Funnel-Based Control for Coupled Spatiotemporal Specifications

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Academic Dissertation which, with due permission of the KTH Royal Institute of Technology, is submitted for public defence for the Degree of Licentiate of Philosophy on Friday 26th January 2024, at 14:00, in V2, Teknikringen 76, Stockholm.
Abstract

In the past decade, the integration of spatiotemporal constraints into control systems has emerged as a crucial necessity, driven by the demand for enhanced performance, guaranteed safety, and the execution of complex tasks. Spatiotemporal constraints involve criteria that are dependent on both space and time, which can be represented by time-varying constraints in nonlinear control systems. Funnel-based control methods provide computationally tractable and robust feedback control laws to enforce time-varying constraints in uncertain nonlinear systems. This thesis begins by exploring the application of funnel-based control designs to address performance specifications in coordinate-free formation control of multi-agent systems. Moreover, we develop new robust feedback control schemes dealing with coupled spatiotemporal constraints in uncertain nonlinear systems that cannot be directly addressed by conventional funnel-based control methods.

In the first part of the thesis, we present a novel coordinate-free formation control scheme for directed leader-follower multi-agent systems, exhibiting almost global convergence to the desired shape. The synthesis of fully decentralized robust controllers for agents is achieved through the application of the Prescribed Performance Control (PPC) method. This method imposes spatiotemporal funnel constraints on each agent’s formation errors, ensuring a predefined transient and steady-state performance while maintaining robustness to system uncertainties. The core idea in this work is the utilization of bipolar coordinates to achieve orthogonal (decoupled) formation errors for each follower agent. This approach not only ensures the global convergence to the desired shape but also facilitates the effective application of the PPC method.

In the second part of the thesis, first, we introduce a novel approach that extends funnel-based control schemes to deal with a specific class of time-varying hard and soft constraints. In this work, we employ an online Constraint Consistent Funnel (CCF) planning scheme to tackle couplings between hard and soft constraints. By satisfying these CCF constraints, we ensure adherence to hard (safety) constraints, while soft (performance) constraints are met only when they do not conflict with the hard constraints. Subsequently, we directly employ the PPC design method to craft a robust, low-complexity control law, ensuring that the system’s outputs consistently stay within the online planned CCF constraints. In subsequent work, we tackle the challenge of satisfying a generalized class of potentially coupled time-varying output constraints. We show that addressing multiple constraints effectively boils down to formulating a single consolidating constraint. Ensuring the fulfillment of this consolidating constraint guarantees both convergence to and invariance of the time-varying output-constrained set within a user-defined finite time. Building on the PPC design method, we introduce a novel, robust low-complexity feedback control framework to handle this issue in uncertain high-order MIMO nonlinear control systems. Additionally, we present a mechanism for online modification of the consolidating constraint to secure a least-violating solution when constraint infeasibilities occur for an unknown time interval.
**Sammanfattning**

Under det senaste decenniet har integrationen av bivillkor i tid och rum för reglersystem framställt som en nödvändighet, driven av efterfrågan på förbättrat prestanda, garanterad säkerhet och utförandet av komplexa uppgifter. Bivillkor i tid och rum för icke-linjära reglersystem kan representeras av tidsvarierande bivillkor. Trattbaserade reglernetoder ("funnel-based control") tillhandahåller beräkningsmässigt hanterbara och robusta återkopplingslagar för att garantera tidsvarierande bivillkor i osäkra icke-linjära system. Denna avhandling börjar med att utforska tillämpningen av trattbaserade kontrollmetoder för att hantera prestandaspecifikationer i koordinatfri formationskontroll av multiagentsystem. Dessutom utvecklar vi nya robusta återkopplingslagar som hanterar kopplade bivillkor i tid och rum för osäkra icke-linjära system som inte direkt kan hanteras av konventionella trattbaserade kontrollmetoder.


