Conceptual Design of an Air-Launched Three-Staged Orbital Launch Vehicle

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Abstract—The objective of this study was to design a launch vehicle capable of deploying a nanosatellite into a Sun-synchronous orbit at 500 km orbital altitude from the JAS 39E/F Gripen fighter aircraft. This was achieved by first performing theoretical calculations for the required nozzles and solid propellant grain configurations for the first two solid stages, followed by the necessary liquid propellant configuration for the third stage. Lastly, two methods were investigated in solving the trajectory ascent problem for the launch vehicle design. First, by stating the trajectory problem as an initial value problem while guessing a Sigmoidal steering law. Secondly, by stating the trajectory problem as a boundary value problem. The latter was solved by transcribing the trajectory problem into a nonlinear program where a parametric steering law was derived using a Sequential quadratic programming algorithm. Ultimately, resulting in a launch vehicle design with a gross lift-off mass of 1,289 kg, capable of launching an 8.4 kg payload into the targeted orbit, with suggested modifications to increase the possible payload mass to 12.9 kg.


Index Terms—Air-launched, Launch Vehicle, Sun-Synchronous orbit, Rocket, Solid propellant, Liquid propellant, Trajectory optimization, Nonlinear programming, JAS 39 Gripen

Acknowledgements

I want to dedicate this master thesis to my father, who always encouraged and supported me in my various space-related endeavours. Additionally, I would like to thank my supervisors Thomas Steiner and Mathias Lindström at Saab Dynamics, and Christer Fuglesang at KTH Royal Institute of Technology for all the help and support I have received along the way.

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I. INTRODUCTION

Traditionally, space launches have been carried out with the use of expendable (and often quite toxic and unstable) launch vehicles (LV) from a stationary location at an eastern coastline close to the equator. Some examples of spaceports around the world utilising such a location are: Korou French Guiana (European Union), Sriharikota Island (India), Cape Canaveral (United States), and Tanegashima (Japan). The reason for this is because Earth rotates eastwards. So, by launching close to the equator at an eastern coastline towards the east, it is possible to maximally utilise the initial velocity gained from Earth’s rotation, as well as reduce chances of collateral damage in the impact area downstream. For example, the European Space Agency’s launches from Korou generally possess 5.9 % of their required orbital velocity just from Earth’s rotation alone.

A second-best alternative would be to launch from a vast, barren and inhabited area to reduce the risk of harming or disturbing any local residence, infrastructure or wildlife. One example would be the Swedish spaceport Esrange, located above the polar circle with an impact area of over 1 % of Sweden’s total surface area. However, for many orbits and applications, it is not always possible or convenient to launch from such a stationary location. Some being due to geopolitical reasons along with three other main drivers: risk, time and cost.

One alternative approach to a conventional ground-based vertical launch system is a so-called Air-Launch System (ALS), which has several advantages concerning its: high manoeuvrability, adaptability and flexibility. An ALS works by having a carrier aircraft, normally using air-breathing propulsion, elevate the LV to a higher altitude and velocity, then releasing the LV horizontally that propels itself up into orbit. This launch system takes advantage of the higher efficiency of the air-breathing engine of the carrier at lower altitudes. However, the main advantage is the flexibility of launch location and adaptable launch window. Since weather factors are reduced together with the risk associated with the impact zone. E.g., the carrier aircraft could transport the LV to launch over an open ocean above the cloud base with an appropriate flight heading angle.

Moreover, considering the state of the rapidly increasing market for space launch systems indicates an evolution towards
nano- and microsatellites, require new and affordable launch systems to be provided. A market study made by SpaceWorks in 2020 showed a forecast over the next 5 years with approximately 1,800 - 2,400 nano/microsatellites (1-50 kg) requiring launch into orbit. The 6 U (1 U = 10x10x10 cm³) microsatellites are gaining popularity as they seem to strike a balance between the cost and capability of the satellite.

Finally, considering the current evolving space market along with the high flexibility and possible reduction of an impact zone and cost, makes an ALS a highly attractive option. As conventional rideshare options on larger launch vehicles introduce several constraints, many of which decided by the primary payload, such as the launch window, final orbit selection and increasing dependence of external payloads and partners.

II. THESIS STRUCTURE

This thesis study was divided into two modules: a Rocket Module and a Trajectory Module. In the former, the structural design of the LV was focused on regarding: the nozzles, grain geometries, combustion chamber, material selection and nose cone shape. In the later module, the designed LV resulting from the first module was implemented into a trajectory
problem to be analysed. Resulting in some modifications that could be iterated once more through the Rocket Module.

In the Trajectory Module, the trajectory problem of steering the LV from the initial position to its final targeted orbit was solved using two methods. First by stating the problem as an Initial Value Problem (IVP) while guessing a steering law, secondly by stating the problem as a Boundary Value Problem (BVP). The BVP was solved by transcribing the trajectory problem into a nonlinear program where a parametric steering law was derived using a gradient descent, sequential quadratic programming algorithm. Lastly, these two methods were compared to motivate the credibility of the guessed steering law with the optimised parametric steering law.

III. Scope

Firstly, in this study, only the LV itself will be considered regarding the envelope, grain and nozzles for each stage. Along with an estimation of the potential payload volume and mass that the LV will be able to operate.

Secondly, since this is a conceptual study of the feasibility of a LV design, the integration of the payload with the LV along with its purpose and application will not be covered here. As a result, no constraints on the acceleration is imposed in the mission requirements since this is highly dependent on the type of payload. Further, the highly intricate integration between the LV and the carrier aircraft is considered to be outside of the scope.

Thirdly, since there is a multitude of interconnected factors that affect the final design of the LV: considering the possible trajectory paths, burn times and thrusts, to mention a few. Only a feasible design of a LV will be investigated, leaving a complete optimisation of the whole ALS to be outside of the scope.

Fourthly, a full economic analysis is also considered to be outside of the scope, even though it has a significant impact on the final design of the LV, as it always does in real-world engineering applications.

Lastly, the time extent of the whole study is limited to fit inside of a 6 month period.

IV. Mission Definition

The objective and goal of the mission that the designed launch vehicle shall fulfil are summarised in the following Mission Statement:

To be able to safely and successfully deploy a nanosatellite into a circular Sun-synchronous orbit around Earth.

A. Mission Objectives

The mission statement can further be broken down into two main objectives:

- Design a launch vehicle capable of being launched in mid-air from the JAS 39E/F Gripen fighter aircraft.
- Design a launch vehicle capable of transporting and deploying a nanosatellite into a circular Sun-synchronous orbit at 500 km altitude.

B. Mission Requirements

The requirements of the mission were based on the capabilities of the JAS 39E/F Gripen fighter aircraft (see Fig. [1]) and the Taurus KEPD 350 air-to-surface missile system, and were further modified based on interviews held with several employees at Saab Aeronautics and Dynamics, including a retired fighter pilot of the JAS 39 Gripen.

The resulting requirements are summarised in Table I and stated in the following bullet list:

- The launch vehicle shall be launched at an altitude of 10 km under an initial flight path angle of 45 degrees at a speed of Mach 0.95.
- The launch vehicle including payload shall not exceed a maximum lift-off mass of 1,300 kg.
- The launch vehicle shall not exceed a total length of 5 m nor a diameter of 1 m.

The initial design of the launch vehicle was based on the results presented by Sigvant [3] for a three-staged launch vehicle using a composite solid propellant consisting of: Ammonium Perchlorate (AP), powdered Aluminium (Al) and Hydroxyl-Terminated Polybutadiene (HTPB). The results are presented in Table II.

TABLE I
MISSION REQUIREMENTS

<table>
<thead>
<tr>
<th>Altitude (km)</th>
<th>Velocity (Mach)</th>
<th>Flight path angle (deg)</th>
<th>MLOM (kg)</th>
<th>Length (m)</th>
<th>Width (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.95</td>
<td>45</td>
<td>1300</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE II
INITIAL LAUNCH VEHICLE CHARACTERISTICS

<table>
<thead>
<tr>
<th>Stage</th>
<th>(P_i) (kN)</th>
<th>(t_{b,i}) (s)</th>
<th>(m_{s,i}) (kg)</th>
<th>(m_{p,i}) (kg)</th>
<th>Separation altitude (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>60.2</td>
<td>41.0</td>
<td>58.9</td>
<td>936.6</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>14.8</td>
<td>51.9</td>
<td>29.2</td>
<td>275.3</td>
<td>60</td>
</tr>
<tr>
<td>3</td>
<td>2.95</td>
<td>42.0</td>
<td>7.36</td>
<td>43.61</td>
<td>260</td>
</tr>
</tbody>
</table>

Fig. 1. JAS 39E Gripen fighter aircraft, credit Saab Aeronautics [4].

V. Rocket Module

A rocket propulsion system is man’s attempt to facilitate the power held within the chemical bonds of matter in order to leave the gravitational prison of Earth.
To accomplish this feat, three main areas need to be considered and designed for: the propellant with appropriate grain geometry (if solid propellant), the shell design and the nozzle design. These areas will be covered in the following sections.

1) Assumptions: In this study, the following assumptions were used (unless otherwise stated) for the working fluid in the nozzle flows:

- Chemical equilibrium in the combustion chamber with chemically-frozen composition everywhere in the flow.
- Steady, homogeneous, quasi-one-dimensional flow.
- Transient effects are short enough to be neglected.
- Optimum expansion at the design point.
- Adiabatic nozzle (i.e. no wall friction or transfer of heat).
- Ideal gas behaviour applies with constant specific heat at constant pressure, and a constant isentropic efficiency from inlet to exit.

A. Nozzle design - governing equations

The thrust \( F \) provided by a rocket propulsion system for one nozzle is given by the general thrust equation \( (1) \), which is derived from Newton’s second law and the conservation of momentum, assuming constant mass flow \( \dot{m} \), axial- and uniform exit velocity \( v_2 \) (for a complete derivation see earlier editions of [5]).

\[
F = \dot{m}v_2 + (p_2 - p_3)A_2 \quad (1)
\]

where \( p_2 \) is the internal pressure of the exhaust at the nozzle exit, \( A_2 \) is the cross-sectional area at the nozzle exit, and \( p_3 \) is the ambient pressure outside of the nozzle. A general supersonic converging-diverging nozzle is shown in Fig. 2 note that the subindex \( t \) will be used from now on to represent the throat of the nozzle.

![Fig. 2. Sketch of a general supersonic converging-diverging nozzle.](image)

The purpose of a nozzle is to increase the exit velocity of a fluid at the expense of pressure, thereby increasing the total thrust, as can be seen from eq. \( (1) \).

Looking at a control volume of a flowing fluid with a single inlet (subindex 1) and outlet (subindex 2), the relationship between the velocity \( v \), pressure \( p \), temperature \( T \) and mass flow \( \dot{m} \) for the inlet and outlet state respectively (and anywhere along the flow), can be derived by considering: thermodynamic steady-flows, conservation of mass, eq. \( (2) \), conservation of energy \( E \) (rate of energy \( E \), eq. \( (3) \)), the first law of thermodynamics, eq. \( (4) \) and the thermodynamic function of state, enthalpy \( h \) [6]. Acknowledging that energy can only be transferred in three ways: by heat \( Q \), work \( W \) and mass flow allows the four equations \( (2) - (5) \) to be stated \( (7) \) as:

\[
\dot{m}_{in} = \dot{m}_{out} \quad (2)
\]

\[
\dot{E}_{in} = \dot{E}_{out} \quad (3)
\]

\[
\Delta \dot{E} = \Delta \dot{Q} + \Delta \dot{W} + \dot{m}\Delta \theta \quad (4)
\]

\[
\Delta \theta = \Delta h + \frac{\Delta v^2}{2} + g\Delta z \quad (5)
\]

where \( \theta \) is the total energy of a flowing fluid per unit mass and \( g\Delta z \) is the potential energy difference resulting from the fluid’s position in Earth’s gravitational field.

Using eq. \( (2) \) which implies that the mass flow is constant and rewriting eq. \( (3) - (5) \) gives

\[
\Delta \dot{Q} + \Delta \dot{W} = \dot{m}(\Delta h + \frac{\Delta v^2}{2} + g\Delta z). \quad (6)
\]

Now, by implementing some valid and helpful assumptions for supersonic nozzles:

1) since the velocity of the flow is high, there is very little time for heat to be transferred through the boundaries of the control volume (the flow is adiabatic, \( \Delta \dot{Q} = 0 \)).

2) since it is a steady flow the flow does no work on an outer boundary (\( \Delta \dot{W} = 0 \)).

3) the potential energy difference is negligible in comparison to the enthalpy and kinetic energy. This allows eq. \( (6) \) to be written as

\[
0 = \Delta h + \frac{\Delta v^2}{2} \quad (7)
\]

and by using the definition of enthalpy for ideal gases \( \Delta h = C_p(T_2 - T_1) \) give that eq. \( (7) \) in turn can be written as

\[
v_2 = \sqrt{\frac{2C_p(T_1 - T_2) + v_1^2}{T_2}} \quad (8)
\]

where \( C_p \) is the specific heat at constant pressure. Eq. \( (8) \) can now provide a relation between temperature and velocity of the inlet and outlet state of the fluid. Normally, the inlet velocity \( v_1 \ll v_2 \) and the inlet kinetic energy is usually neglected [5]. However, if the combustion chambers cross-sectional area is less than four times the throat area, the increase of \( v_1 \) and the following pressure drop can not be ignored. For solid propellant, this also leads to erosive burning: causing positive feedback between chamber pressure and burn rate, which can lead to catastrophic results.

This now allows for the assumption that the inlet temperature and pressure \( T_1 \) and \( p_1 \) can be replaced with the stagnation temperature and pressure (subindex 01) \( T_{01} \) and \( p_{01} \) as internal conditions in the combustion chamber, which will be used from now on.

Isentropic flows are easy to calculate but are rarely found in reality, even for supersonic nozzles that are essentially adiabatic devices. Therefore, it is a good practice to implement the isentropic efficiency coefficient \( \eta_s \) [6], defined as: the actual kinetic energy received at the nozzle exit divided by the theoretical isentropic value of the kinetic energy at the nozzle exit,

\[
\eta_s = \frac{v_{2s}^2}{v_{2s}^2}. \quad (9)
\]
This ratio is usually above 90% \(^6\) and assumed to be constant throughout the flow.

In Fig. in the actual expansion of the fluid (shown in red) is drawn, displaying a higher exit temperature compared to an ideal isentropic expansion, corresponding to a lower exit velocity and kinetic energy.

