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Exactly solved quantum many-body systems in one dimension

Martin Hallnäs

Department of Physics
School of Engineering Sciences
Royal Institute of Technology

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Martin Hallnäs
Department of Physics
School of Engineering Sciences
Royal Institute of Technology
Albanova University Center
SE-106 91 Stockholm
Sweden

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Abstract. This thesis is devoted to the study of various examples of exactly solved quantum many-body systems in one-dimension. It is divided into two parts: the first provides background and complementary results to the second, which consists of three scientific papers. The first paper concerns a particular extension, corresponding to the root system C_N, of the delta-interaction model. We prove by construction that its exact solution, even in the general case of distinguishable particles, can be obtained by the coordinate Bethe ansatz. We also elaborate on the physical interpretation of this model. It is well known that the delta-interaction is included in a four parameter family of local interactions. In the second paper we interpret these parameters as coupling constants of certain momentum dependent interactions and determine all cases leading to a many-body system of distinguishable particles which can be solved by the coordinate Bethe ansatz. In the third paper we consider the so-called rational Calogero-Sutherland model, describing an arbitrary number of particles on the real line, confined by a harmonic oscillator potential and interacting via a two-body interaction proportional to the inverse square of the inter-particle distance. We construct a novel solution algorithm for this model which enables us to obtain explicit formulas for its eigenfunctions. We also show that our algorithm applies, with minor changes, to all extensions of the rational Calogero-Sutherland model which correspond to a classical root system.

Preface

This thesis for the degree Licentiate of Engineering is an account of my research at the Physics Department of the Royal Institute of Technology in Stockholm, Sweden, during the years 2004 and 2005. It is divided into two parts: the first provides background and complementary results, while the second consists of the following:

Appended scientific papers

(1) Hallnäs, Martin and Langmann, Edwin (2005). Exact solutions of two complementary one-dimensional quantum many-body systems on the half-line. *Journal of Mathematical Physics*, vol. 46, num. 5, p. 1-15.

(2) Hallnäs, Martin, Langmann, Edwin and Paufler, Cornelius (2005). Generalized local interactions in 1D: solutions of quantum many-body systems describing distinguishable particles. *Journal of Physics A: Mathematical and General*, vol. 38, p. 4957-4974.

(3) Hallnäs, Martin and Langmann, Edwin (2005). Explicit formulas for the eigenfunctions of the N-body Calogero model. Preprint.

These will henceforth be referred to as Paper 1, Paper 2 and Paper 3.

On my contributions to these scientific papers

The results presented in the three scientific papers listed above have all been obtained in close collaboration between all authors. Regarding my specific contributions they can be summarized as follows:

(1) This paper presents and extends results from my master thesis [Hal04]. The computations were all performed by me and I also wrote a first version of the manuscript.

(2) I developed and wrote the material in Sections 3 and 4, where the later is based on an idea by Cornelius Paufler. I also took part in writing the remaining parts of the paper.

(3) Lemma 3.1, its root system generalisation and the results on the one-particle case is the result of a collaborative effort, while the reamining results were obtained by me. The work of writing the manuscript has been divided equally among the two authors.

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Many people have contributed, in more ways than one, to this thesis and the research it represents: most of all my supervisor Edwin Langmann with his never ending support, guidance and above all inspiration; Cornelius Paufler by supplying imaginative ideas and clever thinking on the project that lead to Paper 2; all the teachers I have had through the years by filling my head with more exciting facts and figures than I ever could remember; colleagues, friends and family with the friendship and support that keeps me going. To all of you I want to extend my warmest Thanks!

Stockholm, November 2005 Martin Hallnäs

Contents

Part 1

Background and complementary results

CHAPTER 1

Introduction

When using a mathematical model for computing properties or predicting the behavior of a physical system we are often confronted with the problem of solving a particular differential equation. As a concrete example, one might think of the Schrödinger equation which determines the energy spectrum of a quantum mechanical model or the phase-space trajectories of a model in classical mechanics as solutions of Hamilton's equations. In most cases of direct physical relevance it is, however, very difficult to obtain the solution of these differential equations. This gives a strong incentive for further studies of the rare cases in which solutions are known and also to look for new examples. Such models can then be used as stepping stones for studying more complex cases and also to shed light on the general theories from which they emerged. A striking example of recent interest is the way in which exactly solved spin chains have been used to extract information from supersymmetric Yang-Mills theories; see e.g. [MZ03]. More classical examples include the exact computation of the energy spectrum for the hydrogen atom and Newton's solution of the Kepler problem. We should also mention that exactly solved models of these and other types have not only drawn from but also supplied inspiration and results to many areas in pure mathematics.

In this thesis we are mainly interested in quantum systems of one or more particles which can be described by an exactly soluble mathematical model. Quite a large number of such systems are known in one dimension, while there are very few in higher dimensions. Our first example is the so called delta-interaction model, which formally can be defined by the Hamiltonian

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + 2c \sum_{1\leq j<k\leq N} \delta(x_j - x_k), \tag{1.1}$$

with $\partial_{x_j} = \partial/\partial x_j$ and c a real coupling constant. Its solution was first obtained in the case of bosonic particles by Lieb and Liniger [LL63], while the general case of distinguishable particles was solved by Yang [Yan67]. The delta-interaction model has since then been further developed in a number of directions. One particular class of extensions are those where the structure of the interactions can be described in terms of so called root systems. This can for example be used to introduce a boundary in the model. In Paper 1 we elaborate this point of view and prove

that Yang's solution method extends to all cases corresponding to a classical root system.

It is well known that there exist in one dimension a four parameter family of interactions which like the delta-interaction are localized at a single point. The coupling constant of the delta-interaction corresponds to one of these parameters, while the remaining three, on a formal level, can be attached to certain momentum dependent interactions. One of the main results in this thesis is the classification of, to which of the resulting quantum many-body systems, Yang's solution method consistently can be applied. A precise formulation and proof of the result is contained in Paper 2, where also the solutions are explicitly constructed in all applicable cases.

Another type of exactly solved quantum many-body systems, which are rather different from the delta-interaction model, are those named after Calogero and Sutherland; see e.g. [Cal71, Sut72]. They can be characterized by the fact that their interaction terms, in some limit, all reduce to those of the so-called rational Calogero-Sutherland model, formally defined by the Hamiltonian

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + \omega^2 \sum_{j=1}^{N} x_j^2 + 2\lambda(\lambda - 1) \sum_{1 \leq j < k \leq N} (x_j - x_k)^{-2}, \tag{1.2}$$

with ω and $\lambda > 0$ real coupling parameters. A special property of the rational Calogero-Sutherland model is that its eigenfunctions are closely related to a natural many-variable generalization of the Hermite polynomials. In Paper 3 an explicit series representation is derived for these polynomials and their generalizations by classical root systems.

In defining a quantum mechanical model it is not sufficient to just write down its Hamiltonian. We must also specify on which domain of functions it is to be defined. This is in the cases discussed above a rather delicate matter due to the singularities present in their interactions. In Chapter 2 we therefore describe how this issue can be resolved and the formal Hamiltonians in (1.1) and (1.2) can be turned into well defined models. In Chapter 3 we then discuss methods by which they can be solved. In particular, we focus on the method of factorisable scattering and elaborate on the relation between the rational Calogero-Sutherland model and the so called generalized Hermite polynomials. Chapter 4 deals with constructions of exactly soluble quantum many-body systems. The root system construction, as well as a connection between multi-variable polynomials and models of Calogero-Sutherland type, is here treated in some detail. It seems that this later construction has not before been fully explored in the literature. The treatment presented here therefore contains certain new results. We conclude in Chapter 5 by an outlook on future research.

CHAPTER 2

From formal Hamiltonians to well-defined models

Although models with singular interactions have been in use since the early days of quantum mechanics, a proposal for how such models can be put on firm mathematical ground was first presented as late as 1963 by Berezin and Faddeev [BF61]. They suggested that the theory of self-adjoint extensions of symmetric operators should be used. Following this approach we here show how the formal Hamiltonians presented in the introduction can be turned into mathematically well-defined models.

1. Self-adjoint extensions of symmetric operators

Observables, such as the Hamiltonian, are in quantum mechanical models required to be described by self-adjoint operators. One reason for this requirement is that the dynamics should be given by a unitary group, so that probabilities are conserved. According to Stone's theorem, see e.g. Section VIII in Ref. [RS75], the infinitesimal generator of such a group, in this case the Hamiltonian, must be self-adjoint. Another important reason for the restriction to self-adjoint operators is the spectral theorem, which ensures a real spectrum and a complete set of eigenfunctions. In a more concrete sense this means that an expectation value of the observable can be associated to all states of the quantum mechanical model and also that this expectation value is a real number.

Physical reasoning, on the other hand, can in most cases only produce a formal Hamiltonian, i.e., a differential operator whose domain of definition is left unspecified. Domain here refers to the set of functions on which the Hamiltonian is allowed to act, and in the cases we shall consider they are all subsets of a given L^2 space of square-integrable functions. It is often rather easy to construct a domain, dense in the given L^2 space, on which the Hamiltonian is well-defined and symmetric. To find one on which it is self-adjoint or indeed to determine whether such a domain exist at all is, on the other hand, often rather difficult. This problem leads to two important questions: under which conditions has a given symmetric operator self-adjoint extensions, and if it does, how can we characterize them. A definite answer to these questions was first obtained by von Neumann [vN55] using his theory of deficiency indices. In this section we give a brief review of his approach. The presentation is based on the monographs by Naimark [Nai68], Reed and Simon [RS75] and Thirring [Thi81], where further details can be found.

We start by recalling the precise distinction between a symmetric and a self-adjoint operator. Suppose that A is an unbounded operator with a domain $D(A)$, dense in some Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$. Then its adjoint A^* has the domain

$$D(A^*) = \{\psi \in \mathcal{H} : (\psi, A\phi) = (\tilde{\psi}, \phi) \text{ for some } \tilde{\psi} \in \mathcal{H} \text{ and all } \phi \in D(A)\}$$

and is defined by the formula $(\psi, A\phi) = (A^*\psi, \phi)$, for all $\phi \in D(A)$ and $\psi \in D(A^*)$. Note that requiring $D(A)$ to be dense in $\mathcal{H}$ is essential for the adjoint of A to be uniquely defined. If $A = A^*$ then A is said to be self-adjoint, while it is called symmetric if $A \subset A^*$, i.e., if $D(A) \subset D(A^*)$ and $A\psi = A^*\psi$ for all $\psi \in D(A)$.

EXAMPLE 2.1. *The momentum operator $p = -i\partial_x$ on $L^2([0,1])$. Taking as the domain of p the subset of functions ψ which are absolutely continuous and obey the boundary condition $\psi(0) = \psi(1) = 0$ it is only a symmetric and not a self-adjoint operator. The reason is that we for all ϕ in $L^2([0,1])$ and ψ in the domain of p have that*

$$(p\psi, \phi) \equiv \int_0^1 \overline{(p\psi(x))}\phi(x)dx = \int_0^1 \overline{\psi(x)}p\phi(x)dx \equiv (\psi, p\phi).$$

Recall that absolute continuity is here required in order for the partial integration to be justified; see e.g. Thirring [Thi81]. If we, on the other hand, extend the domain of p to include all functions ψ in $L^2([0,1])$ such that $\psi(0) = e^{i\alpha}\psi(1)$, for a fixed but arbitrary real number α, we instead find that

$$(p\psi, \phi) = (\psi, p\phi) + i[\overline{\psi}\phi]_0^1$$

with a boundary term which in general is non-zero. We must therefore restrict the domain of the adjoint of p in such a way that $\psi^*(0)\phi(0) - \psi^*(1)\phi(1) = 0$. It is clear that this means that ϕ has to obey the same boundary conditions as ψ. The domain of p hence coincide with its adjoint, meaning that p now is self-adjoint. This is not just a mathematical curiosity but also has a clear physical interpretation. If we view the momentum operator as the infinitesimal generator of translations this is simply a manifestation of conservation of probability. Any probability being translated past one of the endpoints of the interval $[0,1]$ must appear in equal quantity at the other side. The only allowed difference is a phase-shift, parameterized by the real parameter α.

The two different boundary conditions that appeared in Example 2.1 has a rather interesting implication for the spectrum of the adjoint p^* of the momentum operator. In the former case it has certain eigenfunctions $\mathrm{e}^{\mathrm{i}\gamma x}$ with complex eigenvalues γ. These do however not obey the boundary conditions imposed in the second case and then consequently does not appear in the domain of p^*. In fact, the adjoint of an operator which is symmetric but not self-adjoint always has certain complex eigenvalues, while a self-adjoint operator only has real eigenvalues. This distinction lies at the heart of von Neumann's theory of deficiency indices and can

be used to determine if a given symmetric operator is self-adjoint, and, if not, to characterize its possible self-adjoint extensions.

