# Benfords law and the Characteristic Polynomial of a CUE Matrix

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**Abstract**

Benford's Law describes a profound behavior that the leading digits of many quantities arising from mathematics, physics, finance, and engineering exhibit. In this text we prove Benford's Law for the absolute value of the characteristic polynomial $\det(U-\lambda I)$ of the CUE$(N)$ as $N \to \infty$. Our analysis produces an integrable bound for the characteristic function of $\log|\det(U-\lambda I)|$.

**Sammanfattning**

Benfords lag beskriver ett djupgående förhållande, vilket utövas av dem första siffrorna av många kvantiteter som uppstår i matematik, fysik, finansvetenskap och teknik. I den här texten bevisar vi Benfords lag för absolutbeloppet av CUE(N) karakteristiska polynomet $\det(U - \lambda I)$ när $N \to \infty$. Vår analys ger en integrerbar övre gräns för den karakteristiska funktionen av $\log|\det(U - \lambda I)|$.

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# 1 Introduction

Let us start with a simple question: What is the leading digit of $2^n$? In Figure 1 we plotted a histogram for the leading digit of $2^n$ for $n = 1, ..., 1000$. What we see may at first seem surprising. The most common leading digit is 1, and as digits increase, their occurrence decreases rapidly. How can we understand that?

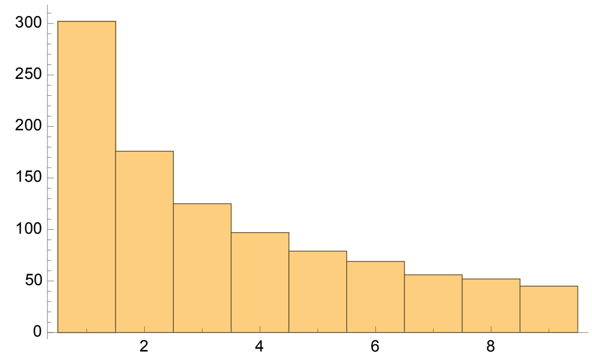


Figure 1: First 1000 samples of $2^n$, proportions of leading digits

## History

In 1938 Benford [1] published his empirical observations on the leading digit of a lot of natural quantities such as the Areas of Rivers, Death Rates, Populations, Cost Data, Pressure, Number of Foot Notes per Page in Books. His observations also included empirical data on mathematical expressions such as $n!, 2^n$. At the time of the publishing, it was a standard practice for scientists to use tables of logarithms, and it had already been observed by Newcomb(1881)[13], that the pages on which numbers began with 1 or 2 were much more worn out than other pages. To Benford(and Newcomb) this suggested that the quantities that were being studied had starting digit 1, or 2, much more often, and he proposed the distribution that we study in this thesis and that carries his name. He further observed that if he combines his data set of seemingly uncorrelated quantities, the evidence for Benford's Law is even stronger. Somehow Benford's Law is a universal, intrinsic property of numeric quantities.

## The Law

It is well known that the Bell Curve appears arises often when we study large samples of data obtained from physics, biology, finance, demography, etc. This is largely due to the Central Limit Theorem. Roughly speaking, in practice we expect to see the Bell Curve when we have large sums of independent quantities obtained in the same way. It is really interesting that in the same way we expect to observe Benford's Law when we have large products, instead of sums [12, 11]. For an overview of the subject see [11, 2].

## Mathematical Sequences

Let us come back to the example that we began with. The key realization is that to understand the leading digit of a number, we only need to study the fractional part of its logarithm. Why? We give an example with a specific number here and treat the matter formally in Section 2. For a number $S$ to begin with, say, 3 means that $3 \times 10^k \leqslant S < 4 \times 10^k$. So $\log_{10} 3 + k < \log_{10} S < \log_{10} 4 + k$ and indeed, $\log_{10} 3 \leqslant \log_{10} S \mod 1 < \log_{10} 4$.

In the case of $2^n$, this is $n \log_{10} 2$. Now as $\log_{10} 2$ is irrational, $n \log_{10}(2) \mod 1$ is dense in $[0, 1]$. Furthermore, as $n$ grows $n \log_{10}(2) \mod 1$ takes values uniformly in the interval $[0, 1]$. Thus, up to a certain $N$ the fraction of terms $n \log_{10} 2 \mod 1$ for $n \leqslant N$ between $\log_{10} 1$ and $\log_{10} 2$, should be about $\log_{10} 2 - \log_{10} 1 \approx 30\%$, and so $\approx 30\%$ of the terms would begin with 1. Now is as good time as any to introduce the definition of Benford's Law.

**Definition 1.1** (Benford Sequence of Real Numbers)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. The sequence $(x_n)_{n\in\mathbb{N}}$ obeys Benford's Law base $B$ if

$$\lim_{N\to\infty} \frac{|\{n \in \mathbb{N} : n \leqslant N, \text{ The leading digit of } x_n \text{ is } k\}|}{N} = \log_B\left(1 + \frac{1}{k}\right), \tag{1}$$

where $k$ is a non-zero digit base $B$. We say the sequence $(x_n)_{n\in\mathbb{N}}$ is Benford base $B$ if it obeys Benford's Law base $B$.

So, by the reasoning preceding the definition, $2^n$ is a Benford sequence base 10. What we should take away from the example is the connection between equidistributed quantities and Benford quantities.

**Definition 1.2** (Equidistributed sequence $\mod 1$)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. $(x_n)$ is equdistributed $\mod 1$, if for $0 \leqslant a < b \leqslant 1$

$$\lim_{N\to\infty} \frac{|\{n \in \mathbb{N} : n \leqslant N, x_n - [x_n] \in [a, b]\}|}{N} = b - a \tag{2}$$

Thus the fractional part of an equidistributed sequence $\mod 1$ may simulate the outcomes of a $\mathcal{U}(0, 1)$ random variable. It is a classical result due to Weyl that $(n\theta)_{n\in\mathbb{N}}$ is equidistributed $\mod 1$ iff $\theta$ is irrational, see Proposition 4.8 (i) in [2] or Theorem 2.6 in [9].

Our reasoning for the Benford behavior of $2^n$ can be made rigorous and generalized into the following theorem:

**Theorem 1.1.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Then $(x_n)$ is Benford base $B$ iff $(\log_B(x_n))_{n\in\mathbb{N}}$ is equdistributed* $\mod 1$.

The proof is analogous to that of Proposition 2.2. Alternatively, see Theorem 4.2 in [2]. The Corollary for Geometric Series is immediate, and formally answers our initial question about $2^n$.

**Corollary 1.2.** *Let $a > 0$. The sequence $a^n$ is Benford base $B$ iff $\log_B a = \frac{\log a}{\log B}$ is irrational.*

So Benford sequences arise as products. So it is not surprising that using Theorem 1.1, we see that the sequences $n!$, $n^n$ are also Benford, [2]. For Benford's Law for triangular arrays see [3]. The histogram of the proportions of leading digits from the first 10000 terms of $n!$ and $n^n$ can be seen correspondingly in Figure 2, and Figure 3. For a formal proof see [2, 3].

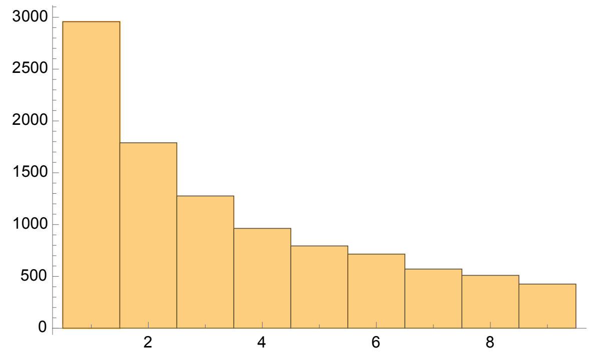


Figure 2: First 10000 samples of $n!$, proportions of leading digits

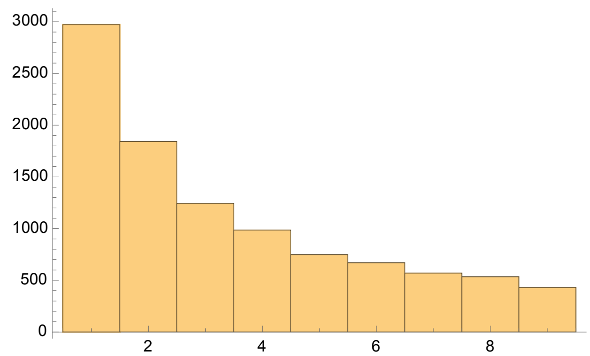


Figure 3: First 10000 samples of $n^n$, proportions of leading digits

## Toy model from Finance

It is known for many financial quantities to exhibit Benford behavior such as tax returns and financial indices. In fact Benford's Law has been used to identify tax fraud. With the motivation built so far we have developed some intuition about why that is.

Often financial quantities are modeled by products of random indices, they come in percentages such as the inflation rate, the percentage growth of stock prices. One way to model such a financial process is by random variables $X_0, X_1, .., X_n$, given by the following. $X_0$ obeys some distribution,

$$X_n = X_{n-1}(1 + \alpha_n), \text{ where } \alpha_1, \alpha_2, ..., \alpha_n \text{ i.i..d}, \alpha_1 \sim \mathcal{U}[0, 0.1]. \tag{3}$$

$\log_{10} X_n$, as a large sum of independent random variables, satisfies a Central Limit Theorem, but with growing variance and $\mod 1$ this will approximate a uniform distribution.

We run a simulation of this process and again write out the digits of the first 10000 outcomes in a histogram in Figure 4. The histogram is almost the same as ones for $2^n, n^n, n!$. We give a more formal interpretation of Benford random variables in Section 2.

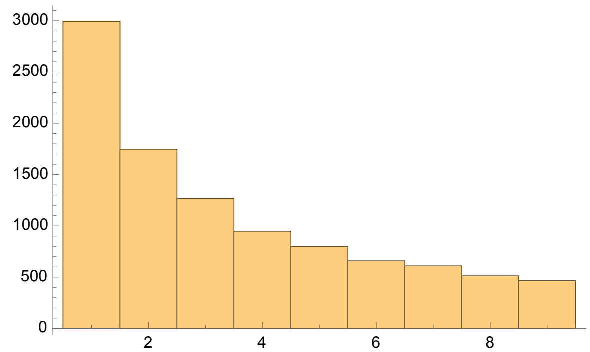


Figure 4: First 10000 samples of $X_n$, proportions of leading digits

## Random Matrix Theory

Random matrix theory was pioneered by Wigner in the 1960s. His goal was to understand the spacings between energy levels for heavy nuclei. His findings suggested that such spacings do not depend on the exact nucleus (as long as it is heavy) and exhibit patterns that are universal. Since these models are too complicated to be solved by hand(or by computer), he suggested that the same patterns should be present in a simpler model: the spacings between eigenvalues of random matrices. Since then random matrix theory has developed into a very active field in both the mathematics and the physics community. It is found that random matrices give rise to a variety of new probabilistic laws that are universal. See [10].
The goal of this thesis is to show that they also gives rise to Benford's law in a natural way.

