# INTRODUCTION TO CELLULAR CLASSES IN THE DERIVED CATEGORY OF A RING

JONAS KIESSLING

## 1. Preliminary Remarks

The purpose of this thesis is to discuss various aspects of the groving area of cellular classes, or more particularly cellular classes of chain complexes. The original setting of cellular classes was topological spaces and spectra. The standard reference is the book of Dror-Farjoun []. Dror-Farjoun and others have used cellular classes to set up a framework for doing unstable homotopy theory in which many traditional theorems have a common generalisation and explanation. A nice example is the paper [**?**] in which the author generalizes a well known theorem of Blakers and Massey.

The basic way to understand a topological space is by comparing it to more familiar spaces, such as the spheres. By studying maps from spheres we obtain the homotopy groups, the most fundamental of topological invariants. If we maps into Eilenberg-Maclane spaces we obtain cohomology, an other important invariant. The idea of cellular classes is the following: Suppose that we have a space $A$. We then study which other spaces that can be constructed out of $A$ using only sums and homotopy push-outs. This class is called the class of $A$-cellular spaces. If $X$ is $A$-cellular then we know that any invariant of $A$ that is preserved by sums and homotopy push-outs (such as connectivity, as in the theorem of Blakers and Massey) will also be an invariant of $X$. The point is the statement "$X$ is $A$-cellular" contains more information then say "$X$ is at least as highly connected as $A$".

Given any space $A$ there is a funtor $CW_A$ which assigns to any space $X$ the "universal" $A$-approximation of $X$ (see [**?**]) in the sense that $CW_AX$ is $A$-cellular and there is a map $CW_A \to X$ which, from the point of view of $A$, is an equivalence. As the name suggests, we should think of $CW_AX$ as a generalization of the classical construction of a $CW$-approximation using $A$ instead of $S^0$. Cellular classes form a lattice ([**?**]) however a complete characterization of this lattice has so far been out of reach.

While cellular classes of topological spaces have been around for over 15 years, it is only recently that people have started to study cellular classes in different settings. In this thesis we focus on cellular classes of chain complexes of modules over a commutative ring. The hope is that by focusing on cellular classes one can shed some light on classical homological invariants and contructions. An indication that this might indeed be a fruitful idea is the paper [**?**]. In this paper the author construcs a complete invariant of $t$-structures using a special kind of cellular classes called "acyclic classes" (see section **??** for a fuller discussion). Acyclic classes are a special kind of cellular classes that appear naturally when studying invariants closed under extensions.

The rest of this introduction is divided into several sections. First we fix notation. Then we introduce the notion of cellular classes and derive some properties. Next we discuss the algebraic verison of the $CW_A$-functor mentioned above. We then

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turn to the much better understood notions of "acyclic classes" and "localizaing subcategories". Localizing subcategories are the stable version of cellular classes and were (in the Noetherian case) classified already in 1992 be Neeman in his paper [**?**]. We close the introduction with a brief discussion of the two papers that make up the rest of this thesis.

## 2. Notation

In this introduction (and in paper 1) a *chain complex* (or simply a *complex*) is a *non-negatively graded* chain complex of $R$-modules with the homological grading, i.e. the differential lowers the degree by one. A *weak equivalence* is a morphism of chain complexes $f$ that induces isomorphism on homology (sometimes called quasi-isomorphism). We denote a weak equivalence from $X$ to $Y$ by $X \xrightarrow{\sim} Y$.

We say that a map $f : X \to Y$ is a *cofibration* if $f$ is injective and the cokernel of $f$ is a chain complex of projective modules. A chain complex is called *cofibrant* if the canonical map $0 \to X$ is a cofibration, or explicitly, that $X$ is a chain complex of projective modules. Recall that any map $f : X \to Y$ can be factored into a cofibration followed by a weak equivalence, i.e. there exist $f'$ and $f''$ such that $f = f''f'$ and $f'$ is a cofibration and $f''$ a weak equivalence (see [**?**] section 7).

