KTH ROYAL INSTITUTE
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Doctoral Thesis in Mathematics

Stacky Modifications and Operations in the Étale Cohomology of Number Fields

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Stockholm, Sweden 2022

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Abstract

This thesis consists of 4 papers. In Paper A we define *stacky building data* for *stacky covers* in the spirit of Pardini and give an equivalence of $(2,1)$-categories between the category of stacky covers and the category of stacky building data. We show that every stacky cover is a flat root stack in the sense of Olsson and Borne–Vistoli and give an intrinsic description of it as a root stack using stacky building data. When the base scheme S is defined over a field, we give a criterion for when a stacky building datum comes from a ramified cover for a finite abelian group scheme over k, generalizing a result of Biswas–Borne.

In Paper B we compute the étale cohomology ring $H^*(X, \mathbb{Z}/n\mathbb{Z})$ for X the spectrum of the ring of integers of a number field K. As an application, we give a non-vanishing formula for an invariant defined by Minhyong Kim. We also give examples of two distinct number fields whose rings of integers have isomorphic cohomology groups but distinct cohomology ring structures.

In Paper C we generalize the results of Paper B to include the case when X is replaced by an open subset $U \subseteq X$, where we have removed a finite number of closed points from X. We show that when U is the complement of two odd primes p and q which are congruent to 1 (mod 4), the Legendre symbol of p over q may be interpreted as a cup product in $H^*(U, \mathbb{Z}/2\mathbb{Z})$.

In Paper D we find formulas for Massey products in étale cohomology of the ring of integers of a number field. Then we use these formulas to, with the help of a computer, find the first ever known examples of imaginary quadratic fields with p-class group of rank 2 for odd p and infinite class field tower. We also compute examples disproving McLeman's $(3,3)$-conjecture.

Sammanfattning

Denna avhandling består av 4 artiklar. I Artikel A definierar vi *stackig byggnadsdata* för *stackiga övertäckningar* i Pardinis anda och visar en ekvivalens av $(2,1)$-kategorier mellan kategorin av stackiga övertäckningar och kategorin av stackig byggnadsdata. Vi visar att varje stackig övertäckning är en platt rotstack i Olsson och Borne–Vistolis mening och vi ger en intrinsisk beskrivning av den som en rotstack med hjälp av stackig byggnadsdata. När basen S är definierad över en kropp ger vi ett kriterium för när ett stackigt byggnadsdatum kommer från en ramifierad övertäckning för ett ändligt abelskt gruppschema över k. Detta generaliserar ett resultat av Biswas–Borne.

I Artikel B beräknar vi den étala kohomologiringen $H^*(X, \mathbb{Z}/n\mathbb{Z})$ då X är spektrumet av ringen av heltal av en talkropp K. Som en tillämpning, ger vi ett kriterium i form av en formel för när en invariant definierad av Minhyong Kim är noll eller ej. Vi ger också exempel på två olika talkroppar vars ringar av heltal har isomorfa kohomologigrupper men olika kohomologiringstrukturer.

I Artikel C generaliserar vi resultaten i Artikel B till att innefatta fallet då X ersätts av en öppen delmängd $U \subseteq X$, där vi tagit bort ett ändligt antal slutna punkter ifrån X. Vi visar att då U är komplementet till två udda primtal p och q, som är kongruenta till 1 (mod 4), så kan Legendre symbolen av p över q betraktas som en kopprodukt i $H^*(U, \mathbb{Z}/2\mathbb{Z})$.

I Artikel D beräknar vi formler för Masseyprodukter i étale kohomologi av ringen av heltal till en talkropp. Vi använder sedan dessa formler för att, med hjälp av en dator, hitta de första kända exemplen på kvadratiskt imaginära talkroppar vars klassgrupp har p-rang 2, för udda p, och oändligt p-klasskroppstorn. Vi beräknar också exempel som motbevisar McLemans $(3,3)$-förmodan.

Contents

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Part I

Introduction and summary of results

1 Introduction

This thesis consists of two main parts:

The first introductory part (kappa) gives some motivation and background. The scientific results are presented in an informal way to invite a broader audience.

The second part consists of four scientific papers, three which are more towards number theory and one which is more towards stacks and moduli.

Paper A treats objects which we call *stacky covers* and the main result is a classification of these in terms of *stacky building data*.

Paper B, C, and D are concerned with étale cohomology of arithmetic curves. Inspired by ideas from algebraic topology and the analogy between arithmetic curves and 3-manifolds, we compute the cup product for étale cohomology with finite coefficients in Paper B and C. In Paper D we find formulas for Massey products in the étale cohomology ring of a proper arithmetic curve. Then we use these formulas to, with the help of a computer, find the first ever known examples of imaginary quadratic fields with p-class group of rank 2, for odd p, and infinite class field tower. We also compute examples showing that McLeman's $(3, 3)$-conjecture is false.

2 Number fields and Class Field Towers

1 Class field towers

A prime number p is the sum of two squared integers

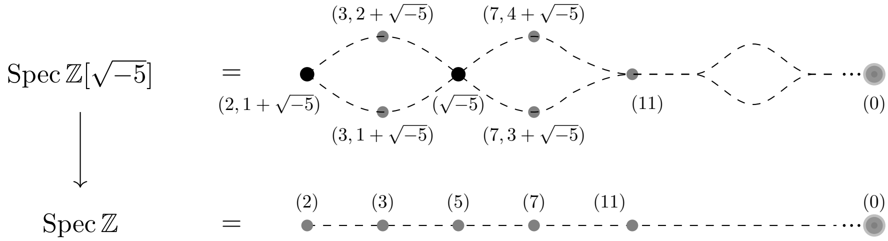
$$p = a^2 + b^2$$

if and only if $p \equiv 1 \mod 4$. This was proved by Fermat in the mid 1600s [Fer94]. If we take the integers $\mathbb{Z}$ and add a square root of -1, we obtain the *Gaussian integers* $\mathbb{Z}[\sqrt{-1}]$. Fermat's proof depends on the fact that all elements in $\mathbb{Z}[\sqrt{-1}]$ can be uniquely (up to units) factored into prime elements, just like any positive integer can be uniquely factored into prime numbers. Fermat also characterized the prime numbers of the form $p = a^2 + 2b^2$ or $p = a^2 + 3b^2$, but was not able to prove which prime numbers are of the form $p = a^2 + 5b^2$. The important difference in the latter case is that the ring $\mathbb{Z}[\sqrt{-5}]$ is no longer a unique factorization domain in contrast to the case where we added $\sqrt{-1}$, $\sqrt{-2}$, or $\sqrt{-3}$.

Before we continue the discussion, let us take a short detour to give an idea of how geometry comes into play. Every ring mentioned so far is an example of a *ring of integers* of a *number field*. For instance, $\mathbb{Z}[\sqrt{-5}]$ is the ring of integers in the number field $\mathbb{Q}(\sqrt{-5})$.

In formal language, a *number field* is a finite field extension $K \supseteq \mathbb{Q}$. The *ring of integers* $\mathcal{O}_K$ of K is the algebraic closure of $\mathbb{Z}$ in K. The ring $\mathcal{O}_K$ is an integrally closed Noetherian integral domain of Krull dimension one, that is, a *Dedekind domain*. In particular, every non-zero prime ideal in $\mathcal{O}_K$ is maximal.

From the point of view of algebraic geometry we now have a *geometric* object $\operatorname{Spec} \mathcal{O}_K$ which we refer to as an *arithmetic curve* and to which we may apply geometric tools such as *étale cohomology*. The ring $\mathcal{O}_K$ is finitely generated and free (since it is torsion free) as a module over $\mathbb{Z}$ and its rank is equal to the degree of $K \supseteq \mathbb{Q}$. This means that the canonical morphism $\operatorname{Spec} \mathcal{O}_K \to \operatorname{Spec} \mathbb{Z}$ is finite, flat, and surjective (since $\mathbb{Z} \subseteq \mathcal{O}_K$ is integral), i.e., a "branched covering".



For every non-zero prime (hence maximal) ideal $P \in \mathcal{O}_K$ we have that $P \cap \mathbb{Z}$ is generated by a prime number p. The residue field of P will be finite of order $q = p^n$ for some $1 \leq n \leq [K : \mathbb{Q}]$ and the surjection $\mathcal{O}_K \to \mathcal{O}_K/P \cong \mathbb{F}_q$ corresponds to a morphism $\operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathcal{O}_K$.

Back to the initial discussion, the failure of unique factorization in a ring of integers $\mathcal{O}_K$ is measured by the *class group* $\mathrm{Cl}(K)$. For example, the ring $\mathbb{Z}[\sqrt{-5}]$ has non-trivial class group. So even though the prime 3 splits as $(3) = (3, 1 + \sqrt{-5})(3, 2+\sqrt{-5})$, we cannot apply Fermat's technique since the ideals $(3, 1+\sqrt{-5})$ and $(3, 2+\sqrt{-5})$ are not principal. In 1902 Hilbert proposed that for every number field K there is a unique unramified abelian extension $H_K \supseteq K$, today known as the *Hilbert class field of* K, such that every element in $\mathrm{Cl}(K)$ becomes trivial once extended to H_K [Hil02]. The existence and uniqueness of the Hilbert class field was proven a few years later by Philipp Furtwängler [Fur06].

The problem with unique factorization may now be overcome by going to the Hilbert class field $H_{\mathbb{Q}(\sqrt{-5})}$ whose ring of integers is a unique factorization domain. But the problem is solved for this example only since there are no new obstructions to unique factorization showing up in $H_{\mathbb{Q}(\sqrt{-5})}$, that is, $\mathrm{Cl}(H_{\mathbb{Q}(\sqrt{-5})}) = 0$. However, this is not always the case. For other examples of number fields K, it may very well happen that the Hilbert class field has non-trivial class group and one has to go one step further and take the Hilbert class field H_{H_K} of H_K and the Hilbert class field of H_{H_K} and so on, until we get a field with trivial class group. Inspired by this phenomenon, Furtwängler asked the following question in 1925: Will the tower

$$K \subseteq H_K \subseteq H_{H_K} \subseteq \dots$$

finally stabilize or are there number fields such that this tower is infinite? This is equivalent to asking if some field in the tower has trivial class group, since $\mathrm{Cl}(L) = 0$ if and only if $H_L = L$ for any number field L.

This was an open problem until 1964 when Golod and Shafarevich found a counter-example [Gv64] giving a group theoretic argument. More precisely, given a prime number p, we may consider the p-class field tower. The p-class field of a number field K is simply the subfield of the Hilbert class field in which every p-torsion element of $\mathrm{Cl}(K)$ vanishes. The Galois group of the limit of the p-class field tower is a pro p-group and Golod and Shafarevich proved a numerical inequality between the number of generators and the number of relations for such groups.

This inequality could then be used to prove that, for instance, the number field $\mathbb{Q}(\sqrt{-3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17})$ has infinite 2-class field tower.