We will look at the characteristic polynomial $\det(U - \lambda I)$, of a random unitary matrix $U$. The characteristic polynomial is the product of $\lambda - e^{i\theta_j}$, where $e^{i\theta_j}$ are the eigenvalues of the unitary $U$, a large product of random variables. Judging from the previous discussion, we may wonder if the characteristic polynomial satisfies Benford's Law. We prove that it does when $|\lambda| = 1$. An important difference here with the finance model is that the eigenvalues are not independent, but on the contrary, highly correlated. This complicates the analysis significantly.In fact the statement can already be found in the literature[9]. The proof there relies on a result in [7] which is not introduced entirely rigorously, as there is an exchange of a pointwise limit with an integral, which needs justification. Our approach provides this justification, which is the subject of the analysis in Section 5 and can be generalized to analyze the same problem for general Circular $\beta-$ensembles.

In Figure 5, we display in a histogram the first digit of 1000 samples of the characteristic polynomial of the CUE(100). The evidence is suggestive.

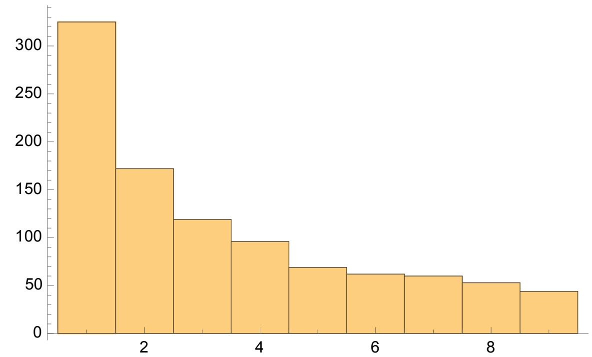


Figure 5: 1000 samples of the Absolute Value of the Characteristic Polynomial of the CUE(100)

## Overview of the text

In Section 2 we introduce a Formal Interpretation of Benford's Law together with some notation. We define what Benford's Law is for a sequence of Random Variables and derive some sufficient conditions for Benford's Law to hold for a sequence of Random Variables. We describe conditions that hold for our models of interest and we deduce conditions for Benford behavior in our setting, that is Theorem 2.8.

In Section 3 we describe Circular $\beta$-Ensembles and introduce the sequence of random variables that we study arising from them, that is the absolute value of the characteristic polynomial of the $N$'th ensemble is the $N'th$ random variable, $X_N$.

In Sections 4, 5 we show that the $X_N$ satisfy the conditions of Theorem 2.8.

In Section 4 we introduce the Selberg Integral, a poweful tool that allows us to find explicitly the characteristic functions of $Y_N = \log(X_N)$, $f_N$, and we compute the characteristic functions explicitly. Using the expressions obtained, we find the variance and expectation of $Y_N$, and we show that $Y_N/\sqrt{\mathbb{V}\mathrm{ar}(Y_N)}$ converge in distribution to a Standard Normal Random Variable.

In Section 5 we use the expression obtained in section 4 to show $f_N(t/\sqrt{\mathbb{V}\mathrm{ar}\, Y_N})$ is bounded above by an integrable function. We show how this completes the proof of Benford behavior.

# 2 Formal Footing for Benford's Law

## 2.1 Benford's law for Random Variables

In the thesis we are mainly interested about proving Benford's Law in the context of random variables. This section discusses in detail how to formulate mathematically Benford's Law for random variables. The extension of this notion to a different context is also natural, but one needs to formally introduce likelihood for deterministic sequences, for a "growing set of data". I.e. a measure needs to be introduced, and although this may be very natural, it may be a little definition heavy. Instead we let the reader see [9, 2, 3] and the examples in the Introduction. Observe that the leading digit does not really depend on the magnitude of the number. This concept we formalize via the Mantisa function.

**Definition 2.1** (Mantissa function base $B$)**.** Given $x > 0$, it can be uniquely represented as

$$x = M_B(x)B^k, \text{ where } M_B(x) \in [1, B). \tag{4}$$

$M_B : \mathbb{R}_+ \to [1, B)$ is the Mantissa function.

**Remark.** Observe that for $x > 0$

$$\log_B(x) = k + \log_B(M_B(x)), \tag{5}$$

where $k \in \mathbb{Z}, 0 \leqslant \log_B(M_B(x)) < 1$, i.e. $\log_B M_B(x)$ is the fractional part of $\log(x)$,

$$\log_B M_B(x) = \log(x) - [\log(x)]. \tag{6}$$

**Remark.** Note that in (4), $B^k \leqslant x < B^{k+1}$. Further, we should note that $M_B(x)$ contains not only the information for the first digit, but for any digit of the number $x$. This is why the form of Benford's law that we introduce here is sometime refered to as Strong Benford's Law, see [3].

**Remark.** We extend $M_B$ to $\mathbb{R}_{\geqslant 0}$ by setting $M_B(0) = 0$.

**Proposition 2.1.** *If $X : \Omega \to \mathbb{R}_{\geqslant 0}$ is measurable, so is $Y = M_B(X)$.*

*Proof.* Observe that:

$$Y^{-1}(\infty, b) = \begin{cases} \{\omega \in \Omega, X(\omega) \geqslant 0\} \text{ if } b > T, \\ \{\omega \in \Omega : X(\omega) \in \{0\} \cup \bigcup_{n \in \mathbb{Z}} [1, bB^n)\} \text{ if } 1 \leqslant b \leqslant T, \\ \{\omega \in \Omega : X(w) = 0\} \text{ if } 0 \leqslant b < 1, \\ \varnothing \text{ if } b < 0. \end{cases} \tag{7}$$

These are all preimages of X of Borel sets, so measurable. Thus Y is measurable. □

Now that we know the Mantissa function is coherent in random variable context, we can give the formal notion of a Benford random variable.

**Definition 2.2** (Benford Random Variable)**.** Let $X : \Omega \to \mathbb{R}_{\geqslant 0}$, be a non-negative random variable. $X$ is Benford base $B$ if

$$P(1 \leqslant (M_B(X)) \leqslant s) = \log_B(s), \text{ where } 1 \leqslant s \leqslant B. \tag{8}$$

First of all, we observe that this definition is coherent with the definition of Benford's Law that we gave in the introduction. Let $k \in \{1, 2, ..., B-1\}$, be a digit base B, that is not 0.

$$P(\text{"The first digit of } X \text{ is } k\text{"}) = P(k \leqslant M_B(X) < k{+}1) = P(\log_B(k) \leqslant \log_B(M_B(X)) < \log_B(k{+}1) = \log_B\left(1 + \frac{1}{k}\right).$$

Here we point out that Definition 2.2 reveals more than just the distribution of the first digit of $X$. For simplicity we give an example in base 10, but we can do the same in any base.

$$P(\text{"The first 5 digit of } X \text{ are 12345 in that order"}) = P(1.2345 \leqslant M_{10}(X) < 1.2346)$$

$$= P(\log_{10}(1.2345) \leqslant \log_{10} M_{10}(X) < \log_{10}(1.2346)) = \log_{10}\left(\frac{1.2346}{1.2345}\right) = \log_{10}\left(1 + \frac{1}{12345}\right).$$

The next Proposition sheds further light on what Benford's Law means for a random variable in our setting.

**Proposition 2.2.** *Let $X$ be a non-negative random variable. Then $X$ is Benford base $B$ iff $\log_B(X)$ is equidistributed modulo 1.*

*Proof.* $X$ − Benford $\iff \log_B(s) = P(1 \leqslant (M_B(X)) \leqslant s) = P(0 \leqslant \log_B M_B(X) \leqslant \log_B s)$.
By (6), $\log_B M_B(x) = \log(X) \mod 1$. □

Proposition 2.2 is a really important equivalence when discussing Benford's Law. See [2, 3, 9].

**Definition 2.3** (Benford Sequence of Random Variables)**.** We say a sequence of random variables $X_n$ is Benford base $B$, if $M_B(X_N)$ converge in distribution to $X$, a random variables which is Benford base $B$.

We end this part of the text with Definition 2.3, as for the rest of the text we will discuss when certain sequences of random variables converge in distribution to a Benford random variable.

## 2.2 Sufficient conditions for Benford behavior in a general setting

The thesis is concerned with sequences of non-negative random variables $(X_N), N \in \mathbb{N}$ s.t. $Y_N = \log(X_N)$ has a finite second moment and:

$$E(Y_N) = 0, \tag{E0}$$

$$\mathbb{V}\text{ar}(Y_N) = T(N)^2, \tag{FV}$$

$$T = T(N), N \in \mathbb{N}, T \to \infty \text{ as } N \to \infty. \tag{GV}$$

From now on we interchangebly index by $T$ and $N$, depending on context. We wish to find sufficient conditions for such sequences to be Benford.

In this subsection we introduce some general conditions for a sequence of random variables to be Benford due to Kontorovich and Miller, [9]. Once we have them, we will apply them to our setting of interest, that we described with equations (E0),(FV), (GV).

- Let $\rho(x)$ be a probability density function. Let $F(x) = \int_{-\infty}^{x} \rho(t)dt$ be it's associated cumulative distribution function.

- Let $X_T$ be a sequence of random variables indexed by $T \in \Omega \subset \mathbb{R}_+$, totally ordered.
- Let $Y_T = \log_B(X_T)$ for fixed $B \in \mathbb{R}, B > 0$.
- Let $F_T(x) = P(Y_T \leqslant x)$ be the cumulative distribution function of $Y_T$, with corresponding density $\rho_T$. Set:
$$F_T(x) = F(x/T) + E_T(x). \tag{9}$$
Note that in the above, the definition of the error, $E_T(x)$ is implicit. We rewrite it in two more ways:
$$E_T(x) = \int_{-\infty}^{x} \left( \rho_T(\xi) - \frac{1}{T}\rho(\xi/T) \right) d\xi, \tag{10}$$
$$E_T(x) = \int_{-\infty}^{x/T} \left( T\rho_T(T\xi) - \rho(\xi) \right) d\xi. \tag{11}$$
The three equations, (9),(10),(11) are equivalent.
- The density $\rho$, satisfies:
$$|\rho(x)| + |\hat{\rho}(x)| \leqslant C \frac{1}{(1+|x|)^{1+\delta}}. \tag{12}$$
The inequality (12) gives a control on the tails of the density and its corresponding characteristic function that allows Poisson summation:
$$\sum_n \rho(n) = \sum_n \hat{\rho}(n). \tag{13}$$

One example of choice of density we compare to is the standard normal density, $\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Then:
$$E_T(x) = P(Y_T \leqslant x) - P(Z_T \leqslant x) \text{ where } Z_T \sim N(0, T^2), \tag{14}$$
$$E_T(x) = P(Y_T/T \leqslant x/T) - P(Z \leqslant x/T) \text{ where } Z \sim N(0,1). \tag{15}$$

We next introduce sufficient conditions in the above setting that will give us Benford behavior.