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Stockholm, Dec 2024
Contents

Acknowledgements vi
Abbreviations x
List of Figures xiii

1 Introduction 1

1.1 Motivation ........................................... 1
1.2 Literature Review and Research Questions ............... 1
1.3 Thesis Outline and Contributions 8

2 Preliminaries 13

2.1 Notation .................... 13
2.2 Graphs and Graph Rigidity ...................... 14
2.3 Dynamical Systems ............... 20
2.4 Prescribed Performance Control ................. 22
2.5 Useful Mathematical Results .................. 26
2.6 Mobile Robot Dynamics ............... 28

I Coordinate-Free Multi-Agent Formation Control 31

3 Leader-Follower Directed Formation Control Based on Bipolar Coordinates with Global Convergence 33

3.1 Introduction ...................... 33
3.2 Contributions ..................... 34
3.3 Problem Formulation ............... 36
3.4 Characterization of the Desired Formation Based on Bipolar Co-ordinates .................. 39
3.5 Controller Design ............... 43
3.6 Stability Analysis ..................... 47
3.7 Formation Control with Orientation Adjustment .......... 49
3.8 Simulations Results .................. 51

viii
## Contents

3.9 Conclusions ................................................. 53
3.10 Proofs of Lemmas, Theorems, and Some Technical Derivations ... 54

II Feedback Control under Coupled Spatiotemporal Constraints ................................................. 63

4 Funnel Control Under Hard and Soft Output Constraints ................................................. 65
4.1 Introduction .................................................. 65
4.2 Contributions ............................................... 66
4.3 Problem Formulation ....................................... 66
4.4 Online Constraint Consistent Funnel Planning ................................................. 69
4.5 Funnel-Based Controller Design ................................................. 74
4.6 Stability Analysis ............................................. 76
4.7 Simulation Results ............................................ 79
4.8 Conclusions .................................................. 80

5 Feedback Control for Uncertain MIMO Nonlinear Systems under Generalized Time-Varying Output Constraints ................................................. 83
5.1 Introduction .................................................. 83
5.2 Contributions ............................................... 84
5.3 Problem Formulation ....................................... 85
5.4 Methodology ................................................... 88
5.5 Low-Complexity Controller Design ................................................. 94
5.6 Stability Analysis ............................................. 97
5.7 Dealing with Potential Infeasibilities in the Constraints ................................................. 100
5.8 Simulation Results ............................................ 105
5.9 Conclusions .................................................. 111
5.10 Proofs of Lemmas and Theorems ................................................. 114

6 Summary and Future Research Directions ................................................. 127
6.1 Summary ..................................................... 127
6.2 Potential Future Research Directions ................................................. 128

Bibliography ...................................................... 131
### Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>BLF</td>
<td>Barrier Lyapunov Function</td>
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<tr>
<td>CCF</td>
<td>Constraint Consistent Funnel</td>
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<tr>
<td>CBF</td>
<td>Control Barrier Function</td>
</tr>
<tr>
<td>CLBF</td>
<td>Control Lyapunov Barrier Function</td>
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<tr>
<td>ERG</td>
<td>Explicit Reference Governor</td>
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<td>FC</td>
<td>Funnel Control</td>
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<td>IVP</td>
<td>Initial-Value Problem</td>
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<td>LFF</td>
<td>Leader-First-Follower</td>
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<td>MIMO</td>
<td>Multi Input Multi Output</td>
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<tr>
<td>MPC</td>
<td>Model Predictive Control</td>
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<td>PPC</td>
<td>Prescribed Performance Control</td>
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<tr>
<td>QP</td>
<td>Quadratic Program</td>
</tr>
<tr>
<td>RG</td>
<td>Reference Governor</td>
</tr>
<tr>
<td>TVBLF</td>
<td>Time-Varying Barrier Lyapunov Function</td>
</tr>
</tbody>
</table>
List of Figures

2.1 Graph (a) is a non-rigid (flexible). Graph (b) a minimally rigid. Graph (c) is a complete graph, hence it is globally rigid. Graph (d) is equivalent but not congruent to graph (b), and in particular, they are flip-ambiguous. (e) shows reflected frameworks (they are congruent but not strongly congruent). ........................................ 15

2.2 The underlying undirected graph in (a) is rigid, yet the shape of the corresponding 2D-formation may not be maintained at all times. For instance, agent 4 has only one distant constraint to fulfill and can move freely on a circle centered on agent 1. When this occurs, it becomes impossible for agent 3 to simultaneously satisfy the three constraints it is responsible for, while the other agents fulfill all their constraints and have no reason to move, as depicted in (b). Such a situation never arises with the formation depicted in (c). Formally, the graph in (a) is rigid but not persistent due to its lack of constraint consistency, whereas (c) is both rigid and constraint consistent, making it persistent. .................................................. 17

