T_x, L, M) > 0$

(iv) $\beta_2(T_x, L, M, R) \leq R^2 \beta_1(T_x, L, M)$

where expressions for $\beta_1$ and $\beta_2$ can be found in (B.43) in Appendix 13, then $V(x(t))$ is bounded according to

\[
V(x(t)) \leq e^{-\beta_1(t-t_0)} V(x(t_0)) + \frac{\beta_2}{\beta_1} \left(1 - e^{-\beta_1(t-t_0)}\right) \tag{B.24}
\]
along the trajectories of (B.1).

The terms $\beta_i$ can be written

$$\beta_1 = \alpha_N - c_{11} - c_{12}, \quad \beta_2 = c_{21} + c_{22}$$

where expressions for $c_{ij}$ can be found in (B.43) in Appendix 13. The coefficients satisfy $c_{11} = O(T_s)$ and $c_{21} = O\left(\frac{T_s}{\sqrt{(L+1)(M+1)}}\right)$ and we conclude that if $T_s$ is small enough and $L$ and $M$ are large enough, then stability of the approximate system (B.23) implies stability of (B.1).

**Proof:** A proof is given in Appendix 13.

**Remark 6.1.** We may also consider the case where the system dynamics (B.10) is approximated using the averaged matrices in (B.21) and there is no truncation. In this case we say that $M = \infty$. The result in Proposition 6.1 still holds although with new coefficients $\beta_1 = \alpha_N - c_{11}(T_s)$ and $\beta_2 = c_{21}(T_s)$.

**Remark 6.2.** We obtain the classical state space averaging result as a special case when $M = N = 0$ and $T_s \to 0$.

**Remark 6.3.** The conclusion (B.24) implies that the solutions of (B.1) are exponentially stable in a practical sense, i.e., the solutions converge exponentially to a neighborhood of the origin, which is smaller the larger $L$ and $M$ are and the smaller $T_s$ is. This type of practical stability conclusion is very common in analysis of systems with discontinuous dynamics, see e.g., [8, 10, 19].

### 6.1 Verification of the Lyapunov Inequality

Condition (ii) in Proposition 6.1 is satisfied if and only if the following nonlinear and infinite dimensional Lyapunov inequality

$$\left< x, (P_N + \alpha_N P_N + 2P_N A_{av,M}) x + 2P_N B_{av,M} \right>_{L_2} \leq 0 \quad (B.25)$$

holds for all $x \in L_2[-T_s, 0]$ such that $\langle x, P_N x \rangle \leq R^2$. The inequality is nonlinear since $A_{av,M}$ and $B_{av,M}$ are functions of the feedback, i.e.,

$$A_{av,M} = A_{av,M}(a(x), d(x)), \quad B_{av,M} = B_{av,M}(a(x), d(x))$$

where $a(x) = (a_1(x), \ldots, a_m(x))$ and $d(x) = (d_1(x), \ldots, d_m(x))$ are vectors containing the feedback functions defined in (B.19).

In this section we derive an inequality which implies (B.25) and which can be stated in terms of LMI. The derivation relies on a partition of the state space and strict griding procedure. The method is adopted from [1] which in turn is inspired by an idea introduced in [7]. The following two observations are crucial.

1. The saturations in the feedback functions $(a(x), d(x))$ provide a partition of the state space. On the saturated regions, $x$ satisfies linear inequalities which can be relaxed using the S-procedure [2, 21].
2. All $x \in \mathbb{L}_2[-T_s, 0]$ such that $(a(x), d(x)) = (\bar{a}, \bar{d})$ for some fixed $(\bar{a}, \bar{d})$ define a subset on which (B.25) corresponds to an inequality which is linear in $P_N$.

We first introduce the partition of the state space. For $i = 1, \ldots, m$ we define

$$S_{i,-1}^1 := \{ x \in \mathbb{L}_2 \mid a_i(x) \leq -1 \}, \quad S_{i,-1}^2 := \{ x \in \mathbb{L}_2 \mid d_i(x) \leq 0 \}$$

$$S_{i,0}^1 := \{ x \in \mathbb{L}_2 \mid -1 < a_i(x) < 1 \}, \quad S_{i,0}^2 := \{ x \in \mathbb{L}_2 \mid 0 < d_i(x) < 1 \}$$

$$S_{i,1}^1 := \{ x \in \mathbb{L}_2 \mid a_i(x) \geq 1 \}, \quad S_{i,1}^2 := \{ x \in \mathbb{L}_2 \mid d_i(x) \geq 1 \}.$$

Thus, $S_{i,1}^1$ and $S_{i,1}^2$ are the regions where $a_i(x)$ and $d_i(x)$ saturate and $S_{i,0}^1$ and $S_{i,0}^2$ are the non-saturated regions. For $j = 1, 2$, let $I_j := (i_{j,1}, i_{j,2}, \ldots, i_{j,m})$ be index vectors where $i_{j,k} \in \{-1, 0, 1\}, k = 1, \ldots, m$. Using this notation, the state space is partitioned as

$$\mathbb{L}_2[-T_s, 0] = \bigcup_{(I_1, I_2) \in \mathcal{P} \times \mathcal{P}} \Omega(I_1, I_2)$$

where $\mathcal{P} = \{-1, 0, 1\}^m$ is the Cartesian product of $m$ copies of the set $\{-1, 0, 1\}$ and where

$$\Omega(I_1, I_2) := \{ x \in \mathbb{L}_2 \mid x \in \bigcap_{k=1}^m S_{i_{1,k}}^1, x \in \bigcap_{k=1}^m S_{i_{2,k}}^2 \}$$

where $(i_{1,k}, i_{2,k}) \in (I_1, I_2)$. The partition implies that we may consider the Lyapunov inequality (B.25) on a number of pairwise disjoint subsets. In other words, we consider (B.25) for all $x \in \Omega(I_1, I_2)$ for all $(I_1, I_2) \in \mathcal{P} \times \mathcal{P}$.

Let us now consider a fixed set $\Omega(I_1, I_2), (I_1, I_2) \in \mathcal{P} \times \mathcal{P}$. Let $I_{s,j}^+$ and $I_{s,j}^-$ be index sets describing the feedbacks that saturate on $\Omega(I_1, I_2)$, i.e., for $j = 1, 2$ let

$$I_{s,j}^+ := \{ k \in \{1, \ldots, m\} \mid I_j \ni i_{j,k} = 1 \}$$

$$I_{s,j}^- := \{ k \in \{1, \ldots, m\} \mid I_j \ni i_{j,k} = -1 \}$$

and let $q_{s,j}^+ = \# I_{s,j}^+$, $q_{s,j}^- = \# I_{s,j}^-$ and $q_s = q_{s,1}^+ + q_{s,1}^- + q_{s,2}^+ + q_{s,2}^-$. To describe the inequalities implied by the saturated feedbacks we introduce operators $F_{s}^+ : \mathbb{L}_2 \to \mathbb{R}^{q_{s,1}^+ + q_{s,2}^+}$ and $F_{s}^- : \mathbb{L}_2 \to \mathbb{R}^{q_{s,1}^- + q_{s,2}^-}$ according to

$$F_{s}^+ x := \begin{bmatrix} \langle F_1^{1}, x \rangle_{\mathbb{L}_2} \\ \langle F_2^{1}, x \rangle_{\mathbb{L}_2} \end{bmatrix}_{k \in I_{s,1}^+, l \in I_{s,2}^+}, \quad F_{s}^- x := \begin{bmatrix} \langle F_1^{2}, x \rangle_{\mathbb{L}_2} \\ \langle F_2^{2}, x \rangle_{\mathbb{L}_2} \end{bmatrix}_{k \in I_{s,1}^-, l \in I_{s,2}^-}$$

and vectors $f_{s}^{0+} \in \mathbb{R}^{q_{s,1}^+ + q_{s,2}^+}$ and $f_{s}^{0-} \in \mathbb{R}^{q_{s,1}^- + q_{s,2}^-}$ according to

$$f_{s}^{0+} := \begin{bmatrix} a_{k}^0 \\ d_{l}^0 \end{bmatrix}_{k \in I_{s,1}^+, l \in I_{s,2}^+}, \quad f_{s}^{0-} := \begin{bmatrix} a_{k}^0 \\ d_{l}^0 \end{bmatrix}_{k \in I_{s,1}^-, l \in I_{s,2}^-}$$
and finally vectors $f_s^+$ and $f_s^-$ according to

$$
\begin{align*}
    f_{s,j}^+ &= 1, & j &= 1, \ldots, q_{s,1}^+ + q_{s,2}^+ \\
    f_{s,j}^- &= \begin{cases} 
        -1, & j &= 1, \ldots, q_{s,1}^- \\
        0, & j &= q_{s,1}^- + 1, \ldots, q_{s,1}^- + q_{s,2}^- 
    \end{cases}
\end{align*}
$$

Using this notation, the inequalities satisfied on $\Omega(I_1, I_2)$ are described as

$$
F_s x + f_s^0 - f_s^- \geq 0 
$$

where

$$
F_s := \begin{bmatrix} F_s^- \\ -F_s^- \end{bmatrix}, \quad f_s^0 := \begin{bmatrix} f_s^0+ \\ -f_s^- \end{bmatrix}, \quad f_s := \begin{bmatrix} f_s^+ \\ -f_s^- \end{bmatrix}.
$$

We now consider the feedbacks that are not saturated on $\Omega(I_1, I_2)$. For $j = 1, 2$, let $I_{ns,j}$ be index sets describing the non-saturated feedbacks, i.e., $I_{ns,j} = \{1, \ldots, m\} \setminus (I^+_s \cup I^-_s)$ and let $q_{ns} = \#I_{ns,1} + \#I_{ns,2}$. The non-saturated feedbacks are collected in a vector $f_{ns}$ according to

$$
f_{ns}(x) = f_{ns}^0 + F_{ns} x \in A_{ns} \times D_{ns}
$$

where $F_{ns} : L_2 \to \mathbb{R}^{q_{ns}}$ and $f_{ns}^0 \in \mathbb{R}^{q_{ns}}$ are defined as

$$
F_{ns} x := \begin{bmatrix} \langle F^1_k, x \rangle_{L_2} \\ \langle F^2_l, x \rangle_{L_2} \end{bmatrix}_{k \in I_{ns,1}, l \in I_{ns,2}}, \quad f_{ns}^0 := \begin{bmatrix} a_k^0 \\ d_l^0 \end{bmatrix}_{k \in I_{ns,1}, l \in I_{ns,2}}
$$

and where $A_{ns} = [-1, 1]^{q_{ns}}$ and $D_{ns} = [0, 1]^{q_{ns}}$.

Let

$$
V_{ns} = \text{span}\{ F^1_k, F^2_l \mid k \in I_{ns,1}, l \in I_{ns,2} \}
$$

be the linear subspace in $L_2$ spanned by the non-saturated feedback functions in (B.16) and let

$$
\gamma(f_{ns} - f_{ns}^0) = F_{ns}^*(F_{ns} F_{ns}^*)^{-1}(f_{ns} - f_{ns}^0)
$$

where $F_{ns}^* : \mathbb{R}^{q_{ns}} \to L_2[-T_s, 0]$ is the adjoint operator of $F_{ns}$. For any point $\bar{f}_{ns} \in A_{ns} \times D_{ns}$, all $x \in L_2$ such that $f_{ns}(x) = \bar{f}_{ns}$ can be written on the form

$$
x = v^+ + \gamma(f_{ns} - f_{ns}^0), \quad v^+ \in V_{ns}^+.
$$

We note that the matrix $F_{ns} F_{ns}^*$ is non-singular if the functions $F^1_i, F^2_j, i \in I_{ns,1}, j \in I_{ns,2}$ in $F_{ns}$ are linearly independent. Any linearly dependent functions in $F_{ns}$ can be removed without affecting the characterization of the set $\Omega(I_1, I_2)$ and thus, we may without loss of generality assume that $F_{ns} F_{ns}^*$ is non-singular.

By relaxing the inequalities (B.26) and $\langle x, P_N x \rangle \leq R^2$ and using the representation (B.27) of the set $\Omega(I_1, I_2)$ one can show that (B.25) is satisfied for all $x \in \Omega(I_1, I_2)$ satisfying $\langle x, P_N x \rangle \leq R^2$ if a parameterized time periodic matrix inequality is satisfied. The result is stated in the following proposition.
Proposition 6.2. Consider the inequality (B.25) for all $x$ such that $\langle x, P_N x \rangle \leq R^2$ and $x \in \Omega(I, I_2)$ for some $(I, I_2) \in \mathcal{P} \times \mathcal{P}$. The inequality is satisfied if there exists a number $\tau \geq 0$, vectors $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and a positive semidefinite matrix $\Lambda$ satisfying the time periodic matrix inequality

$$
\Pi(f_{ns}, \tau) + \Omega(\lambda_1, f_{ns}) + \Psi(\lambda_2, \Lambda) \leq 0 \quad \forall f_{ns} \in \mathcal{A}_{ns} \times \mathcal{D}_{ns}
$$

where

$$
\Pi(f_{ns}, \tau) = \begin{bmatrix}
\Pi_{11}(f_{ns}, \tau) & \Pi_{12}(f_{ns}, \tau) \\
\Pi_{12}(f_{ns}, \tau)' & \Pi_{22}(f_{ns}, \tau)
\end{bmatrix}
$$

$$
\Omega(\lambda_1, f_{ns}) = \begin{bmatrix}
0 & F^*_s \lambda_1 \\
(F^*_s \lambda_1)' & 2\lambda_1 F_s \gamma - 2\lambda_1' (f_s - f_s^0)
\end{bmatrix}
$$

$$
\Psi(\lambda_2, \Lambda) = \begin{bmatrix}
-F^*_s \Lambda F_{ns} & F^*_s \lambda_2 \\
(F^*_s \lambda_2)' & 0
\end{bmatrix}
$$

where (the argument $s$ is suppressed)

$$
\Pi_{11} = A'_{av,M} P_N + P_N A_{av,M} + \hat{P}_N + (\alpha_N - \tau) P_N \\
\Pi_{12} = \Pi_{11} \gamma + P_N B_{av,M} \\
\Pi_{22} = \gamma \Pi_{11} \gamma + \gamma' P_N B_{av,M} + B'_{av,M} P_N \gamma + \tau R^2
$$

and where $\gamma = \gamma(f_{ns} - f_{ns}^0)$.

Proof: A proof is given in Appendix 14.

Remark 6.4. We note that the inequality in (B.28) is a condition on a time-periodic matrix function. By gridding over $\mathcal{A}_{ns} \times \mathcal{D}_{ns}$ it is possible to transform the inequalities (B.28) to a set of LMI by using Lemma 4.1.

7 Harmonic analysis

In the section below we introduce an external disturbance in the system dynamics (B.1) and describe the harmonic influence of the disturbance on the state. To this end we represent the system approximately using the truncated averaged phasor model. The approximate phasor model is linearized and this yields an LTP system. The harmonic transfer function corresponding to the time periodic dynamics gives an approximate description of the response of (B.1) to periodic disturbances.

Consider the presence of an external disturbance $w \in L_\infty[-T_s, \infty]$ in the system (B.1)

$$
\dot{x}(t) = A(t, x)x(t) + B(t, x) + D(t, x)w(t)
$$

where $A(t, x)$ and $B(t, x)$ are defined in (B.2) and

$$
D(t, x) := \sum_{i=1}^m s_i(t, x) D_i
$$
where $s_i(t, x)$ are defined in (B.3) and $D_i$ are constant matrices. As an approximation of the system above we consider the corresponding skew truncated averaged phasor model. Let

$$w(t, \tau) := \sum_{n=-\infty}^{\infty} \langle w \rangle_n \frac{\hat{w}(t+\tau)}{n} e^{in\omega_s(t+\tau)}$$

be the phasor representation of $w$. The skew-truncated averaged phasor system becomes

$$\dot{x}(t, \tau) = A_{av, M}(t, \tau)x(t, \tau) + B_{av, M}(t, \tau) + D_{av, M}(t, \tau)w(t, \tau)$$ \hspace{1cm} (B.29)

where $A_{av, M}$ and $B_{av, M}$ are defined in (B.22) and

$$D_{av, M}(t, \tau) := \pi_M \sum_{i=1}^{m} s_{av, i}(t, \tau) D_i$$

$$= \sum_{i=1}^{m} \sum_{n=-M}^{M} s_{av, i, n}(a_i(t), d_i(t)) e^{in\omega_s(t+\tau)} D_i$$

where $a_i(t)$ and $d_i(t)$ are functions of $x(t, \tau)$ defined in (B.19).

### 7.1 Linearized dynamics

We note that $(x, w) = (0, 0)$ is an equilibrium point of the above skew truncated dynamics. The truncation of the Fourier series implies that the dynamics are differentiable with respect to the feedback variables $a$ and $d$. To express the linearized dynamics we introduce the operators $F^\kappa_i : L_2[-T_s, 0] \to \mathbb{R}$ defined as

$$F^\kappa_i x(t, \tau) := \langle F^\kappa_i(\cdot), x(t, \cdot) \rangle_{L_2[-T_s, 0]}$$

where $F^\kappa_i(\cdot)$ are the feedback function defined in (B.17). Let

$$s_{i, M}(t, \tau) := \pi M s_{av, i}(t, \tau)$$

$$= \sum_{n=-M}^{M} s_{av, i, n}(a_i(t), d_i(t)) e^{in\omega_s(t+\tau)}$$

$$s^0_{i, M}(t, \tau) := \pi M s_{i}(a^0_i, d^0_i)(t, \tau)$$

$$= \sum_{n=-M}^{M} s_{av, i, n}(a^0_i, d^0_i) e^{in\omega_s(t+\tau)}.$$
where

\[ A^0_M(t, \tau) := \sum_{i=1}^{m} s^0_{i,M}(t, \tau) A_i \]
\[ D^0_M(t, \tau) := \sum_{i=1}^{m} s^0_{i,M}(t, \tau) D_i \]
\[ \partial B^0_M(t, \tau) := \sum_{i=1}^{m} B_i \left( \partial s^0_{i,M,o}(t, \tau) F^1_i + \partial s^0_{i,M,d}(t, \tau) F^2_i \right) \]

where

\[ \partial s^0_{i,M,o}(t, \tau) := \frac{\partial}{\partial a} s^0_{i,M}(t, \tau) \bigg|_{(a_i, d_i) = (a^0_i, d^0_i)} = \sum_{n=-M}^{M} c^1_{i,n} e^{jn\omega_s (t+\tau)} \]
\[ \partial s^0_{i,M,d}(t, \tau) := \frac{\partial}{\partial d} s^0_{i,M}(t, \tau) \bigg|_{(a_i, d_i) = (a^0_i, d^0_i)} = \sum_{n=-M}^{M} c^2_{i,n} e^{jn\omega_s (t+\tau)} \]

where the coefficients \( c^\kappa_{i,n} \) are defined as

\[
\begin{align*}
    c^1_{i,n} &:= \frac{1}{T_s} \int_0^{d^0_i T_s} f_i(\tau) e^{-jn\omega_s \tau} d\tau \\
    c^2_{i,n} &:= a^0_i f_i(d^0_i T_s) e^{-jn2\pi d^0_i}.
\end{align*}
\] (B.31)

We note that \( A^0_M \) and \( D^0_M \) are time periodic matrices while \( \partial B^0 \) is a composition of time periodic matrices and the operators \( F^\kappa_i \).

### 7.2 Harmonic transfer function

The solution \( x \) of (B.30) is equivalently expressed using the state space representation of Section 3.1. Let \( \hat{x} \) be the vector of phasor coefficients of \( x \) defined in (B.7) and let \( \hat{w} \) be defined similarly. It holds

\[
\frac{d}{dt} \hat{x} = \left( -j\omega_s N + \hat{A}^0 + \hat{\partial B}^0 \right) \hat{x} + \hat{D}^0 \hat{w}
\] (B.32)

where

\[
\begin{align*}
    \hat{A}^0 &:= \sum_{i=1}^{m} (I \otimes A_i) T[s^0_{i,M}] \\
    \hat{D}^0 &:= \sum_{i=1}^{m} (I \otimes D_i) T[s^0_{i,M}] \\
    \hat{\partial B}^0 &:= \sum_{i=1}^{m} (I \otimes B_i) \left( \hat{\Psi}^1_i F^1_i + \hat{\Psi}^2_i F^2_i \right)
\end{align*}
\]
where

\[
\bar{F}_i^\kappa := \left[ \ldots, F_{i,1}^\kappa, F_{i,0}^\kappa, F_{i,-1}^\kappa, \ldots \right]
\] (B.33)

is an infinite dimensional row vector containing the Fourier coefficients of the feedback function \( F_i^\kappa \) defined in (B.17) and where

\[
\Psi_i^\kappa := \left[ \ldots, c_{i,1}^\kappa, c_{i,0}^\kappa, c_{i,-1}^\kappa, \ldots \right]'
\]

is an infinite dimensional column vector containing the coefficients defined in (B.31).

We note that the coefficients \( F_{i,n}^\kappa \) of order \(|n| > N\) are all zero and that the coefficients \( c_{i,n}^\kappa \) of order \(|n| > M\) are also zero.

Let \( \hat{x}(\omega) := \mathcal{F}(\hat{x})(\omega) \) and \( \hat{\omega}(\omega) := \mathcal{F}(\hat{\omega})(\omega) \) denote the Fourier transforms of \( \hat{x} \) and \( \hat{\omega} \) respectively. By formally applying the Fourier transform to the equations (B.32) and inverting we obtain the following input-to-state relationship between the input signal and the state

\[
\hat{x}(\omega) = \left( j\omega I - ( -j\omega_n N + \hat{A}^0 + \partial \hat{B}^0 ) \right)^{-1} \hat{\Delta}^0 \hat{\omega}(\omega)
\]

where \( \mathcal{H}(\omega) \) is a doubly infinite dimensional matrix. In the case of an LTP system where there is no feedback, the term \( \partial \hat{B}^0 \) disappears and the remaining matrix \( \mathcal{H}(\omega) \) is a skew truncated harmonic transfer function (HTF) [13,16,20,23]. Thus the transfer function \( \mathcal{H}(\omega) \) can be seen as a generalization of the HTF to non-periodic feedback systems. However, it should be noted that \( \partial \hat{B}^0 \) is not block Toeplitz and therefore, the structure of the HTF [13] is not preserved.

8 Examples

8.1 Example 2: Inverted pendulum

In this example we apply Proposition 5.1 to a vibrational control system. We consider the inverted pendulum from [12] where the pivot point is vibrated along the vertical axis by the signal \( \mu \sin(\omega t) \). We define \( x := [\phi \phi]' \) where \( \phi \) is the angle to the vertical axis. The linearized equations describing the upper equilibrium point of the inverted pendulum can then be stated

\[
\dot{x}(t) = (A_0 + \sin(\omega t)A_1)x(t)
\]

where

\[
A_0 = \begin{bmatrix} 0 & 1 \\ g/l & -\lambda \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ -\mu \omega^2/l & 0 \end{bmatrix}
\]

where \( g \) is the gravitational acceleration, \( l \) is the length of the pendulum and \( \lambda \) is the Rayleigh resistance. The numerical values are chosen as \( \omega = 2000\pi, \mu = 0.01, l = 1 \) and \( \lambda = 1 \).
We note that the matrix \( A_0 \) has one positive and one negative eigenvalue and that both eigenvalues of \( A_1 \) are zero. The real part of the eigenvalues of the matrix \( A_0 + \sin(\omega t)A_1 \) as a function of time are shown in Fig. B.2. Here it can be seen that during the first half of the time period the eigenvalues are purely imaginary and during the second half they move into the left and right half plane respectively.

![Figure B.2: Real part of the eigenvalues of the matrix \( A_0 + \sin(\omega t)A_1 \) as a function of time.](image)

Figure B.2: Real part of the eigenvalues of the matrix \( A_0 + \sin(\omega t)A_1 \) as a function of time.

Because the system dynamics have a finite number of harmonics it is sufficient to consider the first two conditions in Proposition 5.1. In other words, there exists a time periodic Lyapunov function of the form (B.11) proving stability of the system iff conditions (i) and (ii) in Proposition 5.1 are satisfied. We choose parameters \( \beta = \)
0.1, $\alpha_N = 0.1$, $M = 5$ and $N = 4$ and state inequalities (i) and (ii) (equivalently) in terms of LMIs using Lemma 4.1. The LMIs are feasible and the solution yields a Lyapunov function $P_N(t)$. The eigenvalues of the Lyapunov function and the Lyapunov equation $\dot{P}_N(t) + A_M(t)^TP(t) + P(t)A_M(t) + \alpha_N P_N(t)$ are plotted as a function of time in Fig. B.3.

### 8.2 Example 3: Step-down converter revisited

In this example we return to the step-down DC-DC converter in Section 2.1 and prove stability of the system in closed loop operation. We consider a constant feedback function $F = [-f_1 0]'$, $f_1 > 0$ so that the duty cycle is determined by sampling the function

$$d(x(t)) = \text{sat}_{[0,1]} \left( d^0 + \frac{1}{T_s} \int_{t-T_s}^{t} F'(x(\tau))d\tau \right).$$

To show stability of the step-down converter with this feedback we take $N = M = 0$ and proceed to verify stability of the truncated averaged system (B.23) where the system matrices are

$$A_{av,M}(t, \tau) = A_{av,0}(t, \tau) = A$$
$$B_{av,M}(t, \tau) = B_{av,0}(t, \tau) = B(d(t) - d^0)$$

where $A$, $B$ are defined in Example 2.1 and where $d(t)$ is a function of the approximate solution defined as

$$d(t) = \text{sat}_{[0,1]} \left( d^0 + \frac{1}{T_s} \int_{-T_s}^{0} F'(x(t, \tau))d\tau \right).$$

To verify stability of the averaged system we proceed as described in Section 6.1, i.e., we want to verify inequality (B.25). In this inequality we consider the matrix function

$$P_N(t) = P_0 = \frac{1}{2} \begin{bmatrix} x_l & 0 \\ 0 & x_c \end{bmatrix}$$

as a Lyapunov candidate. The feedback implies a partition of the state space into three regions. They are

$$S^2_{-1} := \{ x \in L_2 \mid d(x) \leq 0 \}$$
$$S^0_0 := \{ x \in L_2 \mid 0 < d(x) < 1 \}$$
$$S^2_1 := \{ x \in L_2 \mid d(x) \geq 1 \}.$$
We now check inequality (B.25) on each of the three regions. From Proposition 6.2 if follows that (B.25) is satisfied on the saturated regions $S^2_{\lambda_1}$, $S^1_{\lambda_1}$ iff the exists non-negative numbers $\lambda_{1,\pm 1} \geq 0$ such that the LMIs

$$
\begin{bmatrix}
-\frac{v_1}{2} (1 - d^0) - f_1 \lambda_{1,1} & 0 & -2(1 - d^0) \lambda_{1,1} \\
-\frac{v_2}{2} (1 - d^0) - f_1 \lambda_{1,1} & 0 & -2(1 - d^0) \lambda_{1,1} \\
-\frac{v_1}{2} d^0 + f_1 \lambda_{1,-1} & 0 & -2d^0 \lambda_{1,-1}
\end{bmatrix} \leq 0
$$

are satisfied. The two LMI above are satisfied for $\alpha_N$ small enough by choosing the S-procedure multipliers as $\lambda_{1,1} = \frac{v_1}{2f_1} (1 - d^0) \geq 0$ and $\lambda_{1,-1} = \frac{v_1}{2f_1} d^0 \geq 0$.

We now consider (B.25) on the non-saturated set $S^0_\lambda$. By applying Proposition 6.2 and noting that $\gamma(f_{ns} - f_{ns}^0) = \frac{(d - d^0)}{f_{ns}} F$ (where we have used $f_{ns}^0 = d^0$ and $f_{ns} = d \in [0, 1]$) one can show that (B.25) is satisfied for all $x \in S^0_\lambda$ if there exists numbers $\lambda_2 \geq 0$ and $\Lambda \geq 0$ such that for all $d \in [0, 1]$ it holds

$$
\begin{bmatrix}
-\frac{v_1}{2} x_l - f_1^2 \Lambda & 0 & \frac{(\frac{v_1}{2} + \frac{v_2}{2})(d - d^0) - f_1 \lambda_2}{} \\
0 & -\frac{1}{r_o} + \frac{\alpha_N}{2} x_c & 0 \\
(\frac{v_1}{2} + \frac{v_2}{2})(d - d^0) - f_1 \lambda_2 & 0 & -(\frac{v_1}{2f_1} + \frac{v_2}{2})(d - d^0)^2
\end{bmatrix} \leq 0. \ (B.34)
$$

It is easily verified that (B.34) is satisfied for all $d \in [0, 1]$ for $\lambda_2 = 0$ and $\Lambda$ large enough. We conclude that the averaged system is stable. We note that the condition $\langle x, \mathbb{P}_N x \rangle \leq R^2$ was not relaxed in the LMI above and thus, $R$ can be chosen arbitrarily large without affecting validity of the stability conditions.

The discussion above showed stability of the averaged approximate model. To show stability of the actual system we must make sure that conditions (iii) and (iv) in Proposition 6.2 are also satisfied. From the definition of $\beta_1$ and $\beta_2$ in Appendix 13 it is clear that conditions (iii) and (iv) hold for $T_s$ small enough.

## 9 Interpretation of the feedback

The somewhat unorthodox definition of the feedback in (B.16) is motivated because it conforms with the dynamic phasor model. More precisely, if $\dot{x}$ is the vector of phasor coefficients of $x$ and $\tilde{F}_t x$ is the infinite dimensional row vector defined in (B.33), it holds

$$
d_i(x(t)) := \text{sat}_{[0, 1]} \left( d_i^0 + \langle F_t^2(\cdot), x(\cdot) \rangle_{L_2[\tau - T_s, t]} \right)$$

$$= \text{sat}_{[0, 1]} \left( d_i^0 + \tilde{F}_t \dot{x}(t) \right)
$$
and similarly for \( a_i(x(t)) \). The relation above means that we can easily state the closed loop system in the phasor domain. In the section below we show that the definition (B.16) is not simply for convenience. It can be interpreted as an approximation of a feedback which is very common in practice; namely linear feedback with a sample and hold device.

In the definition of the feedback functions in (B.16) two cases are of special interest. The first case is when the coefficients satisfy \( F_{i,k}^\text{c} = 0 \) \( \forall k \neq 0 \). This means that the feedback uses the average value of (part of) the state over the past switch period. This scenario is relevant since the state of pulse-modulated systems contains ripple which one may want to filter out.

The second case of special interest is when all coefficients are identical, \( i.e., \) when

\[
F_{i,k}^\text{c}(t) = \tilde{F}_i^\text{c} \sum_{k=-N}^{N} e^{jk\omega_s t}
\]

for some vector \( \tilde{F}_i^\text{c} \). In this case, the feedback functions \( a_i(x(t)) \), \( d_i(x(t)) \) can be interpreted as approximations of the output of a sample and hold circuit (see Fig. B.4). To see this, we now perform some formal calculations.

Consider the feedback function \( d_i(x(t)) \) and note that

\[
d_i(x(t)) = \frac{1}{T_s} \int_{t-T_s}^{t} F_i^2(\tau) x(\tau) d\tau = \int_{-\infty}^{\infty} h(t-\tau) F_i^2(\tau) x(\tau) d\tau
\]

where

\[
h(t) = \begin{cases} 
\frac{1}{T_s}, & t \in [0, T_s] \\
0, & t \notin [0, T_s]
\end{cases}
\]
has the Fourier transform

\[ H(\omega) := \mathcal{F}\{h\}(\omega) := \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \frac{1 - e^{-j\omega T_s}}{j\omega T_s}. \]

We note that the Fourier transform of \( e^{j\omega_s t} \) is \( 2\pi\delta(\omega - k\omega_s) \) and it follows that

\[
D_i(\omega) := \mathcal{F}\{d_i\}(\omega) = H(\omega)(\mathcal{F}\{F_i^2\} * \mathcal{F}\{x\})(\omega)
\]

\[
= H(\omega) \int_{-\infty}^{\infty} F_i^2 \sum_{k=-N}^{N} 2\pi\delta(\omega - k\omega_s - \eta)X(\eta)d\eta
\]

\[
= 2\pi H(\omega) \tilde{F}_i^2 \sum_{k=-N}^{N} \int_{-\infty}^{\infty} \delta(\omega - k\omega_s - \eta)X(\eta)d\eta
\]

\[
= 2\pi H(\omega) \tilde{F}_i^2 \sum_{k=-N}^{N} X(\omega - k\omega_s)
\]

where \(*\) denotes the convolution and where \( X(\omega) := \mathcal{F}\{x\}(\omega) \).

We now consider the sample and hold circuit in Fig. B.4. The sampled signal \( \tilde{d}_{i,s} \) is represented using an impulse train, i.e.,

\[
\tilde{d}_{i,s} = \tilde{d}_i(t) \sum_{k=-\infty}^{\infty} T_s\delta(t - kT_s).
\]

The corresponding Fourier transform is

\[
\tilde{D}_{i,s}(\omega) := \mathcal{F}\{\tilde{d}_{i,s}\}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \tilde{D}_i(\omega - k\omega_s)
\]

where \( \tilde{D}_i(\omega) := \mathcal{F}\{\tilde{d}_i\}(\omega) \) and where we have used that the Fourier transform of \( \sum_{k=-\infty}^{\infty} T_s\delta(t - kT_s) \) is \( 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \). Since \( D_i(\omega) = \tilde{F}_i^2 X(\omega) \) it follows that \( \tilde{D}_{i,h}(\omega) := \mathcal{F}\{\tilde{d}_{i,h}\}(\omega) \) satisfies

\[
\tilde{D}_{i,h}(\omega) = 2\pi H(\omega) \tilde{F}_i^2 \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).
\]

Comparing this expression to (B.35) we conclude that \( d_i \) is equal to \( \tilde{d}_{i,h} \) in the limit as \( N \to \infty \).