Throughout this section we will consider only closed symmetric operators. This does not limit the generality of our discussion in any essential way since every symmetric operator has a closure and the operator and its closure have the same closed extensions; see e.g. Reed and Simon [RS75].

Let A be a symmetric operator on a Hilbert space $\mathcal{H}$ and let $F_\pm$ denote the following subspaces of $\mathcal{H}$:

$$F_\pm = (A \pm \mathrm{i})D(A).$$

The deficiency spaces $N_\pm$ of A are then defined to be the orthogonal complements of $F_\pm$ in $\mathcal{H}$. Using the definition of the adjoint of A and the fact that it is a symmetric operator it is a straightforward exercise to show that this is equivalent to the statement that

$$N_\pm = \ker(A^* \mp \mathrm{i}),$$

where $\ker(A^* \mp \mathrm{i})$ denotes the kernel of $(A^* \mp \mathrm{i})$. The deficiency indices $n_\pm$ are furthermore defined to be the dimension of the deficiency spaces $N_\pm$. The name deficiency spaces is motivated by the following:

LEMMA 2.1. *Let A be a closed symmetric operator on a Hilbert space $\mathcal{H}$. Then the subspaces $D(A)$, N_+ and N_- are linearly independent and together span $D(A^*)$, i.e.,*

$$D(A^*) = D(A) + N_+ + N_-.$$

REMARK 2.1. *The subspaces $D(A)$, N_+ and N_- are of course not orthogonal in the Hilbert space inner product since $D(A)$ is dense in $\mathcal{H}$. They are however orthogonal in the so called graph inner product of A, see e.g. Section X.1 in the monograph by Reed and Simon [RS75], in which a proof of the lemma also can be found.*

COROLLARY 2.1. *A closed symmetric operator is self-adjoint if and only if its deficiency spaces are empty.*

This result provides us with a simple and concrete criteria for testing whether a given closed symmetric operator is self-adjoint. It does however not supply any information on possible self-adjoint extensions of closed symmetric operators with non-empty deficiency spaces. To attack this problem we introduce the Cayley transform, for each symmetric operator A given by

$$V = (A - \mathrm{i})(A + \mathrm{i})^{-1}.$$

Note that it is well-defined since a symmetric operator has no complex eigenvalues. Before proceeding we summarize some of the basic properties of the operator V in a lemma.

LEMMA 2.2. *The Cayley transform V of a symmetric operator A is an isometric operator with domain F_+ and range F_-. Moreover,*

$$A = i(1 + V)(1 - V)^{-1}.$$

REMARK 2.2. *Also the converse of this result holds true: every isometric operator V for which the set $(1-V)D(V)$ is dense in $\mathcal{H}$ is the Cayley transform of a symmetric operator. The proof of this fact as well as that of Lemma 2.2 can be found in Section 14.3 of Naimark [Nai68].*

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. From Corollary 2.1 then follows that $N_+ = N_- = \emptyset$ and hence that $F_+ = F_- = \mathcal{H}$. This together with Lemma 2.2 implies that the Cayley transform of a self-adjoint operator is a unitary operator on $\mathcal{H}$. Recall that a linear operator is called unitary on a Hilbert space $\mathcal{H}$ if and only if it is a bijective isometry, where the property of being isometric means that $||Vx|| = ||x||$ for all $x \in D(V)$. From the previous Remark and Lemma 2.2 conversely follows that each unitary operator V on a Hilbert space $\mathcal{H}$ is the Cayley transform of a self-adjoint operator. This means that constructing a self-adjoint extension of a closed symmetric operator is equivalent to constructing a unitary extension of its Cayley transform. From the fact that its domain and range is respectively F_+ and F_- furthermore follows that this unitary extension must map their complements, the deficiency spaces N_+ and N_-, isometrically onto each other. This is clearly possible only if they have the same dimension, i.e., if the deficiency index $n_+ = n_-$. Hence,

THEOREM 2.1. *A closed symmetric operator $A = i(1+V)(1-V)^{-1}$ can be extended to a self-adjoint operator $\hat{A}$ if and only if its deficiency indices are equal. In such a case corresponds to each unitary mapping $V' : N_+ \to N_-$ a distinct self-adjoint extension*

$$\hat{A} = i(1 + V \oplus V')(1 - V \oplus V')^{-1}.$$

REMARK 2.3. *It is interesting to note that the domain of the self-adjoint extension $\hat{A}$ differs from that of A by the addition of the elements in $(1-V')N_+$. This follows from the observation that*

$$D(\hat{A}) = (1 - V \oplus V')\mathcal{H} = (1-V)F_+ + (1-V')N_+ = D(A) + (1-V')N_+.$$

EXAMPLE 2.2. *Let us now return to Example 2.1 and apply this general result. Starting with the initial boundary conditions $\psi(0) = \psi(1) = 0$ it is clear that the equations $(p^* \pm i)\psi_\pm = 0$ has the following solutions in $D(A^*)$:*

$$\psi_\pm = ce^{\mp x}, \quad c \in \mathbb{C}.$$

The deficiency indices of p are therefore $(n_+, n_-) = (1, 1)$. It follows that p has self-adjoint extensions, characterized by all unitary operators mapping the deficiency space $N_+ = \{ce^{-x} | c \in \mathbb{C}\}$ onto $N_- = \{ce^{x} | c \in \mathbb{C}\}$. These are clearly defined by

$$V'_\alpha : e^{-x} \to e^{i\alpha}e^{x-1}, \quad \alpha \in \mathbb{R}.$$

From Remark 2.3 follows that the corresponding domains of the self-adjoint extensions of p, here denoted by $\hat{p}_\alpha$, are given by

$$D(\hat{p}_\alpha) = \{\psi + ce^{-x} - ce^{i\alpha}e^{x-1} : \psi \in D(p), c \in \mathbb{C}\}.$$

It is a matter of straightforward computations, see e.g. Example 2.5.12 in Thirring [Thi81], to verify that this indeed coincides with the self-adjoint extensions obtained by more direct methods in Example 2.1. We have thus recovered the results of Example 2.1 and, in addition, shown that these are the only subspaces of $L^2([0,1])$ on which the momentum operator is self-adjoint.

2. The generalized point interaction

We now use the machinery assembled in the previous section to show that the most general point interaction in one dimension is characterized by four real parameters. This so-called generalized point interaction was first introduced by Šeba [Š86]. More recent accounts, as well as extensive references to subsequent and related work, can be found in the monographs by Albeverio et.al. [AGeKH05,AK99].

Consider the free Hamiltonian $h = -d^2/dx^2$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Take as the domain $D(h)$ of h the subspace consisting of all functions $\psi \in \mathcal{H}$, such that ψ' is absolutely continuous, $\psi'' \in \mathcal{H}$ and $\psi(0) = \psi'(0) = 0$. It is clear that this operator is symmetric, i.e., that

$$(\phi, h\psi) = (h\phi, \psi), \quad \forall\ \phi, \psi \in D(h).$$

It is however not self-adjoint since $D(h)$ is a proper subset of the domain $D(h^*)$ of its adjoint h^*. This is a direct consequence of the fact that the functions $\psi \in \mathcal{H}$ such that ψ' is absolutely continuous except possibly at the point $x = 0$ and $\psi'' \in \mathcal{H}$ constitutes $D(h^*)$. To determine whether h can be extended to a self-adjoint operator we follow the general method reviewed in the previous section and compute its deficiency indices. This amounts to solving the differential equations $(-d^2/dx^2 \pm \mathrm{i})\psi = 0$ in $D(h^*)$. It follows by inspection that the space of solutions is spanned by

$$\psi_1^{\pm}(x) = \begin{cases} \mathrm{e}^{-(1\pm\mathrm{i})x/\sqrt{2}}, & x > 0 \\ 0, & x \le 0 \end{cases}, \quad \psi_2^{\pm}(x) = \begin{cases} 0, & x > 0 \\ \mathrm{e}^{(1\pm\mathrm{i})x/\sqrt{2}}, & x \le 0 \end{cases}.$$

The deficiency indices of h are therefore $(2,2)$. From Theorem 2.1 and the fact that the space of unitary 2×2 matrices can be parameterized by four real parameters thereby follows that h has a four parameter family of self-adjoint extensions. Rather than characterizing these extension by a set of unitary matrices it is for our purposes more convenient to represent them in terms of concrete boundary conditions.

Proposition 2.1. *Let $D(h)$ denote the set of all functions $\psi \in L^2(\mathbb{R})$ such that ψ' is absolutely continuous except possibly at the point $x = 0$, $\psi'' \in L^2(\mathbb{R})$ and*

$$\psi(+0) - \psi(-0) = 2\lambda[\psi'(+0) + \psi'(-0)] + (\gamma + \mathrm{i}\eta)[\psi(+0) + \psi(-0)], \tag{2.1a}$$

$$\psi'(+0) - \psi'(-0) = \frac{c}{2}[\psi(+0) + \psi(-0)] - (\gamma - \mathrm{i}\eta)[\psi'(+0) + \psi'(-0)] \tag{2.1b}$$

for some fixed set of real numbers c, λ, γ and η. Then the differential operator $h = -d^2/dx^2$ with domain $D(h)$ is self-adjoint.

SKETCH OF PROOF. Let $\psi \in D(h)$ and let $\phi \in L^2(\mathbb{R})$ be such that ϕ' is absolutely continuous except possibly at $x = 0$ and $\phi'' \in L^2(\mathbb{R})$. By partial integration then follows that

$$\int_{\mathbb{R}} \overline{\psi''}\phi dx = \int_{\mathbb{R}} \overline{\psi}\phi'' dx + [\overline{\psi'}\phi - \overline{\psi}\phi']_{x=+0} - [\overline{\psi'}\phi - \overline{\psi}\phi']_{x=-0}.$$

To prove that h is self-adjoint it is thus necessary and sufficient to verify that

$$[\overline{\psi'}\phi - \overline{\psi}\phi']_{x=+0} = [\overline{\psi'}\phi - \overline{\psi}\phi']_{x=-0}$$

if and only if also $\phi \in D(h)$. Note that the boundary conditions in (2.1) can be rewritten as

$$\psi(+0) = \alpha_1\psi(-0) + \alpha_2\psi'(-0)$$
$$\psi'(+0) = \beta_1\psi(-0) + \beta_2\psi'(-0)$$

for some complex constants α_1, α_2, β_1 and β_2 which can be obtained through straightforward computations. Combine this with the condition for self-adjointness deduced above to obtain the following boundary conditions for ϕ:

$$\phi(+0) = \overline{(\alpha_1\beta_2 - \alpha_2\beta_1)}^{-1}[\overline{\alpha_1}\phi(-0) + \overline{\alpha_2}\phi'(-0)],$$
$$\phi'(+0) = \overline{(\alpha_1\beta_2 - \alpha_2\beta_1)}^{-1}[\overline{\beta_1}\phi(-0) + \overline{\beta_2}\phi'(-0)].$$

Insert the values of the appearing constants to verify that these indeed are equivalent to the boundary conditions for ψ. Hence, $D(h^*) = D(h)$ and h is self-adjoint. □

It is interesting to note that the boundary conditions in (2.1) can be given a natural, albeit heuristic, physical interpretation as arising from the formal Hamiltonian

$$h = -\partial_x^2 + c\delta(x) + 4\lambda\partial_x\delta(x)\partial_x + 2(\gamma + \mathrm{i}\eta)\partial_x\delta(x) - 2(\gamma - \mathrm{i}\eta)\delta(x)\partial_x. \tag{2.2}$$

We recognize the first interaction term as the ordinary delta-interaction while the remaining terms also contain the momentum operator $p = -\mathrm{i}\partial_x$. The boundary conditions in (2.1) can be recovered from this Hamiltonian by twice integrating the implied eigenvalue equation $h\psi = e\psi$: first from $x = -0$ to $x = +0$ and then from $x = -0$ to $x > 0$, followed by an integration from $x = -0$ to $x = +0$. It should be mentioned that this is not the only possible physical interpretation of the boundary conditions in (2.1) and the self-adjoint extension they represent. A discussion of alternative interpretations can, e.g., be found in the introduction to Exner and Grosse [EG] and in Paper 2.

