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Limit Shapes for q^{Volume} Tilings of a Large Hexagon

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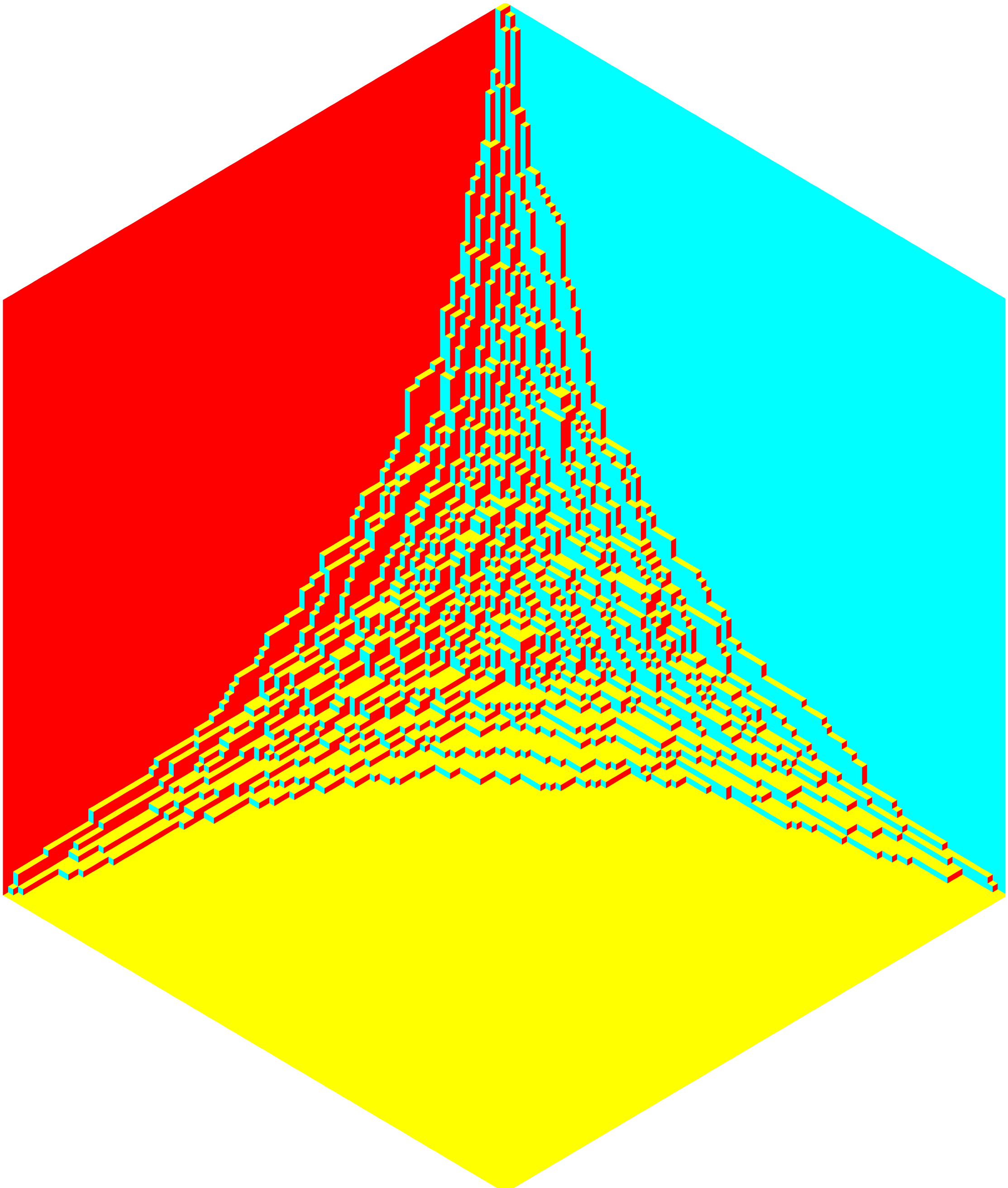


Figure 1: An example of a lozenge tiling of the hexagon (generated using code from Christophe Charlier shared through private correspondence)

Abstract

Lozenges are polygons constructed by gluing two equilateral triangles along an edge. We can fit lozenge pieces together to form larger polygons and given an appropriate polygon we can tile it with lozenges. Lozenge tilings of the semi-regular hexagon with sides A, B, C can be viewed as the 2D picture of a stack of cubes in a $A \times B \times C$ box.

In this project we investigate the typical tiling of the hexagon as the sides A, B, C of the box all grow according to some $N \to \infty$. We consider two cases: In the uniform case all tilings occur with equal probability. This is a special case of the general q^{Volume}-tiling where the probability is proportional to the volume taken up by a corresponding stack of cubes. We transform the problem into a question on families of non-intersecting paths and define a probability function on the paths for the q-Volume case. This probability function can be expressed in terms of the q-Hahn orthogonal polynomials.

We then study the behaviour of non-intersecting paths as the sides of the hexagon grow to infinity by analysing the asymptotic behaviour of the corresponding polynomials. We determine the probability density for where these non-intersecting paths cross. Furthermore we characterise the "Arctic curve" in the hexagon. Outside of this curve the probability density is constant. This result shows that the six corners of the hexagon are (with probability one) tiled with just one type of lozenge.

Gränsformer i q^{Volym}-plattor för stora hexagon

Sammanfattning

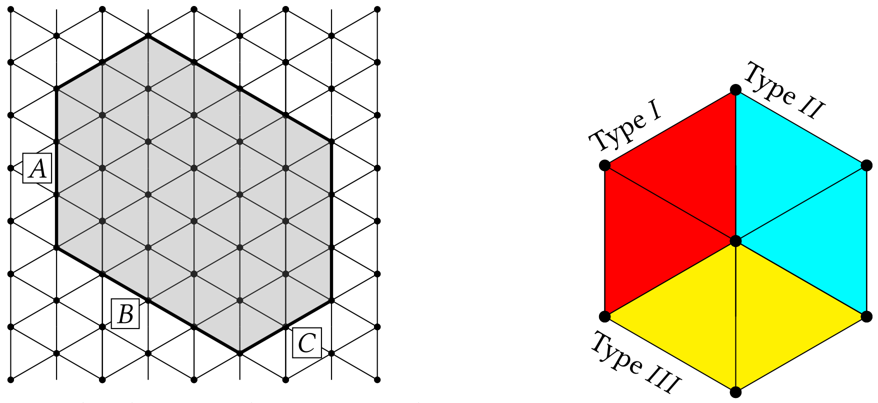
En “Lozenge" är en polygon konstruerad genom att limma två liksidiga trianglar längs en kant. Vi kan montera ihop lozengstycken för att bilda större polygoner och med en lämplig polygon kan vi lozengplatta den. Lozengplattor av den semi-liksidiga hexagonen med sidorna A, B, C kan ses som 2D-bilden av en stapel kuber i en låda med dimensioner $A \times B \times C$. I det här projektet undersöker vi den typiska formen på en platta när sidorna A, B, C på rutan växer till oändlighet och vi analyserar två fall: Det likformiga fallet där alla plattor sker med samma sannolikhet och q^{Volym}-fallet där sannolikheten för en platta är proportionell mot volymen som tas upp av motsvarande kubstaplar. För att undersöka dessa plattor förvandlar vi det till en fråga om samlingar av icke-korsande vägar på en motsvarande graf som representerar hexagonen. Med hjälp av satsen Lindström-Gessel-Viennot kan vi definiera sannolikheten för att en icke-korsande väg går genom en viss punkt i hexagonen både för det enhetliga och q^{Volym}-fallet. I båda fallen är dessa sannolikhetsfunktioner kopplade till Hahn eller q-Hahn ortogonala polynom. Eftersom dessa ortogonala polynom beror på hexagonens sidor så vi betraktar polynomens asymptotiska beteende när sidorna växer till oändlighet genom ett resultat från Kuijlaars och Van Assche. Detta bestämmer densiteten för de icke-korsande vägarna genom varje punkt i det hexagon vi beräknar. Detta bestämmer också också en “arktisk kurva" utanför vilket sannolikheten är konstant vilket visar att hexagonens sex hörn är (med sannolikhet ett) plattade med bara en typ av lozeng.

Introduction: Lozenge Tilings

A jigsaw puzzle is a familiar game in which the player is tasked with assembling variously shaped pieces in a predetermined way, either by restricting the way pieces interlock or suggesting a unique final configuration. We consider a type of puzzle where there are just three different pieces, three types of rhombi with unit sides and internal angles 60° and 120°. We refer to these shapes as *lozenges* and we assemble them into a predetermined shape, such as a hexagonal polygon. We call these configurations *lozenge tilings* (or just tilings) of the hexagon and we consider hexagons with integer sides A, B, C with internal angles measuring 120°, which we call a *semi-regular* hexagon. Figure 1 on the first page is an example of what a tiling might look like for a large hexagon. The figure also reveals lozenge tilings as 2D representations of cubes stacked in an $A \times B \times C$ box. While it is easier to find one tiling of the semi-regular hexagon with just three unique puzzle pieces we will be interested in characterizing *all* the possible tilings and to consider the *typical* tiling. If we draw the hexagon into a uniform grid made up of equilateral triangular pieces with unit sides then lozenges can be identified with two unit triangles attached along one edge, which can be done in three different ways. E.g. hexagon with unit sides can be tiled with three lozenges, one of each type, as in figure 2. A natural question is the total number of possible lozenge tilings of the $A \times B \times C$ semi-regular hexagon. This was answered in 1915 by Percy MacMahon [16], who established the following formula for the number of tilings,

$$\prod_{i=1}^{A}\prod_{j=1}^{B}\prod_{k=1}^{C}\frac{i+j+k-1}{i+j+k-2}.$$

In chapter 1 we give a proof of this result to motivate the development much of the machinery required for the rest of the text. The formula reveals the quick growth of the number of tilings e.g. For an equilateral hexagon with sides $N = 3$ we have 980 different tilings and with $N = 5$ there are already 267227532 different tilings. To count lozenge tilings we describe a bijection between tilings and *families of non-intersecting paths* so that counting lozenge tilings is equivalent to counting families of non-intersecting paths. We consider paths as subgraphs of a larger graph $G = (V, E)$ with vertices V and edges E which corresponds to points and lines contained in the semi-regular hexagon. We can then define a *weight* and a *probability* of a tiling by defining them by proxy on the family of non-interesting paths. By assigning non-negative weights to all the edges in the hexagon we can define the weight of a family of paths. One choice of weights (considered in chapter 1) will constitute the uniform case. In this case every edge, and therefore every family of non-intersecting path, is assigned the same weight. For a set of N points $(t, z_i) \in \mathbb{Z}^2$ determine the joint probability function N and show that the natural way to express these probability functions is in terms of the *Hahn orthogonal polynomials.* In chapter 2 we extend the model, setting weights of the non-intersecting paths so that the weight of a tiling is proportional to q^{volume} where $0 < q < 1$ or $q > 1$ and the volume is proportional to the space taken up by the stacks of cubes in the $A \times B \times C$ box. The solution



(a) Triangular lattice and a semi-regular hexagonal boundary with sides A, B, C and internal angles all 120°.

(b) The three types of lozenges, each made up of two unit triangles.

Figure 2: The hexagonal lattice and the three lozenge types

to this model will correspond to the q-Hahn orthogonal polynomials which are *q-extensions* of the Hahn orthogonal polynomials. A q-extension is a generalization achieved by replacing numbers $n \in \mathbb{R}$ with the $q-$basic number

$$[\, n\,]_q = \frac{1-q^n}{1-q}.$$

One reclaims $[\, n\,]_q \longrightarrow n$ as $q \longrightarrow 1$, therefore this replacement can yield natural generalizations. Indeed there is a corresponding q-calculus with $q-$extended notions of orthogonal polynomials, binomials, factorials, etc. In chapter 3 we consider the Hahn and the $q-$Hahn joint probability functions in greater detail and show that probability of a family of non-intersecting paths passing through a set of points (t, z_i) in the support of the probability with $0 \leq t \leq T$ and $1 \leq i \leq N$ can be expressed as a determinant

$$\mathbf{P}(x_1, x_2, \ldots, x_N) = \det\left(K(x_k, x_l)\right)_{k,l=1}^{N}$$

where $K : \mathfrak{X}\times\mathfrak{X} \longrightarrow \mathbb{R}$ is known as the *kernel* of the determinantal point process. In chapter 4 we investigate the behaviour of the typical tiling as the sides of the hexagon grow using the properties of determinantal process and a result due to Kuijlaars and Van Assche. This will achieve a version of the well-known *Arctic circle* result. The Artic circle (which is a more general curve in the q-Hahn case) is a closed, simple curve which demarcates a change in the behaviour of the typical tiling, in our case the density of paths. This curve always intersects the sides of the hexagon at six places defining six distinct regions; one for each corner of the hexagon. These six regions are tiled (with probability one) with only one type of lozenges. Inside the Arctic curve, the probability of tilings vary and we determine this distribution explicitly.

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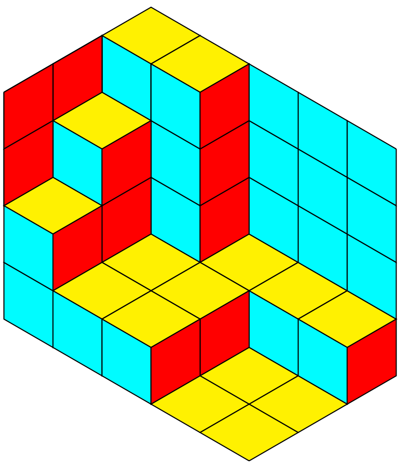
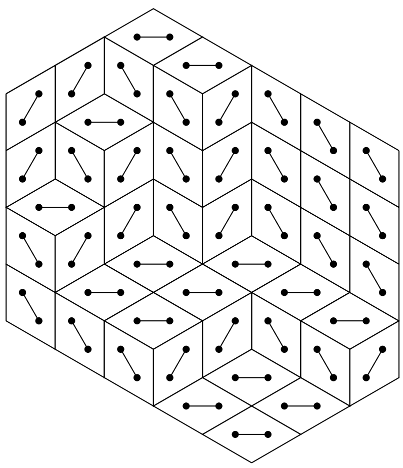
Chapter 1

Lozenge Tilings as Families of Non-intersecting Paths

We describe two alternative and useful ways of viewing lozenge tilings through graph theoretic formalism, adopting the notation and definitions from Diestel [3]. We view lozenge tilings as both *matchings* of a graph and as *families of non-intersecting paths*. We recall some basic



(a) An example of a lozenge tiling with $A = 4, B = 5, C = 3$.



(b) A lozenge tiling as a matching on a graph

Figure 1.1: Viewing lozenge tilings as matchings on a graph

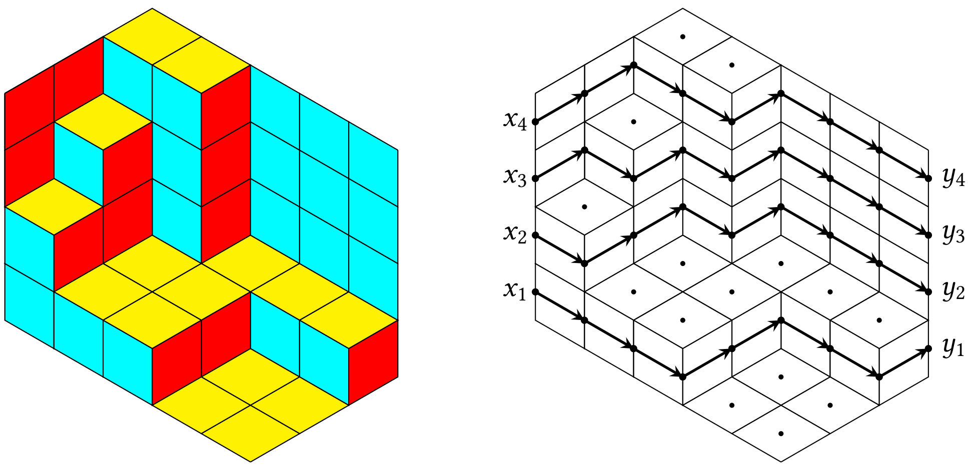
terminology of graph theory: A graph $G = (V, E)$ is a pair of sets: The set $V = \{v_0, \dots, v_1\}$ consists of the vertices and $E \subseteq [V]^2$ consists of the edges, where each edge is a two vertex subsets of V. In other words if $e \in E$ then $e = \{u, v\}$ for two $u, v \in V$ and we usually write $e = uv$ or vu. A vertex $v \in V$ is *incident* with an edge $e \in E$ if $v \in e$ and the two vertices u, v of an edge e are its *end-vertices*. An edge $e = \{u, v\}$ *joins* its end-vertices u, v. Two edges are *adjacent* if they have a common end-vertex and we say that two vertices are *neighbours* if they are joined by an edge. Two vertices $u, v \in V$ are *independent* if they are not end-vertices of any edge $e \in E$ and two edges $e, f \in E$ are said to be independent if they share no common end-vertex. In practice we work with vertices V that are points in the plane $\mathbb{R}^2$. For a graph G we write $V(G)$ for its set of vertices and $E(G)$ for its set of edges.

Definition 1.0.1. *A matching M is a set of independent edges in a graph $G = (V, E)$. We say that M is a matching of $U \subseteq V$ if every vertex in U is incident with an edge in M. If we can find* *a matching of V then M is said to be a perfect matching of the graph G.*

Definition 1.0.2. *A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' = [V']^2$ consists of all the two-element subsets of V'. Conversely we call G the supergraph of G'. We can denote the subgraph and supergraph relationship by $G' \subseteq G$ and $G \supseteq G'$ respectively. Furthermore we define the union of two graphs $P = (V, E)$ and $Q = (V', E')$ and $P \cup Q := (V \cup V', E \cup E')$.*

For any graph $G = (V, E)$ and a subset $V' \subseteq V$ we define $G[V']$ as a sub-graph $G' = (V', E')$ where $E' = \{uv \in E \mid u, v \in V'\}$. We construct a graph G from figure 2a by drawing a vertex in the center of every triangle in the lattice and drawing edges between vertices that belong to triangles that touch on a side. Let V' be the subset of vertices on or inside the semi-regular hexagonal boundary. We define the sub-graph associated with the hexagonal region $H = G[V']$. Every lozenge corresponds to a pair of triangles in the hexagonal boundary; we identify to each lozenge the edge which joins the vertices of the two triangles (see figure 1.1b.) Doing this for all the lozenges in a tiling we have a matching of the graph H. This process is bijective as we can reverse the process, for any matching we can draw lozenges (one of the three types) over the edges in exactly one way.

1.1 Transforming Lozenge Tilings into Non-intersecting Paths



(a) An example of a lozenge tiling with $A = 4, B = 5, C = 3$.

(b) Transforming the lozenge tiling into non-intersecting paths from x_i to y_i for $i = 1, \ldots, N$.

Figure 1.2: Lozenge tilings as non-intersecting paths

A path is a non-empty graph $P = (V, E)$ consisting of a set of vertices $V = \{x_0, x_1, \ldots, x_n\}$ and a set of edges $E = \{x_0x_1, x_1x_2, \ldots, x_{n-1}x_n\}$, where all the x_i's are distinct vertices. More succinctly we write for such a path $P = x_0x_1 \ldots x_n$. We say that P is a path from x_0 to x_n and that x_0, x_n are linked by P. We can define portions of a path, with $i \leq j$,

$$x_iP := x_i \ldots x_n$$
$$Px_j := x_0 \ldots x_j$$
$$x_iPx_j := x_i \ldots x_j.$$

More generally we consider families of paths $\{P_i\}_{i \in I}$ where I is an indexing set (either finite or countably infinite.) We can often view paths as sub-graphs of a larger graph G; we call

P a G-path if $P \subseteq G$. For two G-paths P and Q we say that they meet (or intersect) at each $v \in V(P) \cap V(Q)$. If $V(P) \cap V(Q) = \emptyset$ we say that the paths are *independent*. A family of non-intersecting paths is therefore a family of paths which are all pairwise independent.

We define the *concatenation* of two paths paths $P = x_0x_1 \dots x_n$ and $Q = z_0z_1 \dots z_m$ that meet at a vertex $x_i = u = z_j$ for some $1 \leq i \leq n$ and $1 \leq j \leq m$ by

$$PuQ := Pu \cup uQ = x_0x_1 \dots x_{i-1}uz_{j+1} \dots z_{m-1}z_m.$$

This allows us to define the concept of a tail-swapping procedure. We assume that G below admits a partial order on the vertices such that every path G-path P can be written in a unique way as $P = x_0x_1 \dots x_n$ where $x_i < x_j$ for any $i < j$.

Definition 1.1.1. *For two G-paths, P and Q write the paths in the partial order adopted from G. Let $P = x_0x_1 \dots x_n$ and $Q = z_0z_1 \dots z_m$. Choose the vertex x_i with minimal index such that Px_i meets z_jQ. Tail-swapping generates two new paths by mapping*

$$P \mapsto P' = x_0x_1 \dots x_iz_{j+1} \dots z_m \text{ and } Q \mapsto Q' = z_0z_1 \dots z_jx_{i+1} \dots x_n.$$

We require the paths P and Q in the above G to admit a partial order (as described above) for the tail-swapping swapping procedure to be well-defined. In order to ensure that all G-paths admit such a partial ordering we will work with *acyclic* and *directed* graphs. An acyclic directed graph is a graph that does not contain a path (which is not just a point) which begins and ends at the same point (also referred to as a cycle.)

Definition 1.1.2. *A directed graph $D = (V, E)$ consists of a set of vertices V and a set of edges E with two functions init : $E \to V$ and ter : $E \to V$ so that for any edge $e \in E$, init(e) is the initial vertex and ter(e) is the terminal vertex. We say that these edges are directed and represent them by drawing an arrow from the initial vertex to the terminal vertex.*

Remark. Under this definition there may be multiple directed edges drawn between the same two vertices and we can have edges forming loops if $\mathrm{init}(e) = \mathrm{ter}(e)$. An *orientation* (of a graph G) is a special directed graph D which shares the same vertices and D has no loops or multiple edges between the same pair of vertices. An *oriented path* $P = x_0x_1 \dots x_n$ is a directed graph with vertices $x_0, \dots, x_n$ and edges $e_0, \dots, e_{n-1}$ such that $\mathrm{init}(e_i) = x_i$ and $\mathrm{ter}(e_i) = x_{i+1}$ for $i = 0, 1, \dots, n-1$.

We will only consider with directed graphs D which are orientations of some graph G and paths which are orientations of a G-path.

Remark. Because a family of (oriented) paths $\mathcal{P} = \{P_i\}_{i \in I}$ allows for multiple edges this means that $\mathcal{P}$ can be identified as a graph in its own right (with some abuse of notation) we can identify $\mathcal{P} := (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \bigcup_{i \in I} V(P_i)$ and $\mathcal{E} = \bigcup_{i \in I} E(P_i)$. We define the $\mathrm{init}(e) : \mathcal{E} \to \mathcal{V}$ and $\mathrm{ter}(e) : \mathcal{E} \to \mathcal{V}$ to inherit the mappings of the initial and terminal functions from P_i.

Defining the Bijection

To define the bijection between lozenge tilings and families of non-intersecting paths we bisect every type I and II lozenge with a directed edge (pointed in the left-to-right direction) as in figure 1.2b. We call these edges type *I* and *II* respectively. We see that these edges make up a family $\mathcal{P} = \{P_i\}_{i=1}^{A}$ of non-intersecting and oriented paths.

For directed path $P = x_0x_1 \dots x_n$ x_0 define the initial vertex of the path $\mathrm{init}(P_i) := x_0$ and similarly the terminal vertex $\mathrm{ter}(P_i) := x_n$. For a family of oriented paths $\mathcal{P} = \{P_i\}_{i=1}^{N}$ define $\mathrm{init}(\mathcal{P}) := \bigcup_{i=1}^{A} \mathrm{init}(P_i)$ the initial vertices of all the P_i and similarly for $\mathrm{ter}(\mathcal{P})$ the terminal vertices of all the P_i. Conversely for a directed graph D and two sets of N vertices $X, Y \subseteq V(D)$ we may consider a class $\mathcal{P}(X, Y)$ of all the families of oriented paths $\mathcal{P} = \{P_1, P_2, \ldots, P_A\}$ which link the vertices of X to Y so that each $x \in X$ and each $y \in Y$ have exactly one path incident on them (again, see figure 1.2b.) For each $\mathcal{P} \in \mathcal{P}(X, Y)$ we have $\mathrm{init}(\mathcal{P}) = X$ and $\mathrm{ter}(\mathcal{P}) = Y$.

We define a procedure which produces from a tiling of a semi-regular hexagon A, B, C a family of non-intersecting oriented paths and also defines a coordinate system so that the vertices are identified as points in $\mathbb{R}^2$. We see from figure 1.2b that by drawing directed edges across type I and II lozenges we have A non-intersecting paths $P_1, P_2, \ldots P_A$ that link initial vertices on the left-most edge of the hexagon $X = \{x_1, x_2, \ldots, x_A\}$ (ordered so that x_1 is the bottom-most vertex and x_A the top-most vertex) to the terminal vertices $Y = \{y_1, y_2, \ldots, y_A\}$. We define $x_i = (0, i-1)$. For any edge e with $\mathrm{init}(e) = (i, j)$ we define

$$\mathrm{ter}(e) = \begin{cases} (i+1, j+1) & \text{if } e \text{ is type I} \\ (i+1, j) & \text{if } e \text{ is type II} \end{cases}$$

Having defined $\mathrm{init}(P_i) = \mathrm{init}(x_i) = (0, i-1)$ we can determine a coordinate for every vertex along the path and in particular the terminal points $y_i = (B + C, C + i - 1)$ for $i = 1, 2 \ldots, N$. Redrawing our paths with this choice of coordinate system implies an affine transformation of the hexagon, which we see in figure 1.3a). We define $N = A, T = B + C$ and $S = C$.

Remark. Whether $B > C$ or $B < C$ (equivalently $T - S > S$ or $T - S < S$) decides if the hexagon is wider than it is tall or taller than it is wide. Up to a reflection these two types of hexagon have the same tilings and so all interesting dynamics is preserved if one assumes $B \geq C$ $(T - S \geq S)$.

Therefore every tiling of the A, B, C-hexagon corresponds to a family of non-intersecting paths $X - Y$ paths where

$$X = \{(0, 0), \ldots, (0, N-1)\}$$

and

$$Y = \{(T, S), \ldots, (T, N + S - 1)\}.$$

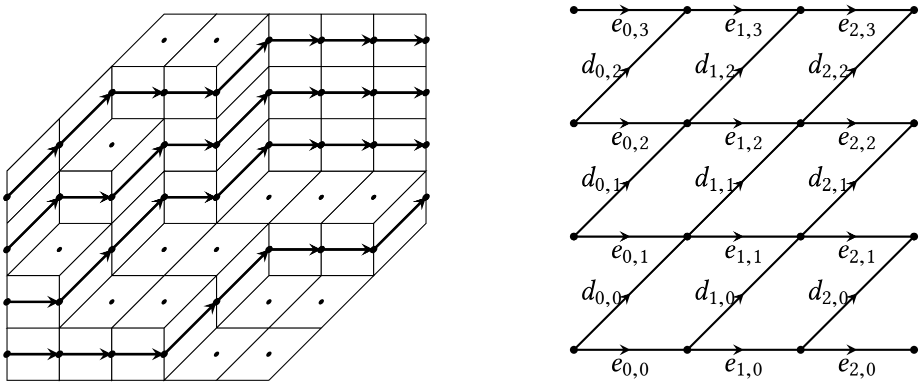
Recall that a family of oriented paths could be viewed as a directed graph $\mathcal{P}$ in its own right. In fact we can consider the family of non-intersecting paths $\mathcal{P}$ as a sub-graph of a directed graph $\mathcal{D} = (\mathbb{Z}^2, E)$ with E all the directed edges $(i, j) \longrightarrow (i+1, j+1)$ and $(i, j) \longrightarrow (i+1, j)$ like in figure 1.3b. (A technical note, to allow for multiple edges in $\mathcal{P}$ we $\mathcal{D}$ should copy N copies of each directed edge.) In this scheme the (affine) hexagon can be identified with the collection of points (and edges) that the edges of $\mathcal{P}$ can possibly meet (and coincide with.) These are the points $\mathcal{V} = \{(t, x)\}$ such that

$$\begin{gathered} 0 \leq t \leq S, 0 \leq x \leq N + t - 1 \\ S < t \leq T - S, 0 \leq x \leq N + S - 1 \\ T - S < t \leq T, S - T + t \leq x \leq N + S - 1. \end{gathered}$$

For every $0 \leq t \leq T$ define $\mathfrak{X}_t \subseteq V$ as the subset of points (t, x) inside the hexagon for a particular t. This hexagon can be identified with the sub-graph of $\mathcal{D}$ generated by the set $\mathcal{V}$, in other words $H := \mathcal{D}[\mathcal{V}]$.