2.3 Generating an acyclic minimally persistent graph under successive application of Henneberg directed type I operations starting from a graph of two vertices connected with a directed edge (a). (b): Primitive LFF structure (L: Leader, FF: First-Follower, F: Follower). (c) Henneberg directed type I operation at step \( i - 2 \). .......................... 19

2.4 (a) Minimally persistent graphs suffer from ambiguous configurations. (b) Use of signed area to distinguish the position of vertex 4 with respect to its reflected version \( 4' \). (c) Use of edge-angle for distinguishing the position of vertex 4 with respect to \( 4' \). ............. 19

2.5 Evolution of the error inside the prescribed performance funnel. ... 23

2.6 Evolution of the \( x(t) \) inside an asymmetric funnel constraint. ... 25

2.7 (a) \( V = \frac{1}{2} \epsilon^2 \) as a candidate BLF w.r.t. \( \dot{x} \). (b) \( V = \frac{1}{2} \epsilon^2 \) as a candidate TVBLF w.r.t. \( x \). ......................................................... 26

2.8 Mobile robot. .......................................................... 28
3.1 (a) edge-angle in a triangular subgraph. (b) example of a desired formation (note that \(d_{31} \neq d_{32} = d_{43} \)).

3.2 (a) The virtual local Cartesian coordinate frame \(\{C_k\}\) uniquely characterizes the position of agent \(k \geq 3\) with respect to its neighbors (agents \(i\) and \(j\)). Instead of using the Cartesian coordinates in \(\{C_k\}\) one can adopt bipolar coordinates \((3.4)\) and \((3.7)\) in \(\{C_k\}\) to determine agent \(k\)'s position. (b) Orthogonal bipolar coordinate basis \(\hat{r}_k\), \(\hat{\alpha}_k\) associated with agent \(k \geq 3\) and some of their isoquant curves.

3.3 Given a desired sensing graph \(G\) as in Fig. 3.1b, in each case the desired formation is characterized by different desired relative positions between agents 2 and 1, whereas the sets of desired edge-angles and ratio of the distances for followers \((i \geq 3)\) are the same. The dashed arrows show the local coordinate frame of agent 2 in which the formation orientation can be characterized by the desired bearing angle \(\beta^*\). \(p_{21,a}^*\) and \(p_{21,b}^*\) have the same orientation but different length while \(p_{21,a}^*\) and \(p_{21,c}^*\) have different orientations with the same length.

3.4 Starting from arbitrary initial positions, agents converge to the desired shape while following the leader’s (agent 1’s) motion. The scale and orientation of the formation is adjusted by agent 2 along the way. In particular, roughly around \(t = 14\) agent 2 starts following a time-varying desired bearing and distance w.r.t. agent 1 that leads the formation to pass through a narrow passage (black curves).

3.5 Agent 2’s desired (time-varying) distance \(d_{21}^*(t)\) and bearing angle \(\beta^*(t)\).

3.6 Evolution of agent 2’s squared distance \((e_d)\) and bearing angle \((e_\beta)\) errors. The magnified subplots provide the details of error evolution in the steady-state.

3.7 Evolution of the edge-angle errors \(\alpha_k, k \geq 3\). The magnified subplots provide the details of error evolution in the steady-state.

3.8 Evolution of the logarithmic ratio of the distance errors \(r_k, k \geq 3\). The magnified subplots provide the details of error evolution in the steady-state.

3.9 (a) Alternative edge-angle calculation. (b) Infinitesimal variation of edge \((k,i)\)'s angle.

3.10 Configuration of \(z_{ki}, z_{kj}\), and \(\hat{r}_k\) in three arbitrary positions of agent \(k\) with respect to its neighbors.

3.11 Mobile robot’s trajectory (blue line) tracking a moving object (dashed line) under hard constraints (red lines) with \(k_c = 3\).

4.1 (a) compatible \((\forall t \geq 0)\), and (b) incompatible hard and soft funnel constraints.