Now let p be an odd prime. For imaginary quadratic fields K, the results of Golod–Shafarevich, together with a result of Furtwängler, almost gives a complete classification, in terms of the p-rank of the class group, of when the p-class field is infinite: if we let $l_p(K)$ be the least number of times we have to take the Hilbert class field before the tower stabilizes, then we have

$$l_p(K) = \begin{cases} 0 & \text{if } p\text{-rank}(\mathrm{Cl}(K)) = 0\,, \\ 1 & \text{if } p\text{-rank}(\mathrm{Cl}(K)) = 1\,, \\ ? & \text{if } p\text{-rank}(\mathrm{Cl}(K)) = 2\,, \\ \infty & \text{if } p\text{-rank}(\mathrm{Cl}(K)) \geq 3 \end{cases}$$

(see [McL08]). This shows that the only remaining case of interest is when $\mathrm{Cl}(K)$ has p-rank 2. Before Paper D, there where no known examples, for odd p, of imaginary quadratic fields K with class group of p-rank 2 and infinite p-class field tower. We prove the following:

Theorem 1.1. *There exists odd prime numbers p and imaginary quadratic fields of discriminant D with class group of p-rank two and infinite p-class field tower. For instance, for prime p and discriminant D, the pairs*

$$\begin{aligned}(p, D) = {} & (3, -3826859), (3, -8187139), (3, -11394591), (3, -13014563)\,, \\ & (5, -2724783), (5, -4190583), (5, -6741407), (5, -6965663)\end{aligned}$$

give examples of such fields: for each pair in the above list, the associated quadratic imaginary field with discriminant D has an infinite p-class field tower.

To prove this theorem we use algebraic geometry and one of our most important results: a formula for *Massey products* in the étale cohomology of arithmetic curves.

McLeman showed in [McL08], using results of Vogel, that the structure of the Galois group G of the Hilbert p-class field tower is controlled by Massey products in the Galois cohomology $H^*(G, \mathbb{Z}/p\mathbb{Z})$. In particular, he shows that when all 3-fold Massey products are zero, the p-class field tower is infinite. Using our formulas for Massey products in étale cohomology and comparing the étale cohomology with the Galois cohomology $H^*(G, \mathbb{Z}/p\mathbb{Z})$, we may use a computer to determine if the arithmetic curve, associated to an imaginary quadratic field, has vanishing 3-fold Massey products and hence, by McLeman's result, an infinite class field tower.

McLeman predicted that the Hilbert p-class field tower of an imaginary quadratic field K with class group of p-rank 2 has finite p-class field tower if and only if K has *Zassenhaus type* $(3, 3)$ [McL08, Conjecture 2.9]. We disprove McLeman's conjecture by finding counter-examples:

Theorem 1.2. *The* $(3,3)$*-conjecture is false. For instance, for* p *a prime and* D *a discriminant, the pairs*

$$(p, D) = (5, -90868), (7, -159592)$$

are counter-examples to the $(3,3)$*-conjecture: for each pair in the above list, the associated quadratic imaginary field with discriminant* D *has Zassenhaus type* $(5,3)$ *or* $(7,3)$*, but the* p*-class field tower is finite.*

In Paper B and C we find formulas for the *cup product* in étale cohomology and in Paper D we find formulas for the *Massey product* in étale cohomology. Both the cup product and the Massey product is part of a multiplicative structure on the étale cohomology of $\operatorname{Spec} \mathcal{O}_K$, which encodes important arithmetic information about the number field K. Both theorems above illustrates how the multiplicative structure on étale cohomology may be used to prove concrete number theoretical statements. To give another example, we state a Proposition from Paper C, which illustrates how the cup product encodes the classical *Legendre symbol*:

Proposition 1.3. *Let* p *and* q *be distinct odd primes both equal to* 1 *(mod 4) and let* $U = \operatorname{Spec} \mathbb{Z} \setminus \{p, q\}$*. Let* x_p *and* x_q *be the elements in* $H^1(U, \mathbb{Z}/2\mathbb{Z})$ *corresponding to the extensions* $\mathbb{Q}(\sqrt{p})$ *and* $\mathbb{Q}(\sqrt{q})$ *respectively. Then* $x_p \smile x_q$ *is completely determined by the Legendre symbol* $\left(\frac{p}{q}\right)$ *and vice versa. In particular,* $x_p \smile x_q = 0$ *if and only if* $\left(\frac{p}{q}\right) = 1$*.*

The theory of étale cohomology of arithmetic curves will be discussed in Section 4 and 5. To prove our results we use ideas from algebraic topology and much inspiration comes from a fascinating analogy between number fields and 3-dimensional manifolds. This analogy is *homotopical* in nature and to give a taste, we first need to introduce the *étale fundamental group*. In the rest of this introduction, we will use a more formal mathematical language.

2 The étale fundamental group

The *étale fundamental group* is an algebro-geometric analogue of the fundamental group in algebraic topology. The étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ of a connected and locally noetherian scheme X with a geometric point $\bar{x}$, is a profinite group. That is, an inverse limit of finite groups, which classifies finite étale covers $Y \to X$ in the following sense: let $(\mathsf{F\acute{E}t}/X)$ be the category of finite étale morphisms $Y \to X$ and let $\pi_1^{\text{ét}}(X, \bar{x})$-sets be the category of finite sets on which $\pi_1^{\text{ét}}(X, \bar{x})$ acts continuously on the left. There is an equivalence of categories

$$(\mathsf{F\acute{E}t}/X) \to \pi_1^{\text{ét}}(X, \bar{x})\text{-sets}$$

which takes $Y \to X$ to the fiber $Y_{\bar{x}}$ over the geometric point $\bar{x}$.

Example 2.1. Let X be the spectrum of a field k, and $\bar{x}\colon \operatorname{Spec}\Omega \to \operatorname{Spec} k$ a geometric point corresponding to a separable closure Ω/k.

Example 2.2. Let X be a smooth projective variety over $\mathbb{C}$ and let X^{an} be the associated complex analytic space (see e.g. [Har77, Appendix B]). Then $\pi_1^{\text{ét}}(X, x)$ is the profinite completion of $\pi_1(X^{an}, x)$.

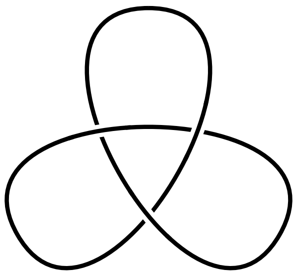
3 An unexpected analogy

In the 60's Barry Mazur started investigating an interesting analogy between knots in 3-manifolds and primes in number rings. According to Mazur's notes [Maz63] from 1963–1964, the idea is originally due to David Mumford. The analogy says that given a number field K with ring of integers $\mathcal{O}_K$ we may think of $\operatorname{Spec}\mathcal{O}_K$ as a three dimensional simply connected real manifold M, and that we may think of a prime

$$\operatorname{Spec}\mathbb{F}_q \to \operatorname{Spec}\mathcal{O}_K$$

as a knot

$$\gamma\colon S^1 \hookrightarrow M\,.$$

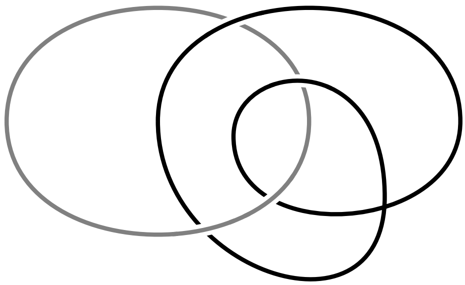


Today, there is a long dictionary which translates concepts, from the arithmetic side of this analogy, to the world of manifolds. To get a hint, we give an example, which is a short version of Section 4 of Morishita's book [Mor12].

Example 3.1. Alexander's theorem states that any connected oriented 3-manifold is a finite covering of the 3-dimensional sphere S^3 branched over a finite union of knots. Similarly, any arithmetic curve is a finite cover of $\operatorname{Spec}\mathbb{Z}$ branched over a finite set of primes.

Let $\gamma, \delta\colon S^1 \hookrightarrow S^3$ be knots and let $X_\delta = S^3 \setminus V_\delta$ where V_δ is a tubular neighborhood of δ. The boundary of V_δ looks like a (knotted) torus and we let α be a meridian which we think of as a knot $\alpha\colon S^1 \to S^3$. Take $x \in X_\delta$. Then one can show that there is a unique surjective morphism $\pi_1(X_\delta, x) \to \mathbb{Z}$ sending α to

1 and if we take the quotient by 2 we get a morphism $\varphi\colon \pi_1(X_\delta, x) \to \mathbb{Z}/2\mathbb{Z}$. By the theory of covering spaces, this corresponds to a covering space $h_2\colon X_2 \to X_\delta$ of degree 2. The knot γ represents a class $[\gamma] \in \pi_1(X_\delta, x)$ and it turns out that $\varphi([\gamma]) \equiv \mathrm{lk}(\gamma, \delta) \pmod 2$, where $\mathrm{lk}(\gamma,\delta)$ denotes the *linking number*, i.e., the number of times the γ wraps around δ.



We get that

$$h_2^{-1}(\gamma) = \begin{cases} \gamma_1 \cup \gamma_2 & \text{if } \mathrm{lk}(\gamma,\delta) \equiv 0 \pmod 2\,, \\ \gamma' & \text{if } \mathrm{lk}(\gamma,\delta) \equiv 1 \pmod 2\,. \end{cases}$$

This can be compared to the analogous arithmetic situation. Let p and q be primes such that $q \equiv 1 \pmod 4$. Let

$$X_q = \operatorname{Spec}\mathbb{Z} \setminus \{(q)\} = \operatorname{Spec}\mathbb{Z}[1/q]$$

and let $G_q = \pi_1^{\text{ét}}(X_q, \bar{x})$ where we choose the base point $\bar{x}\colon \operatorname{Spec}\overline{\mathbb{Q}}_p \to \operatorname{Spec}\mathbb{Z}[1/q]$ corresponding to a fixed embedding $\mathbb{Q} \to \overline{\mathbb{Q}}_p$. Let $\tilde{\alpha}$ be a generator of $\mathbb{F}_q^\times$. Consider the morphism

$$\tilde{\varphi}\colon G_q \to G_q^{ab} \cong \mathbb{Z}_q^\times \to \mathbb{Z}_q^\times/(1+q\mathbb{Z}_q) \cong \mathbb{F}_q^\times \to \mathbb{Z}/2\mathbb{Z}$$

sending $\tilde{\alpha}$ to 1. By the theory of (étale) covering spaces, this corresponds to a quadratic extension $\mathbb{Q}(\sqrt{q}) \supset \mathbb{Q}$ and by taking rings of integers we get an étale double cover $\tilde{h}_2\colon X_2 \to X_q$. Define

$$\mathrm{lk}(p,q) = \begin{cases} 0 & \text{if } x^2 \equiv q \pmod p \text{ has a solution}\,, \\ 1 & \text{otherwise} \end{cases}$$

(this is almost the *Legendre symbol*). To get a result analogous to the 3-manifold case, we have to make sense of an element of $\pi_1^{\text{ét}}(X_q, \bar{x})$ represented by p. This is done as follows. We have a canonical morphism

$$\pi_1^{\text{ét}}(\operatorname{Spec}\mathbb{Q}_p, \bar{x}) = \operatorname{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \pi_1^{\text{ét}}(X_q, \bar{x})$$

and we define σ_p to be the image of the Frobenius automorphism (sending an element to its p:th power). It turns out that $\tilde{\varphi}(\sigma_p) = \mathrm{lk}(p, q)$. From this one can show that

$$\tilde{h}_2^{-1}(p) = \begin{cases} \{p_1, p_2\} & \text{if } \mathrm{lk}(\gamma, \delta) = 0\,, \\ p' & \text{if } \mathrm{lk}(\gamma, \delta) = 1\,, \end{cases}$$

in analogy with the situation for 3-manifolds. For more examples we recommend Morishita's book [Mor12].