1.2 Counting Paths with Lindström–Gessel–Viennot

We consider the well-known Lindström–Gessel–Viennot theorem [5] [6], [14], [18] which will in particular allow us to count the total number of non-intersecting paths. There are many versions of this theorem of varying generality but we consider one enough for our purposes. We



(a) Affine transformation of hexagon, with non-intersecting paths illustrated.

(b) Edges of the directed graph, indexed by their position $(i, j) \in \mathbb{R}^2$.

Figure 1.3: Tilings as non-intersecting paths

work with an acyclic directed graph D of which we saw two examples of in the previous section, either an acyclic directed graph $\mathcal{D}$ on the whole lattice $\mathbb{Z}^2$ or the sub-graph H corresponding to the semi-regular hexagon. In both cases we have two sets of points X and Y of equal cardinality and we consider families of oriented paths $\mathcal{P} \in \mathcal{P}(X, Y)$. In particular if $X = x$ and $Y = y$ we drop the braces and denote $\mathcal{P}(x, y)$ by the set all oriented D-paths from vertices x to y. Note that we allow for repeated edges so $P \in \mathcal{P}(X, Y)$ may contain multiple edges between a pair of vertices. Let $\mathcal{P}_0(X, Y) \subseteq \mathcal{P}(X, Y)$ be the families of non-intersecting paths Calculating the cardinality of $|\mathcal{P}_0(X, Y)|$ would then provide us with a formula similar to MacMahon's, as the number of lozenge tilings of an A, B, C semi-regular are in bijection to the non-intersecting families of paths in the associated graph H. Consider a function $w : E(G) \to \mathbb{R}_{\geq 0}$ which assigns a weight to each edge in a directed graph G. The function is extended multiplicatively to subsets $F \subseteq E(G)$ by

$$w(F) = \prod_{e \in F} w(e),$$

and by convention the empty set is given unit weight. For an n-tuple of paths P,

$$w(P) = \prod_{i=1}^{n} \prod_{e \in E(P_i)} w(e).$$

For X, Y as above we define the quantity

$$h(X, Y) := \sum_{P \in \mathcal{P}(X,Y)} w(P) \tag{1.1}$$

and if $X = \{x\}$ we write just x. We also define a similar quantity with only the non-intersecting paths

$$h_0(X, Y) := \sum_{P \in \mathcal{P}_0(X,Y)} w(P).$$

If we take $w(e) = 1$ for all $e \in E(G)$ then the $h(X, Y)$ measures the number of families of N directed paths that link the vertices of X to Y, while $h_0(X, Y)$ measures the number of such families that are also non-interesting.

Theorem 1.2.1. *Assume that $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are sets of points such that if $i < j$ and $k < l$ then any pair of paths, P linking x_i and y_l, Q linking x_j and y_k, intersects each-other in at least one point then*

$$h_0(X, Y) = \det_{1\leq i,j\leq n} h(x_i, y_j).$$

Proof. We follow the proof of Stembridge [18]. Expanding the determinant by summing over all the permutations σ is a permutation of $\{1, \ldots, n\}$ we have

$$\det h(x_i, y_j) = \sum_{\sigma} \operatorname{sgn}(\sigma) h(x_1, y_{\sigma(1)}) h(x_2, y_{\sigma(2)}) \ldots h(x_n, y_{\sigma(n)}). \tag{1.2}$$

We define an involution on n-tuples called the *Lindström-Gessel-Viennot involution* which is a way of applying the tail-swap procedure defined in 1.1.1. To describe the involution consider an n-tuple of paths $P = (P_1, \ldots, P_n)$ with $P_i \in \mathcal{P}(x_i, y_{\sigma(i)})$ and where at least one pair of paths intersects at some point z. As there are only a finite number of paths (that are themselves of finite length), there is a vertex $z = (t, x)$ where at least two paths intersect and with t the smallest. From all the paths that pass through z choose P_k, P_l, $k < l$ as those with the smallest indices. Applying tail-swapping (see definition 1.1.1) to these paths we generate two new paths $P'_k = P_k z P_l$ and $P'_l = P_l z P_k$. This is a new n-tuple $P' = (P'_1, \ldots, P'_2)$ with $P'_i = P_i$ for all $i \neq k, l$. Furthermore this n-tuple is associated with a permutation of the end-points $\sigma' = (k, l) \circ \sigma$. Tail-swapping reassigns edges without addition or removal therefore the edges of $P'_1, \ldots, P'_n$ are the same as edges of $P_1, \ldots, P_n$. Our assumption that the graph D is acyclic is required for the set of intersection vertices to be preserved under tail-swapping (see [18] for a counter-example when the graph is not acyclic.) Applying the tail-swapping procedure reclaims the original paths. Thus tail-swapping is an involution and affects the contribution to the determinant of an n-tuple $P = (P_1, \ldots, P_n)$ by

$$\begin{aligned}
&\operatorname{sgn}(\sigma') w(P'_1) \ldots w(P'_k) \ldots w(P'_l) \ldots w(P'_n) \\
&= \operatorname{sgn}(\sigma \circ (k, l)) w(P_1) \ldots w(P_k z) w(z P_l) \ldots \\
&\qquad w(P_l z) w(z P_k) \ldots w(P_n) \\
&= -\operatorname{sgn}(\sigma) w(P_1) \ldots w(P_k) w(P_l) \ldots w(P_n)
\end{aligned}$$

This means that if any pair of paths intersect then their contributions cancel in the sum of (1.2). The only contribution to the determinant that remains comes from the non-intersecting n-tuples of paths. Furthermore for these paths, $\sigma = \mathrm{id}$ and so $\operatorname{sgn}(\sigma) = 1$ thus

$$\det_{1\leq i,j\leq n} h(x_i, y_j) = \sum_{(P_1,\ldots,P_n)\in\mathcal{P}_0(X,Y)} w(P_1) \ldots w(P_n) = h_0(X, Y).$$

□

This theorem allows us to calculate quantities of non-intersecting paths by considering the determinant on *all* paths:

$$\sum_{P\in\mathcal{P}_0(X,Y)} w(P) = \det_{1\leq i,j\leq n} \left(\sum_{P\in\mathcal{P}(x_i,y_j)} w(P) \right)$$

This is useful because while determining all the non-intersecting paths of a particular system is complicated, calculating the determinant on the left is easier.

Remark. We introduce some more notation, for two points $u = (t, x)$ and $v = (s, y)$ then we write $\mathcal{P}(t, x; s, y)$ to signify $\mathcal{P}(u, v)$. Furthermore for two sets of points

$$U = \{u_1, \ldots, u_n\} \text{ and } V = \{v_1, \ldots, v_n\}$$

where $u_i = (t, x_i)$ and $v_i = (s, y_i)$ we write $\mathcal{P}(t, X; s, Y) := \mathcal{P}(U, V)$ where

$$X = \{x_1, x_2, \ldots, x_n\} \text{ and } Y = \{y_1, y_2, \ldots, y_n\}$$

or just $\mathcal{P}(X, Y)$ when t and s are clear. We similarly extend the notation for the non-intersecting paths $\mathcal{P}_0(s, X; t, Y)$ and quantity $h(s, X; t, Y)$.

A useful property of the extended notation is that the cardinality of the set $\mathcal{P}(t, x; s, y)$ can be related to its parameters through

$$|\mathcal{P}(t, x; s, y)| = \binom{s-t}{y-x}.$$

The quantity $|\mathcal{P}(x_i, y_j)|$ is simply the number of paths from (t, x_i) to (s, y_j) in the directed graph $\mathfrak{D}$. If $x_i > y_j$ then there are no paths between the two points; if $t > s$ there are no paths. Consider then $s \geq t$ and $y_j \geq x_i$, every path can be uniquely characterized by the which of its vertices are the left (alternatively right) end-points of a diagonal edge. With the left convention this can be viewed as the point where the path "jumps". Therefore the number of paths is equal to the ways of choosing which $y_j - x_i$ vertices out of the $s - t$ possible positions are designated jumping points.

Lemma 1.2.2. *Consider two sets of points in the plane, (t, x_i) and (s, y_i) with $1 \leq i \leq n$, $t < s$. The total number of non-intersecting N-tuples of paths*

$$|\mathcal{P}_0(t, X; s, Y)| = \det_{1 \leq i,j \leq n} \binom{s-t}{y_j - x_i}.$$

Proof. We use theorem 1.2.1 and let $w(P) = 1$ then on the right hand side

$$h_0(t, X; s, Y) = |\mathcal{P}_0(t, X; s, Y)|.$$

On the other hand on the left,

$$\det_{1 \leq i,j \leq n} h(t, x_i; s, y_j) = \det |\mathcal{P}(t, x_i; s, y_j)|$$

$\square$

We will need a lemma from Krattenthaler [13, pg. 7] (this is a corollary of the result A.1.2 in the appendix by taking $q \to 1$.)

Lemma 1.2.3. *Given three sets of indexed placeholders $X_1, \ldots, X_N, A_2, \ldots, A_N$ and $B_2, \ldots, B_N$ the following holds:*

$$\det_{1 \leq i,j \leq N} (X_i + A_N) \ldots (X_i + A_{j+1})(X_i + B_j) \ldots (X_i + B_2)$$
$$= \prod_{1 \leq i < j \leq N} (X_i - X_j) \prod_{2 \leq i \leq j \leq N} (B_i - A_j)$$

Where empty products such as $(X_i + B_j) \ldots (X_i + B_2)$ when $j = 1$ or the product $(X_i + A_N) \ldots (X_i + A_{j+1})$ when $j = N$ is by convention set equal to one.

Lemma 1.2.4. *Let $L_i \in \mathbb{Z}_{\geq 0}$ and assume that $L_j > L_i$ if $j > i$ then*

$$\det_{1\leq i,j\leq N}\begin{pmatrix} T \\ L_i - j + 1\end{pmatrix} = \prod_{i=1}^{N} \frac{(T+i-1)!}{(T+N-1-L_i)!L_i!} \prod_{1\leq i<j\leq N}(L_j - L_i)$$

Proof. Consider more generally $L_i \in \mathbb{Z}_{\geq 0}$ and assume that $L_j > L_i$ if $j > i$, then

$$\det_{1\leq i,j\leq N}\begin{pmatrix} T \\ L_i - j + 1\end{pmatrix} = \det_{1\leq i,j\leq N} \frac{T!}{(L_i - j + 1)!(T + j - 1 - L_i)!}$$

Eliminating dependence on j in the denominator

$$\frac{1}{(L_i - j + 1)!} = \frac{(L_i - j + 2)\dots(L_i - 1)L_i}{L_i!}$$

and

$$\begin{aligned}\frac{1}{(T + j - 1 - L_i)!} &= \frac{(T + j - L_i)\dots(T + N - 1 - L_i)}{(T + N - 1 - L_i)!} \\ &= \frac{(-1)^{N-j}(L_i - (T + j))\dots(L_i - (T + N - 1))}{(T + N - 1 - L_i)!}.\end{aligned}$$

We define $A_j = -(T + j - 1)$, $B_j = 2 - j$ and factor out terms as possible and then use lemma 1.2.3:

$$\begin{aligned}\det_{1\leq i,j\leq N}\begin{pmatrix} T \\ L_i - j + 1\end{pmatrix} &= \prod_{i=1}^{N} \frac{T!(-1)^{N-i}}{(T+N-1-L_i)!L_i!} \\ &\quad \det_{1\leq i,j\leq N}(L_i + A_N)\dots(L_i + A_{j+1})(L_i + B_j)\dots(L_i + B_2) \\ &= \prod_{i=1}^{N} \frac{T!(-1)^{N-i}}{(T+N-1-L_i)!L_i!} \prod_{1\leq i<j\leq N}(L_i - L_j) \prod_{2\leq i\leq j\leq N}(T + 1 + j - i).\end{aligned}$$

In Pochhammer notation we express,

$$\prod_{2\leq i\leq j\leq N}(T + 1 + j - i) = \prod_{i=1}^{N}(T+1)_{i-1}.$$

and interchanging $L_i - L_j = -(L_j - L_i)$ in the product gives another term $\prod_{i=1}^{N}(-1)^{N-i}$ therefore,

$$\det_{1\leq i,j\leq N}\begin{pmatrix} T \\ L_i - j + 1\end{pmatrix} = \prod_{i=1}^{N} \frac{T!(T+1)_{i-1}}{(T+N-1-L_i)!L_i!} \prod_{1\leq i<j\leq N}(L_j - L_i)$$

where $T!(T+1)_{i-1} = (T+i-1)!$. □

This allows us to prove the MacMahon formula for the number of tilings of the A, B, C semi-regular hexagon:

Corollary 1.2.5. *The semi-regular hexagon with sides A, B, C can be tiled in exactly*

$$\prod_{i=1}^{A}\prod_{j=1}^{B}\prod_{k=1}^{C}\frac{i+j+k-1}{i+j+k-2}$$

different ways.

Side-length N =	Number of distinct tilings
1	2
2	20
3	980
4	232848
5	267227532

Figure 1.4: The total number of ways to tile the regular hexagon (with side N) using lozenges of unit size.

Proof. Tiling a hexagon with sides A, B, C corresponds to N-tuples of non-intersecting paths between the two vertical sides of the hexagon of sides T, S, N with $N = A$, $T = B + C$ and $S = C$ and according to lemma 1.2.2 this is exactly $\det \binom{T}{S+i-j}$. Using lemma 1.2.4 with $L_i = S + i - 1$,

$$\det_{1\le i,j\le N} \binom{T}{S+i-j} = \prod_{i=1}^{N} \frac{(T+i-1)!(i-1)!}{(T+N-S-i)!(S+i-1)!} = \prod_{i=1}^{N} \frac{(T+i-1)!(i-1)!}{(T-S+i-1)!(S+i-1)!}.$$

Where we use the equality

$$\prod_{1\le i<j\le N} (j-i) = \prod_{i=1}^{N} (i-1)!$$

On the other hand with $N = A$, $T = B + C$ and $S = C$,

$$\prod_{i=1}^{A} \frac{(B+C+i-1)!(i-1)!}{(B+i-1)!(C+i-1)!} = \prod_{i=1}^{A}\prod_{j=1}^{B} \frac{C+i+j-1}{i+j-1} = \prod_{i=1}^{A}\prod_{j=1}^{B}\prod_{k=1}^{C} \frac{i+j+k-1}{i+j+k-2}.$$

□

1.3 Defining a Probability on the Tilings

We can define a probability on the set of all tilings $\mathcal{T}$ of the semi-regular hexagon with sides A, B, C so that the probability of any particular tiling is proportional to the product of the weights of all the tiles. For any particular tiling $\tau \in \mathcal{T}$ define the weight of a tiling as the weight $w(P)$ of the corresponding N-tuple of non-intersecting paths P. The probability of a tiling $\tau \in \mathcal{T}$ can be identified as the corresponding probability of a non-intersecting path P is defined

$$Prob(P) := \frac{w(P)}{\sum_{P'\in\mathcal{P}_0} w(P')}.$$

The probability above is unaffected by multiplying the weight function w by a non-zero scalar, in other words the probability only depends on the relative weight between different edges. For the special case when the weight of every edge is one the probability of any particular tiling is

$$Prob(P) = \frac{1}{\sum_{P'\in\mathcal{P}_0} 1} = \frac{1}{|\mathcal{P}_0|}$$

so every tiling occurs with equal probability. On the other hand for any general weight w the probability of a particular family of non-intersecting paths $P = (P_1, \ldots, P_N)$ between points

$(0, i-1)$ to $(T, S+i-1)$ with $1 \leq i \leq N$ can be expressed using theorem 1.2.1 as

$$Prob(P) = \frac{w(P)}{\det_{1\leq i,j\leq N}\left(\sum_{P\in\mathcal{P}(x_i,y_j)} w(P)\right)}. \tag{1.3}$$

Using the LGV-theorem 1.2.1 we established the number of non-intersecting paths from the

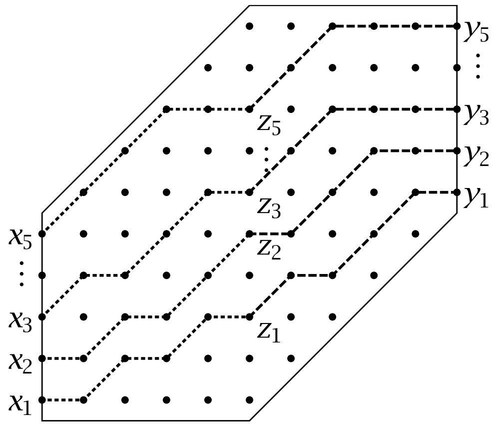


Figure 1.5: Non-intersecting paths, from X to Z and from Z to Y

initial N points X on the left edge of the hexagon to the N end-points Y on the right edge of the hexagon. More generally we may consider any set of N vertices $Z = \{(t, z_i)\}_{i=1}^N$ and consider the number of N non-intersecting X-Z and Z-Y paths. Note that by construction there are no X-Z paths if $t < 0$ and there are no $Y - Z$ paths if $t > T$. Similarly for any $z_i \in Z$ that is not a vertex in the hexagon we do not have either a X-Z path or a Y-Z path. On the other hand consider a set of vertices Z with $0 \leq t \leq T$ and $z_1 < z_2 < \cdots < z_N$ with $0 \leq z_i \leq M$ where M depends on t so that z_i's are in the hexagon. For Z we define the weight of the non-intersecting X-Z and Z-Y paths, this allows us to define the probability of passing through the points Z as

$$\frac{\text{Weight of } X\text{-}Z \text{ paths } \times \text{Weight of } Z\text{-}Y \text{ paths}}{\text{Weight of } X\text{-}Y \text{ paths}}.$$

Proposition 1.3.1. *Consider the affine transformation of the semi-regular hexagon above, with $N = A, T = B + C, S = C$ where $T > S \geq N$. Furthermore consider the initial vertices $X = \{(0, i-1)\}_{i=1}^N$, terminal vertices $Y = \{(T, S+i-1)\}_{i=1}^N$ and a set of N vertices $Z = \{(t, z_i)\}_{i=1}^N$. If $0 \leq t \leq T$ in Z and all the z_i belong to $\mathfrak{X}_t$ (see figure 1.5) then the joint probability of a family of non-intersecting paths passing through $(r, z_1), (r, z_2), \ldots, (r, z_N)$ is defined*

$$\mathbf{P}_t(z_1, z_2, \ldots, z_N) = \frac{\det_{1\leq i,j\leq N}\left(\sum_{P\in\mathcal{P}(x_i,z_j)} w(P)\right)\det_{1\leq i,j\leq N}\left(\sum_{P\in\mathcal{P}(z_i,y_j)} w(P)\right)}{\det_{1\leq i,j\leq N}\left(\sum_{P\in\mathcal{P}(x_i,y_j)} w(P)\right)}. \tag{1.4}$$

If $t < 0$, $t > T$ then the probability is zero, if $0 \leq t \leq T$ but any of the z_i's do not belong to $\mathfrak{X}_t$ then the probability is also zero.

In the special case $w(P) = 1$,

$$\sum_{P\in\mathcal{P}(x_i,y_j)} 1 = |\mathcal{P}(x_i, y_j)| = \binom{T}{y_j - x_i}.$$

Therefore the probability can be written

$$\mathbf{P}_t(z_1,\dots,z_N) = \frac{1}{Z}\det_{1\le i,j\le N}\begin{pmatrix} t \\ z_i+1-j\end{pmatrix}\det_{1\le i,j\le N}\begin{pmatrix} T-t \\ S+i-1-z_j\end{pmatrix} \tag{1.5}$$

where

$$Z = \det_{1\le i,j\le N}\begin{pmatrix} T \\ S+i-j\end{pmatrix} = \prod_{i=1}^{N}\frac{(T+i-1)!(i-1)!}{(T-S+i-1)!(S+i-1)!}.$$

This is the same determinant as the one in the proof of 1.2.5. In the next step we compute the determinants in (1.5).

Computing the Joint Probability Function

Lemma 1.2.4 allows us to calculate the two determinants in (1.5). For the first determinant we just take $\tau = t$ and $L_i = z_i$.

$$\det_{1\le i,j\le N}\begin{pmatrix} t \\ z_i-j+1\end{pmatrix} = \prod_{i=1}^{N}\frac{(t+i-1)!}{(t+N-1-z_i)!z_i!}\prod_{1\le i<j\le N}(z_j-z_i)$$

For the second term, we write

$$\begin{pmatrix} T-t \\ S+i-1-z_j\end{pmatrix} = \begin{pmatrix} T-t \\ T-t-S+z_j+1-i\end{pmatrix}$$

and take $\tau = T-t$ and $L_i = T-t-S+z_i$. Switching i,j (a transpose) and using lemma 1.2.4:

$$\det_{1\le i,j\le N}\begin{pmatrix} T-t \\ T-t-S+z_i-j+1\end{pmatrix} = \prod_{i=1}^{N}\frac{(T-t+i-1)!}{(S+N-1-z_i)!(T-t-S+z_i)!}\prod_{1\le i<j\le N}(z_j-z_i).$$

The joint probability function can be written

$$\mathbf{P}_t(z_1,\dots,z_N) = \frac{1}{Z_t}\prod_{1\le i<j\le N}(z_j-z_i)^2\prod_{i=1}^{N}w(z_i) \tag{1.6}$$

where

$$w(z) = \frac{c}{z!(T-t-S+z)!(t+N-1-z)!(S+N-1-z)!}.$$

This $c>0$ is independent of z and is chosen to factors out anything in $w(z)$ not dependent on z. Furthermore the normalization can be identified

$$\frac{1}{Z_t} = \frac{1}{c}\prod_{i=1}^{N}\frac{(t+i-1)!(T-t+i-1)!(S+i-1)!(T-S+i-1)!}{(T+i-1)!(i-1)!}.$$

At this point we consider the different intervals of support, depending on the parameter t and on S,T,N where we assume $T-S\ge S$. There are three intervals of interest.

The Interval $0 \le t \le S$

The support depends on the value of t as $0 \le z \le t+N-1$ so we define $M = t+N-1$ and write

$$w(z) = \frac{c}{z!(M-z)!} \frac{1}{(T-S+N-1-(M-z))!(S+N-1-z)!}.$$

Eliminating dependence on z in the denominator

$$\begin{aligned}\frac{1}{(S+N-1-z)!} &= \frac{(S+N-1)(S+N-2)\dots(S+N-z)}{(S+N-1)!}\\ &= \frac{(-1)^z(1-S-N)(2-S-N)\dots(z-S-N)}{(S+N-1)!}\end{aligned}$$

and setting $\alpha = -S-N$ we can write in Pochhammer notation

$$\frac{1}{(S+N-1-z)!} = \frac{(-1)^z(\alpha+1)_z}{(S+N-1)!}.$$

where $(\alpha)_k$ which is defined

$$(\alpha)_k := \alpha(\alpha+1)\dots(\alpha+k-1). \tag{1.7}$$

for $k = 1,2,3,\dots$ with $(\alpha)_0 := 1$. In particular $(1)_k = k!$ recoups the standard factorial. In the same way with $\beta = S-T-N$:

$$\frac{1}{(T-S+N-1-(M-z))!} = \frac{(-1)^{M-z}(\beta+1)_{M-z}}{(T-S+N-1)!}.$$

Reinserting these results and simplifying with an appropriate choice of c we can write for (1.6)

$$w(z) = \binom{\alpha+z}{z}\binom{\beta+M-z}{M-z}$$

and

$$\frac{1}{Z_t} = (-1)^{MN}\prod_{i=1}^{N}\frac{(t+i-1)!(T-t+i-1)!(S+i-1)!(T-S-1+i)!}{(T+i-1)!(i-1)!(T-S+N-1)!(S+N-1)!}.$$

The Interval $S < t < T-S$

In practice we take $S = T-S$ (for the the regular hexagon) so this interval not factor into our analysis but we include it for completeness. The support is $0 \le z \le S+N-1$, therefore taking $M = S+N-1$, where $c \in \mathbb{R}$,

$$w(z) = \frac{c}{z!(M-z)!} \frac{1}{(T-t+N-1-(M-z))!(t+N-1-z)!}.$$

Setting $\beta = t-T-N$ then

$$\begin{aligned}\frac{1}{(T-t+N-1-(M-z))!} &= \frac{(T-t+N-(M-z))\dots(T-t+N-1)}{T-t+N-1}\\ &= \frac{(-1)^{M-z}(\beta+1)_{M-z}}{(T-t+N-1)!}.\end{aligned}$$

On the other hand, with $\alpha = -t - N$

$$\frac{1}{(t+N-1-z)!} = \frac{(t+N-z)\dots(t+N-1)}{(t+N-1)!} = \frac{(-1)^z(\alpha+1)_z}{(t+N-1)!}.$$

Therefore we may write,

$$\frac{(-1)^M c}{(T-t+N-1)!(t+N-1)!}\binom{\alpha+z}{z}\binom{\beta+M-z}{M-z}$$

and so with the appropriate choice of c we can write define,

$$w(z) = \binom{\alpha+z}{z}\binom{\beta+M-z}{M-z}$$

where

$$\frac{1}{Z_t} = (-1)^{MN}\prod_{i=1}^{N}\frac{(t+i-1)!(T-t+i-1)!(S+i-1)!(T-S-1+i)!}{(T+i-1)!(i-1)!(T-t+N-1)!(t+N-1)!}.$$

The Interval $T - S \leq t \leq T$

In this case the support of z is $S - T + t \leq z \leq S + N - 1$, writing $z' = z + T - S - t$ the support becomes

$$0 < z' \leq T - t + N - 1.$$

Here z' is a shift to harmonize the support with the previous cases. With z' chosen as above and with $M = T - t + N - 1$ we can express,

$$w(z) = \frac{c}{z'!(M-z')!}\frac{1}{(S+N-1-(M-z'))!(T+N-S-1-z')!}$$

Following the same process as above allows us to write $w(z)$ in the following form,

$$w(z) = \binom{\alpha+z'}{z'}\binom{\beta+M-z'}{M-z'}.$$

where in this case $\alpha = S - T - N$ and $\beta = -S - N$ and with

$$\frac{1}{Z_t} = (-1)^{MN}\prod_{i=1}^{N}\frac{(t+i-1)!(T-t+i-1)!(S+i-1)!(T-S-1+i)!}{(T+i-1)!(i-1)!(T+N-S-1)!(S+N-1)!}.$$

The Joint Probability Function

We have shown that the joint probability function can be expressed in all intervals, depending on some parameters that depend on t, where $0 \leq t \leq T$, by

$$\mathbf{P}_t(z_1,\dots,z_N) = \frac{1}{Z_t}\prod_{1\leq i<j\leq N}\left(z_i - z_j\right)^2\prod_{i=1}^{N}w(z_i). \tag{1.8}$$

with normalization:

$$\frac{1}{Z_t} = \begin{cases}(-1)^{MN}\prod_{i=1}^{N}\frac{(t+i-1)!(T-t+i-1)!(S+i-1)!(T-S+i-1)!}{(T+i-1)!(i-1)!(S+N-1)!(T-S+N-1)!} & \text{if } t \leq S \text{ or } t \geq T-S\\ (-1)^{MN}\prod_{i=1}^{N}\frac{(t+i-1)!(T-t+i-1)!(S+i-1)!(T-S-1+i)!}{(T+i-1)!(i-1)!(T-t+N-1)!(t+N-1)!} & \text{if } S < t < T-S\end{cases}$$

and

$$w(z) = \binom{\alpha + z'}{z'} \binom{\beta + M - z'}{M - z'}. \tag{1.9}$$

The coefficients M, α, β depend on the interval of t:

- If $0 \le t \le T - S$. The probability is supported on $0 \le z' \le M$ where $z' = z$ and $M = t + N - 1$. The parameters $\alpha = -S - N$ and $\beta = S - T - N$.
- If $T - S < t \le S$. The probability is supported on $0 \le z' \le M$ where $z' = z$ and $M = S + N - 1$. The parameters $\beta = t - T - N$ and $\alpha = -t - N$.
- If $S < t \le T$. The probability is supported on $0 \le z' \le M$ where $z' = z + T - S - t$ and $M = T - t + N - 1$. The parameters $\alpha = S - T - N$ and $\beta = -S - N$.