The suspension functor is denoted by $\Sigma$: $(\Sigma X)_i = X_{i-1}$ ($= 0$ if $i = 0$) and $\partial_{\Sigma X} = (-1)\partial_X$. Since we are working with non-negative chain complexes the suspension functor is *not* an equivalence. It does however posses a left inverse: we let $\Omega$ denote the functor that takes $X$ to $\Omega X$, where $(\Omega X)_i = X_{i+1}$ for $i > 0$ and $(\Omega X)_0 = \ker \partial_1$. The differential becomes $\partial_{\Omega X} = (-1)\partial_X$. It follows directly that $\Omega\Sigma \cong 1$. As the notation suggests one should think of $\Sigma$ and $\Omega$ as analogs of the topological suspension and loop functors.

We let $Hom$ denote the *hom*-complex. It is defined as follows: if $X, Y \in Ch_{\geq 0}(R)$ then $Hom(X,Y)_n = \prod_{k\geq 0} hom(X_k, Y_{n+k})$. The differential takes $\{f_k : X_k \to Y_{k+n}\}$ to $\{\partial f_i + (-1)^n f_{i+n}\}$. An important property of the $Hom$-complex is that if $A$ is cofibrant then $Hom(A, \bullet)$ preserves weak equivalences (this follows from Brown's lemma, see [**?**] Lemma 9.9). We also note that $H_0(Hom(X,Y))$ is the set of homotopy classes of maps from $X$ to $Y$.

For two complexes $X$ and $Y$ we let $X \otimes Y$ denote their tensor product. It is a chain complex defined by: $(X \otimes Y)_n = \oplus_{p+q=n} X_p \otimes Y_q$. The differential is determined by the usual sign convention. It is a standard exercise to verify the following adjoint relation:

$$Hom(X \otimes Y, Z) \cong Hom(X, Hom(Y,Z))$$

The *cone* of a map $f : X \to Y$ of chain complexes if a chain complex $C(f)$ defined by: $(C(f))_n = Y_n \oplus X_{n-1}$. The differential maps $(y,x)$ to $(\partial_Y(y) - f(x), \partial_X(x))$. There is a natural inclusion $Y \to C(f)$ and the cokernel of this map is isomorphic to $\Sigma X$.

## 3. Model Categories and Triangulated Categories

## 4. Cellular Classes

The main topic of this thesis is the study of collections of objects called *cellular classes*. In particular we are intrested in cellular classes generated by a given object $A$, the *A-cellular objects*. In this section we give an axiomatic definition of cellular classes and deduce some formal properties. In the next section we show that the $A$-cellular complexes can be thought of as those complexes that can be constructed out of $A$.

We say that a collection $\mathcal{C}$ of chain complexes is *closed under cones* if given a map $f : X \to Y$ of chain complexes such that $X$ and $Y$ belongs to $\mathcal{C}$, then also the cone of $f$ belongs to $\mathcal{C}$.

Analagously we say that $\mathcal{C}$ is closed under sums if given a family $\{X_i\}$ of chain complexes in $\mathcal{C}$ then its sum $\oplus_i X_i$ also belongs to $\mathcal{C}$. The collection $\mathcal{C}$ is closed under weak equivalences if whenever there is a weak equivalence $X \xrightarrow{\sim} X'$ then either $X$ and $X'$ belong to $\mathcal{C}$ or neither.

A collection $\mathcal{C}$ satisfying all these properties is called a *cellular class*:

**Definition 4.1.** A non-empty class of objects $\mathcal{C} \subset Ch_{\geq 0}(R)$ is called a *cellular class* if it is closed under cones, weak equivalences and sums.

Here are some examples of cellular classes:

**Proposition 4.2.** *(1) The collection of all chain complexes is a cellular class.*

*(2) Fix an integer $k \geq 0$. The collection of all chain complexes $X$ such that $H_i X = 0$ for $i \leq k$ is a cellular class.*

*(3) The collection of all complexes weakly equivalent to zero is a cellular class.*

*(4) Fix an $R$-module $M$. The collection of all chain complexes $X$ such that $H_0 X$ is a quotient of a sum of $M$ is a cellular class.*

Any cellular class contains the zero and is closed under suspensions. To see this note that the cone of the identity map $1 : X \to X$ is weakly equivalent to 0. Moreover the complex $\Sigma X$ is isomorphic to the cone of the unique map $X \to 0$.