**Definition 2.4** (Benford- good)**.** If the sequence $(Y_T)_{T\in\Omega}$ is defined as above, it is Benford good, if there exists monotone increasing function $h : \Omega \to \mathbb{R}$ with $\lim_{T\to\infty} h(T) = \infty$ and the following conditions hold:

(i) (Small tails)
$$P(|Y_T| \geqslant Th(T)) = o(1) \text{ as } T \to \infty, \tag{16}$$

(ii) (Rapid decay of the characteristic function)
$$S(T) = \sum_{k\neq 0} \left| \frac{\hat{\rho}(Tk)}{k} \right| = o(1) \text{ as } T \to \infty, \tag{17}$$

(iii) (Small Truncated Translated Error) $a, b \in [0,1]$
$$\mathcal{E}_T(a,b) = \sum_{|k|\leqslant Th(T)} \left( E_T(b+k) - E_T(a+k) \right) = o(1) \text{ as } T \to \infty. \tag{18}$$

The above definition is only useful, because of the theorem that follows:

**Theorem 2.3** (3.2 in [9])**.** *If the sequence $(Y_T)_{T\in\Omega}$ is Benford-good, then $Y_T \mod 1 = Y_T - [Y_T]$ converges in distribution to $Y$, a random variable that is equidistributed $\mod 1$.*

**Remark.** Theorem (2.3) and Proposition (2.2) together imply that $(X_T)_{T\in\Omega}$ is a Benford sequence, if $(Y_T)_{T\in\Omega}$ is Benford good.

*Proof.* Let $a, b \in [0, 1]$

$$P(Y_T \mod (1) \in [a, b]) = \sum_{k\in\mathbb{Z}} P(Y_T \in [a + k, b + k])$$

$$= \sum_{|k|\leqslant Th(T)} P(Y_T \in [a + k, b + k]) + \sum_{|k|>Th(T)} P(Y_T \in [a + k, b + k])$$

By (16)

$$P(Y_T \mod (1) \in [a, b]) = \sum_{k\leqslant Th(T)} P(Y_T \in [a + k, b + k]) + o(1) \tag{19}$$

(where if, concerned with speed of convergence the bound for the $o(1)$ term is $\sum_{|k|>Th(T)} P(Y_T \in [a + k, b + k]) \leqslant P(Y_T > Th(T)))$. Now

$$\sum_{k\leqslant Th(T)} P(Y_T \in [a + k, b + k]) = \sum_{k\leqslant Th(T)} (F_T(b + k) - F_T(a + k))$$

$$= \sum_{k\leqslant Th(T)} (E_T(b + k) - E_T(a + k)) + \sum_{k\leqslant Th(T)} \left(F\left(\frac{b + k}{T}\right) - F\left(\frac{a + k}{T}\right)\right)$$

$$\sum_{k\leqslant Th(T)} P(Y_T \in [a + k, b + k]) = \mathcal{E}_T(a, b) + \sum_{|k|\leqslant Th(T)} \int_a^b \frac{1}{T}\rho\left(\frac{x + k}{T}\right) dx. \tag{20}$$

Set

$$g_x(\xi) = \rho(\frac{x + \xi}{T}).$$

Claim: Since $\rho$ satisfies Poisson summation formula, so does $g_x$. So:

$$\sum_{|k|\leqslant Th(T)} \int_a^b \frac{1}{T}\rho\left(\frac{x + k}{T}\right) dx = \int_a^b \sum_{k\in\mathbb{Z}} \frac{1}{T}\rho\left(\frac{x + k}{T}\right) dx + o(1)$$

$$= \frac{1}{T}\int_a^b \sum_{k\in\mathbb{Z}} g_x(k)dx + o(1) = \frac{1}{T}\int_a^b \sum_{k\in\mathbb{Z}} \widehat{g_x}(k)dx + o(1). \tag{21}$$

We can exchange the order of summation and integration, since (12) guarantees good control of the tails of $\rho$. We compute the Fourier transform of $g_x$.

$$\widehat{g_x}(k) = \int_{-\infty}^{\infty} \rho(\frac{x + \xi}{T})e^{-i2\pi\xi k}d\xi = T\int_{-\infty}^{\infty} \rho(y)\exp\left(-2\pi(Ty - x)k\right) dy = T\widehat{\rho}(Tk)e^{i2\pi xk}.$$

So, using the newly obtained expression, we see that:

$$\frac{1}{T}\int_a^b \sum_{k\in\mathbb{Z}} \widehat{g}_x(k)dx = \sum_{k\in\mathbb{Z}} \widehat{\rho}(Tk) \int_a^b e^{i2\pi xk}dx.$$

So, going back to 21, we obtain

$$\sum_{|k|\leqslant Th(T)} \int_a^b \frac{1}{T}\rho\left(\frac{x+k}{T}\right)dx = b-a+\sum_{k\in\mathbb{Z}\backslash 0} \frac{\widehat{\rho(Tk)}}{i2\pi k}(e^{2\pi ibk}-e^{2\pi iak})+o(1). \tag{22}$$

By (19), (20), (22),

$$P(Y_T \mod (1) \in [a,b]) = b-a+\mathcal{E}_T(a,b)+\sum_{k\in\mathbb{Z}\backslash 0} \frac{\widehat{\rho(Tk)}}{i2\pi k}(e^{2\pi ibk}-e^{2\pi iak})+o(1) = b-a+o(1). \tag{23}$$

□

Now that we have sufficient conditions for a sequence to be Benford in this more general setting, we are ready to apply them to the setting of interest we introduced earlier on, to deduce sufficient conditions.

### 2.3 Sufficient conditions for Benford behavior in the setting of interest

In our analysis, we compare the distributions of $Y_T/T$ to that of a Standard Normal:

$$\rho(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \tag{ND}$$

**Theorem 2.4.** *If we assume* (E0)*,* (FV)*,* (GV)*,* (ND)*, $Y_T$ is Benford good iff for all $M>0$*

$$\mathcal{E}_T(a,b,M) = \sum_{|k|\leqslant TM} (E_T(b+k)-E_T(a+k)) = o(1) \text{ as } T\to\infty. \tag{24}$$

The proof of Theorem 2.4 consists of three lemmas.

**Lemma 2.5.** *The standard normal density $\rho(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ satisfies* (13)*, and* (17)*.*

*Proof.* Remember $\hat{\rho}(t) = e^{-t^2/2}$, so both $\rho$ and $\hat{\rho}$ decay really fast and certainly both (12)($\implies$ (13)), and (17) hold. □

**Lemma 2.6.** *Assume* (E0)*,* (FV)*,*(GV) *hold. Then the sequence of distributions of $Y_T/T$ is a tight sequence, and* (16) *holds for any $h$ of the required kind in Definition 2.4.*

*Proof.* By Chebyshev's inequality:

$P(|\frac{Y_T}{T}| \geqslant M) = P(|Y_T| \geqslant MT) \leqslant \frac{\mathbb{V}\mathrm{ar}\, Y_T}{M^2T^2} = \frac{1}{M^2}$ and this bound holds for all $T$. Tightness follows.

Proving (16) is completely analogous, just replace $M$ with $h(T)$ in the above. □

**Lemma 2.7.** *If the sequence of distributions of $Y_T/T$ is tight, then* (18) *and* (24) *are equivalent.*

*Proof.* Fix $\varepsilon > 0$.

$$\sum_{|k|\leqslant Th(T)} (E_T(b+k) - E_T(a+k)) = \sum_{|k|\leqslant TM} (E_T(b+k) - E_T(a+k)) + \sum_{Th(T)\geqslant |k| > TM} (E_T(b+k) - E_T(a+k)) \,.$$

By (15), and triangle inequality, we get:

$$|\sum_{|k|\leqslant Th(T)} (E_T(b+k) - E_T(a+k))\,| \leqslant |\sum_{|k|\leqslant TM} (E_T(b+k) - E_T(a+k))\,| + P(\frac{|Y_T|}{T} \geqslant M) + P(Z > M, Z \sim N(0,1)) \tag{25}$$

and

$$|\sum_{|k|\leqslant TM} (E_T(b+k) - E_T(a+k))\,| \leqslant |\sum_{|k|\leqslant Th(T)} (E_T(b+k) - E_T(a+k))\,| + P(\frac{|Y_T|}{T} \geqslant M) + P(Z > M, Z \sim N(0,1)). \tag{26}$$

From (25),(26), choosing $M$ so that $P(\frac{|Y_T|}{T} \geqslant M) + P(Z > M, Z \sim N(0,1)) < \varepsilon$, we see that:

$$|\sum_{|k|\leqslant Th(T)} (E_T(b+k) - E_T(a+k))\,| \leqslant |\sum_{|k|\leqslant TM} (E_T(b+k) - E_T(a+k))\,| + \varepsilon, \tag{27}$$

and

$$|\sum_{|k|\leqslant TM} (E_T(b+k) - E_T(a+k))\,| \leqslant |\sum_{|k|\leqslant Th(T)} (E_T(b+k) - E_T(a+k))\,| + \varepsilon. \tag{28}$$

The equivalence follows from (27),(28). □

Now we are in condition to prove the theorem.

*Proof of Theorem 2.4.* By fixing our approximation function of choice to be the standard normal density, by Lemma 2.5 (13) and (17) are satisfied. By Lemma 2.6, (16) is satisfied. And by Lemma 2.7, Lemma 2.6 (18) and (24) are equivalent in our setting. So $Y_T$ is Benford good in the suggested setting. □

So we have reduced the three conditions we needed to show are satisfied initially to just one (24) in our setting. In the latter sections we see that the characteristic functions of $Y_N$ are often objects that we can study and understand, so in the next part of this section we rewrite (24) in terms of characteristic functions.

## 2.4 Shifting to characteristic functions

We can think of an integral of a density over a set, as the inner product of the indicator function of the set and the density in $L^2$. Then we can use the properties of Hilbert spaces. Keeping that in mind, define

$$U_{T,M} = \bigcup_{|k|\leqslant MT} [\frac{a+k}{T}, \frac{b+k}{T}]. \tag{29}$$

Let $r_{T,M}$ be the characteristic function of $\mathbb{1}_{U_{T,M}}$,

$$r_{T,M}(x) = \int_{\mathbb{R}} e^{itx} \mathbb{1}_{U_{T,M}}(x)dx. \tag{30}$$

**Theorem 2.8.** *If we assume* (E0), (FV), (GV), *and*

*(i) we have that*

$$f_N(t/T) \to e^{-t^2/2} \text{ as } N \to \infty, \tag{31}$$

*where $f_N$ is the characteristic function of* $\log_B X_N$.