The quantity $w(z)$ can be identified as the weight in the orthogonality relation of the Hahn orthogonal polynomials. We will discuss this more in chapter 3, but at this point we note one property of the Hahn orthogonal polynomials $p_0, p_1, \dots, p_M$. For any pair p_n, p_m, $n \ne m$ we have an orthogonality relation

$$\sum_{z'=0}^{M} \binom{\alpha + z'}{z'} \binom{\beta + M - z'}{M - z'} p_n(z') p_m(z') = 0.$$

Remark. How do the orthogonal polynomials make an appearances? Consider the quantity in probability (1.8)

$$\Delta_N(x) := \prod_{1 \le j < i \le N} (x_i - x_j).$$

We can also identify $\Delta_N(x)$ with a $N \times N$ Vandermonde determinant,

$$\Delta_N(x) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ \vdots & \vdots & & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{vmatrix}.$$

From here appropriate row operations can transform the rows of $\Delta_N(x)$ to any desired monomial with degree less than N. In particular we can choose them as a set of orthogonal polynomials like the Hahn polynomials. In chapter 3 we will consider the quantity

$$\frac{1}{Z_t} \prod_{1 \le i < j \le N} \left(z_i - z_j\right)^2 \prod_{i=1}^{N} w(z_i)$$

in more generality with $w(z)$ a general positive weight associated with a class of orthogonal polynomials.

Chapter 2

Weighted Hexagonal Tilings

We consider a tilings corresponding to edges which are weighted with a non-uniform weight. In the non-intersecting path formulation from 1.2 this amounted to a weight function which assigns the same weight to every edge in the underlying directed graph. In this section we will consider different non-intersecting collections of paths (n-tuples of paths) where the edges have different weights.

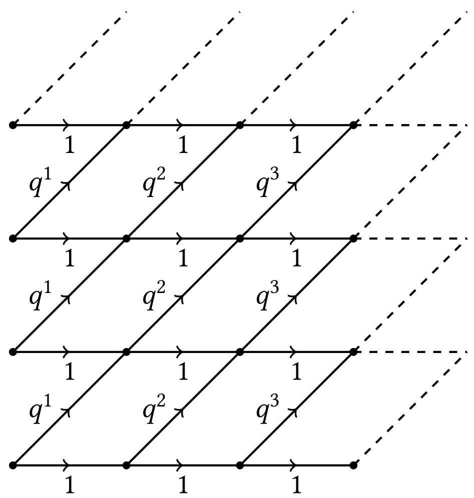


Figure 2.1: Edge-weights on the directed graph, choosing $w(e_{ij}) = 1$ and $w(d_{ij}) = q^{i+1}$.

2.1 Non-intersecting Paths with Weight

Recall the model constructed in 1.2 in which every path was weighed according its constituent edges. Therefore we can define a weight function on paths by describing its action on every edge in the directed graph (as in figure 1.3b). We have two types of edges in the graph, indexed by their left end-point

- horizontal edges $e_{i,j}$ going from $(i, j) \to (i, j + 1)$ and weighted $w(e_{i,j}) = 1$.
- diagonal edges $d_{i,j}$ going from $(i, j) \to (i + 1, j + 1)$ and weighted $w(d_{i,j}) = q^{i+1}$ where $0 < q < 1$ or $q > 0$.

In order to apply the LGV theorem 1.2.1 we will need to calculate the weight of paths from points (t, x) to (s, y) where $t \leq s$ and $y \leq x$, specifically we need to calculate the quantity

$$h(t, x; s, y) = \sum_{P \in \mathcal{P}(t,x;s,y)} w(P)$$

where $\mathcal{P}(x, y)$ denotes all paths $(t, x) \longrightarrow (s, y)$. Note that any path $P \in \mathcal{P}(t, x; s, y)$ is a collection of two types of (directed) edges, horizontal edges e_{ij} and diagonal edges d_{ij}. As the edges are directed and the graph is acyclic, P has exactly $y-x$ diagonal edges. With the same argument as in corollary 1.2.2 we see that the paths correspond to a $y-x$ element subset $\sigma(P) \subseteq \{t, \dots, s-1\}$ where $k \in \sigma(P)$ if k is a left end-point of a diagonal edge in P. The weight of such a path P is, given the choice of edge-weights above,

$$w(P) = \prod_{k \in \sigma(P)} q^{k+1}.$$

Define $\mathcal{S}_{t,s;y-x}$ as the set of all the $y - x$ element subsets of $\{t + 1, \dots, s\}$. By the argument above $|\mathcal{P}(t, x; s, y)| = |\mathcal{S}_{t,s;y-x}|$ and

$$h(t, x; s, y) = \sum_{S \in \mathcal{S}_{t,s;y-x}} q^{\lambda(S)}$$

where

$$\lambda(S) = \sum_{k \in S} k.$$

Remark. Consider the effect of a shift in the parameters $h(t, x; s, y) \longrightarrow h(t - r, x; s - r, y)$ for some $t \geq r \geq 0$. Let S be an n element subset of $\{t + 1, \dots, s\}$ then

$$\sum_{k \in S} k = r|S| + \sum_{k \in S} k - r = r|S| + \sum_{k' \in S'} k'$$

where $k' = k - r$ and S' is a subset of $\{t - r + 1, \dots, s - r\}$. Therefore,

$$h(t, x; s, y) = q^{r(y-x)} h(t - r, x; s - r, y)$$

and in particular, taking $r = t$ and writing $h(t', x; y) := h(0, x; t', y)$ we have,

$$h(t, x; s, y) = q^{t(y-x)} h(s - t, y - x).$$

To calculate the joint probability we consider the points $(r, x_i) = (0, i - 1)$ and $(s, y_i) = (T, S + i - 1)$ for $1 \leq i \leq N$ and furthermore consider a set of points $(t, z_1), \dots, (t, z_N)$ with $r \leq t \leq s$. We also take $z_1 < z_2 < \cdots < z_N$ with every z_i in the support which is defined by the boundary of the hexagon. Therefore from the definition (1.4) we have

$$\mathbf{P}_t(z_1, \dots, z_N) = \frac{\det_{1 \leq i,j \leq N} h(t, z_j - x_i) \det_{1 \leq i,j \leq N} h(T - t, y_j - z_i)}{\det_{1 \leq i,j \leq N} h(T, y_j - x_i)} \prod_{i=1}^{N} q^{t(y_i - z_i)}.$$

To proceed further we need to be able to calculate calculate the sum

$$\sum_{S \in \mathcal{S}_{0,x;t,y}} q^{\lambda(S)} \text{ where } \lambda(S) = \sum_{k \in \sigma(P)} k$$

which require some basic ideas from q-calculus.

2.2 A (q-)Extended Diversion

We selectively review some basic facts of q-calculus needed for our purposes. The q-analogue (or q-extension) is a way to extend the definition of formulas and functions by reflecting that for $0 < q < 1$ or $q > 1$,

$$\lim_{q\to 1} \frac{1-q^{\alpha}}{1-q} = \alpha$$

therefore defining $[\alpha] := \frac{1-q^{\alpha}}{1-q}$ (called the basic number) we can extend other concepts by the replacement $n \longrightarrow [n]$. The q-extension of the Pochhammer-symbol (see 1.7)

$$(\alpha; q)_k := (1-\alpha)(1-\alpha q)(1-\alpha q^2)\dots(1-\alpha q^{k-1}) \tag{2.1}$$

where $(\alpha; q) := 1$ and $k = 1, 2, \dots$. This q-extension is related to the standard case through

$$\lim_{q\to 1} \frac{(q^{\alpha}; q)_k}{(1-q)^k} = (\alpha)_k.$$

We can also define the q-factorial for non-negative integers k where $[\,0\,]_q! = 1$ and for $k > 0$,

$$[\,k\,]_q! := [\,1\,]_q \dots [\,k\,]_q = \frac{(1-q)\dots(1-q^k)}{(1-q)\dots(1-q)} = \frac{(q;q)_k}{(1-q)^k}$$

There is also a q-extension to the binomial coefficient, the q-binomial coefficients are defined for $n \in \mathbb{Z}$ and $k = 0, 1, 2, \dots$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}.$$

If $n > 0$ then the q-binomial coefficient can be written,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[\,n\,]_q!}{[\,n-k\,]_q!\,[\,k\,]_q!}.$$

For $k < 0$ we define $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$. Taking the limit $q \longrightarrow 1$ we recover the classical binomial coefficient. Like for the classical binomial we have

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{bmatrix} n \\ n-j \end{bmatrix}_q.$$

The result we want from q-calculus is the following (from [19].)

Theorem 2.2.1. *Let $S_n = \{1, 2, \dots, n\}$ and let $\mathcal{S}_{n,j}$ be the collection of all subsets of S_n with j elements, $0 \leq j \leq n$. Then*

$$\sum_{S\in\mathcal{S}_{n,j}} q^{\lambda(S)} = q^{j(j+1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q \quad \textit{where } \lambda(S) = \sum_{s\in S} s. \tag{2.2}$$

Proof. We follow the proof in [19]. Consider induction on n. The base cases are $n = 1, j = 0, 1$. If $j = 0$ then the q-binomial equals one while $\mathcal{S}_{n,0} = \{\emptyset\}$ so $\lambda(\emptyset) = 0$ and

$$\sum_{S\in\{\emptyset\}} q^{\lambda(S)} = q^{\lambda(\emptyset)} = 1.$$

While on the right,

$$q^{j(j+1)/2}\begin{bmatrix} n \\ j \end{bmatrix}_q = q^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q = \frac{[n]!}{[n]![0]!} = 1.$$

(Note that the value of n did not factor into the calculation.) If $j = 1$ then, $\mathcal{S}_{1,1} = \{\{1\}\}$ and

$$\sum_{S\in\{\{1\}\}} q^{\lambda(S)} = q^{\lambda(\{1\})} = q$$

and on the right side we have

$$q^{j(j+1)/2}\begin{bmatrix} n \\ j \end{bmatrix}_q = q^1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q = q.$$

For the inductive step assume 2.2 holds for $1 \le n < m$ where $m > 1$ and consider $n = m$. We skip the $j = 0$ case as it holds for all n. If $j > 0$ then separate $\mathcal{S}_{m,j}$ into all sets without m as an element and those with m,

$$\mathcal{S}_{m,j} = \mathcal{B} \cup \mathcal{B}'$$

where $B = \{S \in \mathcal{S}_{m,j} | m \notin S\}$ and $B' = \mathcal{S}_{m,j} \setminus B$. There is a correspondence (or bijection given by the identity function) between the sets of $S \in \mathcal{S}_{m-1,j}$ and B. On the other hand sets in B' correspond to fixing m and choosing $j-1$ elements among $\{1,\dots,m-1\}$, there is a bijection between $\mathcal{S}_{m-1,j-1}$ and B' sending $S \mapsto S \cup \{m\}$ and vice versa. Therefore using our assumption for $m-1$,

$$\begin{aligned}
\sum_{S\in\mathcal{S}_{m,j}} q^{\lambda(S)} &= \sum_{S\in\mathcal{B}'} q^{\lambda(S)} + \sum_{S\in\mathcal{B}} q^{\lambda(S)} \\
&= \sum_{S\in\mathcal{S}_{m-1,j}} q^{\lambda(S)} + \sum_{S\in\mathcal{S}_{m-1,j-1}} q^{\lambda(S\cup\{m\})} \\
&= q^{j(j+1)/2}\begin{bmatrix} m-1 \\ j \end{bmatrix}_q + q^m q^{j(j-1)/2}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q \\
&= q^{j(j+1)/2}\left(\begin{bmatrix} m-1 \\ j \end{bmatrix}_q + q^{m-j}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q\right) \\
&= q^{j(j+1)/2}\begin{bmatrix} m \\ j \end{bmatrix}_q
\end{aligned}$$

where the last equality follows from proposition A.1.1 in the appendix. By induction this is holds for all $0 \le j \le m$. While we demonstrated $j = 0, 1$ for $n = 1$ above assume the identity is true for $n = m-1, 0 \le j \le m-1$ then by the calculation above the case $n = m-1, j = k-1$ and $n = m-1, j = k$ implies the case $n = m, j = k$. This takes care of the cases $n = m, 0 \le j < m$ while in the case $j = m$ the left hand side 2.2 is

$$\sum_{S\in\{\{1,\dots,m\}\}} q^{\lambda(S)} = q^{m(m+1)/2}$$

and the right hand side is $q^{m(m+1)/2}\begin{bmatrix} m \\ m \end{bmatrix}_q = q^{m(m+1)/2}$ so the proposition holds for all j. Finally we induct on n. □

Corollary 2.2.2. *If*

$$h(t,x;s,y) = \sum_{S \in \mathcal{S}_{t,x;s,y}} q^{\lambda(S)}, \text{ where } \lambda(S) = \sum_{k \in S} k.$$

then using theorem 2.2.1:

$$h(s-t, y-x) = q^{(y-x)(y-x+1)/2} \begin{bmatrix} s-t \\ y-x \end{bmatrix}_q$$

2.3 Computing the Joint Probability Function

Recall the joint probability function under consideration

$$\mathbf{P}_t(z_1, \ldots, z_N) = \frac{\det_{1\le i,j\le N} h(t, z_j - x_i) \det_{1\le i,j\le N} h(T-t, y_j - z_i)}{\det_{1\le i,j\le N} h(T, y_j - x_i)} \prod_{i=1}^{N} q^{t(y_i - z_i)} \tag{2.3}$$

where $x_i = i - 1$ and $y_j = S + j - 1$. For simplicity we consider the probability only up to a constant of proportionality not in z. We must therefore compute (using a property of determinants to switch indices i, j)

$$\det_{1\le i,j\le N} h(t, z_i - x_j) \tag{2.4}$$

and

$$\det_{1\le i,j\le N} h(T-t, y_j - z_i). \tag{2.5}$$

We calculate (2.4) first. Using corollary 2.2.2,

$$h(t, z_i - x_j) = q^{(z_i-x_j)(z_i-x_j+1)/2} \begin{bmatrix} t \\ z_i - x_j \end{bmatrix}_q = \frac{q^{(z_i-x_j)(z_i-x_j+1)/2} [\, t \,]_q!}{[\, z_i - x_j \,]_q! \, [\, t + x_j - z_i \,]_q!}. \tag{2.6}$$

Using identity (A.8) which states $[\, k \,]_q! = q^{k(k-1)/2} [\, k \,]_{q^{-1}}!$ we have

$$h(t, z_i - x_j) = \frac{q^{(z_i-x_j)t} [\, t \,]_{q^{-1}}!}{[\, z_i - x_j \,]_q! \, [\, t + x_j - z_i \,]_{q^{-1}}!}. \tag{2.7}$$

Calculating the determinant of (2.4) is similar to the process in the proof of corollary 1.2.5 with all quantities replaced with their q-extended counterparts. We start with (2.7) writing $x_j = j-1$ and eliminate j from the denominator,

$$\frac{q^{t(z_i-j+1)} [\, t \,]_{q^{-1}}!}{[\, t+N-1-z_i \,]_{q^{-1}}! \, [\, z_i \,]_q!} [\, t+N-1-z_i \,]_{q^{-1}} \ldots [\, t+j-z_i \,]_{q^{-1}}$$
$$[\, z_i - j + 2 \,]_q \ldots [\, z_i - 1 \,]_q [\, z_i \,]_q \,. \tag{2.8}$$

Factoring out the negative sign using (A.1), for each $k = j+1, \ldots, N$,

$$[\, t+k-1-z_i \,]_{q^{-1}} = -q [\, z_i - t - k + 1 \,]_q$$

and defining $A_j = -(t+j-1)$ and $B_j = 2-j$ the expression (2.8) becomes

$$\frac{q^{t(z_i-j+1)}\left[\,t\,\right]_{q^{-1}}!(-q)^{N-j}}{\left[\,t+N-1-z_i\,\right]_{q^{-1}}!\left[\,z_i\,\right]_q!}\left[\,z_i+A_N\,\right]_q\dots\left[\,z_i+A_{j+1}\,\right]_q$$
$$\left[\,z_i+B_j\,\right]_q\dots\left[\,z_i+B_3\,\right]_q\left[\,z_i+B_2\,\right]_q. \tag{2.9}$$

Then using lemma A.1.2 (and noting the remark below it) this becomes (up to terms not in z for simplicity)

$$\det_{1\le i,j\le N} h(t, z_i - j + 1) \sim \prod_{i=1}^{N}\frac{q^{tz_i}}{\left[\,t+N-1-z_i\,\right]_{q^{-1}}!\left[\,z_i\,\right]_q!}\prod_{1\le i<j\le N}\left[\,z_i\,\right]_q-\left[\,z_j\,\right]_q. \tag{2.10}$$

Together with a term from (2.3), where

$$\prod_{i=1}^{N} q^{t(y_i-z_i)} \sim \prod_{i=1}^{N} q^{-tz_i}$$

so that $\prod_{i=1}^{N} q^{t(y_i-z_i)} \det_{1\le i,j\le N} h(t, z_i-j+1)$ is proportional to

$$\prod_{i=1}^{N}\frac{1}{\left[\,t+N-1-z_i\,\right]_{q^{-1}}!\left[\,z_i\,\right]_q!}\prod_{1\le i<j\le N}\left[\,z_i\,\right]_q-\left[\,z_j\,\right]_q.$$

Consider now the other determinant

$$h(T-t, y_j-z_i) = q^{(y_j-z_i)(y_j-z_i+1)/2}\begin{bmatrix} T-t \\ y_j-z_i \end{bmatrix}_q \tag{2.11}$$

where with identity (A.8) we can write,

$$h(T-t, y_j-z_i) = \frac{q^{y_j-z_i}\left[\,T-t\,\right]_q!}{\left[\,y_j-z_i\,\right]_{q^{-1}}!\left[\,T-t-y_j+z_i\,\right]_q!}.$$

With $y_j = S+j-1$ we eliminate j in the denominator,

$$\frac{q^{S+j-1-z_i}\left[\,T-t\,\right]_q!}{\left[\,S+N-1-z_i\,\right]_{q^{-1}}!\left[\,T-t-S+z_i\,\right]_q!}$$
$$\left[\,S+N-1-z_i\,\right]_{q^{-1}}\dots\left[\,S+j-z_i\,\right]_{q^{-1}}$$
$$\left[\,T-t-S-j+2+z_i\,\right]_q\dots\left[\,T-t-S+z_i\,\right]_q. \tag{2.12}$$

Factoring out negative signs as before, then writing $A_j = -(S+j-1)$ and $B_j = T-t-S-j+2$ we can apply lemma A.1.2. Neglecting terms not in z (all absorbed into constants of proportionality) we write

$$\det_{1\le i,j\le N} h(t, S+j-1-z_i) \sim \prod_{i=1}^{N}\frac{q^{-z_i}}{\left[\,S+N-1-z_i\,\right]_{q^{-1}}!\left[\,T-t-S+z_i\,\right]_q!}$$
$$\prod_{1\le i<j\le N}\left[\,z_i\,\right]_{q^{-1}}-\left[\,z_j\,\right]_{q^{-1}}. \tag{2.13}$$

Reinserting these results into the joint probability function (2.3) then

$$\mathbf{P}_t(z_1, \ldots, z_N) = \frac{1}{Z_t} \prod_{1 \le i < j \le N} \left([\, z_i \,]_q - [\, z_j \,]_q \right)^2 \prod_{i=1}^{N} w_{q^{-1}}(z_i)$$

where Z_t is a constant without any dependence in z (but with t dependence) and

$$w_{q^{-1}}(z_i) = \frac{c q^{-z_i}}{[\, z_i \,]_q! \, [\, t+N-1-z_i \,]_{q^{-1}}! \, [\, S+N-1-z_i \,]_{q^{-1}}! \, [\, T-t-S+z_i \,]_q!}$$

where we introduce a constant $c \in \mathbb{R}$ to absorb any eventual terms in $w_{q^{-1}}(z)$ not depending on z (but perhaps depending on t.) We now consider as in chapter 1 t in three different intervals, which is due to the changing support of P_t depending on t.

The Interval $0 \le t \le S$

If $0 \le t \le S$ then the support of $\mathbf{P}_t$ is $0 \le z \le t+N-1$ and so defining $M = t+N-1$ we can consider the weight (up to a constant of proportionality)

$$w(z_i) \quad = \quad \frac{c q^{-z_i}}{[\, z_i \,]_q! \, [\, M-z_i \,]_{q^{-1}}! \, [\, S+N-1-z_i \,]_{q^{-1}}! \, [\, T-S+N-1-(M-z_i) \,]_q!}.$$

Consider then the term,

$$\frac{1}{[\, S+N-1-z_i \,]_{q^{-1}}! \, [\, z_i \,]_q!} = \frac{[\, S+N-1 \,]_{q^{-1}} [\, S+N-2 \,]_{q^{-1}} \ldots [\, S+N-z_i \,]_{q^{-1}}}{[\, S+N-1 \,]_{q^{-1}}! \, [\, z_i \,]_q!}$$

Therefore defining $\alpha = -S-N$ and using (A.1) and (A.11):

$$\begin{aligned} \frac{1}{[\, S+N-1-z_i \,]_{q^{-1}}! \, [\, z_i \,]_q!} &= \frac{(-q)^{z_i} [\, 1+\alpha \,]_q [\, 2+\alpha \,]_q \ldots [\, z_i+\alpha \,]_q}{[\, S+N-1 \,]_{q^{-1}}! \, [\, z_i \,]_q!} \\ &= \frac{(-q)^{z_i}}{[\, S+N-1 \,]_{q^{-1}}!} \frac{(q^{\alpha+1}; q)_{z_i}}{(q;q)_{z_i}} = \frac{(-q)^{z_i}}{[\, S+N-1 \,]_{q^{-1}}!} \begin{bmatrix} \alpha+z_i \\ z_i \end{bmatrix}_q \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{1}{[\, T-S+N-1-(M-z_i) \,]_q! \, [\, M-z_i \,]_{q^{-1}}!} &= \frac{[\, -\beta-(M-z_i) \,]_q \ldots [\, -\beta-1 \,]_q}{[\, T-S+N-1 \,]_q! \, [\, M-z_i \,]_{q^{-1}}!} \\ &= \frac{(-q^{-1})^{M-z_i}}{[\, T-S+N-1 \,]_q!} \begin{bmatrix} \beta+M-z_i \\ M-z_i \end{bmatrix}_{q^{-1}}. \end{aligned}$$

Reinserting these results,

$$w(z_i) = \frac{c(-1)^M q^{-M}}{[\, T-S+N-1 \,]_q! \, [\, S+N-1 \,]_{q^{-1}}!} \begin{bmatrix} \alpha+z_i \\ z_i \end{bmatrix}_q \begin{bmatrix} \beta+M-z_i \\ M-z_i \end{bmatrix}_{q^{-1}} q^{z_i}.$$

Therefore taking $c = c_t$ to cancel all terms not in z we can write,

$$\mathbf{P}_t(z_1, \ldots, z_N) = \frac{1}{Z_t} \prod_{1 \le i < j \le N} \left([\, z_i \,]_{q^{-1}} - [\, z_j \,]_{q^{-1}} \right)^2 \prod_{i=1}^{N} w_{q^{-1}}(z).$$

Using (A.10) we have, for $\alpha \in \mathbb{R}$ and $k = 0, 1, \dots$

$$\begin{bmatrix} \alpha + k \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha + k \\ k \end{bmatrix}_{q^{-1}} q^{k\alpha} \tag{2.14}$$

Therefore we may write,

$$w_{q^{-1}}(z) = q^{z(\alpha+1)} \begin{bmatrix} \alpha + z \\ z \end{bmatrix}_{q^{-1}} \begin{bmatrix} \beta + M - z \\ M - z \end{bmatrix}_{q^{-1}}$$

which is the q-Hahn weight of some q-Hahn orthogonal polynomial family found in literature (see e.g. [4] pg. 180.) Note that in the limit $q \to 1$ we get back the q-Hahn N-point probability. Because we end up taking $S = T - S$ we skip the case $S < t < T - S$ and move on immediately to $T - S \le t \le T$, just like in Chapter 1. While the details are left out, we can deduce the form of the answer by comparing with the Hahn case.