Using the first law of thermodynamics, the ideal gas law, the definition of entropy and the ratio of specific heats \(\gamma\); the relationships in eq. \((10)\) can be derived for an isentropic expansion with the isentropic efficiency coefficient implemented

\[
\frac{T_{01}}{T_2} = \left(\frac{p_{01}}{p_2}\right)^{\frac{\gamma(\gamma-1)}{\gamma-1}}, \quad \left(\frac{\nu_2}{\nu_{01}}\right) = \left(\frac{p_2}{p_{01}}\right)^{-\frac{\gamma(\gamma-1)}{\gamma-1}} - 1
\]

where \(\nu_{01}\) and \(\nu_2\) is the inlet and outlet specific volumes (reciprocal of density) \(^6\). For the ideal case when \(\eta_s = 1\), the isentropic flow relations are given from eq. \((10)\).

\[\text{Fig. 3. Temperature } T \text{ and entropy } S \text{ diagram of the actual (red) and isentropic expansion process (black) for a supersonic nozzle.}\]

Following that the mass flow is constant throughout the control volume, equation \((2)\) can be re-written as:

\[
\dot{m} = \frac{A_2 \nu_2}{\nu_2} = \frac{A_1 \nu_1}{\nu_1}
\]

\[
\text{(11)}
\]

Thereby connecting: the velocity, specific volume and the cross-sectional area for any location in the flow.

By combining eq. \((8)\) with: the definition of Mach number

\[
M = \frac{v}{a} = \frac{v}{\sqrt{\gamma RT}}
\]

\[
\text{(12)}
\]

where \(a\) is the speed of sound in the medium, \(\hat{R}\) is the specific gas constant (universal gas constant divided by the molar mass of the gas), \(C_p\) and the assumption that \(v_{01}^2 \approx 0\) results in the relation between \(T, \gamma \) and \(M\) to be derived as

\[
T_2 = T_{01}\left[1 + \frac{\gamma - 1}{2} M_2^2\right]^{-1}
\]

\[
\text{(13)}
\]

Which can be further combined with eq. \((10)\) to give

\[
\nu_2 = \sqrt{\frac{2\gamma}{\gamma - 1}} \frac{\hat{R} T_{01}}{\gamma - 1} \left[1 - \left(\frac{p_2}{p_{01}}\right)^{\frac{\gamma(\gamma - 1)}{\gamma - 1}}\right].
\]

\[
\text{(14)}
\]

Finally, \(\dot{m}\) is given from eq. \((1)\) by designing the nozzle to achieve optimum expansion towards the ambient pressure \((p_2 = p_3)\) \(^5\)

\[
\dot{m} = \frac{F}{v_2}
\]

\[
\text{(15)}
\]

This now allows for \(A_2\) to be calculated for a specific exit pressure \(p_2\), by first solving for: \(v_2, \dot{m}, T_2\) and \(\nu_2\), using eq. \((14), (15), (13)\), the ideal gas law and eq. \((11)\).

The same calculations can be simplified and used to get the throat conditions by using the fact that the Mach number \(M_t = 1\) at the throat, due to choking of the flow.

1) Performance parameters: In order to compare different nozzles and propulsion systems, three main performance parameters are commonly used: characteristic velocity \(c^*\), thrust coefficient \(C_F\) and specific impulse \(I_{sp}\).

The characteristic velocity is given by

\[
c^* = \frac{p_{01} A_t}{\dot{m}}
\]

\[
\text{(16)}
\]

and is independent of the nozzle characteristics, representing the efficiency of the combustion process.

The thrust coefficient is a dimensionless quantity, which for optimum expansion is given by

\[
C_F = \frac{F}{p_{01} A_t}
\]

\[
\text{(17)}
\]

and represents the amplification of thrust provided by the diverging section of the supersonic nozzle.

The specific impulse has the unit s and is given by

\[
I_{sp} = \frac{\int_0^t F \, dt}{g_0 \int_0^t \dot{m} \, dt}
\]

\[
\text{(18)}
\]

representing the thrust provided per unit of propellant weight, which is dependent on both the combustion- and nozzle characteristics. A common analogy is to compare the concept of litres per km used for automobiles \(^5\).

Since the ambient pressure drops as the LV gains altitude, means that the nozzle quickly becomes under-expanded, resulting in a lower \(C_F\). Hence, it is common to design the nozzle for a slight over-expansion. Here it is important not to over-expand too much as this can result in flow separation. To avoid this, a common rule of thumb as presented by Summerfield \(^8\) is to design for an ambient pressure 40 % lower the actual value.

B. Contoured Nozzle

Using the governing equations for a supersonic nozzle, as derived in the previous section, it is possible to calculate the required cross-section area of a nozzle. The length of the nozzle directly corresponds with the selected shape where there are mainly two types used today: conical- and contoured nozzles. The conical nozzle is an old and simple nozzle configuration, used in many applications due to its simplicity and low manufacturing cost. However, one large drawback, as noted by Sigvant \(^3\), is the required length of the nozzle as the ambient pressure reduces at higher altitudes.
Contoured nozzles, often called Rao-, bell- or de Laval nozzles, are the most commonly used nozzle today, as they allow for a more optimal gas expansion. This is because they minimise the energy losses compared to conical nozzles: having a high initial expansion angle followed by a gradual reversal of the contour slope so that the exit angle of the nozzle is small. This way, the exhaust is more effectively expanded uni-axially towards the nozzle exit, resulting in a higher $C_F$.

For this reason, a contoured parabolic (or bell) nozzle was used in this study.

To account for the loss due to the exhaust not being completely axial at the nozzle exit, the correction factor $\lambda$ used in this study.

Brooks [9] as $\lambda$ nozzle, an approximate formula to calculate introduced in the momentum term of eq. (1). For a contoured $C$ uni-axially towards the nozzle exit, resulting in a higher $\frac{\alpha}{\theta}$ small. This way, the exhaust is more effectively expanded

reversal of the contour slope so that the exit angle of the nozzle having a high initial expansion angle followed by a gradual

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was then given by

$\frac{\alpha}{\theta}$

angle. The radius $r$ slope at the exit consisting of the tangent of the selected exit

parabola: the throat radius and exit radius were used with the

a zero degrees slope at the throat. While for the diverging

sections were calculated by fitting a second degree polynomial

parabolic contour for two different exit angles, credit Sutton [5] (modified).

Fig. 4. Shows a conical nozzle along with a general Rao bell-nozzle with parabolic contour for two different exit angles, credit Sutton [5] (modified).

The parabolic shape of the converging- and diverging nozzle sections were calculated by fitting a second degree polynomial to two radii and one angle constraint. For the converging parabola: the chamber radius and throat radius were used with a zero degrees slope at the throat. While for the diverging parabola: the throat radius and exit radius were used with the slope at the exit consisting of the tangent of the selected exit angle. The radius $r_d$ of the diverging parabola (subindex $d$) was then given by

\[
r_d(z) = r_t + \left[ \frac{2(r_2 - r_t)}{L_d} - \tan(\theta_2) \right] \frac{z}{2} + \left[ \frac{(r_t - r_2)}{L_d^2} + \frac{\tan(\theta_2)}{L_d} \right] \left( \frac{z}{L_d} \right)^2 \tag{20}
\]

where $z$ is zero at the throat and $L_d = 0.8L_{cone}$ (see Fig. 4 for the definition of $L_{cone}$).

1) Constrained Nozzle: It is more common than not that an outer constraint on the rockets overall dimensions, be it in mass or size, results in a dimensional constraint for the nozzle. In that case, the desired expansion might not be possible, which results in a higher exit pressure and temperature and lower exit velocity of the exhaust. This results in a lower thrust than desired; since thermal energy is left in the exhaust that has not yet been optimally converted into kinetic energy by the expansion. The increase in the pressure term in eq. (1) is not enough to compensate for the loss in kinetic energy (exit velocity) which has a larger contribution from the momentum term in eq. (1).

One way to design around this is to increase the size of the throat of the nozzle (getting a lower expansion ratio) which increases the mass flow through the nozzle. Thereby, gain in a larger contribution of the momentum term compared to what would have been obtained from using a truncated nozzle designed for optimum expansion with a larger exit area. However, it should be noted that using this approach it is not possible to get a higher thrust than designed for using optimal expansion with a unconstrained nozzle.

By using an iterative approach: where the new constrained expansion ratio $\epsilon = \frac{\lambda_{constrained}}{\lambda_{cone}}$ is used to get the new Mach number of the exhaust for the constrained nozzle, which then is used to calculate the new thermodynamical state of the exhaust, compare the loss in thrust to the designed thrust value, and lastly increase the throat area by an equivalent amount and iterate once more, it is possible to get a thrust close or equal to the desired thrust value, depending on the selected tolerance used in the iteration.

The new Mach number ($M_2$) results from the new expansion ratio by solving eq. (21) numerically,

\[
\epsilon = \frac{1}{M_2} \left[ \frac{2}{\gamma + 1} \left( 1 + \frac{\gamma - 1}{2} \frac{M_1^2}{\epsilon} \right) \right]^{\frac{1}{\gamma - 1}} - \frac{1}{2} \tag{21}
\]

which can be derived from eq. (10), (11), (13) using the fact that the flow has a sonic velocity at the throat, $M_1 = 1$. See Appendix C for the pseudo-code describing this iterative constrained nozzle method.

2) Nozzle casing: Since the exhaust is being rapidly compressed and expanded in a supersonic converging-diverging nozzle, the pressure is also being drastically increased and reduced in the process. Additionally, since the shape of the chosen nozzle has a parabolic contour, it is not trivial to calculate the internal pressure at every cross-section area section of the nozzle.

To estimate the required thickness of the nozzle, the nozzle was divided into infinitesimally small cylinders, which then could be used together with the hoop stress equation for thin-walled cylinders to approximate the required thickness.

Using the fact that the mass flow through a cross-section is constant eq. (11), along with the relations for isentropic expansion eq. (10), velocity of the gas eq. (14) and the ideal
gas law, provided eq. (22) to be derived as

\[
\frac{T_0 \dot{m}}{A_y} = p_y \left( \frac{p_0}{p_y} \right)^{\frac{\gamma-1}{\gamma}} \sqrt{\frac{2\gamma RT_0}{\gamma - 1}} [1 - \left( \frac{p_0}{p_y} \right)^{\frac{\gamma}{\gamma}}]^\gamma
\]

(22)

where \( p_y \) represents the local pressure at location \( y \) in the flow and can be solved using numerical methods by calculating \( A_y = \pi r_y^2 \), where \( r_y \) is given from eq. (20).

Lastly, the mass of the nozzle can be estimated by using the density of the chosen material and integrating the outer and inner radius using the disk method and the wall thickness.

3) Nozzle insulation: Even though the flow is expanded rapidly in a contoured nozzle, the flame temperature can still be between 1000 - 3000 K [5], which is over many metals melting temperatures. One method is to implement cooling channels in the nozzle walls. However, this is only applicable for liquid-propellant engines.

For solid-propellant motors, using some type of insulation is the preferred method. Consequently, the choice of insulation is essential to prevent the nozzle from melting and rupture due to the extreme heat and loads.

The most common way to cool down nozzles and exposed surfaces in the combustion chamber for solid rocket motors is to use ablative materials [5]. These absorb the heat from the hot exhaust gases by undergoing endothermic degradation, turning into char and pyrolysed gas while creating a fuel-rich and protective boundary layer over the inner surface of the nozzle in the process. This implies that the thickness of the insulation depends on the burn time of the motor. Char, consisting almost completely of carbon can withstand temperatures up to 3500 K [5].

For all nozzles in this study, an integral throat/entrance fixed nozzle was chosen consisting of carbon-carbon using an erosion rate of \( r_e = 0.038 \text{ mm/s} \) and a density of 1700 kg/m\(^3\) [5]. The required insulation thickness can be calculated by

\[
t_{ins} = f r_e t_b
\]

(23)

where \( t_b \) is the burn time and \( f \) is a safety factor.

C. Grain design - governing equations

In order to get the internal combustion pressure, temperature and mass flow required to propel the rocket for the given mission constraints (mainly thrust and burn time), the internal ballistics: burn rate \( r_b \), burning surface \( A_b \) and grain geometry needs to be considered and designed for [9].

This usually requires an iterative approach, going back and forth between the outer ballistic requirements and the grain geometry selection and stress & failure analysis of the grain envelope. Burn rate is often the most difficult internal ballistic variable to determine. One common method is to perform sub-scale motor tests, using de Saint Robert’s burning law

\[
r_b = a p_0 n_0, \]

which shows the dependency between \( r_b \) and the pressure in the combustion chamber, where \( a \) and \( n \) are constants that are evaluated empirically [9]. Here \( r_b \) will be assumed to be constant due to the assumptions of constant steady flow in the nozzle and chemical equilibrium in the combustion chamber.

The independent variables that drive the grain design are the mission constraints and the propellant properties: density, specific impulse and burn rate.

Using the independent outer ballistic constraints: \( F \) and \( t_b \), the required mass flow was calculated using eq. (11) which can be used to calculate the required propellant mass \( m_p \) by

\[
m_p = \dot{m} t_b (1 + EF)
\]

(24)

taking into account some extra fuel (EF) \( 0 \leq EF \), to account for transient effects and potential residual slivers.

The burning surface area \( A_b \) and web thickness \( w \) required to provide the required mass flow can be calculated using the given propellant properties by

\[
A_b = \frac{\dot{m}}{\rho_p p_b}
\]

(25)

\[
w = r_b t_b.
\]

(26)

Continuing by combining eq. (25) together with eq. (1) and the definition of specific impulse eq. (18) for steady- and uniform flow, yields (under optimal expansion) that the thrust can be expressed as

\[
F = \dot{m} g_0 I_{sp} = A_b \rho_p r_b g_0 I_{sp}
\]

(27)

showing that \( F \) is directly proportional to both \( A_b \) and \( r \). This fact gives rise to two ”good practise” constraints that are commonly used in the ballistic industry: the burning surface area at burnout should not be below 30 % of the initial value, or above 20 % at any point before that. The motivation for this is to make sure that the pressure does not build up too high inside of the chamber, or too low which could extinguish the burn.

Lastly, the geometrical constraint must be followed and the maximum radius \( R_{max} \) of the solid-propellant grain is easily calculated by

\[
R_{max} = \frac{D_{out}}{2} - t_{ins} - t_{shell}
\]

(28)

where \( D_{out} \) is the outer diameter constraint (see Table 1), \( t_{ins} \) is the required insulation thickness and \( t_{shell} \) the required thickness of the combustion chamber. The radius of the grain \( R_{grain} \) must fulfill \( R_{grain} \leq R_{max} \).

1) N-star perforation: The N-star, radial-burning 2-D grain configuration was investigated due to: its constant thrust-to-time profile, being case bounded (protecting the chamber walls from high temperatures and erosion) and having frequent use in the space industry with a long heritage history. One drawback is the existence of slivers, however, these can be reduced by modifying the star configuration using the seven independent variables, as seen in Fig. 5.

The neutral thrust profile is given by the regressive-burning star wedges being balanced out by the progressive-burning tube [9], see Fig. 6.