Cellular classes are closed under cokernels of monomorphisms: If

$$0 \to X \to Y \to Z \to 0$$

is an exact sequence then the cone of the map $X \to Y$ is weakly equivalent to $Z$. So if $X$ and $Y$ belong to $\mathcal{C}$ then also $Z \in \mathcal{C}$. In fact this argument shows that being a cellular class is equivalent to being closed under weak equivalences, sums and cokernels of monomorphisms. It follows that cellular classes are closed under directed colimits: Given a (possibly transfinite) system of chain complexes:

$$X_0 \to X_1 \to \ldots \to X_\alpha \to \ldots$$

Its colimit is the cokernel of the injective map:

$$\coprod_i X_i \xrightarrow{1-\text{shift}} \coprod_i X_i$$

Hence if for every $i$, $X_i \in \mathcal{C}$ then the colimit $colim_i X_i \in \mathcal{C}$. This last observation will be used in the next section when we construct the cellular approximation functor. We also note that cellular classes are closed under retracts: If $X$ is a retract of $Y$, i.e. there are maps $i : X \to Y$ and $p : Y \to X$ such that $pi = 1_X$, then $X$ is isomorphic to the colimit of the system:

$$Y \xrightarrow{ip} Y \xrightarrow{ip} \cdots$$

so if $Y \in \mathcal{C}$, then also $X \in \mathcal{C}$.

We are also intereted in how cellular classes behave under push-outs:

**Lemma 4.3.** *Let $\mathcal{C}$ be a cellular class. Suppose that we are given a push-out diagram:*

$$X_1 \leftarrow X_2 \to X_3$$

*such that $X_i \in \mathcal{C}$, $i = 1, 2, 3$. If one of the two maps is a monomorphism then the push-out also belongs to $\mathcal{C}$.*

*Proof.* We let $X$ denote the push-out. It is clear that if one of the maps is a monomorphism then we associate to the push-out square an exact sequence of the form:

$$0 \to X_2 \to X_1 \oplus X_3 \to X \to 0$$

The statement now follows from the discussion preceding the lemma. $\square$

If we fix a chain complex $A$ then we can regard the collection of all complexes $X$ such that whenever $A$ belongs to a cellular class, so does $X$. This collection is also a cellular class, in fact it is the smallest cellular class containing $A$. We call members of this collection the $A$-cellular complexes:

**Definition 4.4.** Fix a chain complex $A$. We let $\mathcal{C}(A)$ denote the smallest cellular class containing $A$. Objects in $\mathcal{C}(A)$ are called $A$-cellular. If $X$ is $A$-cellular then we write that $X \gg A$.

This definition introduces a relation between chain complexes and it is the main objective of this thesis to understand this relation. We first make some trivial remarks.

If $I$ is any set then $\oplus_i A \in \mathcal{C}(A)$ because cellular classes are closed under sums. Moreover $A$ is a retract of $\oplus_I A$ so $A \in \mathcal{C}(\oplus_I A)$. Hence there is an equality $\mathcal{C}(A) = \mathcal{C}(\oplus_I A)$.

The relation $\gg$ is transitive: If $X \gg Y$ and $Y \gg Z$ then $X \gg Z$. It also behaves well with respect to the derived tensor product:

**Lemma 4.5.** *Suppose that $X$ is cofibrant and that $Y \gg A$. Then $X \otimes Y \gg X \otimes A$.*

*Proof.* Let $\mathcal{C}$ be the collection of all chain complexes $Y$ such that $X \otimes Y$ is $X \otimes A$-cellular. Of course $A \in \mathcal{C}$. Moreover since $X$ is cofibrant it presereves exact sequences and weak equivalences. The tensor product always preserves sums. Hence $\mathcal{C}$ is a cellular class containing $A$. Be definition it follows that $\mathcal{C}(A) \subset \mathcal{C}$ and we are done. $\square$

## 5. The $A$-cellular approximation

In topology we can construct $CW$-approximations and in algebra cofibrant replacements (or projective resolutions). These are funtorial approximations of a given object by something that we construct out of spheres and disks. In this section we show how a similar approximation can be obtained using other objects then spheres. It turns out that the $A$-cellular objects of the last section are precisely the objects that can be reconstructed (up to weak equivalence) out of $A$.

**Definition 5.1.** Fix a cofibrant replacement $A' \xrightarrow{\sim} A$. A morphism $f : X \to Y$ is called an *$A$-equivalence* if $Hom(A', f) : Hom(A', X) \to Hom(A', Y)$ is a weak equivalence.

Note that this does not depend on the choice of cofibrant replacement $A' \xrightarrow{\sim} A$.