The Interval $T - S \le t \le T$

In this case the support of z is $S + t - T < z \le S + N - 1$, writing $z' = z + T - t - S$ we the support becomes

$$0 < z + T - t - S = z' \le T - t + N - 1.$$

Here z' is a shift in origin to keep the support as in the previous case. With this z' and setting $M = T - t + N - 1$ the weight $w(z)$ is, up to a constant of proportionality,

$$\frac{cq^{-z'}}{[\, z' \,]_q! \, [\, M - z' \,]_{q^{-1}}! \, [\, S + N - 1 - (M - z') \,]_q! \, [\, T + N - S - 1 - z' \,]_{q^{-1}}!}.$$

As in the previous section we simply further, defining $\beta = -S - N$ the term

$$\begin{aligned} \frac{1}{[\, S + N - 1 - (M - z') \,]_q! \, [\, M - z' \,]_{q^{-1}}!} &= \frac{[\, -\beta - (M - z') \,]_q \dots [\, -\beta - 1 \,]_q}{[\, S + N - 1 \,]_q! \, [\, M - z' \,]_{q^{-1}}!} \\ &= \frac{(-q^{-1})^{M - z'}}{[\, S + N - 1 \,]_q!} \begin{bmatrix} \beta + M - z' \\ M - z' \end{bmatrix}_{q^{-1}}. \end{aligned}$$

Defining $\alpha = S - T - N$ we can also write,

$$\begin{aligned} \frac{1}{[\, T + N - S - 1 - z' \,]_{q^{-1}}! \, [\, z' \,]_q!} &= \frac{[\, -\alpha - z' \,]_{q^{-1}} \dots [\, -\alpha - 1 \,]_{q^{-1}}}{[\, T + N - S - 1 \,]_{q^{-1}}! \, [\, z' \,]_q!} \\ &= \frac{(-q)^{z'}}{[\, T + N - S - 1 \,]_q!} \begin{bmatrix} \alpha + z' \\ z' \end{bmatrix}_q. \end{aligned}$$

$$\frac{c(-1)^M q^{-M} q^{z'}}{[\, S + N - 1 \,]_q! \, [\, T + N - S - 1 \,]_q!} \begin{bmatrix} \alpha + z' \\ z' \end{bmatrix}_q \begin{bmatrix} \beta + M - z' \\ M - z' \end{bmatrix}_{q^{-1}}.$$

Therefore using the q-binomial identity (A.10) and setting c to cancel all terms not in z,

$$w_{q^{-1}}(z) = \begin{bmatrix} \alpha + z' \\ z' \end{bmatrix}_{q^{-1}} \begin{bmatrix} \beta + M - z' \\ M - z' \end{bmatrix}_{q^{-1}} q^{z'(\alpha+1)}.$$

The Joint Probability for q-Hahn

In this section we restate the results compiled above. Note that in the literature convention is to consider q-orthogonal polynomials in the variable q^{-z}. For example the q-Hahn polynomials are defined in the notation of hypergeometric functions, (see appendix B.1 for details)

$$Q_n(q^{-z}, \alpha, \beta, M) = {}_3\tilde{\phi}_2 \left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-z} \\ \alpha q, q^{-M} \end{matrix} ; q, q \right)$$

is the q-Hahn polynomial of degree n (in q^{-z}) belonging to a family of polynomials Q_0, Q_1, $\ldots, Q_M$ with parameters α, β which satisfies the orthogonality relation,

$$\sum_{z=0}^{M} Q_n(z) Q_m(z) w_q(z)$$

Here the relation is in q^{-z} and w_q (this is usual convention in the literature) while we have q^z and $w_{q^{-1}}$ in the model. Given that our choice of q is free in $0 < q < 1$ or $q > 1$ we can interchange $q \mapsto q^{-1}$ without issue. Recalling that $S = N$ and $T = 2N$ we can summarize the results as follows:

Proposition 2.3.1. *Consider points $(t, z_1), (t, z_2), \ldots, (t, z_N)$ with $1 \le t \le T$ and $z_1 < \cdots < z_N$ that are contained in the support (in other words every $(t, z_i) \in V(H)$.) The joint probability function that a non-intersecting path crosses all the points is*

$$\mathbf{P}_t(z_1, \ldots, z_N) = \frac{1}{Z_t} \prod_{1 \le i < j \le N} \left([\, z_i \,]_{q^{-1}} - [\, z_j \,]_{q^{-1}} \right)^2 \prod_{i=1}^{N} w_q(z_i) \tag{2.15}$$

where the factor $\dfrac{1}{Z_t}$ is independent of z and the weight is

$$w_q(z) = \begin{bmatrix} \alpha + z' \\ z' \end{bmatrix}_{q^{-1}} \begin{bmatrix} \beta + M - z' \\ M - z' \end{bmatrix}_{q^{-1}} q^{(\alpha+1)z'}. \tag{2.16}$$

The values of M, α, β depend on the interval that t belongs to:

- *If $0 \le t \le N$ then $z' = z$ and the probability is supported on $0 \le z \le M$ where $M = t+N-1$ and $\alpha = \beta = -2N$.*
- *If $N < t \le 2N$ then $z' = N - t + z$ and the probability is supported on $0 \le z \le M$ where $M = 3N - 1 - t$ and $\alpha = \beta = -2N$.*

If $t < 0$ or $t > T$ or if any of the points are outside the support (i.e. $(t, z_i) \notin V(H)$ for any $1 \le i \le N$) then the probability is zero.

As in the Hahn case of chapter 1, the weight w_q is associated with a family of orthogonal polynomials, this time the q-Hahn polynomials. We discuss some of the properties of orthogonal polynomials in the specific case of q–Hahn polynomials in appendix B.

Remark. Taking $q \to 1$ in proposition 2.3.1 we have for the product

$$\lim_{q \to 1} \prod_{1 \le i < j \le N} \left([\, z_i \,]_{q^{-1}} - [\, z_j \,]_{q^{-1}} \right)^2 = \prod_{1 \le i < j \le N} \left(z_i - z_j \right)^2$$

and for the weight

$$\lim_{q \to 1} w_q = \binom{\alpha + z'}{z'} \binom{\beta + M - z'}{M - z'}.$$

which means that taking $q \to 1$ in the joint probability function for q-Hahn we reclaim the Hahn probability function. In other words we can subsume the Hahn case as a special case of q–Hahn.

2.4 The Joint Probability Function as a Determinant

In 2.3.1 we had a joint probability of exactly N points (t, x_i) with $0 \leq t \leq T$ and $x_1 < x_2 < \cdots < x_N$. More generally we may consider the joint probability functions for any m points where $1 \leq m \leq N$, for $m > N$ these the probabilities are of course zero. Define

$$\mathfrak{X}_t \subseteq \{(t, x) \in \mathbb{Z}^2 \mid x \in \mathbb{Z}\}$$

corresponding to points (t, x) in the lattice $\mathbb{Z}^2$ that are also in the support defined by the hexagonal boundary from proposition 2.3.1. In terms of $\mathfrak{X}_t$ these are points (t, x) such that $0 \leq x' \leq M$, where x' and M may depend on t. We then define the m-point joint probability function by

$$\mathbf{P}_t(\{x_1, \ldots, x_m\} \subseteq \mathfrak{X}_t),$$

if the z_i's are not distinct then we consider a m'-point joint probability function where $m' < m$ with distinct z_i. We will show a stronger result: That there exists some kernel $K : \mathfrak{X}_t \times \mathfrak{X}_t \to \mathbb{R}$ such that for any $1 \leq m \leq N$ the m-point joint probability function is a determinant of K:

$$\mathbf{P}_t(\{x_1, \ldots, x_m\} \subseteq \mathfrak{X}_t) = \det_{1 \leq i,j \leq N} [K(x_i, x_j)].$$

We present this result in our particular case of discrete orthogonal polynomials, following Mehta [17] (where it is presented in more generality.) (See [2] for a similar discussion in terms of *determinantal point processes* which are point processes, probabilities on $\mathfrak{X}$, with a *determinantal* property.) We note that the joint probability function can be extended to all points (t, x) where $x \in \mathbb{Z}$ by defining it to be zero if any of z_i's are outside the support. For simplicity define $X = X(x) := [\ x_i\ ]_{q^{-1}}$ and $X_k := X(x_k)$ for $k = 0, 1, \ldots$ so that the N-point joint probability becomes

$$\mathbf{P}_t(\{x_1, \ldots, x_N\} \subseteq \mathfrak{X}) = \frac{1}{Z_t N!} \prod_{1 \leq i < j \leq N} (X_i - X_j)^2 \prod_{k=1}^{N} w(X_k).$$

We have to rescale by $N!$ in the denominator as $\mathbf{P}_t(\{x_1, \ldots, x_N\}$ is now summed independently over the x_i. Define for some $f(x_1, \ldots, x_N) : \mathfrak{X}_t^N \to \mathbb{R}$,

$$\mathcal{I}_2(f) := \sum_{(x_1, \ldots, x_N) \in \mathfrak{X}_t^N} f(x_1, \ldots, x_N) |\Delta_N(X)|^2 \prod_{k=1}^{N} w(X_k) \tag{2.17}$$

where

$$\Delta_N(X) := \prod_{1 \leq i < j \leq N} (X_i - X_j).$$

In particular (noting that we sum over $x_i \in \mathfrak{X}_\mathrm{t}$ independently):

$$\begin{aligned} \mathcal{I}_2(1) &= \sum_{(x_1, \ldots, x_N) \in \mathfrak{X}_t^N} |\Delta_N(X)|^2 \prod_{k=1}^{N} w(X_k) \\ &= N! \sum_{x_1 < \cdots < x_N, x_i \in \mathfrak{X}_t} |\Delta_N(X)|^2 \prod_{k=1}^{N} w(X_k) = Z_t N!. \end{aligned} \tag{2.18}$$

Therefore the average of any function

$$\langle f(x_1, \ldots, x_N) \rangle = \frac{\mathcal{I}_2(f)}{\mathcal{I}_2(1)}.$$

The term $\Delta_N(X)$ can be also be identified as a $N \times N$ Vandermonde matrix where $|\Delta_N(X)|$ is the Vandermonde determinant

$$|\Delta_N(X)| = \begin{vmatrix} 1 & 1 & \dots & 1 \\ X_1 & X_2 & \dots & X_N \\ \vdots & \vdots & & \vdots \\ X_1^{N-1} & X_2^{N-1} & \dots & X_N^{N-1} \end{vmatrix} = \det_{1\le i,j\le N}[X_j^{i-1}].$$

Without changing the determinant we can use linear row-operations can turn the rows of $\Delta_N(X)$ into any desired sequence of N linearly independent, *monic*, polynomials $p_0(X)$, $p_1(X)$ $, \dots, p_{N-1}(X)$ each with degree less than N. Generally if P_n is a degree n, orthogonal polynomial with respect to the the Hahn or q-Hahn weight we can write

$$|\Delta_N(X)| = \gamma_0 \dots \gamma_{N-1} \det_{1\le i,j\le N}\left[\frac{1}{\gamma_{i-1}}P_{i-1}(X_j)\right]$$

where the real numbers $\gamma_0, \dots, \gamma_{N-1}$ are the leading coefficients of $P_0(X), \dots, P_{N-1}(X)$ and the $\frac{1}{\gamma_{i-1}}P_{i-1}$ are monic polynomials. We will specifically consider means of functions $\prod_{k=1}^N f(x_k)$ where $f : \mathfrak{X}_t \to \mathbb{R}$. We define

$$\mathcal{I}_2(f) := \sum_{(x_1,\dots,x_N)\in\mathfrak{X}_t^N} \left(\det_{1\le i,j\le N} p_{i-1}(X_j)\right)^2 \prod_{k=1}^N f(x_k)w(X_k).$$

Let S_N be the set of all permutations on $\{1, 2, \dots, N\}$ then we can expand the determinant as a sum over permutations $\sigma, \pi \in S_N$,

$$\mathcal{I}_2(f) = \sum_{\sigma\in S_N}\sum_{\pi\in S_N} \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi) \prod_{k=1}^N \sum_{x_k\in\mathfrak{X}_t} f(x_k)w(X_k)p_{\sigma(k)-1}(X_k)p_{\pi(k)-1}(X_k).$$

Remark. By the orthogonal property, for any set of orthogonal polynomials $p_0(X), \dots p_N(X)$ with respect to the weight $w(X)$,

$$\sum_{x\in\mathfrak{X}_t} w(X)p_i(X)p_j(X) = c_i\delta_{ij}$$

where $c_i \in \mathbb{R}$ is independent of X. Therefore defining for some $f : \mathfrak{X}_t \to \mathbb{R}$

$$\Phi_{ij}(f) := c_i^{-1} \sum_{x\in\mathfrak{X}_t} f(x)w(X)p_i(X)p_j(X)$$

we have in particular $\Phi_{ij}(1) = \delta_{ij}$ and therefore

$$\mathcal{I}_2(f) = \left(\prod_{i=0}^{N-1} c_i\right) \sum_{\sigma\in S_N}\sum_{\pi\in S_N} \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi) \prod_{k=1}^N \Phi_{\sigma(k),\pi(k)}(f).$$

Consider now the product, rearranging

$$\prod_{k=1}^N \Phi_{\sigma(k),\pi(k)}(f) = \prod_{k=1}^N \Phi_{k,(\pi\sigma^{-1})(k)}(f).$$

Let $\sigma' = \pi\sigma^{-1}$ then as σ^{-1} is a bijection $S_N \to S_N$ the sum over π can be taken over σ' instead, so

$$\mathcal{I}_2(f) = \left(\prod_{i=0}^{N-1} c_i\right) \sum_{\sigma \in S_N} \sum_{\sigma' \in S_N} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma'\sigma) \prod_{k=1}^{N} \Phi_{k,\sigma'(k)}(f).$$

The inner sum is independent of σ, summing $|S_N| = N!$ times. Writing $\operatorname{sgn}(\sigma'\sigma) = \operatorname{sgn}(\sigma')\operatorname{sgn}(\sigma)$ and noting $\operatorname{sgn}(\sigma)^2 = 1$

$$\mathcal{I}_2(f) = \left(\prod_{i=0}^{N-1} c_i\right) N! \sum_{\sigma' \in S_N} \operatorname{sgn}(\sigma') \prod_{k=1}^{N-1} \Phi_{k\sigma'(k)}(f) = \left(\prod_{i=0}^{N-1} c_i\right) N! \det[\Phi_{ij}(f)].$$

In the last line identifying the sum as a determinant over σ'. In particular,

$$\mathcal{I}_2(1) = \left(\prod_{i=0}^{N-1} c_i\right) N! \det[\Phi_{ij}(1)] = N! \det[\delta_{ij}] = N! \prod_{i=0}^{N-1} c_i.$$

Proposition 2.4.1. *The average value of $\prod_{i=1}^{N} f(x_i)$ can be expressed*

$$\left\langle \prod_{i=1}^{N} f(x_i) \right\rangle = \frac{\mathcal{I}_2(f)}{\mathcal{I}_2(1)} = \det_{1 \le i,j \le N} [\Phi_{ij}(f)]$$

where

$$\Phi_{ij}(f) := c_i^{-1} \sum_{x \in \mathfrak{X}_t} f(x) w(X) p_i(X) p_j(X)$$

and c_i is the constant in the orthogonality relation,

$$\sum_{x \in \mathfrak{X}_t} w(X) p_i(X) p_j(X) = c_i \delta_{ij}.$$

Remark. Consider the probability of $\{y_1, \ldots, y_m\} \subseteq \mathfrak{X}_t$ where $1 \le m \le N$. We can write this as the expectation of an indicator function which counts whenever $y_1, \ldots, y_m$ are among the $x_1, \ldots, x_N$. We see that starting from y_1 it can be any of the $x_1, \ldots, x_N$, then y_2 can be any of the remaining $N-1$ and so on... Therefore the m-point probability is

$$\frac{N!}{(N-m)!} \left\langle \prod_{i=1}^{m} \delta(x_i, y_i) \right\rangle$$

where $\delta(x, y) = 1$ if $x = y$ and zero otherwise. Therefore we can define the joint probability function by identifying $y_1 = x_1, y_2 = x_2, \ldots, y_m = x_m$ and summing over the remaining $N-m$ variables, $x_{m+1}, \ldots, x_N$,

$$\mathbf{P}_t(x_1, \ldots, x_m) := \frac{N!}{(N-m)!} \mathcal{I}_2(1)^{-1} \sum_{x_{m+1}, \ldots, x_N} |\Delta_N(X)|^2 \prod_{i=1}^{N} w(X_i).$$

On the other hand note that the same joint probability function can also be obtained by the use of a test function $\prod_{i=1}^{N}(1 + a(x_i))$, writing

$$\left\langle \prod_{i=1}^{N} (1 + a(x_i)) \right\rangle = \det_{1 \le i,j \le N} \Phi_{ij}(1 + a(x_i))$$

then differentiating m times in a and setting $a = 0$. The term $\Phi_{ij}(1 + a(x_i))$ can be expanded out

$$c_i^{-1} \sum_{x_i} (1 + a(x_i)) w(X_i) p_i(X_i) p_j(X_i) = \delta_{ij} + \Phi_{ij}(a)$$

and so

$$\det_{1 \le i,j \le N} \Phi_{ij}(1 + a(x_i)) = \det_{1 \le i,j \le N} \left(\delta_{ij} + \Phi_{ij}(a) \right).$$

We will try to rearrange this determinant as a sum of determinants of $m \times m$ sub-matrices

$$\det_{1 \le k,l \le N} \Phi_{i_k, i_l}(a)$$

where $0 \le i_1 < i_2 < \cdots < i_m \le N - 1$. To that end we expand the determinant out

$$\det_{1 \le i,j \le N} \left(\delta_{ij} + \Phi_{ij}(a) \right) = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \prod_{k=1}^{N} \left(\delta_{k\sigma(k)} + \Phi_{k\sigma(k)}(a) \right) \tag{2.19}$$

and multiply the product out taking $0 \le i_1 < i_2 < \cdots < i_m \le N - 1$ to avoid double counting,

$$\prod_{k=1}^{N} \left(\delta_{k\sigma(k)} + \Phi_{k\sigma(k)}(a) \right) = \sum_{m=0}^{N} \sum_{0 \le i_1 < \cdots < i_m \le N-1} \prod_{i_k \notin \{i_1, \ldots, i_m\}} \delta_{i_k i_{\sigma(k)}} \prod_{i_k \in \{i_1, \ldots, i_m\}} \Phi_{i_k i_{\sigma(k)}}(a).$$

Dragging the sum over σ inside we get

$$1 + \sum_{m=1}^{N} \sum_{0 \le i_1 < \cdots < i_m \le N-1} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \prod_{i_k \notin \{i_1, \ldots, i_m\}} \delta_{i_k i_{\sigma(k)}} \prod_{i_k \in \{i_1, \ldots, i_m\}} \Phi_{i_k i_{\sigma(k)}}(a)$$

for every m and $i_1, \ldots, i_m$ is the summand which is non-zero only if $\sigma(k) = k$ for all $i_k \notin \{i_1, \ldots, i_m\}$. If $\sigma_m \in S_m$ are the permutations on $\{1, 2, \ldots, m\}$ and $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma_m)$ then we can identify the summands as determinants in their own right:

$$\begin{aligned} &1 + \sum_{m=1}^{N} \sum_{0 \le i_1 < \cdots < i_m \le N-1} \sum_{\sigma_m \in S_m} \operatorname{sgn}(\sigma_m) \prod_{l \in \{i_1, \ldots, i_m\}} \Phi_{i_k i_{\sigma_m(k)}}(a) \\ &= 1 + \sum_{m=1}^{N} \sum_{0 \le i_1 < \cdots < i_m \le N-1} \det_{1 \le k,l \le m} \Phi_{i_k, i_l}(a). \end{aligned}$$

What relation does the sum restricting $0 \le i_1 < \cdots < i_m \le N-1$ have with summing independently over $i_k = 0, 1, \ldots, N - 1$ for all i_k? Note that if $i_k = i_l$ for any distinct k, l then we have two equal rows (and columns) so the determinant vanishes. On the other hand interchanging any two indices, for example, if $k > l$ but $i_k < i_l$, we will again have an interchange of two rows and two columns which leaves the determinant without a sign change. Therefore summing independently (and dividing through with the number of duplicates) the determinant in 2.19 can be written,

$$\det_{1 \le i,j \le m} (\delta_{ij} + \Phi_{ij}(a)) = \sum_{m=1}^{N} \frac{1}{m!} \sum_{i_1, \ldots, i_m = 0}^{N-1} \det_{1 \le k,l \le m} \Phi_{i_k, i_l}(a). \tag{2.20}$$

We turn our focus on the quantity $\det_{1\le k,l\le m}(\Phi_{i_k,i_l}(a))$, here

$$\begin{aligned}\det_{1\le i,j\le N}(\Phi_{i_k,i_l}(a)) &= \det_{1\le k,l\le m}\left(c_{i_k}^{-1}\sum_{x_k\in\mathfrak{X}_t} a(x_k)w(X_k)p_{i_k}(X_k)p_{i_l}(X_k)\right)\\ &= \sum_{(x_1,\dots,x_m)\in\mathfrak{X}_t^m}\left(\prod_{k=1}^m a(x_k)w(X_k)c_{i_k}^{-1}p_{i_k}(X_k)\right)\det_{1\le k,l\le m} p_{i_l}(X_k)\\ &= \sum_{(x_1,\dots,x_m)\in\mathfrak{X}_t^m}\left(\prod_{k=1}^m a(x_k)w(X_k)\right)\det_{1\le k,l\le m}\left(c_{i_l}^{-1}p_{i_l}(X_l)p_{i_l}(X_k)\right).\end{aligned}$$

The only dependence on the variables i_l where $1\le l\le m$ is in the determinant. Therefore bringing in the sums over i_l and expanding the determinant we find

$$\sum_{i_1,\dots,i_m=0}^{N-1}\det_{1\le k,l\le m}\left(c_{i_l}^{-1}p_{i_l}(x_l)p_{i_l}(x_k)\right) = \sum_{i_1,\dots,i_m=0}^{N-1}\sum_{\sigma_m\in S_m}\mathrm{sgn}(\sigma_m)\prod_{k=1}^m c_{i_k}^{-1}p_{i_k}(x_k)p_{i_k}(x_{\sigma(k)}).$$

Interchange the order of summation and consider for any permutation σ, the sum

$$\sum_{i_1,\dots,i_m=0}^{N-1}\prod_{k=1}^m c_{i_k}^{-1}p_{i_k}(x_k)p_{i_k}(x_{\sigma(k)})$$

it is possible here to iteratively factor out terms,

$$\begin{aligned}&\sum_{i_1,\dots,i_m=0}^{N-1}\mathrm{sgn}(\sigma_m)\prod_{k=1}^m c_{i_k}^{-1}p_{i_k}(x_k)p_{i_k}(x_{\sigma(k)})\\ &=\sum_{i_2\dots,i_m=0}^{N-1}\mathrm{sgn}(\sigma_m)\prod_{k=2}^m c_{i_k}^{-1}p_{i_k}(x_k)p_k(x_{\sigma(k)})\sum_{i_1=0}^{N-1}c_{i_1}^{-1}p_{i_1}(x_1)p_{i_1}(x_{\sigma(1)})\\ &=\sum_{i_3\dots,i_m=0}^{N-1}\mathrm{sgn}(\sigma_m)\prod_{k=3}^m c_{i_k}^{-1}p_{i_k}(x_k)p_k(x_{j_k})\sum_{i_2=0}^{N-1}c_{i_2}^{-1}p_{i_2}(x_2)p_{i_2}(x_{\sigma(2)})\sum_{i_1=0}^{N-1}c_{i_1}^{-1}p_{i_1}(x_1)p_{i_1}(x_{\sigma(1)})\\ &=\dots=\sum_{\sigma_m}\mathrm{sgn}(\sigma_m)\prod_{k=1}^m\sum_{i=0}^{N-1}c_i^{-1}p_i(x_k)p_i(x_{\sigma(k)}).\end{aligned}$$

After iterating through completely the result can be identified as another determinant and therefore we have shown that

$$\sum_{i_1,\dots,i_m=0}^{N-1}\det_{1\le i,j\le m}\left(c_{i_l}^{-1}p_{i_l}(x_l)p_{i_l}(x_k)\right) = \det_{1\le k,l\le m}\left(\sum_{i=0}^{N-1}c_i^{-1}p_i(x_k)p_i(x_l)\right). \tag{2.21}$$

Reinserting 2.21 into 2.20 we show that the average of the test function $\left\langle\prod_{i=1}^N 1+a(x_i)\right\rangle_2$ can be written as the following integral:

$$1+\sum_{m=1}^N\frac{1}{m!}\sum_{(x_1,\dots,x_m)\in\mathfrak{X}_t^N}\left(\prod_{k=1}^m a(x_k)w(X_k)\right)\det_{1\le k,l\le m}\left(\sum_{i=0}^{N-1}c_i^{-1}p_i(x_k)p_i(x_l)\right).$$

Differentiating m times and setting $a=0$ we have,

$$\sum_{(x_1,\dots,x_m)\in\mathfrak{X}_t^N}\prod_{k=1}^m w(X_k)\det_{1\le k,l\le m}\left(\sum_{i=0}^{N-1}c_i^{-1}p_i(x_k)p_i(x_l)\right)$$

and we identify the summand as the m-point joint probability function so that (factoring the $w(X_i)$'s into the determinant)

$$X_m(x_1, \ldots, x_m) = \det_{1 \leq k,l \leq m} K(x_k, x_l)$$

where we define the *kernel* $K : \mathfrak{X}_t \times \mathfrak{X}_t \to \mathbb{R}$ by

$$K(x, y) := w(X(x)) \sum_{i=0}^{N-1} c_i^{-1} p_i(X(x)) p_i(X(y)). \tag{2.22}$$

Therefore a joint probability that can be written in the form of (2.18) can be expressed as a determinant of a kernel if we can find an appropriate choice of orthogonal polynomials. This kernel is not unique, for example if we had factored $w(X)$ into the determinant in a different way we can find a kernel which is more symmetric:

$$K(x, y) = \sqrt{w(X(x))w(X(y))} \sum_{i=0}^{N-1} c_i^{-1} p_i(X(x)) p_i(X(y)).$$

Furthermore if we choose our p_i orthonormal with respect to the weight $w(X)$, we have $c_i = 1$. This result allows us to express the expectation of any function $\prod_{i=1}^{m} f(x_i)$ as

$$\left\langle \prod_{i=1}^{m} f(x_i) \right\rangle = \sum_{(x_1, \ldots, x_m) \in \mathfrak{X}_t^N} \left(\prod_{i=1}^{m} f(x_i) \right) \det_{1 \leq k,l \leq m} (K(x_k, x_l)). \tag{2.23}$$

where with a choice of orthonormal p_i we write,

$$K(x_k, x_l) = \sqrt{w(X_k)w(X_l)} \sum_{i=0}^{N-1} p_i(X_k) p_i(X_l). \tag{2.24}$$

Chapter 3

Asymptotic Zero Distributions of Orthogonal Polynomials

3.1 The Jacobi Operator

For a sequence of orthonormal q-Hahn orthogonal polynomials (indexed by the number of points N in the support)

$$p_{0,N}(X),\ p_{1,N}(X), \dots,\ p_{N,N}(X)$$

where $X = X(x) := [\,x\,]_{q^{-1}}$ there exists a three term recurrence relation (see appendix B.1.1) with $a_{n,N} \geq 0$ and $b_{n,N} \in \mathbb{R}$ such that

$$Xp_{n,N}(X) = a_{n+1,N}p_{n+1,N}(X) + b_{n,N}p_{n,N}(X) + a_{n,N}p_{n-1,N}(X).$$

For orthogonal polynomial sequences that terminate at some N we take $a_{n,N}$ as zero for $n > N$ while for $0 \leq n \leq N$ we have $a_{n,N} > 0$, $b_{n,N} \in \mathbb{R}$. Furthermore the polynomials $p_0(X) = 1$ and we define by convention $p_{-1}(X) := 0$. Specifically for q-Hahn polynomials, if $0 \leq n \leq N$ we have $a_{n,N} = \sqrt{A_{n-1,N}C_{n,N}}$ and $b_{n,N} = A_{n,N} + C_{n,N}$ where

$$A_{n,N} = \frac{[\,N-n\,]_{q^{-1}}\,[\,-(\alpha+1)-n\,]_{q^{-1}}\,[\,-(\alpha+\beta+1)-n\,]_{q^{-1}}}{[\,-(\alpha+\beta+1)-2n\,]_{q^{-1}}\,[\,-(\alpha+\beta+2)-2n\,]_{q^{-1}}},$$

$$C_{n,N} = \frac{q^{\alpha+2n-N}\,[\,n\,]_{q^{-1}}\,[\,-\beta-n\,]_{q^{-1}}\,[\,-(\alpha+\beta+N+1)-n\,]_{q^{-1}}}{[\,-(\alpha+\beta)-2n\,]_{q^{-1}}\,[\,-(\alpha+\beta+1)-2n\,]_{q^{-1}}}.$$

We suppress the index N when it is clear which q-Hahn polynomials we are considering. These recurrence relation can be encoded in a matrix (see [1]), for any terminated sequence of orthogonal polynomials $p_0(X), p_1(X), \dots p_{N-1}(X), p_N(X)$:

$$\begin{pmatrix} b_0 & a_1 & & & & 0 \\ a_1 & b_1 & a_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{N-2} & b_{N-2} & a_{N-1} \\ 0 & & & & a_{N-1} & b_{N-1} \end{pmatrix} \begin{pmatrix} p_0(X) \\ p_1(X) \\ \vdots \\ \vdots \\ p_{N-2}(X) \\ p_{N-1}(X) \end{pmatrix} = \begin{pmatrix} Xp_0(X) \\ Xp_1(X) \\ \vdots \\ \vdots \\ Xp_{N-2}(X) \\ Xp_{N-1}(X) - a_N p_N(X) \end{pmatrix}.$$

Definition 3.1.1. *The Jacobi operator J_N is an $N \times N$ matrix with entries*

$$[J_N]_{ij} := a_i\delta_{i+1,j} + b_{i-1}\delta_{i,j} + a_{i-1}\delta_{i-1,j} \tag{3.1}$$

where δ_{ij} is the Kronecker delta defined as $\delta_{i,j} = 1$ only if $i = j$ and zero otherwise.