*(ii) There exists a non-negative, integrable $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ s.t.*

$$|f_N(t/T)| \leqslant g(t) \forall t, \tag{32}$$

*then the sequence $X_T$ is a Benford sequence.*

**Remark.** This means $f_N(t/T) \to e^{-t^2/2}$ in $L^1$ sense.

**Proposition 2.9** (Plancherel for Characteristic funtions)**.** *For $h, g \in L^2(\mathbb{R})$, density functions, and $\tilde{h}, \tilde{g}$, their corresponding characteristic functions:*

$$\int_{\mathbb{R}} h(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{\tilde{h}(t)}\tilde{g}(t)dt \tag{33}$$

*Proof of Theorem 2.8.*

$$\mathcal{E}_T(a,b,M) = \sum_{|k| \leqslant TM} \int_{\frac{a+k}{T}}^{\frac{b+k}{T}} \left(T\rho_T(Tx) - \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right) dx = \int_{\mathbb{R}} \mathbb{1}_{U_{T,M}}(x) \left(T\rho_T(Tx) - \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right) dx \tag{34}$$

By Plancherel (Theorem 2.9),

$$\mathcal{E}_T(a,b,M) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{r_{T,M}(x))}\overline{(f_T(t/T) - e^{-t^2/2})}dt. \tag{35}$$

Observe that

$$r_{T,M} \leqslant \int_{-\infty}^{\infty} \mathbb{1}_{U_{T,M}}(x)dx = L(U_{T,M}) \leqslant \frac{b-a}{T}(2MT+1) \leqslant 2M+1.$$

Thus,

$$|r_{T,M}(x))(f_T(t/T) - e^{-t^2/2})| \leqslant (2M+1)\left(g(t) + e^{-t^2/2}\right), \tag{36}$$

and we can apply Dominated convergence to (35), with dominating function $(2M+1)[g(t)+e^{-t^2/2}]$ to obtain that

$$\lim_{T\to\infty} \mathcal{E}_T(a,b,M) = \int_{\mathbb{R}} \lim_{T\to\infty} \left(r_{T,M}(x)(f_T(t/T) - e^{-t^2/2})\right) dt = 0. \tag{37}$$

By Theorem (2.4), the conclusion of the theorem follows. □

Before moving on to use the tools developed in this section, we encounter one more property of Benford's Law: base invariance. Roughly speaking, if we are able to prove Benford behavior for some base, we should expect that we can prove it for any other base, see [6]. The next proposition asserts that this is indeed the case in our setting.

**Proposition 2.10.** $\widetilde{f_N}(t) = f_N\left(t/\sqrt{\mathbb{V}\mathrm{ar}(\log_B Y_N)}\right)$ *is invariant of the choice of* $B$.

*Proof.* Let $\phi_N(t)$ be the characteristic function of $\log X_N$, $f_N(t)$, the characteristic function of $\log_B X_N$. Note $\log_B X_N = \log X_N / \log B$, so

$$f_N(t) = \phi_N(t/\log B). \tag{38}$$

Also,

$$\mathbb{V}\mathrm{ar}\left(\log_B X_N\right) = \frac{1}{\log^2 B}\,\mathbb{V}\mathrm{ar}\left(\log X_N\right). \tag{39}$$

Thus

$$f_N\left(t/\sqrt{\mathbb{V}\mathrm{ar}(\log_B(X_N))}\right) = \phi_N\left(\frac{t}{\log B}\frac{\log B}{\sqrt{\mathbb{V}\mathrm{ar}(\log X_N)}}\right) = \phi_N\left(t/\sqrt{\mathbb{V}\mathrm{ar}(\log X_N)}\right) = \widetilde{f_N}(t). \tag{40}$$

□

**Remark.** Theorem 2.8 together with Proposition 2.10 imply that it is sufficient to prove $X_N$ is Benford for some base $B$ to show $X_N$ is Benford for any base $B$.

In the remaining part of the text we discuss the models which motivated the development of the theory up to this moment, and apply the tool developed in this section, Theorem 2.8 to prove Benford's Law for the characteristic polynomials of the CUE. They are introduced in the next section, Section 3.

# 3 Circular $\beta$-ensembles; CUE

Referring back to the Introduction, one of the main goals of this text is to show how Random Matrices give rise to Benford's Law. In this section, we introduce the $C_\beta$ ensembles, objects of significance in Random Matrix Theory, see [8].

**Definition 3.1** (Circular $\beta$-ensemble)**.** The Circular $\beta$-ensemble is a probability measure on $[0, 2\pi]^N$, $N \in \mathbb{N}$, given by

$$d\mu = \frac{1}{(2\pi)^N C_N(\beta/2)} \prod_{1 \leqslant j < k \leqslant N} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{j=1}^{N} d\theta_j, \tag{41}$$

where $C_N(\beta/2)$ is a normalizing constant s.t.

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1}{(2\pi)^N C_N(\beta/2)} \prod_{1 \leqslant j < k \leqslant N} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{j=1}^{N} d\theta_j = 1. \tag{42}$$

For the special values of $\beta = 1, 2, 4$, the densities arise as the joint eigenvalue distribution of randomly chosen matrices [8]. For instance, for $\beta = 2$, the $e^{i\theta_j}$ can be thought of as the eigenvalues of a $N \times N$ matrix $U$, taken at random from the unitary group $\mathcal{U}(N)$, equipped with its Haar Measure, coming from matrix multiplication. This is called the Circular unitary ensemble, the CUE.

There are very natural interpretations of the Circular $\beta$-ensembles, when $\beta = 1, 4$ arising in a similar fashion, the $COE$ and the $CSE$ correspondingly. For further investigation of these interpretations, see [8].

Now we introduce the sequence of random variables we study here:

$$Z(N, \theta) = \prod_{j=1}^{N} (e^{i\theta_j} - e^{i\theta}). \tag{43}$$

In the interpretations in which $e^{i\theta_j}$ are the eigenvalues of a randomly chosen matrix $U$, $Z(N, \theta)$ can be thought of as the characteristic polynomial of such a matrix, $\det(U - I\lambda)$, where $|\lambda| = 1$. The aim of this thesis is to show that the sequence $|Z(N, \theta)| = X_N$ is Benford. With our set of tools, in order to do that we need to understand the characteristic function of $Y_N = \log(|Z_N|)$. It is given by:

$$f(t) = \mathbb{E}[e^{itY_N}] = \mathbb{E}[|Z_N|^{it}] = \int |Z_N|^{it} d\mu. \tag{44}$$

So in the case of Circular $\beta$-ensembles, the characteristic function of interest is given by:

$$f(t) = \int_{-\pi}^{\pi} \cdots\cdots \int_{-\pi}^{\pi} \frac{|\prod_{j=1}^{N}(e^{i\theta_j} - e^{i\theta})|^{it}}{(2\pi)^N C_N(\beta/2)} \prod_{1 \leqslant j < k \leqslant N} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{j=1}^{N} d\theta_j. \tag{45}$$

Circular $\beta$-ensembles have a nice property. They are rotationally invariant, that is if we make the substitution $\theta_j = \theta'_j + \delta$, for $\in \{1, 2, .., N\}$, the measure does not really change (We rotate by angle $\delta$). This is an important property, as for circular $\beta$-ensembles, we see that the distribution of $Z(N,\theta)$, does not really depend on $\theta$. Having made that remark, we make the substitution $\theta_j = \theta'_j + (\theta + \pi)$ in (45), to obtain:

$$f(t) = \int_{-\pi}^{\pi} \ldots\ldots \int_{-\pi}^{\pi} \frac{\prod_{j=1}^{N} |e^{i\theta}(e^{i(\theta'_j+\pi)} - e^{i\theta})|^{it}}{(2\pi)^N C_N(\beta/2)} \prod_{1\leqslant j<k\leqslant N} |e^{i\theta'_j} - e^{i\theta'_k}|^{\beta} \prod_{j=1}^{N} d\theta'_j.$$

Using that $|e^{i\theta}| = 1$, and that $e^{i(\delta+\pi)} = -e^{i\delta}, \delta \in [0, 2\pi]$, we see that

$$f(t) = \int_{-\pi}^{\pi} \ldots\ldots \int_{-\pi}^{\pi} \frac{1}{(2\pi)^N C_N(\beta/2)} \prod_{j=1}^{N} |1 + e^{i\theta_j}|^{it} \prod_{1\leqslant j<k\leqslant N} |e^{i\theta_j} - e^{i\theta_k}|^{\beta} \prod_{j=1}^{N} d\theta_j. \tag{46}$$

The last expression is the one we proceed to analyze, and the one which we can actually compute via the Selberg integral. For the sake of clarity in the following sections, we focus on the case $\beta = 2$, i.e. the CUE, but we strongly believe that the upcoming analysis does not depend on $\beta$, and can be extended to general $\beta$.

# 4 Selberg Integral

In this section we introduce a tool that allows us to compute explicitly the characteristic function of $\log |Z(N, \theta)|$, the Selberg integral. This is the reason why Circular $\beta$-ensembles are particularly nice to work with in this context. For an overview of the Selberg Integral, see [4]. In this section we also borrow the notation in [4].

**Definition 4.1** (Selberg Integral)**.** Let $n \in \mathbb{N}$, $\alpha, \beta, \gamma \in \mathbb{C}$, with $Re(\alpha) > 0, Re(\beta) > 0$ and $Re(\gamma) > -\min\{(1/n), Re(\alpha)/(n-1), Re(\beta)/(n-1)\}$

$$S_n(\alpha,\beta,\gamma) = \int_0^1 \dots \int_0^1 \prod_{j=1}^{n} t_j^{\alpha-1}(1-t_j)^{\beta-1} \prod_{1 \leqslant j < k \leqslant n} |t_i - t_j|^{2\gamma} dt_1 \dots dt_n \tag{47}$$

is the Selberg Integral.

Recall the definition of the gamma function.

**Definition 4.2** (Gamma Function)**.** $\Gamma : \{z \in \mathbb{C} : \Re(z) > 0\} \to \mathbb{C}$, defined by

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} \tag{48}$$

is the Gamma function.

**Theorem 4.1.** *The Selberg Integral satisfies*

$$S_n(\alpha,\beta,\gamma) = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j\gamma)\Gamma(\beta + j\gamma)\Gamma(1 + (j+1)\gamma)}{\Gamma(\alpha + \beta + (n+j-1)\gamma)\Gamma(1+\gamma)}. \tag{49}$$

*Proof of Theorem 4.1.* See [10]. □

**Remark.** The original proof of Theorem 4.1 is due to Selberg and can be found in [14] in Norwegian. The proof is summarized in [4].

This integral, $S_n(\alpha, \beta, \gamma)$ can be used to evaluate many of the quantities of interest in this project, such as the normalizing constant $C_N$ of the $C_\beta$ Ensembles in (41), and the characteristic function of $\log(|Z_n|)$ via the expression in (46). We proceed to introduce these results.

**Proposition 4.2.** *The normalizing constant from* (41) *is:*

$$C_n(\gamma) = \frac{\Gamma(1+n\gamma)}{\Gamma^n(1+\gamma)}. \tag{50}$$

*Proof.* See [4] . □

**Corollary 4.3.** *In the case of CUE, the normalizing constant in* (41) *is* $n!$*.*

*Proof.* As we are interested in $C_N(\beta/2)$ for $\beta = 2$, Theorem 4.2 gives

$$C_N(1) = \frac{\Gamma(1+n)}{\Gamma^n(2)} = n!. \tag{51}$$

□

The proposition that is to follow, Proposition 4.4, is especially useful when computing the characteristic function in (46). It is a direct consequence of the Selberg integral.