We think of $A$-equivalences as morphisms that from the point of view of $A$ is a weak equivalence. It is easy to show that being an $A$-equivalences automatically translates to being an $X$-equivalence, for all $A$-cellular chain complexes $X$:

**Lemma 5.2.** *If $X$ is $A$-cellular and $f$ is an $A$-equivalence then $f$ is an $X$-equivalance.*

*Proof.* Fix an $A$-equivalence $f$. Let $\mathcal{C}$ denote the collection of all complexes $Y$ such that $f$ is a $Y$-equivalence. By assumption $A \in \mathcal{C}$. Moreover $\mathcal{C}$ is closed under arbitrary sums, cones and weak equivalences. Togehter this yields $\mathcal{C}(A) \subset \mathcal{C}$. $\square$

Later we show that in fact $X$ is $A$-cellular *if and only if* all $A$-equivalences are also $X$-equivalences.

We can now define what we mean by an $A$-cellular approxiamtion of a complex $X$:

**Definition 5.3.** Fix a chain complex $X$. A pair $(X', f)$, where $X'$ is a cofibrant complex and $f : X' \to X$ is a morphism, is called an *$A$-cellular approximation* if $X'$ is $A$-cellular and $f$ is an $A$-equivalence.

An $A$-approximation is terminal among maps from $A$-cellular complexes and initial among $A$-equivalences.

**Lemma 5.4.** *Suppose that $(X', f)$ is an $A$-approximation of $X$ and that $g : Y \to X$ is a map from a cofibrant complex $Y$. If $Y$ is $A$-cellular then there is a map $h : Y \to X'$ making the following diagram commute up to homotopy:*

$$\begin{tikzcd} & X' \arrow[d, "f"] \\ Y \arrow[ur, "h"] \arrow[r, "g"] & X \end{tikzcd}$$

*If $g$ is an $A$-equivalence then there is a map $h : X' \to Y$ making the following diagram commute up to homotopy:*

$$\begin{tikzcd} & X' \arrow[d, "f"] \arrow[dl, "h"'] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

*Proof.* Suppose first that $Y$ is $A$-cellular. By Lemma 5.2 $f$ is $Y$-equivalence, i.e. $Hom(Y, f)$ is a weak equivalence. In particular $H_0(Hom(Y, f))$ is an isomorphism from the set of homotopy classes of maps from $Y$ to $X'$ to the set of homotopy classes of maps from $Y$ to $X$. Hence we can find the desired $h$.

If $g$ is an $A$-equivalence then $H_0(Hom(X', g))$ is an isomorphism and again we can find $h$. □

*Remark* 5.5. A consequence of this lemma is that $A$-cellular approximations are unique up to homotopy.

We now show a construction that given any two complexes $A$ and $X$ will produce a pair $(CW_A X, c_X)$ such that when $A$ is cofibrant, this is an $A$-cellular approximation . We will closely mimick the construction in [**?**].

We let $\gamma$ denote some limit ordinal such that the cofinality of $\gamma$ is bigger then the cardinality of the underlying set of $A$ (i.e. the underlying set of $\oplus A_i$). We also let $\gamma$ denote the category of all ordinal numbers smaller then $\gamma$ with a unique map $i \to j$ if $i \leq j$. The assumption on $\gamma$ means that given any functor indexed by $\gamma$: $F : \gamma \to Ch_{\geq 0}(R)$ and map $A \to colim_\gamma F$, then there is an $i < \gamma$ and a factorization of the map $f$ into: $A \to F_i \to colim_\gamma F$. See [**?**] chapter 5.3 for the existence of such an ordinal $\gamma$.

Let $F : \gamma \to Ch_{\geq 0}(R)$ denote the functor defined inductively by:

(1) For $i = 0$ we let:

$$F_0 = \coprod_{\hom(\Sigma^k A, X), 0 \leq k < \infty} \Sigma^k A$$

There is an induced map $p_0 : F_0 \to X$. It is clear that $F_0$ is $A$-cellular.