Of particular interest is when $p_N(X) = 0$ in which case

$$J_N p(X) = Xp(X)$$

where $p(X)$ is the column matrix with entries $p_0(X), p_1(X), \ldots, p_{N-1}(X)$. This is a sort of eigenvalue correspondence where the condition is on X being a zero of $p_N(X) = 0$. We can extend the definition of the Jacobi operator to sequences of orthogonal polynomials $p_0, p_1, \ldots$ that continue into infinity, given that the associated three-term recurrence relation satisfies extra conditions on the coefficients, (in particular we consider $a_i \leq C$ and $|b_i| \leq C$ for some finite $C > 0$.) Then J can be defined,

$$[J]_{ij} := a_i \delta_{i+1,j} + b_{i-1}\delta_{i,j} + a_{i-1}\delta_{i-1,j} \quad \text{for } i \geq 1$$

For sequences of orthogonal polynomials that terminate at some N we have $a_{n,N} = 0$ for $n > N$. We can write $J = J^{(N)}$ to make the dependence on the index N clear (in practice we write just J for simplicity.) Denoting the projection operator as the block matrix $P_N = \begin{pmatrix} I_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ where I_N is the $N \times N$ identity matrix and the rest of the entries are populated with zeros, we can relate the the Jacobi operator terminated at a positive integer N by $J_N := P_N J P_N$. This is equivalent to $J_N = [J_{ij}]_{1 \leq i,j \leq N}$.

Remark. Consider the inner product $(\cdot, \cdot)$ defined

$$(f_i, g_j) := \sum_{x_k \in \mathfrak{X}_t} f_i(X_k) g_j(X_k) w(X_k)$$

where f, g are functions over the space $\mathfrak{X}_t$. Then in particular for the polynomials $p_i(X)$s orthonormal under the weight on the right hand side we have $(p_i, p_j) = \delta_{ij}$. We may therefore write for the Jacobi operator (3.1)

$$J_{ij} = a_i(p_{i+1}, p_j) + b_{i-1}(p_i, p_j) + a_{i-1}(p_{i-1}, p_j)$$

Using the linearity of the inner product,

$$J_{ij} = (a_i p_{i+1} + b_{i-1} p_i + a_{i-1} p_{i-1}, p_j) = (Xp_i, p_j)$$

Therefore we may identify the the matrix J as associated with an operator $\mathcal{J} : L^2(\mathbb{R}, w) \to L^2(\mathbb{R}, w)$ on the set of orthonormal orthogonal polynomials $\{p_0(X), p_1(X), \ldots\}$ sending $p_i \mapsto Xp_i$.

Conversely if there is a set of polynomials $e_0(X), e_1(X), \ldots$ form a basis of $\mathbb{Z}[X]$ and there is an inner product so that $(e_i, e_j) = \delta_{ij}$ then we may view a linear operator $\mathcal{J} : L^2(\mathbb{R}, w) \to L^2(\mathbb{R}, w)$ as a $J : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ defined

$$J_{ij} = \left(\mathcal{J} p_{i-1}(x), p_{j-1}(x)\right).$$

In general we can define a linear operator $\mathcal{A}$: by

$$\mathcal{A} = \sum_{i,j} a_{ij} e_j (e_i, \cdot)$$

where $A_{ij} = (\mathcal{A} e_i, e_j)$ and the matrix $[A]_{ij} = A_{ij}$ such that $A : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$. In other words $\mathcal{J}$ is equivalent to matrix multiplication by J in a vector space where $e_i's$ make up the basis.

Remark. We show that the composition of the linear operator $\mathcal{A} : L^2(\mathbb{R}, w) \to L^2(\mathbb{R}, w)$ on itself $\mathcal{A}^2 p$ is equivalent to the operator with the matrix A^2. Consider $\mathcal{A}^2 p = \mathcal{A}(\mathcal{A}p)$ then

$$\begin{aligned}
&\sum_{ij} a_{ij} e_j (e_i, \mathcal{A}p) \\
&= \sum_{i,j} A_{ij} e_j \left(e_i, \sum_{k,l} A_{kl} e_k (e_l, p) \right) = \sum_{i,j} \sum_{k,l} A_{ij} A_{kl} e_j (e_i, e_l)(e_k, p) \\
&= \sum_{k,j} \sum_i A_{ki} A_{ij} e_j (e_k, p) = \sum_{k,j} [A^2]_{kj} e_j (e_k, p).
\end{aligned}$$

The average value of some function in just one variable $a(x) : \mathfrak{X}_t \times \mathfrak{X}_t \to \mathbb{R}$ can be determined by setting $m = 1$ and taking $x_1 = x$ in the result of (2.23),

$$\langle a(x) \rangle = \sum_{x \in \mathfrak{X}_t} a(x) K_N(x, x)$$

where we write the kernel with the N index for clarity,

$$K_N(x, x) = w(X) \sum_{i=0}^{N-1} p_i(X) p_i(X).$$

The average value of $a(x)$ is therefore,

$$\langle a(x) \rangle = \sum_{x \in \mathfrak{X}_t} a(x) w(X) \sum_{i=0}^{N-1} p_i(X) p_i(X) = \sum_{i=0}^{N-1} \sum_{x \in \mathfrak{X}_t} a(x) p_i(X) p_i(X) w(X).$$

Note that terms inside the first sum are the inner products $(a(x)p_i, p_i)$ and

$$\langle a(x) \rangle = \sum_{i=0}^{N-1} (a(x) p_i, p_i)$$

If $a(x)$ is a polynomial $\sum_{k=0}^{m} c_k x^k$ with $c_k \in \mathbb{R}, c_m > 0$ then we show that the corresponding operator $\mathcal{A}$ which maps $p_i \mapsto a(x)p_i$ is associated with the matrix $a(J) = \sum_{k=0}^{m} c_k J^k$ (where J is the matrix associated with the operator $\mathcal{J}$ that sends $p_i \to xp_i$, described above.) The matrix A has entries,

$$A_{ij} = \big(\mathcal{A}p_i(x), p_j(x)\big) = \big(a(x)p_i(x), p_j(x)\big)$$

and using linearity,

$$A_{ij} = \sum_{k=0}^{m} c_k \left(x^k p_i(x), p_j(x) \right) = \sum_{k=0}^{m} c_k \left(\mathcal{J}^k p_i(x), p_j(x) \right).$$

Because $[J^i]_{ij} = \big(\mathcal{J}^i p_{i-1}(x), p_j(x)\big)$ then,

$$a_{ij} = \sum_{k=0}^{m} c_k [J^k]_{ij} = \left[\sum_{k=0}^{m} c_k J^k \right]_{ij} = [a(J)]_{ij}.$$

This allows us to write the average value of $a(x)$ as,

$$\langle a(x) \rangle = \sum_{i=1}^{N} [a(J)]_{ii}.$$

Using the projection matrix P_N we can identify this as the trace of a matrix, writing:

$$\langle a(x)\rangle = \mathrm{Tr}\ P_N a(J) P_N.$$

Consider on the other hand the zeros $x_j = X_1, X_2, \ldots, X_N$ of $P_N(X)$ and note in particular that these X_j are the N eigenvalues of the $N \times N$ matrix, J_N (by corollary B.2.3 these eigenvalues are real and distinct.) For any $k = 0, 1, \ldots$ we have for an $N \times N$ matrix M that has N eigenvalues $\lambda_1, \ldots, \lambda_N$ (counted with multiplicity) that

$$\mathrm{Tr}\ M^k = \sum_{j=1}^{N} \lambda_j^k.$$

Using the linearity of the transpose then we therefore write the transpose of $a(J_N)$ as

$$\mathrm{Tr}\ a(J_N) = \sum_{j=1}^{N} a(x_j).$$

The following proposition (from private correspondence with Maurice Duits) connects the two quantities:

Proposition 3.1.2. *Consider a Jacobi operator $J = J^{(N)}$ associated with a sequence of discrete orthogonal polynomials (indexed by N) $p_{0,N}(X), p_{1,N}(X), \ldots, p_{N,N}(X)$ which have the three-term recurrence relations*

$$Xp_n(X) = a_{n+1,N}p_{n+1,N}(X) + b_{n,N}p_{n,N}(X) + a_{n,N}p_{n-1,N}(X)$$

where $a_{n,N} \geq 0$, $b_{n,N} \in \mathbb{R}$, $n = 0, 1, \ldots, N$. Assume that there exists some upper bound $C > 0$ on all the $a_{n,N} \leq C$, $|b_{n,N}| \leq C$, then for any polynomial $a(x)$,

$$\frac{1}{N}\sum_{x\in\mathfrak{X}_t} a(x)K_N(x,x) - \frac{1}{N}\sum_{x,p_{N,N}(X(x))=0} a(x) \to 0 \quad as \quad N \to \infty.$$

Proof. By the previous discussion the limit in the proposition is equivalent to,

$$\frac{1}{N}\mathrm{Tr}(P_N f(J) P_N - f(J_N)) \to 0 \quad \text{as} \quad N \to \infty.$$

Let L_k denote a $k \times k$ lower triangular matrix,

$$L_k = \begin{pmatrix} l_{11} & & 0 \\ \vdots & \ddots & \\ l_{k1} & \cdots & l_{kk} \end{pmatrix}$$

with the l_{ij} taking values in $\mathbb{R}$. We similarly define upper triangular matrices as the transpose of the lower triangular ones, $L_k^\top$. In general for any matrix M we say that M is L_k to mean that $M = L_k$ for some choice of l_{ij}'s. We also define another types of matrix for convenience, with $d_{ij} \in \mathbb{R}$,

$$D_k = \begin{pmatrix} 0 & & d_{1k} \\ & \iddots & \vdots \\ d_{k1} & \cdots & d_{kk} \end{pmatrix}.$$

Furthermore we extend the definition of L_k by considering the block matrix $\mathbf{L}_{N,k}$ for any $N > k$ which is a $N \times N$ matrix consisting of L_k in the lower left corner and padded with zeros everywhere else. We just write $\mathbf{L}_k$ if the dimension is implicitly understood, e.g. due to restrictions imposed by the shape of the block matrix or matrix summation, multiplication with another matrix of known dimension. The matrix $\mathbf{L}_{N,k}$ can be written in block matrix form as,

$$\mathbf{L}_{N,k} = \left(\begin{array}{c|c} \mathbf{0}_{N-k\times k} & \mathbf{0}_{N-k\times N-k} \\ \hline L_k & \mathbf{0}_{k\times N-k} \end{array}\right)$$

where $\mathbf{0}_{r,s}$ is an $r \times s$ zero matrix. We define $\mathbf{D}_{k-1}$ similarly by padding with zeros above and to the left to make an $N \times N$ matrix. With this notation we propose that we can express

$$\mathbf{J}^k = \left(\begin{array}{c|c} J_N^k + \mathbf{D}_{k-1} & \mathbf{L}_k \\ \hline \mathbf{L}_k^\top & J^k + \mathbf{D'}_{k-1} \end{array}\right)$$

where the various dimensions are implicitly determined by J_N, J. We prove this by induction. It clearly holds for $k = 1$ where $\mathbf{D}_0 = 0$ and $E_1 = A_N$. Assume then that it holds for $k = l$ and consider $k = l+1$. Writing $\mathbf{J}^{l+1} = \mathbf{J}\mathbf{J}^k$ we need only show that left multiplication by J preserves the suggested form. Actually by symmetry we know that the cross-diagonal matrices will have the same form, so we can ignore the bottom-right matrix and deduce the top-right one from the bottom-left. Consider therefore,

$$\left(\begin{array}{c|c} J_N & A_N \\ \hline A_N^\top & J \end{array}\right)\left(\begin{array}{c|c} J_N^k + \mathbf{D}_{k-1} & \mathbf{L}_k \\ \hline \mathbf{L}_k^\top & * \end{array}\right)$$
$$= \left(\begin{array}{c|c} J_N^{k+1} + J_N\mathbf{D}_{k-1} + A_N\mathbf{L}_k^\top & A_N^\top J_N^k + A_N^\top\mathbf{D}_{k-1} + J\mathbf{L}_k^\top \\ \hline * & * \end{array}\right).$$

We say that M is a matrix of the same form as M' if for any i, j if $[M']_{ij} = 0$ then $[M]_{ij} = 0$. By symmetry we only need to show that $J_N\mathbf{D}_{k-1} + A_N\mathbf{L}_k^\top$ is in the form specified by $\mathbf{D}_k$ and that

$$A_N^\top J_N^k + A_N^\top\mathbf{D}_{k-1} + J\mathbf{L}_k^\top$$

is of the same shape as $\mathbf{L}_k^\top$. The first case can be confirmed by considering J_N as a sum of three matrices R_{-1}, R_0, R_1, corresponding to the three diagonals; the main diagonal R_0, the off-diagonal R_1 above and the off-diagonal R_{-1} below. These diagonals each act in a different way on a matrix of the form $\mathbf{D}_{k-1}$. We are only concerned about the change in the position of the non-zero values $\mathbf{D}_{k-1}$ rather than its particular values. The main diagonal of J_N does not change the position of non-zero values in $\mathbf{D}_{k-1}$. On the other hand the off-diagonal above the main diagonal shifts the non-zero entries of $\mathbf{D}_{k-1}$ up a step:

$$R_1\begin{pmatrix} 0 & & & & \\ & & & & \\ & & & & * \\ & & & \iddots & \vdots \\ & & * & \cdots & * \\ * & * & & \cdots & * \end{pmatrix} = \begin{pmatrix} 0 & & & & & \\ & & & & & * \\ & & & & \iddots & \vdots \\ & & & * & \cdots & * \\ & & * & * & \cdots & * \\ 0 & 0 & 0 & & \cdots & 0 \end{pmatrix}.$$

On the other hand every off-diagonal below the main diagonal shifts every value one step down. The total contribution of all three operations produces a matrix with shape $\mathbf{D}_k$.

$$R_{-1}\begin{pmatrix} 0 & & & & \\ & & & & \\ & & & & * \\ & & & \iddots & \vdots \\ & & * & \cdots & * \\ & * & * & \cdots & * \end{pmatrix} = \begin{pmatrix} 0 & & & & & \\ & & & & & \\ & & & & & 0 \\ & & & & 0 & * \\ & & \iddots & & \iddots & \vdots \\ & 0 & * & & \cdots & * \end{pmatrix}.$$

On the other hand $A_N\mathbf{L}_k^\top$ is a row of values $i = N, N-k+1 \le j \le N$, which is also of the form $\mathbf{D}_k$ (filling the bottommost rung of the triangle.) Combining all these matrices we see that $A_N\mathbf{L}_k^\top + J_N\mathbf{D}_{k-1}$ is of the form $\mathbf{D}_k$. The same argument can be used to show that $A_N^\top J_N^k + A_N^\top\mathbf{D}_{k-1} + J\mathbf{L}_k^\top$ can be written as a matrix of the form $\mathbf{L}_k^\top$. We have therefore shown that we can express $\mathbf{J}^k$ as a block matrix of the form

$$\mathbf{J}^k = \left(\begin{array}{c|c} J_N^k + \mathbf{D}_{k-1} & \mathbf{L}_k \\ \hline \mathbf{L}_k^\top & J^k + \mathbf{D'}_{k-1} \end{array}\right).$$

Taking the trace,

$$\mathrm{Tr}(P_N\mathbf{J}^kP_N - \mathbf{J}_N^k) = \mathrm{Tr}\left(\begin{array}{c|c} \mathbf{D}_{k-1} & 0 \\ \hline 0 & 0 \end{array}\right) = \mathrm{Tr}(D_{k-1}).$$

(Here we consider D_{k-1} as the actual matrix with entries from $P_N\mathbf{J}^kP_N - \mathbf{J}_N^k$, and not just an lower-right triangular matrix with unspecified entries.) Furthermore note that D_{k-1} is independent of N and has at most $\lceil (k-1)/2 \rceil$ non-zero diagonal entries. Note that a bound on the absolute value of the entries of the matrix $\mathbf{J}^k$ also bounds the absolute value of the entries of D_{k-1}. The absolute values of the entries of J were assumed to be bounded from above by C therefore if T is the matrix with every a_i, b_i replaced with one, i.e.

$$[T]_{ij} = \delta_{i+1,j} + \delta_{i,j} + \delta_{i-1,j}$$

then we have have,

$$\sup_{i,j}[J^k]_{i,j} \le C^k \sup_{i,j}[T^k]_{i,j}.$$

We show that we can bound $\sup_{i,j}[T^k]_{i,j} \le 3^k$ for $k \ge 0$. For $k = 0$, we have the identity matrix so the bound holds. Assume that this holds for T^l, then

$$[T^{l+1}]_{i,j} = \sum_{s\ge 1}[T]_{is}[T^l]_{sj} = (\delta_{i+1,s} + \delta_{i,s} + \delta_{i-1,s})[T^l]_{sj} \le 3\sup_{i,j}[T^l]_{i,j}$$

therefore taking the supremum on the left,

$$\sup_{i,j}[T^{l+1}]_{i,j} \le 3\sup_{i,j}[T^l]_{i,j} \le 33^l = 3^{l+1}.$$

We can therefore conclude that the bound $\sup_{i,j}[J^k]_{i,j} \le C^k 3^k$ holds for all $k \ge 0$ and this means that

$$\mathrm{Tr}(D_{k-1}) \le \lceil (k-1)/2 \rceil \sup_{i,j}[J^k]_{i,j} \le C^k 3^k \lceil (k-1)/2 \rceil.$$

In general for a polynomial $f(J) = \sum_{k=0}^{m} c_k J^k$ we have by linearity

$$\operatorname{Tr}(P_N f(J) P_N - f(J_N)) = \sum_{k=0}^{m} c_k \operatorname{Tr}(P_N J^k P_N - J_N^k).$$

Using the triangle inequality we can bound the magnitude,

$$\left| \sum_{k=0}^{m} c_k \operatorname{Tr}(P_N J^k P_N - J_N^k) \right| \leq \sum_{k=0}^{m} |c_k| \left| \operatorname{Tr}(P_N J^k P_N - J_N^k) \right| = \sum_{k=0}^{m} |c_k| \operatorname{Tr}(D_{k-1}).$$

Let $\gamma = \max_{1 \leq k \leq m} |c_k|$ and use the bound established for $\operatorname{Tr}(D_{k-1})$, then

$$|\operatorname{Tr}(P_N f(J) P_N - f(J_N))| \leq \sum_{k=1}^{m} |c_k| 3^k C^k \lceil (k-1)/2 \rceil \leq \gamma \lceil (m-1)/2 \rceil C^m (3^{m+1} - 1)/2.$$

The bound itself is far from the best achievable, but it suffices as it is importantly only dependent on m (the degree of $a(x)$) and not on the degree of N. Therefore as $N \longrightarrow \infty$,

$$\frac{1}{N} |\operatorname{Tr}(P_N f(J) P_N - f(J_n))| \leq \frac{\gamma \lceil (m-1)/2 \rceil C^m (3^{m+1} - 1)}{2N} \longrightarrow 0.$$

□

3.2 The Asymptotic Zero Distribution of Orthogonal Polynomials

Following Kuijlaars and Van Assche in [15] we consider the normalized zero counting measure that for some positive integer N counts the zeros of $p_{n,N}(X)$,

$$\chi_{n,N} := \frac{1}{n} \sum_{i=1}^{n} \delta(x_k, x),$$

where $x_{k,n}$ for $k = 1, \ldots, n$ are the zeros of $p_{n,N}(X(x_k)) = 0$. Summing over all points in the support

$$\sum_{x_i \in \mathfrak{X}_t} \chi_{n,N}(x_i) = 1$$

so $\chi_{n,N}$ is a probability measure. A probability measure μ is said to be an asymptotic zero distribution of the sequence $p_{0,N}, p_{1,N}, \ldots, p_{N,N}$ if

$$\lim_{n \to \infty} \sum_{x \in \mathfrak{X}_t} f(x) \chi_{n,N}(x) = \int f(x) \mathrm{d}\mu$$

for every continuous function f on $\mathbb{R}$ that vanishes at ∞. We define the notation

$$\lim_{n/N \to t} X_{n,N} = X$$

to mean

$$\lim_{j \to \infty} X_{n_j, N_j} = X$$

whenever $\{n_j\}$ and $\{N_j\}$ are two sequences of numbers such that $N_j \longrightarrow \infty$ and $n_j / N_j \longrightarrow s$ as $j \longrightarrow \infty$. Also from Kuijlaars and Assche [15] we relay the following theorem:

Theorem 3.2.1. *For each $N \in \mathbb{N}$ consider two sequences recurrence coefficients $\{a_{n,N}\}_{n=1}^{\infty}$ and $\{b_{n,N}\}_{n=0}^{\infty}$ with $a_{n,N} \geq 0$, $b_{n,N} \in \mathbb{R}$. Consider also orthonormal polynomials $p_{n,N}$ generated by the recurrence*

$$xp_{n,N}(x) = a_{n+1,N}p_{n+1,N}(x) + b_{n,N}p_{n,N}(x) + a_{n,N}p_{n-1,N}(x),$$

for $n \geq 0$ and the initial conditions $p_{0,N} := 1$ and $p_{-1,N} := 0$. Suppose that there exist two continuous function $a : (a, \infty) \to [0, \infty)$, $b : (0, \infty) \to \mathbb{R}$, such that

$$\lim_{n/N \to s} a_{n,N} = a(s), \qquad \lim_{n/N \to s} b_{n,N} = b(s)$$

whenever $s > 0$. Define the functions

$$\alpha(s) := b(s) - 2a(s), \qquad \beta(s) := b(s) + 2a(s), \qquad s > 0.$$

Then we have for every $s > 0$ the asymptotic zero distribution

$$\mu_s = \lim_{n/N \to s} \chi_{n,N} = \frac{1}{s} \int_0^s \omega_{[\alpha(s),\beta(s)]} ds$$

where $\omega_{[\alpha(s),\beta(s)]}$ has density

$$\frac{d\omega_{[\alpha(s),\beta(s)]}(x)}{ds} = \begin{cases} \frac{1}{\pi\sqrt{(\beta(s)-x)(x-\alpha(s))}} & \text{if } x \in (\alpha(s), \beta(s)) \\ 0 \text{ elsewhere} \end{cases}.$$

and the density of the asymptotic zero distribution is $\mu_s(x)dx$.

Remark. As we are applying theorem 3.2.1 on the discrete q-Hahn polynomials we must be mindful that the sequence of polynomials $p_0, p_1, \ldots, p_N$ terminate for some N and that the corresponding the recurrence relation,

$$Xp_{n,N} = a_{n+1,N}p_n(X) + b_{n,N}p_{n,N}(X) + a_{n-1,N}p_{n,N}(X)$$

terminates at $n = N$. Therefore in we must restrict $s \leq 1$ so that $n_j \leq N_j$ for all j as $j \to \infty$. Taking $s = 1$ we see that the theorem gives us the asymptotic zero distribution of $p_N(X)$.

The orthonormal recurrence relation for q-Hahn orthogonal polynomials $\{p_i(X)\}$ with $X := [\,x\,]_{q^{-1}}$ (see appendix B.2) is

$$[\,x\,]_{q^{-1}}\, p_n = a_{n+1}p_{n+1} + b_np_n + a_np_{n-1}$$

where for $n \leq N$, $a_n = \sqrt{A_{n-1}C_n}$ and $b_n = A_n + B_n$ and

$$A_n = \frac{[\,M - n\,]_{q^{-1}}\,[\,-(\alpha+1) - n\,]_{q^{-1}}\,[\,-(\alpha+\beta+1) - n\,]_{q^{-1}}}{[\,-(\alpha+\beta+1) - 2n\,]_{q^{-1}}\,[\,-(\alpha+\beta+2) - 2n\,]_{q^{-1}}},$$
$$C_n = \frac{q^{\alpha+2n-M}\,[\,n\,]_{q^{-1}}\,[\,-\beta - n\,]_{q^{-1}}\,[\,-(\alpha+\beta+M+1) - n\,]_{q^{-1}}}{[\,-(\alpha+\beta) - 2n\,]_{q^{-1}}\,[\,-(\alpha+\beta+1) - 2n\,]_{q^{-1}}}.$$

We consider a sequence of q-Hahn orthogonal polynomials, indexed by N. Therefore we define $q^{-1} = c^{1/N}$, where $0 < c < 1$ and consider the limit as $N \to \infty$. For any number α we have

$$[\alpha]_{c^{1/N}} = \frac{1 - c^{\alpha/N}}{1 - c^{1/N}} = \frac{1 - c}{1 - c^{1/N}} \left[\, \frac{\alpha}{N} \,\right]_c.$$

We need to rescale x by Nx as $N \to \infty$ so that $[\, Nx \,]_{q^{-1}} = \frac{1-c}{1-c^{1/N}} [\, x \,]_c$. Therefore defining

$$A'_n := \frac{1-c^{1/N}}{1-c} A_n = \frac{\left[\, \frac{M}{N} - \frac{n}{N} \,\right]_c \left[\, -\frac{\alpha+1}{N} - \frac{n}{N} \,\right]_c \left[\, -\frac{\alpha+\beta+1}{N} - \frac{n}{N} \,\right]_c}{\left[\, -\frac{\alpha+\beta+1}{N} - \frac{2n}{N} \,\right]_c \left[\, -\frac{\alpha+\beta+2}{N} - \frac{2n}{N} \,\right]_c}$$

and

$$C'_n = \frac{1-c^{1/N}}{1-c} C_n = \frac{c^{\frac{M-\alpha-2n}{N}} \left[\, \frac{n}{N} \,\right]_c \left[\, -\frac{\beta}{N} - \frac{n}{N} \,\right]_c \left[\, -\frac{\alpha+\beta+M+1}{N} - \frac{n}{N} \,\right]_c}{\left[\, -\frac{\alpha+\beta}{N} - \frac{2n}{N} \,\right]_c \left[\, -\frac{\alpha+\beta+1}{N} - \frac{2n}{N} \,\right]_c}.$$

In other words the recurrence relation can be written for $0 < c < 1$ then

$$[\, x \,]_c \, p_n = a'_{n+1} p_{n+1} + b'_n p_n + a'_n p_{n-1}$$

where $a'_n := \sqrt{A'_{n-1} C'_n}$ and $b'_n := A'_n + C'_n$. Here the p_n's are polynomials in $[\, x \,]_c$. The parameters α, β, M are determined by the model in proposition 2.3.1 and depend on t, where the regular hexagon corresponds to parameters $N, T = 2N, S = N$. There are two distinct cases, $0 \le t \le N$ (case I) and $N < t \le 2N$ (case II.)