4.2 Mobile robot’s trajectory (blue line) tracking a moving object (dashed line) under hard constraints (red lines) with \(k_c = 3\).
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3</td>
<td>Evolution of $x_1(t)$ in its respective CCF under hard and soft constraints with $k_c = 3$ (left). The evolution of the associated modification signals (right).</td>
<td>81</td>
</tr>
<tr>
<td>4.4</td>
<td>Evolution of $x_2(t)$ in its respective CCF under hard and soft constraints with $k_c = 3$ (left). The evolution of the associated modification signals (right).</td>
<td>81</td>
</tr>
<tr>
<td>4.5</td>
<td>Mobile robot’s trajectory (blue line) tracking a moving object (dashed line) under hard constraints (red lines) with $k_c = 0.3$.</td>
<td>82</td>
</tr>
<tr>
<td>4.6</td>
<td>Evolution of $x_1(t)$ in its respective CCF under hard and soft constraints with $k_c = 0.3$ (left). The evolution of the associated modification signals (right).</td>
<td>82</td>
</tr>
<tr>
<td>4.7</td>
<td>Evolution of $x_2(t)$ in its respective CCF under hard and soft constraints with $k_c = 0.3$ (left). The evolution of the associated modification signals (right).</td>
<td>82</td>
</tr>
<tr>
<td>5.1</td>
<td>Snapshots of $\Omega(t)$ and its corresponding inner-approximation under (5.9) for three different examples.</td>
<td>90</td>
</tr>
<tr>
<td>5.2</td>
<td>$\varepsilon_\alpha(\varepsilon_\alpha)$ and $V_1(\varepsilon_\alpha) = \frac{1}{2}\varepsilon_\alpha^2$ when $\nu = 0.4$ (a) and $\nu = 2$ (b).</td>
<td>99</td>
</tr>
<tr>
<td>5.3</td>
<td>(a) Snapshot of the output constraint $\rho(t) &lt; h(x_1) = x_{1,1}^2 + x_{1,2}^2 &lt; p(t)$ for which its corresponding $\alpha(t,x_1)$ does not satisfy Assumption 5.8 due to the existence of a local minimum at $x_1 = [0,0]^T$. (b) surface of $\alpha(t,x_1)$.</td>
<td>100</td>
</tr>
<tr>
<td>5.4</td>
<td>The evolution of $\alpha(t,x_1(t);x(0))$ under the consolidating constraint (5.13), where $\rho_\alpha(t)$ is determined by (5.33). The adaptation of $\rho_\alpha(t)$ (dashed line) based on the evolution of $\hat\alpha(t)$ in (5.31) (dotted line) allows for deviations of $\rho_\alpha(t)$ from the nominal lower bound function $\varrho(t)$ in (5.32). Consequently, satisfaction of (5.13) during the time intervals when $\alpha^<em>(t) &lt; 0$ (shaded intervals) results in a least violating solution. In this illustrative example, $\hat\alpha(t)$ provides a reliable estimate of $\alpha^</em>(t)$, and the parameters used for this illustration are $\mu = 0.45$, $\rho_x = 0.5$, and $T = 4$.</td>
<td>103</td>
</tr>
<tr>
<td>5.5</td>
<td>Cascaded control architecture under the estimation scheme (5.31) and online computation of $\rho_\alpha(t)$ in (5.33).</td>
<td>105</td>
</tr>
<tr>
<td>5.6</td>
<td>Snapshots of the mobile robot’s hand position trajectory $x_1(t)$ along with the time-varying constrained set $\Omega(t)$ in Scenario 1.</td>
<td>108</td>
</tr>
<tr>
<td>5.7</td>
<td>Evolution of $\alpha(t,x_1(t))$ in Scenario 1.</td>
<td>108</td>
</tr>
<tr>
<td>5.8</td>
<td>Tracking errors and performance bounds evolution in Scenario 1.</td>
<td>108</td>
</tr>
<tr>
<td>5.9</td>
<td>Snapshots of the mobile robot’s hand position trajectory $x_1(t)$ along with the time-varying constrained set $\Omega(t)$ in Scenario 2.</td>
<td>109</td>
</tr>
<tr>
<td>5.10</td>
<td>Evolution of $\alpha(t,x_1(t))$ in Scenario 2.</td>
<td>109</td>
</tr>
<tr>
<td>5.11</td>
<td>Tracking errors and performance bounds evolution in Scenario 2.</td>
<td>109</td>
</tr>
</tbody>
</table>
5.12 Time-varying region tracking of the mobile robot (case 1). With an estimator gain of $k_\alpha = 4$ in (5.31a) minimal conservative behavior is achieved in satisfaction of the time-varying constraints (or region tracking), attributed to the estimator’s good performance in estimating (unknown) $\alpha^*(t)$. Moreover, a least violating solution is ensured with a small gap whenever the constraints become infeasible (empty region).