The discussion above implies that in the special case where \( F_{i,k}^2 = \tilde{F}_i^2 \) \( \forall k \) and when \( N \to \infty \), the sampled feedback \( \tilde{d}_i(kT_s) \) satisfies

\[
\tilde{d}_i(kT_s) = \tilde{F}_i^2 x(kT_s)
\]

and similarly for \( a_i(kT_s) \). The equality above would greatly simplify the proof of Proposition 6.1. In particular, to bound the terms \( |d_i(kT_s) - d_i(t)| \) and \( |a_i(kT_s) -
which appear in Lemma 15.4 we would not need to use Lemma 15.5 and 15.6 (which is now the case).

However, in the stability analysis, the system (B.1) with feedback (B.16) is represented by the $L_2$-equivalent Fourier series representation (B.10) with feedback

$$d_i(x(t, \cdot)) = \text{sat}_{[0,1]} \left( d_i^0 + \langle F_i^2(\cdot), x(t, \cdot) \rangle_{L_2[-T,0]} \right)$$

and similarly for $a_i(x(t, \cdot))$. As $N \to \infty$, the gain of the term $\langle F_i^2(\cdot), x(t, \cdot) \rangle_{L_2[-T,0]}$ goes to infinity. Hence, the feedback (B.36) is only possible to represent in the phasor model by using infinite gain.

## 10 Conclusions

The dynamic phasor model was used to describe a general class of pulse-modulated systems in the frequency domain. The infinite dimensional dynamics were approximated via skew truncation and the approximate system was used to compute time periodic quadratic Lyapunov functions and for harmonic analysis.

When applied to LTP systems, the approach showed that the time periodic Lyapunov inequality could be stated approximately in terms of LMIs. In the closed loop case, the approach yielded stability conditions in terms of parameterized matrix inequalities which include the state space averaging result as a special case.

## 11 Appendix: Proof of Lemma 4.1

Periodicity of $P_N(t)$ implies that inequality (i) in Lemma 4.1 holds if and only if

$$\sum_{k=-N}^{N} p_k e^{j\omega_0 t} \geq \beta^2 I, \forall \ t \iff \sum_{k=-N}^{N} p_k z^k \geq \beta^2 I, \forall |z| = 1 \iff$$

$$\begin{bmatrix} I \\ zI \\ \vdots \\ z^N I \end{bmatrix}^* \begin{bmatrix} p_0 - \beta^2 I & p_1 & \cdots & p_N \\ \bar{p}_1 & 0 & \cdots & \bar{p}_N \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \bar{p}_N & z^N I \end{bmatrix} \begin{bmatrix} I \\ zI \\ \vdots \\ z^N I \end{bmatrix} \geq 0, \forall |z| = 1 \iff$$

$$H(z)^* K(p_k, \beta) H(z) \geq 0, \forall |z| = 1$$

where the last inequality is obtained by multiplying both sides with $1/|(z + a)^N|^2$. This function is bounded and nonzero on the unit circle. By the discrete time version of the KYP lemma [15] it can easily be shown that this inequality holds iff there is
a matrix $Q = Q' \in \mathbb{C}^{N \times N}$ satisfying
\[
\begin{bmatrix}
A'QA - Q & A'QB \\
B'QA & B'QB
\end{bmatrix} + \begin{bmatrix} C' \\
D'
\end{bmatrix} K(p_k, \beta) \begin{bmatrix} C & D \end{bmatrix} \geq 0
\]
where $(A, B, C, D)$ is a minimal realization of $H(z)$. This concludes the proof.

12 Appendix: Proof of Proposition 5.1

From Lemma 4.2 we have
\[
\dot{V}(x(t)) = \left\langle x, (\dot{P}_N + 2P_NA)x \right\rangle_{L_2}
\]
\[
= \left\langle x, (\dot{P}_N + 2P_NA_M)x \right\rangle_{L_2} + 2 \left\langle x, P_N(A - A_M)x \right\rangle_{L_2}
\]
\[
\leq -\alpha_N V(x(t)) + 2 \left\langle x, P_N(A - A_M)x \right\rangle_{L_2}
\]
where the inequality follows from assumption $(ii)$. To bound the remaining term above we note that the coefficients of the Fourier series
\[
P_N(t, \tau)(A(t, \tau) - A_M(t, \tau))
\]
of order less than or equal to $M - N \geq 0$ are zero. Let $L := \lfloor (M - N)/2 \rfloor$, the observation above implies that
\[
\left\langle \pi_{Lx}, P_N(A - A_M)\pi_Lx \right\rangle_{L_2} = 0
\]
and thus we have
\[
\left\langle x, P_N(A - A_M)x \right\rangle_{L_2} = \left\langle (I - \pi_L)x, P_N(A - A_M)x \right\rangle_{L_2}
\]
\[
+ \left\langle \pi_Lx, P_N(A - A_M)(I - \pi_L)x \right\rangle_{L_2}.
\]
To bound the first term above we use Lemma 15.2 which implies
\[
\left\langle (I - \pi_L)x, P_N(A - A_M)x \right\rangle_{L_2} \leq \parallel (I - \pi_L)x \parallel_{L_2} \parallel P_N \parallel \sum_{i=1}^{m} \sigma(A_i) \parallel x(I - \pi_M)s_i \parallel_{L_2}
\]
\[
\leq \parallel (I - \pi_L)x \parallel_{L_2} \parallel P_N \parallel \parallel x \parallel_{\infty} \sum_{i=1}^{m} \sigma(A_i) \parallel (I - \pi_M)s_i \parallel_{L_2}
\]
where $\parallel P_N \parallel := \sup_t \bar{\sigma}(P_N(t))$ and where Lemma 15.2 was used in the second inequality. Applying Lemma 15.1 and 15.3 to the inequality above yields
\[
\left\langle (I - \pi_L)x, P_N(A - A_M)x \right\rangle_{L_2}
\]
\[
\leq \frac{2T_s}{\pi^2 \sqrt{(L + 1)(L + 1)}} |A| (1 + T_s |A|) \parallel P_N \parallel \sum_{i=1}^{m} \sigma(A_i) (2 + T_s \parallel \dot{s}_i \parallel) \frac{1}{\beta^2} V
\]
where we have also used $\|x\|_{L^2} \leq \frac{1}{\beta} V(x)$. To bound the second term in (B.39) we note that

$$\langle \pi_L x, P_N (A - A_M)(I - \pi_L)x \rangle_{L^2} = \langle (A - A_M)^\dagger P_N \pi_L x, (I - \pi_L)x \rangle_{L^2}$$

$$\leq \| (I - \pi_L)x \|_{L^2} \sum_{i=1}^m \bar{\sigma}(A_i) \| P_N(\pi_L x)(I - \pi_M)s_i \|_{L^2}$$

$$\leq \| (I - \pi_L)x \|_{L^2} \sup_{\tau} |P_N(t, \tau) \pi_L x(t, \tau) \rangle \| \sum_{i=1}^m \bar{\sigma}(A_i) \| (I - \pi_M)s_i \|_{L^2}.$$  

Using the bound on $|\langle \chi \rangle_n|$ provided in Lemma 15.3 together with the inequality $|\langle x \rangle_0| \leq \|x\|_{L^2} \leq \frac{1}{\beta} V(x)^{1/2}$ it can be shown that

$$\sup_{\tau \in [-T_s, 0]} |P_N(t, \tau) \pi_L x(t, \tau)| \leq \|P_N\| \sum_{n=-L}^L |\langle x \rangle_n (t)|$$

$$\leq \|P_N\| \frac{1}{\beta} \left( 1 + \frac{2T_s|A|}{\pi} \ln(L + 1) \right) V^{1/2}.$$  

This inequality is used together with Lemma 15.3 to bound the second term according to

$$\langle (I - \pi_L)x, P_N (A - A_M)x \rangle_{L^2} \leq \frac{2T_s}{\pi^2 \sqrt{(L + 1)(M + 1)}} |A| \|P_N\|$$

$$\times \left( 1 + \frac{2T_s|A|}{\pi} \ln(L + 1) \right) \sum_{i=1}^m \bar{\sigma}(A_i)(2 + T_s \| \hat{f}_i \|) \frac{1}{\beta^2} V.$$  

In summary we get

$$\langle x, P_N (A - A_M)x \rangle_{L^2} \leq \frac{2T_s|A||P_N||}{\pi^2 \sqrt{(L + 1)(M + 1)}} \left( 2 + T_s|A| \left( 1 + \frac{2}{\pi} \ln(L + 1) \right) \right) \sum_{i=1}^m \bar{\sigma}(A_i)(2 + T_s \| \hat{f}_i \|) \frac{1}{\beta^2} V.$$  

Using this inequality in (B.37) we bound on $\dot{V}$ as

$$\dot{V}(x(t)) \leq \left(-\alpha_N + \frac{C(T_s, L)}{\sqrt{M + 1}} \right) V(x(t)).$$  

By the assumption (iii) we get $\dot{V}(x(t)) \leq -\alpha V(x(t))$ and it follows that

$$V(x(t)) \leq e^{-\alpha(t-t_0)} V(x(t_0))$$

for $t \geq t_0$. This concludes the proof.
13 Appendix: Proof of Proposition 6.1

Consider a solution \( x \) of (B.1). Without loss of generality we assume \( x \) is contained in the set \( \Omega := \{ x \mid V(x) \leq R^2 \} \) during the time interval \([t_0, t_1]\) where \( R \) is an arbitrary positive number, \( t_0 \) is the initial time and \( t_1 > t_0 \). Using the result of Lemma 4.2 the time derivative of \( V(x) \) is bounded for \( t \in [t_0, t_1] \) according to

\[
\dot{V}(x(t)) = \left( x, (\dot{P}_N + 2P_N A)x + 2P_N B \right)_{L_2} \\
= \left( x, (\dot{P}_N + 2P_N A_{av,M})x + 2P_N B_{av,M} \right)_{L_2} \\
+ 2 \left( x, P_N (A_{av} - A_{av,M})x + P_N (B_{av} - B_{av,M}) \right)_{L_2} \\
+ 2 \left( x, P_N (A - A_{av})x + P_N (B - B_{av}) \right)_{L_2} \\
\leq -\alpha_N V(x) + 2 \| x \|_{L_2} \| P_N \| \left( \| (A - A_{av})x \|_{L_2} + \| B - B_{av} \|_{L_2} \right) \\
+ 2 \left( x, P_N (A_{av} - A_{av,M})x + P_N (B_{av} - B_{av,M}) \right)_{L_2}
\]

where the last inequality follows form assumption \( (ii) \). To bound the expression above we need to bound a number of terms. Since many of these terms contain the quantity \( \| s_i - s_{av,i} \|_{L_2} \), we first proceed to bound said quantity: We note that for \( t \in [kT_s, (k + 1)T_s] \), the term \( |a_{i,k} - a_i(t)| \) satisfies

\[
|a_{i,k} - a_i(t)| = \left| \text{sat}_{[-1,1]} \left( a_i^0 + \left( F^{1}_{i}, x \right)_{L_2(k-1)T_s, kT_s} \right) - \text{sat}_{[-1,1]} \left( a_i^0 + \left( F^{1}_{i}, x \right)_{L_2[t-T_s,t]} \right) \right| \\
\leq \left| \left( F^{1}_{i}, x \right)_{L_2(k-1)T_s, kT_s} - \left( F^{1}_{i}, x \right)_{L_2[t-T_s,t]} \right| \\
= \left| \left( F^{1}_{i}, \hat{x}(kT_s) - \hat{x}(t) \right)_{L_2} \right| \\
\leq \| F^{1}_{i} \| \| \hat{x}(kT_s) - \hat{x}(t) \|_{L_2}
\]

where \( \| F^{1}_{i} \| := \left( F^{1}_{i}, F^{1}_{i} \right)^{1/2} \) and analogously it can shown that

\[
|a_{i,k-1} - a_i(t)| = \| F^{1}_{i} \| \| \hat{x}((k - 1)T_s) - \hat{x}(t) \|_{L_2} \\
|d_{i,k} - d_i(t)| = \| F^{2}_{i} \| \| \hat{x}(kT_s) - \hat{x}(t) \|_{L_2} \\
|d_{i,k-1} - d_i(t)| = \| F^{2}_{i} \| \| \hat{x}((k - 1)T_s) - \hat{x}(t) \|_{L_2}
\]

where \( \| F^{2}_{i} \| := \left( F^{1}_{i}, F^{1}_{i} \right)^{1/2} \). The inequalities above are used together with Lemma 15.4 and 15.6 to obtain the bound

\[
\| s_i(t, \cdot) - s_{av,i}(t, \cdot) \|_{L_2[-T_s,0]} \leq C_i(T_s) \left( \| x(t, \cdot) \|_{L_2} + \frac{|B|}{|A|} \right) \tag{B.41}
\]

where

\[
C_i(T_s) := 2(\| F^{1}_{i} \| + \| F^{2}_{i} \|) \left( (2T_s |A| + 1)e^{2T_s |A|} - 1 \right).
\]
The inequality (B.41) is now used to bound a number of terms. First we note that
\[ \| (A - A_{av}) x \|_{L^2} \leq \sum_{i=1}^{m} \sigma(A_i) \| (s_i - s_{av,i}) x \|_{L^2} \leq \| x \|_{\infty} \sum_{i=1}^{m} \sigma(A_i) \| s_i - s_{av,i} \|_{L^2} \]
\[ \leq \left( \sum_{i=1}^{m} \sigma(A_i) C_i(T_s) \right) \left( 1 + T_s |A| \right) \frac{1}{\beta^2} V + \left( \frac{|B|}{|A|} + 2T_s |B| \right) \frac{1}{\beta} V^{1/2} + T_s \frac{|B|^2}{|A|} \]
where the second inequality follows from Lemma 15.2 and where the last inequality follows from (B.41), Lemma 15.1 and \( \| x \|_{L^2} \leq \frac{1}{\beta} V(x)^{1/2} \). Secondly we note that
\[ \| B - B_{av} \|_{L^2} \leq \sum_{i=1}^{m} \sigma(B_i) \| s_i - s_{av,i} \|_{L^2} \]
\[ \leq \left( \sum_{i=1}^{m} \sigma(B_i) C_i(T_s) \right) \left( \frac{1}{\beta} V^{1/2} + \frac{|B|}{|A|} \right) \]
where we again have used (B.41). To bound the remaining two terms above we note (as in the proof of Proposition 5.1) that in the Fourier series (B.38) the coefficients of order less than or equal to \( M - N \) are all zero. In analogy with the proof of Proposition 5.1 we therefore have
\[ \langle x, P_N(A_{av} - A_{av,M}) x \rangle_{L^2} = \langle (I - \pi_L)x, P_N(A_{av} - A_{av,M}) x \rangle_{L^2} \]
\[ + \langle \pi_L x, P_N(A_{av} - A_{av,M})(I - \pi_L)x \rangle_{L^2} \quad (B.42) \]
The first term in (B.42) is bounded according to
\[ \langle (I - \pi_L)x, P_N(A_{av} - A_{av,M}) x \rangle_{L^2} \leq \| (I - \pi_L)x \| \| P_N \| \| x \|_{\infty} \sum_{i=1}^{m} \sigma(A_i) \| (I - \pi_M) s_{av,i} \| \]
\[ \leq 2T_s \frac{\| P_N \|}{\pi^2} \sum_{i=1}^{m} \sigma(A_i) (2 + T_s \| \tilde{f}_i \|) \frac{1}{\sqrt{L + 1}} \frac{1}{\sqrt{M + 1}} \]
\[ \times \left( |A| \frac{1}{\beta^2} V + |B| \frac{1}{\beta} V^{1/2} + T_s \left( |A| \frac{1}{\beta} V^{1/2} + |B| \right)^2 \right) \]
where in the first inequality we used Lemma 15.2 and in the last inequality we used Lemma 15.1 and 15.3. The second term in (B.42) is bounded as
\[ \langle \pi_L x, P_N(A_{av} - A_{av,M})(I - \pi_L)x \rangle_{L^2} = \langle (A_{av} - A_{av,M}) \pi_N \pi_L x, (I - \pi_L)x \rangle_{L^2} \]
\[ \leq \| (I - \pi_L)x \| \sup_{\pi} |P_N \pi_L x| \sum_{i=1}^{m} \sigma(A_i) \| (I - \pi_M) s_{av,i} \| \]
\[ \leq 2T_s \frac{\| P_N \|}{\pi^2} \sum_{i=1}^{m} \sigma(A_i) (2 + T_s \| \tilde{f}_i \|) \frac{1}{\sqrt{L + 1}} \frac{1}{\sqrt{M + 1}} \]
\[ \times \left( |A| \frac{1}{\beta^2} V + |B| \frac{1}{\beta} V^{1/2} + \frac{2T_s}{\pi} \left( |A| \frac{1}{\beta} V^{1/2} + |B| \right)^2 \ln(L + 1) \right) \]
where we have used inequality (B.40) and Lemma 15.1 and 15.3. Finally we note that in analogy with (B.42) it holds

\[ \langle x, P_N(B_{av} - B_{av,M}) \rangle_{L_2} = \langle (I - \pi_L)x, P_N(B_{av} - B_{av,M}) \rangle_{L_2} \]

\[ \leq \| (I - \pi_L)x \| \| P_N \| \sum_{i=1}^m \tilde{\sigma}(B_i) \| (I - \pi_M)s_{av,i} \| \]

\[ \leq 2T_s \frac{\| P_N \|}{\pi^2} \sum_{i=1}^m \frac{\tilde{\sigma}(B_i)(2 + T_s\|\hat{f}_i\|)1}{\sqrt{L + 1}\sqrt{M + 1}} \left( |A| \frac{1}{\beta} V^{1/2} + |B| \right). \]

The inequalities above are combined to bound \( \dot{V} \) according to

\[ \dot{V} \leq \alpha_0 V^{3/2} + (-\alpha_N + \alpha_{11} + \alpha_{12})V + (\alpha_{21} + \alpha_{22})V^{1/2} + \alpha_3 \]

where \( \alpha_0 = \alpha_0(T_s) \) and \( \alpha_{11} = \alpha_{11}(T_s), i = 1, 2 \) are defined as

\[ \alpha_0(T_s) = 2 \frac{\| P_N \|}{\beta^2} (1 + T_s|A|)\eta_A \]

\[ \alpha_{11} = 2 \frac{\| P_N \|}{\beta^2} \left( \left( \frac{|B|}{|A|} + 2T_s|B| \right) \eta_A + \eta_B \right) \]

\[ \alpha_{21} = 2 \frac{\| P_N \|}{\beta} \frac{|B|}{|A|} (T_s|B|\eta_A + \eta_B) \]

and \( \alpha_{12} = \alpha_{12}(T_s, L, M), i = 1, 2 \) and \( \alpha_3 = \alpha_3(T_s, L, M) \) are defined as

\[ \alpha_{12} = 4 \frac{\| P_N \|}{\beta^2 \pi^2} \left( 2 + T_s|A|(1 + \frac{1}{\pi} \ln(L + 1)) \right) \frac{T_s|A|\gamma_A}{\sqrt{(L + 1)(M + 1)}} \]

\[ \alpha_{22} = 4 \frac{\| P_N \|}{\beta^2 \pi^2} \left( 2 + T_s|A|(1 + \frac{1}{\pi} \ln(L + 1)) \right) \frac{2T_s|B|\gamma_A}{\sqrt{(L + 1)(M + 1)}} \]

\[ + 4 \frac{\| P_N \|}{\beta^2 \pi^2} \frac{T_s|A|\gamma_B}{\sqrt{(L + 1)(M + 1)}} \]

\[ \alpha_3 = 4 \frac{\| P_N \|}{\pi^2} \left( 1 + \frac{2}{\pi} \ln(L + 1) \right) \frac{T_s^2|B|^2\gamma_A}{\sqrt{(L + 1)(M + 1)}} \]

\[ + 4 \frac{\| P_N \|}{\pi^2} \frac{T_s|B|\gamma_B}{\sqrt{(L + 1)(M + 1)}} \]

where

\[ \gamma_A := \sum_{i=1}^m \tilde{\sigma}(A_i)(2 + T_s\|\hat{f}_i\|), \quad \gamma_B := \sum_{i=1}^m \tilde{\sigma}(B_i)(2 + T_s\|\hat{f}_i\|) \]

and

\[ \eta_A(T_s) := \sum_{i=1}^m \tilde{\sigma}(A_i)C_i(T_s), \quad \eta_B(T_s) := \sum_{i=1}^m \tilde{\sigma}(B_i)C_i(T_s). \]
We note that the coefficients $\alpha_i$ are small when $T_s$ is small and $L$ and $M$ are large.

The assumption that $x(t) \in \Omega$ for $t \in [t_0, t_1]$ is used together with the inequality $\alpha_2 V^{1/2} \leq \frac{\alpha_1}{2} + \frac{\alpha_2}{2} V$ to bound $\dot{V}$ according to

$$
\dot{V} \leq -\beta_1(T_s, L, M)V + \beta_2(T_s, L, M, R)
$$

where

$$
\begin{align*}
\beta_1(T_s, L, M) &= \alpha_N - \alpha_{11}(T_s) - \alpha_{12}(T_s, L, M) - \frac{1}{2} \alpha_{21}(T_s) - \frac{1}{2} \alpha_{22}(T_s, L, M) \\
&= \alpha_N - c_{11}(T_s) - c_{12}(T_s, L, M) \\
\beta_2(T_s, L, M, R) &= \alpha_0(T_s) R^2 + \frac{1}{2} \alpha_{21}(T_s) + \frac{1}{2} \alpha_{22}(T_s, L, M) + \alpha_3(T_s, L, M) \\
&= c_{21}(T_s, R) + c_{22}(T_s, L, M)
\end{align*}
$$

and this implies that

$$
V(x(t)) \leq e^{-\beta_1(t-t_0)} V(x(t_0)) + \frac{\beta_2}{\beta_1} \left(1 - e^{-\beta_1(t-t_0)}\right).
$$

We derived this expression under the assumption that $V(x(t)) \leq R^2$. It is easy to see that this condition is satisfied if $\beta_2 \leq R^2 \beta_1$. By inspecting the coefficients $\alpha_i$, $i = 0, \ldots, 2$ it is clear that $\beta_1 > 0$ for $M$ and $L$ large enough and $T_s$ small enough. It follows that $\lim_{t \to \infty} V(x(t)) \leq \beta_2/\beta_1$ and it can be verified that the bound $\beta_2/\beta_1$ can be made arbitrarily small for $M$ and $L$ large enough and $T_s$ small enough.

14 Appendix: Proof of Proposition 6.2

The condition that (B.25) is satisfied for all $x \in \Omega(x_1, x_2)$ can be stated: Inequality (B.25) must hold for all $x \in V_{ns}^\perp + \gamma(A_{ns} \times D_{ns} - f_{ns}^0)$ s.t. $F_s x \geq f_s - f_s^0$ and $(x, P_N x)_{L_2} \leq R^2$. By the S-procedure this condition is implied by the existence of $\lambda_1 \geq 0$ and $\tau \geq 0$ such that

$$
\left\langle \begin{bmatrix} v_{ns}^\perp \\ 1 \end{bmatrix}, \left(\Pi(f_{ns}, \tau) + \Omega(\lambda_1, f_{ns})\right) \begin{bmatrix} v_{ns}^\perp \\ 1 \end{bmatrix} \right\rangle \leq 0, \quad \forall v_{ns}^\perp \in V_{ns}^\perp, \quad \forall f_{ns} \in A_{ns} \times D_{ns}.
$$

To see that this is equivalent to (B.28) we consider arbitrary $v \in L_2$, separated into the components $v = v_{ns} + v_{ns}^i$, $v_{ns} \in V_{ns}$, $v_{ns}^i \in V_{ns}^i$. We define the following
function

\[ Q(v) = \left\langle \begin{bmatrix} v_{ns} + v_{ns}^\perp \\ 1 \end{bmatrix}, (\Pi + \Omega + \Psi) \begin{bmatrix} v_{ns} + v_{ns}^\perp \\ 1 \end{bmatrix} \right\rangle_{L^2} \]

\[ = \langle v_{ns}, (\Pi_{11} - F_{ns}^* \Lambda F_{ns})v_{ns} \rangle_{L^2} + 2 \left\langle v_{ns}, \Pi_{11}v_{ns}^\perp + \Pi_{12} + F_{ns}^* \lambda_2 + F_{ns}^* \lambda_1 \right\rangle_{L^2} \]

\[ + \left\langle \begin{bmatrix} v_{ns}^\perp \\ 1 \end{bmatrix}, (\Pi + \Omega) \begin{bmatrix} v_{ns}^\perp \\ 1 \end{bmatrix} \right\rangle_{L^2}. \]

If we introduce the matrix decompositions \( \Pi_{11} - F_{ns}^* \Lambda F_{ns} : \mathcal{V}_n \oplus \mathcal{V}_{ns}^\perp \rightarrow \mathcal{V}_n \oplus \mathcal{V}_{ns}^\perp \)

and \( \left[ \begin{array}{c} \Pi_{11} \\ \Pi_{12} + F_{ns}^* \lambda_2 + F_{ns}^* \lambda_1 \end{array} \right] : \mathcal{V}_n \oplus \mathcal{V}_{ns}^\perp \oplus \mathbb{R} \rightarrow \mathcal{V}_n \oplus \mathcal{V}_{ns}^\perp \)

then the expression for \( Q(v) \) simplifies to

\[ Q(v) = \langle v_{ns}, (\Pi_{11})_{11} - F_{ns}^* \Lambda F_{ns} \rangle_{L^2} \]

\[ + 2 \left\langle v_{ns}, (\Pi_{11})_{11}v_{ns}^\perp + (\Pi_{12} + F_{ns}^* \lambda_1)_{1} + F_{ns}^* \lambda_2 \right\rangle_{L^2} \]

\[ + \left\langle \begin{bmatrix} v_{ns}^\perp \\ 1 \end{bmatrix}, (\Pi + \Omega) \begin{bmatrix} v_{ns}^\perp \\ 1 \end{bmatrix} \right\rangle_{L^2}. \]

By making \( \Lambda \) sufficiently large the function above becomes concave in \( v_{ns} \). Hence, we can maximize \( Q \) over \( v_{ns} \). By completion of squares we get the value

\[ \max_{v_{ns} \in \mathcal{V}_n} Q(v) = \left\langle \begin{bmatrix} v_{ns}^\perp \\ 1 \end{bmatrix}, (\Pi + \Omega - \Upsilon) \begin{bmatrix} v_{ns}^\perp \\ 1 \end{bmatrix} \right\rangle_{L^2} \]

and where

\[ \Upsilon(\Lambda, \lambda_2) = \Gamma(\lambda_2)^*((\Pi_{11})_{11} - F_{ns}^* \Lambda F_{ns})^{-1}\Gamma(\lambda_2) \]

where

\[ \Gamma(\lambda_2) = \left[ (\Pi_{11})_{12} \quad (\Pi_{12} + F_{ns}^* \lambda_1)_{1} + F_{ns}^* \lambda_2 \right]. \]

By increasing the magnitude of \( \Lambda \) the contribution of \( \Upsilon \) becomes small and the maximal quadratic form is negative due to (B.44). Hence, (B.44) implies (B.28). The reverse implication is obvious.
15 Appendix: Auxiliary lemmas and proofs

Proof of Lemma 3.1

Boundedness of $\hat{A}$ follows from the fact that $|s_i| \leq 1$ and the first two equalities follow directly from the Parseval identity. The inequality is shown below. The Parseval identity implies

$$
\|\hat{A}\| := \sup_{\|\hat{y}\|_2 = 1} \|\hat{A}\hat{y}\|_2
= \sup_{\|y\|_2 = 1} \|A(y)\|_{L_2[0, Ts]}$
$$
$$
= \sup_{\|y\|_2 = 1} \left( \frac{1}{T_s} \int_{t-T_s}^{t} |A(x(t), \tau)y(\tau)|^2 d\tau \right)^{1/2}
\leq \sup_{\|y\|_2 = 1} \max_{\tau \in [0, T_s]} \sigma(A(x(t), \tau)) \left( \frac{1}{T_s} \int_{t-T_s}^{t} |y(\tau)|^2 d\tau \right)^{1/2}
= \max_{\tau \in [0, T_s]} \sigma(A(x(t), \tau)) \leq \sum_{i=1}^{m} \sigma(A_i).
$$

We note that equality can also be shown. See paper C of the thesis.

Proof of Lemma 4.2

We note that $V(x(t))$ satisfies

$$
V(x(t)) := \frac{1}{T_s} \int_{t-T_s}^{t} x(\tau)' P_N(\tau) x(\tau) d\tau
:= \frac{1}{T_s} \int_{-T_s}^{0} x(t + \tau)' P_N(t + \tau) x(t + \tau) d\tau.
$$

Thus, the time derivative of $V(x(t))$ can be written

$$
\dot{V}(x(t)) = \left( x(t + \tau), P_N(t + \tau) + 2A(t + \tau) \right)_L, L_2[-T_s, 0] + \left( x(t + \tau), 2P_N(t + \tau) B(t + \tau) \right)_L, L_2[-T_s, 0].
$$

Using the definition $P_N(t, \tau) := P_N(t + \tau)$ and the fact that $x$ and $x$ are $L_2$-equivalent yields the result.

Lemma 15.1. The norm $\|x(t)\|_\infty := \sup_{\tau \in [0, T_s]} |x(\tau)|$ can be estimated as

$$
\|x\|_\infty \leq (1 + T_s |A|) \|x\|_{L_2[-T_s, 0]} + T_s |B|.
$$
Proof of Lemma 15.1

Let \( \tau_0 \in [t - T_s, t] \) be determined by

\[
\tau_0 = \text{argmin}_{\tau \in [t - T_s, t]} |x(\tau)|
\]

(where we have used that \( x \) is continuous). We now use that

\[
x(\tau) = x(\tau_0) + \int_{\tau_0}^{\tau} \dot{x}(\tau) d\tau
\]

and we get

\[
\|x\|_\infty \leq |x(\tau_0)| + \sup_{\tau \in [t - T_s, t]} \left| \int_{\tau_0}^{\tau} \dot{x}(\tau) d\tau \right|
\]

\[
\leq \|x\|_{L_2[t - T_s, t]} + T_s \|\dot{x}\|_{L_2[t - T_s, t]}
\]

where we have used that \( |x(\tau_0)| \leq \|x\|_{L_2[t - T_s, t]} \) and the Cauchy-Schwartz inequality was used in the second term. Finally, it holds

\[
\|x\|_\infty \leq (1 + T_s |A|) \|x\|_{L_2[t - T_s, t]} + T_s \|B\|_{L_2[t - T_s, t]}
\]

\[
\leq (1 + T_s |A|) \|x\|_{L_2[-T_s, 0]} + T_s |B|.
\]

Lemma 15.2. Let \( x \in L_\infty[t - T_s, t] \) be a solution of (B.1) and let \( x(t, \tau) = \sum_{k=-\infty}^{\infty} \langle x \rangle_k e^{jk}\omega_s(t+\tau) \) be the corresponding Fourier series representation. Let \( f \in L_2[t - T_s, t] \) with Fourier series expansion \( f(t, \tau) := \sum_{k=-\infty}^{\infty} \langle f \rangle_k e^{jk}\omega_s(t+\tau) \). It holds

\[
\|(xf)(t, \cdot)\|_{L_2[-T_s, 0]} \leq \|x\|_\infty \|f(\cdot, \cdot)\|_{L_2[-T_s, 0]}
\]

where \( \|x\|_\infty := \sup_{\tau \in [t - T_s, t]} |x(\tau)| \).

Proof of Lemma 15.2

The assumptions imply that \( x \in L_2[t - T_s, t] \cap L_\infty[t - T_s, t] \) and \( f \in L_2[t - T_s, t] \) which in turn implies that \( xf \in L_2[t - T_s, t] \). The fact that \( x \in L_2[t - T_s, t], f \in L_2[t - T_s, t] \) implies that the periodic extension of \( xf \) has a Fourier series expansion with Fourier coefficients

\[
\langle xf \rangle_k = \sum_{l=-\infty}^{\infty} \langle x \rangle_{k-l} \langle f \rangle_l
\]

where the sum converges absolutely \([4]\). This implies that \( xf \) and \( x \) are equivalent in the \( L_2 \) sense, which gives the bound

\[
\|xf\|_{L_2[-T_s, 0]} = \|xf\|_{L_2[t - T_s, t]} \leq \|x\|_\infty \|f\|_{L_2[t - T_s, t]} = \|x\|_\infty \|f\|_{L_2[-T_s, 0]}
\]

where the first equality follows from Parseval theorem.

Next we introduce a lemma concerning the convergence of the phasor coefficients of \( x \) and \( s_i \). It is shown that the phasors converge quadratically and this is key to all results of the paper.
Lemma 15.3. Let \( x \) be a solution of (B.1) and let \( s_i \) be defined as in (B.3). The magnitude of the phasors \( \langle x \rangle_n \) and \( \langle s_i \rangle_n \) are bounded as follows

\[
|\langle x \rangle_n| \leq \frac{T_s}{\pi |n|} (|A|||x||_{L_2} + |B|)
\]

\[
|\langle s_i \rangle_n| \leq \frac{1}{\pi |n|} \left(2 + T_s \|\hat{f}_i\|_{L_2[0,T_s]}\right).
\]

Remark 15.1. If \( d_{i,k} = 1, \forall k \) then the bound on \( \langle s_i \rangle \) can be improved to \( |\langle s_i \rangle_n| \leq \frac{2T_s}{\pi |n|} \|\hat{f}\|_{L_2[0,T]}, \) which is advantageous if \( T_s \) is small.