(2) If $i = j+1$ then let $I_{j+1}$ be the set of all commutative squares:

$$I_{j+1} = \left\{ \begin{array}{ccc} \Sigma^k A & \to & F_j \\ \downarrow & & \downarrow \\ A \otimes D^{k+1} & \to & X \end{array} \right\}$$

The complex $F_{j+1}$ is defined as the push-out of

$$\coprod_{I_{j+1}} A \otimes D^{k+1} \leftarrow \coprod_{I_{j+1}} \Sigma^k A \to F_j$$

Moreover there are maps $p_{j+1} : F_{j+1} \to X$ and $q_j : F_j \to F_{j+1}$ such that $p_j = p_{j+1} q_j$.

From Lemma 4.3 it follows that $F_{j+1}$ is $A$-cellular.

(3) Finally if $i$ is a limit ordinal then $F_i = colim_{j<i} F_j$. We let $p_i : F_i \to X$ be the induced map. We in section 4 that cellularity is presereved by directed colimits, hence $F_i$ is $A$-cellular.

**Definition 5.6.** We define the pair $(CW_A X, c_X)$ by letting $CW_A X = colim_\gamma F$ and $c_X : CW_A X \to X$ be the map induced map.

*Remark* 5.7. If $A$ is cofibrant then inductively all the maps $q_j : F_j \to F_{j+1}$ are cofibrations and all complexes $F_i$ are $A$-cellular. Hence $CW_A X$ is also $A$-cellular.

*Remark* 5.8. It is clear from the construction that $CW_A X$ and $c_X$ are natural in $X$, i.e. that we have in fact constructed a functor $CW_A : Ch_{\geq 0}(R) \to Ch_{\geq 0}(R)$ and a natural transformation $c : CW_A \to Id$. The value of this funtor depends on the choice of ordinal $\gamma$, however, as we will see, its homotopy type does not.

**Theorem 5.9.** *If $A$ is cofibrant then the pair $(CW_A X, c_x)$ constructed above is an $A$-cellular approximation of $X$.*

*Proof.* By definition $CW_A X = colim_\gamma F_i$ and each $F_i$ is $A$-cellular, so $CW_A X$ is $A$-cellular. To see that $c_X$ is an $A$-equivalence we note that by the adjoint of **??** it is enough to find a lift in diagrams on the form:

$$\begin{tikzcd} \Sigma^k A \arrow[r] \arrow[d] & CW_A X \arrow[d] \\ A \otimes D^{k+1} \arrow[r] & X \end{tikzcd}$$

We assumed that $A$ was $\gamma$-small, so there is some $j$ such that the map $\Sigma^k A \to CW_A X$ factors as:

$$\Sigma^k A \to F_j \to F_{j+1} \to CW_A X$$

The definition of $F_{j+1}$ implies directly that there is a map $h : A \otimes D^{k+1} \to F_{j+1}$ such that the following commutes:

$$\begin{tikzcd} \Sigma^k A \arrow[r] \arrow[d] & X_{j+1} \arrow[d] \\ A \otimes D^{k+1} \arrow[r] \arrow[ur, "h"] & X \end{tikzcd}$$

The composition of $h$ with the natural map $X_j \to X$ yields the required lift.

□

**Corollary 5.10.** *Let $A$ and $X$ be chain complexes. Then $X$ is $A$-cellular if and only if every $A$-equivalence is an $X$-equivalence.*

*Proof.* Necessity was shown in Lemma 5.2. Suppose that $X$ has the property that any $A$-equivalence is also an $X$-equivalence. We can assume that $A$ is cofibrant. Let $(CW_A X, c_X)$ the $A$-cellular approximation constructed above. The map $c_X$ is an $A$-equivalence, hence also an $X$-equivalence. It follows from 5.4 that $c_X$ is a weak equivalence. The complex $CW_A X$ is $A$-cellular, hence $X$ is also $A$-celllular. □

We close this section by deducing some consequences of the above theorem.

**Proposition 5.11.** *(1) All chain complexes are $S^0$-cellular.*
*(2) $\Sigma X \gg \Sigma A \Leftrightarrow X \gg A$*

*Proof.* To prove (1) note that since $Hom(S^0, f) \cong f$, a map is an $S^0$-equivalence if and only if it is a weak equivalence. If $X$ is cofibrant and $f$ a weak equivalence then $Hom(X, f)$ is also a weak equivalence and the statement follows.