5.13 Time-varying region tracking of the mobile robot (case 2). A reduced estimator gain of $k_\alpha = 1$ in (5.31a) leads to a least violating solution with a large gap, which adversely impacts the controller’s performance, resulting in a more conservative satisfaction of time-varying constraints.
Chapter 1

Introduction

1.1 Motivation

Over the past decade, the incorporation of spatiotemporal constraints into control systems has become a pivotal aspect in the realm of control engineering. Spatiotemporal constraints entail requirements that are contingent on both space and time. Such constraints emerge in a wide range of problems and applications such as tracking and stabilization under a user defined transient and steady state performance, handling time-dependent safety-critical specifications, and meeting a set of complex tasks that need to be done in a timely manner in autonomous robotic systems. For example, consider a mobile robot that requires to relocate from region A to region B within a certain time limit to perform some task. This simple example demonstrates spatiotemporal constraints on mobile robot’s position.

A natural way to address spatiotemporal constraints is through imposing time-varying state or output constraints in nonlinear control systems. Funnel-based control methods provide computationally tractable (closed form) and robust feedback control laws for enforcing specific class of time-varying output constraints in uncertain nonlinear control systems and thus can be considered as a useful tool to deal with spatiotemporal constraints. However, the application of funnel-based control methods are currently confined to decoupled time-varying output constraints, which limits their applicability in some emerging engineering applications.

This thesis, introduces an application of funnel-based control to tackle a novel class of coordinate-free multi-agent formation control problems. Additionally, inspired by funnel-based control methods, it presents innovative robust and computationally tractable feedback control schemes crafted to address a broader spectrum of time-varying output constraints in uncertain nonlinear control systems.

1.2 Literature Review and Research Questions

The literature relevant to the topics addressed in this thesis is categorized into two subsections: “Coordinate-Free Formation Control of Multi-Agent Systems” and
“Control of Nonlinear Systems under Time-Varying Output Constraints”. Within each subsection, a comprehensive review of existing literature is presented, accompanied by the research questions explored in this thesis.

1.2.1 Coordinate-Free Formation Control of Multi-Agent Systems

Control of multi-agent networks is a growing area of research and recently has attracted much attention due to its applications. A multi-agent system consists of physical entities (agents) that are locally interacting with each other through communications or sensory networks to collectively execute tasks beyond the capability of each agent \[1\]. One particular problem in multi-agent systems is the formation control, which refers to the design of decentralized control laws for stabilizing agents’ positions so that they form and maintain a predefined geometrical configuration.

Formation control of multi-agent systems has been studied extensively during the past decade and depending on the sensing and controlled variables existing works can be mainly categorized into \[2,3\]: position-based \[4\], displacement-based \[5\], distance-based \[6,9\], bearing-based \[10\], and angle-based \[11,14\] methods. For more recent classes of formation control approaches as well as a comparative literature review on issues related to target formation’s constraints, required measurements, and convergence, see \[15–18\]. Among the above categories, the position-based method requires agents to have a common knowledge of a global coordinate system while the displacement-based (also known as consensus-based) and bearing-based methods require agents’ local coordinate frames to have a common orientation (i.e., to be aligned). On the other hand, coordinate-free methods, e.g., distance-, angle-based, etc., \[6,7,12,15\] are more attractive formation control architectures since they impose less implementation issues compared to other control methods. Indeed, in coordinate-free formation control, the desired shape is defined by a set of coordinate-free variables (e.g., distances, angles), which specify formation errors for agents. In addition, agents also require measurements of vectorized relative information of their neighboring agents (e.g., relative positions or bearings) in their local coordinate frames to constitute a control law. Hence, coordinate-free approaches enable us to design formation control laws in agents’ local coordinate frames, which do not require global position measurements (e.g., using GPS) nor the assumption of agents’ aligned local coordinate frames (e.g., using a compass or orientation alignment methods through inter-agent communication) \[2,19,20\].