Remark 15.2. The bound implies that

\[
\| (I - \pi N)x \|_{L_2} \leq \frac{\sqrt{2T_s}}{\pi \sqrt{N + 1}} (|A|||x||_{L_2} + |B|)
\]

\[
\| (I - \pi N)s_i \|_{L_2} \leq \frac{\sqrt{2}}{\pi \sqrt{N + 1}} \left(2 + T_s \|\hat{f}_i\|_{L_2[0,T_s]}\right).
\]

Proof of Lemma 15.3

Consider first the phasors \( \langle x \rangle_n \): Partial integration implies

\[
\langle x \rangle_n (t) := \frac{1}{T_s} \int_{t-T_s}^{t} x(\tau)e^{-jn\omega_s \tau} \, d\tau
\]

\[
= \frac{j}{n\omega_s T_s} \int_{t-T_s}^{t} x(\tau)(-jn\omega_s)e^{-jn\omega_s \tau} \, d\tau
\]

\[
= \frac{j}{n\omega_s T_s} \left( [x(\tau)e^{-jn\omega_s \tau}]_{t-T_s}^{t} - \int_{t-T_s}^{t} \dot{x}(\tau)e^{-jn\omega_s \tau} \, d\tau \right).
\]

From the Schwarz inequality we have

\[
|\langle x \rangle_n (t)| \leq \frac{1}{|n|\omega_s T_s} \left( |x(t) - x(t-T_s)| + \int_{t-T_s}^{t} |\dot{x}(\tau)| \, d\tau \right)
\]

\[
\leq \frac{1}{|n|\pi} \int_{t-T_s}^{t} |\dot{x}| \, d\tau
\]

\[
\leq \frac{T_s}{|n|\pi} \|\dot{x}\|_{L_2[t-T_s,t]}
\]

\[
= \frac{T_s}{|n|\pi} \|A(t)x + B(t)\|_{L_2[t-T_s,t]}
\]

\[
\leq \frac{T_s}{|n|\pi} \left(|A|||x||_{L_2[t-T_s,t]} + |B|\right).
\]

Applying the Parseval identity to this expression yields the bound on \( |\langle x \rangle_n| \).
Consider now the phasors $\langle s_i \rangle_n$. To bound these coefficients we note that for $0 \leq t_1 < t_2$ partial integration implies

$$\phi(t_1, t_2) := \frac{1}{T_s} \int_{t_1}^{t_2} f(\tau)e^{-j\omega_s \tau} \, d\tau$$

$$= \frac{j}{n\omega_s T_s} \left( [f(\tau)e^{-j\omega_s \tau}]_{t_1}^{t_2} - \int_{t_1}^{t_2} f(\tau) e^{-j\omega_s \tau} \, d\tau \right)$$

which by using the Schwarz inequality (second inequality below) gives

$$|\phi(t_1, t_2)| \leq \frac{1}{|n|\omega_s T_s} \left( 2 \sup_{t \in [t_1, t_2]} |f(t)| + \int_{t_1}^{t_2} |f(\tau)| \, d\tau \right)$$

\[\text{(B.45)}\]

Without loss of generality we assume that $t \in [kT_s, (k+1)T_s]$. The integral defining $\langle s_i \rangle_n$ is split as follows

$$\langle s_i \rangle_n (t) := \frac{1}{T_s} \int_{t-T_s}^{t} s_i(\tau)e^{-j\omega_s \tau} \, d\tau$$

$$= \frac{1}{T_s} \int_{kT_s}^{\min(t, (k+d_i)T_s)} a_{i,k} f_i(\tau)e^{-j\omega_s \tau} \, d\tau$$

$$+ \frac{1}{T_s} \int_{t-T_s}^{\max(t, (k-1+d_{i,k-1})T_s)} a_{i,k-1} f_i(\tau)e^{-j\omega_s \tau} \, d\tau.$$ 

Applying the relation (B.45) (with $f = s_i$) to the two integrals above and using the assumptions $|a_i| \leq 1$ and $|f_i| \leq 1$ yields

$$|\langle s_i \rangle_n| \leq \frac{1}{|n| \pi} \left( 2 + T_s \|f_i\|_{L_2} \right).$$

**Lemma 15.4.** Let $s_i(t)$ be a pulse modulation function as defined in (B.3) with feedback $a_i(t)$ and $d_i(t)$ as defined in (B.16) and let $t \in [kT_s, (k+1)T_s]$. The Fourier series representation $s_i(t, \tau)$ of $s_i(t)$ and $s_{av,i}(t, \tau)$ satisfy

$$\|s_i(t, \cdot) - s_{av,i}(t, \cdot)\|_{L_2[-T_s, 0]} \leq |a_{i,k-1} - a_i(t)| + |a_{i,k} - a_i(t)|$$

$$+ |d_{i,k-1} - d_i(t)| + |d_{i,k} - d_i(t)|.$$ 

**Proof of Lemma 15.4**

Recall that $s_{i,(a_i,d_i)}(t)$ is defined by (B.3) where the pulse amplitude and duty cycle are fixed and equal to $a_i$ and $d_i$ respectively so that $a_{i,k} = a_i \forall k$, $d_{i,k} = d_i \forall k$. As was noted in Section 6, if the duty cycle and pulse amplitude of $s_i$ are constant it
holds \( s_i(t, \tau) = s_{av, i, n} \) and this implies that \( s_{av, i}(t, \tau) = s_{i, (a_i, d_i)}(t, \tau) \). Since \( s_i(t, \tau) \) and \( s_{i, (a_i, d_i)}(t, \tau) \) are \( L_2 \)-equivalent to \( s_i(t) \) and \( s_{i, (a_i, d_i)}(t) \) respectively it follows

\[
\|s_i(t, \cdot) - s_{av, i}(t, \cdot)\|_{L_2[-T_s, 0]} = \|s_i(\cdot) - s_{i, (a_i(t), d_i(t))}(\cdot)\|_{L_2[-T_s, t]}.
\]

Let \( s_i(t) = f_i(t) \bar{s}_i(t) \) where \( f_i \) is the \( C^1 \) function defined in (B.3) and where \( \bar{s}_i \) is a square pulse function with amplitudes \( a_{i, k} \) and pulse-widths \( d_{i, k} \). Using \(|f_i| \leq 1\) we have that

\[
\|s_i(\cdot) - s_{i, (a_i(t), d_i(t))}(\cdot)\|_{L_2[-T_s, t]} \leq \|\bar{s}_i(\cdot) - \bar{s}_{i, (a_i(t), d_i(t))}(\cdot)\|_{L_2[-T_s, t]}
\]

\[
\leq |a_{i, k-1} - a_i(t)| + |d_{i, k-1} - d_i(t)|
\]

\[
+ |a_{i, k} - a_i(t)| + |d_{i, k} - d_i(t)|
\]

where the last inequality follows from Fig. B.5. This concludes the proof.

![Diagram](image)

**Figure B.5:** The dashed line shows the periodic function \( \bar{s}_{i, (a_i, d_i)} \) and the solid line shows the modulated function \( \bar{s}_i \). The term \( \|\bar{s}_i(\cdot) - \bar{s}_{i, (a_i(t), d_i(t))}(\cdot)\|_{L_2[-T_s, t]} \) is bounded by the area of the shaded regions.

**Lemma 15.5.** Let \( t \geq t_0 \). It holds

\[
\|x(t, \tau) - x(t, \tau - (t - t_0))\|_{L_2[-T_s, 0]} \leq 2T_s(\|x(t, \tau)\|_{L_2[-T_s, 0]} + |B|).
\]

**Proof of Lemma 15.5**

We first note that the quantity \( x(t, \tau) - x(t, \tau - (t - t_0)) \) is the difference between a periodic function and a shifted version of the same function, see Fig. B.6 for an illustration. Let \( \xi = t - t_0 \mod T_s \). We use the \( L_2 \)-equivalence between the solution \( x \) and the Fourier series representation of \( x \) to establish the following equality (see
Fig. B.6)
\[
\|x(t, \tau) - x(t, \tau - (t - t_0))\|_{L_2[-T_s, 0]} = \|x(t, \tau) - x(t, \tau - \xi)\|_{L_2[-T_s, 0]} = \\
\left( \frac{1}{T_s} \int_{t-T_s}^{t-T_s+\xi} |x(\tau) - x(\tau + T_s - \xi)|^2 d\tau + \frac{1}{T_s} \int_{t-T_s+\xi}^{t} |x(\tau) - x(\tau - \xi)|^2 d\tau \right)^{1/2}.
\]

It then follows
\[
\|x(t, \tau) - x(t, \tau - (t - t_0))\|_{L_2[-T_s, 0]} \\
\leq \max_{\tau \in [t-T_s,t-T_s+\xi]} |x(\tau) - x(\tau + T_s - \xi)| + \max_{\tau \in [t-T_s+\xi,t]} |x(\tau) - x(\tau - \xi)| \\
\leq \max_{\tau \in [t-T_s,t-T_s+\xi]} \int_{\tau}^{\tau+T_s-\xi} |\dot{x}(\sigma)| d\sigma + \max_{\tau \in [t-T_s+\xi,t]} \int_{\tau-\xi}^{\tau} |\dot{x}(\sigma)| d\sigma \\
\leq 2 \int_{t-T_s}^{t} |\dot{x}(\sigma)| d\sigma.
\]

The Schwarz inequality is applied to the last inequality above and we conclude that
\[
\|x(t, \tau) - x(t, \tau - (t - t_0))\|_{L_2[-T_s, 0]} \leq 2T_s \|\dot{x}\|_{L_2[-T_s, t]} \\
\leq 2T_s \left( |A| \|x(t, \tau)\|_{L_2[-T_s, 0]} + |B| \right).
\]

Figure B.6: Solid line: \(x(t, \tau)\). Dashed line: \(x(t, \tau - (t - t_0))\). The shift \(\xi\) is defined as \(\xi := t - t_0 \mod T_s\).

**Lemma 15.6.** For \(t \geq t_0\) the solution \(\dot{x}\) of the phasor dynamics (B.8) satisfies
\[
\|\dot{x}(t) - \dot{x}(t_0)\| \leq \left( 2T_s |A| + 1 \right) e^{|A|(t-t_0)} - 1 \left( \|x(t, \cdot)\|_{L_2[-T_s, 0]} + \frac{|B|}{|A|} \right).
\]
Proof of Lemma 15.6

To prove the claim we move into the time domain. However, it should be noted that the quantity $||\hat{x}(t) - \hat{x}(t_0)||_{L_2}$ is not equal to $||x(t, \tau) - x(t_0, \tau)||_{L_2}$. In fact, the time domain expressions are not directly comparable since they are defined on different sections of the time axis.

Let $t \geq t_0$, we have

$$
\|\hat{x}(t) - \hat{x}(t_0)\|_{L_2} = \|y(\tau) - x(t_0, \tau)\|_{L_2[-T_s, 0]}
= \|y(\tau) - x(t - (t - t_0), \tau)\|_{L_2[-T_s, 0]} 
$$

where

$$
y(\tau) := x(t, t_0 + \tau) = \sum_{n=-\infty}^{\infty} \langle x \rangle_n (t)e^{j\omega_{st}t_0}e^{j\omega_{st}\tau}.
$$

Let $z(\eta) = x(t - \eta, \tau)$, then $z$ satisfies

$$
\dot{z} = -Az - B, \quad z(0) = x(t, \tau).
$$

For $\eta \in [0, t - t_0]$ we have the inequality

$$
\|z(\eta) - y\|_{L_2} = \|z(0) + \int_0^{\eta} \dot{z}(\sigma)d\sigma - y\|_{L_2} \leq \|z(0) - y\|_{L_2} + \int_0^{\eta} \|\dot{z}(\sigma)\|_{L_2}d\sigma.
$$

We use the result in Lemma 15.5 to bound the first term in the sum and we have

$$
\|z(\eta) - y\|_{L_2} \leq 2T_s(|A||x||_{L_2} + |B|) + \int_0^{\eta} \|\dot{z}(\sigma)\|_{L_2}d\sigma
\leq 2T_s(|A||x||_{L_2} + |B|) + |A|\int_0^{\eta} \|z(\sigma)\|_{L_2}d\sigma + \eta|B| 
$$

where in the last inequality we have used that $\|y\|_{L_2} = \|\hat{x}\|_{L_2}$ and $\|A\| = ||A||$.

The result follows by applying the Grönwall inequality to (B.47) and choosing $\eta = t - t_0$. This yields

$$
\|z(t - t_0) - y\|_{L_2} \leq \left((2T_s|A| + 1)e^{[A(t-t_0)]} - 1 \right)\|x(t, \cdot)\|_{L_2[-T_s, 0]}
+ \left((2T_s|A| + 1)e^{[A(t-t_0)]} - 1 \right)\frac{|B|}{|A|}
$$

which together with inequality (B.46) yields the result.
16 References


Harmonic Analysis of
Pulse-Width Modulated Systems

Stefan Almér and Ulf Jönsson

Abstract

The paper considers the so-called dynamic phasor model as a basis for harmonic analysis of a class of switching systems. The analysis covers both periodically switched systems and non-periodic systems where the switching is controlled by feedback. The dynamic phasor model is a powerful tool for exploring cyclic properties of dynamic systems. It is shown that there is a connection between the dynamic phasor model and the harmonic transfer function of a linear time periodic system and this connection is used to extend the notion of harmonic transfer function to describe periodic solutions of non-periodic systems.

Keywords: Pulse-width modulation, Harmonic analysis, Dynamic phasors, Periodic systems, Switched mode circuits.

1 Introduction

The paper investigates the use of the so-called dynamic phasor model (DPM) as a tool for harmonic analysis of a class of switching systems. The systems considered are a class of pulse-width modulated (PWM) systems that switch between subsystems in a given order. In open loop, the switching is periodic and the PWM systems are linear time periodic (LTP). In the closed loop case the switching instants are determined by feedback of sampled values of the state. This type of modulation introduces additional difficulties in the analysis, but in many applications it is the most realistic method. In closed loop, the PWM systems are no longer periodic. However, the pulses that excite the system begin at periodically repeated time instants and the non-periodic systems retain a cyclic property which is explored in the analysis.

The analysis is motivated mainly by switched mode power converters. Such devices can cause excessive harmonics in power systems and may lead to instability, see e.g., [8]. To be able to predict harmonics in switched mode circuits is important
also in micro electronics. Switched converters are used extensively in portable radio frequency (RF) amplifiers (in e.g., cellular phones) to increase efficiency and save power [9]. However, for RF devices there are strict limits on the disturbance brought to neighboring channels. Spurious frequency content caused by switched converters is therefore a major concern in circuit design. Conventionally, harmonic analysis of switched circuits is done with extensive simulations and the FFT. Part of the contribution of this paper is to provide a tool for harmonic analysis which may facilitate circuit verification.

The DPM is a powerful tool for exploring cyclic properties of dynamic systems. It is obtained from a Fourier series expansion of the system state over a moving time window. It yields an L₂-equivalent representation of the system in terms of an infinite dimensional dynamic system which describes the time evolution of the Fourier coefficients. The time dependent Fourier coefficients are also known as dynamic phasors.

To our knowledge, the DPM was introduced in the field of power electronics as a tool for modeling the transients of switched converters, see e.g., [4,11]. It has also been used for stability analysis, see e.g., [14] and for developing efficient simulation tools for switched converters, see [6] and references therein.

The DPM is conceptually appealing but poses some mathematical difficulties. The main problem is that the Fourier series expansion of the system state over a given time window in general does not converge uniformly. These convergence problems and the corresponding differentiability problems are revealed in [13]. The paper [13] derives conditions for the existence of a steady state solution to the phasor system. Our paper also contains results on the existence of solutions to fixed point equations, but for a different situation. We consider the presence of an external disturbance of arbitrary frequency with the intention of establishing an input-output mapping in the frequency domain.

In open loop, the PWM systems are LTP and the corresponding DPM is an infinite dimensional time invariant system. In the closed loop case however, the phasor system is not only infinite dimensional but also nonlinear and it depends on sampled values of the state with a delay. To obtain a tractable model we consider a truncated averaged approximation of the phasor system. The averaging and truncation yields a hierarchy of finite dimensional nonlinear systems that become increasingly more accurate as the size of the truncation increases. It is shown that the approximation error can be made arbitrarily small provided that the switch period is small enough and the truncation is large enough.

For harmonic analysis of the open loop (LTP) systems we review some results on the so-called harmonic transfer function (HTF) [3,7,10,15,17–19] and provide a time domain interpretation. The HTF generalizes the concept of transfer function to LTP systems and is thus an efficient tool for illustrating the frequency coupling between input and output. We show that the HTF and the DPM are connected in the sense that the DPM yields an explicit expression for the HTF.

For harmonic analysis of the closed loop PWM systems we linearize the plant and derive an (approximate) HTF which is analogous to the HTF of an LTP system.
It is not obvious how to linearize the closed loop PWM system, especially since we consider sampled feedback. In our approach we rely on the truncated averaged phasor system. This model accounts for the harmonics generated by switching, but is still continuous and therefore suitable for linearization.

We assume that the approximate phasor model is subjected to a periodic disturbance and we consider the corresponding harmonic balance equations to determine the steady state response of the phasor coefficients. We prove that the harmonic balance equations have a solution provided that the disturbance is small enough and we obtain an approximate solution by linearizing the equations. The linearized harmonic balance equations provide an approximate HTF for the closed loop system. This HTF describes the steady state response of the non-periodic PWM system to periodic disturbances and takes the switched nature of the plant into account.

The analysis techniques discussed in this paper are applied to two realistic examples. The steady state response to a periodic disturbance is approximated for both open and closed loop operation and the predicted responses are verified by simulations in Matlab™. It is shown that the approximated and simulated responses correspond well and that the approximations capture the nonlinear phenomena caused by switching.

**Notation**

In this paper \( L_2[-T, 0] \) denotes the set of square integrable functions \( x : [-T, 0] \rightarrow \mathbb{R}^n \) with inner product

\[
\langle x, y \rangle_{L_2} = \frac{1}{T} \int_{-T}^0 x(\tau)^\top y(\tau) d\tau
\]

and corresponding norm \( \| x \|_{L_2} = \langle x, x \rangle_{L_2}^{1/2} \). \( l_2 \) denotes the set of square summable sequences \( x = \{ x_k \}_{k=-\infty}^\infty \) where \( x_k \in \mathbb{C}^n \) satisfies \( \bar{x}_k = x_{-k} \) where \( \bar{x}_k \) is the complex conjugate of \( x_k \). The set is equipped with inner product

\[
\langle x, y \rangle_{l_2} = \sum_{k=-\infty}^\infty x_k^* y_k
\]

and corresponding norm \( \| x \|_{l_2} = \langle x, x \rangle_{l_2}^{1/2} \). Note that since \( \bar{x}_k = x_{-k} \), \( l_2 \) denotes a smaller set than usual. \( l_{2,N} \) is the finite subspace of \( l_2 \) where \( x_k = 0 \forall |k| > N \). When the vector space is clear from context, the subindices of \( \| \cdot \|_{L_2} \) and \( \| \cdot \|_{l_2} \) are often omitted. The Euclidean norm is denoted \( | \cdot | \). For any bounded signal \( x(t) \) we define \( \| x \|_\infty = \sup_{\tau} | x(\tau) | \) and \( \| x \|_{l_2,\infty} = \sup_{\tau \in [t-T_s,t]} | x(\tau) | \) where \( T_s > 0 \).

\( \pi_N \) denotes both a projection on \( l_2 \) and a truncation as follows: Firstly, \( \pi_N : l_2 \rightarrow l_2 \) is defined by the relation

\[
(\pi_N x)_k = \begin{cases} 
 x_k, & |k| \leq N \\
 0, & |k| > N 
\end{cases}
\]
Secondly, $\pi_N : l_2 \to l_{2,N}$ is defined by the relation $(\pi_N x)_k = x_k, -N \leq k \leq N$. The interpretation of $\pi_N$ will be clear from context. The transformation $\mathcal{T}$ maps a complex valued sequence $\xi = \{\xi_k\}^\infty_{k=-\infty}$ to a doubly infinite dimensional block Toeplitz matrix according to

$$
\mathcal{T}[\xi] = \begin{bmatrix}
\cdots & \cdots & \cdots \\
\xi_0 & \xi_1 & \xi_2 \\
\cdots & \xi_0 & \xi_1 & \cdots \\
\xi_{-2} & \xi_{-1} & \xi_0 \\
\cdots & \cdots & \cdots 
\end{bmatrix}
$$

where $\hat{\xi}_k = \xi_kI_n$ if $\xi_k$ is scalar and $\hat{\xi}_k = \xi_k$ otherwise. For a periodic function $s$, $\mathcal{T}[s]$ is defined as above with the sequence of Fourier coefficients of $s$ in place of $\xi$. $T_N[\xi]$ is a finite dimensional matrix consisting of the $2N + 1$ central blocks of $\mathcal{T}[\xi]$. $\text{col}\{\xi_k\} = [\cdots, \xi_1, \xi_0, \xi_{-1}, \cdots]^*$ is an infinite dimensional column vector where $\xi_k$ appear in descending order. $I$ is used for the identity operator on both finite and infinite dimensional spaces. On finite dimensional spaces we sometimes write $I_q$ (where $q$ is the dimension of the space) for clarity. We use $\sigma$ to denote the maximum singular value of a matrix and $\otimes$ is the Kronecker product.

## 2 A class of switching systems

We consider a class of systems that switch between subsystems in a given order. The systems are of the form

$$
\begin{align*}
\dot{\xi}(t) &= (A_0 + s(t)A_1)\xi(t) + B_0 + s(t)B_1 + (D_0 + s(t)D_1)w(t) \\
\zeta(t) &= C(t)\xi(t)
\end{align*}
$$

(C.1)

where $\xi(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$ is an external disturbance assumed to be in $L_2[l_0, t_1]$ for any finite time interval $[l_0, t_1]$, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^n$, $D_i \in \mathbb{R}^{n \times m}$ are constant matrices, $C(t) \in \mathbb{R}^{p \times n}$ is a $T_s$-periodic matrix and $s$ is the PWM function

$$
s(t) = \begin{cases} 
1, & t \in [kT_s, (k + d_k)T_s) \\
0, & t \in [(k + d_k)T_s, (k + 1)T_s).
\end{cases}
$$

(C.2)

Here, $T_s > 0$ is the period time, $k \in \mathbb{N}$ and $d_k \in [0, 1]$ is the so-called duty cycle. The duty cycle determines the fraction of time each mode is active and thus controls the system dynamics. The unperturbed system (where $w \equiv 0$) is assumed\(^1\) to satisfy the following assumption.

\(^1\) The assumption is natural for switched power converters as such systems are typically designed to have periodic solutions. The assumption can be verified using harmonic balance techniques.
Assumption 2.1. Consider the unperturbed system (C.1) where $w \equiv 0$. There exists at least one point $(\xi_0, d^0) \in \mathbb{R}^n \times [0, 1]$ such that (C.1) attains a $T_s$-periodic solution $\xi^0(t) = \xi^0(t + T_s)$ when the initial condition is $\xi(0) = \xi_0$ and the duty cycle is constant so that $d_k = d^0 \forall k$.

We denote the deviation from $\xi^0$ as $x := \xi - \xi^0$ and in the sequel we consider the error dynamics

$$
\dot{x}(t) = (A_0 + s(t)A_1)x(t) + (s(t) - s_\rho(t)) (A_1\xi^0(t) + B_1) + (D_0 + s(t)D_1)w(t)
$$

$$
y(t) = C(t)x(t)
$$

(C.3)

where $s_d(t)$ is defined according to (C.2) with the duty cycle fixed at $d$ ($d_k = d \forall k$). Note that $s_d(t)$ is a periodic function but $s(t)$ need not be periodic. It can be shown that (C.3) has a unique absolutely continuous solution for every initial condition.

The duty cycle $d_k$ is determined by sampling a weighted average of the state. The feedback is of the form

$$
d_k = \text{sat}_{[0,1]} \left( d^0 + \frac{1}{T_s} \int_{(k-1)T_s}^{kT_s} F(\tau)x(\tau)d\tau \right)
$$

(C.4)

where $\text{sat}_{[0,1]}(\cdot) := \min(\max(\cdot, 0), 1)$ denotes the saturation between zero and one and where the feedback vector $F(t)$ is of the form

$$
F(t) = \sum_{k=-N}^{N} e^{jk\omega_s t} F_k
$$

where $\omega_s = 2\pi/T_s$ and where $F_k \in \mathbb{C}^n$ satisfy $F_k = F_{-k}$. Note that when $F_k = 0 \forall k \neq 0$, the integral in (C.4) gives the average value of $x$ over the past switch period.

Remark 2.1. The averaging integral in (C.4) is motivated since the state of a PWM system contains ripple which needs to be filtered out. Also, the integral implies that the feedback can be expressed as a linear function of the dynamic phasor coefficients defined in (C.5). It is thus easily represented in the DPM (C.7) defined below.

3 The dynamic phasor model

We use the idea of [4,11] to represent the solution of (C.3) in the frequency domain where we can distinguish how the various harmonics develop over time. The $n^{th}$ phasor (Fourier coefficient) of $x$ is defined as

$$
\langle x \rangle_n(t) = \frac{1}{T_s} \int_{t-T_s}^{t} x(\tau)e^{-jn\omega_s \tau} d\tau
$$

(C.5)
where \( \omega_s = 2\pi/T_s \). Note that the phasors are defined over a moving time window and are thus time dependent. Note also that if \( x \) is periodic with period \( T_s \), then \( \langle x \rangle_n(t) \) is constant. As was remarked above, the solution \( x \) of (C.3) is absolutely continuous. This implies that the sequence \( \{ \langle x \rangle_n(t) \}_{n=-\infty}^{\infty} \) is in \( l_2 \) for all \( t \). For brevity, the time dependence of the phasors is often suppressed.

The time domain signal \( x \) is reconstructed on the interval \([ t - T_s, t] \) according to

\[
x(t, \tau) = \sum_{n=-\infty}^{\infty} \langle x \rangle_n(t) e^{jn\omega_s(t+\tau)}, \quad \tau \in [-T_s, 0].
\]  

(C.6)

Note that \( x(t) \neq x(t, \tau) \), but the equality \( x(t+\tau) = x(t, \tau) \) holds a.e. on the set \( \{ \tau \mid \tau \in [-T_s, 0] \} \).

Using partial integration one can show that the phasor coefficients satisfy

\[
\frac{d}{dt} \langle x \rangle_n = \left< \frac{d}{dt} x \right>_n - jn\omega_s \langle x \rangle_n.
\]

Introducing the notation

\[
\hat{x} = \text{col}\{ \langle x \rangle_n \}, \quad \hat{\xi}^0 = \text{col}\{ \langle \xi^0 \rangle_n \}, \quad \hat{w} = \text{col}\{ \langle w \rangle_n \}
\]

\[
\hat{s} = \text{col}\{ \langle s \rangle_n \}, \quad \hat{s}^{0} = \text{col}\{ \langle s^{0} \rangle_n \}
\]

\[
\mathcal{F} = (F_N, \ldots, F_0, \ldots, F_N)
\]

the phasor dynamics can be written in the compact form

\[
\frac{d}{dt} \hat{x} = (-j\omega_s \hat{E}_n + \hat{A}(\hat{s})) \hat{x} + \hat{B}(\hat{s} - \hat{s}^{0}) + \hat{D}(\hat{s}) \hat{w}
\]

\[
\hat{y} = \hat{C} \hat{x}
\]

\[
d_k = \text{sat}_{[0,1]}(d^0 + \mathcal{F} \pi_N \hat{x}(kT_s))
\]

(C.7)

where

\[
\hat{E}_n = \text{blkdiag}(1, 2I_n, I_n, 0, -I_n, -2I_n, \ldots)
\]

\[
\hat{A}(\hat{s}) = I \otimes A_0 + (I \otimes A_1) \mathcal{T}[\hat{s}]
\]

\[
\hat{B} = I \otimes B_1 + (I \otimes A_1) \mathcal{T}[\hat{s}^{0}]
\]

\[
\hat{D}(\hat{s}) = I \otimes D_0 + (I \otimes D_1) \mathcal{T}[\hat{s}]
\]

(C.8)

and where \( \hat{C} = \mathcal{T}[\mathcal{C}(t)] \). Note that the feedback (C.4) corresponds to sampling the \( 2N + 1 \) low order phasors \( \langle x \rangle_n \) and that \( \hat{s} \) depends on these samples.

In the open loop case (where the duty cycle is constant and equal to \( d^0 \)), \( \mathcal{T}[\hat{s}] \) is constant and the affine term \( \hat{B}(\hat{s} - \hat{s}^{0}) \) disappears. In this case the time periodic switched system (C.3) is represented by a linear time invariant system in the frequency domain. However, when \( s \) is determined by the feedback (C.4), the DPM (C.7) is an infinite dimensional, nonlinear system that depends on the sampled
state with a delay. To obtain a tractable model we introduce an approximation in two steps:

In the first step we replace the phasor coefficients \( \langle s \rangle_n \) with the nonlinear averaged approximation

\[
s_{\text{av},n}(d) = \begin{cases} d, & n = 0 \\ \frac{j}{n2\pi} (e^{-j n 2\pi d} - 1), & n \neq 0 \end{cases}
\]

(C.9)

where \( d \in [0,1] \). Note that if the duty cycle is fixed so that \( d_k = d \ \forall k \), then \( \langle s \rangle_n(t) = \langle s_d \rangle_n(t) = s_{\text{av},n}(d) \). This implies that if the duty cycle varies slowly (compared to the switch period \( T_s \)), then \( s_{\text{av},n}(d) \) is a good approximation of \( \langle s \rangle_n \). This claim is formalized in Proposition 3.1 below.

In the second step we truncate the infinite state vector to obtain a finite dimensional system which approximates the low order phasor coefficients. We also describe the dynamics as a function of the deviation \( \delta := d - d^0 \) from the stationary duty cycle. For a fixed integer \( N \geq 0 \) the approximation of the phasors \( \langle x \rangle_{-N}, \ldots, \langle x \rangle_N \) is given by the system

\[
\begin{align*}
\frac{d}{dt} z &= (-j \omega_N + A(\delta)) z + B \mathcal{S}(\delta) + D(\delta) \mathcal{W} \\
\gamma &= C z \\
\delta &= \text{sat}_{[-d^0,1-d^0]} (Fz)
\end{align*}
\]

(C.10)

where

\[
\begin{align*}
A(\delta) &= \pi_N \hat{A}(S_{\text{av}}(\delta)) \pi_N \\
D(\delta) &= \pi_N \hat{D}(S_{\text{av}}(\delta)) \pi_N \\
\mathcal{S}(\delta) &= \pi_N (S_{\text{av}}(\delta) - S_{\text{av}}(0)) \\
S_{\text{av}}(\delta) &= \text{col}\{s_{\text{av},n}(d^0 + \delta)\}
\end{align*}
\]

and where \( C = \pi_N \hat{C} \pi_N \), \( B = \pi_N \hat{B} \pi_N \) are square truncations and \( \mathcal{W} = \pi_N \hat{\mathcal{W}} \) and where \( \hat{A}(\cdot), \hat{\mathcal{S}}(\cdot) \) and \( \hat{D}(\cdot) \) are defined in (C.8).

**Remark 3.1.** It should be noted that \( z \) is an approximation of the \( 2N + 1 \) phasors \( \langle x \rangle_{-N}, \ldots, \langle x \rangle_N \) in (C.7) and thus, \( z \in \mathbb{C}^{(2N+1)n} \). The \( k^{th} \) approximate phasor is denoted \( z[k] \in \mathbb{C}^n \), \( k = -N, \ldots, N \). Analogously, the \( k^{th} \) approximate phasor of the output is denoted \( \hat{\gamma}[k] \).

The system (C.10) is a nonlinear differential equation and is therefore tractable for analysis. The important distinctions from (C.7) is that the state space is finite dimensional and that \( s_{\text{av},n} \) is a continuous function of \( d \) whereas \( \langle s \rangle_n \) is determined by the samples \( d_k \) defined in (C.4).

When the switch period \( T_s \) is small, the system (C.10) provides a good approximation of the DPM. To prove this claim we now show that the solutions \( \hat{x} \)
and $z$ are close on infinite time intervals. Since the approximation $z$ is finite dimensional, it is natural to define the error in two parts. We define the error as 
\[ \hat{e} := (\hat{e}_1, \hat{e}_2) \in l_{2,N} \times l_{2,\bar{N}} \]
where $\hat{e}_1 := \pi_N \hat{x} - z$ and $\hat{e}_2 := (I - \pi_N) \hat{x}$ and where 
\[ l_{2,N} := (I - \pi_N)l_2 \]
is a subspace of $l_2$. We define the norm on $l_{2,N} \times l_{2,\bar{N}}$ as 
\[ \|\hat{e}\| := (|\hat{e}_1|^2 + |\hat{e}_2|_{l_2}^2)^{1/2} \]
and proceed to show that $\|\hat{e}\|$ is small. The result is formalized in the following proposition.