Case I: $0 \le t \le N$

From proposition 2.3.1 the parameters in this interval are

$$\begin{cases} \alpha = -S - N = -2N \\ \beta = S - T - N = -2N \\ 0 \le x \le M = N + N\tau - 1 \end{cases}$$

we scale t in N by writing $N\tau$ where $0 \le \tau \le 2$. Therefore $0 \le t \le N$ corresponds to $0 \le \tau \le 1$. With these choices we take the limits,

$$\lim_{n/N \to s} A'_n := \frac{[\, 1+\tau-s \,]_c [\, 2-s \,]_c [\, 4-s \,]_c}{[\, 4-2s \,]_c [\, 4-2s \,]_c}$$
$$\lim_{n/N \to s} C'_n := \frac{c^{3+\tau-2s} [\, s \,]_c [\, 2-s \,]_c [\, 3-\tau-s \,]_c}{[\, 4-2s \,]_c [\, 4-2s \,]_c}.$$

Case II: $N < t \le 2N$

In this interval the parameters can be written

$$\begin{cases} \alpha = S - T - N = -2N \\ \beta = -S - N = -2N \\ M = T - t + N - 1 = 3N - N\tau - 1 \end{cases}$$

furthermore $t' \le x' \le 2N - 1$ and

$$x' = T - S - t + x = N - N\tau + x.$$

Therefore in the limit we have,

$$\lim_{n/N \to s} A'_n := \frac{[\, 3-\tau-s \,]_c [\, 2-s \,]_c [\, 4-s \,]_c}{[\, 4-2s \,]_c [\, 4-2s \,]_c}$$
$$\lim_{n/N \to s} C'_n := \frac{c^{5-\tau-2s} [\, s \,]_c [\, 2-s \,]_c [\, 1+\tau-s \,]_c}{[\, 4-2s \,]_c [\, 4-2s \,]_c}.$$

Note the correspondence with the previous case if we substitute $\tau \to 2-\tau$ (in other words letting t run in the other direction starting from the rightmost edge.) With this substitution the recurrence coefficients take identical forms, revealing that the model is symmetric with respect to τ about the midpoint $\tau = 1$. Therefore only the case I needs to be considered and symmetry determines case II.

3.2.1 Determining the Arctic Curve

Using theorem 3.2.1, we can find for each $0 < c < 1$, and for $0 \le \tau \le 2$ the $a(s)$ and $b(s)$ which determine the support of the density $\mathrm{d}\omega_{[\alpha(s),\beta(s)]}(x)/\mathrm{d}s$ for every s. We can identify the support of the asymptotic zero distribution for $p_N(x)$ with the choice $s = 1$. Therefore, by writing the c and τ dependence explicitly we can define functions $(\alpha_c(s,\tau), \beta_c(s,\tau))$ such that for each appropriate c, τ the zeros of $p_N(x)$ must lie inside a (closed, simply-connected) region which is bounded by $(\alpha(1,\tau), \beta(1,\tau))$. To signal the dependence here on both c we write $a_c(s,\tau)$ and $b_c(s,\tau)$ which in terms of A'_n and B'_n are

$$a_c(s,\tau) = \lim_{n/N\to s} \sqrt{A'_n C'_n}$$

and

$$b_c(s,\tau) = \lim_{n/N\to s} A'_n + C'_n.$$

Using continuity and setting $u(s) = 2 - s$ to symmetrize:

$$\lim_{n/N\to s} A'_n = \frac{[\,\tau-1+u\,]_c\,[\,u\,]_c\,[\,2+u\,]_c}{[\,2u\,]_c^2} = \frac{[\,2+u\,]_c\,[\,u+\tau-1\,]_c}{(1+c^u)^2\,[\,u\,]_c}$$

where in the last step we used $[\,2u\,]_c^2 = ([\,u\,]_c + c^u\,[\,u\,]_c)^2 = [\,u\,]_c^2\,(1+c^u)^2$, and

$$\lim_{n/N\to s} C'_n = \frac{c^{-1+\tau+2u}\,[\,2-u\,]_c\,[\,u\,]_c\,[\,1-\tau+u\,]_c}{[\,2u\,]_c^2} = \frac{c^{-1+\tau+2u}\,[\,2-u\,]_c\,[\,1-\tau+u\,]_c}{(1+c^u)^2\,[\,u\,]_c}.$$

These limits are finite and non-zero for $0 \le \tau \le 2$ and $0 < c < 1$ therefore we can write

$$a_c(u,\tau) = \frac{c^{u-1/2+\tau/2}\sqrt{[\,2-u\,]_c\,[\,2+u\,]_c\,[\,u+\tau-1\,]_c\,[\,u+1-\tau\,]_c}}{(1+c^u)^2\,[\,u\,]_c}$$

and

$$b_c(u,\tau) = \frac{[\,2+u\,]_c\,[\,u+\tau-1\,]_c + c^{-1+\tau+2u}\,[\,2-u\,]_c\,[\,u+1-\tau\,]_c}{(1+c^u)^2\,[\,u\,]_c}.$$

For every $0 \le \tau \le 2$ we can apply theorem 3.2.1 to define an asymptotic zero distribution ω_τ of the zeros of $p_N(x)$. From the theorem statement we see that the distribution changes only inside the interval $(\alpha(1), \beta(1))$ and outside of which the probability is constant. Setting $s = 1$, then $u = 1$ and

$$a_c(\tau) = a_c(1,\tau) = \frac{c^{\frac{1+\tau}{2}}\sqrt{[\,3\,]_c}}{(1+c)^2}\sqrt{[\,\tau\,]_c\,[\,2-\tau\,]_c},$$
$$b_c(\tau) = b_c(1,\tau) = \frac{1}{(1+c)^2}\left([\,2-\tau\,]_c\,c^{\tau+1} + [\,3\,]_c\,[\,\tau\,]_c\right).$$

The Arctic curve is defined piecewise by curves $\alpha_c(\tau) = b_c(\tau) - 2a_c(\tau)$ and $\beta_c(\tau) = b_c(\tau) + 2a_c(\tau)$ with respect to $0 \le \tau \le 2$. For $0 < \tau < 2$ the term $a(1) > 0$ and $a(0) = a(2) = 0$. Therefore $\beta(0) = \alpha(0)$, $\beta(2) = \alpha(2)$ and otherwise $\alpha(\tau) < \beta(\tau)$. Therefore these two curves together define one closed, simply-connected region. See figure 3.1 for the Arctic circle drawn for a few $c \in (0,1)$.

Remark. The Hahn recurrence coefficients are reclaimed in the limit $c \longrightarrow 1$ therefore we can reclaim the Hahn Arctic circle from the q-Hahn case by taking the limit $c \longrightarrow 1$ in $b(u)$ and $a(u)$ and then evaluating for $u = 1$. We have,

$$\lim_{c \to 1} b(u)\Big|_{u=1} = \frac{\tau + 1}{2}$$

and

$$\lim_{c \to 1} a(u)\Big|_{u=1} = \frac{\sqrt{3}}{4}\sqrt{(2-\tau)\tau}.$$

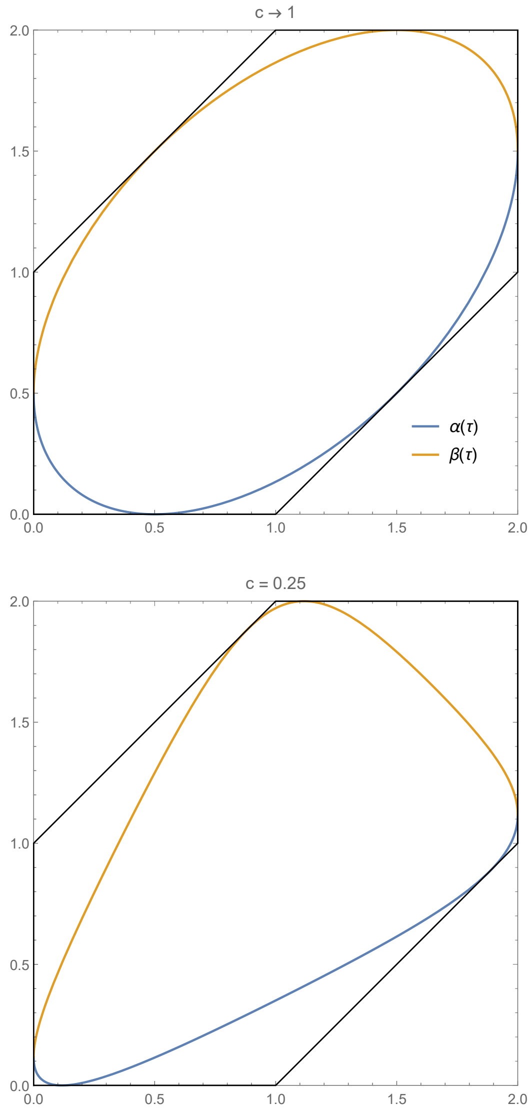


Figure 3.1: Arctic curve for different values of c, as $c \longrightarrow 1$ we reclaim the Arctic curve for the Hahn case.

3.2.2 Points of Intersections Between the Arctic Circle and Hexagon

The Arctic curve can be viewed as the points (τ, x) which satisfy either $[\,x\,]_c = \alpha_c(\tau)$ or $[\,x\,]_c = \beta_c(\tau)$, alternatively we may view the curve as the set of points that satisfy the implicit equation,

$$([\,x\,]_c - \alpha_c(\tau))([\,x\,]_c - \beta_c(\tau)) = 0$$

which can also be written as,

$$([\,x\,]_c - b_c(\tau))^2 = 4a_c(\tau)^2.$$

After expanding and making some simplifications the implicit equation can be written,

$$\left(c^{x-\tau/2} - \frac{1+c^2}{1+c}\left(\frac{c^{\tau/2}+c^{1-\tau/2}}{1+c}\right)\right)^2 = \frac{4c\left(1+c+c^2\right)}{(1+c)^2}\left(1-\left(\frac{c^{\tau/2}+c^{1-\tau/2}}{1+c}\right)^2\right).$$

The implicit equation can therefore be written

$$\left(c^{x-\tau/2} - \eta\Theta(\tau)\right)^2 + (\lambda^2-1)\Theta(\tau)^2 = 0$$

with

$$\begin{cases} \Theta(\tau) & := \frac{c^{\tau/2}+c^{1-\tau/2}}{1+c} \\ \eta & := \frac{1+c^2}{1+c} \\ \lambda^2 & := \frac{4c\left(1+c+c^2\right)}{(1+c)^2}. \end{cases}$$

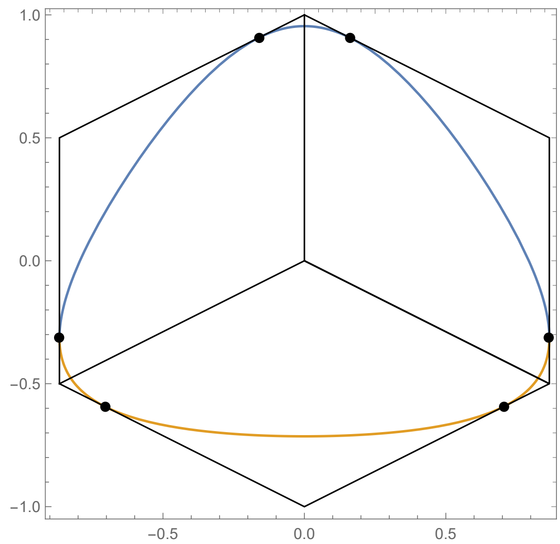


Figure 3.2: The Arctic curve for $c = 1/E$, symmetrized and highlighting points where the curve intersects the corners of the hexagon.

Inspecting the above equation suggests a more symmetric equation, consider the coordinate transformation defined by

$$(\tau', x') = \left(\frac{\sqrt{3}(\tau-1)}{2}, x - \frac{\tau+1}{2}\right)$$

which inscribes the curve in a regular hexagon, of side length 1, centered at $(0,0)$. Note that this simply undoes the affine transformation on the regular hexagon that we considered at the start. With this transformation the implicit equation can be written

$$\left(c^{x'} - \eta\Theta_{\text{sym}}(\tau')\right)^2 + \lambda^2\Theta_{\text{sym}}(\tau')^2 = \lambda^2/c$$

where

$$\Theta_{\rm sym}(\tau) = \frac{c^{\tau'/\sqrt{3}} + c^{-\tau'/\sqrt{3}}}{1+c}.$$

Alternatively, as this is a linear transformation it can be achieved as a coordinate transformation in $\mathbb{R}^2$, the new coordinates for the symmetric hexagon (t', x') relate to the former (t, x) by

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} t-1 \\ x-1 \end{pmatrix}.$$

and conversely

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We can find the six intersection of the curve with the right side of the hexagon by considering the solutions of $F_c(t, x) = 0$ where

$$F_c(t,x) = \left(c^x - \eta\Theta_{\rm sym}(\tau)\right)^2 + \lambda^2\Theta_{\rm sym}(\tau)^2 - \lambda^2/c.$$

We can find one such point by considering the intersection of curve $F(t_0, x_0) = 0$ and the right edge $t_0 = \frac{\sqrt{3}}{2}$. Because $\Theta\left(\frac{\sqrt{3}}{2}\right) = 1/\sqrt{c}$ the solutions reduce to x_0 such that $c^{x_0} = \eta/\sqrt{c}$ and so:

$$x_0 = -\frac{1}{2} + \log_c\left(\frac{1+c^2}{1+c}\right)$$

We can now obtain the other 5 showing that $(\frac{\sqrt{3}}{2}, x_0)$ remains a solution under rotation and reflection by $2\pi/3$ and $4\pi/3$. First note that the curve is symmetric under reflections $\tau \mapsto -\tau$ then, as τ only occurs in $F(\tau, x)$ in terms of $\Theta_{\rm sym}(\tau)$ where

$$\Theta_{\rm sym}(-\tau) = \frac{c^{-\tau/\sqrt{3}} + c^{\tau/\sqrt{3}}}{1+c} = \Theta_{\rm sym}(\tau)$$

Setting $\eta_0 = \log_c(\eta) = \log_c\left(\frac{1+c^2}{1+c}\right)$ we consider a rotation by $2\pi/3$, sending

$$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2} + \eta_0\right) \to \left(-\frac{\sqrt{3}\eta_0}{2}, 1 - \frac{\eta_0}{2}\right)$$

and after some calculation $F_c\left(-\frac{\sqrt{3}\eta_0}{2}, 1 - \frac{\eta_0}{2}\right) = 0$ if

$$\left(c - \eta\frac{1+\eta}{1+c}\right)^2 + \lambda^2\left(\frac{1+\eta}{1+c}\right)^2 = \lambda^2\frac{\eta}{c}.$$

Substituting in definitions of $\eta(c)$ and $\lambda(c)$ and expanding the terms cancel to zero. In a similar way we can rotate by $-2\pi/3$,

$$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2} + \eta_0\right) \to \left(\frac{\sqrt{3}}{2}(\eta_0 - 1), -\frac{1}{2}(\eta_0 + 1)\right)$$

then

$$\left(c^{-\frac{\eta_0}{2}} - \eta\frac{c^{\frac{\eta_0}{2}-\frac{1}{2}} + c^{-\frac{\eta_0}{2}+\frac{1}{2}}}{c^{1/2} + c^{-1/2}}\right)^2 + \lambda^2\left(\frac{c^{\frac{\eta_0}{2}-\frac{1}{2}} + c^{-\frac{\eta_0}{2}+\frac{1}{2}}}{c^{1/2} + c^{-1/2}}\right)^2 - \lambda^2.$$

Symmetric hexagon		Affine-transformed hexagon	
τ	x	τ	x
$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\eta_0$	2	$1+\eta_0$
$\frac{\sqrt{3}}{2}\eta_0$	$1-\frac{1}{2}\eta_0$	$1+\eta_0$	2
$-\frac{\sqrt{3}}{2}\eta_0$	$1-\frac{1}{2}\eta_0$	$1-\eta_0$	$2-\eta_0$
$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\eta_0$	0	η_0
$\frac{-\sqrt{3}}{2}(\eta_0-1)$	$-\frac{1}{2}(\eta_0+1)$	$2-\eta_0$	$1-\eta_0$
$\frac{\sqrt{3}}{2}(\eta_0-1)$	$-\frac{1}{2}(\eta_0+1)$	η_0	0

Figure 3.3: The points of intersection between the Arctic curve and bounding hexagon, for both the symmetric and affine-transformed hexagon, listed counter-clockwise from the intersection at the right edge, with $\eta_0 = \log_c\left(\frac{1+c^2}{1+c}\right)$.

Simplifying,

$$\left(1-\eta\frac{\eta+c}{1+c}\right)^2+\lambda^2\left(\frac{\eta+c}{1+c}\right)^2-\lambda^2\eta$$

Which expanding out can be revealed to be equal to zero, therefore rotation by $2\pi/3$ and reflection in the vertical axis produces in total six intersection of the Arctic curve with the corners of the hexagon. From the solution for the symmetric case we can undo the affine transformation and find the corresponding intersection points, see the table in figure 3.3.

3.3 Calculating the Asymptotic Zero Distribution

From theorem 3.2.1 the density of the asymptotic zero distribution $\omega_{[\alpha(s),\beta(s)]}$ is

$$\frac{\mathrm{d}\omega_{[\alpha(s),\beta(s)]}(s)}{\mathrm{d}s} = \begin{cases} \frac{1}{\pi\sqrt{(\beta(s)-s)(s-\alpha(s))}} & \text{if } s\in(\alpha(s),\beta(s)) \\ 0 \text{ elsewhere} \end{cases}$$

and the asymptotic zero distribution of the q-Hahn polynomial $p_N(X)$ as as we take $N\to\infty$ in the particular way described above is

$$\mu(x)=\int_0^1 \omega_{[\alpha(s),\beta(s)]}\mathrm{d}s.$$

Here the quantities $\alpha(u,\tau)=\alpha(2-s,\tau)$

$$a_c(u,\tau)=\frac{c^{u-1/2+\tau/2}\sqrt{[\,2-u\,]_c\,[\,2+u\,]_c\,[\,u+\tau-1\,]_c\,[\,u+1-\tau\,]_c}}{(1+c^u)^2\,[\,u\,]_c}$$

and

$$b_c(u,\tau)=\frac{[\,2+u\,]_c\,[\,u+\tau-1\,]_c+c^{-1+\tau+2u}\,[\,2-u\,]_c\,[\,u+1-\tau\,]_c}{(1+c^u)^2\,[\,u\,]_c}.$$

Remark. To simplify calculations define

$$C(t):=\frac{1}{2}(c^t+c^{-t}) \text{ and } S(t):=\frac{1}{2}(c^t-c^{-t}).$$

These functions can be identified with the hyperbolic functions by $c^t = e^{\gamma t}$. They share many of the same properties, e.g. differentiating $C'(t) = \gamma S(t)$ and $S'(t) = \gamma C(t)$ where $\gamma = \log(c)$. Furthermore we can relate $C(t)$ and $S(t)$ through $S(t)^2 = C(t)^2 - 1$ and we also have the double-angle identities $C(2t) = 2C(t)^2 - 1$, $S(2t) = 2C(t)S(t)$.

The q-numbers are related to $C(t), S(t)$ above by identifying $[\, u\,]_c = \frac{-2c^{u/2}}{1-c} S(u/2)$ and $[\, u+v\,]_c\, [\, u-v\,]_c = \frac{2c^u}{(1-c)^2}(C(u) - C(v))$. Therefore we will work with

$$a_c(u,\tau) = \frac{-c^{(1+\tau)/2}\sqrt{(C(2)-C(u))(C(u)-C(\tau-1))}}{2(1-c)(1+C(u))S(u/2)}$$

and (after some rearrangement)

$$b_c(u,\tau) = \frac{1}{1-c} - \frac{2c^{(1+\tau)/2}C(1)C\left(\frac{\tau-1}{2}\right)}{(1-c)\,(1+C(u))}.$$

3.3.1 Calculating The q-Hahn Asymptotic Zero Distribution

Having considered the support of the measure $\omega_{[\alpha,\beta]}$ we can now calculate the asymptotic zero distribution of the q-Hahn polynomials $p_n(x)$ where $n/N \to 1$. From theorem 3.2.1 the asymptotic zero distribution $\mu_w = \mu_{c,\tau}$ is, for any $0 \le \tau \le 2$ and $1 > c > 0$ or $c > 1$,

$$\mu_{w,c,\tau}(x) = \int_0^1 \omega'_{[\alpha_c(s,\tau),\beta_c(s,\tau)]}(x)ds = \frac{1}{\pi}\int_{s_-}^{\min(1,s_+)} \frac{1}{\sqrt{(\beta_c(s,\tau)-x)(x-\alpha_c(s,\tau))}}ds$$

while the density of the measure $\mu_{w,c,\tau}$ is

$$\mu_{w,c,\tau}(x)dx = \left(\frac{1}{\pi}\int_{s_-}^{\min(1,s_+)} \frac{1}{\sqrt{(\beta_c(s,\tau)-x)(x-\alpha_c(s,\tau))}}ds\right)dx.$$

The integration limits $s_\pm(x)$ are defined so that $s_+ > s_-$ and $\alpha(s_\pm(x),\tau) = x$ or $\beta(s_\pm(x),\tau) = x$. Before this integral is amenable to calculation it will need some simplification.

Remark. Substituting x by $\frac{1-c^x}{1-c}$ so that dx becomes $d\left(\frac{1-c^x}{1-c}\right) = \frac{-\gamma c^x}{1-c}dx$ where $\gamma = \log(c)$ then the density is

$$\mu_{w,c,\tau}([\, x\,]_c)\frac{-\gamma c^x}{1-c}dx = \left(\frac{-\gamma c^x}{\pi(1-c)}\int_{s_-}^{\min(1,s_+)} \frac{1}{\sqrt{(\beta(s)-[\, x\,]_c)([\, x\,]_c-\alpha(s))}}ds\right)dx. \tag{3.2}$$

Also note that $s_\pm(x)$ is now defined so that $s_+ > s_-$ and $\alpha_c(s_\pm(x),\tau) = [\, x\,]_c$ or $\beta_c(s_\pm(x),\tau) = [\, x\,]_c$.

Remark. The equation

$$(\beta(s) - [\, x\,]_c)([\, x\,]_c - \alpha(s)) = 0$$

can be identified with an implicit equation for the Arctic curve. We can achieve some simplification by relating this to symmetrized Arctic curve in c^x. For some $\beta'(s), \alpha'(s)$ this will be the solutions to an equation,

$$(\beta'_{c,\tau'}(u) - c^x)(c^x - \alpha'_{c,\tau'}(u)) = 0$$

We achieve this transformation by defining

$$\beta'(s)c^{(\tau+1)/2} = 1 - (1-c)\beta_{c,\tau}(s)$$
$$\alpha'(s)c^{(\tau+1)/2} = \alpha_{c,\tau}(s)(1-c) - 1$$

which after some simplification yields (suppressing indices and writing $u = 2 - s$:)

$$\beta'(2-s) = \beta'(u) := \frac{1}{1+C(u)}\left(2C(1)C\left(\frac{\tau-1}{2}\right) + \frac{\sqrt{(C(2)-C(u))(C(u)-C(\tau-1))}}{S(u/2)}\right)$$
$$\alpha'(2-s) = \alpha'(u) := \frac{1}{1+C(u)}\left(2C(1)C\left(\frac{\tau-1}{2}\right) - \frac{\sqrt{(C(2)-C(u))(C(u)-C(\tau-1))}}{S(u/2)}\right).$$

Furthermore setting $\tau' = \tau - 1$ and $x' = x - (\tau+1)/2 = x - 1 - \tau'/2$ we can write

$$\sqrt{(\beta(s) - [\,x\,]_c)([\,x\,]_c - \alpha(s))} = \frac{c^{1-\tau'/2}}{|1-c|}\sqrt{(\beta'(2-s) - c^{x'})(c^{x'} - \alpha'(2-s))}$$

and so the integral in (3.2) can be written

$$\frac{|\gamma|c^{x'}}{\pi}\int_{s_-}^{\min(1,s_+)} \frac{1}{\sqrt{(\beta'(2-s) - c^{x'})(c^{x'} - \alpha'(2-s))}}ds \tag{3.3}$$

where we calculated

$$\frac{-\gamma c^x}{\pi(1-c)}\frac{|1-c|}{c^{1-\tau'/2}} = \frac{|\gamma|c^{x'}}{\pi}.$$

Setting $u = 2 - s$ we focus on the expression under the square root,

$$(\beta'(u) - c^{x'})(c^{x'} - \alpha'(u))$$

and collect terms in $u = 2 - s$. We start by multiplying out

$$-\beta'(u)\alpha'(u) + c^{x'}\left(\beta'(u) + \alpha'(u)\right) - c^{2x'}$$

which after some calculation of terms yields,

$$(\beta'(u) + \alpha'(u))S(u)^2 = 4(C(u) - 1)C(1)C(\tau'/2)$$

and

$$-\beta'(u)\alpha'(u)S(u)^2 = -4S(1)^2 S\left(\tau'/2\right)^2 + 2(1 - C\left(u\right)).$$

Factoring out $S(u)^2$ the integrand

$$\frac{|S(u)|ds}{\sqrt{-4S(1)^2 S\left(\frac{\tau'}{2}\right)^2 + 2(1 - C\left(u\right))(1 - 2C(1)C\left(\frac{\tau'}{2}\right)c^{x'}) - S(u)^2 c^{2x'}}}.$$

Collecting terms by powers of $C(u)$, we define

$$\begin{aligned}
\mu_2 &= -c^{2x'} \\
\mu_1 &= -2\left(1 - 2C(1)C\left(\frac{\tau'}{2}\right)c^{x'}\right) \\
\mu_0 &= c^{2x'} + 2\left(1 - 2C(1)C\left(\frac{\tau'}{2}\right)c^{x'}\right) - 4S(1)^2 S\left(\frac{\tau'}{2}\right)^2 \\
&= \left(c^{x'} - 2C(1)C\left(\frac{\tau'}{2}\right)\right)^2 - 2C(2)C\left(\tau'\right).
\end{aligned}$$

so that the integrand becomes,

$$\left(\frac{c^{x'}}{\pi}\int_{s_-}^{\min(1,s_+)}\frac{|\gamma S(2-s)|ds}{\sqrt{\mu_0+\mu_1 C(2-s)+\mu_2 C(2-s)^2}}\right)dx.$$

This is an integral amenable to substitution $v = C(2-s)$, $dv = -\gamma S(2-s)ds$. Note that for $0 \leq s \leq 1$ that negative signs cancel so that $\gamma(c)S(2-s) \geq 0$ for both $0 < c < 1$ and $c > 1$ so that $\gamma S(2-s) = |\gamma S(2-s)|$. The function $C(2-s)$ is a decreasing function of s therefore the lower integration limit $v_+ = C(2-s_-)$ and the upper integration limit becomes the greater of $v_- = C(2-s_+)$ and $C(1)$. Swapping integration bounds we get a minus sign which cancels the other one from the substitution

$$\frac{c^{x'}}{\pi}\int_{\max(C(1),v_-)}^{v_+}\frac{dv}{\sqrt{\mu_0+\mu_1 v+\mu_2 v^2}}.$$

Referring to a table of integrals such as [7, pg. 94] we can evaluate this,

$$\frac{c^{x'}}{\pi}\int_{\max(C(1),v_-)}^{v_+}\frac{dv}{\sqrt{\mu_0+\mu_1 v+\mu_2 v^2}} = \frac{-1}{\pi}\arcsin\left(\frac{2\mu_2 v+\mu_1}{\sqrt{\mu_1^2-4\mu_0\mu_2}}\right)\Bigg|_{\max(C(1),v_-)}^{v_+}.$$

Furthermore the integration limits $v_+ > v_-$ correspond to the zeros of the polynomial $\mu_0 + \mu_1 v + \mu_2 v^2$ and so the quadratic formula yields,

$$v_\pm = \frac{-\mu_1 \mp \sqrt{\mu_1^2 - 4\mu_0\mu_2}}{2\mu_2}$$

(Recall that $v_\pm$ depends on c, x, τ.) Reinserting,

$$\arcsin\left(\frac{2\mu_2 v_\pm+\mu_1}{\sqrt{\mu_1^2-4\mu_0\mu_2}}\right) = \arcsin(\mp 1) = \mp\frac{\pi}{2}.$$

We can now express the asymptotic measure $\mu_w(x)dx$, depending on whether $v_- = C(u_-) > C(1)$ or $v_- = C(u_-) \leq C(1)$.