The burn through zone 1 is progressive and depends on \( r_2 \). Since \( r_2 \) is usually small, the burn through zone 1 is rapid and the majority of time spent of the burn is in zone 2. Therefore,
the focus of the design should be on zone 2 and limiting the progressive end burn in zone 3 and residual slivers in zone 4.

To eliminate any progressive burning after zone 2 and ensure minimal sliver, requires setting \( w \) equal to \( Y^* \) (see Fig. 5), which eliminates burning in zone 3 [9]. Using trigonometry relations and Fig. (5), allows \( Y^* \) to be expressed as

\[
Y^* = (R - w - r_1) \frac{\sin \xi}{\cos \eta} \tag{29}
\]

which further allows \( \xi \) to be calculated for by setting \( Y^* = w \), yielding

\[
\xi = \arcsin \left( \frac{w + r_1}{R - w - r_1} \cos \eta_0 \right) < \frac{\pi}{N}. \tag{30}
\]

The burning perimeter \( P_b \) for zone 2 can be derived by adding up the sum of the arcs and sides of the star perforation (note all angels are in radians), using the fact that the angle \( \alpha \) from Fig. [5] can be expressed as \( \alpha = \frac{\pi}{2} - \eta + \xi \) gives

\[
P_b = 2N \left[ (R - w - x)\left( \frac{\pi}{N} - \xi \right) + (R - w - r_1) \frac{\sin \xi}{\sin \eta} \right] + (r_1 + x)(\frac{\pi}{2} - \eta + \xi) - (r_1 + x)\tan(\frac{\pi}{2} - \eta) \right]. \tag{31}
\]

Differentiating \( P_b \) with respect to the burnt distance \( x \) normal to the burning surface gives

\[
\frac{dP_b}{dx} = 2N \left[ \frac{\pi}{2} - \eta - \frac{\pi}{N} - \tan(\frac{\pi}{2} - \eta) \right]. \tag{32}
\]

Now, by setting eq. (32) equal to zero gives a constant burning perimeter, resulting in a constant burning area \( A_b \), which gives a constant thrust and burn profile. This gives the following non-linear equation (using \( \eta_0 \) to emphasise the initial perforation of the grain)

\[
\eta_0 + \tan(\frac{\pi}{2} - \eta_0) = \frac{\pi}{N} + \frac{\pi}{2}. \tag{33}
\]

So, by choosing \( N \), \( r_1 \) and \( r_2 \) it is therefore possible to use eq. (33) and eq. (30) to solve for \( \eta_0 \) numerically.

The initial port area \( A_p \) of the grain can be derived by considering Fig. [5] and using common trigonometry as

\[
A_p = N \left[ (R - w)\left( \frac{\pi}{N} - \xi \right) + (R - w - r_1) \frac{\sin \xi}{\sin \eta} \right] + w^2(\frac{\pi}{2} - \eta + \xi) + 2r_1(R - w - r_1)\frac{\sin \xi}{\sin \eta} - r_1^2\tan(\frac{\pi}{2} - \eta) \right]. \tag{34}
\]

To prevent erosive burning and risk of choking the flow leading to critical pressure build-up, \( A_p \) should be four times larger than the cross-section area of the throat \( A_t \).

Lastly, the length of the grain \( L_{\text{grain}} \) can be calculated using

\[
L_{\text{grain}} = \frac{A_b}{P_b}. \tag{35}
\]

2) Internal-burning tube: The internal-burning tube, see Fig. 7 is a simple and powerful design that is suitable for a neutral burn profile when the length over diameter fraction \( L/D \) of the grain is less than two, and the web fraction \( w_f = 2w/D \) is between 0.5 – 0.9 [9].

It also posses the characteristic of not producing any residual slivers, assuming that the final pressure does not drop too low, which can extinguish the burn.

This type of grain formation also possesses the advantage that it reduces production costs since it can be more easily manufactured in a case-bounded fashion than e.g. the N-star perforation.

For an internal-burning tube, the end surfaces can either be inhibited or not. The latter helps reduce the progressive burn caused by the internal radial burning grain (since the burning surface area increases as the web burns towards the outer rim).

The burning surface \( A_b \) can be easily formulated as a function of burnt web distance \( w_z \), shown by Brooks [9] as

\[
A_b(w_z) = \pi(d + 2w_z)(L - nw_z) + \pi\frac{N}{4} \left[ D^2 - (d + 2w_z)^2 \right]. \tag{36}
\]
where \( n \) represents the number of open-ended surfaces of the internal-burning tube.

The length \( L \), outer diameter \( D \) and inner diameter \( d \) of the grain can be calculated by: 1) setting the internal-burning tube’s volume equal to the required propellant volume. 2) setting the internal-burning tube’s burning area equal to the required burning area. And 3) setting the web thickness to match the required burn time. This then results in three equations \((37) - (39)\):

\[
A_b = \frac{\dot{m}}{\rho_p r_b} = n\frac{\pi}{4}(D^2 - d^2) + \pi dL \tag{37}
\]

\[
V = \frac{m_p}{\rho_p} = \frac{\pi}{4}(D^2 - d^2)L \tag{38}
\]

\[
w = t_b r_b = \frac{(D - d)}{2} \tag{39}
\]

with the three unknowns: \( L, D \) and \( d \) that can be solved for, while fulfilling the constraints of fitting inside of the envelope of the LV (see Table 1).

To prevent choking the flow, the port area (the internal-radial burning grain) must fulfil \( A_P \geq 4A_b \) to make sure that all the gases produced upstream has enough internal space to expand and exit the chamber through the nozzle. Thus, avoiding any erosive burning and rapid increase of chamber pressure.

Having extra voids at the forward and or aft section, before the nozzle, acts as a buffer towards erosive burning, at an expense of lower volumetric loading fraction, \( V_f \): which is the propellant volume divided by the available volume in the chamber.

To get the initial area equal the final area of the grain: equal to having the same initial- and final thrust values (neglecting the pressure term in \((1))\), \( L/D \) needs to fulfil

\[
\frac{L}{D} = \frac{4 - w f}{2}. \tag{40}
\]

Further, the maximum value of \( A_b \) is given by setting the first derivative \( \frac{dA_b}{dw_x} \) equal to zero, and solving for \( w_x \) which yields

\[
w_x = \frac{L - nd}{3n} \tag{41}
\]

which is a maximum point due to the second derivative being negative: \( \frac{d^2A_b}{dw_x^2} = -6n\pi \).

The burning surface area of an internal-burning tube is therefore a parabola with a maximum point that depends on \( L, d \) and \( n \). The final burning surface area can either be larger or smaller than the initial burning surface area, depending on the independent variables: \( r_b, \rho_p \) (given from the choice of propellant), \( t_b \) and \( F \) that are given from the outer ballistics (trajectory analysis).

D. Solid propellant

The choice of propellant is a critical factor in the final performance of the solid rocket motor, not only deciding the burn rate, specific impulse and size of the fuel grain but also the thrust-to-time profile. Figure 8 shows how different types of solid propellants specific impulse vary with the burn rate.

Since a typical solid propellant contains around 4 to 12 different ingredients, a full thermochemical analysis of the propellant is considered to be outside of the scope of this study.

Several software tools exist that can solve large sets of chemical equation systems; involving mass and energy balances of the propellant composition that together with thermodynamical states and the use of chemical equilibrium, most commonly minimising Gibbs free energy, allows for several important gas-mixture parameters to be calculated. Such as the molar mass of the gas mixture \( M_p \), the average molar specific heat \( C_p \), the specific heat ration \( \gamma \), which in turn allows for \( c^* \) and \( R \) to be calculated along with other performance parameters. One common and popular software tool is CEARUN by NASA [10].

1) Composite solid propellant: In Sutton [5], already calculated results for a conventional solid composite propellant are presented, more specifically for an Aluminized Ammonium Perchlorate Composite Propellant, consisting of: 60 % ammonium perchlorate \( \text{NH}_4\text{ClO}_4 \), 20 % aluminium powder and 20 \% of an organic polymer \( \text{C}_{3.1}\text{ON}_{0.86}\text{H}_{3.8} \), which was calculated using shifting equilibrium at a standardized combustion
pressure of 1000 psi (6.985 MPa) for optimum expansion at sea-level. Using shifting equilibrium instead of a frozen flow approximation makes more enthalpy available for conversion to kinetic energy, thus providing a higher specific impulse, combustion efficiency and exit flame temperature. A shortened collection of the performance parameters are displayed in Table [III]

<table>
<thead>
<tr>
<th>TABLE III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Performance Parameters for an Aluminized Ammonium Perchlorate Composite Propellant [5]</td>
</tr>
<tr>
<td>Chamber pressure (MPa)</td>
</tr>
<tr>
<td>Chamber temperature (K)</td>
</tr>
<tr>
<td>Exit molecular mass (kg/mol)</td>
</tr>
<tr>
<td>Specific heat ratio</td>
</tr>
<tr>
<td>Propellant density (kg/m$^3$)</td>
</tr>
<tr>
<td>Specific impulse, sea-level expansion (sec)</td>
</tr>
</tbody>
</table>

The advantages of a composite propellant are the high specific impulse, high density, wide burn rate control, stable combustion, good storage stability, long experience heritage and class 1.3 hazard with a medium-cost. The disadvantages are that they tend to be moisture sensitive with toxic and smoky exhaust. For the propellant used in this study, the three largest parts of the exhaust plume consists of: 32.4 % H$_2$, 22.4 % CO and 12 % HCL [5]. For a full version of the mole fractions of the exhaust gases, the reader is referred to Appendix D.

To get a real-valued solution to the equation system (37) - (39) that provides a satisfactory thrust-to-time profile, the burn rate $r_g$ and propellant density $\rho_p$ need to be chosen accordingly. These can be tailored in the manufacturing process of the propellant and depend mainly on: the particle sizes and distribution, the ingredients, and lastly, the initial pressure and temperature in the combustion chamber.

For composite propellants, it is common that $r_g$ can be made to vary between: 7 to 20 mm/s with a density between: 1750 to 1810 kg/m$^3$ [5]. A higher density is the desired quality since it reduces the required volume of the grain and the structural mass fraction.

Alternatively, $r_g$ can be of a higher value and $A_b$ can be increased by adding: a slotted tube, finocyl or conocyl configuration.

E. Combustion chamber

After the solid-propellant grain has been calculated and designed for, according to the mission requirements and outer ballistics, the combustion chamber can be designed, along with calculating its mass after a selection of material for the shell and insulation have been made.

There are many external loads and stresses that need to be considered in designing a LV: vibrations, shocks, aerodynamic forces, axial forces resulting from the thrust etc. In this study, only the internal pressure will be considered in calculating the required shell thickness, assuming no bending and that all loads are in tension.

The geometry of the solid-propellant combustion chamber consisted of: a simple cylinder with a semi-ellipsoidal head on the forward end and a converging hyperboloid at the aft end, closing into the nozzle throat.

1) Insulation: One of the advantages of using solid propellant is that the propellant itself acts as an insulator: due to it being a poor conductor of heat. If the grain is completely internally burning, one can afford to use less insulation on the interior of the shell structure. For a grain with open ends, more insulation is required. To simplify the analysis, the insulation thickness was assumed to be uniform at the interior of the combustion chamber. This is usually not ideal since the insulation has to be thicker at locations with higher flow velocities and temperatures, e.g. at the nozzle throat and inlet area. The estimated insulation thickness was again calculated using eq. (23).

Lastly, the insulation material chosen for the interior of the combustion chamber was the same as for the nozzle insulation.

2) Shell thickness: Considering the load exerted on the shell by the maximum expected internal operating pressure. The required shell thickness of a cylinder $t_{cyl}$ can be evaluated by considering the hoop stress (or tangential stress) equation, given by

$$t_{cyl} = \frac{j}{\sigma} \frac{p_01 R_i}{R_c - R_i}$$  (42)

where $\sigma$ is the tensile strength of the material chosen, $p_01$ the internal pressure and $R_i = R_{grain} + t_{ins}$.

There are several options of end-closing heads to choose from for a pressure vessel: spherical, semi-ellipsoidal, tori-spherical, conical and flat-ended. In this study, the semi-ellipsoidal configuration was selected due to it having a shorter height than the spherical as well as not possessing as large discontinuity stresses that are present in the torispherical head configuration [11].

The required thickness for the semi-ellipsoidal head $t_{head}$ was calculated using the tangential stress equation for the point at the centre of the head, where the tangential and longitudinal stress are equal, given by Bednar [11] as

$$t_{head} = \frac{j}{\sigma} \frac{R_i^2 - h^2}{2\pi h}$$  (43)

where $R_i$ is the major axis and $h$ the minor axis. Note that when $h$ limits towards $R_i$, eq. (43) yield the hoop stress equation for a sphere. Thus, there is a small penalty in increasing wall thickness by using a semi-ellipsoidal head instead of a spherical head, which is somewhat counterbalanced with the reduced surface area.

The shape of the outer surface of the semi-ellipsoid is given in three dimensions by

$$1 = \left( \frac{x}{R} \right)^2 + \left( \frac{y}{R} \right)^2 + \left( \frac{z}{h} \right)^2$$  (44)

where $h$ is the height of the semi-ellipsoid and $R = R_{grain} + t_{ins} + t_{cyl}$.

3) Shell mass: To calculate the mass of the shell the wall volume had to be estimated. This was done by using eq. (43) for the cylindrical shell.
\[ V_{cyl} = L \pi \left( \frac{D_{out}}{2} \right)^2 - (R + t_{ins})^2. \]  

(45)

The wall volume of the semi-ellipsoidal head was calculated by using the volume formula for an ellipsoid, dividing it by two to get the volume for a semi-ellipsoid, then calculating the volume of the outer surface and subtracting the volume of the inner surface. Yielding the wall volume of the head as

\[ V_{head} = \frac{2\pi}{3} (hR^2 - h_i R_i^2) \]  

(46)

which finally can be multiplied with the density of the chosen wall material to provide the mass.

4) Shell material: For cylindrical solid rocket motors, the most common design choice for the shell is to use: a steel monolithic (one-piece steel case), fibre monolithic or segmented shell design [5]. For this study, the one-piece case design was used for the shell.

Two different materials were considered for the shell of the combustion chamber: Aluminium alloy 7068 and Stainless steel SAE 300M. Another possible material with high tensile strength and lower density than steel is Titanium. However, due to the amount of material required and the price of titanium, it was not considered in this study, as it would increase the total cost of the expendable launch vehicle considerably. Their respective physical properties can be seen in Table IV.

### Table IV: Physical properties of materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Tensile Strength [N/mm²]</th>
<th>Density [kg/m³]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminium alloy 7068</td>
<td>655</td>
<td>2850</td>
</tr>
<tr>
<td>ASTM/SAE 300M</td>
<td>1900</td>
<td>7840</td>
</tr>
<tr>
<td>S-glass epoxy (45%)</td>
<td>870</td>
<td>1800</td>
</tr>
</tbody>
</table>

F. Third stage

To perform a successful orbit insertion, the third stage should be of high performance to reduce weight and have a lower thrust compared to the first and second stage since the mass of the stage is much lower, thereby keeping the acceleration down.