If one takes the analogy between number rings and 3-manifolds seriously, one may hope to be able to use techniques from algebraic topology and apply them in number theory. Paper B, C, and D are very much in this spirit as we will explain in the next section.

4 Étale cohomology of number fields

Given a number field K with ring of integers $\mathcal{O}_K$ and $X = \operatorname{Spec} \mathcal{O}_K$, we may define the étale cohomology groups

$$H^i(X, \mathbb{Z}/n\mathbb{Z})\,.$$

Barry Mazur computed these groups in the 60's [Maz73] except for the case $i = 2$. In the second paper we compute the cup product in étale cohomology, revealing the ring structure of

$$\bigoplus_i H^i(X, \mathbb{Z}/n\mathbb{Z})\,.$$

Inspired by the analogy between number rings and manifolds we use methods and ideas coming from algebraic topology. The étale cohomological dimension of X is three and the cup product is graded commutative and unital. This means that it is enough to compute the cup product

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \otimes H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^2(X, \mathbb{Z}/n\mathbb{Z})$$

and

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \otimes H^2(X, \mathbb{Z}/n\mathbb{Z}) \to H^3(X, \mathbb{Z}/n\mathbb{Z})\,,$$

or in other words, it is enough to know how to take the cup product with an element $x \in H^1(X, \mathbb{Z}/n\mathbb{Z})$. But $H^1(X, \mathbb{Z}/n\mathbb{Z})$ classifies $\mathbb{Z}/n\mathbb{Z}$-torsors, so we may choose a $\mathbb{Z}/n\mathbb{Z}$-torsor $Y \to X$ representing x. Thinking of X just as a manifold with a covering space $Y \to X$, we may hope to find a "transfer sequence"

$$0 \to \mathbb{Z}/n\mathbb{Z} \to P \to \mathbb{Z}/n\mathbb{Z} \to 0 \tag{4.1}$$

such that cup product with the element x is given as the connecting homomorphism of (4.1). This is exactly what we will do.

Every $\mathbb{Z}/n\mathbb{Z}$-torsor $p\colon Y \to X$ is induced from a $\mathbb{Z}/d\mathbb{Z}$-torsor of the form

$$\operatorname{Spec} \mathcal{O}_L \to X$$

for some $d|n$ and some Galois extension $L \supseteq K$ of degree d, unramified at all finite primes. Since p is finite étale we have that p_* is *left* adjoint to p^* and hence there is a counit map

$$N: p_* p^* \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

which we call the *norm* (often called *trace* in the algebraic geometry literature). Hence we have an exact sequence (the "norm sequence")

$$0 \to \ker N \to p_* p^* \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

and we want to find an appropriate morphism $\ker N \to \mathbb{Z}/n\mathbb{Z}$ along which we can push the norm sequence to obtain a transfer sequence.

To be able to make computations, we use the equivalence between the category of locally constant abelian sheaves split by p (that is, a locally constant sheaf on X whose pullback to Y is constant) and the category of $\mathbb{Z}/n\mathbb{Z}$-modules. When Y is *connected* this takes a locally constant sheaf $\mathcal{F}$ and sends it to the $\mathbb{Z}/n\mathbb{Z}$-module $\mathcal{F}(Y)$ where $g \in \mathbb{Z}/n\mathbb{Z}$ acts via $\mathcal{F}(g^{-1})$, and the inverse functor takes a $\mathbb{Z}/n\mathbb{Z}$-module M and sends it to the abelian group scheme

$$(Y \times M)/(\mathbb{Z}/n\mathbb{Z}),$$

where $\mathbb{Z}/n\mathbb{Z}$ acts diagonally. Under this equivalence, the transfer sequence takes the form

$$\begin{array}{c} 0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}[G]/I^2 \xrightarrow{\varepsilon} \mathbb{Z}/n\mathbb{Z} \to 0 \\ 1 \mapsto g-1 \qquad\quad g \mapsto 1 \end{array}$$

where $G \cong \mathbb{Z}/n\mathbb{Z}$ is the Galois group of Y over X, g is a fixed generator, and I is the augmentation ideal, i.e., the kernel of ε.

In Paper C, K is assumed to be a general number field with no restrictions and X an open subscheme of $\operatorname{Spec} \mathcal{O}_K$, but for simplicity of exposition, let us here assume that K is totally imaginary and that $X = \operatorname{Spec} \mathcal{O}_K$. Central to the theory of étale cohomology of number fields is Artin–Verdier duality, which states that there is a non-degenerate pairing

$$H^i(X, \mathbb{Z}/n\mathbb{Z}) \times \operatorname{Ext}^{3-i}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z},$$

where $\mathbb{G}_m$ is the sheaf of units. The cohomology groups can then be computed as duals of Ext-groups, using class field theory and the Grothendieck spectral sequence. One obtains the list

$$\operatorname{Ext}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \cong \begin{cases} \mu_n(K) & \text{if } i = 0 \\ Z_1/B_1 & \text{if } i = 1 \\ \operatorname{Cl}(K)/n & \text{if } i = 2 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 3, \end{cases}$$

where

$$Z_1 = \{(a, I) \in K^\times \oplus \mathrm{Div}(K) : \mathrm{div}(a) + nI = 0\}, \text{ and}$$
$$B_1 = \{(b^{-n}, \mathrm{div}(b)) \in K^\times \oplus \mathrm{Div}(K) : b \in K^\times\}.$$

Here $\mathrm{Div}(K)$ is the group of fractional ideals of K.

Now we are in a position to give a formula for the cup product. Let $x \in H^1(X, \mathbb{Z}/n\mathbb{Z})$ be represented by a torsor induced from an unramified Galois extension L/K of degree $d|n$ and fix a generator $\sigma \in \mathrm{Gal}(L/K)$. We identify $H^i(X, \mathbb{Z}/n\mathbb{Z})$ with the dual group $\mathrm{Ext}^{3-i}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)^\sim$. The formulas we obtain for the cup product now look as follows:

Proposition 4.1. *For $y \in H^2(X, \mathbb{Z}/n\mathbb{Z})$ we have that $x \smile y \in H^3(X, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Ext}^0(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)^\sim \cong \mu_n(K)^\sim$ satisfies the formula*

$$\langle x \smile y, \xi\rangle = \langle y, (a, I)\rangle$$

where $b \in L^\times$ is an element satisfying $\sigma(b)/b = \xi^{n/d}$, and $a \in K^\times$ and $I \in \mathrm{Div}(K)$ are elements such that $b^{-n} = a$ and $\mathrm{div}(b) = I\mathcal{O}_L$.

Proposition 4.2. *For $y \in H^1(X, \mathbb{Z}/n\mathbb{Z})$ we have that $x \smile y \in H^2(X, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)^\sim \cong (Z_1/B_1)^\sim$ satisfies the formula*

$$\langle x \smile y, (b, \mathfrak{b})\rangle = \langle y, \frac{n}{d} N_{L/K}(I) + \frac{n^2}{2d}\mathfrak{b}\rangle,$$

where $I \in \mathrm{Div}(L)$ satisfies

$$\mathfrak{b}^{n/d}\mathcal{O}_L = I - \sigma(I) + \mathrm{div}(t)$$

for some $t \in L^\times$ such that $N_{L/K}(t) = b^{-1}$.

Using the formulas above one can find examples of number fields whose rings of integers have isomorphic étale cohomology groups in all degrees but where the ring structure of the étale cohomologies are distinct.

5 Massey products

Let (A, d) be a differential graded algebra and let $x, y, z \in H^*(A)$ be homogeneous elements of degree a, b, c respectively, such that $x \smile y = y \smile z = 0$. Then one may define a subset $\langle x, y, z\rangle \subseteq H^n(A)$, called the (3-fold) *Massey product*, where $n = a+b+c-1$. This is done as follows: By abuse of notation, we also write x, y, z for some elements of A representing the classes above. If $w \in A$ is a homogeneous element, we also write $\overline{w} = (-1)^{\deg(w)+1}w$. Since $x \smile y = y \smile z = 0$, there exist elements $k_{xy} \in A_{a+b-1}$ and $k_{yz} \in A_{b+c-1}$ such that $dk_{xy} = -\overline{x} \smile y$ and

$dk_{yz} = -\overline{y} \smile z$. The sign convention here is chosen to match Dwyer's definition [Dwy75]. Hence we get an element

$$\overline{k_{xy}} \smile z + \overline{x} \smile k_{yz} \in A_{a+b+c-1}\,.$$

The Leibniz rule implies that

$$d(\overline{k_{xy}} \smile z + \overline{x} \smile k_{yz}) = (-1)^{a+b+1}\overline{x} \smile y \smile z + (-1)^{(a)+(b+1)+1}\overline{x} \smile y \smile z = 0\,.$$

and hence $\overline{k_{xy}} \smile z + \overline{x} \smile k_{yz}$ defines an element in $H^n(A)$. Hence we may define

$$\langle x, y, z\rangle = \{\overline{k_{xy}} \smile z + \overline{x} \smile k_{yz} : dk_{xy} = \overline{x} \smile y\,, dk_{yz} = \overline{y} \smile z\} \subseteq H^n(A)\,.$$

Now suppose that x', y', z' are some different choices of representatives, and suppose that $dk'_{x'y'} = \overline{x}' \smile y'$ and $dk'_{y'z'} = \overline{y}' \smile z'$. Writing $x' = x + dr_x, y' = y + dr_y, z' = z + dr_z$ for some coboundaries dr_x, dr_y, dr_z, we see that the elements $k_{xy} - k'_{x'y'}$ and $k_{yz} - k'_{y'z'}$ are cocycles and hence $\langle x, y, z\rangle$ gives a well-defined element

$$\langle x, y, z\rangle \in H^n(A)/(H^{a+b-1}(A) \smile z + x \smile H^{b+c-1}(A))\,.$$

Now let p be an odd prime and let us restrict to the case when we have a number field K and hence an arithmetic curve $X = \operatorname{Spec}\mathcal{O}_K$. Then we take A to be the differential graded algebra $C^*(\acute{\mathsf{E}}\mathsf{t}(X), \mathbb{Z}/p\mathbb{Z})$ associated to the *étale topological type* $\acute{\mathsf{E}}\mathsf{t}(X)$ of X [Fri82]. Then we have $H^*(A) = H^*(X, \mathbb{Z}/p\mathbb{Z})$. In Paper D we find formulas for computing 3-fold Massey products in $H^*(X, \mathbb{Z}/p\mathbb{Z})$ without any restriction on the field K.