**Proposition 3.1.** Consider the DPM (C.7) for the case where $(I - \pi_N)\hat{w} = 0$ and let $\hat{x}(t)$ be the solution with $\hat{x}(t_0) = 0$. Let $z(t)$ be the solution of the approximate system (C.10) with $z(t_0) = 0$. Define the approximation error as 
\[ \hat{e} := (\hat{e}_1, \hat{e}_2) \in l_{2,N} \times l_{2,\bar{N}} \]
where $\hat{e}_1 := \pi_N \hat{x} - z$ and $\hat{e}_2 := (I - \pi_N) \hat{x}$ and let the error norm be 
\[ \|\hat{e}\| := (|\hat{e}_1|^2 + |\hat{e}_2|_{l_2}^2)^{1/2} \]
Assume that the unforced approximate system 
\[ \frac{d}{dt} z = (-j \omega_0 N + A(\delta)) z + BS(\delta) \]
\[ \delta = \text{sat}_{[-\gamma, 1-\gamma]} (\mathcal{F} z) \]
is (locally) exponentially stable. Under these assumptions there exists a number 
$r_w > 0$ s.t. if $\sup_t |\mathcal{W}(t)| < r_w$, then 
\[ \forall \epsilon > 0 \ \exists \ T_0 > 0 \ s.t. \ \|\hat{e}(t)\| \leq \epsilon \ \forall t \text{ if } T_s \leq T_0 \]
where $T_s$ is the switch period.

**Proof:** A proof is given in Appendix 7.

**Remark 3.2.** The assumption $(I - \pi_N)\hat{w} = 0$ is made simply for convenience. In the general case the proof is longer but not essentially different.

In the proposition above we assume that the unforced system (C.10) is exponentially stable. A systematic procedure to verify stability of (C.10) can be adopted from [1] and [2].

## 4 Harmonic analysis

In Section 4.1 below we review some results on the HTF [3, 7, 10, 15, 17–19] and provide a time domain interpretation. We also show that the DPM provides an explicit formula for the HTF. In Section 4.2 we consider (C.3) in closed loop (then the system is non-periodic) and we use the approximate phasor model to derive a harmonic transfer function which is analogous to the HTF of an LTP system.
4.1 The harmonic transfer function; a review

LTP systems do not have the property of frequency separation which is characteristic for LTI systems. If the input to an LTP system is a sinusoid with angular frequency \( \omega \), then the steady state output will be a sum of sinusoids with angular frequencies \( \omega + k\omega_s \) where \( k \in \mathbb{Z} \) and \( \omega_s \) is the angular frequency of the system. The HTF generalizes the concept of transfer function to LTP systems and is thus a powerful tool for illustrating the coupling between the frequencies in the input and output signals.

Consider the LTP system

\[
\begin{align*}
\dot{x}(t) &= A_p(t)x(t) + D_p(t)w(t) \\
y(t) &= C_p(t)x(t)
\end{align*}
\]

(C.12)

where \( A_p, D_p \) and \( C_p \) are \( T_s \)-periodic matrices. Under weak assumptions [10], the impulse response \( h \) of (C.12) can be expanded in a Fourier series and the response \( y \) to the input \( w \) can be expressed as the convolution

\[
y(t) = \int_{0}^{t} \sum_{k=-\infty}^{\infty} h_k(t - \tau)e^{jk\omega_s\tau}w(\tau)d\tau
\]

(C.13)

\[
= \sum_{k=-\infty}^{\infty} \left( h_k(\cdot)e^{jk\omega_s(\cdot)} \ast w(\cdot)e^{jk\omega_s(\cdot)} \right)(t)
\]

where \( h_k \) are the Fourier coefficients of \( h \) and \( \ast \) denotes the convolution operation. Let \( Y(\omega) := (\mathcal{F}y)(\omega) \), \( W(\omega) := (\mathcal{F}w)(\omega) \) be the Fourier transform of the output and input respectively. By applying the Fourier transform to (C.13) one can express \( Y(\omega + n\omega_s) \), \( n \in \mathbb{Z} \) as a function of \( W(\omega + k\omega_s) \), \( k \in \mathbb{Z} \) as shown in (C.14) below. Here, the doubly infinite dimensional matrix \( \mathcal{H}(\omega) \) is the harmonic transfer function and the entries \( H_k(\omega) = (\mathcal{F}h_k)(\omega) \) are the Fourier transform of the Fourier coefficients of \( h \).

\[
\begin{bmatrix}
\vdots \\
Y(\omega + \omega_s) \\
Y(\omega) \\
Y(\omega - \omega_s) \\
\vdots
\end{bmatrix} =
\begin{bmatrix}
\vdots \\
H_0(\omega + \omega_s) & H_1(\omega) & H_2(\omega - \omega_s) \\
\cdots & H_{-1}(\omega + \omega_s) & H_0(\omega) & H_1(\omega - \omega_s) & \cdots \\
H_{-2}(\omega + \omega_s) & H_{-1}(\omega) & H_0(\omega - \omega_s) \\
\vdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\begin{bmatrix}
\vdots \\
W(\omega + \omega_s) \\
W(\omega) \\
W(\omega - \omega_s) \\
\vdots
\end{bmatrix}
\]

(C.14)

As was stated above, the HTF extends the notion of transfer function to LTP systems. From the transfer function of an LTI system one can immediately determine the response to a sinusoidal input. Next we show that the HTF has the corresponding property for LTP systems. Let the input signal be \( w(t) = \sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t}) \).
and assume that this signal has been applied since time \( t = -\infty \). The output becomes

\[
y(t) = \sum_{k=-\infty}^{\infty} \int_{0}^{\infty} h_{k}(\tau) w(t - \tau) d\tau e^{j\omega_{s}t}
\]

\[
= \sum_{k=-\infty}^{\infty} \int_{0}^{\infty} h_{k}(\tau) \frac{1}{2j} \left( e^{j\omega(t-\tau)} - e^{-j\omega(t-\tau)} \right) d\tau e^{j\omega_{s}t}
\]

\[
= \sum_{k=-\infty}^{\infty} \left( \int_{0}^{\infty} h_{k}(\tau)e^{-j\omega_{s}\tau} \frac{1}{2j} e^{j\omega_{s}t} - \int_{0}^{\infty} h_{k}(\tau)e^{j\omega_{s}\tau} \frac{1}{2j} e^{-j\omega_{s}t} \right) e^{j\omega_{s}t}
\]

\[
= \sum_{k=-\infty}^{\infty} H_{k}(\omega) \frac{1}{2j} e^{j(\omega + k\omega_{s})t} - H_{k}(-\omega) \frac{1}{2j} e^{-j(\omega + k\omega_{s})t}.
\]

We now use the relation \( H_{k}(-\omega) = \overline{H_{-k}(\omega)} \) and reorder the sum above. Denoting the real and imaginary parts of \( H_{k} \) by \( H_{k}^{R} \) and \( H_{k}^{I} \) respectively we have

\[
y(t) = \sum_{k=-\infty}^{\infty} H_{k}(\omega) \frac{1}{2j} e^{j(\omega + k\omega_{s})t} - \overline{H_{k}(\omega)} \frac{1}{2j} e^{-j(\omega + k\omega_{s})t}
\]

\[
= \sum_{k=-\infty}^{\infty} H_{k}^{R}(\omega) \sin((\omega + k\omega_{s})t) + H_{k}^{I}(\omega) \cos((\omega + k\omega_{s})t)
\]

\[
= \sum_{k=-\infty}^{\infty} \Re\{H_{k}(\omega)e^{j(\omega + k\omega_{s})t}\}
\]

\[
= \sum_{k=-\infty}^{\infty} |H_{k}(\omega)| \sin((\omega + k\omega_{s})t + \phi_{k})
\]

where \( \phi_{k} = \arg H_{k}(\omega) \). The expression above shows that if the input to an LTP system is a sinusoid with angular frequency \( \omega \), then the output is a sum of sinusoids with shifted frequencies \( \omega + k\omega_{s} \). Furthermore, the gain and phase shift of each term in the sum is given by the HTF.

An explicit formula for the HTF can be obtained from the DPM corresponding to (C.12). To see this, let \( \langle w \rangle_{n} \) be the \( n \)th phasor coefficient of \( w \). One can show that the Fourier transform of \( \langle w \rangle_{n} \) satisfies

\[
\langle \mathcal{F} \langle w \rangle_{n} \rangle(\omega) = W(\omega + n\omega_{s}) \frac{1 - e^{-j\omega T_{s}}}{j\omega T_{s}}.
\]

Using the relation (C.13) we can also derive the equality

\[
\langle \mathcal{F} \langle y \rangle_{n} \rangle(\omega) = \sum_{k=-\infty}^{\infty} H_{k}(\omega + (n - k)\omega_{s})W(\omega + (n - k)\omega_{s}) \frac{1 - e^{-j\omega T_{s}}}{j\omega T_{s}}.
\]
By identifying $\mathcal{H}(w)_{n-k}$ in this expression we recognize that the relation between $\mathcal{F}\langle w \rangle_n$ and $\mathcal{F}\langle y \rangle_n$ is given by the HTF, i.e.,

$$
\begin{bmatrix}
\vdots \\
(\mathcal{F}\langle y \rangle_1)(\omega) \\
(\mathcal{F}\langle y \rangle_0)(\omega) \\
(\mathcal{F}\langle y \rangle_{-1})(\omega) \\
\vdots 
\end{bmatrix}
= \mathcal{H}(\omega)
\begin{bmatrix}
\vdots \\
(\mathcal{F}\langle w \rangle_1)(\omega) \\
(\mathcal{F}\langle w \rangle_0)(\omega) \\
(\mathcal{F}\langle w \rangle_{-1})(\omega) \\
\vdots 
\end{bmatrix}.
$$

The relation above implies that the HTF of an LTP system can be derived by applying the Fourier transform to the corresponding DPM.

Consider the DPM corresponding to the open loop system (C.3), i.e.,

$$
\begin{align*}
\frac{d}{dt} \tilde{x} &= \left( -j \omega_n \tilde{E}_n + \hat{A}(\tilde{s}, \tilde{\phi}) \right) \tilde{x} + \hat{D}(\tilde{s}, \tilde{\phi}) \tilde{w} \\
\hat{y} &= \hat{C} \tilde{x}.
\end{align*}
$$

(C.16)

By formally applying the Fourier transform to (C.16) we obtain an explicit formula for the HTF $\mathcal{H}(\omega)$ corresponding to the open loop system (C.3)

$$
\mathcal{H}(\omega) = \hat{C}(j \omega I - (-j \omega_n \tilde{E}_n + \hat{A}(\tilde{s}, \tilde{\phi})))^{-1} \hat{D}(\tilde{s}, \tilde{\phi}).
$$

(C.17)

Our derivation of the expression (C.17) is strictly formal and is merely used to show an analogy between the HTF of an LTP system and the transfer function derived in Section 4.2 below. Conditions for well-posedness and convergence can be found in [10,18]. In this paper, Proposition 3.1 implies that (C.16) is approximated arbitrarily well by square truncations. Thus, (C.17) can be interpreted as a limit of truncated HTFs.

In Section 4.2 we consider (C.3) in closed loop. In this case, Proposition 3.1 implies that for small disturbances $w$, the truncated phasor model (C.10) approximates the DPM arbitrarily well. The approximate model (C.10) can thus be used to extend the notion of HTF to describe periodic solutions of the non-periodic closed loop system (C.3).

### 4.2 A HTF approximation of the closed loop system

To estimate the effect of the disturbance on the system (C.3) we consider the corresponding truncated averaged phasor model (C.10). We assume that the approximate phasor system is subjected to a periodic disturbance $w$ and we assume that the steady state response of the phasor coefficients is also periodic. To determine the response of the phasor coefficients we state the corresponding harmonic balance equations and suggest an approximate solution to the nonlinear equations. A first order approximation where all frequencies except the zero and first order terms are dropped results in a harmonic transfer function that maps periodic disturbances to the corresponding (approximate) periodic output.
We assume that the steady state response of (C.10) to a periodic disturbance \( \nu(t) \) with angular frequency \( \omega \) is periodic with period \( T = 2\pi / \omega \). In other words, we assume that

\[
\nu(t) = \sum_{k=-\infty}^{\infty} \nu_k e^{j \omega t}, \quad z(t) = \sum_{k=-\infty}^{\infty} z_k e^{j \omega t}
\]

\[
\delta(t) = \sum_{k=-\infty}^{\infty} \delta_k e^{j \omega t}, \quad y(t) = \sum_{k=-\infty}^{\infty} y_k e^{j \omega t}
\]

is a periodic solution of (C.10). In Section 4.2.2 we show that this assumption is valid for small disturbances \( \nu \). Since \( \delta(t) \) is periodic, \( A(\delta(t)), S(\delta(t)) \) and \( D(\delta(t)) \) are also periodic and can be represented by the Fourier series

\[
A(\delta(t)) = \sum_{k=-\infty}^{\infty} A_k e^{j \omega t}, \quad S(\delta(t)) = \sum_{k=-\infty}^{\infty} S_k e^{j \omega t}
\]

\[
D(\delta(t)) = \sum_{k=-\infty}^{\infty} D_k e^{j \omega t}
\]

where the Fourier coefficients \( A_k, S_k, D_k \) are functions of \( \{ \delta_k \}_{k=-\infty}^{\infty} \). We assume that the disturbance \( \nu \) is small enough so that the feedback in (C.10) does not saturate. We thus ignore the saturation in (C.10) and consider the harmonic balance equations associated with the periodic solution (C.18). By introducing the notation

- \( \hat{z} = \text{col}\{z_k\}, \quad \hat{y} = \text{col}\{y_k\}, \quad \hat{\delta} = \text{col}\{\delta_k\} \)
- \( \hat{\nu} = \text{col}\{\nu_k\}, \quad \hat{S}(\hat{\delta}) = \text{col}\{S_k\} \)

the harmonic balance equations can be stated as

\[
\begin{align*}
\hat{j} \omega \hat{E}_q \hat{z} &= \left( -j \omega \hat{N} + \hat{A}(\hat{\delta}) \right) \hat{z} + \hat{B} \hat{S}(\hat{\delta}) + \hat{D}(\hat{\delta}) \hat{\nu} \\
\hat{y} &= \hat{C} \hat{z} \\
\hat{\delta} &= \hat{F} \hat{z}
\end{align*}
\]

where

\[
\hat{A}(\hat{\delta}) := \begin{bmatrix} \cdots & \cdots & \cdots \\
\cdots & A_0 & A_1 & A_2 \cdots \\
\cdots & A_{-1} & A_0 & A_1 & \cdots \\
\cdots & A_{-2} & A_{-1} & A_0 \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}, \quad \hat{D}(\hat{\delta}) := \begin{bmatrix} \cdots & \cdots & \cdots \\
\cdots & D_0 & D_1 & D_2 \cdots \\
\cdots & D_{-1} & D_0 & D_1 & \cdots \\
\cdots & D_{-2} & D_{-1} & D_0 \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

are the infinite dimensional block Toeplitz matrices determined by the Fourier coefficients in (C.19) and where \( \hat{N} = I \otimes \hat{N}, \hat{C} = I \otimes \hat{C}, \hat{F} = I \otimes \hat{F}, \hat{B} = I \otimes \hat{B} \) and

\[
\hat{E}_q = \text{blkdiag}(\ldots, 2I_q, I_q, 0, -I_q, -2I_q, \ldots)
\]
where \( q = (2N + 1)n \). To find an approximate solution to the (highly nonlinear) harmonic balance equations (C.20) we use a first order approximation of the term \( S_{av}(\delta) \) defined in (C.11). We have for \( n \neq 0 \)

\[
s_{av,n}(d^0 + \delta(t)) \approx s_{av,n}(d^0) + e^{-jn2\pi d^0} \sum_{k=0}^{\infty} \delta_k e^{jk\omega t}
\]

where we used the approximation \( e^x \approx 1 + x \). Since \( s_{av,0}(\delta(t)) = d^0 + \delta(t) \) the approximation of \( S_{av}(\delta) \) is written

\[
S_{av}(\delta) \approx S_{av}(0) + \Psi \sum_{k=0}^{\infty} \delta_k e^{jk\omega t} = S_{av}(0) + \Psi \delta(t) \tag{C.21}
\]

where

\[
\Psi = \text{col}\{e^{-j2\pi d^0 n}\}.
\]

The approximation above is used to derive linear approximations (linear in \( \delta_k \)) of the Fourier coefficients \( \mathcal{A}_k, \mathcal{S}_k, \mathcal{D}_k \). By using the linearity of \( T_N[\cdot] \) one can show that (details are found in Appendix 9)

\[
\mathcal{A}_k(\hat{\delta}) \approx \begin{cases} 
A(0) + \delta_0(I \otimes A_1)T_N[\Psi], & k = 0 \\
\delta_k(I \otimes A_1)T_N[\Psi], & k \neq 0 
\end{cases}
\]

\[
\mathcal{S}_k(\hat{\delta}) \approx \delta_k \Psi_N
\]

\[
\mathcal{D}_k(\hat{\delta}) \approx \begin{cases} 
D(0) + \delta_0(I \otimes D_1)T_N[\Psi], & k = 0 \\
\delta_k(I \otimes D_1)T_N[\Psi], & k \neq 0 
\end{cases}
\]

where

\[
\mathcal{A}(0) = \pi_N \hat{\mathcal{A}}(S_{av}(0)) \pi_N = \pi_N \hat{\mathcal{A}}(\hat{s}, \theta) \pi_N \\
\mathcal{D}(0) = \pi_N \hat{\mathcal{D}}(S_{av}(0)) \pi_N = \pi_N \hat{\mathcal{D}}(\hat{s}, \theta) \pi_N
\]

and \( \Psi_N = \pi_N \Psi \). The Fourier coefficients in (C.20) are replaced by the linear approximations above and all cross terms are dropped. Thus, all products of \( \delta_k, z_l \) and \( \omega_m \) are removed from the equation. In doing so, we remove the connection between Fourier coefficients of different order and obtain the block diagonal system of equations

\[
\begin{align*}
& j\omega \hat{E}_q \dot{z} = \left(-j\omega \hat{N} + \hat{A}_0\right) \dot{z} + \hat{B} \hat{\Psi} \dot{\delta} + \hat{D}_0 \dot{\nu} \\
& \dot{\nu} = \hat{C} \dot{z} \\
& \dot{\delta} = \hat{F} \dot{z}
\end{align*}
\tag{C.22}
\]

where \( \hat{\Psi} = I \otimes \Psi_N, \hat{A}_0 = I \otimes A(0) \) and \( \hat{D}_0 = I \otimes D(0) \).
The matrices in (C.22) are block diagonal and there is no coupling between the Fourier coefficients. In other words, the approximation of the $k^{th}$ Fourier coefficient $v_k$ is a linear function of (only) the $k^{th}$ Fourier coefficient $w_k$ of the disturbance. The relation is written

$$v_k = \mathcal{H}_{cl}(k\omega)w_k \quad \forall k \in \mathbb{Z} \quad (C.23)$$

where

$$\mathcal{H}_{cl}(\omega) = C(j\omega I - (-j\omega_N + A(0)) - B\Psi_N F)^{-1} D(0)$$

is a frequency response of dimension $(2N+1)n$. As was noted above, $A(0) = \pi_N \bar{A}(\delta_{\theta})\pi_N$ and $D(0) = \pi_N \bar{D}(\delta_{\theta})\pi_N$ and $N = \pi_N \bar{E}_n \pi_N$. In light of the expression (C.17) for the HTF, $\mathcal{H}_{cl}$ can be seen as a truncated version of $\mathcal{H}$ with the additional term $B\Psi_N F$ representing the effect of the feedback. In the section below we use the individual entries of $\mathcal{H}_{cl}$. They are indexed as

$$\mathcal{H}_{cl}(\omega) = \begin{bmatrix} \cdots & \cdots & \cdots \\ H_{cl,1,1}(\omega) & H_{cl,1,0}(\omega) & H_{cl,1,-1}(\omega) \\ \cdots & \cdots & \cdots \\ H_{cl,-1,1}(\omega) & H_{cl,-1,0}(\omega) & H_{cl,-1,-1}(\omega) \\ \cdots & \cdots & \cdots \end{bmatrix} \quad (C.24)$$

where $H_{cl,0,0}$ denotes the central block of the matrix. It should be noted that $\mathcal{H}_{cl}$ does not have the same structure as (C.14). This structure is lost because of the feedback and pulse modulation.

### 4.2.1 Connection to the time domain

Equation (C.23) gives an approximate steady state response of the approximate phasor model (C.10) in terms of a transfer function from Fourier coefficients $w_k$ to $v_k$. In the section below we express the corresponding time domain representation. For simplicity, we only consider the case of a sinusoidal disturbance.

Let the disturbance in (C.3) be $w(t) = \sin(\omega t)$ where $\omega << \omega_s$. The phasors of $w(t)$ are approximated as $\langle w \rangle_0(t) \approx \sin(\omega t)$ and $\langle w \rangle_n(t) \approx 0$ for $n \neq 0$. In other words, the truncated phasor representation of $w(t)$ is approximately

$$w(t) = w_{-1}e^{-j\omega t} + w_1e^{j\omega t}$$

where $w_{\pm 1} = [0, \ldots, 0, \pm \frac{1}{2j}, 0, \ldots, 0]'$. The frequency separation property of (C.22) implies that the only nonzero coefficients in $\hat{v}$ are $\hat{v}_1$ and $\hat{v}_{-1}$. The approximate
response of (C.3) to the sinusoidal disturbance is therefore given by

\[ y(t) \approx \sum_{n=-N}^{N} y[n] e^{jn\omega_s t} \]

\[ \approx \sum_{n=-N}^{N} \left\{ y_1 e^{j\omega t} + y_{-1} e^{-j\omega t} \right\} [n] e^{jn\omega_s t} \]

\[ = \sum_{n=-N}^{N} \left\{ \mathcal{H}_{cl}(\omega)W_1 e^{j\omega t} + \mathcal{H}_{cl}(-\omega)W_{-1} e^{-j\omega t} \right\} [n] e^{jn\omega_s t} \]

where \( y[n] \) denotes the \( n \)th approximate phasor coefficient of \( y \) (see Remark 3.1). We now use that only the zero coefficient of \( w_{\pm 1} \) is non-zero and that \( H_{cl,n,0}(-\omega) = \overline{H_{cl,-n,0}(\omega)} \). It follows

\[ y(t) \approx \sum_{n=-N}^{N} H_{cl,n,0}(\omega) e^{j(\omega + n\omega_s)t} \frac{2j}{2j} - H_{cl,n,0}(\omega) e^{-j(\omega + n\omega_s)t} \frac{2j}{2j} \]

\[ = \sum_{n=-N}^{N} H_{cl,n,0}(\omega) e^{j(\omega + n\omega_s)t} \frac{2j}{2j} - \overline{H_{cl,n,0}(\omega)} e^{-j(\omega + n\omega_s)t} \frac{2j}{2j} \]  \hspace{1cm} (C.25)

\[ = \sum_{n=-N}^{N} |H_{cl,n,0}(\omega)| \sin((\omega + n\omega_s)t + \phi_{n,0}) \]

where \( H_{cl,n,k} \) is the \((n, k)\)-block of \( \mathcal{H}_{cl} \) (see (C.24)) and where \( \phi_{n,0} = \arg H_{cl,n,0}(\omega) \). The expression above is analogous to the expression given in Section 4.1 for the response of an LTP system to a sinusoidal input.

4.2.2 Existence of solution to the harmonic balance equations

To justify the assumption that the harmonic balance equations have a solution we show that for small disturbances \( w \) this is indeed the case. The harmonic balance equations (C.20) have a solution if there is a solution \( \delta \) to

\[ \delta = \mathcal{F} \mathcal{H}_1(\omega) \mathcal{D}_0 \hat{w} + \mathcal{F} \mathcal{H}_1(\omega) \left( \hat{\Delta}_1(\delta) \hat{w} + \hat{\Delta}_2(\delta) \right) \]

where

\[ \mathcal{H}_1(\omega) = \left( j\omega \hat{E}_q - (-j\omega_s \hat{N} + \hat{A}_0) - \hat{B} \hat{\Psi} \hat{F} \right)^{-1} \]  \hspace{1cm} (C.26)

\[ \hat{\Delta}_1(\delta) = \left( I - \left( \hat{A}(\delta) - \hat{A}_0 \right) \mathcal{H}_1(\omega) \right)^{-1} \hat{D}(\delta) - \hat{D}_0 \]

\[ \hat{\Delta}_2(\delta) = \left( I - \left( \hat{A}(\delta) - \hat{A}_0 \right) \mathcal{H}_1(\omega) \right)^{-1} \hat{B} \left( \hat{S}(\delta) - \hat{\Psi} \delta \right). \]
From (C.22) it is clear that the first term $\hat{\mathcal{H}}_1(\omega)\hat{D}_0\hat{\nu}$ is the approximate solution given by the linearized harmonic balance equations while the second term is a higher order function of $\delta$. We note that theoperator $\mathcal{H}_1(\omega)$ is block diagonal and can be written

$$\mathcal{H}_1(\omega) = \text{blkdiag}(\ldots, \mathcal{H}_2(\omega), \mathcal{H}_2(0), \mathcal{H}_2(-\omega), \ldots) \quad (C.27)$$

where $\mathcal{H}_2(\omega) = (j\omega I - (-j\omega N + A(0)) - B\Psi_N F)^{-1}$. It follows that the operator $j\omega \hat{E}_q \mathcal{H}_1$ is also block diagonal and the induced $l_2$-norms satisfy

$$\|\mathcal{H}_1(\omega)\| := \|\mathcal{H}_1(\omega)\|_{l_2} = \sup_{k \in \mathbb{Z}} \tilde{\sigma}(\mathcal{H}_2(k\omega))$$

$$\|j\omega \hat{E}_q \mathcal{H}_1(\omega)\| := \|j\omega \hat{E}_q \mathcal{H}_1(\omega)\|_{l_2} = \sup_{k \in \mathbb{Z}} \tilde{\sigma}(j\omega k \mathcal{H}_2(k\omega)).$$

Finally we note that the same definition and equalities hold for $\hat{\mathcal{H}}_1$.

Let

$$H(\hat{\delta}, \hat{\nu}) = \hat{\mathcal{H}}_1(\omega) \left( (\hat{D}_0 + \hat{\Delta}_1(\hat{\delta})) \hat{\nu} + \hat{\Delta}_2(\hat{\delta}) \right). \quad (C.28)$$

There exists a solution to the harmonic balance equations (C.20) iff there exists a solution to the fixed point equation $\hat{\delta} = H(\hat{\delta}, \hat{\nu})$. Clearly, for $\hat{\nu} = 0$ there is the solution $0 = H(0, 0)$. We will show that there is a solution also for nonzero disturbances $\hat{\nu}$, provided they are small enough. The claim is formalized in the following proposition.

**Proposition 4.1.** Let $r_w > 0$ and let $r > 0$ be such that

$$\sup_{|\hat{\delta}| < r} \tilde{\sigma}(A(\hat{\delta}) - A(0)) < 1/(2\|\mathcal{H}_1\|) \quad (C.29)$$

$$C_1 r_w + C_3(r)r^2 - (1 - C_2(r)r_w)r < 0 \quad (C.30)$$

where $C_i > 0$ are defined as

$$C_1 = \|\hat{\mathcal{H}}_1\| \tilde{\sigma}(D(0))$$

$$C_2(r) = \|\hat{\mathcal{H}}_1\|\sqrt{2} \left( \frac{\gamma^2}{\sqrt{2}} + \left( \gamma_1(\gamma_2 r + \tilde{\sigma}(D(0))) + \frac{c_5}{\sqrt{2}} + 2\pi c_3 \right)^2 \right)^{1/2}$$

$$C_3(r) = \|\hat{\mathcal{H}}_1\|2\sqrt{2}c_1 \tilde{\sigma}(B) \left( 1 + \left( \frac{1}{2\sqrt{2}} + \gamma_1 r \right)^2 \right)^{1/2}$$

where

$$\gamma_1 = c_4 \left( \|\mathcal{H}_1\|^2 + T^2\|j\omega \hat{E}_q \mathcal{H}_1\|^2 \right)^{1/2} + c_2 T\|j\omega \hat{E}_q \mathcal{H}_1\|$$

$$\gamma_2 = 2\left( c_3 + c_2\|\mathcal{H}_1\|\tilde{\sigma}(D(0)) \right)$$
and where \( c_i > 0, i = 1, \ldots, 5 \) are constants satisfying

\[
\begin{align*}
\sup_{|\delta|<r} |S(\delta) - \Psi_N\delta| &< c_1 r^2 \\
\sup_{|\delta|<r} |S'(\delta) - \Psi_N'| &< c_1 r \\
\sup_{|\delta|<r} \sigma(\mathcal{A}(\delta) - \mathcal{A}(0)) &< c_2 r \\
\sup_{|\delta|<r} \sigma(\mathcal{D}(\delta) - \mathcal{D}(0)) &< c_3 r \\
\sup_{|\delta|<r} \sigma(\mathcal{A}'(\delta)) &< c_4 \\
\sup_{|\delta|<r} \sigma(\mathcal{D}'(\delta)) &< c_5.
\end{align*}
\]  

Then the harmonic balance equations (C.20) have a solution for all \( \hat{\nu} \) such that

\[ 2(\|\hat{\nu}\|_{l_2}^2 + T^2\|j\omega E_q\hat{\nu}\|_{l_2}^2) \leq r_w^2. \]

**Proof:** A proof is given in Appendix 8.

### 4.3 Summary of results

In the sections above we considered the problem of harmonic analysis of the PWM system (C.3). Given a disturbance \( w \), we wanted to determine the spectral content of the output \( y \). To this end we first introduced the DPM (C.7) which gave us an \( L_2 \)-equivalent system description where the input and output are represented in terms of time varying Fourier coefficients (phasors) which were collected in the infinite vectors \( \hat{\nu} \) and \( \hat{\gamma} \). In a second step, we introduced a truncated averaged approximation of the DPM. We obtained the finite dimensional ODE (C.10) where the vector \( \nu \) contains the \( 2N + 1 \) low order phasors of \( w \) and where \( \gamma \) is the approximation of the \( 2N + 1 \) low order phasors of \( y \). In Proposition 3.1 it was shown that for small disturbances \( w \), the approximation can be made arbitrarily good for truncation \( N \) large enough and switch period \( T \) small enough.

It was assumed that for a periodic vector \( \nu \), the system (C.10) attains a periodic solution. The periodic signals \( \nu \) and \( \gamma \) were expressed as Fourier series (see (C.18)) where the Fourier coefficients \( \nu_k \) and \( \gamma_k \) satisfy harmonic balance equations. The Fourier coefficients \( \nu_k \) and \( \gamma_k \) were collected in the infinite dimensional vectors \( \hat{\nu} \) and \( \hat{\gamma} \) and the harmonic balance equations were stated as an infinite dimensional nonlinear equation. This equation is represented by the block schedule in Fig. C.1.

We note that the Fourier coefficients \( \nu_k \) and \( \gamma_k \) contribute to all the \( 2N + 1 \) low order (approximate) phasors of \( w \) and \( y \) respectively and that the time domain signals which are reconstructed from \( \hat{\nu} \) and \( \hat{\gamma} \) contain frequencies \( n\omega + k\omega_s, n \in \mathbb{Z}, k \in \{-N, \ldots, N\} \).

If the terms \( \Delta_i \) in Fig. C.1 are removed, we obtain an approximate solution to the harmonic balance equations. The approximate solution is represented by the infinite dimensional block diagonal matrix \( \hat{CH}_1(\omega)\hat{D}_0 \) which maps \( \hat{\nu} \) to \( \hat{\gamma} \). If the
terms $\Delta_i$ are small compared to the norm of $\mathcal{H}_1$, the linear approximation will be accurate. This claim is a consequence of Proposition 4.1. We note that since $\mathcal{H}_1$ is block diagonal, the $k^{th}$ Fourier coefficient $y_k$ of the linear approximation of $y$ will depend on (only) the $k^{th}$ Fourier coefficient $\mathcal{W}_k$.

In Section 4.2.1 above we considered the case where the disturbance was a sinusoid with frequency $\omega << \omega_*$. We approximated the corresponding vector of Fourier coefficients $\hat{\mathcal{W}}$ with a vector where only the terms $\mathcal{W}_1$ and $\mathcal{W}_{-1}$ were non-zero. These non-zero elements “pick out” the blocks $\mathcal{H}_2(\omega)$ and $\mathcal{H}_2(-\omega)$ from the infinite dimensional matrix $\mathcal{H}_1(\omega)$ in Fig. C.1 and yield an output $\hat{y}$ where only the terms $y_1$ and $y_{-1}$ are non-zero. These terms were plugged in to the Fourier series (C.18) to yield the approximation of the low order phasors of $y$. In Section 4.2.1 we use the approximate phasors to reconstruct the time domain expression of $y$.

5 Examples

To illustrate the theory presented in the paper we consider two simple numerical examples; the synchronous step-down (buck) converter and the synchronous step-up (boost) converter depicted in Fig. C.2 and C.8 respectively. In the section below we use the results of Section 4 to investigate the harmonic properties of these systems in both open and closed loop.