We now prove (2). Assume that $X \gg A$ and let $f$ be a $\Sigma A$-equivalence. As always we may assume that $X$ and $A$ are cofibrant. We need to show that $f$ is an $\Sigma X$-equivalence. But $Hom(\Sigma A, f) \cong Hom(A, \Omega f)$ so $\Omega f$ is an $A$- and therefore also an $X$-equivalence. This is enough since $Hom(X, \Omega f) \cong Hom(\Sigma X, f)$. One show the converse in the same fashion using the fact that $\Omega\Sigma g \cong f$. □

## 6. Acyclic Classes and Localizaing Subcategories

We say that a collection of chain comeplexes $\mathcal{C}$ is closed under *extensions* if given a map $f : X \to Y$ such that $X$ and the cone $C(f)$ belong to $\mathcal{C}$ then so does $Y$. Among all cellular classes there are those which are also closed under extensions. These are important enough to deserve a name of there own:

**Definition 6.1.** A cellular class $\mathcal{A}$ is called an *acyclic class* if $\mathcal{A}$ is closed under extensions.

If we fix a chain complex $\mathcal{A}$ then the intersection of all acyclic classes containing $A$ is an acyclic class. This is the collection of all $A$-acyclic complexes:

**Definition 6.2.** Let $A \in Ch_{\geq 0}(R)$ be a chain complex. We let $\mathcal{A}(A)$ denote the smallest acyclic class containing $A$. The elements of $\mathcal{A}(A)$ are called the $A$-acyclic complexes. If $X$ is $A$-acyclic then we write that $X > A$.

As an acylic class by definition is cellular, we have by definition that $\mathcal{C}(A) \subset \mathcal{A}(A)$. This inclusion is in general strict as we will see shortly. Hence $>$ provides a coarser invariant then $\gg$. Hence it should be easier to classify $>$. This is indeed the case. Th classification of acyclic classes of finite suspension spaces by Bousfield in [**?**] was considered an important acievement in topology. Acyclic classes of finite chain complexes of modules over a Noetherian ring have also been classified by Stanley in [**?**] and the author in paper two of this thesis (see section 7 of paper two). The classification is in terms of the support of the homology of the chain complexes: if $A$ and $X$ are finite (i.e. the $R$-modules $\oplus_i H_i A$ and $\oplus_i H_i X$ are finitely generated) then $X > A$ if and only if for every $i$:

$$\text{Supp } H_i X \subset \text{Supp } \oplus_{j \leq i} H_j A$$

As for $\mathcal{C}(A)$ there is an alternative description of $\mathcal{A}(A)$ in terms of the $Hom$-complex.

**Definition 6.3.** Fix a chain complex $A$. We say that a complex $N$ is $A$-trivial if $Hom(A, N)$ is acyclic (its homology is trivial).

We can regard the collection of all complexes $X$ such that if $Y$ is $A$-trivial then $Y$ is $X$-trivial. It is straightforeward to verify that this is an acyclic class. In fact, as we show below, these are precisely the $A$-acylclic complexes.

**Definition 6.4.** Fix a chain complex $A$ and a map $f : X \to Y$. We say that $f$ is *$A$-local* if $Hom(f, N)$ is a weak equivalence for all $A$-trivial complexes $A$.

**Definition 6.5.** A pair $(X', c_X)$, where $X'$ is a complex and $c_X : X \to X'$ a map, is called an *$A$-localization* if $X'$ is $A$-trivial and $c_X$ is $A$-local.

*Remark* 6.6. An $A$-localization is unique up to homotopy in the sense that if $(X', c_X)$ and $(X'', d_X)$ are $A$-localizations then $X'$ and $X''$ are homotopic under $X$.

Given $A$ we can construct a functor $P_A : Ch_{\geq 0}(R) \to Ch_{\geq 0}(R)$ and a natural transformation $c : 1 \to P_A$ such that for every $X$ the pair $(P_A X, c_X)$ is an $A$-localization of $X$. The construction of the pair $(P_A, c)$ is similar to the construction of $(CW_A, c)$ in section 5 so we omit the details (the interested reader can find more details in [**?**]).

With the aid of $(P_A, c)$ it is now not hard to prove:

**Theorem 6.7.** *Let $A$ and $X$ be chain complexes. Then $X$ is $A$-acyclic if and only if all $A$-trivial complexes are $X$-trivial.*

KIESSLING: DEPARTMENT OF MATHEMATICS, THE ROYAL INSTITUTE OF TECHNOLOGY, S – 100 44 STOCKHOLM, SWEDEN

*E-mail address*: jonkie@kth.se