To get a hint on what the formulas may look like, let us assume that K is an imaginary quadratic field. Then the conjugation action gives splittings

$$H^n(X, \mathbb{Z}/p\mathbb{Z}) = H^n(X, \mathbb{Z}/p\mathbb{Z})^+ \oplus H^n(X, \mathbb{Z}/p\mathbb{Z})^-$$

for all n, and similarly for the dual groups $H^n(X, \boldsymbol{\mu}_p)$. Suppose that (a, I) represents a class in $H^1(X, \boldsymbol{\mu}_p) \cong \operatorname{Hom}(H^2(X, \mathbb{Z}/p\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, that is, $\operatorname{div}(a) + pI = 0$. Let $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$ be the generator. Then $N_{K/\mathbb{Q}}(I) = I + \sigma(I) = \operatorname{div}(b)$ for some $b \in \mathbb{Q}^\times$, since $\operatorname{Cl}(\mathbb{Q}) = 0$. On the other hand, if $(a, I) = (\sigma(a), \sigma(I))$, then $I - \sigma(I) = \operatorname{div}(c)$ for some $c \in K^\times$ and hence $2I = \operatorname{div}(bc)$. Since 2 is invertible modulo p we get that I is principal. Hence we may assume that I is the trivial ideal and a is a unit. But the units of K are just -1 and 1 which are both a pth power since p is odd. Hence we see that $(a, I) = 0$ in cohomology and we conclude that $H^1(X, \boldsymbol{\mu}_p)^+ = 0$ and since $H^2(X, \mathbb{Z}/p\mathbb{Z})^+$ is Pontryagin dual to $H^1(X, \boldsymbol{\mu}_p)^+$, we get that $H^2(X, \mathbb{Z}/p\mathbb{Z})^+ = 0$. An almost identical argument shows that $H^1(X, \mathbb{Z}/p\mathbb{Z})^+ = 0$. Since the cup product is compatible with the Galois action, we see that the cup product

$$H^1(X, \mathbb{Z}/p\mathbb{Z}) \otimes H^1(X, \mathbb{Z}/p\mathbb{Z}) = H^1(X, \mathbb{Z}/p\mathbb{Z})^- \otimes H^1(X, \mathbb{Z}/p\mathbb{Z})^- \to H^2(X, \mathbb{Z}/p\mathbb{Z})$$

lands in $H^2(X, \mathbb{Z}/p\mathbb{Z})^+$ which is zero. This implies that if $x, y, z \in H^1(X, \mathbb{Z}/p\mathbb{Z})$, then $\langle x, y, z\rangle \subseteq H^2(X, \mathbb{Z}/p\mathbb{Z})$ is *always* defined and contains a *unique* element which, by abuse of notation, we denote by $\langle x, y, z\rangle$. When two of the elements x, y, z are equal, the Massey product has a simple formula:

Theorem 5.1. *Let $L_x \supset K$ be an unramified extension of degree p representing an element $x \in H^1(X, \mathbb{Z}/p\mathbb{Z})$, where p is an odd prime. Let $y \in H^1(X, \mathbb{Z}/p\mathbb{Z})$ be an element such that $x \smile y = 0$. Then, for any $(a', J) \in H^1(X, \boldsymbol{\mu}_p)$, the equality*

$$\langle\langle x, x, y\rangle, (a', J)\rangle = \begin{cases} \langle y, N_x(I') + J\rangle & \text{if } p = 3\,, \\ \langle y, N_x(I')\rangle & \text{if } p > 3\,, \end{cases}$$

holds, where

- *$(b, a, J, I) \in H^1(X, D(P_x))$ is a lift of (a', J), that is, $N_x(a) = a'$, and*
- *$I' \in \mathrm{Div}(L_x)$ is a fractional ideal such that $I = \mathrm{div}(u) - i_x(\frac{p-1}{2}J) + (1 - \sigma_x)I'$ for some element $u \in L_x^\times$ such that $i_x(N_x(u)) = \Gamma_x a - b$.*

Using this formula, we wrote a C program, using the library PARI [The22], to find the examples of imaginary quadratic fields in Theorem 1.1 and Theorem 1.2, in Section 1.

3 Stacky covers and root stacks

1 Ramified Galois covers

From now on the introduction turns towards Paper A. In the setting of number fields we saw that, taking a number field $K \subseteq \mathbb{Q}$ one gets a ramified cover $\operatorname{Spec} \mathcal{O}_K \to \operatorname{Spec} \mathbb{Z}$ together with an action of the Galois group $\operatorname{Gal}(K/\mathbb{Q})$ which we may view as a *constant* sheaf of groups, which is not always abelian. The covers treated in Paper B and Paper C are first of all always abelian, and will come with an action of $D(A)$ (where $D(A)$ denotes the diagonalizable group associated to an abstract abelian group A). These covers are in some sense much easier to understand. Even when $A = \operatorname{Gal}(K/\mathbb{Q})$ is abelian, it might not be the case that $\operatorname{Spec} \mathcal{O}_K \to \operatorname{Spec} \mathbb{Z}$ is a $D(A)$-cover. For example, taking K to be the 5th cyclotomic field gives such an example, even though it is a ramified cover with an action of $A = \operatorname{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/5\mathbb{Z})^\times \cong \mathbb{Z}/4\mathbb{Z}$. Note that $\mathbb{Z}/4\mathbb{Z} \ncong D(\mathbb{Z}/4\mathbb{Z})$ since the latter is ramified over the prime (2).

Actions by diagonalizable groups

Let S be a scheme. A *diagonalizable group scheme* over S is a group scheme of the form $D(A) = \operatorname{Spec} \mathcal{O}_S[A]$ where A is an abstract abelian group (which we write additively). Here $\mathcal{O}_S[A]$ is the group ring, i.e., the free $\mathcal{O}_S$-algebra with local sections

$$\mathcal{O}_S[A](U) = \left\{ \sum_{\lambda \in A} a_\lambda x^\lambda : a_\lambda \in \mathcal{O}_S(U)\,, \forall \lambda \in A \right\}$$

with component-wise addition and multiplication given by $x^\lambda x^{\lambda'} = x^{\lambda+\lambda'}$. The multiplicative unit is obtained by taking $a_0 = 1$ and all other coefficients 0 and the additive unit is obtained by taking all $a_\lambda = 0$. The structure morphism $\mathcal{O}_S \to \mathcal{O}_S[A]$ sends 1 to x^0.

If $f \colon X \to S$ is affine, then an action of $D(A)$ on X over S is equivalent to a coaction of the Hopf $\mathcal{O}_S$-algebra $\mathcal{O}_S[A]$ on $f_*\mathcal{O}_X$:

$$\begin{aligned}\Delta\colon f_*\mathcal{O}_X &\to f_*\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_S[A] \\ a &\mapsto \sum_\lambda p_\lambda(a) \otimes x^\lambda \,.\end{aligned}$$

The two axioms for a coaction implies that $\sum_\lambda p_\lambda = \mathrm{id}_{f_*\mathcal{O}_X}$ and $p_{\lambda'} \circ p_\lambda = \delta_{\lambda',\lambda} p_\lambda$, where $\delta_{\lambda',\lambda}$ is the Kronecker delta function. This means that we get a splitting

$$f_*\mathcal{O}_X \cong \bigoplus_{\lambda \in A} p_\lambda(f_*\mathcal{O}_X)\,.$$

Covers

Let B be a ring. Suppose that R is a B-algebra graded by a finite abelian group A making R a finite free B-module which is isomorphic to the regular representation $B[A]$ as a module, i.e.,

$$R \cong \bigoplus_{\lambda \in A} R_\lambda$$

where $R_\lambda \cong B$ for all λ. The canonical morphism $\operatorname{Spec} R \to \operatorname{Spec} B$ is an example of a *ramified* $D(A)$-*cover* (see below).

Choose a generator $x_\lambda \in R$ for each graded piece R_λ. Then we have multiplication morphisms

$$R_\lambda \otimes_B R_{\lambda'} \to R_{\lambda+\lambda'}$$

sending $x_\lambda \otimes x_{\lambda'}$ to $x_\lambda x_{\lambda'} = s_{\lambda,\lambda'} x_{\lambda+\lambda'}$ with $s_{\lambda,\lambda'} \in B$. We may describe R as a coequalizer of free algebras:

$$\begin{aligned}B[\mathbb{N}^{A\times A}] \rightrightarrows B[\mathbb{N}^A/(e_0)] \to R \\ x_{\lambda,\lambda'} \mapsto x_\lambda x_{\lambda'} \\ x_{\lambda,\lambda'} \mapsto s_{\lambda,\lambda} x_{\lambda+\lambda'}\end{aligned}$$

(here e_0 is the basis element corresponding to $0 \in A$).

The properties that the multiplication in R is

1. unital;
2. commutative;
3. associative;

translates to the following equalities

1. $s_{0,\lambda} = 1$ for all $\lambda \in A$;

2. $s_{\lambda,\lambda'} = s_{\lambda',\lambda}$ for all $\lambda, \lambda' \in A$;

3. $s_{\lambda,\lambda'} s_{\lambda+\lambda',\lambda''} = s_{\lambda',\lambda''} s_{\lambda'+\lambda'',\lambda}$ for all $\lambda, \lambda', \lambda'' \in A$:

In a more general setting, we replace $\operatorname{Spec} B$ with a scheme S and we replace $\operatorname{Spec} R$ by a finite, locally free scheme $f\colon X \to S$, together with an action of $D(A)$ on X such that, locally in the Zariski topology, $f_*\mathcal{O}_X$ is isomorphic as a comodule to the regular representation $\mathcal{O}_S[A]$ (as in [Ton14]). If $f\colon X \to S$ satisfies the properties just described, then we call it a *ramified $D(A)$-cover* (or just a *$D(A)$-cover*).

As we saw, this gives a splitting

$$f_*\mathcal{O}_X \cong \bigoplus_{\lambda \in A} \mathcal{L}_\lambda$$

and since $f_*\mathcal{O}_X$ is locally isomorphic to $\mathcal{O}_S[A]$ as a comodule, it follows that each $\mathcal{L}_\lambda$ is a line bundle and $\mathcal{L}_0 \cong \mathcal{O}_S$. We have multiplication morphisms

$$\mathcal{L}_\lambda \otimes \mathcal{L}_{\lambda'} \to \mathcal{L}_{\lambda+\lambda'}$$

which we think of as global sections

$$s_{\lambda,\lambda'} \in \Gamma(S, \mathcal{L}_\lambda^{-1} \otimes \mathcal{L}_{\lambda'}^{-1} \otimes \mathcal{L}_{\lambda+\lambda'})\,.$$

A *generalized effective Cartier divisor* on S is a pair $(\mathcal{L}, s)$ consisting of a line bundle $\mathcal{L}$ with a global section $s \in \Gamma(S, \mathcal{L})$. These form a category which admits a symmetric monoidal structure given by taking tensor products of line bundles and sections. The pair

$$(\mathcal{L}_\lambda^{-1} \otimes \mathcal{L}_{\lambda'}^{-1} \otimes \mathcal{L}_{\lambda+\lambda'}, s_{\lambda,\lambda'})$$

forms a generalized effective Cartier divisor. The relations

1. $s_{0,\lambda} = 1$ for all $\lambda \in A$;

2. $s_{\lambda,\lambda'} = s_{\lambda',\lambda}$ for all $\lambda, \lambda' \in A$;

3. $s_{\lambda,\lambda'} s_{\lambda+\lambda',\lambda''} = s_{\lambda',\lambda''} s_{\lambda'+\lambda'',\lambda}$ for all $\lambda, \lambda', \lambda'' \in A$:

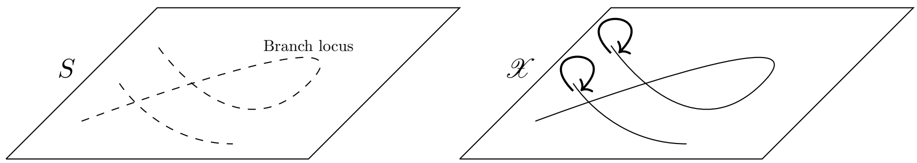
still hold if one interprets the equality symbol as $a = b$ if a is sent to b under the corresponding canonical isomorphism of line bundles. For instance, we have a canonical isomorphism

$$\mathcal{L}_\lambda^{-1} \otimes \mathcal{L}_{\lambda'}^{-1} \otimes \mathcal{L}_{\lambda+\lambda'} \cong \mathcal{L}_{\lambda'}^{-1} \otimes \mathcal{L}_{\lambda}^{-1} \otimes \mathcal{L}_{\lambda+\lambda'}$$

sending $s_{\lambda,\lambda'}$ to $s_{\lambda',\lambda}$.