Most of the existing results on coordinate-free formation control are developed under the assumption of bidirectional sensing (undirected sensing graph) among agents, which usually rely on different types of graph rigidity notions (e.g., distance, angle, ratio of the distances, weak, hybrid rigidity, etc.) \[3,12,14,17\]. However, it is often more practical to consider directed sensing among agents since: (i) the sensing limitations of the agents may enforce such structures, and (ii) it inherently avoids the issue of measurement mismatches in undirected formation control problems.
In this respect, the notion of persistent graphs was developed as the directed counterpart of distance rigidity. Some earlier control designs for persistent formations include. Unfortunately, most coordinate-free formation control methods only guarantee local, not global, convergence to the desired shape. Indeed, they rely on controlling the agents to satisfy certain shape constraints. But the minimal number of shape constraints may allow for multiple (but finite) shapes. In this respect, depending on the initial positions of the agents, meeting the formation constraints may not necessarily lead the agents to the correct shape. This is widely known as reflection, flip and flex ambiguities in distance-based formation control literature. In particular, distance rigidity theory (especially when the target formation is not globally rigid) cannot distinguish shapes under reflections, flip or flex ambiguities merely with distance constraints between agents, thus convergence to the desired shape specified only with inter-agent distances is not guaranteed. These ambiguity and local convergence issues remain for angle, ratio of the distances, and weak rigidity notions as well. To tackle these issues, in the context of undirected and directed 2-D distance-based formations some recent works have employed extra types of formation constraints (e.g., signed area and edge/signed angle) along with the inter-agent distances for characterizing the desired formation uniquely to establish (almost) global shape convergence. However, it turns out that imposing additional formation constraints leads to unwanted equilibria. In particular, since the distance and the auxiliary formation constraint (i.e., signed area/angle) interfere with each other at certain agent positions, new undesirable equilibria emerge, limiting the existing results to desired shapes under certain conditions. In addition, these approaches usually lead to tedious control gain tuning, which complicates controller design process and its extension to more practical formation control problems, where agents may have more complex tasks or dynamics, etc. In contrast to these works, recently used orthogonal error variables for characterizing 2-D directed distance-based formations with (almost) global shape convergence. An extension of this approach for 3-D directed distance-based formations is proposed in. A completely different method for global convergence of directed distance-based formations is proposed in that relies on calculating desired target points online and then asking agents to track them. Moreover, very recently has proposed an angle based formation control scheme with (almost) global convergence for undirected 2-D triangularly angle rigid formations, nevertheless, the proposed method requires all agents to obtain relative position measurements with respect to their neighbors.

Another advantage of coordinate-free formation control approaches concerns reducing agents’ costs since they require less complex equipment for sensing and local interactions. Up to now, most of the coordinate-free formation control methods require relative position measurements for all agents whereas a few of them (e.g.,) only require bearing (or vision-based) measurements. In this respect, since bearing information is easier to obtain through onboard cameras, it is more favorable in practical applications.
formation control methods that have been developed to deal with global shape convergence also require all agents to measure the relative positions.

In practical formation control problems, agents are not only required to maintain the desired shape but also need to cooperatively move (maneuver) while obtaining certain formation orientations and scalings. Nevertheless, most results on coordinate-free formation control focus on stabilization of stationary formations \[2, 14, 15\]. On the other hand, most of the existing formation maneuvering results are mainly limited by the assumption of aligned local coordinate frames of agents (or equivalently existence of inter-agent communications to share velocity-related information) and tracking constant reference velocities \[27, 41–45\], whereas only a few results are developed to discard these limitations \[9, 46\]. Moreover, the problems of orientation and scaling control are usually handled separately \[47, 48\] and are not integrated into the maneuvering task. Recently, \[49\] has proposed an angle-based formation maneuvering control with orientation and scaling adjustment, however, this result is applied to undirected formations with local shape convergence and (piece-wise) constant reference velocities. Furthermore, \[50\] has considered a layered affine formation maneuver problem for directed \((d+1\)-rooted) graphs in which the formation scale, orientation, and also shearing can be adjusted by changing the configuration among \(d+1\) leaders in \(d\)-dimensional space formations. However, this result still relies on the relative position and velocity measurements and requires a sufficiently large number of edges (sensing links) in the directed graph modeling inter-agent interactions, which is far from minimal.