These results can now be extended to a many-body system, where the delta-interaction model in (1.1) is contained as a special case. Consider the free Hamiltonian for N particles, $H_N = -\sum_{j=1}^N \partial_{x_j}^2$, on the Hilbert space $\mathcal{H}_N = L^2(\mathbb{R}^N)$. Require of all functions ψ in the domain of H_N that $\partial_{x_j}\psi$ is absolutely continuous except possibly at the points $x_j = x_k$ and that $\partial_{x_j}^2 \in \mathcal{H}_N$, for all $j, k = 1, \ldots, N$ such that $k \neq j$. There now remains to specify appropriate boundary conditions

at the boundaries $x_j = x_k$. For this we first consider the two-particle Hamiltonian $H_2 = -\partial_{x_1}^2 - \partial_{x_2}^2$. Observe that it in the two coordinates $x = x_1 - x_2$ and $y = x_1 + x_2$ takes the form $H_2 = -2\partial_x^2 - 2\partial_y^2$ with the boundary now located at $x = 0$. In these variables it is clear that the boundary conditions in (2.1) render H_2 self-adjoint. Reversing the previous coordinate change we can rewrite these in the variables x_1 and x_2 as follows:

$$\begin{aligned}\psi|_{x_1=x_2+0} - \psi|_{x_1=x_2-0} &= \lambda(\partial_{x_1} - \partial_{x_2})[\psi|_{x_1=x_2+0} + \psi|_{x_1=x_2-0}] \\ &\quad +(\gamma + \mathrm{i}\eta)[\psi|_{x_1=x_2+0} + \psi|_{x_1=x_2-0}], \\ (\partial_{x_1} - \partial_{x_2})[\psi|_{x_1=x_2+0} - \psi|_{x_1=x_2-0}] &= c[\psi|_{x_1=x_2+0} - \psi|_{x_1=x_2-0}] \\ &\quad -(\gamma - \mathrm{i}\eta)(\partial_{x_1} - \partial_{x_2})[\psi|_{x_1=x_2+0} - \psi|_{x_1=x_2-0}].\end{aligned}$$

Now let us turn to the general N-body case. Since each boundary involves only two coordinates x_j and x_k it is clear that boundary conditions which render also the general N-body Hamiltonian self-adjoint are obtained from the two-body case by substituting x_j for x_1 and x_k for x_2.

3. A singular long ranged interaction

As a second application of the theory of defect indices we consider in this section the long ranged singular interaction defined by the potential $v(x) = g/x^2$, with real coupling constant g. We will in particular show that, for most values of g, there exists a unique domain on which the associated Hamiltonian is self-adjoint.

Let $h = -d^2/dx^2 + g/x^2$, with $g \in \mathbb{R}$ and domain $D(h)$ consisting of all functions $\psi \in L^2(\mathbb{R})$ such that ψ' is absolutely continuous; $\psi', \psi'' \in L^2(\mathbb{R})$; and $\psi(0) = \psi'(0) = 0$. The domain of its adjoint h^* is then rather constrained by the conditions that ψ' should be absolutely continuous except possibly at the point $x = 0$ and that $\psi', (-\psi'' + (g/x^2)\psi) \in L^2(\mathbb{R})$. Whether or not these two domains coincide is a priori not clear. Situations where neither ψ'' nor $(g/x^2)\psi$ are in $L^2(\mathbb{R})$ but where their difference is could occur, thereby rendering the conditions defining $D(h^*)$ weaker than those of $D(h)$. To determine precisely when this happens we turn once again to the tools developed in Section 1. The first step is to solve the differential equations $(h \pm \mathrm{i})\psi = 0$, or equivalently $\psi'' = (g/x^2 \pm \mathrm{i})\psi$. It is a well known fact that all solutions are linear combinations of

$$\psi_1^{\pm}(x) = \begin{cases} x^{1/2}\mathcal{C}_\nu(\sqrt{\mp\mathrm{i}}x), & x > 0 \\ 0, & x \le 0 \end{cases}$$

and

$$\psi_2^{\pm}(x) = \begin{cases} 0, & x > 0 \\ (-x)^{1/2}\mathcal{C}_\nu(-\sqrt{\mp\mathrm{i}}x), & x \le 0 \end{cases},$$

with $\nu = \sqrt{g + 1/4}$ and $\mathcal{C}_\nu$ any of the two Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$; see e.g. Abramowitz and Stegun [AS65]. To determine which of these solutions that actually are contained in $D(h^*)$ we recall that

$$H_\nu^{(1)}(x) \sim \sqrt{2/(\pi x)}\mathrm{e}^{\mathrm{i}(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \quad H_\nu^{(1)}(x) \sim \sqrt{2/(\pi x)}\mathrm{e}^{-\mathrm{i}(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)},$$

as $|x| \to \infty$. In each case we thereby conclude that at most two of the four solutions listed above are contained in $D(h^*)$. As $|x| \to 0$ we furthermore have that

$$H_\nu^{(1)}(x) \sim H_\nu^{(2)}(x) \sim \frac{\mathrm{i}2^\nu \Gamma(\nu)}{\pi} x^{-\nu}.$$

This means that the Hankel functions are square integrable only if $\nu < 1/2$, and the functions $\psi_1^\pm$ and $\psi_2^\pm$ only if $\nu < 1$. Collecting the information above we see that the differential equations $(h \pm \mathrm{i})\psi = 0$ has two linearly independent solutions if $g < 3/4$ and no solutions if $g \geq 3/4$. Using Theorem 2.1 we conclude that the Hamiltonian h, as defined above, is self-adjoint for $g \geq 3/4$ and although not self-adjoint for $g < 3/4$ that it in this case admits a four-parameter family of self-adjoint extensions. Finding these self-adjoint extensions is equivalent to determining appropriate boundary conditions at $x = 0$. It follows from Proposition 2.1 that the allowed choices are parameterized by the boundary conditions in (2.1). From a physical point of view this means that extending the Hamiltonian h to a self-adjoint operator in the case of $g < 3/4$ corresponds to adding a generalized point interaction at $x = 0$.

An important fact for the further discussion is that these results hold true also under the addition of a harmonic oscillator potential $v(x) = \omega^2 x^2$, for any real coupling parameter ω. This can be verified either by repeating the analysis above or by directly verifying that the resulting Hamiltonian indeed is self-adjoint. A precise mathematical definition of the rational N-body Calogero-Sutherland model, defined in the introduction by the formal Hamiltonian in (1.2), can now be obtained by following the latter steps of the previous section.

CHAPTER 3

Exact solution methods

Two important notions in describing the physics of many-body systems are those of scattering and bound states. The former describe collision processes involving any number of particles, while the latter represents situations where one or more particles are bound to each other. In this chapter we first discuss the property of factorisable scattering, in which many-body systems it arise and how it can be used to exactly compute their scattering states. We then proceed to discuss bound states of the rational Calogero-Sutherland models and to prove that they are given by a natural multi-variable generalization of the Hermite polynomials.

1. Factorisable scattering

Scattering of particles is in one dimension a very special phenomena since particles can pass each other only by going through each other. This is a fact which carries great technical simplifications. A scattering process involving an arbitrary number of particles N is nevertheless, even in one dimension, very complicated to describe. We must therefore resort to approximations, or consider many-body systems where the scattering processes are simpler than in the generic case. One example of such systems are those which support so-called factorisable scattering. Factorisable here refers to the property that each scattering event can be factorized into a sequence of two-body events. We will in this section give a brief review, based on the paper by A. B. Zamolodchikov and Al. B. Zamolodchikov [ZZ79], of how the scattering data can be extracted from a many-body system with this property.

Let us start by asking for which systems we should expect the scattering processes to factorize in this manner. As the section progresses it will become apparent that one particular property which is found in all such systems is that the particle momenta are conserved in collisions. To show that such systems with non-trivial interactions indeed exist we consider two examples which are particularly important for this thesis: the rational Calogero-Sutherland model and the delta-interaction model. In these cases we can derive the conservation of particle momenta as a direct consequence of their integrability, i.e., that their Hamiltonians are contained in a set of at least N commuting differential operators. We will construct these differential operators only on a formal level, without referring to their domains, a fact which leaves the discussion rather heuristic. It does however provide valuable physical intuition for the mathematically precise constructions which will follow.

EXAMPLE 3.1. *The rational Calogero-Sutherland model in the absence of a confining harmonic oscillator potential,*

$$H = \sum_{j=1}^{N} p_j^2 + g \sum_{1 \le j < k \le N} (x_j - x_k)^{-2}.$$

To obtain a set of integrals of motion we will use the notion of Dunkl operators, introduced by Dunkl [Dun88] in the context of generalized spherical harmonics. The construction we now describe is due to Polychronakos [Pol92]. Let P_j denote the generalized momentum operators

$$P_j = p_j + i \sum_{\substack{j,k=1 \\ k \neq j}}^{N} W(x_j - x_k) T_{jk}, \quad W(x) = \lambda x^{-1},$$

with T_{jk} the exchange operator permuting particles j and k, i.e., $T_{jk} p_j = p_k T_{jk}$ and similarly for x_j. By straightforward computations now follows that these generalized momentum operators all mutually commute. We therefore obtain a set of commuting operators I_n by the simple prescription

$$I_n = \sum_{j=1}^{N} P_j^n.$$

Note that the generalized momentum operators P_j in the asymptotic limit where all particles are far apart from each other reduce to the ordinary particle momenta p_j. It is also easily verified that I_2 equals the Hamiltonian of the rational Calogero-Sutherland model. This implies that the particle momenta in the rational Calogero-Sutherland model indeed are conserved in scattering processes.

EXAMPLE 3.2. *The delta-interaction model, as defined by the formal Hamiltonian in (1.1). Although somewhat more involved we can also in this case obtain an adequate set of integrals of motion by following the approach of Polychronakos. The appropriate generalized momentum operators P_j are obtained from those in the case of the rational Calogero-Sutherland model by setting $W(x) = c\,\mathrm{sgn}(x)$, where $\mathrm{sgn}(x)$ denotes the sign function, and by adding the term $c \sum_{k \neq j} T_{jk}$. It is now a matter of straightforward but somewhat tedious computations to verify that the resulting operators $I_n = \sum_{j=1}^{N} P_j^n$ indeed commute among themselves. One also verifies that the Hamiltonian of the delta interaction is contained in this set as I_2. Further details on this computation, which is complicated by the fact that the generalized momentum operators themselves no longer mutually commute, can be found in the paper by Polychronakos [Pol92]. That the asymptotic momenta are conserved also in the delta interaction model now follows from the fact that the operators I_n do not contain any terms with sign functions. They are consequently linear combinations of terms containing only the ordinary momentum operators p_j, except at points of coinciding particle coordinates.*

Let us now take a more general view on one-dimensional N-body systems which has this property of conserved asymptotic momenta. We start by observing that the configuration space of an N-body system which supports scattering can be divided into $N!$ asymptotic regions where the particles are free, corresponding to the different orderings of the particles. Each such region can be characterized by an inequality

$$x_{Q(1)} < x_{Q(2)} < \cdots < x_{Q(N)},$$

where the permutation $Q \in S_N$ refers to the ordering of the particles on the real line. We henceforth denote the asymptotic region associated with the permutation $Q \in S_N$ by Δ_Q and employ the short hand notation $x_Q = (x_{Q(1)}, \ldots, x_{Q(N)})$. Now consider a scattering process where the incident particles have momenta $k_1 > k_2 > \cdots > k_N$. Conservation of momenta then implies that the wave function in each region Δ_Q must be a linear combination of plane waves of the form $\mathrm{e}^{\mathrm{i}k_P \cdot x_Q}$ with $P \in S_N$. This motivates the so called Bethe Ansatz

$$\psi_Q(x) = \sum_{P \in S_N} A_P(Q) \mathrm{e}^{\mathrm{i}k_P \cdot x_Q}, \quad x \in \Delta_Q, \tag{3.1}$$

for the wave function in each asymptotic region Δ_Q. This means, in particular, that the amplitude of the incoming wave in the region Δ_Q is given by $A_I(Q)$, while the amplitude for the outgoing is given by $A_{\tilde{I}}(Q)$, where $\tilde{I}$ refers to the permutation $(N, N-1, \ldots, 1)$. To determine the values of the coefficients $A_P(Q)$ we must now extrapolate the wave function between the different regions of free motion. The direct way of doing this would be to solve the full interacting model, something which in general is very hard to do. There is, however, one way around this problem and that is to extrapolate the wave function across regions where only two particles interact. The virtue of this approach is that it only requires that we obtain the full solution in the case of two particles.