In [Par91] Pardini studies ramified $D(A)$-covers $X \to S$ that are generically torsors and gives *building data* for such covers when S is smooth and X is normal. Tonini studied the stack of ramified G-covers [Ton14] for a fixed group G (also without the assumption that the group G is diagonalizable).

Given a ramified $D(A)$-cover, one may ask what part of this data is forgotten when passing to the quotient $\mathscr{X} = [X/D(A)]$. It turns out that the quotient $\mathscr{X}$ will remember the data $(\mathcal{L}_\lambda^{-1} \otimes \mathcal{L}_{\lambda'}^{-1} \otimes \mathcal{L}_{\lambda+\lambda'}, s_{\lambda,\lambda'})$ but will forget the line bundles $\mathcal{L}_\lambda$. When $|A|$ is invertible in $\Gamma(S, \mathcal{O}_S)$, we may think of $[X/D(A)]$ as "the least we have to modify S in order to make the ramified cover $X \to S$ an étale cover".



The pullback of the inertia stack $\mathcal{I}_{\mathscr{X}} = \mathscr{X} \times_{\mathscr{X} \times_S \mathscr{X}} \mathscr{X}$ along the canonical morphism $X \to [X/G]$ is isomorphic to the stabilizer group $\operatorname{Stab}(X)$ which is defined as the pullback

$$\begin{array}{ccc} \operatorname{Stab}(X) & \longrightarrow & G \times_S X \\ \downarrow & & \downarrow{\scriptstyle \sigma, \mathrm{pr}_X} \\ X & \xrightarrow{\Delta} & X \times_S X\,. \end{array}$$

This means that the points in $\mathscr{X}$ lying over the branch locus in S will have non-trivial automorphism groups and the image of $x \in X$ in $\mathscr{X}$ will have automorphism group $\operatorname{Stab}(x)$ in $\mathscr{X}$ sitting in a cartesian square

$$\begin{array}{ccc} \operatorname{Stab}(x) & \longrightarrow & G \times_S X \\ \downarrow & & \downarrow{\scriptstyle \sigma, \mathrm{pr}_X} \\ \operatorname{Spec} k & \xrightarrow{(x,x)} & X \times_S X\,. \end{array}$$

Deligne–Falting data from ramified covers

The presentation

$$D(A) \times_S X \rightrightarrows X \to \mathscr{X}$$

gives a universal commutative diagram on $\mathscr{X}$:

$$\begin{array}{ccc} X & \longrightarrow & X \times_S \mathscr{X} \\ & \searrow & \downarrow \\ & & \mathscr{X} \end{array} \tag{1.1}$$

and the data of a morphism $t\colon T \to \mathscr{X}$ is equivalent to the data of the pullback of 1.1 to T along t. From the group A we construct two monoids P_A and Q_A together with a morphism $\gamma_A\colon P_A \to Q_A$. We have a morphism $\varphi\colon \mathbb{N}^{A\times A} \to \mathbb{Z}^A/(e_0)$ defined on the basis by $e_{\lambda,\lambda'} \mapsto e_\lambda + e_{\lambda'} - e_{\lambda+\lambda'}$. The monoid $\mathbb{N}^{A\times A}$ has a basis of elements

$e_{\lambda,\lambda'}$ for $(\lambda, \lambda') \in A \times A$. We define $P_A = \mathbb{N}^{A\times A}/R$ where R is the congruence relation generated by the relations

$$\begin{aligned} e_{\lambda,\lambda'} &\sim e_{\lambda',\lambda} \\ e_{0,\lambda} &\sim 0 \\ e_{\lambda,\lambda'} + e_{\lambda+\lambda',\lambda''} &\sim e_{\lambda',\lambda''} + e_{\lambda'+\lambda'',\lambda} \,. \end{aligned}$$

We then define $Q_A = P_A \times_{e_{(-,-)}} A$ to be the monoid with underlying set $P_A \times A$ and with addition given by

$$(p, \lambda) + (p', \lambda') = (p + p' + e_{\lambda,\lambda'}, \lambda + \lambda')\,.$$

Finally, we define $\gamma_A \colon P_A \to Q_A$ to be the canonical inclusion into the first factor. The induced action of P_A on Q_A is free and we get that $\operatorname{Spec}\mathbb{Z}[Q_A] \to \operatorname{Spec}\mathbb{Z}[P_A]$ is a $D(A)$-cover. With these definitions we get a *free extension of* A *by* P_A:

$$0 \to P_A \to Q_A \to A \to 0$$

which is *universal* in the sense that it maps uniquely to any other free extension of A by a monoid P. Free extensions of A with values in a monoid P are in bijection with *(commutative) 2-cocycles* $A \times A \to P$ and the universal extension $0 \to P_A \to Q_A \to A \to 0$ corresponds to the *universal 2-cocycle* $e_{(-,-)} \colon A\times A \to P_A$ sending (λ, λ') to $e_{\lambda,\lambda'}$.

The cover X gives a symmetric monoidal functor

$$\mathcal{L} \colon P_A \to [\mathbb{A}^1_S/\mathbb{G}_{m,S}]$$

and the data of Diagram (1.1) is equivalent to the data of a diagram

$$\begin{array}{ccc} P_A & \longrightarrow & [\mathbb{A}^1_{\mathscr{X}}/\mathbb{G}_{m,\mathscr{X}}] \\ \big\downarrow & \swarrow\!\!\swarrow \simeq \;\nearrow & \\ Q_A & & \end{array} \quad .$$

This leads us to the language of root stacks as we will explain in the next section. After replacing A by an étale sheaf $\mathcal{A}$ constructed intrinsically on $\mathscr{X}$ and replacing the monoids P_A and Q_A by étale sheaves of monoids associated to $\mathcal{A}$ we arrive at a setting which is convenient for gluing such quotient stacks.

2 Log structures and root stacks

Classical root stacks

Consider a generalized effective Cartier divisor $(\mathcal{L}, s)$ on S, i.e., a line bundle $\mathcal{L}$ on S with a global section $s \colon \mathcal{O}_S \to \mathcal{L}$. One may ask if there is a morphism $f \colon X \to S$

such that $(f^*\mathcal{L}, f^*s)$ has an nth root for some positive integer $n \geq 2$. That is, a generalized effective Cartier divisor $(\mathcal{E}, \varepsilon)$ on X together with an isomorphsm $\mathcal{E}^{\otimes n} \cong f^*\mathcal{L}$ sending $\varepsilon^{\otimes n}$ to f^*s. The nth root stack is simply the universal such object.

Let $\mathscr{D}iv(T)$ be the fiber of $[\mathbb{A}^1/\mathbb{G}_m]$ over an S-scheme T. The *nth root stack* associated to $(\mathcal{L}, s)$ is the stack $S_{(\mathcal{L},s),n}$ over S with objects $(T, \mathcal{E}, \varepsilon, \tau)$ consisting of

- a scheme $t\colon T \to S$,
- a line bundle $\mathcal{E}$ on T with a global section $\varepsilon \in \Gamma(T, \mathcal{E})$, and
- an isomorphism $\tau\colon \mathcal{E}^{\otimes n} \to t^*\mathcal{L}$ sending $\varepsilon^{\otimes n}$ to t^*s.

It sits in a 2-cartesian diagram

$$\begin{array}{ccc} S_{(\mathcal{L},s),n} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow \\ S & \xrightarrow{(\mathcal{L},s)} & [\mathbb{A}^1/\mathbb{G}_m]\,, \end{array} \tag{2.1}$$

where the right vertical arrow is given by sending the coordinate to its nth power. We can also think of it as follows: The morphism $S \to [\mathbb{A}^1/\mathbb{G}_m]$ induces a symmetric monoidal functor $\mathcal{L}\colon \mathbb{N} \to \mathscr{D}iv(S)$. Let $n\colon \mathbb{N} \to \mathbb{N}$ be the morphism sending 1 to n. Then $S_{(\mathcal{L},s),n}$ is the stack with objects $(T, \mathcal{E}, \tau)$ where

1. $t\colon T \to S$,
2. $\mathcal{E}\colon \mathbb{N} \to \mathscr{D}iv(T)$ is a symmetric monoidal functor, and
3. $\tau\colon \mathcal{E} \circ n \to t^* \circ \mathcal{L}$ is an isomorphism of symmetric monoidal functors.

Example 2.1. In some cases, the root stack $S_{(\mathcal{L},s),n}$ may be constructed as a quotient $[X/\mu_n]$ where $X \to S$ is a scheme. This happens when there exists a line bundle $\mathcal{L}^{1/n}$ on S and an isomorphism $(\mathcal{L}^{1/n})^{\otimes n} \cong \mathcal{L}$. In this case we may define

$$X = \operatorname{Spec} \bigoplus_{i=0}^{n-1} (\mathcal{L}^{1/n})^{-i}$$

with μ_n-action given by the grading.

This is however, not always possible since $\mathcal{L}$ might not have an nth root on S. For instance, consider $S = \mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x, y]$ and $\mathcal{L} = \mathcal{O}(1)$ with the global section x. Then it is not possible to construct the root stack as a quotient of a μ_n-cover since $\mathcal{O}(1)$ does not have an nth root when $n \geq 2$.

There is a universal generalized effective Cartier divisor $(\mathcal{E}^{\text{univ}}, \varepsilon^{\text{univ}})$ on $S_{(\mathcal{L},s),n}$ obtained by pulling back the universal $\mathbb{G}_m$-torsor $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ along the projection $S_{(\mathcal{L},s),n} \to [\mathbb{A}^1/\mathbb{G}_m]$. Let $p\colon P \to S$ be the $\mathbb{G}_m$-torsor on S corresponding to $\mathcal{L}$. Then

$$p_*\mathcal{O}_P \cong \bigoplus_{n\in\mathbb{Z}} \mathcal{L}^n$$

as an $\mathcal{O}_S$-module. Consider the cartesian square (2.1). If we write $S = [P/\mathbb{G}_{m,S}]$ and let φ_n be the morphism sending x to x^n, we have that $S_{(\mathcal{L},s),n} \cong [\operatorname{Spec} \mathcal{A}_{\mathcal{L}}/\mathbb{G}_{m,S}]$ where

$$\begin{aligned}\mathcal{A}_{\mathcal{L}} &\cong p_*\mathcal{O}_P \otimes_{\mathcal{O}_S[x],\varphi_n} \mathcal{O}_S[x] \\ &\cong p_*\mathcal{O}_P[x]/(x^n - s) \\ &\cong \dots x^{n-1}\mathcal{L}^{-1} \oplus \mathcal{O}_S \oplus x\mathcal{O}_S \oplus \dots \oplus x^{n-1}\mathcal{O}_S \oplus \mathcal{L} \oplus \dots .\end{aligned}$$

Since $S_{(\mathcal{L},s),n}$ is a quotient under a $\mathbb{G}_m$-action, the universal line bundle $\mathcal{E}^{\text{univ}}$ corresponds to a $\mathbb{G}_m$-equivariant line bundle on $\operatorname{Spec} \mathcal{A}_{\mathcal{L}}$, which in turn is equivalent to a $\mathbb{Z}$-graded $\mathcal{A}_{\mathcal{L}}$-line bundle. It is the shifted module $\mathcal{E}^{\text{univ}} = \mathcal{A}_{\mathcal{L}}[1]$, where we use the convention

$$(\mathcal{A}_{\mathcal{L}}[1])_i = (\mathcal{A}_{\mathcal{L}})_{i+1} .$$

Indeed, the pullback of $\mathcal{L}$ to E is just $\mathcal{L} \otimes \mathcal{A}_{\mathcal{L}} = \mathcal{A}_{\mathcal{L}}[n]$ and hence $\mathcal{A}_{\mathcal{L}}[1]$ is an nth root.