5.1 Example 1: The synchronous buck converter

The first example we consider is the synchronous buck converter depicted in Fig. C.2. The system is of the form (C.1) with state $\xi = [i_l v_c]'$ where $i_l$ is the inductor current and $v_c$ is the capacitor voltage. The system matrices are

$$A_0 = \begin{bmatrix}
-\frac{1}{x_l} (r_l + \frac{r_o r_c}{r_o + r_c}) & -\frac{1}{x_l} \frac{r_o}{x_c r_o + r_c} \\
\frac{1}{x_c} \frac{r_o}{x_c r_o + r_c} & -\frac{1}{x_c} \frac{1}{x_c r_o + r_c}
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
v_o \\
x_l
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
\frac{1}{x_l} \\
0
\end{bmatrix}$$

Figure C.1: Block schedule representation of the harmonic balance equations.
and $A_1$, $B_0$ and $D_0$ are zero. We choose $C(t) = [0,1]$, i.e., we take the capacitor voltage as the output signal. The dynamics have been scaled to obtain switch period $T_s = 1$ and the parameters are expressed in the per unit system. They are $x_l = 3/4\pi$ p.u., $x_c = 70/10\pi$ p.u., $r_l = 0.05$ p.u., $r_c = 0.005$ p.u., $r_o = 1$ p.u. and the source voltage is $v_s = 1.8$ p.u. It should be noted that the inductance and capacitance are chosen quite small to make the effect of the disturbance appear more clearly.

The reference output voltage is $v_{ref} = 1$. The stationary duty cycle $d^0$ is chosen to make the average output voltage equal $v_{ref}$ and the corresponding periodic stationary solution is denoted $\xi^0$. The dynamics of the error $x := \xi - \xi^0$ is considered in both open loop (i.e., $d_k = d^0 \forall k$) and in closed loop with the linear feedback

$$d_k = \text{sat}_{[0,1]}(d^0 + F \pi_0 \dot{x}(kT)), \quad F = [-0.1 \ 0].$$

Note that we only use the average value $\pi_0 \dot{x}$ of the state in the feedback.

In the open loop case we consider the DPM (C.16) corresponding to the buck converter. The DPM is truncated and we apply the Fourier transform to obtain a truncated HTF $H(\omega)$. The gains $|H_k(\omega)|$, $k = -2, \ldots, 2$ (see (C.14) for the definition) are plotted in Fig. C.3.

In the closed loop case we consider the averaged dynamic phasor system (C.10) corresponding to the buck converter and derive the corresponding closed loop harmonic transfer function $H_{cl}(\omega)$. The gains $|H_{cl,k,0}(\omega)|$, $k = -2, \ldots, 2$ (see (C.24) for the definition) are plotted in Fig. C.3.

The plots in Fig. C.3 can be interpreted in light of the formulas (C.15) and (C.25) which give the steady state response to a sinusoidal disturbance. The formulas (C.15) and (C.25) imply that when the system is subjected to a sinusoidal disturbance with angular frequency $\omega$, the output will be a sum of sinusoids with shifted frequencies $\omega + k\omega_s$. The central plots in Fig. C.3 (which have index zero) show the amplification of the fundamental term $\sin(\omega t)$ and the off-diagonal plots (with index $k$) show the amplitude of the additional shifted frequencies $\sin((\omega + k\omega_s)t)$. The fact that the off-diagonals in Fig. C.3 are non-zero explains why the steady state responses plotted below are not purely sinusoidal but contain ripple.

The harmonic transfer functions of the open and closed loop systems are used in formulas (C.15) and (C.25) respectively to approximate the steady state response to a sinusoidal disturbance. We consider disturbances $w(t) = a \sin(2\pi ft)$ where $a = 0.1$.
Figure C.3: Harmonic transfer function $\mathcal{H}(\omega)$ (left) and closed loop harmonic transfer function $\mathcal{H}_{cl}(\omega)$ (right) of the buck converter. The left plot shows the coefficients $H_k, k = -2, \ldots, 2$ as a function of angular frequency and the right plot shows the coefficients $H_{cl,k}, k = -2, \ldots, 2$.

and we consider two different frequencies. They are $f = 0.1$ Hz and $f = 0.48$ Hz. It should be noted that the higher frequency is close to half the switching frequency.

The approximate steady state responses predicted by the HTFs are verified by simulations in Matlab\textsuperscript{TM}. The predicted and simulated steady state responses of the capacitor voltage are shown in Fig. C.4 and C.6. In the open loop case, Fig. C.4 and C.6 show that the HTF provides a perfect match with the simulation. In Fig. C.4 the harmonics caused by the switching are clearly visible. These harmonics correspond to the off-diagonal elements of the HTF in Fig. C.3.

In the closed loop case, the response to the lower frequency disturbance (see Fig. C.4) is predicted almost perfectly. However, for the higher frequency disturbance (see Fig. C.6), the HTF does not predict the simulated response equally well. This is not surprising since (C.25) is an approximation where we have assumed $\omega << \omega_s$ which is not satisfied in this case. We also note that the high frequency disturbance ($f = 0.48$) seemingly yields a slow frequency in the output. This is the phenomena known as beat which appears when two sinusoids with slightly different frequencies are superimposed.

The FFT is used to determine the spectrum of the two simulated steady state responses. The results are shown in Fig. C.5 and C.7. From these plots it can be seen that the open loop responses only contain frequencies $\omega + k\omega_s$ as predicted by the theory. Consider for example Fig. C.7 which shows the spectrum of the open and closed loop responses to the disturbance with frequency $f = 0.48$ Hz. In the open loop case (left) there are peaks at frequencies $0.48 = f + 0f_s$ Hz and $-0.52 = f - f_s$ Hz and also small peaks at frequencies $1.48 = f + f_s$ Hz and $-1.52 = f - 2f_s$ Hz. In other words, the peaks are at frequencies $f + kf_s$ Hz (recall that the switching frequency is $f_s = 1$ Hz). In general the closed loop response may contain frequencies other than $f + kf_s$. In this case however, no such frequencies are detected.
Figure C.4: Steady state response of the voltage $v_c$ of the buck converter to the disturbance $w(t) = a \sin(2\pi ft)$ with amplitude $a = 0.1$ and frequency $f = 0.1$ Hz. The upper left figure shows the open loop response and the upper right figure shows the closed loop response. The two figures below show close-ups of the figures above.

Figure C.5: Spectrum of the response of the voltage $v_c$ of the buck converter to the disturbance $w(t) = a \sin(2\pi ft)$ with amplitude $a = 0.1$ and frequency $f = 0.1$ Hz. The left figure shows the spectrum of the open loop response and the right figure shows the spectrum of the closed loop response.
Figure C.6: Steady state response of the voltage $v_c$ of the buck converter to the disturbance $w(t) = a \sin(2\pi ft)$ with amplitude $a = 0.1$ and frequency $f = 0.48$ Hz. The upper left figure shows the open loop response and the upper right figure shows the closed loop response. The two figures below show close-ups of the figures above.

Figure C.7: Spectrum of the response of the voltage $v_c$ of the buck converter to the disturbance $w(t) = a \sin(2\pi ft)$ with amplitude $a = 0.1$ and frequency $f = 0.48$ Hz. The left figure shows the spectrum of the open loop response and the right figure shows the spectrum of the closed loop response.
5.2 Example 2: The synchronous boost converter

The second example we consider is the synchronous boost converter depicted in Fig. C.8. The system is of the form (C.1) with state \( \xi = [i_l \ v_c]' \) where \( i_l \) is the inductor current and \( v_c \) is the voltage. The system matrices are

\[
A_0 = \begin{bmatrix}
\frac{1}{x_l} (1 + \frac{r_o r_c}{r_o + r_c}) & -\frac{1}{x_c} \frac{r_o}{r_o + r_c} \\
\frac{1}{x_l} \frac{r_o}{r_o + r_c} & -\frac{1}{x_c} \frac{1}{r_o + r_c}
\end{bmatrix}, \quad
B_0 = \begin{bmatrix}
v_s/x_l \\
0
\end{bmatrix}
\]

\[
A_1 = \begin{bmatrix}
\frac{1}{x_l} \frac{r_c}{r_o + r_c} & \frac{1}{x_c} \frac{r_o}{r_o + r_c} \\
\frac{1}{x_c} \frac{1}{r_o + r_c} & 0
\end{bmatrix}, \quad
D_0 = \begin{bmatrix}
1/x_l \\
0
\end{bmatrix}
\]

and \( B_1 \) and \( D_1 \) are zero. We choose \( C(t) = [0, 1], \) i.e., we take the capacitor voltage as the output signal. As in the buck example, the dynamics have been scaled to obtain switch period \( T_s = 1 \) and the parameter values are expressed in the per unit system. They are \( x_l = 3/10\pi \) p.u., \( x_c = 70/10\pi \) p.u., \( r_l = 0.05 \) p.u., \( r_c = 0.005 \) p.u., \( r_o = 1 \) p.u. and the source voltage is \( v_s = 0.75 \) p.u.. It should be noted that the inductance and capacitance are chosen quite small. This is to make the effect of the disturbance appear more clearly.

The reference output voltage is \( v_{ref} = 1 \). The stationary duty cycle \( d^0 \) is chosen to make the average output voltage equal \( v_{ref} \) and the corresponding periodic stationary solution is denoted \( \xi^0 \). The dynamics of the error \( x := \xi - \xi^0 \) is considered in both open loop (i.e., \( d_k = d^0 \forall k \)) and in closed loop with the linear feedback

\[
d_k = \text{sat}_{[0,1]}(d^0 + F\pi_0 \hat{x}(kT_s)), \quad F = [-0.1021 \ 0.1555].
\]

Note that we only use the average value \( \pi_0 \hat{x} \) of the state in the feedback.

In analogy with the example above we derive the open and closed loop harmonic transfer functions \( \mathcal{H} \) and \( \mathcal{H}_{cl} \) corresponding to the boost converter. The gains \( |H_k(\omega)| \) and \( |H_{cl,k,0}(\omega)|, \ k = -2, \ldots, 2 \) are plotted in Fig. C.9.

As in the previous example, the harmonic transfer functions of the open and closed loop systems are used to approximate the steady state response to a sinusoidal disturbance. We again consider disturbances \( w(t) = a \sin(2\pi ft) \) where \( a = 0.1 \) and we consider two different frequencies \( f = 0.1 \) Hz and \( f = 0.48 \) Hz.
Figure C.9: Harmonic transfer function $\mathcal{H}(\omega)$ (left) and closed loop harmonic transfer function $\mathcal{H}_{cl}(\omega)$ (right) of the boost converter. The left plot shows the coefficients $H_k$, $k = -2, \ldots, 2$ as a function of frequency and the right plot shows the coefficients $H_{cl,k,0}$, $k = -2, \ldots, 2$.

The approximate steady state responses predicted by the HTFs are verified by simulations in Matlab™. The predicted and simulated steady state responses of the capacitor voltage are shown in Fig. C.10 and C.12.

The figures show that in the open loop case, the HTF provides a perfect match with the simulated response for both the low and high frequency disturbance. In the closed loop case, the HTF provides a fairly good approximation for the response to the low frequency disturbance. The response to the high frequency disturbance contains sharp peaks that are not accounted for by the HTF.

The FFT is used to determine the spectrum of the two steady state responses. The results are shown in Fig. C.11 and C.13. As in the example above, it can be verified that the open loop responses only contain frequencies $\omega + k\omega_s$. However, Fig. C.13 (which shows the spectrum of the responses to the disturbance with frequency $f = 0.48$ Hz) shows that the closed loop response contains additional frequencies.

To see this, consider first the open loop case (left). There are peaks at frequencies $0.48 = f + 0f_s$ Hz, $-0.52 = f - f_s$ Hz, $1.48 = f + f_s$ Hz and $-1.52 = f - 2f_s$ Hz. In other words, at frequencies $f + kf_s$ Hz. We then consider the closed loop case (right). We find peaks at the same frequencies as in the open loop case, but there are also peaks that appear to be at frequencies $nf + kf_s$ Hz for $n \in \mathbb{Z}$, $k \in \mathbb{Z}$. For example, we find peaks at frequencies $0 = 0f + 0f_s$ Hz, $0.04 = -2f + f_s$ Hz, $1 = 0f + f_s$ Hz and $1.04 = -2f + 2f_s$ Hz.
Figure C.10: Steady state response of the voltage \( v_c \) of the boost converter to the
disturbance \( w(t) = a \sin(2\pi ft) \) with amplitude \( a = 0.1 \) and frequency \( f = 0.1 \) Hz.
The upper left figure shows the open loop response and the upper right figure shows
the closed loop response. The two figures below show close-ups of the figures above.

Figure C.11: Spectrum of the response of the voltage \( v_c \) of the boost converter to
the disturbance \( w(t) = a \sin(2\pi ft) \) with amplitude \( a = 0.1 \) and frequency \( f = 0.1 \) Hz.
The left figure shows the spectrum of the open loop response and the right
figure shows the spectrum of the closed loop response.
Figure C.12: Steady state response of the voltage $v_c$ of the boost converter to the disturbance $w(t) = a \sin(2\pi ft)$ with amplitude $a = 0.1$ and frequency $f = 0.48$ Hz. The upper left figure shows the open loop response and the upper right figure shows the closed loop response. The two figures below show close-ups of the figures above.

Figure C.13: Spectrum of the response of the voltage $v_c$ of the boost converter to the disturbance $w(t) = a \sin(2\pi ft)$ with amplitude $a = 0.1$ and frequency $f = 0.48$ Hz. The left figure shows the spectrum of the open loop response and the right figure shows the spectrum of the closed loop response.
6 Conclusions

We have shown how the dynamic phasor model can be used for harmonic analysis of pulse-width modulated systems. A connection between the dynamic phasor model and the HTF was shown and the dynamic phasor model was used to extend the notion of HTF to closed loop, non-periodic systems. The method was applied to two realistic examples and it was shown that the approximation given by the closed loop HTF correspond well with simulated results.

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7 Appendix: Proof of Proposition 3.1

The proof presented in this section is rather lengthy since one must consider both the truncation of the infinite dimensional state and the averaging approximation. The main difficulty in the proof is to provide a norm estimate of the difference $\tilde{s} - S_{\omega}(\delta)$, between the phasor vector of the PWM switching function in (C.2) and its corresponding nonlinear approximation in (C.9). The difficulty lies in that the former is determined by the duty cycle function (C.4), which is determined at the sampling times $kT_s$ while the latter is determined by the duty ratio function in (C.10) which is a continuous function of the state.

In the proof below we switch repeatedly between the frequency domain representation of the phasor dynamics in (C.7) and the following time domain representation

$$\dot{x}(t, \tau) := \frac{d}{dt} \chi(t, \tau) = A(t, \tau)x(t, \tau) + B(t, \tau) + D(t, \tau)w(t, \tau)$$

where $x$ is defined in (C.6) and (here $\tau \in [-T_s, 0]$)

$$A(t, \tau) = A_0 + s(t, \tau)A_1, \quad D(t, \tau) := D_0 + s(t, \tau)D_1$$

$$B(t, \tau) = (A_1\xi^0(t, \tau) + B_1) (s(t, \tau) - s_0(t, \tau))$$

$$\xi^0(t, \tau) = \sum_{n=\infty}^{\infty} \langle \xi_0 \rangle_n e^{jn\omega_s(t+\tau)}$$

$$w(t, \tau) = \sum_{n=\infty}^{\infty} \langle w \rangle_n(t)e^{jn\omega_s(t+\tau)}$$

$$s(t, \tau) = \sum_{n=\infty}^{\infty} \langle s \rangle_n(t)e^{jn\omega_s(t+\tau)}$$

and similarly for $s_d(t, \tau)$. We will make frequent use of the Parseval relation

$$\langle \dot{x}(t), \tilde{y}(t) \rangle_{L_2} = \langle x(t, \cdot), y(t, \cdot) \rangle_{L_2[-T,0]} = \langle x(\cdot), y(\cdot) \rangle_{L_2[-T,T]}$$
where we use that $x(t, \tau) = x(t + \tau)$ a.e. on $\{\tau|\tau \in [-T_s, 0]\}$. The proof makes use of the following lemmas. Proofs are found in Section 10.

**Lemma 7.1.** For any time $t$, $\hat{A} = \hat{A}(\hat{s}(t))$ is a bounded linear operator on $l_2$ with induced norm

$$
\|\hat{A}(\hat{s}(t))\|_{l_2 \to l_2} = \|A(t, \cdot)\|_{L_2[-T_s, 0] \to L_2[-T_s, 0]}
= \|A(\cdot)\|_{L_2[t-T_s, t] \to L_2[t-T_s, t]}
= \max_{\tau \in [t-T_s, t]} \sigma(A(\tau))
\leq \sup_{s} \sigma(A(s(t))) =: \|A\|
$$

where the last optimization is over all possible $T_s$-periodic on-off sequences of the form (C.2). Analogously, it holds

$$
\|\hat{D}(\hat{s}(t))\|_{l_2 \to l_2} = \|D(t, \cdot)\|_{L_2[-T_s, 0] \to L_2[-T_s, 0]}
= \|D(\cdot)\|_{L_2[t-T_s, t] \to L_2[t-T_s, t]}
= \max_{\tau \in [t-T_s, t]} \sigma(D(\tau))
\leq \sup_{s} \sigma(D(s(t))) =: \|D\|
$$

and

$$
\|\hat{B}(\hat{s}(t) - \hat{s}_d)\|_{l_2} = \|B(t, \cdot)\|_{L_2[-T_s, 0]}
= \|B(\cdot)\|_{L_2[t-T_s, t]}
\leq \max_{\tau \in [t-T_s, t]} \sigma(B(\tau))
\leq \sup_{s} \sigma(B(s(t))) =: \|B\|.
$$

**Lemma 7.2.** Let $x$ be a solution of (C.3) and suppose the disturbance satisfies

$$
\sup_t |w(t)| \leq r_w
$$
where $r_w > 0$. The corresponding phasor vector $\hat{x}$ satisfies

$$
\|(I - \pi_N)\hat{x}\| \leq (\|A\|\|\hat{x}\| + \|B\| + \|D\|)R_w \frac{\sqrt{2}T_s}{\pi \sqrt{N + 1}}.
$$

**Lemma 7.3.** Suppose the solution $\hat{x}$ of (C.7) remains in the set $\Omega := \{\hat{x} \in l_2 | \|\hat{x}\| \leq R\} \forall t \in [t_0, t_1]$ where $R > 0$ and $[t_0, t_1]$ is any closed interval. Also assume that $\sup_{t} |\hat{w}(t)| \leq r_w$ where $r_w > 0$. For $t \in [t_0, t_1]$ it holds

$$
\frac{d}{dt} \|(I - \pi_N)\hat{x}\|^2 = 2R\{\langle (I - \pi_N)\hat{x}, \hat{A}(\hat{s})\hat{x} + \hat{B}(\hat{s} - \hat{s}_d) + \hat{D}(\hat{s})\hat{w} \rangle \}
$$

and thus, the derivative is well defined.
Main part of proof

To facilitate the proof we introduce some notation. Let

\[ f(t, x, w) = (-j\omega_s \hat{E}_n + \hat{A}(\hat{s})) \dot{x} + \hat{B}(\hat{s} - \hat{s}_0) + \hat{D}(\hat{s}) \dot{w} \]

\[ f_{av}(z, w) = (-j\omega_n N + A(\delta)) z + BS(\delta) + D(\delta) w \]

be the vector fields of the DPM and the truncated averaged approximation respectively. By \( z_u(t) \) we denote the solution to the unforced system \( \dot{z} = f_{av}(z, 0) \) and \( z(t) \) denotes the solution to the forced system \( \dot{z} = f_{av}(z, w) \).

Let \( r' = \gamma'/\|\mathcal{F}\| \) where \( \gamma' = \min\{d^0, 1 - d^0\} \). Then the feedback of the approximate phasor model does not saturate on the set \( \{ z \in \mathbb{C}^{(2N+1)n} \mid |z| < r' \} \) and thus, the vector field \( f_{av}(z, 0) \) is continuously differentiable on this set. By assumption, there exists numbers \( k \geq 1, \lambda > 0 \) and \( r \in (0, r'/k) \) such that the solution to the unforced system satisfies

\[ |z_u(t)| \leq k|z_u(t_0)|e^{-\lambda(t-t_0)} \quad \forall z_u(t_0) \in \Omega, \forall t \geq t_0 \]

where \( \Omega := \{ z \in \mathbb{C}^{(2N+1)n} \mid |z| < r \} \). By using the arguments of the converse Lyapunov theorem, see e.g., Theorem 4.14 in [5], one can show that there exists a Lyapunov function \( V_{av} : \Omega \to \mathbb{R} \) satisfying

\[ c_1|z|^2 \leq V_{av}(z) \leq c_2|z|^2 \]

\[ \partial V_{av} f_{av}(z, 0) \leq -c_3|z|^2 \quad \text{for all } z \in \Omega \quad \text{where } c_i, i = 1, \ldots, 4 \text{ are positive constants.} \]

Remark 7.1. The phasor dynamics is complex valued. However, the dynamics is highly structured and the proof in Theorem 4.14 in [5] goes through with slight modifications. For example, the vector field \( f_{av} \) has the Fréchet derivative \( \frac{df_{av}}{dz}(z, 0) : l_{2,N} \to l_{2,N} \) defined by multiplication with the matrix

\[ \frac{df_{av}}{dz}(z, 0) = (-j\omega_n N + A(\delta)) + (A'(\delta) + BS'(\delta)) z\mathcal{F} \]

where \( A' \) and \( S' \) are the usual derivatives with respect to the real variable \( \delta \). A converse Lyapunov function may as in [5] be constructed by the formula

\[ V_{av}(z) = \int_t^{t+\eta} |\phi(\tau; t, z)|^2d\tau = \int_0^\eta |\phi(\tau; 0, z)|^2d\tau \]

where \( \phi(\tau; t, x) \) is the solution of the system that starts at \( (t, x) \). The same arguments as in [5] show that \( V_{av} \) satisfies the conditions in (C.37). Note, however,
that derivatives must be interpreted correctly. For example, the left hand side in the second inequality becomes

\[ \frac{\partial V_{av}}{\partial z} f_{av}(z, 0) = 2\Re \int_0^t \phi^*(\tau; 0, z) \phi'_z(\tau; 0, z) d\tau f_{av}(z, 0). \]

The existence of a Lyapunov function implies that the forced system 
\[ \dot{z} = f_{av}(z, \omega) \] is locally input-to-state stable [5, 12]. Thus, there exists numbers \( r'_w > 0, r_0 \in (0, r) \) and functions \( \beta \in K_{\infty}, \gamma \in K_{\infty} \) such that for all initial values \( z(t_0) \in \Omega_0 := \{ z \in \mathbb{C}^{(2N+1)n} \mid |z| \leq r_0 \} \) it holds

\[ |\dot{z}(t)| \leq \beta(|z(t_0)|, t - t_0) + \gamma \left( \sup_{\tau \in [t_0, t]} |\omega(\tau)| \right) \quad \forall t \geq t_0 \]

provided \( \sup_t |\omega(t)| < r'_w \). Since we only consider the case when the initial condition is zero, it follows that for any number \( \epsilon_1 > 0 \) there exists a number \( r_w > 0 \) such that

\[ |\dot{z}(t)| \leq \epsilon_1 \forall t \text{ if } \sup_t |\omega(t)| \leq r_w. \]

We now make two assumptions:

(a1) \(|\dot{z}(t)| \leq \epsilon_1 \forall t \) for some \( \epsilon_1 > 0 \)

(a2) \(||\dot{e}(t)|| \leq r_1 \forall t \) for some \( r_1 \in (0, r) \)

According to the discussion above, (a1) can always be satisfied for some \( r_w > 0 \). Assumption (a2) will be verified at the end of the proof.

The dynamics of the error terms \( \dot{e}_1 \) and \( \dot{e}_2 \) are written

\[ \frac{d}{dt} \dot{e}_1 = f_{av}(\dot{e}_1, 0) + \Delta(z, \dot{e}_1) + g_1(t, \dot{x}, \omega) \]
\[ \frac{d}{dt} \dot{e}_2 = g_2(t, \dot{x}, \dot{w}) \]

where

\[ \Delta(z, \dot{e}_1) := f_{av}(z + \dot{e}_1, 0) - f_{av}(z, 0) - f_{av}(\dot{e}_1, 0) \]
\[ g_1(t, \dot{x}, \omega) := \pi_N \hat{A}(I - \pi_N) \dot{x} + \left( \pi_N \hat{A} \pi_N - \Delta(\delta) \right) \pi_N \dot{x} + \pi_N \hat{B}(\delta - \delta_{0\epsilon}) - BS(\delta) \]
\[ + \left( \pi_N \hat{D} \pi_N - \hat{D}(\delta) \right) \omega \]
\[ g_2(t, \dot{x}, \dot{w}) := (I - \pi_N) f(t, \dot{x}, \dot{w}) \]

and where we have used the assumption \((I - \pi_N) \dot{w} = 0\) (see Remark 3.2). To prove that the error is small we consider the Lyapunov candidate

\[ V(\dot{e}) := V_{av}(\dot{e}_1) + c_1 \|\dot{e}_2\|^2 : \Omega \times l_{2, \bar{N}} \to \mathbb{R}. \]

We note that assumption (a2) implies that \( \dot{e} \) is in the domain of definition of \( V \).
To bound the derivative of $V$ we make use of a number of inequalities. Firstly, it can be shown analogously with the proof of Theorem 9.1 in [5] that there exist constants $L_1 > 0$ and $L_2 > 0$ such that

$$|\Delta(z, \hat{e}_1)| \leq L_1|\hat{e}_1|^2 + L_2|z||\hat{e}_1| \leq (L_1r_1 + L_2\varepsilon_1)||\hat{e}||$$  \hspace{1cm} (C.38)

where we have used assumptions (a1) and (a2). Secondly, we note that Lemma 7.2 implies that

$$||\hat{e}_2||^2 = ||(I - \pi_N)\hat{x}||^2 \leq \alpha_1||\hat{x}||^2 + \alpha_2$$  \hspace{1cm} (C.39)

where

$$\alpha_1(T_*) := \|A\|^2 \frac{4T_s^2}{\pi^2(N+1)}$$

$$\alpha_2(T_*) := (\|B\| + \|D\|r_w)^2 \frac{4T_s^2}{\pi^2(N+1)}.$$  

Thirdly, we bound the term $g_1$. To this end we note the inequalities

$$|\pi_N\hat{A}(I - \pi_N)\hat{x}| \leq ||A||(I - \pi_N)\hat{x}|$$  \hspace{1cm} (C.40)

and

$$|\pi_N\hat{B}(\hat{s} - \hat{s}_d) - BS(\delta)| \leq ||B||||\hat{s} - S_{av}(\delta)||_{l_2}$$  \hspace{1cm} (C.41)

where we have used that $\hat{s}_d = S_{av}(0)$. To bound the remaining two terms of $g_1$ we note the following, it holds

$$\|(T[\hat{s}] - T[S_{av}(\delta)])\hat{x}\|_{l_2} = \||(s(t, \tau) - s_{d^0 + \delta}(t, \tau))x(t, \tau)||_{L_2[-T_s, 0]}$$

$$= \||(s(t) - s_{d^0 + \delta}(\tau))x(\tau)||_{L_2[-T_s, 0]}$$

$$\leq ||x||_{l_\infty}||s(t, \tau) - s_{d^0 + \delta}(t, \tau)||_{L_2[-T_s, 0]}$$

$$= ||x||_{l_\infty}||\hat{s} - S_{av}(\delta)||_{l_2}$$

where in the first equality we have used the Parseval identity and the fact that $s_{av,k}(d) = \langle s_d \rangle_k$ and where $||x||_{l_\infty} := \sup_{\tau \in [T_s, t]} |x(\tau)|$. This inequality implies

$$\left| \left(\pi_N\hat{A}\pi_N - \hat{A}(\delta)\right)\pi_N\hat{x}\right| = \left|\pi_N \left(\hat{A} - \hat{A}(S_{av}(\delta))\right)(\hat{x} - (I - \pi_N)\hat{x})\right|$$

$$\leq ||A_1||(T[\hat{s}] - T[S_{av}(\delta)])\hat{x}||$$

$$+ ||A_1||(T[\hat{s}] - T[S_{av}(\delta)])(I - \pi_N)\hat{x}||$$

$$\leq ||A_1||\||x||_{l_\infty}||\hat{s} - S_{av}(\delta)|| + 2||A_1||(I - \pi_N)\hat{x}||$$  \hspace{1cm} (C.42)

where the inequality $||(T[\hat{s}] - T[S_{av}(\delta)])|| \leq 2$ can be proved in analogy with Lemma 7.1. Analogously it holds

$$\left| \left(\pi_N\hat{D}\pi_N - \hat{D}(\delta)\right)\psi\right| = \left|\pi_N \left(\hat{D} - \hat{D}(S_{av}(\delta))\right)\pi_N\psi\right| \leq ||D_1||r_w||\hat{s} - S_{av}(\delta)||$$  \hspace{1cm} (C.43)
where we have used \( \nu = \pi_N \hat{w} \). We now combine Lemma 10.2 and 10.3 to bound \( \| \hat{s} - S_{\nu}(\delta) \| \) as
\[
\| \hat{s} - S_{\nu}(\delta) \| \leq \| F \| \left( (2T_s \| A \| + 1) \left( e^{\| A \| T_s} + e^{\| A \| 2T_s} \right) - 2 \right) \\
\times (\| \hat{x}(t) \| + (\| B \| + \| D \| r_w) / \| A \| ).
\]
This inequality is used together with the inequalities (C.40)-(C.43) and the bounds on the terms \( \| (I - \pi_N) \hat{x} \| \) and \( \| x \|_{t, \infty} \) provided by Lemma 7.2 and 10.1 respectively to yield the bound
\[
| g_1(t, \hat{x}, \nu) | \leq \alpha_3 \| \hat{x} \|^2 + \alpha_4 \| \hat{x} \| + \alpha_5
\]  
(C.44)
where
\[
\alpha_3(T_s) := \| A \|(1 + T_s \| A \|) C_1(T_s)
\]
\[
\alpha_4(T_s) := (2\| A \| + \| A \|) \| A \| \frac{\sqrt{2} T_s}{\pi \sqrt{N + 1}} + \| A \|(1 + T_s \| A \|) C_2(T_s)
\]
\[
\alpha_5(T_s) := (2\| A \| + \| A \|) (\| B \| + \| D \| r_w) \frac{\sqrt{2} T_s}{\pi \sqrt{N + 1}}
\]
\[
\quad + ((1 + T_s \| A \|) \| B \| + (T_s \| A \| \| D \| + \| D \| r_w) C_1(T_s)
\]
and where
\[
C_1(T_s) := \| F \| \left( (2T_s \| A \| + 1) \left( e^{\| A \| T_s} + e^{\| A \| 2T_s} \right) - 2 \right)
\]
\[
C_2(T_s) := C_1(T_s)(\| B \| + \| D \| r_w) / \| A \|.
\]
Finally, we use Lemma 7.2 and 7.3 to bound the derivative of \( c_1 \| \hat{e}_2 \|^2 = c_1 \| (I - \pi_N) \hat{x} \|^2 \) as
\[
\frac{d}{dt} c_1 \| \hat{e}_2 \|^2 = 2c_1 \Re \left\{ \langle (I - \pi_N) \hat{x}, g_2(t, \hat{x}, \nu) \rangle \right\} \leq \alpha_6 \| \hat{x} \|^2 + \alpha_7
\]  
(C.45)
where
\[
\alpha_6(T_s) := 4c_1 \| A \|^2 \frac{\sqrt{2} T_s}{\pi \sqrt{N + 1}}
\]
\[
\alpha_7(T_s) := 4c_1(\| B \| + \| D \| r_w) \frac{\sqrt{2} T_s}{\pi \sqrt{N + 1}}.
\]
The inequalities (C.39), (C.44) and (C.45) are now written in terms of the error \( \hat{e} \). For (C.39) and (C.45) we use the inequality \( \| \hat{x} \|^2 \leq 2(\| x \|^2 + \| \hat{e} \|^2) \) and for (C.44) we
use $\|\dot{\hat{x}}\| \leq |z| + \|\ddot{\varepsilon}\|$. These inequalities are used together with the bounds $|z| \leq \varepsilon_1$ and $\|\ddot{\varepsilon}\| \leq r_1$ and we get

$$\|\ddot{\varepsilon}\|^2 \leq L_3(T_s)\|\dot{\varepsilon}\|^2 + L_4(T_s, \varepsilon_1)$$  \hspace{1cm} \text{(C.46)}

$$|g_1(t, \dot{x}, \nu)| \leq L_5(T_s, r_1)\|\dot{\varepsilon}\|^2 + L_6(T_s, \varepsilon_1)$$  \hspace{1cm} \text{(C.47)}

$$\frac{d}{dt}c_1\|\dot{\varepsilon}\|^2 \leq L_7(T_s)\|\dot{\varepsilon}\|^2 + L_8(T_s, \varepsilon_1)$$  \hspace{1cm} \text{(C.48)}

where

$$L_3(T_s) := 2\alpha_1, \quad L_4(T_s, \varepsilon_1) := 2\alpha_1\varepsilon_1^2 + \alpha_2$$

$$L_5(T_s, r_1) := 2\alpha_3r_1 + \alpha_4$$

$$L_6(T_s, \varepsilon_1) := 2\alpha_3\varepsilon_1^2 + \alpha_4\varepsilon_1 + \alpha_5$$

$$L_7(T_s) := 2\alpha_6, \quad L_8(T_s, \varepsilon_1) := 2\alpha_6\varepsilon_1^2 + \alpha_7.$$  

Using the inequalities (C.46), (C.47) and (C.48) we bound $\dot{V}$ according to

$$\dot{V}(\ddot{\varepsilon}) = \frac{\partial V_{\text{av}}}{\partial z}(f_{\text{av}}(\dot{\varepsilon}_1, 0) + \Delta(z, \dot{\varepsilon}_1) + g_1(t, \dot{x}, \nu)) + \frac{d}{dt}c_1\|\dot{\varepsilon}\|^2$$

$$\leq -c_3|\dot{\varepsilon}_1|^2 + c_4|\dot{\varepsilon}_1|(|\Delta| + |g_1|) + \frac{d}{dt}c_1\|\dot{\varepsilon}\|^2$$

$$\leq -\frac{c_3}{c_2}V_{\text{av}}(\dot{\varepsilon}_1) + \left( (c_4(L_1r_1 + L_2\varepsilon_1 + L_5) + L_7)\|\dot{\varepsilon}\|^2 + c_4L_6\|\dot{\varepsilon}\| + L_8 \right)$$

where we have used inequalities (C.38), (C.44), (C.45) and $c_2\|\dot{\varepsilon}_1\|^2 \geq V_{\text{av}}(\dot{\varepsilon}_1)$. We now use that $V_{\text{av}}(\dot{\varepsilon}_1) = V(\ddot{\varepsilon}) - c_1\|\dot{\varepsilon}\|^2$ and the bound (C.39) to obtain

$$\dot{V}(\ddot{\varepsilon}) \leq -\frac{c_3}{c_2}V(\ddot{\varepsilon}) + \kappa_1\|\dot{\varepsilon}\|^2 + \kappa_2\|\ddot{\varepsilon}\|^2 + \kappa_3$$

$$\leq -\left( \frac{c_3}{c_2} - \frac{\kappa_1}{\kappa_2} \frac{2c_1}{\kappa_2} \right) V(\ddot{\varepsilon}) + \frac{1}{2}\kappa_2 + \kappa_3$$

where

$$\kappa_1(T_s, \varepsilon_1, r_1) := \frac{c_1c_3}{c_2}L_3(T_s) + L_7(T_s)$$

$$+ c_4(L_1r_1 + L_2\varepsilon_1 + L_5(T_s, r_1))$$

$$\kappa_2(T_s, \varepsilon_1) := c_4L_6(T_s, \varepsilon_1)$$

$$\kappa_3(T_s, \varepsilon_1) := \frac{c_1c_3}{c_2}L_4(T_s, \varepsilon_1) + L_8(T_s, \varepsilon_1).$$

In summary we have

$$\dot{V}(\ddot{\varepsilon}) \leq -\kappa_4(T_s, \varepsilon_1, r_1)\dot{V}(\ddot{\varepsilon}) + \kappa_5(T_s, \varepsilon_1)$$

where

$$\kappa_4(T_s, \varepsilon_1, r_1) := \frac{c_3}{c_2} - \frac{\kappa_1(T_s, \varepsilon_1, r_1)}{\kappa_2(T_s, \varepsilon_1)} - \frac{\kappa_2(T_s, \varepsilon_1)}{2c_1}$$

$$\kappa_5(T_s, \varepsilon_1) := \frac{1}{2}\kappa_2(T_s, \varepsilon_1) + \kappa_3(T_s, \varepsilon_1).$$
We note that \( \kappa_4(T_s, \epsilon_1, r_1) \rightarrow c_3/c_2 - (c_4/c_1)(L_1r_1 + L_2\epsilon_1) \) and \( \kappa_5(T_s, \epsilon_1, r_1) \rightarrow 0 \) as \( T_s \rightarrow 0 \) and we pick \( \epsilon_1, r_1 \) small enough to satisfy
\[
\frac{c_3}{c_2} - \frac{c_4}{c_1}(L_1r_1 + L_2\epsilon_1) =: \bar{c} > 0.
\]
This implies that for any \( \alpha \in (0, \bar{c}) \), and \( \epsilon \in (0, r_1) \) there exists \( T_0 > 0 \) such that \( \kappa_4(T_s, \epsilon_1, r_1) < \alpha \) and \( \kappa_5(T_s, \epsilon_1) < c_1\alpha\epsilon \) for all \( T_s \leq T_0 \). The first inequality implies
\[
V(\dot{\epsilon}(t)) \leq (1 - e^{-\alpha t})\frac{1}{\alpha}\kappa_5(T_s, \epsilon_1)
\]
which together with the second inequality implies that \( ||\dot{\epsilon}(t)|| \leq \epsilon \ \forall t \). Note that \( \epsilon < r_1 \) and thus, assumption (a2) is satisfied. This concludes the proof.