To proceed further along this line we now assume that the many-body systems under consideration contain only one type of particles interacting through a translation invariant two-body interaction, i.e., that they can be associated with a Hamiltonian

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{1 \leq j < k \leq N} \hat{V}(x_j - x_k) \tag{3.2}$$

for some interaction $\hat{V}$. The requirement that the particles be identical means that $\hat{V}$ must be even, i.e., $\hat{V}(-x) = \hat{V}(x)$. Note that this prescription allows not only even potentials but for example also the momentum-dependent interaction $\hat{V}(x) = \partial_x \delta(x) \partial_x$. The full generalized point interaction discussed in Chapter 2 is however not allowed, unless the coupling parameter $\gamma = \eta = 0$, since it otherwise is a model of non-identical particles. This case is treated in detail in Paper 2.

With this in mind consider the extrapolation of the Bethe ansatz in (3.1) from the region Δ_I to Δ_{T_1}, where T_1 is taken to denote the elementary transposition

interchanging particle one and two. The only relevant interaction term is then $\hat{V}(x_1 - x_2)$. The extrapolation therefore requires only the solution of the two-particle case involving particle one and two. It is furthermore clear that this applies to any pair of asymptotic regions Δ_Q and Δ_{QT_i}, differing only by the interchange of two particles. Now recall that the permutation group S_N of N objects is generated by the elementary transpositions T_i, interchanging objects i and $i+1$. This suggest that all the information required to obtain the asymptotics for the eigenfunctions of a Hamiltonian of the same form as in (3.2) is the solution of the two-particle case. In fact, this reduces in the center of mass frame even further to a one-particle problem.

LEMMA 3.1. *Let $\hat{V}$ be an even interaction and let h denote the one-particle Hamiltonian*

$$h = -2\partial_x + \hat{V}(x).$$

Suppose that h for each $e = k^2$ in its continuous spectrum has an eigenfunction $\psi(x;k)$ with the asymptotic form

$$\psi(x;k) \sim \begin{cases} e^{-ikx} + S_R(2k)e^{ikx}, & \text{as } x \to \infty \\ S_T(2k)e^{-ikx}, & \text{as } x \to -\infty \end{cases}$$

for some functions S_R and S_T. Then the coefficients $A_P(Q)$ in the Bethe ansatz for the eigenfunctions of a Hamiltonian of the same form as in (3.2) must obey the recursion relation

$$A_{PT_i}(Q) = S_R(k_{P(i)} - k_{P(i+1)})A_P(Q) + S_T(k_{P(i)} - k_{P(i+1)})A_P(QT_i) \tag{3.3}$$

for all $P, Q \in S_N$.

PROOF. We first prove the statement for $N = 2$. In the coordinates $x = x_1 - x_2$ and $y = x_1 + x_2$ the Hamiltonian in (3.2) is given by

$$H = h - 2\partial_y^2.$$

Also introducing the momenta $k_x = 1/2(k_1 - k_2)$ and $k_y = 1/2(k_1 + k_2)$ the Bethe ansatz in (3.1) is written as

$$\psi(x,y) = \begin{cases} A_I(T_1)\mathrm{e}^{\mathrm{i}(-k_x x + k_y y)} + A_{T_1}(T_1)\mathrm{e}^{\mathrm{i}(k_x x + k_y y)}, & x \to \infty \\ A_{T_1}(I)\mathrm{e}^{\mathrm{i}(-k_x x + k_y y)} + A_I(I)\mathrm{e}^{\mathrm{i}(k_x x + k_y y)}, & x \to -\infty \end{cases}.$$

It is thereby clear that the coefficients $A_I(I)$, $A_{T_1}(I)$, $A_{T_1}(T_1)$ and $A_I(T_1)$ depend only on the eigenfunctions of the one-particle Hamiltonian h.

From the fact that $\hat{V}$ is even follows that also $\psi|_{x\to -x}$ is an eigenfunction of the Hamiltonian h with the same eigenvalue $e = k^2$ as ψ. Note that

$$\psi|_{x\to -x}(x;k) \sim \begin{cases} S_T(2k)\mathrm{e}^{\mathrm{i}kx}, & \text{as } x \to \infty \\ \mathrm{e}^{\mathrm{i}kx} + S_R(2k)\mathrm{e}^{-\mathrm{i}kx}, & \text{as } x \to -\infty \end{cases}.$$

It is hence clear that ψ and $\psi|_{x\to -x}$ are linearly independent and therefore span the set of eigenfunctions of h with eigenvalue $e = k^2$ in its continuous spectrum.

Any such eigenfunction ϕ is thus a linear combination

$$\begin{aligned}\phi(x;k) &= a_1\psi(x;k) + a_2\psi|_{x\to -x}(x;k)\\ &\sim \begin{cases} a_1 \mathrm{e}^{-\mathrm{i}kx} + [S_R(2k)a_1 + S_T(2k)a_2]\mathrm{e}^{\mathrm{i}kx}, & \text{as } x\to\infty \\ [S_R(2k)a_2 + S_T(2k)a_1]\mathrm{e}^{-\mathrm{i}kx} + a_2\mathrm{e}^{\mathrm{i}kx}, & \text{as } x\to -\infty \end{cases}.\end{aligned}$$

for some constants a_1 and a_2. Compare this with the Bethe ansatz as written above to obtain the statement for $N = 2$.

Fix a permutation $Q \in S_N$, an elementary transposition $T_i \in S_N$ and consider the two regions Δ_Q and Δ_{QT_i}. Then only the interaction term $\hat{V}(x_{Q(i)} - x_{Q(i+1)})$ in a Hamiltonian of the same form as in (3.2) is relevant. It is hence sufficient to consider the effective Hamiltonian

$$H_{eff} = -\sum_{j=1}^{N} \partial_{x_j}^2 + \hat{V}(x_{Q(i)} - x_{Q(i+1)}),$$

whose eigenfunctions are of the form

$$\psi(x_1,\ldots,x_N) = \phi(x_{Q(i)}, x_{Q(i+1)})\mathrm{e}^{\mathrm{i}(k_3 x_{Q(3)} + \cdots + k_N x_{Q(N)})}$$

with $k_3,\ldots,k_N \in \mathbb{R}$ and ϕ an eigenfunction of the two-particle Hamiltonian

$$H = -\partial_{x_{Q(i)}}^2 - \partial_{x_{Q(i+1)}}^2 + \hat{V}(x_{Q(i)} - x_{Q(i+1)}).$$

It is thereby clear that the statement for an arbitrary number of particles N follows from the two-particle case. □

The recursion relation in (3.3) can now be used to generate all coefficients $A_P(Q)$ once they are fixed for some permutation $P \in S_N$. From a physical point of view, this amounts to specifying the amplitude of the incoming wave in each region Δ_Q. At this point it is important to note that there is a possible inconsistency in this recursion relation, arising from the fact that the generators T_i of the permutation group S_N obey the defining relations

$$T_iT_i = 1, \quad T_iT_j = T_jT_i, \quad \text{for } |i-j| > 1,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}.$$

This implies that the recursion relation in (3.3) is consistent if and only if

$$A_{PT_iT_i}(Q) = A_P(Q), \tag{3.4a}$$

$$A_{PT_iT_j}(Q) = A_{PT_jT_i}(Q), \quad \text{for } |i-j| > 1, \tag{3.4b}$$

$$A_{PT_iT_{i+1}T_i}(Q) = A_{PT_{i+1}T_iT_{i+1}}(Q), \tag{3.4c}$$

for all $P, Q \in S_N$. These consistency conditions can be written in a somewhat different and more practical form by using the so-called right regular representation of S_N, which we now consider. For each permutation $R \in S_N$ define the linear operator $\hat{R}$ by the condition that $(\hat{R})_{Q,Q'} = \delta_{QR,Q'}$. This enables us to write

$$A_P(QR) = \sum_{Q'\in S_N} (\hat{R})_{Q,Q'} A_P(Q) = (\hat{R}A_P)(Q),$$

where the second equality is taken as the definition of the action of $\hat{R}$ on $A_P(Q)$. It is clear that this defines a representation of the symmetric group S_N in the space of coefficients $A_P(Q)$. The recursion relation in (3.3) can now be written in the following simple form:

$$A_{PT_i} = Y_i(k_{P(i)} - k_{P(i+1)})A_P, \tag{3.5}$$

with Y_i the linear operator

$$Y_i(u) = S_R(u) + S_T(u)\hat{T}_i.$$

Rewriting the consistency conditions in (3.4) for the coefficients $A_P(Q)$ in terms of these operators gives

THEOREM 3.1. *The Bethe ansatz in (3.1) for the eigenfunctions of a Hamiltonian of the same form as in (3.2) is consistent if and only if*

$$Y_i(-u)Y_i(u) = I, \tag{3.6a}$$

$$Y_i(u)Y_j(v) = Y_j(v)Y_i(u), \quad \textit{for } |i-j| > 1, \tag{3.6b}$$

$$Y_i(u)Y_{i+1}(u+v)Y_i(v) = Y_{i+1}(v)Y_i(u+v)Y_{i+1}(u), \tag{3.6c}$$

for all real u and v. In that case

$$A_P = \mathcal{Y}_P(k_1, \ldots, k_N)A_I,$$

where $\mathcal{Y}_P$ is a product of the operators Y_i, obtained by repeatedly using the recursion relation in (3.5).

REMARK 3.1. *The consistency conditions in (3.6) are commonly referred to as the Yang-Baxter equations. They first appeared in the work of Yang [Yan67] on the delta-interaction model. Although they here were derived from the structure of the symmetric group they also have a clear physical interpretation. The first can be interpreted as the requirement that two consecutive collisions each interchanging the momenta of the same two particles is to be the same as no collision at all, while the second and third states that the order in which the two-particle collisions occur should not matter.*

The Yang-Baxter equations in (3.6) puts rather sever restrictions on the set of models for which factorisable scattering is applicable. Indeed, one might wonder if any such models exist. For this reason we now demonstrate that both the delta-interaction and rational Calogero-Sutherland model comply with all the requirements of factorisable scattering. We also derive the explicit form of the operators Y_i for these two models.

EXAMPLE 3.3. *The delta-interaction model, formally defined by the Hamiltonian in (1.1). Following the general framework laid out in the discussion above we start by computing the scattering amplitudes for the one-particle Hamiltonian*

$$h = -\partial_x^2 + c\delta(x), \quad c \in \mathbb{R}.$$

From Chapter 2 we know that this is equivalent to solving the free Schrödinger equation subject to the boundary conditions

$$\psi(+0) = \psi(-0),$$
$$\psi'(+0) - \psi'(-0) = c\psi(+0).$$

By straightforward computations follows that one solution is given by

$$\psi(x;k) = \begin{cases} e^{-ikx} + \frac{c}{i2k-c}e^{ikx}, & x > 0 \\ \frac{i2k}{i2k-c}e^{-ikx}, & x < 0 \end{cases}.$$

Hence,

$$Y_i(u) = S_R(u) + S_T(u)\hat{T}_i = \frac{iu\hat{T}_i + c}{iu - c},$$

which by a straightforward computation can be shown to fulfill the Yang-Baxter equations in (3.6).

EXAMPLE 3.4. *The rational Calogero-Sutherland model in (1.2) with $\omega = 0$. Sutherland [Sut71b] applied the Bethe ansatz to this model in the case of bosons and fermions, while Calogero [Cal71] treated the case of distinguishable particles using somewhat different methods. It is well known that those eigenfunctions of the Hamiltonian*

$$h = -\partial_x^2 + \frac{g}{x^2}$$

which are regular at the origin are linear combinations of

$$\psi_1(x;k) = \begin{cases} x^{1/2}J_\nu(sgn(k)kx), & x > 0 \\ 0, & x < 0 \end{cases}$$

and

$$\psi_2(x;k) = \begin{cases} 0, & x > 0 \\ x^{1/2}J_\nu(-sgn(k)kx), & x < 0 \end{cases}$$

with k any real number, $sgn(k)$ the sign of k and $\nu = \sqrt{g + 1/4}$; see e.g. Abramowitz and Stegun [AS65]. The asymptotic behavior of the Bessel function J_ν implies that

$$\psi_1(x;k) \sim \frac{1}{\sqrt{2\pi}}\left[e^{i(sgn(k)kx - \nu\pi/2 - \pi/4)} + e^{-i(sgn(k)kx - \nu\pi/2 - \pi/4)}\right] \text{ as } x \to \infty,$$

and similarly for ψ_2. Referring back to Lemma 3.1 we see that the transmission amplitude $S_T = 0$ and that the reflection amplitude $S_R(2k) = e^{-sgn(k)i\pi(\nu+1/2)}$, a phase shift depending only on the sign of k. The Yang-Baxter equations in (3.6) are therefore clearly fulfilled.