Using root stacks

To illustrate how root stacks can be used in a classical setup, let us consider the following situation (see [Cad07] for more details). Suppose that we have a curve C with a distinguished smooth point $P \in C$. If $D \subseteq \mathbb{P}^2 = X$ is a curve then we might ask if there is a morphism $f\colon C \to \mathbb{P}^2$ such that C intersects D with a certain tangency condition at the point P. Here *tangency condition* means the following: fixing positive integers d and r, we could ask that $f^*D = rZ + dP$ for some effective Cartier divisor $Z \subseteq C$ (which is not fixed on beforehand). This question can be rephrased using root stacks as: does there exist a commutative diagram

$$\begin{array}{ccc} C_{P,n} & \xrightarrow{F} & X_{D,r} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ C & \longrightarrow & X , \end{array}$$

where

1. $n = r/\gcd(r,d)$,
2. $C_{P,n}$ is the nth root stack of $\mathcal{O}_C(P)$ with its canonical section,
3. $X_{D,r}$ is the rth root stack of $\mathcal{O}_X(D)$ with its canonical section,

4. F is representable of *contact type* d.

We need to explain what point (4) means here. The morphism F is given by a triple $(f, (\mathcal{M}, t), \alpha)$ where $f\colon C \to X$ is a morphism, $(\mathcal{M}, t)$ a generalized effective Cartier divisor on $C_{P,n}$, and $\alpha\colon (\mathcal{M}, t)^{\otimes r} \to (\pi^* f^* \mathcal{O}_X(D), \pi^* f^* s_D)$ an isomorphism of generalized effective Cartier divisors. We have that $\mathcal{M} \cong \pi^* \mathcal{L} \otimes \mathcal{E}^{\otimes k}$ for some integer $1 \leq k \leq n-1$, where $\mathcal{E}$ is the universal nth root of $\pi^* \mathcal{O}_C(P)$ and $\mathcal{L}$ is a line bundle on C. The morphism F is representable if and only if for every geometric point in $C_{P,n}$, the induced morphism on stabilizers is injective. It turns out (see [Cad07, Proposition 3.3.3]) that F is representable if and only if $n|r$ and $\gcd(k, n) = 1$. We have

$$\mathcal{M}^{\otimes r} \cong \pi^* \mathcal{L}^{\otimes r} \otimes \pi^* \mathcal{O}_C(P)^{\otimes d}$$

where $nd = kr$ and we call d the *contact type* of the morphism F.

For point (1), note that $\gcd(r, d) = \gcd(n(r/n), k(r/n)) = (r/n)\gcd(n, k) = r/n$ and hence $n = r/\gcd(r, d)$.

This example is supposed to illustrate that root stacks can be used in Gromov–Witten theory when counting curves in X with a certain tangency condition with respect to a divisor $D \subseteq X$. This was done by Cadman–Chen when $X = \mathbb{P}^2$ [CC08].

Log structures à la Fontaine–Illusie

It is sometimes useful to consider a scheme X together with the data of a divisor $D \subset X$. For instance, one may want to allow differential forms to admit poles along D. One example to keep in mind is when X is a compactification of $U = X \setminus D$. Then X is a proper scheme and hence often easier to deal with than U. But since one is originally interested in the (non-proper) scheme U, it is necessary to keep track of what happens near D. This is somehow analogous to a manifold with boundary. This leads to the theory of *logarithmic geometry*. It has applications for instance in moduli theory and p-adic Hodge theory.

A *logarithmic structure (log structure)* on a scheme S is a pair $(\mathcal{M}, \alpha)$ where $\mathcal{M}$ is an étale sheaf of commutative monoids on S and $\alpha\colon \mathcal{M} \to \mathcal{O}_S$ is a morphism of monoids with respect to multiplication in $\mathcal{O}_S$, such that α restricts to an isomorphism $\alpha^{-1}(\mathcal{O}_S^\times) \cong \mathcal{O}_S^\times$. A scheme S with a log structure $(\mathcal{M}, \alpha)$ is called a *log scheme* and is denoted $(S, \mathcal{M})$. A morphism of log schemes $(S, \mathcal{M}) \to (T, \mathcal{N})$ consists of a morphism of schemes $f\colon S \to T$ and a morphism of sheaves of monoids $f^{-1}\mathcal{N} \to \mathcal{M}$ such that the diagram

$$\begin{array}{ccc} f^{-1}\mathcal{N} & \longrightarrow & f^{-1}\mathcal{O}_T \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{O}_S \end{array}$$

commutes. Here f^{-1} denotes the usual pullback of étale sheaves to avoid confusion with other types of pullbacks. Note that $f^{-1}\mathcal{N} \to f^{-1}\mathcal{O}_T \to \mathcal{O}_S$ need not be a log structure.

Let S be a regular scheme and D a reduced divisor on S with normal crossings and $U = S \setminus D$. Then we have an open immersion $j\colon U \hookrightarrow S$ and the *canonical log structure* with respect to D is given by $\mathcal{M} = j_*\mathcal{O}_U^\times \cap \mathcal{O}_S \hookrightarrow \mathcal{O}_S$, i.e., those sections of $\mathcal{O}_S$ that are invertible outside D.

Given a morphism of log schemes $(S, \mathcal{M}) \to (T, \mathcal{N})$ we may define the *(relative) module of log differentials*

$$\Omega^1_{S/T}(\log(\mathcal{M}/\mathcal{N})) = \Omega^1_{S/T} \oplus (\mathcal{O}_S \otimes_{\mathbb{Z}} \mathcal{M}^{gp})/R$$

where R is the $\mathcal{O}_S$-submodule generated by local sections of the form

1. $(d\alpha(m), 0) - (0, \alpha(m) \otimes m)$ for $m \in \mathcal{M}$, and
2. $(0, 1 \otimes n)$ for $n \in \operatorname{im}(f^{-1}(\mathcal{N}) \to \mathcal{M})$,

where $d\colon \mathcal{O}_S \to \Omega^1_{S/T}$ is the universal derivation.

In particular, if S is a regular scheme, D a reduced divisor on S with simple normal crossings, and $\mathcal{M}_D$ the canonical log structure, then we have the following interpretation of $\Omega^1_S(\log \mathcal{M})$: the divisor D gives a line bundle $\mathcal{O}_S(D)$ with a global section $s_D\colon \mathcal{O}_S \to \mathcal{O}_S(D)$. This yields a morphism $S \to [\mathbb{A}^1/\mathbb{G}_m]$ and when D is smooth, we have an isomorphism

$$\Omega^1_S(\log \mathcal{M}) \cong \Omega^1_{S/[\mathbb{A}^1/\mathbb{G}_m]}\,.$$

More generally, Olsson has constructed a moduli stack of log structures $\mathscr{L}og_S$ [Ols03] and interpreted many of the common notions of log geometry in this stack-theoretic framework.

For a great account on log geometry, we recommend either [Ogu18] or [Kat89].

Deligne–Faltings structures

A concept very similar to that of log structures is the theory of *Deligne–Faltings structures* developed in [BV12]. If $\mathcal{P}$ is an étale sheaf of commutative monoids on a scheme S we may think of it as a symmetric monoidal stack with objects given by the sections of the sheaf and a single identity morphism for every object. The symmetric monoidal structure is given by the binary operation. Hence it makes sense to talk about a symmetric monoidal functor from $\mathcal{P}$ to a symmetric monoidal stack over S.

Let $\mathscr{D}iv_{S_{\text{ét}}}$ denote the restriction of $[\mathbb{A}^1_S/\mathbb{G}_{m,S}]$ to the small étale site. A *Deligne–Faltings structure on* S consists of an étale sheaf of commutative monoids $\mathcal{P}$ and a symmetric monoidal functor $\mathcal{L}\colon \mathcal{P} \to \mathscr{D}iv_{S_{\text{ét}}}$ with trivial kernel, i.e., if $p \in \mathcal{P}(U)$ is a local section and $\mathcal{L}_p \cong (\mathcal{O}_S, 1)$, then $p = 0$. Note that $\mathcal{L}$ may have trivial kernel and still be non-injective.

There is a symmetric monoidal functor $\mathcal{O}_S \to \mathscr{D}iv_{S_{\text{ét}}}$ sending a local section $f \in \mathcal{O}_S(U)$ to $(\mathcal{O}_U, f)$. Given a Deligne–Faltings structure $(\mathcal{P}, \mathcal{L})$, the fiber product (of fibered categories) $\mathcal{M} = \mathcal{P} \times_{\mathscr{D}iv_{S_{\text{ét}}}} \mathcal{O}_S$ turns out to be a monoid and the projection $\alpha\colon \mathcal{M} \to \mathcal{O}_S$ is a log structure with the property that the action of $\mathcal{O}_S^\times$ on $\mathcal{M}$ is free, i.e., α is *quasi-integral.*

Conversely, a quasi-integral log structure $\alpha\colon \mathcal{M} \to \mathcal{O}_S$ is $\mathcal{O}_S^\times$-equivariant and the induced morphism on quotient stacks is $\overline{M} = \mathcal{M}/\mathcal{O}_S^\times \to \mathscr{D}iv_{S_{\text{ét}}}$. Hence we get:

Theorem 2.2 ([BV12, Theorem 3.6]). *The category of Deligne–Faltings structures on S is equivalent to the category of quasi-integral log structures on S.*

If S is a regular scheme and D is an effective divisor with normal crossings, then the *canonical Deligne–Faltings structure* with respect to D is the one associated with the canonical log structure associated to D.

General root stacks

From now on we write $\mathscr{D}iv_S = [\mathbb{A}^1_S/\mathbb{G}_{m,S}]$. The classical root stack can be described as follows: Consider the diagram

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\mathcal{L}} & \mathscr{D}iv_S \\ \big\downarrow n & & \\ \mathbb{N} & & \end{array},$$

where $\mathcal{L}$ is the symmetric monoidal functor induced by sending 1 to the generalized effective Cartier divisor $(\mathcal{L}, s)$. Then the classical root stack is the stack with objects $(T, \mathcal{E}, \tau)$ consisting of

- a scheme $t\colon T \to S$,
- a symmetric monoidal functor $\mathcal{E}\colon \mathbb{N} \to \mathscr{D}iv\, T$, and
- an isomorphism $\tau\colon \mathcal{E} \circ n \to t^*\mathcal{L}$ of symmetric monoidal functors.

This construction can now be generalized by replacing the constant monoids in the diagram by (possibly distinct) étale sheaves of commutative monoids $\mathcal{P}$ and $\mathcal{Q}$.