8 Appendix: Proof of Proposition 4.1

To prove the claim we consider (C.28) as an operator on a Sobolev space and apply the Schauder fixed point theorem [16]. The reason for embedding the solution in a Sobolev space and for defining the inner product with a factor 2 as is done below is that we need the norm on the space to be an upper bound on the supremum norm (see Lemma 8.1). This is crucial to show that the inequalities (C.31)-(C.36) imply the inequalities (C.49)-(C.54) in Lemma 8.3. The inequalities (C.49)-(C.54) are essential to the proof, but they are nontrivial to verify. The inequalities (C.31)-(C.36) on the other hand are finite dimensional and straightforward to check.

The underlying space is defined as follows: Let \( T := [0, T] \) where \( T = 2\pi/\omega \). We define \( L_2(T) \) as the space of complex valued square integrable functions on \( T \) with inner product \( \langle x, y \rangle_{L_2(T)} := \frac{1}{T} \int_0^T x(\tau)^*y(\tau)d\tau \). We introduce the Sobolev space
\[
W_2 := \{ f \in L_2(T) \mid f \text{ periodic, } \hat{f} \in L_2(T) \}
\]
and on this space we introduce the inner product
\[
\langle f, g \rangle_{W_2} := 2\left( \langle f, g \rangle_{L_2(T)} + T^2 \langle \hat{f}, \hat{g} \rangle_{L_2(T)} \right)
\]
with the corresponding norm \( \|f\|_{W_2} := \langle f, f \rangle_{W_2}^{1/2} \). The space \( W_2 \) is isometrically isomorphic with the space
\[
G := \{ \hat{f} \in l_2 \mid \sum_{n=-\infty}^{\infty} (1 + (2\pi n)^2)|\hat{f}(n)|^2 < \infty \}
\]
(where \( \hat{f} \) is a sequence of Fourier coefficients) equipped with the inner product
\[
\langle \hat{f}, \hat{g} \rangle_{G} := 2\left( \langle \hat{f}, \hat{g} \rangle_{l_2} + T^2 \langle j\omega E_q \hat{f}, j\omega E_q \hat{g} \rangle_{l_2} \right)
\]
and corresponding norm \( \|\hat{f}\|_G := \langle \hat{f}, \hat{f} \rangle_G^{1/2} \).

In the proof we will make use of the following lemmas. Proofs are found below.
Lemma 8.1. Let \( \|f\|_\infty := \max_{t \in \mathbb{T}} |f(t)| \) be the maximum norm defined on \( W_2 \). The norm satisfies
\[
\|f\|_\infty \leq \|f\|_{W_2} = \|\hat{f}\|_G.
\]

Lemma 8.2. The operator \( \mathcal{H}_1(\omega) \) defined in (C.26) is compact on \( G \).

Lemma 8.3. Suppose there are numbers \( c_i > 0, i = 1, \ldots, 5 \) such that (C.31)-(C.36) are satisfied for some \( r > 0 \). Then for any \( \delta \in W_2 \) such that \( \|\delta\|_{W_2} < r \) it holds
\[
\begin{align*}
&\sup_{\tau \in \mathbb{T}} |S(\delta(\tau)) - \Psi_N \delta(\tau)| < c_1 r^2 \quad \text{(C.49)} \\
&\sup_{\tau \in \mathbb{T}} |S'(\delta(\tau)) - \Psi_N| < c_1 r \quad \text{(C.50)} \\
&\sup_{\tau \in \mathbb{T}} \tilde{\sigma}(A(\delta(\tau)) - A(0)) < c_2 r \quad \text{(C.51)} \\
&\sup_{\tau \in \mathbb{T}} \tilde{\sigma}(D(\delta(\tau)) - D(0)) < c_3 r \quad \text{(C.52)} \\
&\sup_{\tau \in \mathbb{T}} \tilde{\sigma}(A'(\delta(\tau))) < c_4 \quad \text{(C.53)} \\
&\sup_{\tau \in \mathbb{T}} \tilde{\sigma}(D'(\delta(\tau))) < c_5. \quad \text{(C.54)}
\end{align*}
\]

Corollary 8.1. The bounds in (C.49)-(C.54) have implications in the frequency domain. For example, (C.49) implies that
\[
\sup_{\|\delta\|_G < r} \|\hat{S}(\delta) - \hat{\Psi}(\delta)\|_{L_2(\mathbb{T})} = \sup_{\|\delta\|_{W_2} < r} \|S(\delta) - \Psi_N \delta\|_{L_2(\mathbb{T})} \leq \sup_{\|\delta\|_{W_2} < r} \sup_{\tau \in \mathbb{T}} |S(\delta(\tau)) - \Psi_N \delta(\tau)| < c_1 r^2.
\]

Analogous bounds hold for (C.50)-(C.54).

Corollary 8.2. The inequality (C.51) implies that for \( r > 0 \) small enough it holds
\[
\sup_{\|\delta\|_G < r} \|(A(\delta) - A_0) \mathcal{H}_1\|_{t_2 - t_2} < 1/2. \quad \text{This implies}
\]
\[
\sup_{\|\delta\|_G < r} \left\| (I - (A(\delta) - A_0) \mathcal{H}_1)^{-1} \right\|_{t_2 - t_2} \leq \frac{1}{\sup_{\|\delta\|_G < r} 1 - \|(A(\delta) - A_0) \mathcal{H}_1\|_{t_2 - t_2}} < 2.
\]

Main part of proof

We first note that \( |S(\delta) - \Psi_N \delta| \) (which is a finite dimensional nonlinear function) is \( O(\delta) \) and thus, there is indeed a constant \( c_1 > 0 \) such that (C.31) holds for \( r \) small enough. Similar arguments apply to (C.32)-(C.36).
Let \( r_w > 0 \) and let \( r > 0 \) be such that there are constants \( c_1, \ldots, c_5 > 0 \) satisfying (C.31)-(C.36) and such that (C.29) and (C.30) hold. Let

\[
\begin{align*}
\Omega_w &= \{ \hat{\nu} \in G \mid \| \hat{\nu} \|_G \leq r_w \} \\
\Omega &= \{ \hat{\delta} \in G \mid \| \hat{\delta} \|_G \leq r \}.
\end{align*}
\]

We will now show that the fixed point equation \( \hat{\delta} = H(\hat{\delta}, \hat{\nu}) \) has a solution for all \( \hat{\nu} \in \Omega_w \). To do this we show the following claims:

(i) \( H(\Omega, \hat{\nu}) \subseteq \Omega \ \forall \ \hat{\nu} \in \Omega_w \)

(ii) \( H(\cdot, \hat{\nu}) \) is a compact operator on \( \Omega \ \forall \ \hat{\nu} \in \Omega_w \).

Since \( \Omega \) is clearly a nonempty, closed, bounded and convex subset of a Banach space the Schauder fixed point theorem [16] implies that the fixed point equation \( \hat{\delta} = H(\hat{\delta}, \hat{\nu}) \) has a solution \( \forall \ \hat{\nu} \in \Omega_w \).

We begin to show (i): We first note that

\[
\| \hat{\mathcal{F}} \mathcal{H}_1(\omega) \|_{G \rightarrow G} := \sup_{\|\hat{y}\|_G = 1} \| \hat{\mathcal{F}} \mathcal{H}_1(\omega) \hat{y} \|_G
\]

\[
\begin{align*}
&= \sup_{\|\hat{y}\|_G = 1} \sqrt{2} \sqrt{\| \hat{\mathcal{F}} \mathcal{H}_1(\omega) \hat{y} \|_{l_2}^2 + T^2 \| j \omega \hat{E}_q \hat{\mathcal{F}} \mathcal{H}_1(\omega) \hat{y} \|_{l_2}^2} \\
&\leq \sup_{\|\hat{y}\|_G = 1} \| \hat{\mathcal{F}} \mathcal{H}_1 \| \sqrt{2} \sqrt{\| \hat{y} \|_{l_2}^2 + T^2 \| j \omega \hat{E}_q \hat{y} \|_{l_2}^2} \\
&= \| \hat{\mathcal{F}} \mathcal{H}_1 \|
\end{align*}
\]

where we have used that \( \hat{\mathcal{F}} \mathcal{H}_1 \) is block diagonal so that \( j \omega \hat{E}_q \hat{\mathcal{F}} \mathcal{H}_1 \hat{y} = \hat{\mathcal{F}} \mathcal{H}_1 j \omega \hat{E}_q \hat{y} \) (see the discussion in Section 4.2.2). It follows that

\[
\| H(\hat{\delta}, \hat{\nu}) \|_G \leq \| \hat{\mathcal{F}} \mathcal{H}_1 \| (\hat{D}_0 + \hat{\Delta}_1(\hat{\delta})) \hat{\nu} + \hat{\Delta}_2(\hat{\delta}) \|_G. \tag{C.55}
\]

To bound the right hand side above we use a number of inequalities. Firstly we note that the term \( \| \hat{D}_0 \hat{\nu} \|_G \) satisfies

\[
\| \hat{D}_0 \hat{\nu} \|_G = \| D(0) \hat{\nu} \|_{W_2} \leq \hat{\sigma}(D(0)) r_w. \tag{C.56}
\]

Secondly, we note that for all \( \hat{\delta} \in \Omega \) it holds

\[
\| \hat{\Delta}_2(\hat{\delta}) \|_{l_2} = \left\| \left( I - (\hat{\mathcal{A}}(\hat{\delta}) - \hat{A}_0) \mathcal{H}_1 \right)^{-1} \hat{B} \left( \hat{S}(\hat{\delta}) - \hat{\Psi} \hat{\delta} \right) \right\|_{l_2}
\]

\[
\begin{align*}
&\leq \left\| \left( I - (\hat{\mathcal{A}}(\hat{\delta}) - \hat{A}_0) \mathcal{H}_1 \right)^{-1} \right\|_{l_2 \rightarrow l_2} \hat{\sigma}(\hat{B}) \left\| \left( \hat{S}(\hat{\delta}) - \hat{\Psi} \hat{\delta} \right) \right\|_{l_2}
\end{align*} \tag{C.57}
\]

where we have used Corollary 8.2 and in the last inequality we used (C.49) in Lemma 8.3. Thirdly we bound the term \( \hat{\Delta}_1(\hat{\delta}) \hat{\nu} \). We again use Corollary 8.2 and
for all $\hat{\delta} \in \Omega$ it holds

\[
\|\hat{\Delta}_1(\hat{\delta})\hat{\nu}\|_{l_2} = \left\| \left( I - (\hat{\mathcal{A}}(\hat{\delta}) - \hat{\mathcal{A}}_0)\mathcal{H}_1 \right)^{-1} \left( \hat{\mathcal{D}}(\hat{\delta}) - \hat{\mathcal{D}}_0 + (\hat{\mathcal{A}}(\hat{\delta}) - \hat{\mathcal{A}}_0)\mathcal{H}_1 \hat{\mathcal{D}}_0 \right) \hat{\nu} \right\|_{l_2} \\
\leq \left\| \left( I - (\hat{\mathcal{A}}(\hat{\delta}) - \hat{\mathcal{A}}_0)\mathcal{H}_1 \right)^{-1} \right\|_{l_2 \to l_2} \\
\times \left( \|\hat{\mathcal{D}}(\hat{\delta}) - \hat{\mathcal{D}}_0\|_{l_2 \to l_2} + \|(\hat{\mathcal{A}}(\hat{\delta}) - \hat{\mathcal{A}}_0)\mathcal{H}_1 \hat{\mathcal{D}}_0\|_{l_2 \to l_2} \right) \|\hat{\nu}\| \\
\leq 2 \left( \|\hat{\mathcal{D}}(\hat{\delta}) - \hat{\mathcal{D}}_0\|_{l_2 \to l_2} + \sigma(\mathcal{D}(0))\|\hat{\mathcal{A}}(\hat{\delta}) - \hat{\mathcal{A}}_0\|_{l_2 \to l_2} \right) \|\hat{\nu}\|_G \\
\leq 2(c_3 + c_2\sigma(\mathcal{D}(0)))\|\mathcal{H}_1\| \|\hat{\nu}\|_G \\n\text{(C.58)}
\]

where in the last inequality we used (C.51) and (C.52) in Lemma 8.3. Finally, we need to bound the terms $\|j\omega \hat{\mathcal{E}}_q \hat{\Delta}_2(\hat{\delta})\|_{l_2}$ and $\|j\omega \hat{\mathcal{E}}_q \hat{\Delta}_1(\hat{\delta})\hat{\nu}\|_{l_2}$. We use the bounds in Lemma 8.4 and 8.5.

Let $C_1$, $C_2$ and $C_3$ be defined as in Proposition 4.1. Using Lemma 8.4 and 8.5 and the inequalities (C.57), (C.58) we bound the terms $\|\hat{\Delta}_2(\hat{\delta})\|_G$ and $\|H(\hat{\delta}, \hat{\nu})\|_G$. These bounds are combined with inequalities (C.56) and (C.55) and we get

\[
\|H(\hat{\delta}, \hat{\nu})\|_G \leq C_1 r_w + C_2 r_w r + C_3(r) r^2 < r
\]

where the last inequality follows from the assumption (C.30). The inequality above shows that $H(\Omega, \hat{\nu}) \subseteq \Omega$ and thus we have shown (i).

We now show (ii): The operator $H(\cdot, \hat{\nu})$ can be written as the composition of two operators $H(\cdot, \hat{\nu}) = H_1 H_2(\cdot, \hat{\nu})$ where $H_1 = \mathcal{F}_{\mathcal{H}_1}$ and $H_2 = (\mathcal{D}(0) + \hat{\Delta}_1(\hat{\delta})) \hat{\nu} + \hat{\Delta}_2(\hat{\delta})$. The inequalities above show that there is a constant $\tilde{\gamma} > 0$ such that

\[
\|H_2(\hat{\delta}, \hat{\nu})\|_G < \tilde{\gamma} \quad \forall (\hat{\delta}, \hat{\nu}) \in \Omega \times \Omega_w.
\]

Since $H_1$ is block-diagonal it is easy to show that it is compact (see Lemma 8.2 for a proof). It follows that the composite operator $H(\hat{\delta}, \hat{\nu}) = H_1 H_2(\hat{\delta}, \hat{\nu})$ maps bounded sets into relatively compact sets. Since $H$ is also continuous, it follows that $H$ is compact. This concludes the proof.

**Proof of Lemma 8.1:**

We may without loss of generality assume that $|f(0)| = \min_{t \in \mathbb{T}} |f(t)|$ (we may otherwise translate the time axis since we consider periodic functions) and we note
that this assumption implies $|f(0)| \leq \|f\|_{L_2(T)}$. It holds
\[
\|f\|_{\infty} = \max_{t \in \mathbb{T}} |f(0)| + \int_{0}^{T} |\dot{f}(\tau)| d\tau \\
\leq |f(0)| + \int_{0}^{T} |\dot{f}(\tau)| d\tau \\
\leq \|f\|_{L_2(T)} + T\|\dot{f}\|_{L_2(T)} \\
\leq \sqrt{2} \sqrt{\|f\|_{L_2(T)}^{2} + T^2 \|\dot{f}\|_{L_2(T)}^{2}} \\
= \|f\|_{W_2}
\]
where on the second line we used the Schwarz inequality.

**Proof of Lemma 8.2:**

Since $\mathcal{H}_1$ is block diagonal (see (C.27)) and the diagonal entries are bounded, it is clear that $\mathcal{H}_1$ maps $G$ into $G$. To prove that $\mathcal{H}_1$ is compact we show that there is a sequence of finite rank bounded linear operators that is uniformly operator convergent to $\mathcal{H}_1$.

Let
\[
\mathcal{H}_{1,N}(\omega) := \pi_N \mathcal{H}_1(\omega) \pi_N.
\]
Clearly, $\mathcal{H}_{1,N}$ is a finite rank bounded linear operator on $l_2$. For any $\hat{y} \in l_2$ it holds
\[
\| (\mathcal{H}_1 - \mathcal{H}_{1,N}) \hat{y} \|_{l_2}^{2} = \sum_{|k| > N} |(j \omega k I - A(0) - B \Psi_N \mathcal{F})^{-1} y_k|^{2} \\
= \sum_{|k| > N} \left| \left( I - \frac{1}{j \omega k} (A(0) + B \Psi_N \mathcal{F}) \right)^{-1} y_k \right|^{2} \frac{1}{\omega^2 k^2} \\
\leq \sum_{|k| > N} \left| \left( I - \frac{1}{j \omega k} (A(0) + B \Psi_N \mathcal{F}) \right)^{-1} \right|^{2} \frac{1}{\omega^2 k^2} |y_k|^{2} \\
\leq \sum_{|k| > N} \left( \frac{1}{1 - \frac{1}{\omega^2 k^2} \|A(0) + B \Psi_N \mathcal{F}\|} \right)^{2} \frac{1}{\omega^2 k^2} |y_k|^{2} \\
\leq \left( \frac{1}{\omega - \frac{1}{N} \|A(0) + B \Psi_N \mathcal{F}\|} \right)^{2} \sum_{|k| > N} \frac{1}{k^2} |y_k|^{2} \\
\leq \left( \frac{2}{N + 1} \right) \frac{2 \|\hat{y}\|^{2}}{N + 1} 
\]
where we have used that \( \| (\frac{1}{j\omega k} (A(0) + B\Psi_N F) \| < 1 \) for \( |k| > N \) and \( N \) large enough. Similarly it holds
\[
\| j\omega \hat{K}\big(\mathcal{H}_1 - \mathcal{H}_{1,N}\big) \|_2^2 = \| (\mathcal{H}_1 - \mathcal{H}_{1,N}) j\omega \hat{K} \|_2^2 \\
\leq \left( \frac{1}{\omega - \frac{1}{N} \| A(0) + B\Psi_N F \|} \right)^2 \frac{2}{N+1} \| j\omega \hat{K} \|_2^2
\]
and thus
\[
\| (\mathcal{H}_1 - \mathcal{H}_{1,N}) \|_2^2 \leq \left( \frac{1}{\omega - \frac{1}{N} \| A(0) + B\Psi_N F \|} \right)^2 \frac{2}{N+1} \| \hat{g} \|_G^2 \to 0, \quad N \to \infty.
\]
The last inequality above shows that \( \mathcal{H}_{1,N} \) converges to \( \mathcal{H}_1 \) uniformly.

**Proof of Lemma 8.3:**

Here we give a proof of (C.49). The proofs of (C.50)-(C.54) are analogous. As stated in Lemma 8.1, the Sobolev norm satisfies \( \| \delta \|_\infty \leq \| \delta \|_{W_2} \). It follows that for any \( \delta \in W_2 \) satisfying \( \| \delta \|_{W_2} < r \) it holds
\[
\sup_{\tau \in \mathbb{T}} |S(\delta(\tau)) - \Psi_N \delta(\tau)| \leq \sup_{\| \delta \|_{W_2} < r} \sup_{\tau \in \mathbb{T}} |S(\delta(\tau)) - \Psi_N \delta(\tau)| \\
\leq \sup_{\| \delta \|_\infty < r} \sup_{\tau \in \mathbb{T}} |S(\delta(\tau)) - \Psi_N \delta(\tau)| \\
= \sup_{|\delta| < r} |S(\delta) - \Psi_N \delta| \\
< c_1 r^2
\]
where the last inequality is by assumption (C.31). Note that in the first two lines, \( \delta \) is a function defined on \( \mathbb{T} \) while in the last inequality \( \delta \) is a scalar. It can be shown that the finite dimensional nonlinear function \( |S(\delta) - \Psi_N \delta| \) is \( O(\delta) \) and indeed there exists a constant \( c_1 > 0 \) such that (C.31) is valid for small \( r \).

**Lemma 8.4.** *For all \( \hat{\delta} \in \Omega \), the term \( j\omega \hat{E}_q \hat{\Delta}_2(\hat{\delta}) \) satisfies*
\[
\| j\omega \hat{E}_q \hat{\Delta}_2(\hat{\delta}) \|_{L_2} \leq \frac{2c_1 \bar{\sigma}(B)}{T} \left( \frac{1}{2\sqrt{2}} + c_2 T \| j\omega \hat{E}_q \mathcal{H}_1 \| r + c_4 \left( \| \mathcal{H}_1 \|^2 + T^2 \| j\omega \hat{E}_q \mathcal{H}_1 \|^2 \right)^{1/2} r \right) r^2.
\]

**Proof of Lemma 8.4:**

We note that \( \hat{v}_2 := \hat{\Delta}_2(\hat{\delta}) \) satisfies
\[
\hat{v}_2 = (\hat{A}(\hat{\delta}) - \hat{A}_0) \mathcal{H}_1 \hat{v}_2 + \hat{B} \left( \hat{S}(\hat{\delta}) - \hat{\Psi} \hat{\delta} \right)
\]
and the time domain representation \( v_2 \) satisfies

\[
v_2 = (A(\delta) - A(0))y_2 + B(S(\delta) - \Psi_N \delta)
\]

where \( y_2 \) is the time domain representations of \( H_1 \hat{v}_2 \). For any \( \delta \in \Omega \) it holds

\[
\| j\omega \hat{E}_q \hat{\Delta}_2(\delta) \|_{l_2} = \| j\omega \hat{E}_q \hat{v}_2 \|_{l_2} = \| \hat{v}_2 \|_{L_2(T)} \\
\leq \| A'(\delta) \delta y_2 \|_{L_2(T)} + \| (A(\delta) - A(0)) \dot{y}_2 \|_{L_2(T)} \\
+ \sigma(B) \| (S'(\delta) - \Psi_N) \hat{\delta} \|_{L_2(T)}.
\]

The terms in the sum above are bounded as follows. First we note that

\[
\| A'(\delta) \delta y_2 \|_{L_2(T)} \leq \sup_{\tau \in T} \| A'(\delta(\tau)) \| \sup_{\tau \in T} \| y_2(\tau) \| \| \delta \|_{L_2(T)} \\
\leq c_4 \frac{2c_1 \sigma(B)}{T} \left( \| H_1 \|^2 + T^2 \| j\omega \hat{E}_q \hat{H}_1 \|^2 \right)^{1/2} \frac{1}{r^3}
\]

where we have used inequality (C.53) in Lemma 8.3, the fact that \( \| \delta \|_{L_2} < r/(\sqrt{2}T) \) and

\[
\sup_{\tau \in T} \| y_2(\tau) \| \leq \| H_1 \hat{v}_2 \|_{C} \\
= \sqrt{2} \left( \| H_1 \hat{v}_2 \|^2_{l_2} + T^2 \| j\omega \hat{E}_q \hat{H}_1 \hat{v}_2 \|^2_{l_2} \right)^{1/2} \\
\leq \sqrt{2} \left( \| H_1 \|^2 + T^2 \| j\omega \hat{E}_q \hat{H}_1 \|^2 \right)^{1/2} \| \hat{v}_2 \|_{l_2} \\
\leq \sqrt{2} \left( \| H_1 \|^2 + T^2 \| j\omega \hat{E}_q \hat{H}_1 \|^2 \right)^{1/2} 2c_1 \sigma(B) r^2
\]

where Lemma 8.1 and the bound on \( \| v_2 \|_{l_2} = \| \hat{\Delta}_2 \|_{l_2} \) in (C.57) were used. Secondly we note that

\[
\| (A(\delta) - A(0)) \dot{y}_2 \|_{L_2(T)} \leq \sup_{\tau \in T} \| A(\delta) - A(0) \| \| \dot{y}_2 \|_{L_2(T)} \\
\leq c_2 c_1 \sigma(B) \| j\omega \hat{E}_q \hat{H}_1 \| r^3
\]

where we have used inequality (C.51) in Lemma 8.3 and

\[
\| \dot{y}_2 \|_{L_2(T)} = \| j\omega \hat{E}_q \hat{H}_1 \hat{v}_2 \|_{l_2} \leq \| j\omega \hat{E}_q \hat{H}_1 \| \| \hat{v}_2 \|_{l_2} \quad \text{and the bound on} \quad \| \hat{v}_2 \|_{l_2} = \| \hat{\Delta}_2 \|_{l_2} \quad \text{in} \quad (C.57).
\]

Finally we note that

\[
\| (S'(\delta) - \Psi_N) \hat{\delta} \|_{L_2(T)} \leq \sup_{\tau \in T} \| S'(\delta(\tau)) - \Psi_N \| \| \hat{\delta} \|_{L_2(T)} \\
\leq \frac{1}{\sqrt{2} \frac{1}{T^2} r^2}
\]

where we have used inequality (C.50) in Lemma 8.3 and the fact that \( \| \hat{\delta} \|_{L_2} < r/(\sqrt{2}T) \). Combining these bounds we have the result above.
Lemma 8.5. For all \( \hat{\delta} \in \Omega \), the term \( j \omega \hat{E}_q \hat{\Delta}_1(\hat{\delta}) \hat{v} \) satisfies

\[
\| j \omega \hat{E}_q \hat{\Delta}_1(\hat{\delta}) \hat{v} \|_{L_2} \leq \frac{r_w r}{T} \left( \frac{c_0}{\sqrt{2}} + 2\pi c_3 + \left( 2c_3 r + (1 + 2c_2 \| H_1 \| r) \bar{\sigma}(D(0)) \right) \right) \times \left( c_2 T \| j \omega \hat{E}_q H_1 \| + c_4 \left( \| H_1 \|^2 + T^2 \| j \omega \hat{E}_q H_1 \|^2 \right)^{1/2} \right).
\]

Proof of Lemma 8.5:

We note that \( \hat{v}_1 := \hat{\Delta}_1(\hat{\delta}) \hat{v} \) satisfies

\[
\hat{v}_1 = (\hat{A}(\hat{\delta}) - \hat{A}(0)) \hat{H}_1 \hat{v}_1 + \left( \hat{D}(\hat{\delta}) - \hat{D}(0) \right) \hat{v} + \left( \hat{A}(\hat{\delta}) - \hat{A}(0) \right) \hat{H}_1 \hat{D}(0) \hat{v}
\]

and the time domain representation \( v_1 \) satisfies

\[
v_1 = (A(\delta) - A(0)) (y_1 + y_w) + (D(\delta) - D(0)) w
\]

where \( y_1 \) and \( y_w \) are the time domain representations of \( H_1 \hat{v}_1 \) and \( H_1 \hat{D}(0) \hat{v} \) respectively. It holds

\[
\| j \omega \hat{E}_q \hat{\Delta}_1(\hat{\delta}) \hat{v} \|_{L_2} = \| j \omega \hat{E}_q \hat{v}_1 \|_{L_2} = \| \hat{v}_1 \|_{L_2(T)}
\]

\[
\leq \| A'(\delta) \hat{\delta}(y_1 + y_w) \|_{L_2(T)} + \| (A(\delta) - A(0))(\hat{y}_1 + \hat{y}_w) \|_{L_2(T)}
\]

\[
+ \| D'(\delta) \hat{\delta} \hat{w} \|_{L_2(T)} + \| (D(\delta) - D(0)) \hat{v} \|_{L_2(T)}.
\]

The terms in the sum above are bounded as follows. First we note that

\[
\| A'(\delta) \hat{\delta}(y_1 + y_w) \|_{L_2(T)} \leq \sup_{\tau \in T} \bar{\sigma}(A'(\delta)(\tau)) \left( \sup_{\tau \in T} |y_1(\tau) + y_w(\tau)| \| \hat{\delta} \|_{L_2(T)} \right)
\]

\[
\leq \frac{c_4}{T} \left( \| H_1 \|^2 + T^2 \| j \omega \hat{E}_q H_1 \|^2 \right)^{1/2}
\]

\[
\times \left( 2c_3 r + (1 + 2c_2 \| H_1 \| r) \bar{\sigma}(D(0)) \right) r_w r
\]

where we have used inequality (C.53) in Lemma 8.3, the fact that \( \| \hat{\delta} \|_{L_2(T)} < r/(\sqrt{2} T) \) and

\[
\sup_{\tau \in T} |y_1(\tau) + y_w(\tau)| \leq \| y_1 \|_\infty + \| y_w \|_\infty
\]

\[
\leq \sqrt{2} \left( \| H_1 \hat{v}_1 \|^2_{L_2} + T^2 \| j \omega \hat{E}_q H_1 \hat{v}_1 \|^2_{L_2} \right)^{1/2}
\]

\[
+ \sqrt{2} \left( \| H_1 \hat{D}(0) \hat{v} \|^2_{L_2} + T^2 \| j \omega \hat{E}_q H_1 \hat{D}(0) \hat{v} \|^2_{L_2} \right)^{1/2}
\]

\[
\leq \sqrt{2} \left( \| H_1 \|^2 + T^2 \| j \omega \hat{E}_q H_1 \|^2 \right)^{1/2}
\]

\[
\times \left( 2(c_3 + c_2 \| H_1 \| \bar{\sigma}(D(0))) r_w r + \bar{\sigma}(D(0)) r_w \right)
\]
where we have used inequality (C.51) in Lemma 8.3 and the bound on $\|\hat{v}_1\|_2 = \|\hat{\Delta}^1\hat{v}\|_2$ in (C.58). Secondly we note the inequality
\[
\|\varepsilon(\delta(\tau))\|_{L^2(T)} \leq \sup_{\tau \in T} \sigma(\varepsilon(\delta(\tau))) \|\hat{v}_1 + \hat{y}\|_{L^2(T)} \\
\leq c_2\|\hat{y}\|_{L^2(T)} + (1 + 2c_2\|\hat{H}_1\|\|\hat{r}\|_{L^2(T)}) r_w r
\]
where we have used inequality (C.51) in Lemma 8.3. Thirdly we note that
\[
\|D'(\delta)\hat{v}\|_{L^2(T)} \leq \sup_{\tau \in T} \sigma(D'(\delta(\tau))) \sup_{\tau \in T} |\hat{v}(\tau)| \|\hat{\delta}\|_{L^2(T)} \\
\leq c_3 r_w r / (\sqrt{2T})
\]
where we have used inequality (C.54) in Lemma 8.3 and $\|\hat{\delta}\|_{L^2(T)} \leq r / (\sqrt{2T})$. Lastly we note that
\[
\|D(\delta) - D(0)\|_{L^2(T)} \leq \sup_{\tau \in T} \sigma(D(\delta(\tau)) - D(0)) \|\hat{v}\|_{L^2(T)} \\
\leq c_3 r_w r
\]
where we have used inequality (C.52) in Lemma 8.3. Combining these bounds we have the result above.