Since the atmosphere is close to non-existing at higher altitudes, low-thrust propulsion for a longer period can be implemented since the aerodynamical drag is almost none. Additionally, there is no constraint on the orbital insertion time which could otherwise pose as a constraint on the required thrust. This allows for higher accuracy in the orbit injection compared to having a short burn time with a high thrust. Additionally, the third stage should also be throttleable and restartable with thrust vector control to take into account potential orbit corrections.

1) Thruster: As a result of the above-mentioned reasons, the Swedish made 200N High-Performance Green Propulsion (HPGP) thruster from Bradford ECAPS [12] for orbit-raising applications was selected (see Fig. 9), which at the time of writing has a technology readiness level of 3. The physical- and targeted life performance properties of the HPGP thruster are displayed in Table V.

2) Feed system: To inject the propellant into the combustion chamber a feed system has to be chosen. Two types are common amongst liquid-propellant engines: pressurized feed systems and turbopumps. For low values of total impulse, a pressurized feed system provides better results since it: reduced system complexity and has a higher reliability [5]. Therefore, a pressurized feed system was used. A simplified schematic of a pressurized feed system is shown in Fig. 10.

<table>
<thead>
<tr>
<th>Expansion Ratio</th>
<th>150:1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thrust range</td>
<td>55 - 220 N</td>
</tr>
<tr>
<td>Specific Impulse</td>
<td>253 s</td>
</tr>
<tr>
<td>Length</td>
<td>390 mm</td>
</tr>
<tr>
<td>Structural Mass</td>
<td>6 kg</td>
</tr>
<tr>
<td>Propellant Throughput</td>
<td>25 kg</td>
</tr>
<tr>
<td>Propellant density</td>
<td>1240 kg/m³</td>
</tr>
<tr>
<td>Firing Time</td>
<td>400 s</td>
</tr>
</tbody>
</table>
Fig. 10. Simplified schematic of a general pressurized liquid monopropellant feed system, credit Sutton [5] (modified).

3) Gas- and propellant tank: A simple analysis was made regarding the size of the gas- and propellant tank required to feed the propellant into the combustion chamber. To simplify the analysis the expansion process was assumed to be adiabatic, which is a good approximation for burn times around a few minutes, moreover, ideal gas behaviour with no evaporation, sloshing or vortexing of the liquid-propellant was assumed [5]. The pressurized gas chosen was helium which is the lightest of the inert gases, having a specific heat ratio \( \gamma = \frac{5}{3} \).

By combining the ideal gas law with mass continuity between the gas- and propellant tank for a reversible adiabatic (isentropic) expansion yields

\[
p G V_G^\gamma = p_p (V_G + V_p)^\gamma
\]

where the subscript "G" indicates the initial condition of the gas tank and "p" the propellant tank. Solving eq. (47) for the volume \( V_G \) yield

\[
V_G = \frac{V_p}{\left(\frac{p_G}{p_p}\right)^{\frac{1}{\gamma}} - 1}
\]

which then gives the required gas mass as

\[
m_G = \frac{p_G V_G}{\hat{R} T}
\]

where \( \hat{R} = 2079 \) J/kgK for helium and \( T \) is the ambient temperature.

To reduce the structural mass of the third stage, spherical shells were used for the tanks consisting of ASTM/SAE 300M (see Table [IV], which has the property of being compatible with both helium and LMP-103s [12]. The required wall thickness of a sphere is half of what is required for a cylinder (see eq. (42)). Lastly, the mass of the tank can be calculated by multiplying the surface area of the sphere with the required wall thickness (since thin-walled conditions apply). All piping and auxiliary structures were assumed to be included in the structural mass of the HPGP thruster.

G. Nose cone

The purpose of a nose cone is to minimize the aerodynamical drag, mainly pressure drag which is proportional to the square of the relative velocity of the incoming airflow that a body travelling through compressible fluids experiences. To know the frictional drag, which is the second part comprised in the aerodynamical drag, one would have to know the complete LV configuration with the roughness of the wetted surface area. This is usually small compared to the pressure drag, therefore, no frictional drag was assumed.

1) Shape: Several types of nose cones exist: conical, bi-conical, power series, tangent ogive, secant ogive, elliptical, parabolic and Haack series; some are displayed in Fig. [11]. The Haack series type stands out among the others by not being derived from a geometric shape but derived mathematically to minimize drag [13].

Fig. 11. Comparison of drag characteristics of various nose configurations in the transonic Mach region. Rankings range from superior (1), good (2), fair (3) and inferior (4), credit Crowell [13].

Since the launch vehicle is being launched close to Mach 1 at high altitude and will thereafter quickly increase its velocity, reaching its highest aerodynamical force within seconds, the Von Karman Ogive configuration was chosen for the nose cone, which belongs to the Haack series family. The general shape of the Haack series is given by

\[
y = \frac{R}{\sqrt{\pi}} \sqrt{\theta - \frac{\sin(2\theta)}{2} + C\sin^3\theta}
\]

\[
\theta = \arccos\left(1 - \frac{2x}{L}\right)
\]

where \( 0 \leq x \leq L \) is the distance measured from the tip of the nose cone, \( R \) is the radius at the base of the nose cone, \( L \) is the length and \( C \geq 0 \) is a constant. Depending on the choice of \( C \), there is a continuous set of Haack shapes available. Setting \( C = 0 \) yields the Von Karman Ogive.

The specific drag coefficient \( C_d \) can normally not be determined analytically. The most common method to determine \( C_d \)
is to perform sub-scale wind tunnel experiments. Furthermore, the fitness ratio \( L/D \) also affects the drag significantly at supersonic speeds. However, below Mach 0.8 the pressure drag is essentially zero independent of the nose cone shape [13].

2) Material: The material chosen for the nose cone was glass fibre composite since it is a strong- and lightweight material that has the advantage of being transparent to radio signals. Thus, allowing for potential communication- and guiding instruments to be situated there. Further, a thickness of 1 mm was assumed, which is a common thickness on the wings for many types of glider planes.

3) Aerodynamic effects: The aerodynamical effects that the nose cone will have to be able to withstand as it ploughs the way forward through the atmosphere is the dynamical pressure which is given by

\[
Q = \frac{1}{2} \rho r^2 \text{rel} \tag{52}
\]

along with the aerothermal flux \( q_f \) or heat rate

\[
q_f = \frac{1}{2} \rho r^3 \text{rel} SC_f \tag{53}
\]

where \( S \) is the cross sectional surface area and \( C_f \) is the mean friction coefficient. It is first when the aerodynamical pressure and heat rate are sufficiently low that the fairing can be jettisoned.

VI. Trajectory Module

As noted earlier, one major driver for a rocket design is the outer ballistics, namely what trajectory will be implemented to go from the initial position to the final targeted orbit. The mission requirements often put some additional limits on the ballistic performance; e.g. constraints on maximal acceleration and ascent time. However, three crucial variables determine the trajectory and are dominant factors in the grain design. Namely: Thrust, burn time (alternatively impulse) and the steering law for each stage. So, to arrive in a complete rocket design, an integrated ballistic analysis needs to be performed and ideally optimized [9].

To accurately model an ascent trajectory, one first needs to make an: idealization, select an appropriate frame of reference and derive the governing dynamical equations that are consistent with the idealization and the chosen reference frame [14].

The trajectory problem to be solved is a boundary value problem where the initial and final state is known, but the steering law that steers between the initial and final position is unknown. If the solution would be known for all the internal points the problem reduces to an initial value problem that can be easily forward propagated using e.g. Runge-Kutta integration to solve the governing dynamical equations: consisting of ordinary differential equations.

This was the first approach for this trajectory optimization where the steering law for the thrust of the rocket was assumed to follow a Sigmoidal function.

The second approach was to solve the boundary value problem by transcribing the trajectory problem into a nonlinear program where a parametric steering law was derived using a local gradient search algorithm, which is covered in section VI-D Optimization.

A. Dynamical model

1) Idealization: The main idealizations (or assumptions) used in the trajectory problem are summarized in the following bullet list:

- Particle idealisation (point-mass) - no torques or moments
- Ascent and one orbit propagation
- Zero bank angle, \( \mu = 0 \)
- Side slip force is zero
- Constant drag coefficient, \( C_d = 0.25 \)
- Zero-lift, \( L = 0 \)
- Aspherical gravitational field, \( J_2 \)
- Two-body system (Earth and LV)
- Exponential atmospheric model
- Constant thrust and mass flow

2) Frame of reference: For high-altitude trajectories, it is common to use an Earth-Centered, Earth-Fixed reference frame (ECEF) to express the position of the LV. E.g. such as the International Terrestrial Reference Frame (ITRF) which is a pseudo-inertial reference frame where the small noninertial effects can be assumed to be negligible during the propagation time of only one orbit [14], [15].

To express the velocity of the LV it is common to use the Body-fixed Local Horizon frame of the LV. The ECEF frame and local horizon frame used for this model are displayed in Fig. 12.

3) Gravity: To model Earth’s aspherical gravitational field an oblate Earth model was used. More specifically, the second zonal spherical harmonics \( J_2 \), which is around 1000 times larger than the third zonal harmonics \( J_3 \) and is sufficient to model secular perturbations for Low-Earth orbits. The gravitational potential \( U \) including \( J_2 \) is given by Vallado [15] as

\[
U = -\frac{\mu_{\oplus}}{r} - \frac{3J_2 \mu_{\oplus}}{2r} \left( \frac{R_{\oplus}}{r} \right)^2 \left( \sin^2(\phi) - \frac{1}{3} \right) \tag{54}
\]

where \( \mu_{\oplus} \) is the standard gravitational parameter of Earth and \( R_{\oplus} \) is Earth’s radius. Lastly, eq. (54) can be derived with respect to \( r \) and \( \phi \) to get the gravitational acceleration in spherical coordinates.

4) Atmosphere: To model the atmosphere, the 21-layered, U.S. standard Atmosphere of 1976 was implemented, which model Earths atmosphere using an exponential relation depending on the altitude [15]

\[
\rho = \rho_H e^{\left( \frac{-h}{H} \right)} \tag{55}
\]

where \( \rho_H \) is the density for a specific scale height \( H \) provided from tabulated data and \( h \) is the altitude.

5) Drag: The aerodynamical force \( D \) acting upon the LV is mainly due to its form and the skin friction, also known generally as drag, which can be calculated by

\[
D = -\frac{1}{2} C_d A \rho r^2 \text{rel} \tag{56}
\]

where \( C_d \) is the drag coefficient and \( A \) is the cross-sectional area. There are also other effects such as wave drag that arise as the LV reaches transonic speeds. However, for simplicity, all such effects are assumed to be accounted for in the drag coefficient.

\[
\]
The relative velocity vector \( v_{rel} \) differs from the LV’s velocity at ascent because Earth’s atmosphere has a mean motion due to Earth’s rotation. However, since the frame of reference is ECEF implies that the relative velocity is simply given by

\[
v_{rel} = || \vec{v}_{ECEF} ||.
\]

Lastly, a zero-lift assumption was made which in general is a valid assumption for an early design of launch vehicles, especially considering the 3-DOF approximation and the fact that the launch vehicle is already starting at a high altitude.

6) Equations of motion: For this study, the general dynamic equations as derived by Tewari [14] were used:

\[
\dot{r} = v \sin \gamma \tag{57}
\]
\[
\dot{\phi} = \frac{vcos\gamma cos\psi}{r} \tag{58}
\]
\[
\dot{\lambda} = \frac{vcos\gamma sin\psi}{rcos\phi} \tag{59}
\]
\[
\dot{v} = \frac{F \cos \alpha - D}{m} - g \frac{sin \gamma + gc \cos \gamma \cos \psi}{r} + w_\infty^2 \frac{rcos\phi}{sin\gamma sin\psi \cos\phi + cos\gamma cos\phi} \tag{60}
\]
\[
\dot{\gamma} = \frac{F \sin \alpha + L}{mv} + \frac{vcos \gamma - g \frac{cos \gamma}{v}}{r} - \frac{g \sin \gamma \cos \psi}{v} + 2w_\infty \frac{sin \psi}{w_\infty \cos \psi} \tag{61}
\]
\[
\dot{\psi} = \frac{F \sin \alpha + L}{mv \cos \gamma} + \frac{v}{w_\infty \sin \psi} \cos \gamma + \frac{ru^2}{w_\infty \sin \psi \cos \phi \sin \phi} \tag{62}
\]
\[
\dot{m} = -\frac{F}{I_{sp} g_0} \tag{63}
\]

which are idealized for a 3-DOF, point-mass vehicle in translational flight inside of the atmosphere, using aspherical gravity written in an ECEF and local horizon frame.

Equation (57) - (59) represent the change of the positional state of the LV in spherical coordinates: radial distance \( r \), geocentric latitude \( \phi \) and longitude \( \lambda \) in the ECEF frame.

Equation (60) - (62) represent the change of the velocity state of the LV in spherical coordinates: velocity \( v \), flight path angle \( \gamma \) and flight heading angle \( \psi \) in the local, body-fixed frame of the vehicle. \( \alpha = \theta - \gamma \) is the angle of attack.

Figure [12] shows how the spherical angles representing the state of the LV are defined.

A LV is an obvious variable mass system as propellant is being converted into energy. By rewriting eq. (27) to eq. (63), the mass variation over time can be represented.

Since the LV is air-launched from the highly agile and manoeuvrable Gripen fighter aircraft, the initial flight path angle \( \gamma \) can be chosen up to 45 degrees at an altitude of 10 km with a velocity of Mach 0.95, as given by the mission requirements. Furthermore, the flight heading angle \( \psi \) can be chosen as desired. By choosing the correct initial flight heading angle and assuming no roll, the steering problem can be reduced from three dimensions to one, only requiring to control the pitch angle of the thrust to reach the desired orbit. This is possible since Earth is approximated with the zero-order zonal harmonics that is double symmetric.

Figure [13] shows how the pitch angle is defined along with the acting forces (aerodynamic, gravitational and thrust) on the rocket while inside of the atmosphere.

7) Multi-phased problem: To account for the discrete changes in mass as the stages separate, the trajectory problem
needs to be further divided into a multi-phase problem \cite{16}. This is done by performing a discretization at every stage separation (also called the event of the phase) and introducing connecting boundary conditions between each phase: coupling the separate trajectory segments. This allows the problem to be treated as continuous and avoids any disturbing discontinuities.

8) State: The state of the LV \( x \) was stated in vector form as

\[ x = \begin{bmatrix} r \phi \lambda v \gamma \psi \mu \end{bmatrix}^T \]  \hspace{1cm} (64)

where \( r \) in m, \( v \) in m/s, \( m \) in kg, \( F \) in N, \( t \) in s and all angles are in radians.