**Proposition 4.4.** *Let*

$$M_n(a,b,\gamma) = \frac{1}{2\pi^n}\int_{-\pi}^{\pi}...\int_{-\pi}^{\pi}\prod_{j=1}^{n} e^{\frac{1}{2}i\theta_j(a-b)}\left|1+e^{i\theta_j}\right|^{a+b}\prod_{1\leqslant j<k,\leqslant n}\left|e^{i\theta_j}-e^{i\theta_k}\right|^{2\gamma}d\theta_1...d\theta_n, \tag{52}$$

*where* $Re(a+b+1)>0$*,* $Re(\gamma)>-\min\{1/n, Re(a+b+1)/(n-1)\}$*. Then,*

$$M_n(a,b,\gamma) = \prod_{j=0}^{n-1}\frac{\Gamma(1+a+b+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(1+a+j\gamma)\Gamma(1+b+j\gamma)\Gamma(1+\gamma)}. \tag{53}$$

*Proof.* See [4]. □

The consequence of Proposition 4.4 is that we can find our characteristic function of interest.

**Theorem 4.5.** *The characteristic function of* $\log|Z(N,\theta)|$ *is*

$$f(t) = \prod_{j=1}^{N}\frac{\Gamma(it+j)\Gamma(j)}{\Gamma(j+it/2)^2}. \tag{54}$$

*Proof.* By Proposition 4.4 when when $\gamma=1$, $a=b=it/2$,

$$M_n(it/2,it/2,1) = \frac{1}{2\pi^n}\int_{-\pi}^{\pi}...\int_{-\pi}^{\pi}\prod_{j=1}^{n}|1+e^{i\theta_j}|^{it}\prod_{1\leqslant j<k\leqslant n}|e^{i\theta_j}-e^{i\theta_k}|^2 d\theta_1...d\theta_n = f(t)C_N(1) \tag{55}$$

But also $M_n(it/2,it/2,1) = \prod_{j=1}^{n}\frac{\Gamma(it+j)\Gamma(j+1)}{\Gamma(j+it/2)^2}$. So

$$f(t) = \frac{1}{N!}\prod_{j=1}^{N}\frac{\Gamma(it+j)\Gamma(j+1)}{\Gamma(j+it/2)^2}.$$

□

Thus to understand the characteristic function of $\log|Z(N,\theta)|$, we need to understand a product of Gamma functions.

## 4.1 Expression for characteristic function

We do a few computations to put (54) into a form we can analyze. Using the factorial property of the Gamma function,

$$\Gamma(z+1) = z\Gamma(z) \text{ for } z \text{ s.t. } \Re(z)\geqslant 1, \tag{56}$$

we see that:

$$\Gamma(it+j) = (it+j-1)(it+j-2).....(it+1)(it)\Gamma(it) \text{ for } t>0. \tag{57}$$

Thus our expression for the characteristic function (54) becomes

$$f_N(t) = \prod_{j=0}^{N}\left(\frac{(it+j)j}{(\frac{it}{2}+j)^2}\right)^{N-j}\left(\frac{\Gamma(it)}{\Gamma(it/2)^2}\right)^N. \tag{58}$$

Of particular use to us will be the following result.

**Theorem 4.6** (Weierstrass Definition of the Gamma Function)**.** *For* $z \in \mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}, \tag{59}$$

*where* $\gamma$ *is the Euler-Mascheroni constant, is a meromorphic expansion of the Gamma function on* $\mathbb{C}$.

*Proof.* This is one of the implications of Weierstrass Factorization Theorem. See page 57 of [5]. □

**Proposition 4.7** ($f_N$ as an infinite product)**.** *Let* $f_N$ *be the characteristic function of* $Y_N = \log|Z_N(\theta)|$*, then*

$$f_N(t) = \prod_{j=1}^{\infty} \left( \frac{(1 + i\frac{t}{2j})^2}{(1 + i\frac{t}{j})} \right)^{\min(N,j)} \tag{60}$$

*Proof.* By Theorem 4.6,

$$\frac{\Gamma(it)}{\Gamma(it/2)^2} = \frac{(it/2)^2}{it} \prod_{n=1}^{\infty} \frac{(1 + \frac{it}{2n})^2}{1 + \frac{it}{n}} = \prod_{j=0}^{\infty} \frac{(j + it/2)^2}{(j + it)j}. \tag{61}$$

Combining (58) and (61) we obtain the desired expression (60). □

## 4.2 Expectation and Variance of the logarithm of the $N$'th characteristic polynomial

In this section, we use the (60), to compute the expectation and variance of $Y_N$. As (60) is a large product, it is easier to work with $\log f_N$ to find the expectation. If we continue the approach onward we can explicitly compute all the cumulants of $f_N$(moments of $\log f_N(t)$).

**Proposition 4.8.** *Let* $Z(N, \theta)$ *be the characteristic polynomial of the CUE. Then* $Y_N = \log|Z_N|$ *satisfies:*

$$\mathbb{E}(Y_N) = 0, \tag{62}$$

$$\mathbb{V}\text{ar}(Y_N) = \frac{1}{2} \sum_{j=1}^{N} \frac{1}{j} + \frac{N}{2} \sum_{j=N+1}^{\infty} \frac{1}{j^2}. \tag{63}$$

Recall that in our set up, (E0), (FV),(GV), this means that

$$T = \sqrt{\frac{1}{2} \sum_{j=1}^{N} \frac{1}{j} + \frac{N}{2} \sum_{j=N+1}^{\infty} \frac{1}{j^2}}. \tag{64}$$

*Proof of Proposition 4.8.* Here, we again exploit (60).

$$\log(f(t)) = \sum_{j=1}^{\infty} \min(N, j) \left( 2(\log(1 + i\frac{t}{2j}) - \log(1 + i\frac{t}{j}) \right) \tag{65}$$

The right hand side is locally uniformly convergent for $t$ sufficiently small, so differentiating both sides we obtain:

$$\frac{f'(t)}{f(t)} = \sum_{j=1}^{\infty} \min(N,j) \left( \frac{i}{j} \left( \frac{1}{1+i\frac{t}{2j}} - \frac{1}{1+i\frac{t}{j}} \right) \right). \tag{66}$$

So, $f'(0) = 0 = i\mathbb{E}(Y_N)$, so $\mathbb{E}(Y_N) = 0$. Note RHS of (66) is uniformly convergent on compact intervals. So we can compute the second derivative of both sides:

$$\frac{f''(t)}{f(t)} - \frac{f'(t)^2}{f(t)^2} = \sum_{j=1}^{\infty} \min(N,j) \frac{1}{j^2} \left( \frac{1}{2} \frac{1}{(1+i\frac{t}{2j})^2} - \frac{1}{(1+i\frac{t}{j})^2} \right). \tag{67}$$

RHS of (67) is again uniformly convergent on compact intervals. Also evaluating (67) at 0 gives:

$$f''(0) = \frac{-1}{2} \sum_{j=1}^{\infty} \min(N,j) \frac{1}{j^2} = -\mathbb{E}(Y_N^2).$$

□

Here we were able to get the exact value of the variance of the $Y_N$. In fact, the technique we use here allows for a more vague understanding: it suffices to understand how the variance behaves as $N$ grows($\approx \frac{\log(N)}{2}$ in this case).

## 4.3 Gaussian behavior

**Proposition 4.9** (Weak Convergence to Standard Normal)**.** *The characteristic function of $Y_T/T$, satisfies:*

$$f_N(t/T) \to e^{-t^2/2} \text{ as } T \to \infty. \tag{68}$$

In particular the sequence of distributions is tight.

*Proof.* Define for $|z| < 1$,

$$\alpha(z) = \frac{1}{|z|^3} \sum_{n=3}^{\infty} (-1)^{n-1} \frac{z^n}{n}. \tag{69}$$

Fix the branch of log on $D(1,1) = \{z \in \mathbb{C}, |z-1| < 1\}$:

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + |z|^3 \alpha(z). \tag{70}$$

It is not difficult to see that $|\alpha(z)| \leqslant \frac{1}{3} \sum_{k=0}^{\infty} |z|^k$ (bounding term by term), so:

$$|\alpha(z)| < 2/3 \text{ for } |z| \leqslant 1/2. \tag{71}$$

Remember that we are interested in $\widetilde{f_N}(t) = f_N(\frac{t}{T})$, and that $T = \sqrt{\mathbb{V}\text{ar}\, Y_N} \to \infty$ as $N \to \infty$. For fixed $t$, and $N$ sufficiently large s.t. $|t/T| \leqslant 1/2$, (60) becomes:

$$\widetilde{f_N}(t) = \exp\left( \sum_{j=1}^{\infty} \min(N,j) \left( 2\log(1+i\frac{t}{2Tj}) - \log(1+i\frac{t}{Tj}) \right) \right). \tag{72}$$

Observe that:

$$2\log\left(1+i\frac{t}{2Tj})-\log(1+i\frac{t}{Tj}\right) = \frac{-t^2}{4T^2j^2}+\frac{|t|^3T^{-3}}{j^3}\left(\frac{1}{4}\alpha(\frac{tT^{-1}}{2j})-\alpha(\frac{tT^{-1}}{j})\right). \tag{73}$$

Denote the sum of the higher order terms

$$\beta(t)=\sum_{j=1}^{\infty}\min(N,j)\frac{|t|^3T^{-3}}{j^3}\left(\frac{1}{4}\alpha(\frac{tT^{-1}}{2j})-\alpha(\frac{tT^{-1}}{j})\right), \tag{74}$$

and plug the right hand side of (73) into (72), and take (74) into account to obtain:

$$f_N(t/T)=\exp\left(\frac{-t^2}{2T^2}\frac{1}{2}\left(\sum_{j=1}^{N}\frac{1}{j}+\sum_{j=N+1}^{\infty}\frac{1}{j^2}\right)+\beta(t)\right). \tag{75}$$

Now remember (71), and it implies that $|\frac{1}{4}\alpha(\frac{tT^{-1}}{2j})-\alpha(\frac{tT^{-1}}{j})|\leqslant\frac{1}{4}\frac{2}{3}+\frac{2}{3}=\frac{5}{6}<1$ for any $j\in\mathbb{N}$. Thus

$$|\beta(t)|<|t|^3T^{-3}\sum_{j=1}^{N}\frac{1}{j^2}+N|t|^3T^{-3}\sum_{j=N+1}^{\infty}\frac{1}{j^3}\leqslant 2|t|^3T^{-3}+\frac{1}{N}|t|^3T^{-3}\to 0 \text{ as } N\to\infty, \tag{76}$$

where we used that $\sum_{j=N+1}^{\infty}\frac{1}{j^3}<\frac{1}{N^2}$. By continuity, since,

$$\beta(t)\to 0 \text{ as } T\to\infty, \text{ by (76)}$$

and

$$\frac{\sum_{j=1}^{N}\frac{1}{j}+\sum_{j=N+1}^{\infty}\frac{1}{j^2}}{T^2}=1 \text{ by (64)},$$

it follows that

$$f_N(t/T)=\exp(-t^2/2)\exp(\beta(t))\to e^{-t^2/2} \text{ as } N\to\infty.$$

This completes the proof. □

# 5 An Integrable Upper Bound of the Characteristic functions

In this section we understand the characteristic function in much greater debth through detailed analysis of the explicit expression that we obtained in the previous section and very specific computations. This analysis is unique to our approach in the proof of Benford behavior for the characteristic polynomials of the CUE. The most difficult condition of Theorem 2.8, comes out of it:

**Theorem 5.1.** *There exists a function* $g \in L^1(\mathbb{R})$ *s.t.* $|f_N(t)| \leqslant g(t) \forall t$.

We postpone the proof of Theorem 5.1 for the end of the section. Assuming it, Benford behavior follows:

**Theorem 5.2.** *The sequence* $X_N = |Z_N(\theta, N)|$ *is Benford.*

*Proof of Theorem 5.2.* Let $Y_N = \log |Z_N(\theta, N)|$. By Proposition 4.8, (E0),(FV),(GV) hold for $Y_N$. Proposition 4.9 from the previous section gives (31), and Theorem 5.1 gives (32). Thus all the conditions of Theorem 2.8 hold, and so the sequence $X_N$ is Benford. □

## 5.1 Expression for the absolute value of the characteristic function

By obtaining the explicit expression for the characteristic function, (60), we already have an expression that is in good shape to analyze as we saw in Section 4, where we proved weak convergence of $Y_T/T$ to a Gaussian. In this subsection we use (60) to obtain two expressions. We use the first one when bounding the characteristic function $f_N(t)$ for $|t| \leqslant N$. It follows directly form (60), by taking absolute value on both sides.

$$|f_N(t)|^2 = \prod_{k=1}^{N} \left( \frac{(1+t^2/4k^2)^2}{1+t^2/k^2} \right)^k \prod_{j=N+1}^{\infty} \left( \frac{(1+t^2/4j^2)^2}{1+t^2/j^2} \right)^N . \tag{77}$$

The equation which we use for $|t| > N$, requires a few extra computations. We recall some identities for the Gamma function.
For $b \in \mathbb{N}$,

$$|\Gamma(it+b+1)|^2 = \frac{\pi t}{\sinh(\pi t)} \prod_{k=1}^{b} (k^2+t^2), \tag{78}$$

$$\Gamma(j) = (j-1)!, \tag{79}$$

$$|\Gamma(1+it)|^2 = \frac{\pi t}{\sinh(\pi t)}. \tag{80}$$

Recall the expression (54). Taking into consideration (78) and (79), we rewrite it as:

$$|f_N(t)|^2 = \left( \frac{\pi t}{\pi^2 t^2/4} \frac{\sinh^2(\pi t/2)}{\sinh \pi t} \right)^N \prod_{j=1}^{N-1} \prod_{k=1}^{j} \frac{(k^2+t^2)k^2}{(k^2+t^2/4)^2}. \tag{81}$$

We clean this expression up to obtain:

$$|f_N(t)|^2 = \left( \frac{4}{\pi} \right)^N \frac{1}{t^N} \left( \frac{\sinh^2(\pi t/2)}{\sinh \pi t} \right)^N \prod_{j=1}^{N-1} \prod_{k=1}^{j} \frac{(k^2+t^2)k^2}{(k^2+t^2/4)^2}. \tag{82}$$

Finally, using the simple trigonometric identity, $\sinh^2(x/2)/\sinh(x) = \frac{1}{2}\tanh(x/2)$, we obtain

$$|f_N(t)|^2 = \left(\frac{2}{\pi}\right)^N \frac{1}{t^N} \tanh^N(\pi t/2) \prod_{k=1}^{N-1} \left(\frac{(k^2+t^2)k^2}{(k^2+t^2/4)^2}\right)^{N-k}. \tag{83}$$

The above expression, (83), is the second expression of importance to the rest of the thesis in this subsection. It it is very useful for us to bound the characteristic function when $|t| \geqslant N$. Now we are in good shape to proceed with our analysis.

## 5.2 Estimation for $t \leqslant N$

The aim of this subsection is to give a good upper bound on the characteristic function when $t \leqslant N$. In order to do that, we make use of the first expression (77) for $|f_N(t)|^2$ .

**Theorem 5.3.** *For $|t| \leqslant N$,*

$$|f_N(t)|^2 \leqslant \exp\left(\frac{-t^2}{16}\log(\frac{2\sqrt{2}(N+1)^2}{e|t|([|t|]+1)})\right). \tag{84}$$

We postpone the proof, and first show its implications, which we will use when bounding $f_N(t/T)$ from above. The first corollary that we introduce, gives us a Gaussian bound on the characteristic function of $Y_N$, for $|t| \leqslant N$, independent of $N$. This suggests that the convergence of $f_N(t)$ looks a lot like the characteristic function of some Gaussian centered at 0, and gives us an appropriate upper bound for $f_N(t/T)$ for $t \geqslant T^2$.

**Corollary 5.4.** *There exists $C > 0$, independent of $N$ s.t.*

$$|f_N(t)| \leqslant \exp(-Ct^2) \text{ for } |t| \leqslant N. \tag{85}$$

*Proof.* For $t \leqslant N$, taking the following inequality into account,

$$\log\left(\frac{2\sqrt{2}}{e}\frac{(N+1)^2}{|t|([|t|]+1)}\right) \geqslant \log\left(\frac{2\sqrt{2}}{e}\frac{(N+1)^2}{N(N+1)}\right) \geqslant \log(2\sqrt{2}/e),$$

equation (84) gives us

$$|f_N(t)|^2 \leqslant \exp\left(-\frac{t^2}{16}\log(\frac{2\sqrt{2}}{e})\right) = \exp(-ct^2),$$

where $c = \log(2\sqrt{2}/e)/16 > 0$, and so

$$|f_N(t)|^2 \leqslant \exp(-ct^2) \text{ for } |t| \leqslant N. \tag{86}$$

□

However, one may notice that Corollary 5.4 does not give a satisfying bound in the interval, say $[-T, T]$. Observe that

$$\exp(-c|t/T|^2) \geqslant \exp(-c) \text{ for } |t| \leqslant T, \tag{87}$$

and that in turn $\int_{-T}^{T} \exp(-c)dt \to \infty$ as $T \to \infty$. So the values of $f_N(t/T)$ near zero require more careful treatment. This is given by the following Corollary of Theorem 5.3.

**Corollary 5.5.** *For* $|t| \leqslant N^{1/2}$,

$$|f_N(t)| \leqslant \exp\left(\frac{-\log(N)}{32}t^2\right). \tag{88}$$

*Proof.* For $|t| \leqslant N^{1/2} < N$, Theorem 5.3 applies, and using that as $|\frac{2\sqrt{2}(N+1)^2}{e|t|([|t|]+1)})| \geqslant N$, (84) gives us:

$$|f_N(t)|^2 \leqslant \exp\left(\frac{-t^2}{16}\log(N)\right).$$

□

From Corollary 5.5, it follows that

$$f_N(t/T) \leqslant \exp\left(-\frac{1}{32}\frac{\log N}{T^2}t^2\right) \text{ for } |t| \leqslant N^{1/2} \text{ (and in particular } |t| \leqslant T^2\text{)},$$

and remembering the exact expression for $T^2$, (63), we can see that $T^2 \leqslant 1 + \frac{1}{2}\log(N)$, so in fact

$$f_N(t/T) \leqslant \exp\left(-\frac{1}{32}t^2\right) \text{ for } |t| \leqslant N^{1/2}. \tag{89}$$

This is an appropriate upper bound near 0 of $f_N(t/T)$, as it can be bounded independently of $N$ by an $L^1$ non-negative function. So now that we understand the implications of Theorem 5.3, let us dive into its proof.

*Proof of Theorem 5.3.* Let $s = s(k) = t/k$, and note that:

$$\frac{1+s^2}{(1+s^2/4)^2} = \frac{1+s^2}{(1+s^2/2+s^4/16)} \begin{cases} \leqslant 1 \text{ if } s^2 \geqslant 8 \iff k \leqslant t/2\sqrt{2}, \\ > 1 \text{ if } s^2 < 8 \iff k > t/2\sqrt{2}. \end{cases} \tag{90}$$

Rewrite the expression for the control of $|t| \leqslant N$,(77), as:

$$|f_N(t)|^2 = \prod_{k=1}^{N}\left(\frac{(1+s(k)^2/4)^2}{1+s(k)^2}\right)^k \prod_{j=N+1}^{\infty}\left(\frac{(1+s(j)^2/4)^2}{1+s(j)^2}\right)^N. \tag{91}$$

Following (90), we separate this product into three parts, based on the absolute value of $s(k)$:

$$|f_N(t)|^2 = \prod_{k\leqslant|t|/2\sqrt{2},s^2\geqslant 8}\left(\frac{(1+s(k)^2/4)^2}{1+s(k)^2}\right)^k \prod_{k>|t|/2\sqrt{2},s^2<8}\left(\frac{(1+s(k)^2/4)^2}{1+s(k)^2}\right)^k \prod_{j=N+1}^{\infty}\left(\frac{(1+s(j)^2/4)^2}{1+s(j)^2}\right)^N. \tag{92}$$

Let the three parts of (92) be:

$$P_1 = \prod_{k\leqslant|t|/2\sqrt{2},s^2\geqslant 8}\left(\frac{(1+s(k)^2/4)^2}{1+s(k)^2}\right)^k, \tag{93}$$

$$P_2 = \prod_{N\geqslant k>|t|/2\sqrt{2},s^2<8}\left(\frac{(1+s(k)^2/4)^2}{1+s(k)^2}\right)^k, \tag{94}$$

$$P_3 = \prod_{j=N+1}^{\infty} \left( \frac{(1+s(j)^2/4)^2}{1+s(j)^2} \right)^N . \tag{95}$$

We are analyzing expressions for which $|t| \leqslant N$, so for $k > N$, $|t/k| \leqslant 1 < 2\sqrt{2}$, and by (90), an easy bound for the third product (95) is

$$P_3 < 1. \tag{96}$$

Thus, to prove Theorem 5.3, it suffices to prove

$$P_1 P_2 \leqslant \exp\left( \frac{-t^2}{16} \log\left( \frac{2\sqrt{2}(N+1)^2}{e|t|([|t|]+1)} \right) \right). \tag{97}$$

Now let us do a rough estimation for $P_2$. By (90), as all of the terms of the product have absolute value less than 1,

$$P_2 < \prod_{N \geqslant k \geqslant |t|, s^2 \leqslant 1} \left( \frac{(1+s(k)^2/4)^2}{1+s(k)^2} \right)^k . \tag{98}$$