9 Appendix: Approximation of Harmonic Balance Equations

The approximation of the Fourier coefficient $S_k$ of $S$ is immediate from (C.21). Again using the approximation (C.21) of $S_{av}(\delta)$, the Fourier coefficients $A_k$ and $D_k$ are approximated by linear functions of $\hat{\delta}$ as

\[
A_k = \frac{1}{T} \int_0^T A(\tau) e^{-j k \omega \tau} d\tau \\
= \frac{1}{T} \int_0^T (I \otimes A_0 + (I \otimes A_1) T_N[S_{av}(\delta(\tau))] e^{-j k \omega \tau} d\tau \\
\approx \frac{1}{T} \int_0^T \left( I \otimes A_0 + (I \otimes A_1) \left( T_N[S_{av}(0)] + T_N[\Psi] \sum_{n=-\infty}^\infty \delta_n e^{j n \omega \tau} \right) \right) e^{-j k \omega \tau} d\tau \\
= \begin{cases} 
I \otimes A_0 + (I \otimes A_1) (T_N[S_{av}(0)] + \delta_0 T_N[\Psi]) , & k = 0 \\
\delta_k (I \otimes A_1) T_N[\Psi] , & k \neq 0 
\end{cases}
\]
and

\[ D_k = \frac{1}{T} \int_0^T D(\delta(\tau)) e^{-j\omega \tau} d\tau \]

\[ = \frac{1}{T} \int_0^T (I \otimes D_0 + (I \otimes D_1) T_N[S_{av}(\delta(\tau))] e^{-j\omega \tau} d\tau \]

\[ \approx \frac{1}{T} \int_0^T \left( I \otimes D_0 + (I \otimes D_1)(T_N[S_{av}(0)] + T_N[\Psi] \sum_{n=-\infty}^{\infty} \delta_n e^{jn\omega \tau}) \right) e^{-j\omega \tau} d\tau \]

\[ = \begin{cases} 
I \otimes D_0 + (I \otimes D_1) (T_N[S_{av}(0)] + \delta_0 T_N[\Psi]), & k = 0 \\
\delta_k (I \otimes D_1) T_N[\Psi], & k \neq 0.
\end{cases} \]

Here we have used the linearity of \( T_N[\cdot] \).

## 10 Appendix: Auxiliary lemmas and proofs

**Proof of Lemma 7.1:**

Boundedness of \( \hat{A} \) follows from the fact that \( s \) and \( s_{\delta^0} \) take value in the set \( \{0, 1\} \) and the first two equalities follow directly from the Parseval identity. The inequality is shown below.

The fact that \( s \) and \( s_{\delta^0} \) are piecewise constant guarantees the existence of the maxima defined in the lemma. The Parseval identity implies

\[ ||\hat{A}|| := \sup_{\hat{x} \in l_2, ||\hat{x}||_{l_2}} ||\hat{A}\hat{x}||_{l_2} \]

\[ = \sup_{x \in L_2, ||x||_{L_2}=1} ||Ax||_{L_2[t-T_s,t]} \]

Firstly, we bound this expression from above according to

\[ \sup_{x \in L_2, ||x||_{L_2}=1} ||Ax||_{L_2} = \sup_{x \in L_2, ||x||_{L_2}=1} \left( \frac{1}{T_s} \int_{t-T_s}^{t} |A(\tau)x(\tau)|^2 d\tau \right)^{1/2} \]

\[ \leq \sup_{x \in L_2, ||x||_{L_2}=1} \max_{\tau \in [t-T_s,t]} \tilde{A}(\tau) \left( \frac{1}{T_s} \int_{t-T_s}^{t} |x(\tau)|^2 d\tau \right)^{1/2} \]

\[ = \max_{\tau \in [t-T_s,t]} \tilde{A}(\tau) \]

Then we bound the expression from below as follows: Since \( s - s_{\delta^0} \) is piecewise constant, there is an interval \([a, b] \subset [t-T_s,t], b > a\) such that

\[ \tilde{A}(\tau) = \max_{s \in [t-T_s,t]} \tilde{A}(s) \]
for all \( \tau \in [a, b] \). Let \( A = U \Sigma V' \) be the singular value decomposition of \( A \) and let \( u_1 \) and \( v_1 \) be the first column of \( U \) and \( V \) respectively. Define \( x^* := v_1 \delta \) where

\[
\delta(\tau) = \begin{cases} 
(T_s/(b-a))^{1/2}, & \tau \in [a, b] \\
0, & \text{otherwise}
\end{cases}
\]

It holds

\[
\sup_{x \in L_2, \|x\|_{L_2}=1} \|Ax\|_{L_2} \geq \|Ax^*\|_{L_2} \geq \|\sigma \delta u_1\|_{L_2} = \left( \frac{1}{T_s} \int_a^b \sigma(\tau)^2 \frac{T_s}{b-a} d\tau \right)^{1/2} = \max_{\tau \in [t-T_s, t]} \sigma(A(\tau))
\]

Combining the inequalities above yields the result. The proof of the inequalities involving \( B \) is analogous.

**Proof of Lemma 7.2:**

Let \( x \) be a solution of (C.3). Partial integration implies

\[
\langle x \rangle_n(t) = \frac{j}{n \omega_s T_s} \left( [x(t) e^{-jn \omega_s \tau}]_{t-T_s}^t - \int_{t-T_s}^t \dot{x}(\tau) e^{-jn \omega_s \tau} d\tau \right)
\]

From the Schwarz inequality we have

\[
\| \langle x \rangle_n(t) \| \leq \frac{1}{|n| \omega_s T_s} \left( \|x(t) - x(t - T_s)\| + \int_{t-T_s}^t \|\dot{x}(\tau)\| d\tau \right) \leq \frac{1}{|n| \pi} \int_{t-T_s}^t \|\dot{x}\| d\tau \leq \frac{T_s}{|n| \pi} \|\dot{x}\|_{L_2[t-T_s, t]} \leq \frac{T_s}{|n| \pi} (\|A\| \|x\|_{L_2[t-T_s, t]} + \|B\| + \|D\| r_w)
\]

Applying the Parseval identity \( \|x\|_{L_2[t-T_s, t]} = \|\dot{x}\|_{L_2} \) yields the result.

**Proof of Lemma 7.3:**

For any \( t \in [t_0, t_1] \) the quantity \( \|(I - \pi_N) \dot{x}(t)\|^2 \) is the limit of a sequence \( \{f_M(t)\} \) where

\[
f_M(t) := \|(\pi_M - \pi_N) \dot{x}(t)\|^2
\]
are differentiable functions on $[t_0, t_1]$. Consider the sequence \( \{ f'_M \} \) where
\[
f'_M(t) = 2\mathcal{R}\{ (\pi_M - \pi_N) \hat{x}, (j\omega_K + \hat{A}) \hat{x} + \hat{B} + \hat{D}\hat{w} \}
\]
\[
= 2\mathcal{R}\{ (\pi_M - \pi_N) \hat{x}, \hat{A}\hat{x} + \hat{B} + \hat{D}\hat{w} \}
\]
are the derivatives of \( f_M \). The sequence \( \{ f'_M \} \) converges uniformly on \([t_0, t_1]\). This can be seen by bounding \( |f'_{M_2} - f'_{M_1}| \) according to
\[
|f'_{M_2} - f'_{M_1}| = 2\left| \mathcal{R}\{ (\pi_{M_2} - \pi_{M_1}) \hat{x}, \hat{A}\hat{x} + \hat{B} + \hat{D}\hat{w} \} \right|
\]
\[
\leq 2\left( (\pi_{M_2} - \pi_{M_1}) \hat{x}, (\|A\|\|\hat{x}\| + \|B\| + \|D\|r_w)^2 \right) \left| \frac{1}{M_1} - \frac{1}{M_2} \right|^{1/2}
\]
where we have used the result of Lemma 7.2 and the assumption that \( \hat{x} \in \Omega \). Since \( \{ f'_M \} \) converges uniformly we conclude that
\[
\frac{d}{dt}\| (I - \pi_N) \hat{x} \|^2 = \lim_{M \to \infty} f'_M = 2\mathcal{R}\{ (I - \pi_N) \hat{x}, \hat{A}\hat{x} + \hat{B} + \hat{D}\hat{w} \}
\]

**Lemma 10.1.** Let \( x \) be a solution of (C.3) and assume \( \sup_{t} |w(t)| \leq r_w \). The norm
\[
\|x\|_{t, \infty} := \sup_{t \in [t - T_s, t]} |x(\tau)|
\]
can be estimated as
\[
\|x\|_{t, \infty} \leq (1 + T_s \|A\|) \|\hat{x}\| + T_s (\|B\| + \|D\|r_w).
\]

**Proof of Lemma 10.1:**

Let \( \tau_0 \in [t - T_s, t] \) be determined by
\[
\tau_0 = \arg\min_{\tau \in [t - T_s, t]} |x(\tau)|
\]
(where we have used that \( x \) is continuous). It holds
\[
x(\tau) = x(\tau_0) + \int_{\tau_0}^{\tau} \hat{x}(\tau) d\tau
\]
and we get
\[
\|x\|_{t, \infty} \leq |x(\tau_0)| + \sup_{\tau \in [t - T_s, t]} \left| \int_{\tau_0}^{\tau} \hat{x}(\tau) d\tau \right|
\]
\[
\leq \|x\|_{L^2[t - T_s, t]} + T_s \|\hat{x}\|_{L^2[t - T_s, t]}
\]
where we have used that $|x(\tau_0)| \leq \|x\|_{L_2[0, T_s]}$ and the Cauchy-Schwartz inequality was used in the second term. Finally, it holds
\[
\|x\|_{t, \infty} \leq (1 + T_s \|A\|) \|x\|_{L_2[t-T_s, t]} + T_s (\|B\| + \|D\|) r_w
\leq (1 + T_s \|A\|) \|\dot{x}\| + T_s (\|B\| + \|D\|) r_w).
\]

**Lemma 10.2.** Let $\dot{x}$ be a solution of the DPM (C.7) and assume $\sup_t \|\dot{w}(t)\| \leq r_w$. For $t \geq t_0$ it holds
\[
\|\dot{x}(t) - \dot{x}(t_0)\| \leq \left( (2T_s \|A\| + 1) e^\|A\|(t-t_0) - 1 \right) \left( \|\dot{x}(t)\| + (\|B\| + \|D\|) r_w/\|A\| \right).
\]

**Proof of Lemma 10.2:**

To prove the claim we move into the time domain. However, it should be noted that the quantity $\|\dot{x}(t) - \dot{x}(t_0)\|_{L_2}$ is not equal to $\|x(t, \tau) - x(t_0, \tau)\|_{L_2}$. In fact, the time domain expressions are not directly comparable since they are defined on different sections of the time axis.

Let $t \geq t_0$, we have
\[
\|\dot{x}(t) - \dot{x}(t_0)\|_{L_2} = \|y(\tau) - x(t_0, \tau)\|_{L_2[-T_s, 0]}
= \|y(\tau) - x(t - (t - t_0), \tau)\|_{L_2[-T_s, 0]},
\]
where
\[
y(\tau) := x(t, t_0 + \tau) = \sum_{n=-\infty}^{\infty} \langle x_n(t) \rangle e^{in\omega_0 t} e^{in\omega \tau}.
\]

Let $z(\eta) = x(t - \eta, \tau)$, then $z$ satisfies
\[
\dot{z} = -Az - B - Dw, \quad z(0) = x(t, \tau)
\]
For $\eta \in [0, t-t_0]$ we have the inequality
\[
\|z(\eta) - y\|_{L_2} = \|z(0) + \int_0^\eta \dot{z}(\sigma)d\sigma - y\|_{L_2}
\leq \|z(0) - y\|_{L_2} + \int_0^\eta \|\dot{z}(\sigma)\|_{L_2}d\sigma.
\]

We use the result in Lemma 10.4 to bound the first term in the sum and we have
\[
\|z(\eta) - y\|_{L_2} \leq 2T_s (\|A\| \|\dot{x}\|_{L_2} + \|B\| + \|D\| r_w) + \int_0^\eta \|\dot{z}(\sigma)\|_{L_2} d\sigma
\leq 2T_s (\|A\| \|\dot{x}\|_{L_2} + \|B\| + \|D\| r_w)
+ \|A\| \int_0^\eta \|z(\sigma)\|_{L_2} d\sigma + \eta(\|B\| + \|D\| r_w)
\leq (2T_s + \eta) (\|A\| \|\dot{x}\|_{L_2} + \|B\| + \|D\| r_w) + \|A\| \int_0^\eta \|z(\sigma) - y\|_{L_2} d\sigma
\]
where we have used that \( \|y\|_{L_2} = \|\dot{x}\|_{L_2} \) and \( \|A\| = \|A\| \). The result follows by applying the Grönwall inequality to (C.60) and choosing \( t = t_0 \). This yields

\[
\|z(t - t_0) - y\|_{L_2} \leq \left( (2T_s\|A\| + 1)e^{\|A\|(t-t_0)} - 1 \right) \|\dot{x}(t)\|
+ \left( (2T_s\|A\| + 1)e^{\|A\|(t-t_0)} - 1 \right) (\|B\| + \|D\|r_w)/\|A\|
\]

which together with inequality (C.59) yields the result.

**Lemma 10.3.** Let \( t \in [kT_s, (k + 1)T_s] \) and let \( d(t) = \text{sat}(d^0 + \mathcal{F}_N \dot{x}(t)) \) and \( \delta(t) = d(t) - d^0 \), it holds

\[
\|\dot{s}(t) - S_{av}(\delta(t))\|_{L_2} \leq |d_{k-1} - d(t)| + |d_k - d(t)|.
\]

**Proof of Lemma 10.3:**

It holds

\[
\|\dot{s}(t) - S_{av}(\delta(t))\|_{L_2} \leq \left\| \sum_{n=-\infty}^{\infty} \langle s, n \rangle (t)e^{jn\omega t} - \sum_{n=-\infty}^{\infty} s_{av,n}(d(t))e^{jn\omega t} \right\|_{L_2[t-T_s,t]}
\leq \|s(\tau) - s_{d(t)}(\tau)\|_{L_2[(k-1)T_s,kT_s]} + \|s(\tau) - s_{d(t)}(\tau)\|_{L_2[kT_s,(k+1)T_s]}
= |d_{k-1} - d(t)| + |d_k - d(t)|
\]

where the last inequality follows immediately from Fig. C.14.

![Diagram](image)

Figure C.14: The pulse of the periodic switching function \( s_{d(t)} \) goes down at a time instance determined by \( d(t) \) while the pulse of \( s \) goes down at time instances defined by the \( d_k \), which are determined at the sampling times. We have that \( \|s(\tau) - s_{d(t)}(\tau)\|_{L_2[kT_s,(k+1)T_s]} = |d_k - d(t)| \) and similarly for the interval \( [(k-1)T_s, kT_s] \).

**Lemma 10.4.** Let \( x \) be the Fourier series representation of a solution \( x \) of (C.3) where sup \( t \) \( |w(t)| \leq r_w \). For \( t - t_0 \geq 0 \) it holds

\[
\|x(t, \tau) - x(t, \tau - (t - t_0))\|_{L_2[-T_s,0]} \leq 2T_s(\|A\|\|\dot{x}(t)\| + \|B\| + \|D\|r_w).
\]
Proof of Lemma 10.4:

The quantity $x(t, \tau) - x(t, \tau - (t - t_0))$ is the difference between a periodic function and a shifted version of the same function, see Fig. C.15 for an illustration. Let $\xi := t - t_0 \mod T_s$. We use the $L_2$-equivalence between the solution $x$ and the Fourier series representation $x$ to establish

$$\|x(t, \tau) - x(t, \tau - (t - t_0))\|_{L_2[-T_s, 0]} = \|x(t, \tau) - x(t, \tau - \xi)\|_{L_2[-T_s, 0]}$$

$$= \left(\frac{1}{T_s} \int_{t-T_s}^{t-T_s+\xi} |x(\tau) - x(\tau + T_s - \xi)|^2 d\tau \right)^{1/2} + \frac{1}{T_s} \int_{t-T_s+\xi}^{t} |x(\tau) - x(\tau - \xi)|^2 d\tau \right)^{1/2}.$$

It then follows

$$\|x(t, \tau) - x(t, \tau - (t - t_0))\|_{L_2[-T_s, 0]} \leq \max_{\tau \in [t-T_s, t-T_s+\xi]} |x(\tau) - x(\tau + T_s - \xi)| + \max_{\tau \in [t-T_s+\xi, t]} |x(\tau) - x(\tau - \xi)|$$

$$\leq 2 \left(\frac{1}{T_s} \int_{t-T_s}^{t} \int_{t-T_s+\xi}^{\tau+T_s-\xi} |\dot{x}(\sigma)| d\sigma + \max_{\tau \in [t-T_s+\xi, t]} \int_{\tau-\xi}^{\tau} |\dot{x}(\sigma)| d\sigma \right)$$

Applying the Schwarz inequality to the last term above yields the result.

Figure C.15: Solid line; $x(t, \tau)$. Dashed line; $x(t, \tau - (t - t_0))$. The shift is defined as $\xi := t - t_0 \mod T_s$.

11 References


Sampled Data Control of DC-DC Converters

Stefan Almér

Abstract

The paper considers control synthesis for switched mode DC-DC converters based on a sampled data model. We first give a short introduction to DC-DC converters. We then introduce the sampled data model which gives an exact description of the converter state at the switching instances and also includes a lifted signal which represents the inter sampling behavior. Within the sampled data framework we consider $\mathcal{H}_\infty$ design to achieve robustness to disturbances and load variations. The suggested controller is applied to two benchmark examples and performance is tested in both simulations and in experiments.

Keywords: Switched mode DC-DC converters, Sampled data modeling, $\mathcal{H}_\infty$ control synthesis

1 Introduction

The paper summarizes theoretical and practical work which has been done on the problem of control of switched mode DC-DC converters. The control approach considered is based on the sampled data model which was developed in cooperation with Hisaya Fujioka and Chung-Yao Kao in the papers [5–9]. In this paper we introduce the basic concept of the sampled data model and motivate why it is useful. Most of the technical detail of [5–9] is omitted.

In particular, we consider the $\mathcal{H}_\infty$ sampled data control approach for robust tracking developed in [5,9]. The $\mathcal{H}_\infty$ sampled data approach results in a linear controller. This controller is augmented with additional control structure which provides nonlinear compensation. The additional control structure is motivated mainly because the $\mathcal{H}_\infty$ sampled data problem formulation does not take state constraints into account.

The resulting augmented controller was applied to two benchmark examples (a step-down and a step-up converter) that were defined by Georgios Papafotiou and
Manfred Morari as part of a project within the HYCON network. Simulation results are found in the papers [1,2] which compared control performance of a number of different control approaches. The controller has also been applied in experimental setups of a step-down and a step-up converter. This work was done in cooperation with Sébastien Mariéthoz and is presented in the papers [16,17].

The paper is organized as follows. Section 2 gives a brief introduction to switched mode DC-DC converters and gives some comment on the step-down and step-up topology. Section 3 gives a review of the most common ways of modeling DC-DC converters and introduces the basic concept of the sampled data model. Section 4 introduces the $H_\infty$ sampled data problem formulation which is used for control synthesis. Finally, in Section 5 and 6 the controller is applied to a step-down and a step-up converter and performance is tested in simulations and experiments.

2 Switched mode DC-DC converters

Switched mode DC-DC converters [12,19] are used in a wide variety of applications. Their low weight and high efficiency have been utilized in applications ranging from computer power supplies and cellular phones to electrical motor drives.

The purpose of a DC-DC converter is to transform a given DC voltage (the source) into a DC voltage (the output) at a given reference level. A switched mode DC-DC converter transforms the source voltage by switching between different circuit topologies in a cyclic manner. Ideally, periodic switching results in a periodic solution. For sufficiently high switching frequency the amplitude of this solution will be small and thus, the output voltage will have a dominant DC component and small ripple.

All switched mode DC-DC converters consist of two basic components; a switching stage which is operated at high frequency and a low pass LC stage which filters out the DC component of the output voltage. Switched DC-DC converters are commonly classified by the following two properties of the switching stage:

1. The switching stage is either fixed or variable frequency.

2. The switching stage induces a discontinuous conduction mode or it does not.

In a fixed frequency converter the switch is operated periodically. The time axis is partitioned into intervals of equal length $T_s > 0$ as illustrated in Fig. D.1. For each time interval a parameter $d_k \in [0, 1]$ is chosen. This parameter is called the duty cycle and it determines the fraction of the time interval the switch is in the on position. In the examples below, the switch position is represented by a switch function $s(t)$ which takes values zero (representing off-mode) or one (representing on-mode).

In a variable frequency converter (or resonant converter) the switching is not necessarily periodic. The switch changes position whenever a control trajectory crosses zero.
The advantage of variable frequency switching is that the switching losses can be made smaller compared to fixed frequency switching. The control trajectory can be chosen such that the current passing through the switch will be small when the switch changes position. On the other hand, the variable switching frequency makes electro magnetic interference (EMI) design and output filter design more difficult [20].

![Diagram of fixed frequency switching](image)

Figure D.1: Operation principle of fixed frequency switching. The switch position is represented by the switch function \( s(t) \) where \( s = 1 \) represents the on-position and \( s = 0 \) represents the off-position.

Depending on how the switching stage is implemented, a switched DC-DC converter may or may not have a discontinuous conduction mode. If the switch is such that the inductor current cannot be negative, this gives rise to a third mode of operation; the discontinuous conduction mode. The dynamics of the system when restricted to one of the switch modes (on or off) becomes a hybrid system where the vector field has variable structure depending on the state.

If the switching stage is such that the inductor current can flow in either direction there is no discontinuous conduction mode and the dynamics of each switch mode are affine and time invariant. Such converters are called synchronous or bi-directional.

In this paper we consider control design for fixed frequency synchronous converters. The principle of operation of such devices is further illustrated by the following examples.

**Example 1: Synchronous step-down (buck) converter**

The first example we consider is the synchronous buck converter. The topology is depicted in Fig. D.2. The purpose of the buck converter is to provide the load with a DC voltage which is lower than the source. If the duty cycle is constant and equal to \( d \in [0, 1] \) so that \( d_k = d \ \forall k \), then the switching stage produces a voltage in the shape of a periodic rectangular wave which takes values zero and \( v_s \) and which has the time average value \( dv_s \). An LC stage is added to the switching stage to filter out the average value. In the ideal case where the parasitic elements \( r_l \) and \( r_c \) are
zero, the DC part of the output voltage will be

\[ \bar{v}_o = dv_s. \]

A periodic switch function \( s(t) \) and the corresponding periodic steady state solution of the buck converter is illustrated in Fig. D.3. The left plot shows the evolution of the inductor current and the right plot shows the capacitor voltage. We note that the capacitor voltage is not in phase with the current. This is because of the parasitic elements \( r_c \) and \( r_l \).

**Example 2: Synchronous step-up (boost) converter**

The second example we consider is the synchronous boost converter. The topology is depicted in Fig. D.4. The purpose of the boost converter is to provide the load with a DC voltage which is higher than the source. When the switch is in the on position \((s = 1)\), the current in the inductor increases rapidly. When the switch
turns to the off position \((s = 0)\), the current shoots into the capacitor which can thus sustain a voltage which is higher than the source. A periodic switch function \(s(t)\) and the corresponding periodic steady state solution of the boost converter is illustrated in Fig. D.5. The left plot shows the evolution of the inductor current and the right plot shows the capacitor voltage.

Figure D.5: Periodic switch function \(s\) and corresponding periodic solution of the boost converter. \(i_i\) is the inductor current and \(v_c\) is the capacitor voltage.

Suppose the duty cycle is constant and equal to \(d \in [0, 1]\) so that \(d_k = d\ \forall k\). In the ideal case where the parasitic elements \(r_l\) and \(r_c\) are zero, the DC part of the output voltage will be

\[
\bar{v}_o = \frac{1}{1 - d} v_s.
\]

In other words, when the duty cycle approaches unity it is possible to have arbitrarily high output voltage. However, when the parasitic elements are included, the DC component of the output voltage becomes [20]

\[
\bar{v}_o = \frac{r_o(r_o + r_c)(1 - d)}{r_o^2(1 - d)^2 + r_o r_c(1 - d) + r_l(r_o + r_c)} v_s.
\]
The gain $\bar{v}_o/v_s$ for the non-ideal case is plotted in Fig. D.6 for some different values of the load resistance $r_o$. The ideal case is included for comparison. From Fig. D.6 it is clear that including the parasitic elements results in an essentially different model as compared to the ideal case. We note that after the point [20]

$$d_{\text{max}} = 1 - \sqrt{\frac{r_l}{r_o} \left( 1 + \frac{r_c}{r_o} \right)}$$

the slope of the curves change sign. This means that there are two different duty cycles which yield the same average output voltage. Thus, including the parasitic elements reveals the fact that in a boost converter, for any reference output voltage $v_{\text{ref}} > v_s$ there are two periodic solutions with average output voltage equal to $v_{\text{ref}}$. One of these solutions (the one corresponding to the higher duty cycle) has a large inductor current. This implies higher switching losses and the high current equilibrium should therefore be avoided.

The existence of multiple equilibria is a difficulty in the control of the boost converter. The change of sign in the relationship between $d$ and $\bar{v}_o$ at the point $d_{\text{max}}$ is also a difficulty as it may cause instability in closed loop operation. From the observations above we conclude that it is essential to include the parasitic elements in the model of the boost converter.

As a final remark on the boost converter we note that for duty cycles in the interval $[0, d_{\text{max}}]$, the dynamics are non-minimum phase with respect to the output voltage. That is, when the duty cycle is increased, the output voltage will temporar-
ily decrease before it increases. This is because when the duty cycle increases, there is less time to transfer the inductor current to the capacitor.

The primary objective of a DC-DC converter is to keep the output voltage constant at the given reference level. Because the source voltage is unregulated and the load may vary significantly, there is a need for a controller which regulates the duty cycle. In the section below we review some common modeling approaches used in control synthesis for switched DC-DC converters.

3 Modeling switched mode DC-DC converters

The dynamics of fixed frequency synchronous DC-DC converters can be described on the form

\[
\dot{x}(t) = \begin{cases} 
A_1 x(t) + B_1, & t \in [kT_s, (k + d_k)T_s) \\
A_2 x(t) + B_2, & t \in [(k + d_k)T_s, (k + 1)T_s) 
\end{cases}
\]  

(D.1)

where \(x \in \mathbb{R}^n\), \(A_1, B_1\) are constant matrices, \(T_s > 0\) is the period time and \(d_k \in [0, 1]\) is the duty cycle for the \(k^{th}\) time period. The hybrid nature of these equations makes control design a difficult problem and this has motivated a number of approximate models. In this section we review the two most common modeling approaches and introduce the basic concept of the sampled data model which we use for control design.

3.1 The state space averaged model

Conventionally, control design for DC-DC converters relies on the so-called state space averaged model [13,14,18]. In the state space averaged model the discontinuous switch function is replaced with a continuous function which represents the time average value of the duty cycles. The averaged system equations corresponding to (D.1) are

\[
\dot{x}(t) = (dA_1 + (1 - d)A_2)x(t) + dB_1 + (1 - d)B_2
\]

where \(d \in [0, 1]\) is the continuous control variable. The averaged dynamics are typically linearized around some operating point and control tools from linear systems theory are applied.

The state space averaged model is widely used for control design. However, the averaged model only describes the slow scale dynamics of the converter and thus, a controller based on the averaged model may have reduced performance. Also, it has been noted, see [4] and references therein, that the state space averaged model need not capture the stability properties of the system. More precisely, the averaged model can be stable even though the actual system is not.
3.2 The discrete time model

As an alternative to the state space averaged model some authors, see e.g., [3,12,15] have considered a discrete time model which describes the state of the system at the switching instants. Defining

\[ \tilde{A}_1 := \begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_2 := \begin{bmatrix} A_2 & B_2 \\ 0 & 0 \end{bmatrix} \]

and using the relation

\[ e^{\tilde{A}_i t} = \begin{bmatrix} e^{A_i t} & \int_0^t e^{A_i (t-\tau)} B_i d\tau \\ 0 & 1 \end{bmatrix} \]

one can show that \( x_k := x(kT_s) \) satisfies

\[ x_{k+1} = \Phi_{d_k} x_k + \Gamma_{d_k} \tag{D.2} \]

where \( \Phi_{d_k} : [0, 1] \to \mathbb{R}^{n \times n} \) and \( \Gamma_{d_k} : [0, 1] \to \mathbb{R}^{n \times 1} \) are (nonlinear) functions of the duty cycle defined by

\[ [\Phi_{d_k} \ \Gamma_{d_k}] = [I \ 0] e^{\tilde{A}_2 (1-d_k) T_s} e^{\tilde{A}_1 d_k T_s}. \]

The discrete time model (D.2) takes the switching into account and gives an exact description of the state at the switching instants. The discrete time model is thus more accurate than the averaged model and it represents the fast scale dynamics of the converter. The drawback is that the discrete time model is highly nonlinear and the structure of the system matrices \( A_i, B_i \) is lost. Because the model is nonlinear, it is typically linearized around some operating point.

3.3 The sampled data model

The discrete time model gives an exact representation of (D.1) at the switching instants. However, it does not account for the state in between the switching instances. As a consequence, design based on the discrete time model may result in subharmonic oscillations as shown in Section 3.4 below. To remedy this problem we introduce a sampled data model which can be seen as an extension of the discrete time model (D.2). The sampled data model gives an exact representation of the state at the switching instants, but also includes an infinite dimensional lifted output signal which describes the inter sample behavior. For a fixed value \( d \in [0, 1] \) we define

\[ \Omega_d(\theta) := \begin{cases} e^{\tilde{A}_1 \theta}, & \theta \in [0, dT_s] \\ e^{\tilde{A}_2 (\theta-dT_s)} e^{\tilde{A}_1 dT_s}, & \theta \in [dT_s, T_s] \end{cases} \]

which is used to define functions \( \Phi_d(\theta) : [0, T_s] \to \mathbb{R}^{n \times n}, \ \Gamma_d(\theta) : [0, T_s] \to \mathbb{R}^{n \times 1} \) according to

\[ [\Phi_d(\theta) \ \Gamma_d(\theta)] = [I \ 0] \Omega_d(\theta). \]
We also note that $\Phi_d = \Phi_d(T_s)$, $\Gamma_d = \Gamma_d(T_s)$ where $\Phi_d$ and $\Gamma_d$ are defined in the previous section. The sampled data model is written
\[
  x_{k+1} = \Phi_{d_k} x_k + \Gamma_{d_k}, \\
  \hat{x}_k(\theta) = \Phi_{d_k(\theta)} x_k + \Gamma_{d_k(\theta)}.
\]
The system above gives an exact description of the system state $x_k := x(kT_s)$ at the switching instants, but it also provides a sequence of lifted signals $\hat{x}_k(\theta) : [0, T_s] \to \mathbb{R}^n$ which gives an exact description of the inter sampling behavior. The first part of the sampled data model describes the evolution of the state at the switching times $kT_s$ as illustrated to the left in Fig. D.7. The second part uses the sampled state $x_k$ to reproduce the trajectory on the interval $[kT_s, (k + 1)T_s]$. This is illustrated to the right in Fig. D.7. The sampled data model is an equivalent representation of the switched dynamics (D.1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image}
\caption{Sampled signal $x_k$ (left) and lifted signal $\hat{x}_k(\theta)$ (right)}
\end{figure}

A model which extends the discrete time model (D.2) to account for the inter sampling behavior has also been considered in [10]. This paper introduces the so-called $\nu$-resolution model which provides a piecewise affine approximation of the state at an arbitrary number of points in between the switching instants. The $\nu$-resolution model has been used successfully to formulate model predictive control (MPC) problems where hard state constraints can be imposed over the $\nu$-resolution grid. However, the increased accuracy of the $\nu$-resolution model comes at the price of higher dimension of the systems model.

In the sampled data model, continuous time cost criteria can be equivalently expressed in discrete time and the dimension of the system state is not increased. However, state constraints are not directly accounted for in this framework.