We have in this section encountered the notion of factorisable scattering in its most simple setting: identical particles interacting through a translation invariant two-body interaction. There are a number of ways of extending this setting. One possibility is to add one or more boundaries to the system. A detailed discussion of one such example, local interactions on the half-line, can be found in Paper 1. Another direction of increasing complexity is to consider a system of non-identical

particles. In general this is a very hard problem which leads to consistency conditions vastly more complicated than the ones considered here; see e.g. A. B. Zamolodchikov and Al. B. Zamolodchikov [ZZ79]. There are, however, examples where the deviation from identical particles in some sense is rather small. The N-body version of the generalized point interaction, considered in Paper 2, is one such example.

2. Generalized Hermite polynomials as bound states

We now turn to the rational Calogero-Sutherland model in the presence of its harmonic oscillator potential. It has in this case a purely discrete spectrum with corresponding bound states given by a natural many-variable generalization of the Hermite polynomials. To make this connection clear we will start by discussing its one particle restriction, the well known model of a single particle in a harmonic oscillator potential. This will also allow us to introduce, with a minimum of technical complications, many of the ideas and methods which we later shall apply to the full N-body case.

In the case of one particle the Hamiltonian in (1.2), of the rational Calogero-Sutherland model, reduce to

$$h = -\partial_x^2 + \omega^2 x^2.$$

Observe that this one-particle Hamiltonian can be factorized according to

$$h = q^* q + e_0, \quad q = \partial_x + \omega x, e_0 = \omega,$$

where q^* is the adjoint of q in the Hilbert space $L^2(\mathbb{R})$. Since q^*q is a positive operator it follows that the spectrum of h is bounded from below by e_0; see e.g. Section 72 in Halmos [Hal74]. If there furthermore exists a square integrable function ψ_0 such that $q\psi_0 = 0$ then this is the ground state of h with energy eigenvalue e_0. In the present case it is clear that such a function is given by

$$\psi_0(x) = \mathrm{e}^{-\frac{1}{2}\omega x^2}.$$

To obtain the remaining bound states it is convenient to consider not the Hamiltonian h but rather the differential operator obtained after conjugation by the ground state,

$$\tilde{h} := \psi_0(h - e_0)\psi_0^{-1} = -\partial_x^2 + 2\omega x \partial_x. \tag{3.7}$$

The reason is that this differential operator maps any given polynomial to a polynomial of equal or lower degree, i.e., it has a triangular structure if written in the polynomial basis $\{z^n : n = 0, 1, \ldots\}$. This suggests that we can obtain an arbitrary number n of its eigenfunctions by restricting it to a fixed polynomial module $\mathcal{P}_n = \langle 1, x, \ldots, x^n \rangle$, in this way transforming a complicated analytical problem into that of diagonalizing finite-dimensional triangular matrices. Let us now exploit this triangular structure to construct a set of eigenfunctions of the differential operator

in (3.7), complete in the space of polynomials in x. Consider therefore its action on $p_n = \sum_{k=0}^{n} a_k x^{n-k}$,

$$\tilde{h}p_n = -\sum_{k=0}^{n}(n-k)(n-k-1)a_k x^{n-k-2} + 2\omega\sum_{k=0}^{n}(n-k)a_k x^{n-k}.$$

Requiring p_n to be an eigenfunction of $\tilde{h}$ clearly implies that its eigenvalue be $e'_n = 2\omega n$ and that the coefficients a_k should obey the recursion relation

$$a_k = -\frac{(n-k+1)(n-k+2)}{2\omega k}a_{k-2}.$$

Setting $a_0 = 2^n$ hence gives

$$p_n(x) = \sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{k!(n-2k)!}(2\omega^{-1/2}x)^{n-2k},$$

where $[n/2]$ denotes the largest integer smaller than or equal to $n/2$. These polynomials are in fact, apart from a factor $\sqrt{\omega}$, the Hermite polynomials. To be more precise, $p_n(x) = H_n(\sqrt{\omega}x)$, with H_n the Hermite polynomial of degree n. This follows from a direct comparison with the series representation for the Hermite polynomials; see e.g. Section 6.1 in Andrews et. al. [AAR99]. It is also a consequence of the observation that the differential operator $\tilde{h}$ has a non-degenerate spectrum and that it is self-adjoint with respect to the inner product with weight function $\psi_0^2 = \mathrm{e}^{-\omega x^2}$. The eigenfunctions of $\tilde{h}$ are therefore orthogonal with respect to this inner product, a fact which together with a choice of normalization can be taken as the definition of the Hermite polynomials.

We now extend the results deduced above to an arbitrary number of particles. The essential difference is then the presence of the singular two-body interaction. We will in the following discussion restrict our attention to those values of its coupling constant for which there is a unique domain on which the Hamiltonian of the rational Calogero-Sutherland model is self-adjoint. Recall from the previous chapter that in these cases all wave-functions vanish at points of coinciding particle coordinates. This together with the one-dimensional nature of the model prevents the particles from overtaking each other. We shall therefore in the following discussion restrict our attention to the part of configuration space defined by the inequality

$$x_1 < x_2 < \cdots < x_N.$$

Once the eigenfunctions are determined for this particular ordering they are extended to all others by whichever exchange statistics one considers. For further details on this point see e.g. Calogero [Cal71].

We proceed to prove a proposition generalizing the factorization of the one-particle Hamiltonian to the full N-particle case. This also yields the ground-state of the rational Calogero-Sutherland model.

PROPOSITION 3.1. *The rational Calogero-Sutherland Hamiltonian in (1.2) admits a factorization* $H = \sum_{j=1}^{N} Q_j^* Q_j + E_0$ *for* $E_0 = \omega N(1 + \lambda(N-1))$ *and*

$$Q_j = \partial_{x_j} + \omega x_j - \lambda \sum_{\substack{j,k=1 \\ k\neq j}}^{N} (x_j - x_k)^{-1}.$$

Moreover, its ground-state is

$$\Psi_0(x_1, \ldots, x_N) = \prod_{j=1}^{N} e^{-\frac{1}{2}\omega x_j^2} \prod_{1\leq j<k\leq N} (x_j - x_k)^{\lambda}$$

with energy eigenvalue E_0.

PROOF. It is clear that $Q_j \Psi_0 = 0$ for all $j = 1, \ldots, N$. Furthermore,

$$\begin{aligned} Q_j^* Q_j = &- \partial_{x_j}^2 + \omega^2 x_j^2 - \omega - 2\omega\lambda \sum_{\substack{k=1 \\ k\neq j}}^{N} x_j (x_j - x_k)^{-1} - \lambda \sum_{\substack{k=1 \\ k\neq j}}^{N} (x_j - x_k)^{-2} \\ &+ \lambda^2 \sum_{\substack{k,l=1 \\ k,l\neq j}}^{N} (x_j - x_k)^{-1}(x_j - x_l)^{-1} \end{aligned}$$

Observe that

$$2\omega\lambda \sum_{\substack{j,k=1 \\ k\neq j}}^{N} x_j (x_j - x_k)^{-1} = \omega\lambda \sum_{\substack{j,k=1 \\ k\neq j}}^{N} (x_j - x_k)(x_j - x_k)^{-1} = \omega\lambda N(N-1)$$

and that

$$\begin{aligned} \sum_{\substack{j,k,l=1 \\ k,l\neq j, k\neq l}}^{N} (x_j - x_k)^{-1}(x_j - x_l)^{-1} = -\frac{1}{3} \sum_{\substack{k,l=1 \\ k,l\neq j, k\neq l}}^{N} [&(x_k - x_j)^{-1}(x_j - x_l)^{-1} \\ &+ (x_l - x_k)^{-1}(x_k - x_j)^{-1} + (x_j - x_l)^{-1}(x_l - x_k)^{-1}]. \end{aligned}$$

Use this fact together with the identity $(xy)^{-1} + (xz)^{-1} + (yz)^{-1} = 0$ for $x+y+z = 0$ to deduce that

$$\sum_{\substack{j,k,l=1 \\ k,l\neq j}}^{N} (x_j - x_k)^{-1}(x_j - x_l)^{-1} = \sum_{\substack{j,k=1 \\ k\neq j}}^{N} (x_j - x_k)^{-2}.$$

Combine these results to obtain the statement. □

Having established the existence of a square-integrable ground-state the next step is to construct the remaining bound states. Let us therefore consider the

differential operator obtained after conjugating the Hamiltonian of the rational Calogero-Sutherland model by its ground-state,

$$\begin{aligned}\tilde{H} &:= \Psi_0^{-1}(H - E_0)\Psi_0 \\ &= \sum_{j=1}^{N}[-\partial_{x_j}^2 + 2\omega x_j \partial_{x_j}] - 2\lambda \sum_{1\leq j<k\leq N} (x_j - x_k)^{-1}(\partial_{x_j} - \partial_{x_k}).\end{aligned} \tag{3.8}$$

We previously observed that its one variable restriction maps any given polynomial to a polynomial of equal or lower degree, and that it consequently has a complete set of polynomial eigenfunctions. A natural first guess at the generalization of this notion to the many variable case would be to consider linear combinations of monomials of the form $x_1^{n_1} \cdots x_N^{n_N}$, with n_j a non-negative integer for all $j = 1, \ldots, N$. Such polynomials are, however, in general not mapped to polynomials by the differential operator in (3.8), e.g., $\tilde{H}x_1 = 2\omega x_1 - 2\lambda \sum_{1<k\leq N}(x_1 - x_k)^{-1}$ which clearly is not a polynomial. To remedy this problem we restrict attention to symmetric polynomials. From our point of view this restriction implies no loss of generality since we have assumed that $x_1 < \cdots < x_N$.

To discuss this in further detail we need to introduce some notation from the theory of symmetric functions; see e.g. Macdonald [Mac95] for a comprehensive account. Recall that an N-tuple $\alpha = (\alpha_1, \ldots, \alpha_N)$ of non-negative integers commonly is referred to as a multi-index. In the following discussion we will furthermore refer to the value of N as the length of the multi-index α and denote the sum of its parts α_j by $|\alpha| = \sum_{j=1}^{N} \alpha_j$. For notational convenience let e_j denote the particular multi-index defined by the condition $(e_j)_k = \delta_{jk}$. With a partition of length N is meant any multi-index $n = (n_1, \ldots, n_N)$ such that $n_1 \geq n_2 \geq \cdots \geq n_N$. A particular basis for the space of all polynomials in N variables is given by the monomials $x^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_N}$ as α runs through all multi-indices of length N. This particular basis has a natural counterpart in the space of symmetric polynomials, parameterized by partitions rather than multi-indices. To each partition n of length N we associate a so called monomial symmetric polynomial

$$M_n(x_1, \ldots, x_N) = \sum_{P\in S_N} x^{Pn}.$$

It is clear that these monomials constitutes a basis for the space of symmetric polynomials in N variables. We mention in passing that these monomials are usually normalized such that the sum only extends over distinct permutations, but we find this less convenient for our purposes.

By considering the action of the differential operator in (3.8) on these monomials we now proceed to prove that it indeed maps the space of symmetric polynomials into itself. We also show that there is a triangular structure present which allows its eigenfunctions to be computed by algebraic means.