Now for $|s| < 1$,

$$\frac{1+s^2/2+s^4/16}{1+s^2} \leqslant \frac{1+\frac{9}{16}s^2}{1+s^2} = 1 - \frac{\frac{7}{16}s^2}{1+s^2} \leqslant 1 - \frac{\frac{7}{16}s^2}{2} = 1 - \frac{7}{32}s^2.$$

Thus

$$\frac{1+s^2/2+s^4/16}{1+s^2} \leqslant 1 - \frac{1}{8}s^2. \tag{99}$$

We apply this to (98), to see that

$$P_2 < \prod_{N \geqslant k \geqslant |t|} (1 - \frac{1}{8}s^2(k))^k = \exp\left( \sum_{k=[|t|]+1}^{N} k \log(1-(s(k)^2/8) \right). \tag{100}$$

Note that for $0 < x < 1$, $\log(1-x) = -\sum_{k=1}^{\infty} x^k/k < -x$, so by the above computation, exp is increasing,

$$P_2 < \exp\left( -\frac{1}{8} \sum_{k=[|t|]+1}^{N} \frac{t^2}{k} \right) = \exp\left( -\frac{1}{8}t^2 \sum_{k=[|t|]+1}^{N} \frac{1}{k} \right) \leqslant \exp\left( -\frac{1}{8}t^2 \int_{[|t|]+1}^{N+1} \frac{1}{x}dx \right) = \exp\left( -\frac{1}{8}t^2 \log\left(\frac{N+1}{[t]+1}\right) \right).$$

For clarity we write the conclusion of the above computation in a separate equation:

$$P_2 < \exp\left( -\frac{1}{8}t^2 (\log(\frac{N+1}{[t]+1}) \right). \tag{101}$$

Let us now analyze $P_1$, see (93). Again, we first bound each term of the product separately:

$$\frac{(1+s^2/4)^2}{1+s^2} = \frac{1+s^2/2+s^4/16}{1+s^2} = \frac{1+7s^2/16+\frac{s^2}{16}(1+s^2)}{1+s^2} = \frac{s^2}{16} + \frac{1+7s^2/16}{1+s^2}.$$

As $\frac{1+7s^2/16}{1+s^2} < 1$, the above computations yield:

$$\frac{(1+s^2/4)^2}{1+s^2} \leqslant 1 + s^2/16 \leqslant \exp(s^2/16). \tag{102}$$

We apply (102) to each term of the product (93) to obtain:

$$P_1 \leqslant \prod_{k\leqslant |t|/2\sqrt{2}} \exp\left(s(k)^2/16)^k = \exp(\sum_{k\leqslant t/2\sqrt{2}} k\frac{t^2}{16k^2}\right) = \exp\left(\sum_{k\leqslant t/2\sqrt{2}} \frac{t^2}{16k}\right). \tag{103}$$

Now, using a dissection of the interval $[2, |t|/2\sqrt{2}]$, we see that

$$\sum_{k\leqslant |t|/2\sqrt{2}} \frac{1}{k} \leqslant 1 + \int_1^{|t|/2\sqrt{2}} \frac{1}{x} dx.$$

Using that the exponential function is increasing we see that:

$$P_1 \leqslant \exp\left(\frac{t^2}{16} \sum_{k\leqslant |t|/2\sqrt{2}} \frac{1}{k}\right) \leqslant \exp\left(\frac{t^2}{16}(1 + \int_1^{|t|/2\sqrt{2}} \frac{1}{x} dx\right) = \exp\left(\frac{t^2}{16}(1+\log(|t|/2\sqrt{2}))\right). \tag{104}$$

Combining (101) and (104) we obtain:

$$P_1 P_2 \leqslant \exp\left(\frac{-t^2}{16}\left(-1 - \log(|t|/2\sqrt{2}) + 2\log(\frac{N+1}{[t]+1})\right)\right) = \exp\left(\frac{-t^2}{16} \log\left(\frac{2\sqrt{2}(N+1)^2}{e|t|([|t|]+1)}\right)\right). \tag{105}$$

□

## 5.3 Estimation for $|t| \geqslant N$

It remains for us to analyze the case $|t| \geqslant N$. Recall the second expression for the absolute value of the characteristic function of $Y_N$ that we obtained, equation (83).

Let $s(k) = t/k$ as in (90). If we define

$$P(N) = P = \prod_{k=1}^{N} \left( \frac{(k^2+t^2)k^2}{(k^2+t^2/4)^2} \right)^{N-k} = \prod_{k=1}^{N} \left( \frac{1+t^2/k^2}{(1+t^2/4k^2)^2} \right)^{N-k} = \prod_{k=1}^{N} \left( \frac{1+s^2}{(1+s^2/4)^2} \right)^{N-k}, \tag{106}$$

we may rewrite (83) as:

$$|f_N(t)|^2 = \tanh^N(\pi|t|/2)(\frac{2}{\pi})^N |t|^{-N} P(N). \tag{107}$$

In order to control $|f_N|$, we wish to estimate $P$. Our analysis allows us to obtain an upper bound for $P$, which is key to our proof, and which is given in the proposition to follow.

**Proposition 5.6.** *Let $W : [0,\infty) \to \mathbb{R}$, be defined by*

$$W(\xi) = \log(4\xi^2+1) - \xi \arctan(2\xi). \tag{108}$$

*Then, with notation as in* (107),

$$P \leqslant \exp\left(-t^2 W(\frac{N}{|t|})\right) \exp(3N). \tag{109}$$

*Proof of Proposition 5.6.* Again, as in the previous section we analyze the separate terms of the product. Observe that

$$0 \leqslant \frac{1+s^2}{(1+s^2/4)^2} = \frac{1+s(k)^2}{1+s(k)^2/2+s(k)^4/16} \leqslant 2.$$

Subtract 1, from each part of the inequality to see that:

$$\frac{1+s^2}{1+s^2/2+s^4/16} - 1 = \frac{s^2/2-s^4/16}{1+s^2/2+s^4/16} \in [-1,1].$$

Applying the standard bound $0 \leqslant 1+x \leqslant e^x \forall x \in [-1,1]$, to each term in the product, we see that:

$$P \leqslant \prod_{k=1}^{N} \exp\left( \frac{s(k)^2 - s(k)^4/16}{1+s(k)^2/2+s(k)^4/16} \right)^{N-k} = \exp\left( \sum_{k=1}^{N} (N-k) \frac{s^2/2-s^4/16}{1+s^2/2+s^4/16} \right). \tag{110}$$

For the moment we focus on the series within the exponential function on the RHS of (110). If we substitute $s = t/k$, back for each term of the series we see that:

$$\frac{s^2/2-s^4/16}{1+s^2/2+s^4/16} = \frac{t^2/2k^2 - t^4/16k^4}{1+t^2/2k^2+t^4/16k^4} = \frac{k^2t^2/2 - t^4/16}{k^4 + \frac{k^2t^2}{2} + t^4/16}$$

$$= \frac{\frac{t^2}{2}(k^2+t^2/4) - t^4/8 - t^4/16}{(k^2+t^2/4)^2} = \frac{t^2}{2} \frac{1}{k^2+t^2/4} - \frac{3}{16} t^4 \frac{1}{(k^2+t^2/4)^2}.$$

So, using this expression we come back to (110):

$$P \leqslant \exp\left( (\sum_{k=1}^{N} (N-k)(\frac{t^2}{2}\frac{1}{k^2+t^2/4} - \frac{3}{16}t^4\frac{1}{(k^2+t^2/4)^2})\right). \tag{111}$$

If we set

$$A = A(N) = \sum_{k=1}^{N}(N-k)\frac{t^2}{2}\frac{1}{k^2+t^2/4}, \tag{112}$$

and

$$B = B(N) = \sum_{k=1}^{N}(N-k)\frac{3}{16}t^4\frac{1}{(k^2+t^2/4)^2}, \tag{113}$$

then we can rewrite the newly obtained bound of $P$, (111), as

$$P \leqslant \exp(A-B). \tag{114}$$

and our strategy is to find a good upper bound on $A$ and a good lower bound on $B$ to bound $P$ above.
Observe $x \to \frac{t^2}{2}\frac{N-x}{x^2+t^2/4}$ is a decreasing function on $[0,N)$, thus:

$$A \leqslant \int_0^N \frac{t^2}{2}\frac{N-x}{x^2+t^2/4}dx = \frac{Nt^2}{2}\int_0^N \frac{dx}{x^2+t^2/4} - \frac{t^2}{2}\int_0^N \frac{xdx}{x^2+t^2/4} = \frac{Nt^2}{2}I_1 - \frac{t^2}{2}I_2. \tag{115}$$

$$I_2 = \int_0^N \frac{xdx}{x^2+t^2/4} = \frac{1}{2}\int_0^{N^2}\frac{dy}{y+t^2/4}, \text{ where we substituted } y = x^2, dy = 2x.$$

So, writing $\xi = y + t^2/4, d\xi = dy$,

$$I_2 = \frac{1}{2}\int_{t^2/4}^{N^2+t^2/4}\frac{d\xi}{\xi} = \frac{1}{2}\left[\log(\xi)\right]_{t^2/4}^{N^2+t^2/4} = \frac{1}{2}\log\left(\frac{N^2+t^2/4}{t^2/4}\right). \tag{116}$$

$$I_1 = \int_0^N \frac{dx}{x^2+t^2/4}.$$

Substitute $x = \frac{t}{2}y, dx = \frac{t}{2}dy$, to obtain:

$$I_1 = \frac{2}{t}\int_0^{\frac{2}{t}N}\frac{dy}{y^2+1} = \frac{2}{t}[\arctan(y)]_0^{\frac{2}{t}N}\frac{2}{t}\arctan(\frac{2}{t}N). \tag{117}$$

Thus, plugging (117) and (116) back into (115), we obtain an upper bound for $A$:

$$A \leqslant Nt\arctan\left(\frac{2}{t}N\right) - \frac{t^2}{4}\log\left(\frac{N^2+t^2/4}{t^2/4}\right). \tag{118}$$

Now let us take care of of $B$. For a matter of convenience we write

$$B = \tilde{B} - 3N, \tag{119}$$

where

$$\tilde{B} = \sum_{k=1}^{N-1} (N-k)\frac{3}{16}t^4 \frac{1}{(k^2+t^2/4)^2}. \tag{120}$$

As

$$x \mapsto \frac{3}{16}(N-x)t^4 \frac{1}{(x^2+t^2/4)^2},$$

is decreasing in $x$,

$$\tilde{B} = \sum_{k=0}^{N-1} (N-k)\frac{3}{16}t^4 \frac{1}{(k^2+t^2/4)^2} \geqslant \int_0^N \frac{3}{16}(N-x)t^4 \frac{1}{(x^2+t^2/4)^2}. \tag{121}$$

We proceed to evaluate the integral on the RHS of (121):

$$= \frac{3}{16}Nt^4 \int_0^N \frac{dx}{(x^2+t^2/4)^2} - \frac{3}{16}t^4 \int_0^N \frac{xdx}{(x^2+t^2/4)^2}.$$