Further, the state dynamics can be represented in vector form by

\[ \dot{x} = f(x, t, \theta, F) \]  \hspace{1cm} (65)

where \( f \) are the equations of motion given by eq. (57) - (63).

9) Celestial reference frame: Since the dynamical equations are defined in an ECEF frame rotating with Earth, the final trajectory can be further transformed from an ECEF to a Geocentric Celestial Reference Frame (GCRF) by transforming each position with a reverse transformation around the Z-axis by Earth rotation angle \( \theta_{ERA} = w_G t \): where \( t \) is the time from launch in seconds and \( w_G \) is the angular rotational speed of Earth in rad/s.

Lastly, only the ascent is considered, smaller perturbation effects such as Bias-Precession Nutation, Sidereal Rotation and Polar Motion are neglected in the transformation \cite{15}.

B. Target orbit

The targeted final orbit for the payload is a circular Sun-Synchronous orbit (SSO) at 500 km altitude, which is a polar orbit that maintains a constant orientation towards the Sun at all times. The orbit is perturbed in such a way that the satellites nodal rate matches the average rate of the Sun’s motion projected on the equator \cite{15].

The reason for selecting this particular orbit is because it is a suitable orbit when launching rockets into orbit at high latitudes, e.g. for countries located in Scandinavia: considering impact risk zones and political factors. Where the advantage of the lesser rotational speed of the Earth as one gets closer to the poles can be utilized. This is beneficial when targeting polar orbits since less propellant is required to counteract the initial rotational speed of Earth.

In addition, the SSO itself is of high interest for scientific reasons since the payload will circle the Earth from pole to pole, thereby covering the whole Earth during the orbits propagation. Further, by selecting an appropriate local time of the launch, a SSO perpendicular towards the Sun can be targeted which provides a constant sunlight state for the satellite.

1) Velocity budget: The velocity budget, also referred to as Delta-V budget, consists of velocity losses (atmospheric drag and gravity losses) and velocity contributions (thrust and initial velocities), that together need to result in the final orbital velocity required to stay in the targeted orbit \cite{17}.

The final orbital velocity for a circular orbit can easily be calculated by

\[ v_f = \sqrt{\frac{\mu_E}{r}} \]  \hspace{1cm} (66)

which for a 500 km orbital altitude gives \( v_f = 7.612 \text{ km/s} \). The drag losses arise as a result due to the atmosphere and can for a local horizon reference frame be calculated \cite{17} as

\[ \Delta v_{\text{drag}} = \int_0^r D(t) \frac{m(t)}{m_L} \, dt \]  \hspace{1cm} (67)

where \( D \) is given by eq. (66) acting parallel to the velocity vector, \( t \) is time and \( m \) is the mass of the LV.

The gravitational loss can be calculated using

\[ \Delta v_{\text{grav}} = \int_0^r g(t) \sin(\gamma) \, dt \]  \hspace{1cm} (68)

where \( \sin(\gamma) \) represents the vertical component of the gravitational acceleration \( g \), and as \( \gamma \to 0 \) the gravitational loss also goes towards zero.

Lastly, the velocity contribution is mainly given by the thrust from the propulsion system and can be calculated by

\[ \Delta v_{\text{thrust}} = \int_0^r F(t) \frac{\cos(\alpha)}{m(t)} \, dt \]  \hspace{1cm} (69)

where the factor \( \cos(\alpha) \) accounts for the thrust contribution parallel to the velocity vector.

Additional contributions from the initial launch velocity of the carrier aircraft also exist, as from Earth’s rotation (if an eastward launch is performed).

2) Orbital elements: The governing equation that an SSO needs to fulfill is

\[ \frac{d\Omega}{dt} = \dot{\Omega}_{SSO} = \frac{360^\circ}{365.2421897\text{day}} = 0.0172079 \text{rad/day} \]  \hspace{1cm} (70)

which states that the nodal rate of the orbit \( \frac{d\Omega}{dt} \) matches the average rate of the Sun’s motion \( \dot{\Omega}_{SSO} \), projected on the equator.

Using the fact that the dominant secular motion of the Right Ascension of the Ascending Node \( \Omega \) is due to \( J_2 \) gives the required inclination for a SSO, given by Vallado \cite{15} as

\[ i_{SSO} = \arccos\left(-\frac{2\alpha^4/7\dot{\Omega}_{SSO}^2(1-e^2)^2}{3R_\oplus J_2}\sqrt{\mu_E}\right) \]  \hspace{1cm} (71)

where \( a \) is the semi-major axis and \( e \) is the eccentricity of the orbit. For a circular orbit \( e = 0 \) which give that \( a = 500\text{km} + R_\oplus \).

Ultimately, the final targeted orbit in orbital elements can be written as:

\[ a_f = 6878.136 \text{ km} \]
\[ e_f = 0 \]
\[ i_f = 97.402^\circ \]
\[ \Omega_f = 0^\circ \]
\[ 0 \leq u_f \leq 360^\circ \]
where \( i_f \) was calculated using eq. (71) and \( u_f \) is the argument of latitude which is defined as the angle between
the ascending node and the satellite's position vector in the
direction of its motion. No constraint is imposed on where in
the orbit the insertion should take place, therefore \( u_f \) can be
between 0° and 360°. Additionally, the right ascension of the
ascending node \( \Omega_f \) is determined on the local time that orbit
insertion takes place. The launch date and time used for this
study was set to 2021-06-01, 08:00 am (GMT+2).

When the targeted orbits inclination and initial latitude for
the launch are known, the initial flight heading angle for the
launch can be calculated using the relationship for a spherical
triangle [14]

\[
\cos(i) = \cos(\phi) \sin(\psi).
\]

(72)

3) Transformations: Transforming the position between
orbital elements and the GCRF can be done by first determine
the position in the Perifocal reference frame (PQW) [15] and
then perform three rotations to get the final position in the
dynamic coordinate system, e.g. by using Algorithm 10 as
stated by Vallado [15]. Using the semi-parameter \( p = a(1-e^2) \)
and the true anomaly \( \nu \), the positional vector \( \mathbf{r}_{PQW} \) can be
written in the PQW frame as

\[
\mathbf{r}_{PQW} = \begin{bmatrix}
\frac{pcos(\nu)}{1+ecos(\nu)} \\
\frac{psin(\nu)}{1+ecos(\nu)} \\
0
\end{bmatrix}
\]

(73)

and further transformed to the GCRF by the transformation matrix \( C_{GCRF}^{PQW} \), which then can be transformed to ECEF
(if desired). See Appendix B for the transformation matrix
\( C_{GCRF}^{PQW} \).

To see how to transform vectors from a GCRF or ECEF to
orbital elements the reader is referred to Vallado [15].

4) Orbit insertion: After the second burn is complete and
the structure mass has been dropped, the third stage and
payload are in a ballistic trajectory: which is an elliptical orbit
without requiring any extra thrust to be applied. Furthermore,
the third and final burn needs to be performed so that the
flight path angle \( \gamma = 0 \) at burnout, since for circular orbits the
velocity vector is always parallel to the local horizon.

This is somewhat equivalent to performing a Hoffman
transfer, which uses two impulsive burns when the flight path
angle is zero: one at apoapsis and one at periapsis. Thereby,
performing a transfer manoeuvre between two coplanar orbits
while using the lowest possible amount of propellant [15].

5) Initial launch location: The initial launch location was
chosen to be over the Norwegian sea in order to fully exploit
the advantage of using an ALS. Reducing the risks associated
with having a vertical launch from the ground. Lastly, many
countries do not possess an eastward coastline with a suffi-
ciently large area of open water, thus demonstrating another
advantage with an ALS.

The initial launch state was chosen (in ECEF coordinates) as:

\[
\begin{align*}
\mathbf{r}_i &= 10 \text{ km} \\
\phi_i &= 66° \\
\lambda_i &= 10° \\
v_i &= 284.5 \text{ m/s} \\
\gamma_i &= 30° \\
\psi_i &= -18.5°
\end{align*}
\]

where \( \psi_i \) was calculated using eq. (72) with \( \phi_i \) and \( i_f \).

C. Simulation

To stay within the scope of this study, and reduce the accumulative truncation error, only the ascent was simulated.
The final state of the payload was thereafter transformed into
orbital elements to make sure that the payload reached its
targeted final orbit.

The governing equations can be categorized as a boundary
value problem consisting of several continuous, non-linear,
coupled, ordinary differential equations (ODE). Finding a
closed-form solution to these equations is generally not pos-
sible.

To get insight into the trajectory problem it was first solved
as an initial value problem using a numerical integration
approach to propagate the LV trajectory forward in time
while guessing the steering law for all intermediate points.
The simple and popular fourth-order Runge-Kutta method was
implemented, which has proven to be sufficiently accurate
for many trajectory propagations [19, 15], using the SciPy
integrate.solve_ivp integrator for Python [20].

Lastly, the BVP was solved using a multi-phased direct
shooting approach, described in the Optimization section.

D. Optimization

The goal of a trajectory optimization is to find a control
\( \mathbf{u} \) that optimizes a trajectory where the state vectors \( \mathbf{x} \) obeys
some dynamical system \( \dot{\mathbf{x}} \) and satisfies some set of constraints
while minimizing a cost function \( J \) [21].

1) Nonlinear problem: In reality, the trajectory optimiza-
tion problem is continuous in both time and state, also called
being of a single-phase, and is therefore infinite-dimensional.
This makes it almost impossible to optimize since one needs
to account for all the infinitesimal points along the trajectory:
in both state and control between the initial and final times, \( t_i \)
and \( t_f \). A general continuous trajectory optimization problem
can be written as
\[
\begin{align*}
\min_{x, u, t_i, t_f} & \quad J(t_i, t_f, x(t_i), x(t_f)) + \int_{t_i}^{t_f} w(\tau, x(\tau), u(\tau))d\tau \\
\text{subject to} & \quad \dot{x} = f(t, x(t), u(t)) \\
& \quad g(t_i, t_f, x(t_i), x(t_f)) \leq 0 \\
& \quad h(t, x(t), u(t)) \leq 0 \\
& \quad x_{\text{lower}} \leq x \leq x_{\text{upper}} \\
& \quad u_{\text{lower}} \leq u \leq u_{\text{upper}} \\
& \quad t_i \leq t \leq t_f
\end{align*}
\] (74)

where \( J \) is in a Bolza form: containing both a boundary objective term and a path integral term, \( f \) represents equality constraints, given by the system dynamics, \( g \) represent boundary inequality constraints and \( h \) path constraints.

2) Transcription: To solve the optimization problem, a \textit{Transcription} needs to be made. This is done by discretizing the original infinite-dimensional problem into a sequence of finite-dimensional constrained optimization problems; generally called a \textit{Nonlinear Programming Problem}, or NLP.

The most common and widely used approach to solve the launch vehicle trajectory optimization problem (with one stage) is to use a \textit{direct shooting method} \cite{19, 22} to find an open-loop solution. In this case, a parametric control law that steers the LV into its final target orbit. This is done by discretizing all of the continuous functions of the original optimization problem by polynomial splines: a function that is made up of a sequence of polynomial segments \cite{21}. For this study, first-order (linear) polynomial splines were used, also called \textit{trapezoidal collocation}, using trapezoidal quadrature to discretize the continuous functions and approximate integrals.

For a multi-stage launch vehicle that has discrete jumps in mass as stages separate, require that the trajectory problem is stated as a \textit{multi-phased NLP}: consisting of several single phased NLPs with boundary constraints that connect the phases continuously at the end of each phase, also called event \cite{16}.

This allows for the state dynamics of the system to be different in each phase without causing any discontinuity problems, e.g. discrete mass and thrust changes. Thus, allowing the use of gradient search methods to find a locally optimal solution. Another advantage is that it reduces the sensitivity of the solution on the initial guess \cite{21, 23, 24}.

3) Other studies: A similar air-to-air trajectory problem with air-breathing propulsion was optimized by Mukundan et al. \cite{25}: using a direct method to discretize the problem and solving it using a gradient-free metaheuristic optimization algorithm called \textit{harmony search}. The final boundary constraints used in the problem were introduced into the objective function \( J \) and the problem was propagated until the desired altitude was reached.

In his doctoral thesis, Balesdent \cite{22} sets up a ground-to-space three-staged ascent trajectory problem as a sub-problem to a larger \textit{Multi-disciplinary Design Optimization Problem} (MDO), where a total launch vehicle design optimization was considered. Taking into account numerous disciplines such as aerodynamics, structure, trajectory, mass budget, propulsion and cost. Considering all these disciplines the objective was to minimize the GLOM, comparing three different optimization algorithms: \textit{Nelder & Mead, Genetic Algorithm} and \textit{Efficient Global Optimization Algorithm}. This was achieved by first using a gradient-free optimization to quickly get to the vicinity of an optimum, then switching to a gradient-based optimization algorithm in a limited search space to converge towards an optimal solution. Four different trajectory NLP formulations were used; resulting in having the final boundary constraints incorporated into the objective function provided the best convergence to an optimal solution at a sub-system level.

In his doctoral thesis, Benson \cite{26} optimizes a ground-to-space ascent trajectory of the Delta III launch vehicle using an integral \textit{Gauss pseudospectral method}. Posing the problem as multi-phased with four phases by introducing phase boundary constraints at the end of each phase, thus avoiding the discrete jumps in mass when stages are separated (as suggested by Stryk \cite{16}). The objective was to maximize the final mass of the payload by minimizing the negative mass of the final stage, having a free final burnout time variable for the final stage. The solver used was the commercial optimization software SNOPT with a total solution time of approx. 2 min.

4) Problem Formulation: Taking inspiration from these three articles presented in the previous section, the NLP for an air-to-space trajectory was set up by dividing it into four phases, introducing a coast phase after the second stage has been separated \cite{18, 24}. The state and control variables was discretized as: \( x_i = (x_0, x_1, \ldots, x_{N-1}) \) and \( u_i = (u_0, u_1, \ldots, u_{N-1}) \) with respect to time where \( N \) is the number of distinct time nodes (also called grid or knot points) for each phase \( i \), where the sum of all grid points is \( M \).

The cost function \( J \) was written as

\[
J = -w_m m_f + w_a \left( \sum_{j=0}^{M-1} (\alpha_{j+1} - \alpha_j)^2 \right) + w_r (r_f - r^*)^2 + w_v (v_f - v^*)^2 + w_\gamma (\gamma_f - \gamma^*)^2
\] (75)

with the objective to minimize the negative mass of the final state (subscript \( f \)) while reaching the targeted orbit. This can be enforced by introducing three penalty terms including the final boundary constraints (superscript \( * \)) and a summation term of the angles of attack differences; this helps the solver find smooth solutions to the parametric steer law. \( w_m, w_a, w_r, w_v, w_\gamma \) are weights that can be manipulated so that all the terms are of the same order of magnitude, having the reciprocal unit of its respective term.