Remark. Note that the points inside the Arctic circle are points (τ', x') such that for $u = 1$, $1 > c > 0$,

$$\alpha_c'(1,\tau') > c^{x'} > \beta_c'(1,\tau')$$

and if $c > 0$ then

$$\beta_c'(1,\tau') > c^{x'} > \alpha_c'(1,\tau').$$

Generally $\alpha_c'(1,\tau') > \beta_c'(1,\tau')$ for all $2 > u > 0$, because

$$\alpha_c'(u,\tau') - \beta_c'(u,\tau') = \frac{-2\sqrt{(C(2)-C(u))(C(u)-C(\tau-1))}}{(1+C(u))S(u/2)} > 0$$

as $S(u/2) < 0$ for $1 > c > 0$. On the other hand if $c > 1$ then

$$\beta_c'(u,\tau') - \alpha_c'(u,\tau') = \frac{2\sqrt{(C(2)-C(u))(C(u)-C(\tau'))}}{(1+C(u))S(u/2)} > 0.$$

Furthermore $\alpha_c'(u,\tau') = \beta_c'(u,\tau')$ if $u = 2$ or $u = \pm\tau'$, where $-1 \leq \tau' \leq 1$. In other words the two curves form a closed region (as a curve in u) where $u \in [|\tau'|, 2]$. This means that for any point (τ', x') inside the Arctic circle, $c^{x'}$ intersects $\beta_c'(u,\tau')$ or $\alpha_c'(u,\tau')$ for some $u < 1$ or in other words $v_- \leq C(1)$.

Therefore for points inside the Arctic circle we can write

$$\mu_w(x)dx = \frac{1}{\pi}\left(\arcsin\left(\frac{2\mu_2 C(1) + \mu_1}{\sqrt{\mu_1^2 - 4\mu_0\mu_2}}\right) + \frac{\pi}{2}\right)$$

where,

$$\begin{aligned}
\mu_2 &= -c^{2x-\tau-1} \\
\mu_1 &= -2\left(1 - 2C(1)C\left((\tau-1)/2\right)c^{x-(\tau-1)/2}\right) \\
\mu_0 &= \left(c^{x-(\tau-1)/2} - 2C(1)C\left(\frac{\tau-1}{2}\right)\right)^2 - 2C(2)C\left(\tau-1\right).
\end{aligned}$$

Furthermore as the Arctic curve intersects the corners of the hexagon at six places, it forms six regions. For points outside the Arctic curve the probability density should be constant and are determined by boundary properties. On of the two vertical corners, left and right side, we require the probability density to be equal to one as every path will begin and terminate there. On the other hand on the top and bottom corners there will be no paths (with probability 1.) See figure 3.4 which displays this behaviour.

We can also interpret the figures of 3.4 in terms of tilings. Note that paths run across type *I* and *II* lozenges while gaps between paths correspond to type *III* lozenges. This shows therefore that our pictures of the asymptotic zero distribution corresponds well to the distribution of lozenges in simulated samples, see e.g. figure 1.

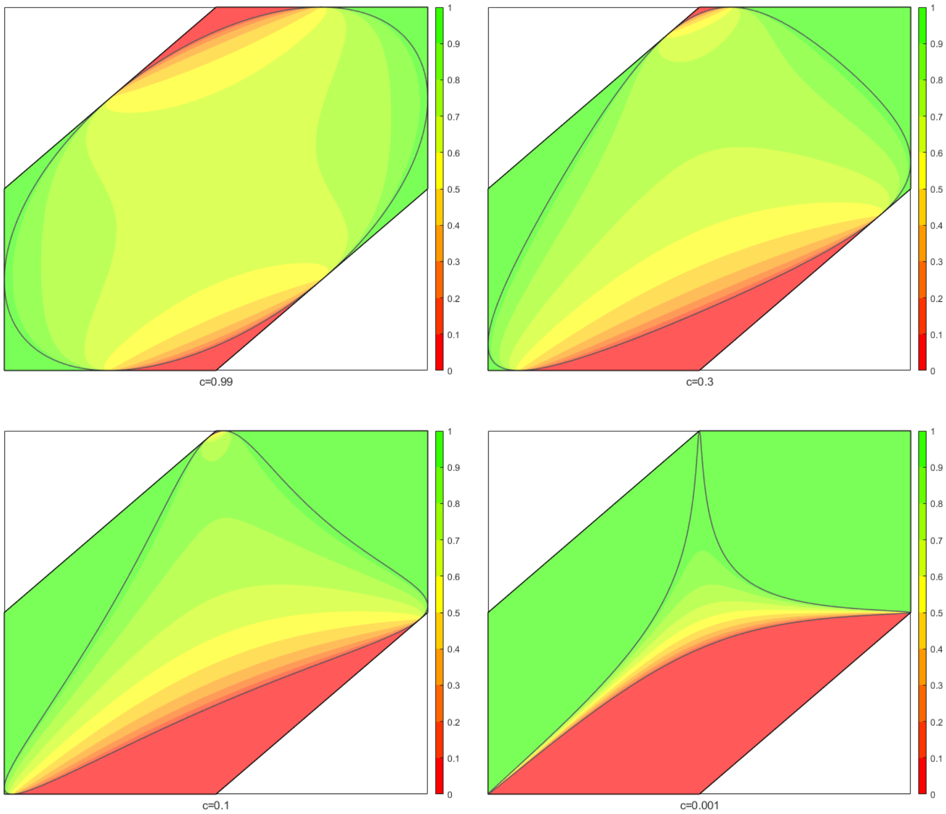


Figure 3.4: The asymptotic zero distribution for $1 > c > 0$, with the Arctic curve displayed, generated in Matlab.

Appendix A

Properties of q-Extensions

A.1 A Few Identities for q-Extensions

We list some properties of the q-extended numbers and related quantities which can aid in calculations containing them. The following identities hold for q-numbers

$$[\,-\alpha\,]_q = -q^{-1}\,[\,\alpha\,]_{q^{-1}} \tag{A.1}$$

$$[\,\alpha\,]_q = q^{\alpha-1}\,[\,\alpha\,]_{q^{-1}} \tag{A.2}$$

$$[\,-\alpha\,]_q = -q^{-\alpha}\,[\,\alpha\,]_q\,. \tag{A.3}$$

We also consider the behaviour of q-numbers under addition, subtraction, multiplication.

$$[\,\alpha+\beta\,]_q = [\,\alpha\,]_q + q^{\alpha}\,[\,\beta\,]_q \tag{A.4}$$

$$[\,\alpha-\beta\,]_q = [\,\alpha\,]_q - q^{\alpha-\beta}\,[\,\beta\,]_q \tag{A.5}$$

$$[\,x+\alpha\,]_q - [\,y+\alpha\,]_q = q^{\alpha}([\,x\,]_q - [\,y\,]_q) \tag{A.6}$$

The q-factorial can have its definition extended to negative integers, let $k > 0$ then

$$[\,-k\,]_q! := [\,-k\,]_q\,[\,-k+1\,]_q \ldots [\,-1\,]_q = \frac{(1-q^{-k})\ldots(1-q^{-1})}{(1-q)^k}.$$

Factoring out $q^{-1}, q^{-2}, \ldots q^{-k}$ from each term in the numerator and simplifying

$$[\,k\,]_q! = (-1)^k q^{k(k+1)/2}\,[\,-k\,]_q!. \tag{A.7}$$

and using (A.2)

$$[\,k\,]_q! = q^{k(k-1)/2}\,[\,k\,]_{q^{-1}}! \tag{A.8}$$

and with (A.1)

$$[\,k\,]_q! = (-1)^k q^k\,[\,-k\,]_{q^{-1}}!. \tag{A.9}$$

The following hold for q-binomials, for real α and $k = 1, 2, \ldots$

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_{q^{-1}} q^{k\alpha-k^2} \tag{A.10}$$

$$\begin{bmatrix} \alpha+k \\ k \end{bmatrix}_q = \frac{[\,\alpha+k\,]_q!}{[\,\alpha\,]_q\,[\,k\,]_q} = \frac{(q^{\alpha+1};q)_k}{(q;q)_k}. \tag{A.11}$$

We establish the proposition which is used in theorem 2.2.1, following the proof in [19]:

Proposition A.1.1. *There are two q-Pascal rules, for $n \in \mathbb{Z}$ and $k = 1, 2, \ldots, n-1$.*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q,$$

where $1 \leq k \leq n-1$.

Proof. By (A.4) we can write

$$[\, n \,]_q = [\, k \,]_q + q^k [\, n-k \,]_q$$

if $1 \leq k \leq n-1$ and therefore,

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]!}{[k]![n-k]!} = \frac{[n-1]!([k] + q^k[n-k])}{[k]![n-k]!} \\ &= \frac{[n-1]!}{[k-1]![n-k]!} + q^k \frac{[n-1]!}{[k]![n-k-1]!} \\ &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q . \end{aligned}$$

For the second identity, by symmetry

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q &= \begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_q \\ &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q . \end{aligned}$$

□

Lemma A.1.2. *Given three sets of indexed placeholders $X_1, \ldots, X_N, A_2, \ldots, A_N$ and $B_2, \ldots, B_N$ the following holds:*

$$\begin{aligned} &\det_{1 \leq i,j \leq N} [\, X_i + A_N \,]_q \ldots \left[\, X_i + A_{j+1} \,\right]_q \left[\, X_i + B_j \,\right]_q \ldots [\, X_i + B_2 \,]_q \\ &\qquad = \left(\prod_{2 \leq i \leq j \leq N} [\, B_i \,]_q - \left[\, A_j \,\right]_q \right) \prod_{1 \leq i < j \leq N} \left([\, X_i \,]_q - \left[\, X_j \,\right]_q \right). \end{aligned}$$

Note that in the determinant we use the convention that empty products such as $\left[\, X_i + B_j \,\right]_q \ldots [\, X_i + B_2 \,]_q$ when $j \leq 1$ or the product $[\, X_i + A_N \,]_q \ldots \left[\, X_i + A_{j+1} \,\right]_q$ when $j = N$ is equal to one.

Proof. Following Krattenthaler [12, pg. 177] consider the columns $C_1, \ldots, C_N$ where $[C_l]_i = [M_1]_{il}$ for $l = 1, 2, \ldots, N$. Denoting the column operation that adds $\sum_{k \neq l} a_k C_k$ to the C_l'th column by $C_l + \sum_{k \neq l} a_k C_k$ we begin with column operations $C_N - C_{N-1}, C_{N-1} - C_{N-2}, \ldots C_2 - C_1$ (leaving C_1 unchanged.) For the first column operation,

$$\begin{aligned} C_N - C_{N-1} &= [\, X_i + B_N \,]_q \ldots [\, X_i + B_2 \,]_q - [\, X_i + A_N \,]_q [\, X_i + B_{N-1} \,]_q \ldots [\, X_i + B_2 \,]_q \\ &= ([\, X_i + B_N \,]_q - [\, X_i + A_N \,]_q) [\, X_i + B_{N-1} \,]_q \ldots [\, X_i + B_2 \,]_q \\ &= q^{X_i}([\, B_N \,]_q - [\, A_N \,]_q) [\, X_i + B_{N-1} \,]_q \ldots [\, X_i + B_2 \,]_q \end{aligned}$$

where we have used (A.6) in the last line. In the same way the other columns become,

$$\begin{aligned}
C_{N-1} - C_{N-2} &= [\, X_i + A_N \,]_q [\, X_i + B_{N-1} \,]_q \dots [\, X_i + B_2 \,]_q \\
&\quad - [\, X_i + A_N \,]_q [\, X_i + A_{N-1} \,]_q [\, X_i + B_{N-2} \,]_q \dots [\, X_i + B_2 \,]_q \\
&= q^{X_i}([\, B_{N-1} \,]_q - [\, A_{N-1} \,]_q) [\, X_i + A_N \,]_q [\, X_i + B_{N-2} \,]_q \dots [\, X_i + B_2 \,]_q \\
&\quad \vdots \\
C_2 - C_1 &= [\, X_i + A_N \,]_q \dots [\, X_i + A_3 \,]_q [\, X_i + B_2 \,]_q \\
&\quad - [\, X_i + A_N \,]_q \dots [\, X_i + A_3 \,]_q [\, X_i + A_2 \,]_q \\
&= q^{X_i}([\, B_2 \,]_q - [\, A_2 \,]_q) [\, X_i + A_N \,]_q [\, X_i + A_{N-1} \,]_q \dots [\, X_i + A_3 \,]_q
\end{aligned}$$

Therefore we can factor out the terms $[\, B_k \,]_q - [\, A_k \,]_q$ from the kth columns $k = 2, \dots, N$:

$$\begin{aligned}
&\det_{1 \le i,j \le N} [\, X_i + A_N \,]_q \dots \left[\, X_i + A_{j+1} \,\right]_q \left[\, X_i + B_j \,\right]_q \dots [\, X_i + B_2 \,]_q \\
&\qquad\qquad = \prod_{i=2}^{N}([\, B_i \,]_q - [\, A_i \,]_q) \det_{1 \le i,j \le N} [M_2]_{ij}.
\end{aligned}$$

where we defining the matrix M_k with entries

$$[M_k]_{ij} = \begin{cases} q^{(k-1)X_i} [\, X_i + A_N \,]_q \dots \left[\, X_i + A_{j+1} \,\right]_q \left[\, X_i + B_{j+1-k} \,\right]_q \dots [\, X_i + B_2 \,]_q & \text{if } j > k \\ q^{(j-1)X_j} [\, X_i + A_N \,]_q \dots \left[\, X_i + A_{j+1} \,\right]_q & \text{if } j \le k. \end{cases}$$

with the convention that empty products $\left[\, X_i + B_{j+1-k} \,\right]_q \dots [\, X_i + B_2 \,]_q$ are equal one. Note that setting $k = 1$ we reclaim the original determinant through $\det_{1 \le i,j \le N} M_1$. The procedure that produced M_2 from M_1 can be repeated to produce M_3 through column operations $C_N - C_{N-1}, \dots, C_3 - C_2$ (note now that C_1, C_2 are unchanged.) For $l = N, \dots, 3$ we have at each l

$$\begin{aligned}
C_l - C_{l-1} &= q^{X_i} [\, X_i + A_N \,]_q \dots [\, X_i + A_{l+1} \,]_q [\, X_i + B_{l-1} \,]_q [\, X_i + B_{l-2} \,]_q \dots [\, X_i + B_2 \,]_q \\
&\quad - q^{X_i} [\, X_i + A_N \,]_q \dots [\, X_i + A_{l+1} \,]_q [\, X_i + A_l \,]_q [\, X_i + B_{l-2} \,]_q \dots [\, X_i + B_2 \,]_q \\
&= q^{2X_i}([\, B_{l-1} \,]_q - [\, A_l \,]_q) [\, X_i + A_N \,]_q \dots [\, X_i + A_{l+1} \,]_q [\, X_i + B_{l-2} \,]_q \dots [\, X_i + B_2 \,]_q .
\end{aligned}$$

Therefore we can factor out $(B_{N-1} - A_N) \dots (B_2 - A_3)$. We now have a determinant with terms

$$[M_3]_{ij} = \begin{cases} q^{2X_i} [\, X_i + A_N \,]_q \dots \left[\, X_i + A_{j+1} \,\right]_q \left[\, X_i + B_{j-2} \,\right]_q \dots [\, X_i + B_2 \,]_q & \text{if } j > 3 \\ q^{(j-1)X_j} [\, X_i + A_N \,]_q \dots \left[\, X_i + A_{j+1} \,\right]_q & \text{if } j \le 3. \end{cases}$$

This process can be continued until we have $\det_{1 \le i,j \le N} M_N$ and the B_j terms are completely reduced out, the determinant becomes

$$\left(\prod_{2 \le i \le j \le N} [\, B_i \,]_q - \left[\, A_j \,\right]_q \right) \det_{1 \le i,j \le N} M_N. \tag{A.12}$$

where

$$\det_{1 \le i,j \le N} M_N = \det_{1 \le i,j \le N} q^{(j-1)X_i} [\, X_i + A_N \,]_q \dots \left[\, X_i + A_{j+1} \,\right]_q .$$

Using (A.4), then (A.2) we have

$$[\, X + A \,]_q = [\, X \,]_q + q^X [\, A \,]_q = -q^{X-1}([\, X \,]_{q^{-1}} - q [\, A \,]_q)$$

therefore

$$\begin{aligned}
&\det_{1\le i,j\le N} M_N \\
&= \det_{1\le i,j\le N} q^{(j-1)X_i}(-q^{X_i-1})^{N-j}([\,X_i\,]_{q^{-1}} - q\,[\,A_N\,]_q)\dots([\,X_i\,]_{q^{-1}} - q\left[\,A_{j+1}\,\right]_q) \\
&= (-1)^{N(N-1)/2} q^{-N(N-1)/2} \prod_{i=1}^{N} q^{(N-1)X_i} \\
&\qquad \det_{1\le i,j\le N}([\,X_i\,]_{q^{-1}} - q\,[\,A_N\,]_q)\dots([\,X_i\,]_{q^{-1}} - q\left[\,A_{j+1}\,\right]_q).
\end{aligned}$$

The (i,j) entry can be viewed as a monic polynomial $P_j([\,X_i\,]_{q^{-1}})$ in $[\,X_i\,]_{q^{-1}}$ of degree $N-j$. Note that a column operation makes the entries

$$P_{j-1}([\,X_i\,]_{q^{-1}}) + q\left[\,A_j\,\right]_q P_j([\,X_i\,]_{q^{-1}}) = P_j([\,X_i\,]_{q^{-1}})\,[\,X_i\,]_{q^{-1}}$$

therefore applying this as as a sequence of column operations on $C_{N-1}, C_{N-2}, \dots, C_1$ and noting that $C_N = 1$ we have,

$$\begin{pmatrix} P_1([\,X_N\,]_{q^{-1}}) & \dots & P_{N-1}([\,X_1\,]_{q^{-1}}) & 1 \\ \vdots & & \vdots & 1 \\ P_1([\,X_N\,]_{q^{-1}}) & \dots & P_{N-1}([\,X_N\,]_{q^{-1}}) & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} P_1([\,X_1\,]_{q^{-1}}) & \dots & [\,X_1\,]_{q^{-1}} & 1 \\ \vdots & & \vdots & \vdots \\ P_1([\,X_N\,]_{q^{-1}}) & \dots & [\,X_N\,]_{q^{-1}} & 1 \end{pmatrix}$$

$$\rightsquigarrow \dots \rightsquigarrow \begin{pmatrix} P_2([\,X_1\,]_{q^{-1}})\,[\,X_1\,]_{q^{-1}} & \dots & P_{N-1}([\,X_1\,]_{q^{-1}})\,[\,X_1\,]_{q^{-1}} & [\,X_1\,]_{q^{-1}} & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ P_2([\,X_N\,]_{q^{-1}})\,[\,X_N\,]_{q^{-1}} & \dots & P_{N-1}([\,X_N\,]_{q^{-1}})\,[\,X_N\,]_{q^{-1}} & [\,X_N\,]_{q^{-1}} & 1 \end{pmatrix}.$$

Repeating the process for the $P_{N-1}, \dots, P_2$ and so on we have,

$$\det \begin{pmatrix} [\,X_1\,]_{q^{-1}}^{N-1} & \dots & [\,X_1\,]_{q^{-1}} & 1 \\ \vdots & & \vdots & \vdots \\ [\,X_N\,]_{q^{-1}}^{N-1} & \dots & [\,X_N\,]_{q^{-1}} & 1 \end{pmatrix} = \det_{1\le i,j\le N} [\,X_i\,]_q^{N-j}.$$

With $\binom{N}{2}$ column interchanges we get a factor of $(-1)^{N(N-1)/2}$ so that we have

$$q^{-N(N-1)/2} \prod_{i=1}^{N} q^{(N-1)X_i} \left(\prod_{2\le i\le j\le N} [\,B_i\,]_q - \left[\,A_j\,\right]_q \right) \det_{1\le i,j\le N} [\,X_i\,]_q^{j-1}.$$

The determinant in this case is the Vandermonde which can also be written in the following form,

$$\prod_{1\le i<j\le N} \left(\left[\,X_j\,\right]_{q^{-1}} - [\,X_i\,]_{q^{-1}} \right).$$

Therefore (A.12) can be written

$$q^{-N(N-1)/2} \prod_{i=1}^{N} q^{(N-1)X_i} \left(\prod_{2\le i\le j\le N} [\,B_i\,]_q - \left[\,A_j\,\right]_q \right) \prod_{1\le i<j\le N} \left(\left[\,X_j\,\right]_{q^{-1}} - [\,X_i\,]_{q^{-1}} \right).$$

Furthermore note that,

$$[\,X_i\,]_q - \left[\,X_j\,\right]_q = q^{X_i+X_j-1} \left(\left[\,X_j\,\right]_{q^{-1}} - [\,X_i\,]_{q^{-1}} \right)$$

and

$$\prod_{1\le i<j\le N} q^{X_i+X_j-1} = \prod_{i=1}^{N} q^{(N-1)X_i-(N-i)} = q^{-N(N-1)/2}\prod_{i=1}^{N} q^{(N-1)X_i}$$

therefore the determinant takes the form,

$$\left(\prod_{2\le i\le j\le N} [\,B_i\,]_q - \left[\,A_j\,\right]_q\right)\prod_{1\le i<j\le N}\left([\,X_i\,]_q - \left[\,X_j\,\right]_q\right).$$

□

Remark. In practice we would prefer to have an expression of the previous theorem in terms of differences $B_i - A_j$ we have

$$[\,B_i\,]_q - \left[\,A_j\,\right]_q = q^{B_i-1}\frac{q^{A_j-B_i}-1}{q^{-1}-1} = q^{B_i-1}\left[\,B_i - A_j\,\right]_{q^{-1}}$$

and

$$\prod_{2\le i\le j\le N} [\,B_i\,]_q - \left[\,A_j\,\right]_q = \prod_{i=1}^{N} q^{(B_{i+1}-1)(N-i)}\prod_{2\le i\le j\le N}\left[\,B_i - A_j\,\right]_{q^{-1}}.$$

Therefore,

$$\begin{aligned}\det_{1\le i,j\le N} [\,X_i+A_N\,]_q\ldots\left[\,X_i+A_{j+1}\,\right]_q\left[\,X_i+B_j\,\right]_q\ldots[\,X_i+B_2\,]_q \\ = \prod_{i=1}^{N} q^{(B_{i+1}-1)(N-i)}\prod_{2\le i\le j\le N}\left[\,B_i-A_j\,\right]_{q^{-1}}\prod_{1\le i<j\le N}[\,X_i\,]_q - \left[\,X_j\,\right]_q.\end{aligned}$$

On the other hand if we write,

$$[\,B_i\,]_q - \left[\,A_j\,\right]_q = q^{A_j}\frac{1-q^{B_i-A_j}}{1-q} = q^{A_j}\left[\,B_i - A_j\,\right]_q$$

then

$$\begin{aligned}\det_{1\le i,j\le N} [\,X_i+A_N\,]_q\ldots\left[\,X_i+A_{j+1}\,\right]_q\left[\,X_i+B_j\,\right]_q\ldots[\,X_i+B_2\,]_q \\ = \prod_{i=1}^{N-1} q^{iA_{i+1}}\prod_{2\le i\le j\le N}\left[\,B_i-A_j\,\right]_q\prod_{1\le i<j\le N}[\,X_i\,]_q - \left[\,X_j\,\right]_q.\end{aligned}$$

Appendix B

Orthogonal Polynomials

A sequence of polynomials $\{p_1(X), p_2(X), \ldots, p_n(X), \ldots\}$ with degree $n = 1, 2, \ldots$ are called orthogonal polynomials on an interval $[a, b]$ if there exists some measure $\mu(X) = w(X)dX$, where $w(X)$ is a non-negative weight function, such that

$$\int_a^b p_n(X)p_m(X)d\mu(X) = 0$$

([8], pg. 122.) In particular, if $d\mu(X)$ is a discrete measure the integral becomes a sum

$$\sum_X p_n(X)p_m(X)w(X) = 0.$$

If we introduce the Kronecker delta δ_{nm} which is 1 if $n = m$ and 0 otherwise, we can write

$$\sum_X p_n(X)p_m(X)w(X)dx = c_n\delta_{nm}$$

where c_n is determined by the particular choice of the $p_n(X)$. (there are many references on orthogonal polynomials, some relevant ones are [4, 9–11].)

B.1 Hahn and q-Hahn Orthogonal Polynomials

The Hahn and q-Hahn polynomials are defined using the Askey-scheme of hypergeometric orthogonal polynomials [11]. The hypergeometric function ${}_r\tilde{F}_s$ is defined by

$${}_r\tilde{F}_s\left(\begin{matrix}a_1, \ldots, a_r \\ b_1, \ldots, b_s\end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k}\frac{z^k}{k!}$$

where using Pochhammer notation

$$(a_1, \ldots, a_r)_k := (a_1)_k \ldots (a_M)_k.$$

The Hahn polynomials $H_n(x, \alpha, \beta, M)$ can be written as a hypergeometric function, terminating the series at M,

$$H_n(x, \alpha, \beta, M) = {}_3\tilde{F}_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -x, \\ \alpha+1, -M\end{matrix}; 1\right) = \sum_{k=0}^{M} \frac{(-n, n+\alpha+\beta+1, -x)_k}{(\alpha+1, -M)_k}\frac{1}{k!}$$

where $n = 0, 1, \ldots, M$. The q-Hypergeometric function are defined similarly

$$ {}_r\tilde{\phi}_s \left(\begin{matrix} q^{-n}, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right) := \sum_{k=0}^{M} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q;q)_k} $$

where we use the q-Pochhammer notation

$$ (a_1, \ldots, a_r; q)_k := (a_1; q)_k \ldots (a_N; q)_k. $$

The q-Hahn polynomials are defined

$$ Q_n(q^{-x}, \alpha, \beta, M) = {}_3\tilde{\phi}_2 \left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x} \\ \alpha q, q^{-M} \end{matrix} ; q, q \right) $$

with $n = 0, 1, \ldots, N$.

Remark. The q-Hahn polynomials are related to the Hahn by setting $\alpha \to q^\alpha$ and $\beta \to q^\beta$ and taking $q \to 1$ then

$$ \lim_{q \to 1} Q_n(q^{-x}, \alpha, \beta, M) = H_n(x, \alpha, \beta, M) $$

This property means that we can analyse all of the properties of the first model as a special case of the q-Hahn model.