LEMMA 3.2. *Let $n = (n_1, \dots, n_N)$ be a partition of length N. Then*

$$\begin{aligned}\tilde{H}M_n =& 2\omega|n|M_n - \sum_{j=1}^{N} n_j(n_j-1)M_{n-2e_j} \\ &- \lambda \sum_{1\le j<k\le N} (2-\delta_{2\nu,n_j-n_k})(n_j-n_k) \sum_{\nu=1}^{[(n_j-n_k)/2]} M_{n-(\nu+1)e_j+(\nu-1)e_k} \\ &+ 2\lambda \sum_{1\le j<k\le N} n_k M_{n-e_j-e_k}.\end{aligned}$$

PROOF. Observe that the statement follows if

$$\begin{aligned}&(x-y)^{-1}(\partial_x-\partial_y)(x^m y^n + x^n y^m) \\ &= (m-n)\sum_{k=1}^{m-n-1} x^{m-1-k}y^{n-1+k} - n(x^{m-1}y^{n-1} + x^{n-1}y^{m-1})\end{aligned}$$

for any $x, y \in \mathbb{R}$ and nonnegative integers m and n such that $m \ge n$. For $m = n$ this is easily verified by direct computations. Suppose $m \ne n$. Then

$$\begin{aligned}&(x-y)^{-1}(\partial_x-\partial_y)(x^m y^n + x^n y^m) \\ &= mx^n y^n \frac{x^{m-n-1}-y^{m-n-1}}{x-y} - nx^{n-1}y^{n-1}\frac{x^{m-n+1}-y^{m-n+1}}{x-y}.\end{aligned}$$

Expand the two fractions in geometric series and collect terms to obtain the statement. □

REMARK 3.2. *Note that the action of the differential operator in (3.8) on the monomial symmetric polynomials as it is written above might contain terms M_α parameterized by multi-indices α which are not partitions, e.g., the action on the monomial $M_{(3,2)}$ contains a term parameterized by $(3,2) - 2e_1 = (1,2)$. Each such multi-index α can, however, be mapped to a unique partition n by permuting its parts α_j. Upon identifying M_α with the resulting M_n it is clear that the differential operator in (3.8) maps the space of symmetric polynomials into itself.*

The so called partial dominance ordering is in the space of partitions defined as follows:

$$m \le n \Leftrightarrow m_1 + \cdots + m_j \le n_1 + \cdots + n_j, \quad \forall j = 1, 2 \dots, N.$$

The word 'partial' refers to the fact that not all partitions can be compared in this ordering, e.g., $n = (3,0,0)$ and $m = (2,2,0)$. If $m \le n$ and $|m| \ne |n|$ we will furthermore write $m < n$. It follows from Lemma 3.2 that the action of the differential operator in (3.8) on the monomial symmetric polynomials in this ordering is triangular, i.e., for all partitions n we have that

$$\tilde{H}M_n = 2\omega|n|M_n + \sum_{m<n} c_{nm}M_m,$$

where the constants c_{nm} in principle can be obtained from Lemma 3.2. This means that we in constructing the eigenfunctions of the differential operator $\tilde{H}$ even in the general N-variable case can restrict it to a polynomial module $\mathcal{P}_n := \langle M_m : m \leq n \rangle$. The explicit construction of the eigenfunctions can be performed much like in the one-particle case. Start by for each partition n making an ansatz $P_n = M_n + \sum_{m<n} a_{nm} M_m$. By Lemma 3.2 then follows that the eigenvalue of P_n is $2\omega|n|$ and furthermore a recursion relation for the coefficients a_{nm}. Although these recursion relations in principle can be inverted they have a rather complicated structure. This construction is elaborated in Paper 3. It follows from their triangular structure that the eigenfunctions of the differential operator in (3.8) are linearly independent and hence provide a basis for the space of symmetric polynomials in N variables. That there in fact also exist an L^2 inner product over $\mathbb{R}^N$ for which they are mutually orthogonal is, due to a high level of degeneracy in the spectrum of the differential operator in (3.8), rather hard to prove. One approach to obtaining this result is to simultaneously diagonalize a family of commuting differential operators which breaks this degeneracy; see e.g. Chapter 9 in the forthcoming monograph by Forrester [For]. Another method, which we now describe, is elaborated in a paper by van Diejen [vD97]. He essentially reversed the argument and started by defining a family of symmetric polynomials through an orthogonality relation and then showed that they are eigenfunctions of the differential operator in (3.8). For all $f, g \in L^2(\mathbb{R}^N, W dx)$, with weight function $W(x_1, \ldots, x_N) = \prod_{j=1}^N \mathrm{e}^{-\omega x_j^2} \prod_{1 \leq j < k \leq N} (x_j - x_k)^{2\lambda}$, let (f, g) denote the inner product

$$(f, g) = \int_{\mathbb{R}^N} \overline{f(x)} g(x) W(x) dx. \tag{3.9}$$

To each partition n we can now associate a unique symmetric polynomial P_n by the two conditions

$$P_n = M_n + \sum_{m<n} a_{nm} M_m, \quad a_{nm} \in \mathbb{C}, \tag{3.10a}$$

$$(P_n, M_m) = 0, \quad \forall m < n. \tag{3.10b}$$

In case n and m are two partitions such that $m < n$ or $n < m$ it follows directly from their definition that the corresponding polynomials P_n and P_m are orthogonal. This orthogonality relation does in fact hold true for any two distinct partitions. In general this is, however, rather hard to prove. The first proof was provided by Baker and Forrester [BF97] by using a limiting procedure relating these polynomials to a particular type of multi-variable Jacobi polynomials for which the corresponding orthogonality relation is known to be valid. A similar proof, exploiting a limit of the multi-variable hypergeometric continuous Hahn polynomials, can be found in van Diejen [vD97]. Kakei [Kak96] has provided a rather different proof using algebraic structures of the rational Calogero-Sutherland model as well as a mapping to its trigonometric counterpart

For future reference we state the orthogonality relation for the polynomials defined by the two conditions in (3.10) as a proposition.

PROPOSITION 3.2. *Let n and m be any two partitions of equal length. Then*

$$(P_n, P_m) \equiv \int_{\mathbb{R}^N} \overline{P_n(x)} P_m(x) W(x) dx = 0 \quad \text{if } n \neq m.$$

REMARK 3.3. *This orthogonality property implies that the polynomials P_n in the one variable case become proportional to the Hermite polynomials upon setting $\omega = 1$. They are for this particular value of the coupling parameter ω therefore commonly referred to as the generalized Hermite polynomials.*

We now proceed to show that the P_n indeed are eigenfunctions of the differential operator in (3.8). Using the fact that $\tilde{H} = \sum_{j=1}^{N} [-\partial_{x_j}^2 - 2(\log \Psi_0)_{x_j} \partial_{x_j}]$ and the fact that $W = \Psi_0^2$, with Ψ_0 the ground-state of the rational Calogero-Sutherland model, it is straightforward to verify that the differential operator in (3.8) is self-adjoint with respect to the inner product in (3.9). We have by Lemma 3.2 that

$$\tilde{H} P_n = 2\omega |n| M_n + \sum_{k<n} c_{nk} M_k + \sum_{m<n} a_{nm} \tilde{H} M_m = 2\omega |n| M_n + \sum_{m<n} d_{nm} M_m$$

for some coefficients d_{nm}. Suppose that n and m are two partitions of equal length such that $m < n$. By the fact that $\tilde{H}$ is self-adjoint and the defining relations of the polynomials P_n then follows that

$$(\tilde{H} P_n, M_m) = (P_n, \tilde{H} M_m) = (P_n, 2\omega |m| M_m + \sum_{k<m} c_{mk} M_k) = 0.$$

Hence,

THEOREM 3.2. *Let $n = (n_1, \ldots, n_N)$ be a partition of length N and let*

$$\psi_n = \psi_0 P_n.$$

Then ψ_n is an eigenfunction of the rational Calogero-Sutherland Hamiltonian in (1.2) with eigenvalue

$$E_n = 2\omega |n| + E_0, \quad E_0 = \omega N (1 + \lambda (N - 1)).$$

There are also other ways of constructing the eigenfunctions of the rational Calogero-Sutherland model. Rather than the methods employed here one can for example use creation and annihilation operators, in principle also in this case much like the standard construction for the one-particle Hamiltonian with just a harmonic oscillator potential. This was first done in the case of four particles by Perelomov [Per71] and shortly thereafter for five particles by Gambardella [Gam75], while the general case of an arbitrary number of particles has been treated by Brink et.al. [BHV92] and also by Kakei [Kak96]. Yet another approach is elaborated in Paper 3. The idea is there to replace the monomial symmetric polynomials by more complicated symmetric polynomials, which themselves contain more of the eigenfunctions structure.

CHAPTER 4

Constructions of exactly soluble many-body systems

In this chapter we take a more systematic view on the type of exactly soluble many-body systems we have encountered so far. In particular, we discuss two methods by which rather large families of such systems can be constructed. The first relies on the notion of root systems and so-called reflection groups. This approach does not only supply powerful tools for constructing and solving many-body systems in great generality but also reveals a number of interesting connections with pure mathematics. By exploiting the connection between symmetric polynomials and models of Calogero-Sutherland type the second method produces many-body systems which by construction are exactly soluble.

1. Root systems and reflection groups

In order to use root systems for constructing and studying one dimensional many-body systems we now briefly review their definition and basic properties. For an in depth discussion of their algebraic properties we refer to the monograph by Benson and Grove [BG85] and for their connection with Lie algebras to Humphreys [Hum72]. Consider $\mathbb{R}^N$, for a fixed positive integer N, equipped with the ordinary inner product $(x, y) = \sum_{j=1}^N x_j y_j$ for all $x, y \in \mathbb{R}^N$. The resulting euclidean space is henceforth denoted by E. For each $x, \alpha \in \mathbb{R}^N$ the reflection $\sigma_\alpha(x)$ of x along α is defined by

$$\sigma_\alpha(x) = x - \langle x, \alpha\rangle\alpha, \quad \langle x, \alpha\rangle = 2\frac{(x, \alpha)}{(\alpha, \alpha)}.$$

It is a simple exercise to verify that $\sigma_u^2 = 1$, that the $N-1$ dimensional hyperplane $H_\alpha = \{\beta \in \mathbb{R}^N : (\alpha, \beta) = 0\}$ is left invariant by the action of the reflection σ_α, and also that $\sigma_\alpha(\alpha) = -\alpha$.

With this in mind we can now proceed to define a root system $\mathcal{R}$ in $\mathbb{R}^N$ as a finite set of non-zero vectors such that $\sigma_\alpha(\beta) \in \mathcal{R}$ for all $\alpha, \beta \in \mathcal{R}$. The root systems we will encounter obey, with one exception, two additional conditions: firstly, $\langle\alpha, \beta\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \mathcal{R}$ and secondly, for a given $\alpha \in \mathcal{R}$ we have $k\alpha \in \mathcal{R}$ if and only if $k = \pm 1$. A root system that complies with the first condition is commonly referred to as crystallographic and one that obeys the second as reduced. Unless otherwise stated we will in the following discussion assume all root systems

to be crystallographic and reduced. Since each root system $\mathcal{R}$ contains only a finite number of roots β there exists a vector $\alpha_0 \in \mathbb{R}^N$ such that $(\beta, \alpha_0) \neq 0$ for all $\beta \in \mathcal{R}$. A vector α_0 with this property is commonly referred to as regular in $\mathcal{R}$ and can be used to write the root system as a union $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ of positive roots $\mathcal{R}_+ = \{\beta \in \mathcal{R} | (\beta, \alpha_0) > 0\}$ and negative roots $\mathcal{R}_- = -\mathcal{R}_+$. Among the positive roots there always exist a unique set of linearly independent roots that span E and by which all other positive roots can be written as linear combinations with non-negative coefficients. Such a set is called a base and its constituents simple roots. The number of simple roots is furthermore referred to as the rank of $\mathcal{R}$. The reflection or Coxeter group $W_{\mathcal{R}}$ of a root system $\mathcal{R}$ is defined as the group generated by all reflections σ_α with $\alpha \in \mathcal{R}$. We conclude this section by a short digression on the structure of Coxeter groups and the classification of root systems. We start by stating the fact that a Coxeter group is generated by any set of simple reflections of the associated root system. This result is originally due to Coxeter [Cox34], while a more recent and comprehensive discussion can, e.g, be found in the monograph by Benson and Grove [BG85].

THEOREM 4.1. *Let $\mathcal{R}$ be a root system of rank N. Fix a basis $\Pi = \{\alpha_1, \ldots, \alpha_N\}$ of $\mathcal{R}$ and define the simple reflections σ_j by the relation $\sigma_j = \sigma_{\alpha_j}$. The Coxeter group $W_{\mathcal{R}}$ of the root system $\mathcal{R}$ is then generated by the σ_j. Moreover, $(\sigma_{\alpha_j}\sigma_{\alpha_k})^{m_{jk}} = I$ for all $j, k = 1, \ldots, N$ and integers m_{jk} such that*

$$\langle \alpha_j, \alpha_k \rangle \langle \alpha_k, \alpha_j \rangle = 4\cos^2(\pi/m_{jk}).$$

A root system which can not be written as a union of other mutually orthogonal root systems is said to be irreducible. There are only four infinite series of root systems, as well as five exceptional ones, which in addition to being irreducible also are crystallographic and reduced. These four infinite series are denoted A_{N-1}, B_N, C_N and D_N. In addition there exist a unique non-reduced irreducible root system $BC_N = B_N \cup C_N$. These five infinite series constitutes the so called classical root systems.