Using eq. (79) the continuous state dynamic equations eq. (57) - (63) can be transcribed using trapezoidal collocation to the following collocation constraints

\[
0 = x_{k+1} - x_k - \frac{h_k}{2} (f_{k+1} + f_k) = \zeta_k
\] (79)
\[
k = 0, 1, 2, \ldots, M - 1
\]
where $h_k = t_{k+1} - t_k$; solving the nonlinear problem means driving the defect $\zeta_k$ to zero \cite{19}. Eq. (79) is enforced at each grid point for all state variables in $x$, resulting in $7M - 7$ equality constraints.

The phase boundary constraints are defined at the end of each phase with the beginning of the next by eq. (79) with eq. (80)

$$0 = m_{i+1} - m_i - \frac{h_i}{2} (m_{i+1} + m_i) + \Delta m_i = \zeta_i$$

$$i = 1, 2, 3, 4$$

where $\Delta m_i$ is the structural mass for each stage. Note that no structural mass is dropped for phase 3 which is the coasting phase.

Since variable step size is being used, with time as a free variable, inequality constraints for the 8th state variable, time, were introduced for the first three phases that had fixed final times to enforce that the grid points are more or less evenly distributed

$$t_k - t_{k+1} + \Delta t \leq 0$$

where $\Delta t$ is the minimum desired time step. This leads to $M - 1$ inequality constraints for the NLP.

The bounds for the NLP was set up as:

$$r_i \leq r_k \leq r_f$$
$$-2\pi \leq \phi_k \leq 2\pi$$
$$-2\pi \leq \lambda_k \leq 2\pi$$
$$v_i \leq v_k \leq v_f$$
$$-2\pi \leq \gamma_k \leq 2\pi$$
$$-2\pi \leq \psi_k \leq 2\pi$$
$$m_{pl} \leq m_k \leq m_i$$
$$0 \leq t_k \leq t_f$$
$$k = 0, 1, \ldots, M - 1$$

where the bounds on time are enforced for the three first phases to be equal to the burnout time for each stage (or coast time for the third phase), keeping the time variable free for the fourth phase. This allows for the possibility to choose a shorter burnout time for the last stage, leaving unburnt propellant that could be converted into additional payload mass.

5) Initial guess: As an initial guess to the solution to the problem, a linear guess between the initial and final state was used

$$x_k = x_0 + \frac{t_k}{t_{M-1}} x_{M-1}$$

which is the simplest guess one can make between the initial and final point. With the exception of the steering law, which was guessed to be zero at all points.

6) Algorithm: There is a multitude of different optimization algorithms that can search the feasible region of a solution space, looking for an optimal solution. Ranging from: gradient-free, heuristic and probabilistic, machine learning algorithms to gradient descent algorithms; calculating or estimating the first- and second-order derivatives of the objective function and following it downhill \cite{27}. It is desirable that, if possible, choose a gradient-based local optimization algorithm compared to a heuristic algorithm \cite{28}.

For constrained nonlinear optimization, a suitable gradient-based algorithm to implement is the Sequential Quadratic Programming (SQP) algorithm, which is one of the most powerful and widely used NLP algorithms used to solve constrained differentiable optimization problems; such as the ascent trajectory problem of the form presented in eq. (74) \cite{22, 23}.

The SQP algorithm implements quadratic approximations to the Lagrangian and a linear approximation on the constraints, using Newton’s method to search the feasible region in the solution space \cite{23}.

Lastly, there are several optimization software available \cite{28}, some commercial: FMINCON, GPOPS2, SOCS, POST, SNOPT, GTS and others that are open-sourced: SciPy, IPOPT and NLOPT to name a few.

For this study, the Python implemented Python Parallel Global Multiobjective Optimizer library (PyGMO) was used to set up the NLP \cite{28}. PyGMO was developed within the European Space Agency to evolve interplanetary trajectories as design spacecraft parts and more. Included in the PyGMO library are many different optimization algorithms; the one used in this study was an open-sourced, gradient descent, local optimizer from NLopt called Sequential Least-Squares Quadratic Programming algorithm (SLSQP) \cite{29}.

VII. RESULTS

For both the first and the second stage, aluminized ammonium perchlorate composite propellant (see Table \ref{table:1}) was used together with an isentropic efficiency coefficient of $\eta_s = 0.95$: which accounts for energy losses from the gas expansion, such as wall-friction, solid- and/or liquid particles in the gas, existence of boundary layers etc. Additionally, a safety factor of 1.5 was used along with the optimal expansion design point being 40% lower than the ambient pressure at the chosen altitude.

The thrusts and burn times from previous work done by Sigvant \cite{3} were used as initial guesses for the first and second stage, designing an initial LV design for a payload mass of 20 kg.

To account for additional structural mass neglected in the LV design, such as an outer skirt, thrust vector control gimbal actuators and internal structures. An additional 10% of the calculated structural mass was used for each stage.

A. Stage 1

To start the design, the first stage thrust and burn time from previous work done by Sigvant \cite{3} was used: $F_1 = 60$ kN and $t_{b_1} = 41$ s, where the optimum expansion was set in accordance with an initial launch altitude of 10 km.
1) Nozzle: The nozzle properties for the first stage are presented in Table VI and VII. Since the ambient pressure is highest for the first stage resulted in a nozzle configuration that did not have to be truncated, due to the outer diameter constraint of the envelope.

<p>| TABLE VI |
| STAGE 1 - THERMODYNAMICAL PROPERTIES |</p>
<table>
<thead>
<tr>
<th>Pressure (Mpa)</th>
<th>Temperature (K)</th>
<th>Velocity (m/s)</th>
<th>Area (m²)</th>
<th>Diameter (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.895</td>
<td>3322.7</td>
<td>~ 0</td>
<td>0.520</td>
<td>81.38</td>
</tr>
<tr>
<td>3.386</td>
<td>3112.5</td>
<td>991.95</td>
<td>0.00509</td>
<td>8.05</td>
</tr>
<tr>
<td>0.0265</td>
<td>1771.7</td>
<td>2694.18</td>
<td>0.156</td>
<td>45.54</td>
</tr>
</tbody>
</table>

The required thickness of the parabolic nozzle considering only the internal pressure gives a thickness of less than 1 mm. Therefore, a divergent nozzle thickness of 1 mm was set, with the motivation that the wall will have some internal rigidity against vibrations, shocks and axial forces. Further, the total diverging length of the parabolic nozzle was 0.477 m.

Finally, the insulation thickness required was \( t_{\text{ins}} = 2.34 \) mm, with a total insulation mass of \( m_{\text{ins}} = 16.3 \) kg, including insulation for both the nozzle and combustion chamber. The total insulation mass was included in the structural mass of the final first stage.

| TABLE VII |
| STAGE 1 - NOZZLE PROPERTIES |
| \( \epsilon \) | \( \lambda \) | \( \epsilon^* \) (m/s) | \( C_F \) | \( \dot{m}_i \) (kg/s) | \( \theta_1 \) (°) | \( \theta_2 \) (°) |
| 30.59          | 0.989          | 1357.8         | 1.709     | 22.537         | 31.9            | 8.0              |

2) N-star perforation: To simplify the N-star perforation analysis: \( r_1, r_2 \) was set to zero and \( r_b = 7 \) mm/s, with only the number of legs \( N \) being varied. This resulted in the lowest total mass for the first stage with \( N = 7 \), fulfilling the choking constraint \( A_p \geq 4A_l \). Using ASTM/SAE 300M instead of Aluminium 7068 for the shell led to a weight reduction of 9 kg. The results are displayed in Table VIII and Fig. 14.

| TABLE VIII |
| STAGE 1 - STAR PERFORATION |
| L (m) | D (m) | \( m_{s} \) (kg) | \( m_p \) (kg) | \( m_{0} \) (kg) | \( \varepsilon \) (%) |
| 1.50 | 1.0   | 102.97          | 1083.52     | 1196.59         | 9.45             |

The mass budget left for the second and third stage (assuming a 20 kg payload) was then: \( m_{Q2} = 1300 \) kg \(- (1196.6 + 20) \) kg = 73.4 kg. Furthermore, looking at the required propellant mass, with 3 % extra fuel: \( m_{p2} = t_b m_{dot} 1.03 = 951.74 \) kg. This indicates that the amount of residual propellant using the 7-star perforation equals to: 131.78 kg.

Given the above, the star grain perforation was not pursued as an option for the first or second stage since it did not provide a feasible solution for the complete LV design.

3) Internal-burning tube: Since the density range for a solid propellant is narrow and has less impact on the burn area than changing the burn rate, the density of the propellant was fixed to 1800 kg/m³ for the first and second stage. Looking at eq. (37) - (39), the only independent variable left to decide to solve the equation system is then the burn rate \( r_b \).

Figures 15 shows the burn profiles for the first stage using an internal-burning tube with both ends uninhibited (\( n = 2 \)) as it stays within the limits of having a maximum of 20 % of the desired burn area while not going below 30 %.
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Fig. 16. The final configuration for the first stage with an internal-burning tube grain with both ends uninhibited, the red areas indicate burning surfaces.

Fig. 17. Thrust profile for the first stage with an internal-burning tube uninhibited at both ends, neglecting the pressure term.

4) Shell material: Looking at the shell material choice between using: Aluminium alloy 7068 or Steel ASTM/ASE 300M for the internal-burning tube with both ends uninhibited and \( r = 7 \text{ mm/s} \), gave the result presented in Table IX.

<table>
<thead>
<tr>
<th>Shell material</th>
<th>( t_{\text{cyl}} ) (mm)</th>
<th>( t_{\text{head}} ) (mm)</th>
<th>( m_s ) (kg)</th>
<th>( \varepsilon ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminium alloy 7068</td>
<td>6.46</td>
<td>13.22</td>
<td>95.37</td>
<td>9.11</td>
</tr>
<tr>
<td>ASTM/ASE 300M</td>
<td>2.23</td>
<td>4.56</td>
<td>93.44</td>
<td>8.94</td>
</tr>
</tbody>
</table>

Given the above, it is clear that less structure material is required by using stainless steel instead of high strength aluminium for the shell: with around 2 kg of mass being saved.

Lastly, to account for the simplifications made in the structure of the stage: an additional 10% of the calculated structural weight was added. The final stage configuration is presented in Table X.

| Stage 1 - Final Configuration |
|-----------------------------|-----------------|-----------------|-----------------|
| \( L \) (m) | \( D \) (m) | \( m_0 \) (kg) | \( m_s \) (kg) | \( m_p \) (kg) | \( \varepsilon \) (%) |
| 1.79 | 0.823 | 1054.54 | 102.78 | 951.75 | 11.12 |

B. Stage 3

For the third stage, ECAPS 200N HPGP liquid monopropellant thruster was used; having a structural mass of 6 kg for the thruster where the required valves and piping was assumed to be included.

Using the data from Table V with a designed propellant throughput of 25 kg, a constant maximum thrust of 220 N and an \( I_{sp} = 255 \text{ s} \), provided a burn time of 284 s. Additionally, by dividing the propellant mass with the propellant density gave the required propellant tank volume: \( V_p = 0.0202 \text{ m}^3 \).

The propellant tank pressure was set to \( p_p = 25 \text{ bar} \) according to the HPGP thruster specifications (see Table V) and the gas tank pressure was set to \( p_G = 150 \text{ bar} \), assuming an ambient temperature \( T = 298 \text{ K} \).

Using this required volume for the propellant, along with the assumption of isentropic expansion between the gas tank and the propellant tank, provided the gas- and propellant tank designs presented in Table XI.

| Stage 3 - Spherical Tanks |
|--------------------------|-----------------|-----------------|
| Gas Tank | Propellant Tank |
| Pressure (MPa) | 15.0 | 2.5 |
| Volume (m³) | 0.0105 | 0.0202 |
| Radius (cm) | 13.6 | 16.9 |
| Gas mass (kg) | 0.253 | 25.0 |
| Shell material | ASTM/ASE 300M | ASTM/ASE 300M |
| Wall thickness (mm) | 0.8 | 0.5 (0.2) |
| Tank mass (kg) | 1.45 | 1.40 |

Since the required wall thickness for the propellant tank was less than 0.5 mm, the minimum value of 0.5 mm was used.

1) Nose cone: The base of the nose cone was chosen to be equal to the width of the first stage outer diameter. Thus, implementing a single diameter outer skirt for the whole launch vehicle. Additionally, using a fitness ratio \( L/D = 1 \) gave a length \( L = 0.82 \text{ m} \) for the nose cone.

To keep the launch vehicle axisymmetric with an even distribution of mass around the centerline, a stacked configuration was chosen for the third stage, having the third stage partly encapsulated within the vicinity of the nose cone.

Numerically calculating the rotational surface area and volume of the nose cone using the disk method with eq. 50 and assuming a wall thickness of 1 mm, give the results for the nose cone as displayed in Table XII.

| Stage 3 - Nose Cone |
|---------------------|-----------------|
| Length (m) | 0.82 |
| Base radius (m) | 0.41 |
| Volume (m³) | 0.218 |
| Mass (kg) | 2.49 |

Considering the space available for the payload \( V_{pl} \) in front of the third stage, is then the volume from the tip of the nose cone to the front end of the gas tank, which resulted in \( V_{pl} = 0.0612 \text{ m}^3 \).

Figure 18 shows the final configuration of the third stage with the payload fairings still attached, although, in reality, the fairing would most likely have been ejected before the third stage ignites.

No information could be found regarding the geometry of the thruster except its total length. For that given length, a nozzle was included along with a 10x20x10 cm³ box, to illustrate the combustion chamber. Furthermore, a 20x20x30 cm³ cuboid was fitted to represent a possible payload: possessing
the same volume as 12 U CubeSats, which equals ~20 % of the available volume for the payload.

Lastly, to account for the simplifications made in the structure of the stage, an additional 10 % of the calculated structural weight was added. The third stage configuration, excluding the nose cone, is presented in Table [XIII] including the helium mass into \( m_p \). Where the largest tanks diameter provided the minimum width, and the length consists of both diameters of the tanks plus the length of the thruster (see Table [V]).

![Fig. 18. The final configuration for the third stage with the nose cone attached for reference, including: Payload (yellow), gas tank (blue), propellant tank (green), thruster (red).](image)

**TABLE XIII**

<table>
<thead>
<tr>
<th>L (m)</th>
<th>D (m)</th>
<th>( m_s ) (kg)</th>
<th>( m_p ) (kg)</th>
<th>( m_{t0} ) (kg)</th>
<th>( \varepsilon ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.34</td>
<td>9.75</td>
<td>25.25</td>
<td>35.0</td>
<td>27.85</td>
</tr>
</tbody>
</table>

This led to a throat diameter that was 2 % larger with a thrust coefficient \( C_F \) that was 3.4 % lower compared to an optimally expanded nozzle. Three iterations were required to reach the set tolerance of 0.001, resulting in a final thrust of \( F_2 = 14999 \) N with a diverging nozzle length of 0.640 m. The nozzle properties are presented in Table [XV] and [XVI].