Thus,

$$\tilde{B} \geqslant= \frac{3}{16}Nt^4 J_1 - \frac{3}{16}t^4 J_2, \tag{122}$$

where

$$J_1 = \int_0^N \frac{dx}{(x^2+t^2/4)^2}, \tag{123}$$

and

$$J_2 = \int_0^N \frac{xdx}{(x^2+t^2/4)^2}. \tag{124}$$

Let us first deal with $J_2$. Substituting $y = x^2, dy = 2xdx$, we obtain:

$$J_2 = \frac{1}{2}\int_0^{N^2} \frac{dy}{(y+t^2/4)^2} = \frac{1}{2}\int_{t^2/4}^{N^2+t^2/4} \frac{d\xi}{\xi^2} = \frac{1}{2}\left[-\frac{1}{\xi}\right]_{t^2/4}^{N^2+t^2/4} = \frac{2}{t^2}\frac{N^2}{N^2+t^2/4}. \tag{125}$$

Now, let us evaluate $J_1$. We substitute $x = \frac{t}{2}y, dx = \frac{t}{2}dy$ to obtain:

$$J_1 = \frac{t/2}{t^4/16}\int_0^{\frac{2}{t}N} \frac{dy}{(y^2+1)^2} = \frac{8}{t^3}\left[\frac{1}{2}\frac{y}{y^2+1} + \frac{1}{2}\arctan(y)\right]_0^{\frac{2}{t}N} = \frac{4}{t^3}\left(\frac{2N}{t}\frac{1}{4N^2/t^2+1} + \arctan(\frac{2}{t}N)\right)$$

$$= \frac{8N}{t^4}\frac{t^2/4}{N^2+t^2/4} + \frac{4}{t^3}\arctan(\frac{2}{t}N) = \frac{2N}{t^2}\frac{1}{N^2+t^2/4} + \frac{4}{t^3}\arctan(\frac{2}{t}N). \tag{126}$$

We plug in the newly obtained expressions for $J_1$ and $J_2$, (126), (125), into (122), to see that:

$$\tilde{B} \geqslant \frac{3}{16}Nt^4\left(\frac{2N}{t^2}\frac{1}{N^2+t^2/4} + \frac{4}{t^3}\arctan(\frac{2}{t}N)\right) - \frac{3}{16}t^4\frac{2}{t^2}\frac{N^2}{N^2+t^2/4}$$

$$= \frac{3}{8}t^2\frac{N^2}{N^2+t^2/4} + \frac{3}{4}tN\arctan(\frac{2}{t}N) - \frac{3}{8}t^2\frac{N^2}{N^2+t^2/4} = \frac{3}{4}tN\arctan(\frac{2N}{t})$$

Remembering that $B = \tilde{B} - 3N$, and using the bound for $\tilde{B}$ from the above computation, we see that:

$$B \geqslant \frac{3}{4} tN \arctan(\frac{2N}{t}) - 3N. \tag{127}$$

Thus, recall $P \leqslant \exp(A - B)$ and apply the bounds for $A$ and $B$, (118) and (127) correspondingly to obtain:

$$\begin{aligned} P &\leqslant \exp\left( Nt \arctan(\frac{2}{t}N) - \frac{t^2}{4}\log(\frac{N^2 + t^2/4}{t^2/4}) - \frac{3}{4}tN\arctan(\frac{2N}{t}) + 3N \right) \\ &= \exp\left( \frac{1}{4} Nt \arctan(\frac{2}{t}N) - \frac{t^2}{4}\log(\frac{N^2 + t^2/4}{t^2/4}) + 3N \right) \\ &= \exp\left( \frac{-|t|}{4}(|t|\log(\frac{4N^2+t^2}{t^2}) - N|\arctan(\frac{2}{t}N)|) + 3N \right). \end{aligned} \tag{128}$$

Writing $\xi = \frac{N}{|t|}$ in (128), we obtain:

$$P \leqslant \exp(\frac{-t^2}{4}[\log(4\xi^2+1) - \xi\arctan(2\xi)])\exp(3N),$$

and this concludes the proof of the proposition. □

In order to make use of Proposition 5.6, we need to analyze $W$.

**Understanding $W$**

The proposition that follows is sufficient to obtain a good upper bound.

**Proposition 5.7.** *$W$ as defined in* (108) *is strictly increasing on* $[0, 1]$. *In particular* $W(0) = 0, W(\xi) > 0 \forall \xi \in (0, 1]$.

*Proof.* The line of argument which we apply is straightforward. We compute the first and second derivative of $W$, $W'$ and $W''$ correspondingly, to understand the behavior of $W$.

$$W'(\xi) = \frac{6\xi}{4\xi^2+1} - \arctan(2\xi). \tag{129}$$

$$W''(\xi) = \frac{4}{4\xi^2+1} - \frac{48\xi^2}{(4\xi^2+1)^2} = \frac{4(1-8\xi^2))}{(4\xi^2+1)^2}. \tag{130}$$

We observe that

$$W(0) = W'(0) = 0,$$

and

$$W''(\xi) \begin{cases} > 0 \text{ for } 0 \leqslant \xi < 1/2\sqrt{2}, \\ \leqslant 0 \text{ for } \xi > 1/2\sqrt{2}. \end{cases} .$$

So $W'$ is strictly increasing on $[0, 1/2\sqrt{2})$ and decreasing on $[1/2\sqrt{2}, 1]$. In particular,

$$W'(\xi) > \max(W'(0), W'(1)) = \max(0, W'(1)) \text{ for } \xi \in [0, 1]. \tag{131}$$

We claim that $W'(1) > 0$. This is sufficient for the proof as this means that $W$ is strictly increasing on $(0, 1]$. Observe that if the claim holds true $W$ is strictly increasing of $[0, 1]$.

To prove the claim we use the following identities for arctan:

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ for } 0 \leqslant x \leqslant 1, \tag{132}$$

$$\arctan(x) = \pi/2 - \arctan(1/x) \forall x. \tag{133}$$

From (132), (134), it follows that:

$$\arctan x = \pi/2 - \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \frac{1}{x^{2n+1}} \text{ for } x \geqslant 1. \tag{134}$$

By (134),

$$\arctan(2) = \pi/2 - \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \frac{1}{2^{2n+1}} \leqslant \frac{\pi}{2} - \frac{1}{2} + \frac{1}{24}. \tag{135}$$

It follows that

$$\arctan(2) \leqslant \pi/2 - 11/24 < 3.15\frac{12}{24} - \frac{11}{24} = \frac{37.8 - 11}{24} \leqslant 9/8. \tag{136}$$

By (129),

$$W'(1) \geqslant 6/5 - 9/8 = 3/40 > 0,$$

and the claim is proved.

□

**Conclusions for $|t| > N$**

We end this section about the bound of the characteristic function for $|t| \geqslant N$ with two propositions.

**Proposition 5.8** (Bound for proof)**.** *For $N$, sufficiently large (i.e. $N \geqslant e^6$),*

$$|f_N(t)| \leqslant |t|^{-N/4} \leqslant |t|^{-3} \textit{ for } |t| \geqslant N. \tag{137}$$

*Proof.* By (5.7), for $|t| > N$, $P(N) \leqslant \exp(3N)$, and so

$$|f_N(t)|^2 \leqslant |t|^{-N} \exp(3N) = t^{-N/2} \exp(3N - \frac{N}{2} \log(|t|)) \leqslant |t|^{-N/2} \exp(N(3 - \frac{\log(N)}{2})) \leqslant |t|^{-N/2}.$$

□

**Remark.** It is actually not difficult to show this bound really with $|t|^{-N/2}$ if we study more closely the behavior of $W$ near 0.

**Proposition 5.9** (Understanding bound)**.** *For any fixed $M > 0$, and $N$ sufficiently big, there exists $c > 0$ s.t.*

$$|f_N(t)| \leqslant \exp(-ct^2) \textit{ for } |t| \leqslant MN. \tag{138}$$

*Proof.* For $|t| \leqslant N$, by (5.4), there exist $C > 0$ s.t. $|f_N(t)| \leqslant \exp(-Ct^2)$.
For $N \leqslant |t| \leqslant MN$, by (5.6) and (5.7), letting $W(1/M)/4 = C' > 0$,

$$|f_N(t)|^2 \leqslant \exp\left(\frac{-|t^2|}{4} W(\frac{N}{NM})\right)\exp(3N) = \exp\left(-C't^2 + 3N\right) \leqslant \exp(-C''t^2).$$

So

$$|f_N(t)| \leqslant \exp\left(-t^2 \min(C''/2, C)\right) \text{ for } |t| \leqslant MN.$$

□

**Remark.** Proposition 5.8 cannot take be improved to give a bound similar to Proposition 5.9, because $W = O(\xi^2)$ near 0, so then (5.6) gives us that $P \leqslant\approx \exp(-2N^2/4)$. This is not because the bound we have is not optimal; for very big $t$,

$$\frac{k^2(k^2+t^2)}{(k^2+t^2/4)(k^2+t^2/4)} \approx \frac{16}{t^2},$$

and there are $\approx N^2/2$ such terms in $P$, so $P \approx |t|^{-2N^2}$, which is a polynomial and not an exponential expression in $|t|$, as in (5.8).

## 5.4 Integrable Bound

Now we get to the proof of Theorem 5.1. In the context that we introduced, it comes as a straightforward consequence of Proposition 5.8 and Proposition 5.9. We state it in a more explicit fashion.

**Proposition 5.10.** *There exist positive constants $c_1, c_2, c_3, K$ s.t. for $N > K$,*

$$f_N(t/T) \leqslant \begin{cases} \exp(-c_1 t^2) \textit{ for } |t| \leqslant N^{1/2}, \\ \exp(-c|t|) \textit{ for } N^{1/2} \leqslant |t| \leqslant N, \\ \frac{1}{|t|^{N/8}} \textit{ for } |t| > N. \end{cases} \tag{139}$$

*In particular,*

$$f_N(t/T) \leqslant \frac{c_3}{t^2} \textit{ for } N \textit{ sufficiently large.} \tag{140}$$

*Proof of Theorem 5.1 and Proposition 5.10.* By Corollary 5.5, and the comments after it,

$$|f_N(t/T)| \leqslant \exp\left(-\frac{1}{32}t^2\right) \text{ for } |t| \leqslant N^{1/2}. \tag{141}$$

By Corollary 5.4, as $N^{1/2} \geqslant T^2 \approx \frac{1}{2}\log(N)$,

$$f_N(t/T) \leqslant \exp(-ct^2/T^2) \leqslant \exp(-c|t|) \text{ for } N^{1/2} \leqslant |t| \leqslant N. \tag{142}$$

And finally, by Proposition 5.8, using again $|t| \geqslant N \geqslant T^2$,

$$|f_N(t/T)| \leqslant \frac{T^{N/4}}{|t|^{N/4}} \leqslant \frac{1}{|t|^{N/8}} \text{ for } t > N, N \text{ sufficiently large.} \tag{143}$$

□

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