EXAMPLE 4.1. *The root system A_{N-1}. Let E be the $N-1$ dimensional subspace of $\mathbb{R}^N$ orthogonal to the vector $e_1 + \cdots + e_N$ and let $A_{N-1} = \{\pm(e_j - e_k) : j \neq k\}$. Since $e_j - e_k = (e_j - e_{j+1}) + \cdots + (e_{k-1} - e_k)$ for all $j, k = 1, \ldots, N$ such that $j < k$ it is clear that the vectors $\alpha_j = e_j - e_{j+1}$ for $1 \leq j \leq N-1$ constitute a basis of A_{N-1}. Note that $\langle \alpha_j, \alpha_j \rangle = 2$, $\langle \alpha_j, \alpha_{j+1} \rangle = 1$ and that $\langle \alpha_j, \alpha_k \rangle = 0$ for $|j-k| > 1$. By Theorem 4.1 thereby follows that the simple reflections $\sigma_j := \sigma_{\alpha_j}$ obey the relations*

$$\sigma_j\sigma_j = 1, \quad \sigma_j\sigma_k = \sigma_k\sigma_j, \quad \text{for } |j-k| > 1,$$
$$\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}.$$

It follows that the Coxeter group generated by these simple reflections is isomorphic to the symmetric group S_N. It is indeed straightforward to verify that the simple reflection σ_j interchanges e_j and e_{j+1} while leaving the remaining vectors e_k invariant.

EXAMPLE 4.2. *The root system* $C_N = \{2e_j, \pm(e_j \pm e_k) : j,k = 1,\dots,N, j \neq k\}$ *in* $E = \mathbb{R}^N$. *A base is provided by the vectors* $\alpha_j = e_j - e_{j+1}$ *for* $1 \leq j \leq N-1$ *and* $\alpha_N = 2e_1$. *Through the use of Theorem 4.1 follows that the Coxeter group in this case not only contains permutations of the vectors* e_i *but also the reflections* $e_i \to -e_i$. *Further details on the root system* C_N *can, e.g, be found in Section 5.3 of the monograph by Benson and Grove [BG85].*

2. Constructing many-body systems using root systems

We now briefly discuss how root systems can be used to construct and study models of quantum many-body systems. A more comprehensive account can be found in the review by Olshanetsky and Perelemov [OP83].

Let us start by demonstrating how one of our standard example, the rational Calogero-Sutherland model, can be formulated using root systems.

EXAMPLE 4.3. *The translation invariant Calogero-Sutherland model, formally defined by the Hamiltonian*

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{1\leq j<k\leq N} V(x_j - x_k), \quad V(x) = \omega^2 x + 2\lambda(\lambda-1)x^{-2}.$$

Let x *denote the vector* $x = (x_1,\dots,x_N)$. *The trick is now to observe that the potential depends only on the inter-particle distances and that they can be written as inner products between* x *and particular constant vectors in* $\mathbb{R}^N$, *e.g.,* $x_1 - x_2 = (x,\alpha)$ *with* $\alpha = e_1 - e_2$. *Upon comparing this fact with Example 4.1 we see that these constant vectors can be taken as the positive roots of the root system* A_{N-1}. *This means that our Hamiltonian also can be written as*

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{\alpha\in(A_{N-1})_+} V(x_\alpha), \quad x_\alpha = (x,\alpha).$$

By the fact that the potential V *is even now follows that the Coxeter group of* A_{N-1}, *the symmetric group* S_N, *leaves the Hamiltonian invariant. This role of the Coxeter group, as a symmetry group of the associated Hamiltonian, is a property which is central to the root system construction of quantum many-body systems.*

Now that we have seen that the rational Calogero-Sutherland model can be reformulated using the root system A_{N-1} there is nothing stopping us from replacing it with another root system. Indeed, let $\mathcal{R}$ be any root system and $\mathcal{R}_+$ its positive roots. Then we can construct a Hamiltonian

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{\alpha\in\mathcal{R}_+} g_\alpha \hat{V}(x_\alpha), \quad g_\alpha \in \mathbb{R}, \tag{4.1}$$

where the interaction $\hat{V}$ intentionally has been left unspecified. Suppose that the interaction is even and furthermore, that the coupling constants g_α are equal for all roots α in the same orbit of $W_{\mathcal{R}}$. Then the Hamiltonian is invariant under the

action of the Coxeter group $W_{\mathcal{R}}$. One virtue of this root system construction is that it can be used to prove results for many models in one go, by using general results on root systems and leaving a specific choice for later. It is clear from the example above that the root system A_{N-1} is useful from a physical point of view. Whether the same can be said about the remaining root systems might be less clear. To convince the doubtful reader we therefore consider another important example which will play an important role in Paper 1.

EXAMPLE 4.4. *Quantum many-body systems related to the root system C_N. Substituting $\mathcal{R} = C_N$ in (4.1) we arrive at the Hamiltonian*

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + g_1 \sum_{j=1}^{N} \hat{V}(2x_j) + g_2 \sum_{1\leq j<k\leq N} [\hat{V}(x_j - x_k) + \hat{V}(x_j + x_k)].$$

Note that the coupling constants have been chosen such that they indeed are constant in each orbit of the Coxeter group W_{C_N}. We know from Example 4.2 that $W_{C_N} = S_N \rtimes (\mathbb{Z}/(2\mathbb{Z}))^N$, i.e., it contains not only permutations of the coordinates but also reflections thereof. The semi-direct product refers to the fact that the two subgroups, permutations and reflections, do not commute. It follows that the Hamiltonian indeed is invariant under the action of this group if $\hat{V}$ is even. Many-body systems which in this sense are related to the root system C_N can in many cases be given a natural, but admittedly heuristic, physical interpretation as describing particles confined to the half-line. To clarify this interpretation it is instructive to recall a standard example from electrodynamics: a point charge moving in space bounded by a grounded plane. This system can also be modeled by the particle and a mirror-particle, placed on the opposite side of the plane with the reverse charge, moving in the whole space. In this way one ensures that the potential is zero at all points on the plane. The model defined by the C_N Hamiltonian above can be interpreted in much the same way, with the first and last interaction terms modeling interactions between particles and mirror-particles. If the interaction $\hat{V}$ is repulsive enough the particles can furthermore not pass each other and they are effectively confined to the half-line. This suggests that the root system C_N can be used to introduce a boundary, at $x = 0$, into a quantum many-body system on the real line. A mathematically precise formulation of these ideas in the context of the delta-interaction is provided in Paper 1.

Let us now for a moment focus our attention on the classical root systems. The corresponding quantum many body systems, defined by substituting $\mathcal{R}$ for the appropriate root system in (4.1), are all contained either in the A_{N-1} or the BC_N model. The latter is defined by the Hamiltonian

$$H = \sum_{j=1}^{N} \left[-\partial_{x_j}^2 + g_1 \hat{V}(x_j) + g_2 \hat{V}(2x_j)\right] + g_3 \sum_{1\leq j<k\leq N} \left[\hat{V}(x_j - x_k) + \hat{V}(x_j + x_k)\right],$$

from which the B_N model is obtained by setting $g_2 = 0$, the C_N model by requiring that $g_1 = 0$ and the D_N model by the prescription $g_1 = g_2 = 0$. Suppose that $\hat{V}$

is a homogeneous interaction, i.e., that there exist a constant d such that $\hat{V}(ax) = a^d \hat{V}(x)$ for all $a \in \mathbb{C}$. The BC_N Hamiltonian is then equal to the B_N Hamiltonian with coupling constants $\tilde{g}_1 = (g_1 + 2^d g_2)$ and $\tilde{g}_2 = g_3$. In the case of a homogeneous interaction it is thus sufficient to consider only the A_{N-1} and the B_N Hamiltonian. The delta interaction and rational Calogero-Sutherland model are two examples of models which qualify into this category.

3. Calogero-Sutherland models from symmetric polynomials

Although the root system construction discussed in the previous section provides a powerful tool for constructing and studying quantum many-body systems, it does not directly provide any information on their solubility. In this section we will therefore take the somewhat different approach of starting with the solution and then constructing the quantum many-body system which it solves.

Recall that the eigenfunctions of the rational Calogero-Sutherland Hamiltonian, bar a ground state factor and a rescaling of the particle coordinates, are equal to the generalized Hermite polynomials. Having excited states described by symmetric polynomials is a property which it shares with many other models of Calogero-Sutherland type.

EXAMPLE 4.5. *The trigonometric Calogero-Sutherland model, formally defined by the Hamiltonian*

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + 2\lambda(\lambda - 1) \sum_{1 \le j < k \le N} \sin^{-2}(x_j - x_k)$$

for $0 < x_j \le \pi$. *Sutherland [Sut71a] showed that it has the ground state*

$$\Psi_0(x_1, \ldots, x_N) = \prod_{1 \le j < k \le N} \sin^{\lambda}(x_j - x_k)$$

with energy eigenvalue $E_0 = \frac{1}{3}\lambda^2 N(N^2 - 1)$. *After conjugation of the Hamiltonian by the ground state we obtain the differential operator*

$$\tilde{H} := \Psi_0^{-1}(H - E_0)\Psi_0 = -\sum_{j=1}^{N} \partial_{x_j}^2 - i2\lambda \sum_{1 \le j < k \le N} \frac{e^{i2x_j} + e^{i2x_k}}{e^{i2x_j} - e^{i2x_k}} (\partial_{x_j} - \partial_{x_k}).$$

By introducing the variables $y_j = e^{i2x_j}$ *we can rewrite it in the more suggestive form*

$$\tilde{H} = \sum_{j=1}^{N} \left[y_j^2 \partial_{y_j}^2 + (1 - \lambda(N-1)) y_j \partial_{y_j} \right] - 2\lambda \sum_{k \ne j} \frac{y_j^2}{y_j - y_k} \partial_{y_j}.$$

It follows from the discussion of the rational Calogero-Sutherland model in the previous Chapter that also this differential operator maps the space of symmetric polynomial into itself and furthermore that this mapping has a triangular structure. In fact, already Sutherland [Sut72] showed how to construct a complete set of polynomial eigenfunctions for this differential operator. It was later realized that they

are identical to the so called Jack polynomials, introduced by Jack [Jac] in a purely mathematical context; see also the paper by Stanley [Sta89] and the forthcoming monograph by Forrester [For].

Now let us reverse the argument and start with a complete set of polynomials and see whether we can construct a quantum many-body system which they solve. For simplicity we start with just one particle and limit our attention to models that can be defined by a Schrödinger operator. The second order differential operators which preserve the polynomial modules $\mathcal{P}_n = \langle 1, x, \ldots, x^n \rangle$ and thereby have a chance of having a complete set of polynomial eigenfunctions are all of the form

$$d = \alpha(x)\partial_x^2 + \beta(x)\partial_x + \gamma, \tag{4.2}$$

with α a polynomial of degree not higher than two, β with a degree less than or equal to one and γ a constant. It has been known for a long time that such a differential operator by a change of independent variable x and conjugation by a particular function ψ_0 always can be transformed into a Scrödinger operator; see e.g. Szegö's [Sze75] classical book on orthogonal polynomials. A precise formulation of this fact, which is well suited for our purposes, follows from a more general result obtained by Kamran and Olver [KO90].

PROPOSITION 4.1. *Suppose that p_n is an eigenfunction of the differential operator in (4.2) with eigenvalue e_n. Let*

$$\psi_0(x) = e^{\int^x \chi(y)dy}, \quad \chi = \frac{2\beta - \alpha'}{4\alpha}$$

and let $\tilde{x}$ denote the independent variable

$$\tilde{x} = \tilde{x}(x) = \int^x \frac{dy}{\sqrt{\alpha(y)}}.$$

The function $\psi_n = \psi_0 p_n$ is then an eigenfunction, with eigenvalue $-e_n$, of the Schrödinger operator

$$h := -\psi_0 d \psi_0^{-1} = -\partial_{\tilde{x}}^2 + v(\tilde{x}), \quad v = \frac{4\beta^2 + 3\alpha'^2 - 8\alpha'\beta}{16\alpha} + \frac{\beta'}{2} - \frac{\alpha''}{4} - \gamma.$$

EXAMPLE 4.6. *The Hermite polynomials H_n, eigenfunctions of the differential operator*

$$d = \partial_x^2 - 2x\partial_x.$$

Following the prescription in Proposition 4.1 we obtain $\psi_0 = e^{-\frac{1}{2}x^2}$, $\tilde{x} = x$ and moreover that the functions $\psi_n = \psi_0 H_n$ are eigenfunctions of the Schrödinger operator

$$h \equiv -e^{-\frac{1}{2}x^2}(\partial_x^2 - 2x\partial_x)e^{\frac{1}{2}x^2} = -\partial_x^2 + x^2 - 1.$$

This is recognized as the Hamiltonian describing a single particle trapped in a harmonic oscillator potential.