![The required thickness of the parabolic nozzle, considering the internal pressure, led to a thickness of less than 1 mm. Therefore, a divergent nozzle thickness of 1 mm was set, with the motivation that the wall will have some internal rigidity against vibrations, shocks and axial forces.](image)

The required thickness of the parabolic nozzle, considering the internal pressure, led to a thickness of less than 1 mm. Therefore, a divergent nozzle thickness of 1 mm was set, with the motivation that the wall will have some internal rigidity against vibrations, shocks and axial forces.

The insulation thickness required was \( t_{ins} = 1.83 \) mm, with a total insulation mass of \( m_{ins} = 6.13 \) kg, including insulation for both the nozzle and combustion chamber. The total insulation mass is included in the structural mass of the final second stage.

The final constrained nozzle configuration for the second stage is presented in Table [XVI].

![The final constrained nozzle configuration for the second stage is presented.](image)

**TABLE XVI**

<table>
<thead>
<tr>
<th>Chamber</th>
<th>Thrust</th>
<th>Exit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure (Mpa)</td>
<td>6.895</td>
<td>3.868</td>
</tr>
<tr>
<td>Temperature (K)</td>
<td>3322.7</td>
<td>3112.5</td>
</tr>
<tr>
<td>Velocity (m/s)</td>
<td>~ 0</td>
<td>991.94</td>
</tr>
<tr>
<td>Area (m²)</td>
<td>0.243</td>
<td>0.00109</td>
</tr>
<tr>
<td>Diameter (cm)</td>
<td>57.05</td>
<td>3.73</td>
</tr>
</tbody>
</table>

Given the above, a truncated nozzle, as presented in Section 19.6 [Constrained Nozzle] was used. The limiting constraints were set so that the parabolic nozzle’s exit diameter would not go outside of the outer diameter of the second stage.

![Integrating the area under the thrust curve in Fig. 20 assuming a constant mass flow during the burn time, resulted in a specific impulse of \( I_{sp2} = 307 \) s. Figure 21 shows the final configuration of the second stage.](image)
3) Shell material: Looking at the two different shell materials for the second stage using the internal-burning tube with one side inhibited and a burn rate of 7 mm/s, provided the results as presented in Table XVII.

<table>
<thead>
<tr>
<th>Shell material</th>
<th>( t_{\text{cyf}} ) (mm)</th>
<th>( t_{\text{head}} ) (mm)</th>
<th>( m_s ) (kg)</th>
<th>( \varepsilon ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminium alloy 7068</td>
<td>4.51</td>
<td>6.51</td>
<td>26.46</td>
<td>14.3</td>
</tr>
<tr>
<td>ASTM/ASE 300M</td>
<td>1.56</td>
<td>2.24</td>
<td>29.0</td>
<td>15.4</td>
</tr>
</tbody>
</table>

Here it is clear that Aluminium alloy 7068 provides the lowest structural mass and is therefore to be preferred compared to using ASTM/ASE 300M, saving around 2.5 kg.

Lastly, to account for the simplifications made in the structure of the stage, an additional 10% of the calculated structural weight was added. The second stage configuration is given in Table XVIII.

<table>
<thead>
<tr>
<th>Stage 2 - final configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>L (m)</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>1.203</td>
</tr>
</tbody>
</table>

D. Complete launch vehicle

The final design of the LV is shown in Fig. 22 where: 3% extra fuel, 10% extra structural weight for each stage, isentropic expansion coefficient of \( \mu_s = 0.95 \) and a safety factor of 1.5 were used. The final LV configuration is presented in Table XIX & XX where the structural mass of the nose cone has been added to the structural mass of the second stage, with an available payload volume of \( V_{pl} = 61 \) litres, or 61 U. Additionally, the ideal specific impulse \( I_{sp_i} \) (assuming a constant thrust) is also given in Table XIX.

Lastly, a thrust-to-weight ratio in multiples of the gravitational acceleration at sea-level \( g_0 \) was calculated for the initial (subindex \( i \)) and final thrust values (subindex \( f \)), given from the thrust profiles for each stage.

E. Trajectory analysis

Using the designed launch vehicle as presented in Table XIX. The trajectory problem was solved in two ways. First, as an IVP, assuming a steering law having the shape of a sigmoid function and secondly, as a BVP that was solved by transcribing the trajectory problem into a NLP that was solved using a SLQSP algorithm, where the initial and final guess for the state of the LV was as given by the solution given from solving the IVP.
Fig. 23. The flight parameters during the ascent of the optimized launch vehicle with a GLOM of 1300 kg, \( t_b = 365 \) s, carrying a 12.9 kg payload where: Stage 1 (red), Stage 2 (yellow), drift (blue), Stage 3 (red), Payload (green).

F. Initial value problem

To solve the IVP, the semi-optimizing approach as suggested in Benson [26] was implemented; where the final stage’s thrust time was held as a free variable, being either longer or shorter than the designed burn time, corresponding to converting payload mass to propellant and vice versa. Implementing a method of iteration using trial and error, provided a solution for a 500 km SSO, as presented in Fig. 23, using the steering law \( \theta(t) = \frac{\theta_0}{1 + e^{\alpha t + \beta}} \) (83) to control the pitch

where \( \alpha = 0.10, \beta = -4.105, \theta_0 = 45^\circ \) and \( t \) is time in seconds from launch. The initial launch velocity was set to \( M = 0.95 \).

Using eq. (72) along with the initial latitude and final targeted orbit inclination provided the initial flight heading angle \( \psi_i = -18.47^\circ \).

1) Designed launch vehicle: Using the LV configuration as given in Table XIX with a payload mass of 20 kg, did not provide a feasible solution. Therefore, the payload mass was reduced until a feasible solution was obtained, keeping the burn time for the third stage fixed at 284 s. This resulted in a final possible payload of 8.4 kg with an apogee of 509 km and a perigee of 503 km using a drift time of 278 s (see Appendix A). The GLOM of the designed LV is then 1.289 kg.

2) Optimized launch vehicle: Since the designed LV configuration as given in Table XIX did not provide a feasible solution with a 20 kg payload. An attempt to optimize the payload to its maximum value was made by keeping the third stage burn time open as a free variable, which was increased until a feasible solution was obtained. This resulted in a final third stage burn time of 365 s with a drift time of 213 s and a final maximal possible payload mass of 12.9 kg.

Thus, by adding 7.1 kg of propellant to the third stage a longer burn time of 365 s could be obtained; resulting in a possible payload mass of 12.9 kg by using the steering law as presented in eq. (83) while having a GLOM of 1,300 kg.

The velocity losses from the ascent along with the initial and final velocities, expressed in the inertial GCRF frame of reference, are given in Table XXI. Note here that the required orbital velocity, in an inertial reference frame, for a circular orbit at 500 km altitude as given by eq. (66), is \( v_{\text{orbit}} = 7.61 \) km/s.

The 500 km SSO final orbit written in orbital elements is presented in Table XXII, where the eccentricity has been rounded down to zero from 0.0012. Thus, the orbit is not perfectly circular but fluctuates between a perigee of 493 km and an apogee of 508 km altitude above Earth’s surface. During the ascent, the LV experienced a maximal dynamic pressure of 63 kPa after 18.4 s from launch (or 17.8 km altitude) with an overall ascent time of 651 s, or \( \sim 11 \) min.

---

**TABLE XXI**  
**VELOCITY BUDGET - OPTIMIZED LV**

<table>
<thead>
<tr>
<th>( v_i )</th>
<th>( v_f )</th>
<th>( \Delta v_{\text{thrust}} )</th>
<th>( \Delta v_{\text{grav}} )</th>
<th>( \Delta v_{\text{drag}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.62 km/s</td>
<td>0.30 km/s</td>
<td>8.72 km/s</td>
<td>-0.53 km/s</td>
<td>-0.34 km/s</td>
</tr>
</tbody>
</table>

**TABLE XXII**  
**ORBITAL ELEMENTS - IVP 500 KM SSO, OPTIMIZED LV**

<table>
<thead>
<tr>
<th>a (km)</th>
<th>e</th>
<th>( i ) (deg)</th>
<th>( \Omega ) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6886</td>
<td>0</td>
<td>97</td>
<td>125</td>
</tr>
</tbody>
</table>

The ascent is shown in Fig. 24 in a GCRF with a consecutive orbit after the third stage burnout.
Fig. 24. The ascent of the optimized launch vehicle with a GLOM of 1,300 kg, \( t_{b3} = 365 \) s, carrying a 12.9 kg payload, plotted in a GCRF for one orbit, where green is the starting point, and yellow indicates the current position of the payload.

By plotting the ascent for a further 2.8 orbits in the ECEF (see Fig. 25), the special property of a SSO can be seen as the orbits nodal rate matches the average rate of the Sun’s motion projected on the equator. Thus, the orientation towards the sun as the orbit is perturbed is conserved due to the dominant secular motion caused by Earth’s aspherical gravitational field.

Fig. 25. The ascent of the optimized launch vehicle with a GLOM of 1,300 kg, \( t_{b3} = 365 \) s, carrying a 12.9 kg payload, plotted in an ECEF for \( \sim 3 \) orbits, where green is the starting point, and yellow indicates the current position of the payload.

G. Boundary value problem

The NLP trajectory problem was solved for the optimised launch vehicle configuration with a total starting mass of 1,300 kg, using the same semi-optimisation approach by keeping the third burn time open as a free variable, setting a maximal burn time of 400 s for the third stage, as given in Table XXIII.

To keep the dimension of the NLP to a reasonable size, a drift time of 9 s was used with 3 grid points for the drift phase. Using this configuration resulted in a global dimension of 342 (297 equality and 37 inequality constraints) with 101,574 expected number of gradients to be estimated for the Jacobian matrix. The algorithm used was NLopt’s open-source Sequential Least-Squares Algorithm \[29\] \[30\], which uses a dense matrix approximation for the Jacobian.

The weights of the cost function \( J \) were set as

\[
\begin{align*}
    w_m &= -0.1, \\
    w_v &= 0.01, \\
    w_{\alpha} &= w_\gamma = 10^3, \\
    w_r &= 10^{-6}
\end{align*}
\]

and where chosen so that the size of the decision variables would be of approximately the same order of magnitude.

The initial and final state from the IVP 500 km SSO solution was used as an initial and final guess for the NLP. The solution took 4392 s to compute on an Intel(R) Core(TM) 9i-10885H 2.40 GHz CPU, using the grid point distribution as presented in Table XXIII with a converging tolerance set to 1e-8. The results of the trajectory can be seen in Fig. 26 with a final payload mass of 11.2 kg, burning the third stage for a total of 387.7 s, reaching the final orbit given in Table XXIV.

Fig. 26. Trajectory optimization results with \( t_{b3}^{\text{max}} = 400 \) s. For the third subfigure: Pitch angle (green) and flight path angle (red). The vertical dashed red lines indicate the start of a new phase. Note that the third stage structure mass has not been ejected at the third stage burnout.

The final orbit had an apogee of 646 km and a perigee of 139 km altitude above Earth’s surface with a maximum dynamic pressure of 52 kPa after 16.5 s from launch (or 17.6 km altitude). With an overall ascent time of 471 s, or \( \sim 8 \) min.
The number of objective function evaluations.

**FIG. 27.** Convergence of the objective function and constraints violation versus the number of objective function evaluations.

![Convergence of the objective function and constraints violation versus the number of objective function evaluations](image-url)

**TABLE XXIV**

<table>
<thead>
<tr>
<th>$a$ (km)</th>
<th>$e$</th>
<th>$i$ (deg)</th>
<th>$\Omega$ (deg)</th>
<th>$\omega$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6771</td>
<td>0.037</td>
<td>97</td>
<td>125</td>
<td>333</td>
</tr>
</tbody>
</table>

Figure 27 displays the convergence of the solver; requiring a total of 497 fitness evaluations and 342 gradient evaluations reaching a final objective function value $J = 84.4$.

**VIII. SUMMARY AND DISCUSSION**

The objective of this study (see section IV-A Mission Objectives) was to design a launch vehicle capable of being launched from the JAS 39E/F Gripen fighter aircraft at 10 km altitude with an initial flight path angle of $45^\circ$ at Mach 0.95, and be able to transport and deploy a nanosatellite into a final 500 km SSO orbit.

**A. Launch vehicle**

A conceptual design of an air-launched, three-staged launch vehicle was derived as presented in Table XIX & XX and Fig. 22. Consisting of two solid-propellant stages using an aluminized ammonium perchlorate composite propellant, with a density of 1800 kg/m$^3$ and a burn rate of 7 mm/s to provide a sufficiently steady burn profile and a valid grain geometry.

For both the solid-propellant stages, an internal-burning tube was considered as the best alternative compared to a grain configuration with a star perforation, since the star resulted in too much additional propellant being required, due to the star’s property of possessing residuals slivers.

By using a slower burn rate of 7 mm/s, a sufficiently neutral thrust profile was provided by the internal-burning tube configuration, with both sides uninhibited for the first stage and the forward end inhibited for the second stage. Another advantage gained from using the internal-burning tube grain is the regressive burn received at the end of the burn; this helps to minimise the terminal acceleration, acting as a natural throttle-back procedure as the mass of the launch vehicle reduces during the burn. Looking at the thrust-to-weight ratios in Table XX have a maximal acceleration of 13.5 g, which is slightly higher compared to conventional launchers designed for larger payloads. E.g. SpaceX Falcon 9 payload user guide states an axial acceleration range between -4 g to 8.5 g for light payloads (< 4’000 lb) [31].

The preferred shell material, from a minimum mass standpoint, was ASTM/ASE 300M for the first stage and Aluminium alloy 7068 for the second stage, saving around 2 kg of mass in both cases from the choice of material alone.

For both solid stages, a bell nozzle with an exit angle of $8^\circ$ was implemented, resulting in an 80 % shorter diverging length of the nozzle compared to using a conical nozzle contour, having an initial expansion angle of $32^\circ$ and a correction factor of $\lambda = 0.989$. The first stage could be expanded optimally to 40 % of the ambient pressure at 10 km altitude without the need for a truncated nozzle, while the second stage required a truncated nozzle design to adhere to the outer constraint of not expanding outside of the outer width of the second stage. This resulted in a nozzle with a 3.4 % lower thrust coefficient than what would have been obtained with an optimal expansion. However, as can be seen in Table XIV the overall length of the unconstrained nozzle is not feasible regarding the mission constraints, in addition to the increased structural mass that such a nozzle would require. Thus, it does not outweigh the slight reduction in $C_F$.