The existence of external disturbances that affect the agents’ dynamics is a significant issue of practical interest for multi-agent formation applications. It is noteworthy to mention that in coordinate-free formation control problems, only a few recent works have taken into account external disturbances and uncertainties in agents’ dynamics \[9, 51, 52\], with these results only applying to local shape convergence. Finally, another crucial issue concerns the transient response of the multi-agent formations. In this regard, Prescribed Performance Control (PPC) \[53, 54\], proposes a simple and constructive procedure based on which the transient performance of the closed-loop system is predetermined by certain user defined performance bounds. Recently, PPC has been utilized for displacement-based and tree structure formation control problems \[55, 57\], as well as distance-based formation control \[9\].

Motivated by the existing literature, this thesis addresses the following key questions:

- Is it feasible to devise decentralized, coordinate-free formation control laws ensuring global convergence to the desired shape with a minimal number of directed (leader-follower) interactions among agents, and preferably without (or with a minimal number of) relative position measurements? This involves characterizing the desired formation shape uniquely and distributing formation errors to each agent effectively.

- How can the PPC method (as a funnel-based control approach) be utilized
in the proposed formation control scheme to synthesize decentralized formation control laws to guarantee a user-defined transient and steady-state performance along with ensuring robustness w.r.t. to modeling uncertainties and/or external disturbances affecting each agent dynamics? This entails verifying the applicability of PPC for the proposed formation control scheme and imposing appropriately designed spatiotemporal constraints on the evolution of formation errors.

- In the context of the globally converging leader-follower formation control scheme, how can formation maneuvering, scaling, and orientation adjustments be seamlessly integrated?

### 1.2.2 Control of Nonlinear Systems under Time-Varying Output Constraints

During the past decades, reference/trajectory tracking, as well as stabilization of complex and uncertain nonlinear dynamical systems, has attracted considerable research effort. Constraints are ubiquitous in controller design of practical nonlinear systems and they mainly emerge as performance and safety specifications. Constraint violation may result in performance degradation, system damage and hazards, therefore, owing to practical needs and theoretical challenges, the rigorous handling of constraints in the control design of nonlinear systems has become a dominant research topic during the past decade. Common existing methods in dealing with different types of constraints for nonlinear systems include: Model Predictive Control (MPC) [58], Reference Governors (RGs) [59] and Explicit Reference Governors (ERGs) [60], Barrier Lyapunov Functions (BLFs) [61] and Time-Varying Barrier Lyapunov Functions (TVBLFs) [62], Funnel Control (FC) [63, 64], Prescribed Performance Control (PPC) [53, 54], and set invariance based approaches such as Control Lyapunov Barrier Functions (CLBFs) [65] and Control Barrier Functions (CBFs) [66, 67].

Among the aforementioned methods, both typical MPC and RG implementations depend on model information and involve an optimization step for control law synthesis. Similarly, employing the CBF concept usually requires accurate knowledge of system dynamics and solving an online Quadratic Programming (QP) problem. In contrast, designs based on ERGs, BLFs, and CLBFs offer closed-form feedback control laws that are free from optimization but still rely on the system model. Additionally, these methods are limited to handling time-invariant constraints.

Existing feedback control approaches for addressing time-varying output constraints fall into three primary categories: FC, PPC, and designs based on TVBLFs. All these methods can be classified as funnel-based control approaches as they only deal with time-varying funnel constraint on certain system outputs. Notably, FC and PPC, often offer simpler and more constructive controller designs with inherent robustness to system uncertainties. Typically, control designs based on FC, PPC, and TVBLF are commonly employed to achieve user-defined transient and steady-
state performance for tracking and stabilization errors. These designs restrict the evolution of error signals within a set of user-defined time-varying funnels, serving as the sole time-varying output constraints. For example, funnel constraints of the form of $-\rho_i(t) < \varepsilon_i = x_i - x_i^d(t) < \rho_i(t)$ are frequently used on independent tracking errors, where $x_i$ represents independent state variables, $x_i^d(t)$ denotes desired trajectories, and $\rho_i(t)$ signifies bounded, strictly positive time-varying functions that model the evolving behavior of these constraints. To ensure the desired transient and steady-state performance of tracking errors, $\rho_i(t)$ is often selected as a strictly positive exponentially decaying function, approaching a small neighborhood of zero \[53, 63\]. Roughly speaking, the control design idea behind FC builds on the adaptive high-gain control, where a time-varying and state-dependent function is replaced by a monotonically increasing control gain. In PPC, initially, a transformation that incorporates the desired performance specifications is defined. Then, an appropriate control action is designed that establishes the uniform boundedness of the transformed system and gives the necessary and sufficient conditions for the satisfaction of the predefined performance (funnel) constraint. Designs based on TVBLFs rely on constructing appropriate time-varying (Lyapunov-like) barrier functions, which approach infinity when the constrained (error) signals approach the time-varying bounds of the funnel constraints. As we will show later in the thesis (see Subsection \[2.4.3\]) one can directly relate the PPC design method to the controller design based on BLFs and TVBLFs.