3 Stacks and representations of groups

Let k be a field and G an affine algebraic group over k. Let us consider $* = \operatorname{Spec} k$ with the trivial action of G. Then the stack $BG = [*/G]$ classifies (left) G-torsors for the fppf topology (or principal G-bundles), i.e., a morphism $\eta\colon \mathscr{X} \to BG$ is the same as a G-torsor $P \to \mathscr{X}$. The stack BG comes equipped with a canonical

G-torsor $t\colon * \to BG$ corresponding to the trivial torsor $G \to *$ and we have a cartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle t} \\ \mathscr{X} & \xrightarrow{\eta} & BG\,. \end{array}$$

The data of a vector bundle on BG is equivalent to the data of a vector bundle $\mathcal{E}$ on $*$ together with an k-linear action of G. In other words we have an equivalence of categories

$$\operatorname{QCoh} BG \simeq \operatorname{Rep} G\,.$$

Under this equivalence, we have that

1. $\mathcal{O}_{BG}$ corresponds to k with the trivial action,
2. the algebra of the canonical torsor $* \to BG$ corresponds to the regular representation, and
3. the algebra of the inertia stack $\mathcal{I}_{BG}$ corresponds to $\mathcal{O}_G$ with the action of itself by conjugation.

We will give some intuition for point 3: We think of the inertia stack $\mathcal{I}_{BG}$ as the fibered category with objects (T, E, α) where

- T is a k-scheme,
- $E \to T$ is a G-torsor, and
- $\alpha\colon E \to E$ is an automorphism of G-torsors.

We have a morphism $G \to \mathcal{I}_{BG}$ defined on the level of objects by sending $g \in G(T)$ to $(T, G \times_S T, r_g)$ where $r_g\colon G \times_S T \to G \times_S T$ is right translation by g. For every $h \in G(T)$, we have a commutative diagram

$$\begin{array}{ccc} G \times_S T & \xrightarrow{r_h} & G \times_S T \\ \downarrow{\scriptstyle r_g} & & \downarrow{\scriptstyle r_{h^{-1}gh}} \\ G \times_S T & \xrightarrow{r_h} & G \times_S T\,, \end{array}$$

which says that we have an isomorphism

$$(T, G \times_S T, r_g) \cong (T, G \times_S T, r_{h^{-1}gh})$$

in $\mathcal{I}_{BG}(T)$. Hence we have an induced morphism $[G/G] \to \mathcal{I}_{BG}$ where G acts on itself by conjugation. It remains to check that this morphism is an isomorphism which we leave to the reader.

Given the morphism $\eta\colon \mathscr{X} \to BG$ (or equivalently the G-torsor $P \to \mathscr{X}$), we have a pullback functor

$$\eta^*\colon \operatorname{Rep} G \to \operatorname{QCoh} \mathscr{X}\,.$$

This functor is a k-linear exact faithful tensor functor, that takes a finite representation of rank n to a finite locally free sheaf of rank n.

It turns out that the following converse statement is also true (see [Nor82, Proposition 2.9]): Given a k-linear exact faithful tensor functor $F\colon \operatorname{Rep} G \to \operatorname{QCoh} \mathscr{X}$, that takes a finite representation of rank n to a finite locally free sheaf of rank n, we get that $\operatorname{Spec} F(k[G])$ is a G-torsor. Let $\operatorname{Func}^{\otimes}(\operatorname{Rep} G, \operatorname{QCoh} \mathscr{X})$ be the category of such functors F and let $\operatorname{Tors} \mathscr{X} = \operatorname{Hom}(\mathscr{X}, BG)$ be the category of G-torsors on $\mathscr{X}$. Then we have an equivalence of categories

$$\operatorname{Tors} \mathscr{X} \simeq \operatorname{Func}^{\otimes}(\operatorname{Rep} G, \operatorname{QCoh} \mathscr{X})\,.$$

4 Stacky covers

Stacky covers are the main objects of study in Paper A. Similarly to the case of ramified covers these are built up from a combinatorial set of data involving line bundles with global sections. It turns out that all stacky covers are obtained as flat (general) root stacks and vice versa.

Definition 4.1. Let $\mathscr{X}$ be a Deligne–Mumford stack with finite diagonalizable stabilizers at closed points and let S be a scheme. We say that $\pi\colon \mathscr{X} \to S$ is a *stacky cover* if it is

1. flat, proper, of finite presentation,
2. a coarse moduli space, and
3. for any morphism of schemes $T \to S$, the base change $\pi|_T\colon \mathscr{X}_T \to T$ has the property that $(\pi|_T)_*$ takes line bundles to line bundles.

We denote by $\mathscr{S}t\mathscr{C}ov$ the $(2,1)$-category of stacky covers.

When $G = D(A)$, $f\colon X \to S$ is a G-cover, and $\mathscr{X} = [X/G]$ we have a canonical G-torsor $p\colon X \to \mathscr{X}$ and we may ask what the algebra $p_*\mathcal{O}_X$ looks like. We have a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathscr{X} & \longrightarrow & BG\,, \end{array}$$

where $BG = [S/G]$ with G acting trivially on S. We have that $\mathcal{O}_{BG}$ corresponds to the A-graded $\mathcal{O}_S$-algebra with $\mathcal{O}_S$ in degree zero and 0 elsewhere, i.e.,

$$\mathcal{O}_S \oplus 0 \oplus 0 \oplus \dots\,,$$

and we have

$$q_*\mathcal{O}_S \cong \bigoplus_{\lambda\in A} \mathcal{O}_{BG}[\lambda]$$

with multiplication given by the canonical isomorphisms $\mathcal{O}_{BG}[\lambda] \otimes \mathcal{O}_{BG}[\lambda'] \to \mathcal{O}_{BG}[\lambda+\lambda']$. Similarly, $\mathcal{O}_{\mathscr{X}}$ corresponds to the $f_*\mathcal{O}_X$-algebra $f_*\mathcal{O}_X$ together with the data of the A-grading induced by the action. Hence we conclude that

$$p_*\mathcal{O}_X \cong \bigoplus_{\lambda\in A} \mathcal{O}_{\mathscr{X}}[\lambda]\,,$$

with multiplication given by the canonical isomorphisms

$$\mathcal{O}_{\mathscr{X}}[\lambda] \otimes \mathcal{O}_{\mathscr{X}}[\lambda'] \to \mathcal{O}_{\mathscr{X}}[\lambda+\lambda']\,.$$

Example 4.2. If $f\colon X \to S$ is a ramified $D(A)$-cover, then $\pi\colon \mathscr{X} = [X/D(A)] \to S$ is a stacky cover. Indeed, if S is connected then any line bundle on $\mathscr{X}$ is of the form $\mathcal{L} \otimes \mathcal{O}_{\mathscr{X}}[\lambda]$ for $\mathcal{L}$ a line bundle on S. The algebra

$$f_*\mathcal{O}_X \cong \bigoplus_{\lambda\in A} \mathcal{L}_\lambda$$

and the degree zero part of $\mathcal{O}_{\mathscr{X}}[\lambda]$ is $\mathcal{L}_\lambda$. The pushforward π_* sends $\mathcal{L} \otimes \mathcal{O}_{\mathscr{X}}[\lambda]$ to its degree zero part $\mathcal{L} \otimes \mathcal{L}_\lambda$ which is a line bundle. Now consider one connected component of S at a time to conclude that $\mathscr{X} \to S$ is a stacky cover.

Example 4.3. If $\mathcal{L}$ is a line bundle on S and $s \in \Gamma(S, \mathcal{L})$ is a global section, then the root stack $S_{(\mathcal{L},s,n)}$ is an example of a stacky cover. Indeed, it looks étale locally like $[X/D(\mathbb{Z}/n)]$ as in the previous example.

Remark 4.4. Let $\pi\colon \mathscr{X} \to S$ be a stacky cover. Then there exists an étale cover $\{U_i \to S\}$, abelian groups A_i, and ramified $D(A_i)$-covers $X_i \to U_i$ with isomorphisms $\mathscr{X} \times_{U_i} S \simeq [X_i/D(A_i)]$. Indeed, by standard limit arguments one reduces to the case when S is the spectrum of a strictly henselian ring. Let $X' \to \mathscr{X}$ be an étale cover. Then $X' \to S$ is quasi-finite and there is a connected component X of X' such that $X \to S$ is finite and $X \to \mathscr{X}$ is still an étale cover. This means that $f\colon X \to S$ is finite flat of finite presentation. One may check that $p\colon X \to \mathscr{X}$ is a $D(A)$-torsor for a diagonalizable group $D(A)$ where $D(A)$ is the stabilizer group of the closed point of $\mathscr{X}$. Thus $p_*\mathcal{O}_X \cong \bigoplus_{\lambda\in A} \mathcal{O}_{\mathscr{X}}[\lambda]$ and since π_* takes line bundles to line bundles we get that $f_*\mathcal{O}_X \cong \bigoplus_{\lambda\in A} \mathcal{L}_\lambda$ for some line bundles $\mathcal{L}_\lambda$. Hence $f_*\mathcal{O}_X$ is locally isomorphic to the regular representation and we conclude that $X \to S$ is a ramified cover.

Lemma 4.5. *Let $\pi\colon \mathscr{X} \to S$ be a stacky cover and let $\mathrm{Pic}_{\mathscr{X}/S}$ be the relative Picard functor. Then $\mathrm{Pic}_{\mathscr{X}/S}$ is representable by an étale algebraic space. Furthermore, if $\mathscr{X}$ is Deligne–Mumford, then there exists a canonical isomorphism $\mathrm{Pic}_{\mathscr{X}/S} \cong \pi_*D(\mathcal{I}_{\mathscr{X}})$.*

So from the stack $\mathscr{X}$ we get an étale group scheme $\mathcal{A} = \mathrm{Pic}_{\mathscr{X}/S}$ and we may define two commutative monoids $P_{\mathcal{A}}$ and $Q_{\mathcal{A}}$ and a morphism $\gamma\colon P_{\mathcal{A}} \to Q_{\mathcal{A}}$. We then construct a symmetric monoidal functor

$$\mathcal{L}\colon P_{\mathcal{A}} \to \mathscr{D}iv_{S_{\text{ét}}} \ .$$

The datum $(\mathcal{A}, \mathcal{L})$ will be referred to as a *stacky building datum* and any such datum gives rise to a root stack $S_{(\mathcal{A},\mathcal{L})}$.

Theorem 4.6. *Let $\mathscr{X} \to S$ be a stacky cover and $(\mathcal{A}, \mathcal{L})$ its associated stacky building datum. Then we have an isomorphism*

$$\mathscr{X} \simeq S_{(\mathcal{A},\mathcal{L})}$$

where the right hand side is the root stack associated to $(\mathcal{A}, \mathcal{L})$.

There is a (2,1)-category $\mathscr{S}t\mathscr{D}ata$ of stacky building data which is equivalent to the (2,1)-category $\mathscr{S}t\mathscr{C}ov$ of stacky covers.

Theorem 4.7. *There exists an equivalence*

$$\mathscr{S}t\mathscr{C}ov \simeq \mathscr{S}t\mathscr{D}ata$$

between the (2,1)-category of stacky covers and the (2,1)-category of stacky building data.