The required wall thickness for both the first and second stages nozzles to withstand the internal pressure from the exhaust was under 1 mm in both cases. However, to be able to withstand the aerodynamic- and axial forces along with shocks and vibrations would most likely require additional thickness and support rings.

For the third stage, a green liquid propellant thruster was considered to be the best alternative. This was chosen considering the orbit injection process, which would most likely require several minor adjustments, taking into account perturbations caused by the atmosphere and launch condition. The choice of helium for the pressurised feed system resulted in 0.25 kg required at 150 bar of pressure with a gas tank diameter of 27.2 cm. Choosing a higher initial pressure or a heavier gas, such as nitrogen, can provide a smaller volume at the expense of an increased mass.

The final designed LV has a total mass of 1,280 kg, a diameter of 0.82 m and total length of 4.48 m, which is 0.52 m below the constraint of 5 m. This allows for longer nozzles, grains or higher ellipsoidal head height to be implemented. Alternatively, a slimmer and longer nose cone configuration could be used, e.g. a LV-Hack shape with $C = 1/3$. Here a complete CFD analysis would also be required to reduce the drag and be able to withstand the dynamical pressure and the aerothermal heat rate. Lastly, by increasing the length or width of the LV would also inevitably result in a higher mass.

1) Comparison: The LV presented in this study differs from previous results by Sigvant [3] with the predominant factor being a possible payload mass of 12.9 kg instead of 22.0 kg as suggested by Sigvant. This is most likely due to the 100 kg lower MLOM, different choice of propellant and grain configuration for the first two stages, having a liquid propellant third stage with lower specific impulse, the use of a rotating ECEF with aspherical gravitation, lower initial launch velocity of Mach 0.95 and initial launch angle of $45^\circ$ at 10 km altitude.
The air-launched LV presented by Kesteren et al. [32] designed to be launched from the F-16 fighter aircraft, was derived using a Multi-disciplinary Optimisation (MDO) approach, considering several design variables such as release altitude and velocity, initial flight path angle, number of stages, burn times and cost. Resulting from the study was a three-stage solid-propellant LV with a GLOM of 931 kg, a length of 5.42 m and 0.56 m in diameter, launched under 50° with an initial velocity of Mach 0.85 at 15 km altitude, and capable of launching a 10 kg payload to a 780 km circular orbit.

Another study made by Young et al. [33] derived an air-launched LV by using a MDO approach, investigating several factors such as subsonic and supersonic launch velocities as well as hybrid and solid propulsion systems. The resulting optimized LV had a GLOM of 1,245 kg, a total length of 6.36 m, an outer diameter of 0.60 m and capable of launching a 7.5 kg payload to a 700 km circular orbit. The initial launch conditions were set to an altitude of 12 km and a velocity of Mach 1.5. The initial launch angle used was not mentioned.

The two aforementioned studies resulted in longer, thinner and lighter LV designs than presented in this study, with payload capabilities under 10 kg and targeting circular orbits at 700 km or higher. Considering these results might suggest some pessimistic assumptions made regarding the initial launch conditions and mission constraints used in this study; mainly the MLOM, initial launch altitude and velocity. However, the highly nonlinear and intricate nature of designing a complete LV makes it extremely difficult to pinpoint any specific factor that is the cause of the different results. Lastly, one should note the confidentiality aspect regarding the conversion error as seen in Fig. 27 suggest that this steering law should only be used as an illustrative solution for a possible shape of an optimized steering law.

**B. Trajectory analysis**

The trajectory problem stated as an IVP and solved using a Sigmoidal steering law to control the pitch showed that it was possible to send up 8.4 kg into a 500 km SSO using the designed LV and a burn time of 284 s for the third stage. Moreover, adding 7.1 kg of propellant to the third stage (neglecting the additional structural masses resulting from this) and increasing the burn time to 365 s, provided a possible payload of 12.9 kg.

One possible modification that could be made is to add an equivalent amount of $\Delta v$ to the second stage instead, thus staying below the maximum designed propellant throughput for the third stage. However, this will increase the parabolic shape of the burn area profile for the second stage, requiring a slower burn rate to compensate. One solution would be to add additional surface area, e.g. by using a finocyl configuration in the grain. However, this needs to be verified by performing another iteration of the LV, which was not pursued due to time limitations for the study itself.

No other comparable study using a Sigmoidal steering law to control the LV was found. Due to the close relation towards military applications and confidentiality, many reports seem to choose not to present their full results. Some studies using vertical ground-launched systems were found where the steering laws seemed to follow an exponential steering law, e.g. Balesdent [22]. This was mainly the reason why another method was pursued to solve the trajectory problem: by transcribing it into a NLP and solve for a parametric steering law using a gradient descent sequential quadratic programming algorithm.

Comparing the two steering laws provided by each method was the drift time which makes it hard to compare them accurately. The drift time was largely neglected in the case of the NLP since it posed a problem too large for the solver to compute.

An interesting result was discovered from the NLP solution; after the first stage, it was more optimal to perform a slight pitch up manoeuvre during for the second stage, then pitching down again for the third stage. This gave rise to an increasingly negative pitch angle for the extent of the third stage. Indicating that due to the lack of sufficient drift time, the third stage needs to overcompensate with a negative pitch angle to bring down the flight path angle towards zero for the third stage burnout to achieve a final circular orbit. This behaviour was also found by Balesdent [22] for a larger ground-launched system, launching towards a Geostationary Transfer Orbit.

The final orbit given from the NLP solution resulted in an apogee of 646 km and a perigee at 139 km. Thus, if this steering law were to be used, further orbital analyses should be made to investigate how long the payload could stay in that orbit before reentering Earth’s atmosphere. Lastly, the convergence errors as seen in Fig. 27 suggest that this steering law should only be used as an illustrative solution for a possible shape of an optimized steering law.

**IX. Conclusion**

To conclude, the objective of designing a launch vehicle capable of being launched from the JAS 39E/F Gripen fighter aircraft at 10 km altitude with an initial flight path angle of 45° at Mach 0.95, and be able to transport and deploy a nanosatellite into a final 500 km SSO orbit, was achieved.

A LV design capable of launching a payload of 8.4 kg to the targeted orbit was found. Additionally, it was showed that adding 7.1 kg of propellant to the third stage would result in a higher possible payload of 12.9 kg, which is in the same range as comparative studies.

Lastly, a Sigmoidal steering law was shown to provide a possible solution to the trajectory ascent problem along with a parametric steering law defined by linear interpolation between 38 discretization points.

**A. Future work**

There are many aspects that can be improved and investigated further in this study. The most predominant would be to investigate the burn time and thrust for each stage and how these variables correlate with the initial launch angle and velocity along with the GLOM and possible payload mass.
These factors correlate in a highly non-linear fashion and would be most appropriately studied using a Multi-disciplinary Analysis, e.g. as the one presented by Balesdent [22], Kesteren et al. [32] and Young et al. [33].

In addition, the 3-DOF assumption point mass assumption for the ascent should be expanded to 9-DOF, including lift, varying thrust with the ambient pressure, along with a proper structural analysis: taking into account the bending forces and heat rate experience by the LV during the ascent. Considerations towards: aerodynamical effects such as maximal aerodynamical pressure and aerothermal heat rate, a nonconstant drag coefficient depending on angle of attack and Mach number, along with maximal axial- and lateral acceleration should also be considered.

A thermochemical analysis of the aluminized ammonium perchlorate composite propellant could also be performed, to see if a 7 mm/s burn rate together with a density of 1,800 kg/m$^3$ is possible to acquire. Additionally, more grain geometries could be investigated, alternatively looking into slotted configurations or a N-star grain perforation with both ends uninhibited.

Regarding the NLP, a more suitable gradient algorithm using the sparsity that exists in the Jacobian matrix should be used. For example, the commercial SNOPT algorithm, which in hindsight to the author, provide free trials for students at the time of writing. This would allow for a larger amount of grid points to be used, resulting in higher accuracy. Since the equality constraint errors did not converge towards zero (see Fig. 27) indicates that a better solution can be found. Lastly, the weights used for the objective function could be manipulated further to result in a less eccentric final orbit.

Lastly, a complete cost analysis regarding the full life cycle of the LV should be performed in order to provide a motivated business case.

**APPENDIX A**

Results from solving the trajectory problem as an IVP using the designed LV configuration with a payload mass of 8.4 kg, $t_{b3} = 284$ s and a GLOM of 1,289 kg.

<table>
<thead>
<tr>
<th>TABLE XXV</th>
<th>ORBITAL ELEMENTS - IVP 500 KM, DESIGNED LV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ (km)</td>
<td>6890</td>
</tr>
<tr>
<td>$e$</td>
<td>0</td>
</tr>
<tr>
<td>$i$ (deg)</td>
<td>97</td>
</tr>
<tr>
<td>$\Omega$ (deg)</td>
<td>125</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TABLE XXVI</th>
<th>VELOCITY BUDGET - DESIGNED LV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_f$</td>
<td>7.61 km/s</td>
</tr>
<tr>
<td>$v_i$</td>
<td>0.30 km/s</td>
</tr>
<tr>
<td>$\Delta v_{\text{thrust}}$</td>
<td>8.73 km/s</td>
</tr>
<tr>
<td>$\Delta v_{\text{grav}}$</td>
<td>-0.46 km/s</td>
</tr>
<tr>
<td>$\Delta v_{\text{drag}}$</td>
<td>-0.35 km/s</td>
</tr>
</tbody>
</table>

Fig. 28. The flight parameters during the ascent of the designed launch vehicle with a GLOM of 1,289 kg, $t_{b3} = 284$ s, carrying a 8.4 kg payload where: Stage 1 (red), Stage 2 (yellow), drift (blue), Stage 3 (red), Payload (green).

**APPENDIX B**

Transformation matrix $C_{\text{GCRF}}^{\text{PQW}}$ transforming vectors from a Perifocal reference frame (PQW) to a Geocentric Celestial Reference frame (GCRF).

$$C_{\text{GCRF}}^{\text{PQW}} = \begin{bmatrix} A & B & C \end{bmatrix}$$

$$A = \begin{bmatrix} \cos(\Omega)\cos(w) - \sin(\Omega)\sin(w)\cos(i) \\ \sin(\Omega)\cos(w) + \cos(\Omega)\sin(w)\cos(i) \\ \sin(w)\sin(i) \end{bmatrix}$$

$$B = \begin{bmatrix} -\cos(\Omega)\sin(w) - \sin(\Omega)\cos(w)\cos(i) \\ -\sin(\Omega)\sin(w) + \cos(\Omega)\cos(w)\cos(i) \\ \cos(w)\sin(i) \end{bmatrix}$$

$$C = \begin{bmatrix} \sin(\Omega)\sin(i) \\ -\cos(\Omega)\sin(i) \\ \cos(i) \end{bmatrix}$$
APPENDIX C

Pseudo-code describing an iterative method for designing a constrained supersonic nozzle, written in Python using the Numpy package (core package in the SciPy library) [20].

```
import numpy as np
def constrained_nozzle(P01, P_amb, T01, gamma, eta_s, A_2_max, A_t, M_2, mass_flow, F):
i = 0
tol = 0.01 # 1 % tolerance of F_design
F_new = F + F*(tol*2)
M_new = M_2
mass_flow_new = mass_flow
A_t_new = A_t
epsilon = A_2_max / A_t
while np.abs(1-F_new/F) > tol and i < 10:
    M_new = # Solve for M_new given the new area ratio epsilon
    T_new = T01 / (1 + (M_new**2)*(gamma -1)/2)
    P_new = P01 * (T_new/T01) ** (gamma / (eta_s*(gamma-1)))
    u_new = np.sqrt(2*Cp*(T01 - T_new))
    F_new = lambd*mass_flow_new*u_new + A_2_max * (P_new - P_amb)
    factor = 1 - F_new/self.F
    A_t_new = A_t_new * (1 + factor)
    mass_flow_new = A_t_new * rho_t * u_t
    epsilon = A_2_max / A_t_new
    i += 1
return [A_t_new, mass_flow_new, T_new, P_new, u_new, F_new]
```

APPENDIX D

Exhaust gas composition from burning an Aluminized Ammonium Perchlorate Composite Propellant at a combustion pressure of 6.895 MPa.

<table>
<thead>
<tr>
<th>Ingredient</th>
<th>Mole Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>0.00009</td>
</tr>
<tr>
<td>AlCl</td>
<td>0.00499</td>
</tr>
<tr>
<td>AlCl₂</td>
<td>0.00167</td>
</tr>
<tr>
<td>AlCl₃</td>
<td>0.00023</td>
</tr>
<tr>
<td>AlH</td>
<td>0.00002</td>
</tr>
<tr>
<td>AlO</td>
<td>0.00009</td>
</tr>
<tr>
<td>AlOCl</td>
<td>0.00065</td>
</tr>
<tr>
<td>AlOH</td>
<td>0.00032</td>
</tr>
<tr>
<td>Al₂O₃</td>
<td>0.00026</td>
</tr>
<tr>
<td>Al₂O₅</td>
<td>0.00004</td>
</tr>
<tr>
<td>Al₂O₅(solid)</td>
<td>0.00000</td>
</tr>
<tr>
<td>Al₂O₃(liquid)</td>
<td>0.09378</td>
</tr>
<tr>
<td>CO</td>
<td>0.22374</td>
</tr>
<tr>
<td>CO₂</td>
<td>0.00001</td>
</tr>
<tr>
<td>CO₃</td>
<td>0.00790</td>
</tr>
<tr>
<td>Cl</td>
<td>0.00620</td>
</tr>
<tr>
<td>Cl₂</td>
<td>0.00001</td>
</tr>
<tr>
<td>H</td>
<td>0.02525</td>
</tr>
<tr>
<td>HCl</td>
<td>0.11900</td>
</tr>
<tr>
<td>HCN</td>
<td>0.00002</td>
</tr>
<tr>
<td>HCO</td>
<td>0.00002</td>
</tr>
<tr>
<td>H₂</td>
<td>0.32380</td>
</tr>
<tr>
<td>H₂O</td>
<td>0.08937</td>
</tr>
<tr>
<td>NH₃</td>
<td>0.00001</td>
</tr>
<tr>
<td>NH₄</td>
<td>0.00003</td>
</tr>
<tr>
<td>NO</td>
<td>0.00021</td>
</tr>
<tr>
<td>N₂</td>
<td>0.09886</td>
</tr>
<tr>
<td>O</td>
<td>0.00014</td>
</tr>
<tr>
<td>OH</td>
<td>0.00297</td>
</tr>
<tr>
<td>O₂</td>
<td>0.00001</td>
</tr>
</tbody>
</table>

Fig. 29. Mole fraction of the exhaust gas resulting from burning an Aluminized Ammonium Perchlorate Composite Propellant at a combustion pressure of 6.895 MPa, adapted and reproduced from Sutton [5].
REFERENCES