Remark. We see above that our q-Hahn polynomial is defined in terms of the q-hypergeometric function in q^{-x}, we can equivalently view such polynomials as functions in $[\, x \,]_{q^{-1}}$, recall the definition of the q-Hahn polynomial

$$ Q_n(q^{-x}, \alpha, \beta, M) = \sum_{k=0}^{M} \frac{(q^{-n}; q)_k (\alpha\beta q^{n+1}; q)_k (q^{-x}; q)_k}{(\alpha q; q)_k (q^{-M}; q)_k} \frac{q^k}{(q;q)_k} $$

here we can rewrite the term

$$ (q^{-x}; q)_k = \prod_{i=1}^{k} (1 - q^{-x+i-1}) = \prod_{i=1}^{k} (1 - q^{-1}) \, [\, x - k + 1 \,]_{q^{-1}} $$

Using (A.4) we have $[\, x + k - 1 \,]_{q^{-1}} = q^{k-1} [\, x \,]_{q^{-1}} + [\, k - 1 \,]_{q^{-1}}$, therefore $(q^{-x}; q)_k$ is a polynomial in $[\, x \,]_{q^{-1}}$ of degree k. Using $[\, -\alpha \,]_q = -q^{-1} [\, \alpha \,]_{q^{-1}}$ from (A.1) we can see this as a polynomial in $[\, -x \,]_{q^{-1}}$. We can define a new polynomial in $[\, x \,]_{q^{-1}}$ with $S_n([\, x \,]_{q^{-1}}) = Q_n(q^{-x})$ and of degree n but in practice $Q_n(q^{-x})$ is easier to work with.

B.1.1 Recurrence Relations

For any sequence of orthogonal polynomials $p_0, p_1, \ldots$ there exists a three-term recurrence relation [4] (pg. 175)

$$ X p_n(X) = A_n p_{n+1}(X) + B_n p_n(X) + C_n p_{n-1}(X) $$

where $A_n, B_n, C_n \in \mathbb{R}$, $A_n C_{n+1} \geq 0$ and $p_0(X) = 1$. We define by convention $p_{-1}(X) := 0$. (If the polynomials have infinite number of points in the support then we can $A_n C_n > 0$.) If the orthogonal polynomials are defined on a finite number of points then the sequence of orthogonal polynomials terminates at some p_N then we have $A_n = 0$ for $n > N$. Conversely, a recurrence relation defines a sequence of orthogonal polynomials which terminates at the first

N for which $A_N C_N = 0$. The three term recurrence relation determine p_{n+1} using p_n, p_{n-1}, we can write

$$A_n p_{n+1}(X) = (X - B_n)p_n(X) - C_n p_{n-1}(X).$$

We may consider sequences with specific properties, for example the *monic* orthogonal polynomials have corresponding recurrence relations where $p_0(X) = 1$ and $A_n = 1$ if $A_n > 0$. The recurrence relation becomes

$$p_{n+1}(X) = (X - B_n)p_n(X) - C_n p_{n-1}(X)$$

which shows that (for n where $A_n > 0$) the leading coefficients of all the (non-trivial) p_n's are one. One may also consider sequences of *orthonormal* polynomials which have the property

$$\sum_X p_n(X)p_m(X)w(X) = \delta_{nm}.$$

In this case the recurrence relation can be written

$$Xp_n(X) = a_{n+1}p_{n+1}(X) + b_n p_n(X) + a_n p_{n-1}(X)$$

where $a_n > 0, b_n \in \mathbb{R}$. For terminated sequences up to N we again let $a_n > 0$ for $n \leq N$ and $a_n = 0$ for $n > N$.

Lemma B.1.1. *Consider a sequence of monic orthogonal polynomials $p_0(X), p_1(X), \dots$ orthogonal under the weight $w(X)$, with the following recurrence relation*

$$Xp_n(X) = p_{n+1}(X) + B_n p_n(X) + A_n p_{n-1}(X),$$

$B_n \in \mathbb{R}$ and $A_n > 0$. By rescaling the p_n's we can also consider a sequence of orthonormal polynomials $r_0(X), r_1(X), \dots$ with recurrence relation

$$Xr_n(X) = a_{n+1}r_{n+1}(X) + b_n r_n(X) + a_n r_{n-1}(X)$$

where $b_n \in \mathbb{R}$ and $a_n > 0$. Then if $\gamma_n > 0$ are the leading coefficients of $r_n(X)$,

$$\gamma_n = \frac{1}{a_2 \dots a_n a_{n+1}}$$

and the recurrence coefficients are related by $A_n = a_n^2$ and $B_n = b_n$.

Proof. Comparing the leading coefficient on either side of the orthonormal recurrence relation,

$$Xr_n(X) = a_{n+1}r_{n+1}(X) + b_n r_n(X) + a_n r_{n-1}(X)$$

we have $\gamma_n = a_{n+1}\gamma_{n+1}$ for all $n = 0, 1, \dots$. On the other hand if $n = 0$ we have defined $p_0(X) = 1$ therefore $\gamma_2 = 1/a_2$. Furthermore

$$\gamma_{n+1} = \frac{1}{a_{n+1}}\gamma_n = \frac{1}{a_{n+1}a_n}\gamma_1 = \dots = \frac{1}{a_{n+1}a_n \cdots a_2}.$$

Because $r_n(X) = \gamma_n p_n(X)$ in the orthonormal recurrence relation,

$$X\gamma_n p_n = a_n\gamma_{n+1}p_{n+1} + b_n\gamma_n p_n + a_n\gamma_{n-1}p_{n-1}$$

and dividing through by γ_n,

$$\begin{aligned} Xp_n &= a_{n+1}\frac{\gamma_{n+1}}{\gamma_n}p_{n+1} + b_n p_n + a_n\frac{\gamma_{n-1}}{\gamma_n}p_{n-1} \\ Xp_n &= p_{n+1} + b_n p_n + a_n a_n p_{n-1}. \end{aligned}$$

In the last line we compare with the monic recurrence relation in the theorem statement and identify $B_n = b_n$ and $A_n = a_n^2$. □

Remark. For sequences of orthogonal polynomials $p_1, p_2, \dots, p_N$ that terminate the result above still holds for $n \leq N$ and for $n > N$ we take a_n, γ_n to be zero.

B.2 Recurrence Relations for Hahn and q-Hahn

The three term recurrence relation for q-Hahn polynomials can be found in resources on orthogonal polynomials such as [4, 9–11]. We consider a sequence of orthogonal polynomials $Q_{0,N}, Q_{1,N}, \ldots, Q_{N,N}$ where $Q_{n,N}(q^{-x}) = Q_{n,N}(q^{-x}, q^{\alpha}, q^{\beta}, N; q)$ are the q-Hahn orthogonal polynomials (we suppress the index N just writing Q_n) and which have the recurrence relation

$$q^{-x}Q_n(q^{-x}) = A_n Q_{n+1}(q^{-x}) + (1 - (A_n + C_n))Q_n(q^{-x}) + C_n Q_{n-1}(q^{-x}) \tag{B.1}$$

for $n = 0, 1, \ldots, N$ and where

$$A_n = \frac{\left(1-q^{n-N}\right)\left(1-q^{\alpha+n+1}\right)\left(1-q^{\alpha+\beta+n+1}\right)}{\left(1-q^{\alpha+\beta+2n+1}\right)\left(1-q^{\alpha+\beta+2n+2}\right)}$$
$$C_n = \frac{q^{\alpha+2n-N}\left(1-q^{-n}\right)\left(1-q^{\beta+n}\right)\left(1-q^{\alpha+\beta+N+n+1}\right)}{\left(1-q^{\alpha+\beta+2n}\right)\left(1-q^{\alpha+\beta+2n+1}\right)}.$$

(See [10] for A_n, C_n with more general coefficients.) From any three term recurrence relation we can extract the corresponding recurrence relation for the monic sequence of orthogonal polynomials by dividing through by A_n in (B.1). We consider the substitution

$$Q_n(q^{-x}) = \frac{\left(q^{\alpha+\beta+n+1};q\right)_n}{\left(q^{\alpha+1},q^{-N};q\right)_n} p_n(q^{-x})$$

into (B.1) which gives the recurrence relation

$$\begin{aligned} q^{-x}p_n(x) =& \frac{\left(q^{\alpha+1},q^{-N};q\right)_n}{\left(q^{\alpha+\beta+n+1};q\right)_n} \frac{\left(q^{\alpha+\beta+n+2};q\right)_{n+1}}{\left(q^{\alpha+1},q^{-N};q\right)_{n+1}} A_n p_{n+1}(x) \\ &+ (1-(A_n+C_n))p_n(x) \\ &+ \frac{\left(q^{\alpha+1},q^{-N};q\right)_n}{\left(q^{\alpha+\beta+n+1};q\right)_n} \frac{\left(q^{\alpha+\beta+n};q\right)_{n-1}}{\left(q^{\alpha+1},q^{-N};q\right)_{n-1}} C_n p_{n-1}(x). \end{aligned}$$

Simplifying,

$$\frac{\left(q^{\alpha+1},q^{-N};q\right)_n}{\left(q^{\alpha+\beta+n+1};q\right)_n} \frac{\left(q^{\alpha+\beta+n+2};q\right)_{n+1}}{\left(q^{\alpha+1},q^{-N};q\right)_{n+1}} = \frac{(1-q^{\alpha+\beta+2n+1})(1-q^{\alpha+\beta+2n+2})}{(1-q^{\alpha+n+1})(1-q^{n-N})(1-q^{\alpha+\beta+n+1})}$$

which is simply $\frac{1}{A_n}$ and

$$\frac{\left(q^{\alpha+1},q^{-N};q\right)_n}{\left(q^{\alpha+\beta+n+1};q\right)_n} \frac{\left(q^{\alpha+\beta+n};q\right)_{n-1}}{\left(q^{\alpha+1},q^{-N};q\right)_{n-1}} = \frac{(1-q^{\alpha+n})(1-q^{n-1-N})(1-q^{\alpha+\beta+n})}{(1-q^{\alpha+\beta+2n-1})(1-q^{\alpha+\beta+2n})}$$

which is A_{n-1} the recurrence relation can be written

$$q^{-x}p_n(q^{-x}) = p_{n+1}(q^{-x}) + (1-(A_n+C_n))p_n(q^{-x}) + A_{n-1}C_n p_{n-1}(q^{-x}) \tag{B.2}$$

An application of B.1.1 gives the following corollary:

Corollary B.2.1. *The recurrence relation for a sequence of orthonormal q-Hahn polynomials $r_0(q^{-x}), r_1(q^{-x}), \ldots$ can be written,*

$$q^{-x}r_n(q^{-x}) = a_{n+1}r_{n+1}(q^{-x}) + b_n r_n(q^{-x}) + a_n r_{n-1}(q^{-x})$$

where $a_n = \sqrt{A_{n-1}C_n}$ and $b_n = (1-(A_n+C_n))$

Remark. Alternatively consider the polynomials in q^{-x} as polynomials in $[\,x\,]_{q^{-1}}$, by writing $X := [\,x\,]_{q^{-1}}$ and defining polynomials $u_n(X) = u_n([\,x\,]_{q^{-1}}) := (-1)^n r_n(q^{-x})$ so that the orthogonal recurrence relation becomes,

$$(1-q^{-x})u_n(X) = a_{n+1}u_{n+1}(X) + (1-b_n)u_n(X) + a_n u_{n-1}(X)$$

where $u_1(X) := 1$ and $u_0(X) := 0$. By dividing through by $1-q^{-1}$ and identifying $[\,x\,]_{q^{-1}} = \frac{1-q^{-x}}{1-q^{-1}}$,

$$[\,x\,]_{q^{-1}}\, u_n(X) = D_{n+1}u_{n+1}(X) + E_n u_n(X) + D_n u_{n-1}(X)$$

where $D_n = \sqrt{\frac{A_{n-1}}{1-q^{-1}}\frac{C_n}{1-q^{-1}}}$, $E_n = \frac{A_n}{1-q^{-1}} + \frac{C_n}{1-q^{-1}}$ and

$$\begin{aligned}\frac{A_n}{1-q^{-1}} &= \frac{[\,M-n\,]_{q^{-1}}\,[\,-(\alpha+1)-n\,]_{q^{-1}}\,[\,-(\alpha+\beta+1)-n\,]_{q^{-1}}}{[\,-(\alpha+\beta+1)-2n\,]_{q^{-1}}\,[\,-(\alpha+\beta+2)-2n\,]_{q^{-1}}},\\ \frac{C_n}{1-q^{-1}} &= \frac{q^{\alpha+2n-M}\,[\,n\,]_{q^{-1}}\,[\,-\beta-n\,]_{q^{-1}}\,[\,-(\alpha+\beta+M+1)-n\,]_{q^{-1}}}{[\,-(\alpha+\beta)-2n\,]_{q^{-1}}\,[\,-(\alpha+\beta+1)-2n\,]_{q^{-1}}}.\end{aligned}$$

Remark. With the above representation we can get the Hahn recurrence relation back in the limit $q \longrightarrow 1$, recall that we have

$$\lim_{q\to 1} Q_n(q^{-x}, q^{\alpha}, q^{\beta}, M|q) = H_n(x,\alpha,\beta,M)$$

where $H_n(x)$ are the Hahn polynomials. Furthermore the recurrence coefficients,

$$\begin{aligned}\lim_{q\to 1}\frac{A_n}{1-q^{-1}} &= \lim_{q\to 1}\frac{[\,M-n\,]_{q^{-1}}\,[\,-(\alpha+1)-n\,]_{q^{-1}}\,[\,-(\alpha+\beta+1)-n\,]_{q^{-1}}}{[\,-(\alpha+\beta+1)-2n\,]_{q^{-1}}\,[\,-(\alpha+\beta+2)-2n\,]_{q^{-1}}}\\ &= \frac{(M-n)(\alpha+n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}\end{aligned}$$

and

$$\begin{aligned}\lim_{q\to 1}\frac{C_n}{1-q^{-1}} &= \lim_{q\to 1}\frac{q^{\alpha+2n-M}\,[\,n\,]_{q^{-1}}\,[\,-\beta-n\,]_{q^{-1}}\,[\,-(\alpha+\beta+M+1)-n\,]_{q^{-1}}}{[\,-(\alpha+\beta)-2n\,]_{q^{-1}}\,[\,-(\alpha+\beta+1)-2n\,]_{q^{-1}}}\\ &= \frac{n(\beta+n)(\alpha+\beta+M+n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}\end{aligned}$$

which are exactly the coefficients of the Hahn recurrence relation (see [10].)

B.2.1 The Christoffel-Darboux Formula

Proposition B.2.2. *For three-term recurrence relation we have the Christoffel-Darboux formula,*

$$\sum_{i=0}^{n-1} p_i(X)p_i(Y) = \begin{cases} \frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(X)p_{n-1}(Y)-p_{n-1}(X)p_n(Y)}{X-Y} & \text{if } X \neq Y \\ \frac{\gamma_{n-1}}{\gamma_n}(p_n'(X)p_{n-1}(X) - p_n(X)p_{n-1}'(X)) & \text{if } X = Y \end{cases} \tag{B.3}$$

Furthermore, $\frac{\gamma_{n-1}}{\gamma_n} = a_n$.

Proof. Consider first $X \neq Y$. We induct on n, for $n=1$ the left sum is $p_0(X)p_0(Y) = 1$ and the right hand side,

$$\frac{\gamma_0}{\gamma_1}\frac{p_1(X)p_0(Y) - p_0(X)p_1(Y)}{X-Y} = \frac{1}{\gamma_1}\frac{p_1(X)-p_1(Y)}{X-Y} = \frac{1}{\gamma_1}\frac{\gamma_1(X-Y)}{X-Y} = 1.$$

Therefore assume the formula holds for $n = k - 1$ and consider $n = k$. The recurrence relation can also be written:

$$a_n p_n(X) = (X - b_{n-1}) p_{n-1}(X) - a_{n-1} p_{n-2}(X)$$

Consider therefore the right hand side,

$$\frac{1}{X - Y}(a_n p_n(X) p_{n-1}(Y) - a_n p_{n-1}(X) p_n(Y)).$$

Using the recurrence relation,

$$\begin{cases} a_n p_n(X) p_{n-1}(Y) & = (X - b_{n-1}) p_{n-1}(X) p_{n-1}(Y) - a_{n-1} p_{n-2}(X) p_{n-1}(Y), \\ a_n p_n(Y) p_{n-1}(X) & = (Y - b_{n-1}) p_{n-1}(X) p_{n-1}(Y) - a_{n-1} p_{n-1}(X) p_{n-2}(Y). \end{cases}$$

Substituting these two expressions we will see,

$$\begin{aligned} &\frac{1}{X - Y}(a_n p_n(X) p_{n-1}(Y) - a_n p_{n-1}(X) p_n(Y)) \\ &= (p_{n-1}(X) p_{n-1}(Y) - p_{n-1}(X) p_{n-1}(Y)) \\ &\quad + a_{n-1} \frac{p_{n-1}(X) p_{n-2}(Y) - p_{n-2}(X) p_{n-1}(Y)}{X - Y} \\ &= a_{n-1} \frac{p_{n-1}(X) p_{n-2}(Y) - p_{n-2}(X) p_{n-1}(Y)}{X - Y}. \end{aligned}$$

This concludes the induction step. For $X = Y$ we consider two cases, in the first situation $X = Y = x$ where $x \in \mathbb{R}$. We set $X = x + \epsilon$ and $Y = x$ and take $\epsilon \longrightarrow 0$ so that,

$$\begin{aligned} &\lim_{\epsilon \to 0} \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x + \epsilon) p_{n-1}(x) - p_{n-1}(x + \epsilon) p_n(x)}{x + \epsilon - x} \\ &= \lim_{\epsilon \to 0} \frac{\gamma_{n-1}}{\gamma_n} \left(p_{n-1}(x) \frac{p_n(x + \epsilon) - p_n(x)}{\epsilon} - p_n(x) \frac{p_{n-1}(x + \epsilon) - p_{n-1}(x)}{\epsilon} \right) \\ &= \frac{\gamma_{n-1}}{\gamma_n} \left(p_{n-1}(x) \lim_{\epsilon \to 0} \frac{p_n(x + \epsilon) - p_n(x)}{\epsilon} - p_n(x) \lim_{\epsilon \to 0} \frac{p_{n-1}(x + \epsilon) - p_{n-1}(x)}{\epsilon} \right) \\ &= \frac{\gamma_{n-1}}{\gamma_n} \left(p_{n-1}(x) p_n'(x) - p_n(x) p_{n-1}'(x) \right). \end{aligned}$$

In the situation where $X = Y = [\, x \,]_q$ we can consider and alternative derivation of the same result by the use of the q-derivative defined by taking $D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$ so that $\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}$. We write $X = s [\, x \,]_q$ and find that,

$$\begin{aligned} &\frac{\gamma_{n-1}}{\gamma_n} \frac{p_n\left(s [\, x \,]_q\right) p_{n-1}([\, x \,]_q) - p_{n-1}\left(s [\, x \,]_q\right) p_n([\, x \,]_q)}{s [\, x \,]_q - [\, x \,]_q} \\ &= \frac{\gamma_{n-1}}{\gamma_n} \left(p_{n-1}(x) \frac{p_n(q [\, x \,]_q) - p_n([\, x \,]_q)}{(s - 1) [\, x \,]_q} - p_n(x) \frac{p_{n-1}(q [\, x \,]_q) - p_{n-1}([\, x \,]_q)}{(s - 1) [\, x \,]_q} \right) \\ &= \frac{\gamma_{n-1}}{\gamma_n} \left(p_{n-1}([\, x \,]_q) D_q p_n([\, x \,]_q) - p_n(x) D_q p_{n-1}([\, x \,]_q) \right). \end{aligned}$$

Taking $q \longrightarrow 1$ we obtain the same result. Therefore the Christoffel-Darboux is well-defined for q-orthogonal polynomials too. $\square$

For any orthogonal polynomial $p_n(X)$ the zeros are real, simple and lie in the support of the associated weight function $w(X)$. Zeros of successive polynomials interlace: If $X_1 < X_2$ and $p_{n+1}(X_1) = p_{n+1}(X_2) = 0$ and p_{n+1} is non-zero on the interval (X_1, X_2) then there exists only one $X \in (X_1, X_2)$ for every $m < n+1$ such that $p_m(X) = 0$.

Corollary B.2.3. *The roots of polynomials* $\{p_n(X)\}$ *are all real and distinct.*

Proof. We prove the claim by contradiction, assume there exists a zero $X \in \mathbb{C}$ of an orthogonal polynomial $p_n(X)$ of degree n. Let $X \in \mathbb{C}$ be root, in which case $\overline{X}$ is another root (because p_n is a polynomial in real coefficients so $\overline{p_n(X)} = p_n(\overline{X}) = 0$.) From the Christoffel-Darboux formula (B.3) we have on the left side of the equation,

$$\sum_{i=0}^{n-1} p_i(X)p_i(\overline{X}) = \sum_{i=0}^{n-1} p_i(X)\overline{p_i(X)} > 0$$

(the inequality is strict as $p_0 = 1$.) Meanwhile on the right hand side,

$$\begin{cases} \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(X)\overline{p_{n-1}(X)} - p_{n-1}(X)\overline{p_n(X)}}{X-Y} & \text{if } X \notin \mathbb{R} \\ \frac{\gamma_{n-1}}{\gamma_n}(p_n'(X)p_{n-1}(X) - p_n(X)p_{n-1}'(X)) & \text{if } X \in \mathbb{R} \end{cases}$$

Therefore if $X \notin \mathbb{R}$ then the right hand side is zero, a contradiction which shows that the root must be real. Furthermore if $X \in \mathbb{R}$ but is a repeated root, then we must have $p_n(X) = 0$ and $p_n'(X) = 0$, which means the right hand side is again zero, so we cannot have repeated roots. □

Corollary B.2.4. *The roots of polynomials* $\{p_n(X)\}$ *are all distinct.*

Proof. We prove this by contradiction, assume there exists a repeated zero X of an orthogonal polynomial $p_n(X)$ of degree n. Because X is repeated root both $p_n(X) = 0$ and $p_n'(X) = 0$ and so by the Christoffel-Darboux formula:

$$\sum_{i=0}^{n-1} p_i(X)^2 = a_n(p_n'(X)p_{n-1}(X) - p_n(X)p_{n-1}'(X)) = 0.$$

This implies that $p_i(X) = 0$ for all $i < n$ while on the other hand $p_0(X) = 1$ so by contradiction X cannot be a repeated root. □

Appendix C

Matlab Code for Generating Pictures of the Asymptotic Zero Distribution

```
function qhahn_dist(C, N, type)
        % C is a number or a list of values c between (0,1)
        % Adding a third argument 'sym' produces symmetric plot
        % N controls the fineness of plot, there will be NxN points.
        arguments
        C (1,:) double
        N (1,1) double = 250
        type char = 'sym'
        end
        l = 1/N;
        t = 0:l:2;
        for c = C
            if c <= 0 || c == 1
                error("c must 0>c>1 or c>1")
            end
        end
        if N<-1 || N > 1000
            error("N negative or too large for memory")
        end

        for c = C
            % meshgrid for all points in the plane
            fullgrid = meshgrid(0:l:2);
            %fullgrid = meshgrid(0:l:2);
            % points inside the hexagon
            inhex = sparse((fullgrid >= max(0,-1:l:1)') &  (fullgrid <=
                min(2,1:l:3)'));
            % discarding points outside the bounding hexagon
            hexgrid = (fullgrid .* inhex);

            % boolean matrix that determines values below and above the
                artic curve
            if c<1 && c>0
```

```
[L,U] = bounds(c,t);
else
[U,L] = bounds(c,t);
end
lower_instit = (hexgrid < L');
upper_instit = (hexgrid > U');
lowerbound = (hexgrid > L');
upperbound = (hexgrid < U');

% calculating the four corners, North/South and East/West,
    which depend on
% intersection points and the arctic curve, we only need to
    calculate
n = length(fullgrid);
t = linspace(0,2,n);
isect = isecpts(c);

% the values for which
EW_upper_t = sparse((t < isect(2) | t > isect(3)));
EW_lower_t = sparse((t < isect(1) | t > isect(4)));

% interstitial values between
interstitial_meas = (upper_instit .* EW_upper_t') + (
    lower_instit .* EW_lower_t');

% boolean matrix for values inside the arctic curve
boundgrid = (upperbound) .* (lowerbound);
% x-values inside of the arctic curve
boundgrid = boundgrid ./ boundgrid;
zgrid = measure(c, boundgrid);

qhahn_meas = (inhex ./ inhex) .* (fillmissing(zgrid, 'constant
    ', 0) + interstitial_meas');

%qmeas=full(qhahn_meas);qmeas=fillmissing(qmeas,'constant',0);
%qhahn_meas = qhahn_meas ./ sum(qmeas);

f = figure;
f.Resize = 'off';

[X,Y] = meshgrid(0:l:2,0:l:2);
switch type
    case 'sym'
        Z = (Y-X/2-1/2);
        s = pcolor(X,Z,qhahn_meas);
        s.EdgeColor = 'none';
        colorn = 10;
        co = hsv(3*colorn);
        colormap(co(1:colorn,:))
```

```
            xlabel("c = " + string(c))
            xticks([])
            yticks([])
            set(gca,'Xlim',[0,2],'Ylim', [-1,1])


            hold on;
            p = plot(t,L-t/2-1/2,'k',t,U-t/2-1/2,'k');
            p(1).LineWidth = 1;
            p(2).LineWidth = 1;

            %bounding hexagon
            hexagon = polyshape([0      1       2       2       1
                      0       ], ...
            [0.5 0 0.5 1.5 2 1.5]-1);
            p2=plot(hexagon,'FaceColor','white');
            p2(1).LineWidth = 1;
            colorbar('Location', 'eastoutside')
        case 'affine'
            s = pcolor(X,Y,qhahn_meas);
            s.EdgeColor = 'none';
            co = hsv(30);
            colormap(co(1:10,:))
            xlabel("c=" + string(c))
            xticks([])
            yticks([])

            hold on;
            p = plot(t,L,'k',t,U,'k');
            p(1).LineWidth = 1;
            p(2).LineWidth = 1;

            %bounding hexagon
            hexagon = polyshape([0      1       2       2       1
                      0       0], ...
            [0  0       1       2       2       1       0 ]);
            p2=plot(hexagon,'FaceColor','white');
            p2(1).LineWidth = 1;
            colorbar('Location', 'eastoutside')
    end
    end
function isec = isecpts(c)
    % first intersection occurs at isec0
    isec0 = log( (1+c^2)/(1+c) ) / log(c);
    % the other intersections are related to isec0
    isec = [isec0, 1-isec0, 1+ isec0, 2-isec0];
end

function [L,U] = bounds(c,t)
```

```
            ct = c.^t;
            a = 1/(1+c)^2 * sqrt( c*(1+c+c^2)*ct.*(1 - ct) .* (1 - c^2 *
                1./ ct) );
            b = c*(1+c^2)/(1+c)^2 * (1 + ct / c);
            L = log(b + 2*a)/log(c);
            U = log(b - 2*a)/log(c);
        end

        % calculate the value of the asymptotic probability measure
        % c, and list of x-values and t-values and gives the probability
            values
        function meas = measure(c,bound)
            bound = full(bound);
            n = length(bound);
            t = linspace(0,2,n);
            x = linspace(0,2,n);
            ct = c.^(t);
            cx = c.^(x);
            T = meshgrid(ct);
            X = meshgrid(cx);
            T = T .* bound;
            X = (X .* bound)';
            num = ((1+c^2)*(-X.*(X-c))+T.*(-2*c^2 + (1+c^2)*X));
            denom = 2*sqrt( c*(X-1).*(X-c^2).*(c*X-T).*(X-c*T) );
            % where denom is 0, prod will produce NaN, which
            % prevents the missing entries from producing output later
            prod = num ./ denom ;
            meas = (asin(prod) + pi/2 ) / pi;
        end

end
```

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