5 Ramification and root stacks

Given a ramified cover $X \to S$ we may associate to each component of the branch locus, the corresponding ramification index. Conversely, suppose that we are given a scheme S and a finite collection of Cartier divisors $\{D_i\}_{i\in I}$ and positive integers $\{r_i\}_{i\in I}$ with $r_i \geq 2$ for all $i \in I$. We refer to the collection $\{D_i\}_{i\in I}$, $\{r_i\}_{i\in I}$ as a birational building datum. One may ask if there is a ramified cover $X \to S$ giving rise to this birational building datum. The covers suitable in this setting are those of *Kummer type*, i.e., they are fppf locally cut out by equations $x^{r_i} = s_i$.

One may want to refine the notion of birational building datum to include other types of covers as well. Here is an example.

Example 5.1. Let $R = \mathbb{C}[s,t]$, $S = \operatorname{Spec} R$, $X_1 = \operatorname{Spec} R[x,y]/(x^2 - s, y^4 - t)$, and $X_2 = \operatorname{Spec} R[x,y]/(x^2 - s, y^2 - xt)$. Then $X_1 \to S$ is a ramified $\mu_2 \times \mu_4$-cover and $X_2 \to S$ is a ramified μ_4-cover. The ramification indices of X_1 and X_2 along the two axis $s = 0$ and $t = 0$ agree but one can distinguish the two cases in that the stabilizer group of X_1 over the origin is $\mu_2 \times \mu_4$ and the corresponding stabilizer for X_2 is μ_4.

Hence a more refined notion of birational building datum is to associate to each divisor, instead of a number, a group, and we also associate a group G_J to each intersection $D_J := \bigcap_{i\in J} D_i$ (for $J \subseteq I$), and for every $J' \subseteq J$, a group homomorphism $G_{J'} \to G_J$. But a better way to package this is to say that we have an étale sheaf of abelian groups whose support is $|\bigcup_{i\in I} D_i|$ and which is constant along each stratum $D_J^\circ = D_J \setminus \bigcup_{i\notin J} D_i$.

But we will need a further refinement in order to distinguish covers (or really the root stacks) from their ramification data if we allow the divisors to be non-reduced.

Example 5.2. Consider the following two μ_3-covers. Take the spectrum of $\mathbb{C}[s] \to \mathbb{C}[s,x,y]/(x^2 - sy, y^2 - sx, xy - s^2)$ where x has weight 1 and y has weight 2 and take the spectrum of $\mathbb{C}[s] \to \mathbb{C}[s,t]/(t^3 - s^2)$. These two covers have the same stabilizer group along the divisor $s = 0$.

To be able to distinguish between such covers we need a decomposition of the branch locus. The notion of birational building datum that we will use is the following:

Definition 5.3. Let S be a scheme. A *birational building datum* is a building datum $(\mathcal{A}, \mathcal{L})$ which is *regular*. That is,

- $\mathcal{A}$ is an étale sheaf of abelian groups of finite type on S, and
- a symmetric monoidal functor $\mathcal{L}\colon P_{\mathcal{A}} \to \mathscr{D}iv_{S_{\text{ét}}}$ whose image consists of generalised Cartier divisors $(\mathcal{L}_p, s_p)$ with s_p *regular*, and

such that the subgroup

$$\mathcal{A}^\perp = \{\lambda \in \mathcal{A} : \mathcal{L}_{\lambda,\lambda'} \simeq (\mathcal{O}_S, 1)\,, \forall \lambda' \in \mathcal{A}(U)\,,\ U \to S \text{ étale}\}$$

is trivial.

To each birational building datum $(\mathcal{A}, \mathcal{L})$ we may associate a root stack $\mathscr{X} = S_{(\mathcal{A},\mathcal{L})}$ and by considering quasi-coherent sheaves on $\mathscr{X}$ we arrive at a notion of *parabolic sheaves* with respect to $(\mathcal{A}, \mathcal{L})$.

Then we have the following theorem.

Theorem 5.4. *Let S be a scheme proper over a field k and assume that S is geometrically connected and geometrically reduced. Let $(\mathcal{A}, \mathcal{L})$ be a birational building datum and $(P_{\mathcal{A}}, Q_{\mathcal{A}}, \mathcal{L})$ the associated Deligne–Faltings datum. Then the following are equivalent:*

1. *There exists a finite abelian group scheme G over k and a ramified G-cover $X \to S$ with birational building datum $(\mathcal{A}, \mathcal{L})$;*
2. *For every geometric point $\bar{s}$ in the branch locus, we have that*

    (i) the map $\Gamma(S, \mathcal{A}) \to \mathcal{A}_{\bar{s}}$ is surjective, and

(ii) for every $\lambda \in \mathcal{A}_{\bar{s}}$, there exists an essentially finite, basic, parabolic vector bundle (E, ρ) on S, with respect to $(\mathcal{A}, \mathcal{L})$, such that the morphism

$$\bigoplus_{\lambda'} E_{e_\lambda - e_{\lambda'}}|_{\bar{s}} \xrightarrow{(E(e_{\lambda'})|_{\bar{s}})_{\lambda'}} E_{e_\lambda}|_{\bar{s}}$$

is not surjective, where the direct sum is over all $\lambda' \in \Gamma(S, \mathcal{A})$ such that $\lambda'_{\bar{s}} \neq 0$.

Example 5.5. Let $\pi\colon \mathscr{X} \to \mathbb{A}^2_{\mathbb{C}} = S$ be the root stack obtained by taking a square root of each of the two coordinate axis in the affine plane over the complex numbers. In this case we know that $\mathscr{X}$ is the quotient of a Kummer cover under the action of $G = \mu_2^2 = D(A)$ with $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. There is a canonical G-torsor $X \to \mathscr{X}$ which is the spectrum over $\mathscr{X}$ of the $\mathcal{O}_{\mathscr{X}}$-algebra $\mathcal{O}_X := \bigoplus_{\lambda \in (\mathbb{Z}/2\mathbb{Z})^2} \mathcal{O}_{\mathscr{X}}[\lambda]$. This corresponds to a parabolic sheaf E on $(S, \mathcal{A}, \mathcal{L})$ which is the one we will consider. We have $E_q = \pi_*(\mathcal{O}_X \otimes \mathcal{E}_q)$ where $\mathcal{E}\colon Q_{\mathcal{A}} \to \mathscr{D}iv_{\mathscr{X}_{\text{ét}}}$ is the universal Deligne–Faltings structure on $\mathscr{X}$ which satisfies $\mathcal{E}_{e_\lambda} = (\pi^*\pi_*\mathcal{O}_{\mathscr{X}}[\lambda])^\vee \otimes \mathcal{O}_{\mathscr{X}}[\lambda]$. To get coordinates for X we write $X = \operatorname{Spec} \mathbb{C}[s,t,x,y]/(x^2 - s, y^2 - t)$ where x and y has weight $\lambda_1 := (1,0)$ and $\lambda_2 := (0,1)$ respectively. Let us write $\mathcal{L}_x = (x)$, $\mathcal{L}_y = (y)$, and $\mathcal{L}_{xy} = (xy)$ for the free modules (ideals) of rank 1 generated by x, y and xy respectively.

We have

$$\begin{aligned} E_{e_\lambda} &= \pi_*\left(\Big(\bigoplus_{\lambda' \in A} \mathcal{O}_{\mathscr{X}}[\lambda']\Big) \otimes \Big((\pi^*\pi_*\mathcal{O}_{\mathscr{X}}[\lambda])^\vee \otimes \mathcal{O}_{\mathscr{X}}[\lambda]\Big)\right) \\ &\cong \bigoplus_{\lambda' \in A} \pi_*\Big((\pi^*\pi_*\mathcal{O}_{\mathscr{X}}[\lambda])^\vee \otimes \mathcal{O}_{\mathscr{X}}[\lambda + \lambda']\Big) \\ &\cong \bigoplus_{\lambda' \in A} \mathcal{L}_\lambda^\vee \otimes \mathcal{L}_{\lambda+\lambda'}\,. \end{aligned}$$

To simplify the notation we write $\mathcal{L}\mathcal{L}' := \mathcal{L} \otimes \mathcal{L}'$. In particular,

$$E_{e_{(1,0)}} \cong (1) \oplus (x)^\vee \oplus (x)^\vee(xy) \oplus (x)^\vee(y)\,.$$

Similarly,

$$E_{e_\lambda - e_{\lambda'}} \cong \bigoplus_{\lambda'' \in A} \mathcal{L}_\lambda^\vee \otimes \mathcal{L}_{\lambda'} \otimes \mathcal{L}_{\lambda - \lambda' + \lambda''}\,.$$

We have a morphism $E(e_{\lambda'})\colon E_{e_\lambda - e_{\lambda'}} \to E_{e_\lambda}$ which is A-graded and given in degree $\lambda'' \in A$ by

$$\mathcal{L}_\lambda^\vee \otimes \mathcal{L}_{\lambda'} \otimes \mathcal{L}_{\lambda - \lambda' + \lambda''} \xrightarrow{\mathrm{id}_{\mathcal{L}_\lambda^\vee} \otimes s_{\lambda', \lambda - \lambda' + \lambda''}} \mathcal{L}_\lambda^\vee \otimes \mathcal{L}_{\lambda + \lambda''}\,.$$

We get

$$
\begin{aligned}
E_{e_{(1,0)}-e_{(1,0)}} &= (x)^\vee(x)(1) \oplus (x)^\vee(x)(x) \oplus (x)^\vee(x)(y) \oplus (x)^\vee(x)(xy) \\
E_{e_{(1,0)}-e_{(0,1)}} &= (x)^\vee(y)(xy) \oplus (x)^\vee(y)(y) \oplus (x)^\vee(y)(x) \oplus (x)^\vee(y)(1) \\
E_{e_{(1,0)}-e_{(1,1)}} &= (x)^\vee(xy)(y) \oplus (x)^\vee(xy)(xy) \oplus (x)^\vee(xy)(1) \oplus (x)^\vee(xy)(x)\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
E(e_{(1,0)}) &= (1) \oplus (s) \oplus (1) \oplus (s) \\
E(e_{(0,1)}) &= (t) \oplus (t) \oplus (1) \oplus (1) \\
E(e_{(1,1)}) &= (t) \oplus (st) \oplus (1) \oplus (s)\,.
\end{aligned}
$$

We see that none of these maps is surjective in degree $(1,0)$ and hence if we pick the geometric point in Theorem 5.4 to be the origin, then the criterion is fulfilled for $\lambda = (1,0)$.

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Contribution to Paper B, C, and D

Let us refer to the author of this thesis as A. Papers B, C, and D of this thesis are joint with Magnus Carlson.

A's contribution to Paper B and Paper C can be found in all aspects of the papers. From laying the foundational theoretical framework to carrying out technical computations. To give a few concrete examples: finding formulas for the cohomology groups in the presens of real embeddings, programming examples to gain intuition, computing the cup product formula for punctured arithmetic curves, which was expected both by A and Carlson to have a certain form.

A's contribution to Paper D can be found in all aspects of the paper. From laying the foundational theoretical framework to carrying out technical computations. To give a few concrete examples: Finding small enough resolutions to be able to find formulas for the Massey products and working out maps between all resolutions. Computing the general formula for the Massey product. Writing a program in C using the library PARI to compute Massey products and Zassenhaus matrices. Realizing through programming that the Massey product reduced to a non-connected massey product when two of the elements are equal. This resulted in a program which was extremely much faster. Writing down a much simpler formula for the non-connected case to be used in our computations.

Part II

Scientific papers