EXAMPLE 4.7. *The generalized Bessel polynomials y_n, eigenfunctions of the differential operator*

$$d = x^2\partial_x^2 + (a - bx)\partial_x, \quad a, b \in \mathbb{R}.$$

For a comprehensive discussion of the Bessel polynomials see e.g. the monograph by Grosswald [Gro78]. The general results of Proposition 4.1 in this case gives $\psi_0 = e^{-\frac{a}{2x} - \frac{b+1}{2}\ln x}$ and $\tilde{x} = \ln x$. It also implies that the functions $\psi_n = \psi_0 y_n$ are eigenfunctions of the Schrödinger operator

$$h \equiv -\psi_0 d\psi_0^{-1} = -\partial_{\tilde{x}}^2 + \frac{a^2}{4}e^{-2\tilde{x}} - \frac{a(b+2)}{2}e^{-\tilde{x}} + \frac{1}{4}(b+1).$$

Although not in the standard parameterization this is the so called Morse potential, originally introduced by Morse [Mor29] in describing diatomic molecules.

These one-particle results, and in particular Proposition 4.1, has a simple and natural extension to the many-body case. As special cases we will find the correspondence between generalized Hermite polynomials and the rational Calogero-Sutherland model and also that between its trigonometric counterpart and the Jack polynomials.

THEOREM 4.2. *Fix a differential operator of the same form as in (4.2) and let D denote the partial differential operator*

$$D = \sum_{j=1}^{N} d(x_j) + 2\lambda \sum_{\substack{j,k=1 \\ k\neq j}}^{N} \alpha(x_j)(x_j - x_k)^{-1}\partial_{x_j}$$

with $\lambda > 0$. Suppose that P_n is an eigenfunction of this partial differential operator with eigenvalue E_n. Let

$$\Psi_0(x_1, \ldots, x_N) = \prod_{j=1}^{N} e^{\int^{x_j} \chi(y)dy} \prod_{1\leq j<k\leq N} (x_j - x_k)^{\lambda}, \quad \chi = \frac{2\beta - \alpha'}{4\alpha}$$

and let $\tilde{x}_j$ denote the independent variables

$$\tilde{x}_j = \tilde{x}_j(x_j) = \int^{x_j} \frac{dy}{\sqrt{\alpha(y)}}.$$

The function $\Psi_n = \Psi_0 P_n$ is then an eigenfunction, with eigenvalue $-E_n$, of the Schrödinger operator

$$H := -\Psi_0 D \Psi_0^{-1} = \sum_{j=1}^{N}[-\partial_{\tilde{x}_j}^2 + V(\tilde{x}_j)] + \lambda(\lambda - 1)\sum_{\substack{j,k=1 \\ k\neq j}}^{N} \frac{\alpha(\tilde{x}_j)}{(x_j(\tilde{x}_j) - x_k(\tilde{x}_k))^2},$$

$$V(\tilde{x}) = v(\tilde{x}) + \frac{\lambda}{2}(N-1)\frac{\partial\beta}{\partial x}\Big|_{x=0} - \frac{\lambda^2}{6}(N-1)(N-2)\frac{\partial^2\alpha}{\partial x^2}\Big|_{x=0},$$

where v denotes the potential function in Proposition 4.1.

SKETCH OF PROOF. The proof follows by straightforward computations using Proposition 4.1, with two exceptions: the constant terms added to v and the fact that no three body terms appear. The first constant term is a direct consequence of the observation that

$$2\sum_{\substack{j,k=1\\k\neq j}}^{N}\frac{x_j^n}{x_j-x_k}=\sum_{\substack{j,k=1\\k\neq j}}^{N}\frac{x_j^n-x_k^n}{x_j-x_k}$$

equals zero if $n=0$ and $N(N-1)$ if $n=1$. The second constant term as well as the cancellation of all three body terms follows from the fact that

$$3\sum_{\substack{j,k,l=1\\k,l\neq j,l\neq k}}^{N}\frac{x_j^n}{(x_k-x_j)(x_j-x_l)}=$$

$$-\sum_{\substack{j,k,l=1\\k,l\neq j,l\neq k}}^{N}\frac{x_j^2(x_l-x_k)+x_k(x_j-x_l)+x_l(x_k-x_j)}{(x_k-x_j)(x_j-x_l)(x_l-x_k)}$$

is zero if $n=0,1$ and equal to $N(N-1)(N-2)$ if $n=2$. □

Let us now extend the previous two examples, derived from the Hermite polynomials and generalized Bessel polynomials, to the many-body case.

EXAMPLE 4.8. *The generalized Hermite polynomials H_n, with $n=(n_1,\dots,n_N)$ any partition of a fixed length N. Recall from Chapter two that they are eigenfunctions of the partial differential operator*

$$D=\sum_{j=1}^{N}[\partial_{x_j}^2-2x\partial_{x_j}]+2\lambda\sum_{\substack{j,k=1\\k\neq j}}^{N}(x_j-x_k)\partial_{x_j}.$$

Let

$$\Psi_0(x_1,\dots,x_N)=\prod_{j=1}^{N}e^{-\frac{1}{2}x_j^2}\prod_{1\leq j<k\leq N}(x_j-x_k)^{\lambda}.$$

It then follows from Theorem 4.2 that the functions $\Psi_n=\Psi_0H_n$ are eigenfunctions of the Schrödinger operator

$$H=\sum_{j=1}^{N}[-\partial_{x_j}^2+x_j^2]+2\lambda(\lambda-1)\sum_{1\leq j<k\leq N}(x_j-x_k)^{-2}-N-\lambda N(N-1).$$

We have thereby obtained the rational Calogero-Sutherland model and its relation to the generalized Hermite polynomials as a special case of Theorem 4.2.

EXAMPLE 4.9. *The multi-variable extension of the generalized Bessel polynomials suggested by Theorem 4.2, for any partition n henceforth denoted by Y_n. This leads us to the partial differential operator*

$$D = \sum_{j=1}^{n} [\partial_{x_j}^2 + (a - bx_j)\partial_{x_j}] + 2\lambda \sum_{\substack{j,k=1 \\ k\neq j}}^{N} x_j^2 (x_j - x_k)^{-1} \partial_{x_j}.$$

It directly follows from Example 4.7 that $\tilde{x}_j = \ln x$ and that

$$\Psi_0(x_1, \ldots, x_N) = \prod_{j=1}^{N} e^{-\frac{a}{2x_j} - \frac{b+1}{2} \ln x_j} \prod_{1\leq j<k\leq N} (x_j - x_k)^{\lambda}.$$

We furthermore have that the functions $\Psi_n = \Psi_0 Y_n$ are eigenfunctions of the Schrödinger operator

$$H = \sum_{j=0}^{N} \left[-\partial_{\tilde{x}_j}^2 + \frac{a^2}{4} e^{-2\tilde{x}_j} - \frac{a(b+2)}{2} e^{-\tilde{x}_j} \right] + \lambda(\lambda - 1) \sum_{\substack{j,k=1 \\ k\neq j}}^{N} \frac{e^{2\tilde{x}_j}}{\left(e^{\tilde{x}_j} - e^{\tilde{x}_k}\right)^2} + c(a, b, \lambda),$$

where c, for fixed a,b and λ, is a constant which can be computed using Theorem 4.2. This operator can be rewritten, in a perhaps more familiar form, by first observing that the two-body terms equal $\sum_{1\leq j<k\leq N} [\coth^{-2} \frac{1}{2}(x_j - x_k) - 1/2 \sinh^{-2} \frac{1}{2}(x_j - x_k)]$. This fact together with the identity $\coth^2 x = \sinh^{-2} x + 1$ implies that

$$H = \sum_{j=0}^{N} \left[-\partial_{\tilde{x}_j}^2 + \frac{a^2}{4} e^{-2\tilde{x}_j} - \frac{a(b+2)}{2} e^{-\tilde{x}_j} \right] + 2\lambda(\lambda - 1) \sum_{1\leq j<k\leq N} \frac{1}{4 \sinh^2 \frac{1}{2}(\tilde{x}_j - \tilde{x}_k)} + c'(a, b, \lambda)$$

for some constant c'. In this form it is clear that the Hamiltonian corresponding to the generalized Bessel polynomials defines an extension of the so called hyperbolic Calogero-Sutherland model by the addition of an external Morse potential. The discrete spectrum of this model has been studied by Inozemtsev and Meshcheryakov [IM86], while an investigation of its continuous part, as far as I know, is yet to be undertaken.

Other examples include the Laguerre and Jacobi polynomials, respectively corresponding to the BC_N versions of the rational and trigonometric Calogero-Sutherland models. A more complete discussion of this connection between symmetric polynomials and Calogero-Sutherland models is planned for a future publication in collaboration with Edwin Langmann.

CHAPTER 5

An outlook on future research

We have in this thesis studied as well as constructed explicit solutions of certain specific examples of quantum many-body systems which in some sense are exactly solved. To conclude we now discuss a number of related questions and directions for future research.

In the particular example of the rational Calogero-Sutherland model we have reviewed the construction of its eigenfunctions by generalized Hermite polynomials and in Paper 3 an explicit representation of these polynomials is deduced. The construction of Calogero-Sutherland models from symmetric polynomials outlined in the previous chapter suggests that this result can be lifted to a more general setting. Indeed, preliminary results indicate that explicit formulas for the bound states of all the resulting Calogero-Sutherland models can be obtained by using the same techniques as in Paper 3. It is interesting to note that only certain of these models have a purely discrete spectrum. A particular exception is the extension of the hyperbolic Sutherland model by an external Morse potential constructed in Example 4.9. Since the Morse potential is only weakly confining, it is clear that part of its spectrum indeed is continuous. Particularly interesting are those cases in which it has both bound as well as scattering states. One would then encounter mixtures of such states, describing scattering of particles off one or more particles trapped by the external Morse potential. These types of phenomena are unique to the many-body case and it seems that little is known about the states that describe them. It is therefore natural to expect that a detailed investigation of the models in which they arise will require, as well as provide, new mathematical results and physical intuition.

We have previously mentioned that the spectrum of the rational Calogero-Sutherland model is degenerate, a property it shares with most models of Calogero-Sutherland type. As we have seen, this degeneracy entails a number of technical complications, e.g., the proof of orthogonality for the generalized Hermite polynomials. It also raises certain important questions in connection with the construction presented in Paper 3: it is not a priori clear whether the polynomials obtained really are the generalized Hermite polynomials, or indeed whether they constitute a complete set in the space of polynomials. One way in which this problem can be remedied is to not diagonalize just the Hamiltonian but rather a family of commuting differential operators which breaks the degeneracy. Preliminary computations

indicate that an extended version of the method employed in Paper 3 can be used to diagonalize a certain generating function for such a family of operators in the trigonometric Calogero-Sutherland model and possibly also in other cases. This generating function is commonly referred to as the Sekiguchi operator and it is closely related to the Dunkl operators used in the first two examples of Chapter 3; for further details see e.g. Forrester [For].

Regarding the results presented in Paper 1 and 2 one might wonder about root system extensions of the many-body systems derived from the generalized point interaction and in which cases they can be solved by the coordinate Bethe ansatz. In Paper 1 a partial answer to this question is given but there is certainly more to investigate. Another question which perhaps is more intriguing arises from the duality between the delta-interaction model and a particular model with momentum dependent interactions discussed in Paper 1 and 2. It was recently proved that this latter model arise as the non-relativistic limit of the massive Thirring model and that delta-interaction model can be seen as the non-relativistic limit of ϕ^4 theory in $1+1$ dimensions [GLP04a, GLP04b]. Furthermore, the delta-interaction is invisible for fermions but non-trivial for bosons, while the reverse statement holds true for the momentum dependent interaction. Since we thereby have a striking duality between a bosonic and a fermionic model, it is natural to inquire whether there exists a supersymmetric model which unifies the two, and if that is the case, if it can be obtained as the non-relativistic limit of some quantum field theoretical model.

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Part 2

Scientific papers

PAPER 1

Hallnäs, Martin and Langmann, Edwin (2005). Exact solutions of two complementary one-dimensional quantum many-body systems on the half-line. *Journal of Mathematical Physics*, vol. 46, num. 5, p. 1-15.

PAPER 2

Hallnäs, Martin, Langmann, Edwin and Paufler, Cornelius (2005). Generalized local interactions in 1D: solutions of quantum many-body systems describing distinguishable particles. *Journal of Physics A: Mathematical and General*, vol. 38, p. 4957-4974.

PAPER 3

Hallnäs, Martin and Langmann, Edwin (2005). Explicit formulas for the eigenfunctions of the N-body Calogero model. Preprint.