### 3.4 Example

To illustrate the modeling approaches described above we consider a simple numerical example; the synchronous buck converter depicted in Fig. D.2. Choosing the
state as \( x = [i_t \ v_c] \) where \( i_t \) is the inductor current and \( v_c \) is the capacitor voltage, the converter dynamics are of the form (D.1) with system matrices

\[
A_1 = A_2 = \begin{bmatrix}
\frac{1}{\tau_t} (r_t + \frac{r_o r_c}{r_o + r_c}) & \frac{1}{\tau_t} \frac{r_o}{r_o + r_c} \\
\frac{1}{\tau_c} \frac{r_o}{r_o + r_c} & -\frac{1}{\tau_c} \frac{r_o}{r_o + r_c}
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
v_s \\
\frac{x_t}{\tau_t}
\end{bmatrix}
\]  
(D.3)

and \( B_2 = 0 \). The parameter values are taken from [1]. The dynamics have been scaled to obtain switch period \( T_s = 1 \) and the parameters are expressed in the per unit system. They are \( x_t = 3/\pi \) p.u., \( x_c = 70/2\pi \) p.u., \( r_t = 0.05 \) p.u., \( r_c = 0.005 \) p.u., \( r_o = 1 \) p.u.. The source voltage is \( v_s = 1.8 \) p.u. and the reference output voltage is \( v_{\text{ref}} = 1 \). Finally we note that the output voltage is given by \( v_o = Cx \) where

\[
C = \begin{bmatrix}
r_o r_c \\
r_o + r_c
\end{bmatrix}.
\]

To motivate the sampled data approach we formulate three different LQ optimal control problems for the buck converter. Each problem is solved to obtain a state feedback vector and we compare performance in a simulation. The first controller is based on an averaged continuous time model of the converter. The second is based on the discrete time model where the inter sample behavior is not accounted for and finally, the third controller is based on the sampled data model.

Firstly, we consider control design based on the averaged model. The averaged model of the buck converter is

\[
\dot{x}(t) = Ax(t) + Bd(t)
\]

where \( A = A_1 \) and \( B = B_1 \) are defined in (D.3) and \( d(t) \in [0, 1] \) is the continuous control variable. We assume the existence of a stationary point \((d^0, x^0) \in [0, 1] \times \mathbb{R}^n\) such that \( 0 = Ax^0 + Bd^0 \) and such that the output voltage corresponding to \( x^0 \) is equal to \( v_{\text{ref}} \). In other words, we assume that \( Cx^0 = v_{\text{ref}} \). We define \( z := x - x^0 \) to be the deviation from the stationary solution and \( u := d - d^0 \) to be the deviation from the stationary duty cycle. We relax the constraint \( u \in [-d^0, 1 - d^0] \) and consider the LQ optimal control problem

(C) Continuous time LQ problem

\[
\text{minimize} \quad J = \int_0^\infty (z(t)^T C^T C z(t) + Ru(t)^2) dt
\]

subject to \( \dot{z}(t) = Az(t) + Bu(t) \)

where \( R = 0.02 \).

Secondly, we consider control design based on the discrete time model, i.e., we consider the model (D.2) corresponding to the buck converter. We assume the existence of a stationary point \((d^0, x^0) \in [0, 1] \times \mathbb{R}^n\) such that \( x^0 = \Phi_{d^0} x^0 + \Gamma_{d^0} \). and
such that the corresponding average output voltage is equal to $v_{\text{ref}}$. In other words, we assume that $\frac{1}{T_s} \int_0^{T_s} C x^0(\tau) d\tau = v_{\text{ref}}$ where $x^0(\tau)$ is the periodic solution obtained with initial condition $x^0(0) = x^0$ and $d_k = d^0 \forall k$. We define $z_k := x_k - x^0$ to be the deviation from the stationary point and consider the error dynamics

$$z_{k+1} = \Phi_{dk} z_k + \Gamma_{dk}$$

where

$$\Gamma_{dk} = (\Phi_{d_k} - \Phi_{d^0})x^0 + \Gamma_{d_k} - \Gamma_{d^0}.$$  

The error dynamics are nonlinear and are therefore linearized around the fixed point $(d^0, x^0)$. The linearized error dynamics satisfy

$$z_{k+1} = \Phi z_k + \Gamma u_k$$

where $u_k := d_k - d^0$, $\Phi := \Phi_{d^0}$ and where an explicit expression for $\Gamma := \frac{\partial^2 J}{\partial d \partial d^0}(d^0)$ can be found in [8]. We relax the constraint $u_k \in [-d^0, 1 - d^0]$ and consider the LQ optimal control problem

(D) Discrete time LQ problem

$$\text{minimize} \quad J = \sum_{k=0}^{\infty} z_k^T C^T C z_k$$

subject to

$$z_{k+1} = \Phi z_k + \Gamma u_k.$$  

Thirdly, we consider control design based on the sampled data model. In analogy with the case above we define $z_k := x_k - x^0$, $u_k := d_k - d^0$ and consider the sampled data error model

$$z_{k+1} = \Phi_{dk} z_k + \Gamma_{dk}$$

$$\hat{z}_k(\theta) = \Phi_{d_k}(\theta) z_k + \Gamma_{d_k}(\theta)$$

where $\Phi_{dk}$, $\Gamma_{dk}$ and $\Phi_{dk}(\theta)$ are defined above and where

$$\Gamma_{d_k}(\theta) = (\Phi_{d_k}(\theta) - \Phi_{d^0}(\theta))x^0 + \Gamma_{d_k}(\theta) - \Gamma_{d^0}(\theta).$$

Using the lifted signal $\hat{z}_k(\theta)$ it can be shown that

$$\int_0^\infty z(\theta)^T C^T C z(\theta) d\theta = \sum_{k=0}^{\infty} \int_0^{T_s} \hat{z}_k(\theta)^T C^T C \hat{z}_k(\theta) d\theta = \sum_{k=0}^{\infty} \left[ \begin{array}{c} z_k \\ 1 \end{array} \right]^T Q(d_k) \left[ \begin{array}{c} z_k \\ 1 \end{array} \right]$$

where $Q(d_k)$ is a nonlinear function of $d_k$. An explicit expression for this matrix function can be found in [8]. The equality above implies that the problem of minimizing a continuous time cost criterion (which takes the inter sampling behavior
into account) subject to switching dynamics can be formulated as a discrete time optimal control problem.

The corresponding optimal control problem has a linear quadratic structure but with the duty cycle appearing as a parameter in the system matrices and the cost function. This makes the control design problem highly nonlinear and we simplify it by considering a linear quadratic approximation. The approximate problem is formulated

(S) Sample data LQ problem

\[
\begin{align*}
\text{minimize} & \quad J = \sum_{k=0}^{\infty} L(z_k, u_k) \\
\text{subject to} & \quad z_{k+1} = \Phi z_k + \Gamma u_k
\end{align*}
\]

where \(\Phi\) and \(\Gamma\) are as above and where \(L(z_k, u_k)\) is a quadratic function of \(z_k\) and \(u_k\). An explicit expression can be found in [8].

The optimal control problems (C), (D) and (S) are solved and we obtain three feedback vectors. Figure D.8 shows a simulation of the closed loop behavior with these three controllers where the initial state deviates slightly from the stationary point corresponding to \(v_{\text{ref}}\).

![Figure D.8: Closed loop response of continuous, discrete and sampled data LQ controllers (dashed, dashed-dotted and solid line respectively).](image)

The sampled data controller compares favorably to both the continuous and discrete time LQ controllers. In particular, the discrete time controller yields a subharmonic oscillation which is avoided in the sampled data approach. Note that
the subharmonic oscillation appears even though the sampling frequency is not an multiple of the systems resonance frequency.

It can be argued that the performance of the continuous and discrete time LQ controllers can be improved by further tuning. For example, the subharmonic solution of the discrete time controller can be avoided by replacing the objective function by an appropriate quadratic cost on \( z_k \) and \( u_k \) which possibly includes a cross term between \( z_k \) and \( u_k \). However, the tuning procedure is simplified by directly taking the inter sampling behavior into account and thus capturing the continuous time behavior.

4 Sampled data \( \mathcal{H}_\infty \) control for robust tracking

The section above introduced the concept of the sampled data model. In this section we consider \( \mathcal{H}_\infty \) control synthesis for robust tracking based on the sampled data model. In \( \mathcal{H}_\infty \) control synthesis, an external signal is introduced in the system to model uncertainty and disturbances. The control objective is to minimize the impact of the external signal on a certain output signal. For example, we may include an independent current source in the load to model uncertainty and changes in the resistance and then minimize the gain from the current to the output voltage. A circuit interpretation of the \( \mathcal{H}_\infty \) control approach can be found in [20].

To consider the problem of \( \mathcal{H}_\infty \) control for robust tracking we introduce an external disturbance in the converter model and we also introduce a so-called averaging sampler. This results in a more complex system as compared to (D.1). The system considered is

\[
\begin{align*}
\dot{x}(t) &= \begin{cases} 
A_1 x(t) + B_1 + D_1 w(t), & t \in [kT_s, (k + d_k)T_s) \\
A_2 x(t) + B_2 + D_2 w(t), & t \in [(k + d_k)T_s, (k + 1)T_s) 
\end{cases} \\
\xi(t) &= \begin{cases} 
C_{01} x(t) + D_{01} w(t), & t \in [kT_s, (k + d_k)T_s) \\
C_{02} x(t) + D_{02} w(t), & t \in [(k + d_k)T_s, (k + 1)T_s) 
\end{cases} \\
v(t) &= \begin{cases} 
C_{11} x(t) + D_{11} w(t), & t \in [kT_s, (k + d_k)T_s) \\
C_{12} x(t) + D_{12} w(t), & t \in [(k + d_k)T_s, (k + 1)T_s) 
\end{cases} \\
\psi_k &= \begin{bmatrix} \psi_{1,k} \\ \psi_{2,k} \end{bmatrix} = \left[ \frac{1}{T_s} \int_{(k-1)T_s}^{kT_s} \frac{v(t)dt}{H x(kT_s)} \right]
\end{align*}
\]

where \( x, w, \xi \) and \( v \) are continuous time signals which denote the state, the external disturbance and two output signals respectively. The signal \( v \), which could coincide with a part of \( \xi \), is the signal which should track the reference \( v_{\text{ref}} \) and \( \xi \) is an auxiliary output which is used to formulate the performance index in the \( \mathcal{H}_\infty \) problem formulation. The signal \( \psi \) is a discrete time signal which is input to the linear discrete time controller to be designed. The signal \( \psi \) is composed of two parts: The second part \( \psi_2 \) is determined by usual sampling, while the first part \( \psi_1 \)
is determined by the averaging sampler. We define \( \psi_{1,k} := (S_{\text{ave}}v)[k] \) where

\[
(S_{\text{ave}}f)[k] := \frac{1}{T_s} \int_{(k-1)T_s}^{kT_s} f(\tau)\,d\tau.
\]

The control problem considered is illustrated in Fig. D.9. Here, the block \( P \) represents the switching system (D.4) and the block \( K \) represents the discrete time controller. The discrete time controller generates a sequence of duty cycles \( d_k \) based on a constant reference input \( v_{\text{ref}} \) and the discrete time measurement output \( \psi \). The primal control objective is to regulate the signal \( v \) so that the error between \( v \) and the reference \( v_{\text{ref}} \) is minimized. Furthermore, this is to be achieved while making sure that the gain from the disturbance \( w \) to the output \( \xi \) is bounded by some predetermined value.

\[ v_{\text{ref}} \rightarrow K \rightarrow d \rightarrow P \rightarrow w \rightarrow v \rightarrow \xi \rightarrow \psi \]

Figure D.9: Feedback control configuration.

Note that due to the switching nature of the system, the signal \( v \) is periodic at stationarity and the error between \( v \) and \( v_{\text{ref}} \) cannot converge to zero even in the disturbance free scenario. Therefore it is natural to consider the objective

\[
\lim_{k \to \infty} \frac{1}{T_s} \int_{(k-1)T_s}^{kT_s} v(\tau)\,d\tau = v_{\text{ref}}
\]

which we refer to as the tracking condition. To satisfy the tracking condition, one possible approach [7] is to consider \( v \) at the switching instances and consider the objective \( \lim_{k \to \infty} v(kT_s) = \bar{v}_{\text{ref}} \) where \( \bar{v}_{\text{ref}} \) is chosen such that the tracking condition is satisfied. However, \( \bar{v}_{\text{ref}} \) would be computed using the nominal system parameters and this approach is therefore not robust to parameter variations. To satisfy the tracking condition robustly we use the averaging sampler \( S_{\text{ave}} \) and consider the objective

\[
\lim_{k \to \infty} (S_{\text{ave}}v)[k] = v_{\text{ref}}
\]

which is equivalent to the tracking condition (D.5) and which can be achieved robustly by including integral action on the control error \( \psi_{1,k} - v_{\text{ref}} \).

We assume the existence of a stationary duty cycle \( d^0 \in [0,1] \) such that the system attains a periodic solution \( x^0 \) of period \( T_s \) when \( w \equiv 0 \). The output \( v \)
corresponding to $x^0$ is also periodic and is denoted $v^0$. We further assume that $d^0$ is such that
\[
\frac{1}{T_s} \int_0^{T_s} v^0(t) \, dt = v_{\text{ref}}. \tag{D.6}
\]
In other words, the tracking condition is satisfied at stationarity.

The control objective is to ensure asymptotic convergence to the nominal periodic solution $x^0$ that satisfies the tracking condition (D.5). At the same time, we want to satisfy the robustness criterion
\[
\int_0^t \|\xi(t) - \xi^0(t)\|^2 \, dt + \sum_{k=0}^M q_k \|e_k\|^2 \leq \gamma^2 \int_0^t \|w\|^2 \, dt, \quad \forall t \geq 0, \quad \forall M \geq 0 \tag{D.7}
\]
for all solutions to the dynamics in (D.4). Here, $\xi$ is the auxiliary output that is chosen to tune the controller and $\xi^0$ is the periodic output corresponding to $x^0$. The integral term $e_k$ is defined as
\[
e_k := \sum_{l=0}^{k-1} y_{1,l}
\]
where $y_{1,l} := \psi_{1,l} - v_{\text{ref}}$ and $\gamma > 0$.

The first term in the left hand side of (D.7) is chosen to make a certain part of the state robust to the disturbance. The second term (i.e., the integral term) implies that the resulting controller will have an integrator state. This is a consequence of the LMI synthesis technique outlined in [11].

The problem of satisfying the $H_\infty$ criterion (D.7) subject to the system dynamics (D.4) can be equivalently stated as a discrete time $H_\infty$ control problem using a lifted representation of (D.4). This problem is highly nonlinear and we therefore consider a linear quadratic approximate problem which is solved using LMI techniques. The lifting and the approximation procedure is described in detail in [5,9].

Depending on the measurements available, the solution to the approximate problem yields either a dynamic output feedback controller or a feedback vector.

5 Case study 1: The step-down converter

In this section we consider control synthesis for the synchronous step-down converter illustrated in Fig. D.10. Here we have included an external disturbance in the form of an independent current source at the load. This is done to model uncertainty and variations in the load. Choosing the state as $x = [i_l \ v_c]'$ where $i_l$ is the inductor current and $v_c$ is the capacitor voltage, the converter dynamics are of the form (D.4) with system matrices
\[
A_1 = A_2 = \begin{bmatrix}
-\frac{1}{x_l} (r_l + \frac{r_o r_c}{r_c}) & -\frac{1}{x_l (r_c + r_c)} \\
\frac{1}{x_c} & \frac{1}{x_c (r_c + r_c)} \\
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
v_c \\
i_l \\
\end{bmatrix}.
\]
Figure D.10: Synchronous step-down converter with load disturbance \( w \) in the form of an independent current source.

\[
D_1 = D_2 = \begin{bmatrix}
\frac{1}{x_l} & \frac{r_o r_c}{r_o} \\
-\frac{1}{x_c} & \frac{r_o r_c}{r_o}
\end{bmatrix}
\]

\[
D_{01} = D_{02} = D_{11} = D_{12} = -\frac{r_o r_c}{r_o + r_c}
\]

\[
C_{01} = C_{02} = C_{11} = C_{12} = \left[\frac{r_o r_c}{r_o + r_c}, \frac{r_o}{r_o + r_c}\right]
\]

and \( B_2 = 0 \) and \( H = I \). We note that we have chosen \( v = v_o \) where \( v_o \) is the output voltage and that we measure the full state.

The parameter values are \( x_c = 100 \mu F \), \( x_l = 2 \text{mH} \), \( r_o = 50 \Omega \), \( r_c = 0.5 \Omega \) and \( r_l = 0.25 \Omega \). The switching frequency is \( f_s = 1/T_s = 20 \text{kHz} \) and the nominal source voltage is \( v_s = 50 \text{V} \). The control objective is to keep the output voltage at the reference level \( v_{\text{ref}} = 25 \text{V} \) and to make sure that the inductor current does not exceed the limit \( i_{l,\max} = 2.5 \text{A} \). This is essential because if the current is too large, the inductor may be damaged.

The proposed controller has a two level structure consisting of an inner and an outer loop. The inner loop contains a linear controller designed according to the method outlined in Section 4 while the outer loop provides nonlinear compensation which is inactive during normal operation.

### 5.1 Inner loop

To design the inner loop we follow the \( \mathcal{H}_\infty \) design procedure outlined in Section 4. The control objective is to steer the output voltage \( v_o \) to the reference value \( v_{\text{ref}} = 25 \text{V} \) and we also want \( v_o \) to be robust against variations in the load. We thus consider the \( \mathcal{H}_\infty \) criterion (D.7) with \( v_{\text{ref}} = 25 \), \( \xi = v_o \) and coefficients \( q = 0.17 \) and \( \gamma = 2.2 \). The corresponding linear quadratic approximate problem is solved to obtain the linear controller \( K \). We note that since we have access to the full state, \( K \) will be a feedback vector and that the term \( \sum_{k=0}^M g \| e_k \|^2 \) in (D.7) implies that there will also be an integrator state. The control structure is illustrated in Fig. D.11.
5.2 Outer loop

The linear controller $K$ is surrounded by an outer loop which, if necessary, will adjust the duty cycle computed by $K$. The outer loop is motivated by a number of reasons:

(i) The state constraint (the limit $i_{\text{max}}$ on the inductor current) is not considered in the synthesis of $K$ and needs to be dealt with by some additional control structure.

(ii) The controller $K$ has integral action. The saturation of the duty cycle to lie in the interval between zero and one suggests the need for an anti-windup strategy.

(iii) The controller $K$ is designed for a fixed nominal input voltage. Changes in the input voltage are handled by the integrator state, but the response can be made faster if measurements of the input voltage are used in a feed forward.

Let $v_s$ be the nominal source voltage and let $v_{s,k}$ be the source voltage measured at time $kT_s$. Let $\bar{x}_k = [x'_k, 1]'$ and let

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}$$

and finally let $c > 0$ be a fixed parameter. The output $d_k$ of the linear feedback $K$ and the integrator state are modified to account for the issues (i) – (iii) above as follows:

1. The duty cycle $d_k$ is scaled and saturated to obtain a duty cycle $d_k^*$ according to

$$d_k^* = \text{sat}_{[0,1]} \left( \frac{v_s}{v_{s,k}} d_k \right).$$
2. From \( d_k^s \) a modified duty cycle \( d_k^m \) is determined which is implemented in the plant

\[
d_k^m = \begin{cases} 
d_k^s, & \text{if } i_l(kT_s) + d_k^s T_s C \tilde{A}_1 \tilde{x}_k \leq i_{l,max} \\
sat[0,1]\left(\frac{i_l(kT_s) + d_k^s T_s C \tilde{A}_1 \tilde{x}_k}{i_{l,max}}\right), & \text{else.} 
\end{cases}
\]  
(D.8)

3. The term \( \Delta_k := d_k - d_k^m \) is included in the equations governing the integrator state according to

\[
e_{k+1} = e_k + \psi_{1,k} - v_{\text{ref}} + c \Delta_k.
\]

The first step described above implements a feed forward of the source voltage and makes sure the duty cycle is in the interval from zero to unity. We note that changes in the source voltage are compensated by the integrator state, but that the feed forward speeds up the response. The motivation for this particular form of the feed forward stems from the relation \( \bar{v}_o = dv_s' \) between the DC component of the output voltage and the (time varying) source voltage \( v_s' \). By scaling the duty cycle with the factor \( v_s/v_s' \) (where \( v_s \) is the nominal source voltage and \( v_s' \) is the actual source voltage) this relation is unaffected by variations in the source.

The second step is added to make sure that the current does not exceed the limit \( i_{l,max} \). The nonlinearity (D.8) is based on a one-step prediction of the inductor current. To see this, let

\[
I_k := \max_{t \in [kT_s,(k+1)T_s]} i_l(t)
\]

be the peak value of the current in the time interval \([kT_s,(k+1)T_s]\) where it is understood that the duty cycle \( d_k^s \) is applied. We make the natural assumption \( v_s > v_c \) where \( v_c \) is that capacitor voltage and it follows that

\[
I_k = i_l((k + d_k^s)T_s) = Ce^{\tilde{A}_1 d_k^s T_s} \tilde{x}_k.
\]

The peak value \( I_k \) as a function of \( d_k^s \) is approximated by a Taylor series around \( d_k^s = 0 \) and we have

\[
I_k \approx i_l(kT_s) + d_k^s T_s C \tilde{A}_1 \tilde{x}_k.
\]

We thus have a linear prediction of the peak inductor current in the interval ahead. To make sure that the current limit is satisfied, the scaled duty cycle must satisfy

\[
i_l(kT_s) + d_k^s T_s C \tilde{A}_1 \tilde{x}_k \leq i_{l,max}.
\]  
(D.9)

If this inequality is satisfied, the nonlinearity (D.8) sets \( d_k^m = d_k^s \). Otherwise, (D.8) replaces \( d_k^s \) with a value which (if not saturated) achieves equality in (D.9).

In the third step, the difference \( \Delta := d_k - d_k^m \) is included in the equations of the integrator state to avoid windup.

Finally we note that under normal operation, the outer loop remains inactive and \( d_k^m = d_k \).
5.3 Experiments

To evaluate the performance of the proposed controller we consider a number of scenarios of practical relevance. In all scenarios the converter is started with duty cycle $d = 0$ and zero inductor current and capacitor voltage. Thus, all scenarios include a start-up transient. The scenarios are

1. Load transient: After the start-up transient, the load resistance steps up from the nominal value ($r_o = 50\Omega$) to $100\Omega$ and after a certain time it steps back to the original value. This scenario is considered for three different reference voltages; $20V$, $25V$ and $30V$.

2. Line transient: After the start-up transient, the source voltage drops from the nominal value ($v_s = 50V$) to $35V$ and after a certain time it is restored to the original value.

3. Robustness to parameter variations: We consider three different values of the capacitance $x_c$: $50\mu F$, $100\mu F$ and $200\mu F$ (where $x_c = 100\mu F$ is the nominal value used in the design). The controller is turned on and steers the output voltage to the reference level.

The scenarios described above are tested in both simulations and in an experimental setup [17]. In the experiments and simulations the controller is implemented with a one switch period delay. A delay is necessary because of the computational time of the controller. However, we note that the computational time of the suggested controller is significantly less than one switch period and the delay is larger than what is strictly necessary. We also note that the component values of the experimental setup may differ considerably from the nominal values used in the design and that they may vary over time.

In Fig. D.12-D.14 below we show the experimental results and we also include simulations in Matlab for comparison. In all scenarios the simulations and experimental results correspond well.

In the first scenario (load transient) the voltage reaches the reference value with little or no overshoot and the step in the load is compensated for efficiently. The current limit $i_{t,\text{max}} = 2.5A$ is respected at all time.

In the second scenario (line transient) the step in the source voltage is also compensated efficiently. From Fig. 13(e)-13(f) we see that the duty cycle responds quickly to the source step. This is due to the feed forward.

In the third scenario (robustness) the voltage tracks the reference for all three values of the capacitance and the current limit is respected for all three cases. For large values of the capacitance, the step response is inevitably slower.

6 Case study 2: The step-up converter

In this section we consider control synthesis for the synchronous step-up converter illustrated in Fig. D.15. Here we have included an external disturbance in the form
Figure D.12: Start-up transient and response to a step in the load resistance from $r_o = 50 \Omega$ to $r_o = 100 \Omega$ and back again. The experiment is performed for three different values of the reference output voltage $v_{ref}$. The values are $20 \text{V}$ (light gray), $25 \text{V}$ (gray) and $30 \text{V}$ (black). The left column shows simulations and the right column shows experimental results.
Figure D.13: Start-up transient and response to a step in the source voltage from $v_s = 50\text{V}$ to $v_s = 35\text{V}$ and back again. The left column shows simulations and the right column shows experimental results.
Figure D.14: Start-up transient for three different values of the capacitance $x_c$. The values are $x_c = 0.5x_{c,\text{nom}}$ (black), $x_c = x_{c,\text{nom}}$ (gray) and $x_c = 2x_{c,\text{nom}}$ (light gray) where $x_{c,\text{nom}} = 100\mu\text{F}$ is the nominal value.
of an independent current source at the load. This is done to model uncertainty and variations in the load. Choosing the state as \( x = [i_l \ v_c]^T \) where \( i_l \) is the inductor current and \( v_c \) is the capacitor voltage, the converter dynamics are of the form (D.4) with system matrices

\[
A_1 = \begin{bmatrix}
-\frac{r_l}{x_l} & 0 \\
0 & -1 - \frac{1}{x_c \ r_o + r_c}
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
-\frac{1}{x_l} (r_l + \frac{r_o r_c}{r_o + r_c}) & -\frac{1}{x_c \ r_o + r_c} \\
\frac{1}{x_l} \frac{r_o}{r_o + r_c} & -\frac{1}{x_c \ r_o + r_c}
\end{bmatrix},
\]

\[
B_1 = B_2 = \begin{bmatrix}
\frac{v_o}{x_l} \\
0
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
0 \\
\frac{1}{x_c \ r_o + r_c}
\end{bmatrix}, \quad D_2 = \begin{bmatrix}
\frac{1}{x_l} \frac{r_o}{x_c \ r_o + r_c} \\
\frac{1}{x_c \ r_o + r_c}
\end{bmatrix}.
\]

\[
C_{01} = C_{02} = \begin{bmatrix}
0.5 & 1
\end{bmatrix},
C_{11} = \begin{bmatrix}
0 & \frac{r_o}{r_o + r_c}
\end{bmatrix}, \quad C_{12} = \begin{bmatrix}
\frac{r_o r_c}{r_o + r_c} & \frac{r_o}{r_o + r_c}
\end{bmatrix},
\]

\[
D_{11} = D_{12} = -\frac{r_o r_c}{r_o + r_c}
\]

and \( D_{01} = D_{02} = 0 \) and \( H = I \). We note that we have chosen \( v = v_o \) where \( v_o \) is the output voltage and that the output \( \xi \) which determines the \( H_\infty \) performance index is a linear combination of the inductor current and the capacitor voltage. We also note that we measure the full state.

The parameter values are \( x_c = 100\mu F, \ x_l = 2\text{mH}, \ r_o = 200\Omega, \ r_c = 0.2\Omega \) and \( r_l = 1.4\Omega \). The switching frequency is \( f_s = 1/T_s = 20\text{kHz} \) and the nominal source voltage is \( v_s = 20\text{V} \). The control objective is to keep the output voltage at the reference level \( v_{\text{ref}} = 50\text{V} \) and to make sure that the inductor current does not exceed the limit \( i_{l,\text{max}} = 2.5\text{A} \).

As in case study 1, the proposed controller has a two level structure consisting of an inner and an outer loop. The inner loop contains a linear controller designed using the method outlined in Section 4 while the outer loop provides nonlinear compensation which is inactive during normal operation.

### 6.1 Inner loop

To design the inner loop we follow the \( H_\infty \) design procedure outlined in Section 4. The control objective is to steer the output voltage \( v_o \) to the reference value \( v_{\text{ref}} = \)
50V and we want to do this robustly against variations in the load. We thus consider the $\mathcal{H}_\infty$ criterion (D.7) with $v_{\text{ref}} = 50$, $\xi = [0.5 1]x$ and coefficients $q = 2$ and $\gamma = 3$. The corresponding linear quadratic approximate problem is solved to obtain the linear controller $K$. We note that since we have access to the full state, $K$ will be a feedback vector and that the term $\sum_{k=0}^{M} q\|e_k\|^2$ in (D.7) implies that there will be an integrator state. The control structure is the same as in case study 1. See Fig. D.11.

6.2 Outer loop

The linear controller $K$ is surrounded by an outer loop which, if necessary, will adjust the duty cycle computed by $K$. The outer loop is motivated by the same reasons listed in Section 5 above.

Let $v_{s,k}$, $\bar{x}_k$, $C$, $A_1$ and $c > 0$ be defined as in Section 5. The output $d_k$ of the linear feedback $K$ and the integrator state are modified as follows:

1. The duty cycle $d_k$ is scaled and saturated to obtain a duty cycle $d^s_k$ according to

$$d^s_k = \text{sat}[0,1] \left( 1 - (1 - d_k) \frac{v_{s,k}}{v_s} \right).$$

2. From $d^s_k$ a modified duty cycle $d^m_k$ is determined which is implemented in the plant

$$d^m_k = \begin{cases} 
  d^s_k, & \text{if } i_l(kT_s) + d^s_k T_s C A_1 \bar{x}_k \leq i_{l,\text{max}} \\
  \text{sat}[0,1] \left( \frac{i_{l,\text{max}} - i_l(kT_s)}{T_s C A_1 \bar{x}_k} \right), & \text{else}. 
\end{cases}$$

3. The term $\Delta_k := d_k - d^m_k$ is included in the equations governing the integrator state according to

$$e_{k+1} = e_k + \psi_{1,k} - v_{\text{ref}} + c \Delta_k.$$

The motivation for the three steps above is analogous to the one given in Section 5. The structure of the feed forward is motivated by the relationship $\bar{v}_o = \frac{1}{1-a} v_s$ between the duty cycle and DC output voltage and the derivation of the current limiting nonlinearity is analogous to the one in Section 5.

6.3 Experiments

To evaluate the performance of the proposed controller we consider a number of scenarios of practical relevance. In all scenarios the converter is started from steady state with duty cycle $d = 0$ and zero inductor current and capacitor voltage $v_c$ equal to the source $v_s$. All scenarios include the start-up transient from this state. The scenarios are
1. Load transient: After the start-up transient, the load resistance drops to 50% of the nominal value \( r_o = 200 \Omega \) and after a certain time it steps back to the original value. This scenario is considered for three different values of the source voltage \( v_s \): 15V, 20V and 25V.

2. Line transient: After the start-up transient, the source voltage drops from the nominal value \( (v_s = 20V) \) to 15V and after a certain time it is restored to the original value.

3. Robustness to parameter variations: We consider three different values of the capacitance \( x_c \): 50\( \mu \)F, 100\( \mu \)F and 200\( \mu \)F (where \( x_c = 100\mu F \) is the nominal value used in the design). The controller is turned on and steers the output voltage to the reference level.

The scenarios described above are tested in both simulations and in an experimental setup [16]. In the experiments and simulations the controller is implemented with a one switch period delay. A delay is necessary because of the computational time of the controller. However, we note that the computational time of the suggested controller is significantly less than one switch period and the delay is larger than what is strictly necessary. We also note that the component values of the experimental setup may differ considerably from the nominal values used in the design and that they may vary over time.

In Fig. D.16-D.18 below we show the experimental results and we also include simulations in Matlab\textsuperscript{TM} for comparison. In all scenarios the simulations and experimental results correspond well. However, in the first and third scenario (load transient and robustness) there is one notable difference between the simulations and the experiments. In the experiments there is chattering in the duty cycle which is not present or is less pronounced in the simulations. The chattering is possibly a result of an unmodeled delay which is present in the measurement of the inductor current.

In the first scenario (load transient) the voltage reaches the reference value with 1-2\% overshoot depending on the source voltage. The step in the load is compensated for efficiently and the current limit \( i_{t,\text{max}} = 2.5A \) is respected at all times.

In the second scenario (line transient) the step in the source voltage is compensated efficiently. From Fig. 13(e)-13(f) we see that the duty cycle responds quickly to the source step. This is due to the feed forward.

In the third scenario (robustness) the voltage tracks the reference for all three values of the capacitance and the current limit is respected for all three cases. For large values of the capacitance, the step response is inevitably slower.

7 Conclusions

The paper introduced and motivated the sampled data model for modeling the dynamics of switched mode DC-DC converters. The sampled data model was used
Figure D.16: Start-up transient and response to a step in the load resistance from $r_o = 200\Omega$ to $r_o = 100\Omega$ and back again. The experiment is performed for three different values of the source voltage. The values are $v_s = 15\text{V}$ (black), $v_s = 20\text{V}$ (gray) and $v_s = 25\text{V}$ (light gray). The left column shows simulations and the right column shows experimental results.
Figure D.17: Start-up transient and response to a step in the source voltage from $v_s = 20$V to $v_s = 15$V and back again. The left column shows simulations and the right column shows experimental results.
Figure D.18: Start-up transient for three different values of the capacitance $x_c$. The values are $x_c = 0.5x_{c,\text{nom}}$ (black), $x_c = x_{c,\text{nom}}$ (gray) and $x_c = 2x_{c,\text{nom}}$ (light gray) where $x_{c,\text{nom}} = 100\mu\text{F}$ is the nominal value.
in an $\mathcal{H}_\infty$ problem formulation which was solved to yield a linear controller. The linear controller was augmented with some nonlinear control structure to account for state constraints.

The control approach was applied to two benchmark examples and the control performance was tested in both simulations and experiments. The control performance was excellent and proved to be robust to parameter variations.

8 References