Existing feedback control approaches addressing time-varying output constraints can be categorized into three main types: FC, PPC, and designs based on TVBLFs. These methods are all considered funnel-based control approaches, specifically dealing with time-varying funnel constraints on certain system outputs. Notably, FC and PPC offer simpler and more constructive controller designs with inherent robustness to system uncertainties. Control designs based on FC, PPC, and TVBLF are often employed to achieve user-defined transient and steady-state performance for tracking and stabilization errors. These designs confine the evolution of error signals within user-defined time-varying funnels, serving as the sole time-varying output constraints. For instance, in these methods funnel constraints like $-\rho_i(t) < \varepsilon_i = x_i - x_i^d(t) < \rho_i(t)$ are frequently used on independent tracking errors. Here, $x_i$ represents independent state variables, $x_i^d(t)$ denotes desired trajectories, and $\rho_i(t)$ signifies bounded, strictly positive time-varying functions modeling the evolving behavior of these constraints. To ensure the desired transient and steady-state performance of tracking errors, $\rho_i(t)$ is often chosen as a strictly positive exponentially decaying function, approaching a small neighborhood of zero \[53, 63\]. In FC, the control design builds on the adaptive high-gain control, replacing a time-varying and state-dependent function with a monotonically increasing control gain. In PPC, a transformation that incorporates desired performance specifications is initially defined, followed by the design of a control action ensuring the uniform boundedness of the transformed system and satisfying the predefined performance (funnel) constraint. Designs based on TVBLFs involve constructing appropriate time-varying (Lyapunov-like) barrier functions that approach infinity when error
signals approach the time-varying bounds of the funnel constraints. As demonstrated later in the thesis (see Subsection 2.4.3), the PPC design method is directly related to the controller design based on BLFs and TVBLFs.

Recent years have witnessed significant advancements of FC, PPC, and TVBLF methods, spanning a wide range of applications. These developments extend to the control of high-order systems \[54, 68, 71\], output feedback \[72, 73\], multi-agent systems \[9, 55, 74, 81\], and more.

Furthermore, there is a substantial body of work addressing various considerations. These include unknown control directions \[82, 84\], control input constraints \[85, 91\], actuator faults \[92, 93\], discontinuous output tracking \[94\], event-triggered control \[95\], asymptotic tracking with prescribed performance \[96, 99\], and addressing signal temporal logic specifications \[100, 104\], among others.

Moreover, researchers have dedicated efforts to crafting specific designs for time-varying boundary functions of funnel constraints to ensure finite/fixed time (practical) tracking and stabilization \[105, 106\]. Considerations such as asymmetric funnel constraints \[107\], introducing monotone tube boundaries to enhance control precision \[108\], and addressing compatibility between output and state constraints \[109\] and reach-avoid specifications \[110\] have been explored in this direction. For comprehensive surveys on FC and PPC, refer to \[111, 112\].

Despite the successful applications and extensions of FC, PPC, and TVBLF approaches, they still encounter limitations in handling couplings between multiple time-varying constraints. Specifically, these control design methods inherently focus on time-varying funnel constraints applied to independent states or error signals, resulting in decoupled time-varying funnel constraints. In simpler terms, these methods implicitly assume that the satisfaction of one funnel constraint does not impact the satisfaction of others \[54, 70, 107\]. Additionally, these methods are confined to nonlinear systems with an equal number of inputs and outputs.

While conventional CBF-based control synthesis typically addresses constant constraints, a recent contribution by \[113\] introduces control synthesis using time-varying CBFs to manage time-varying output constraints. In a similar vein, \[114\] employs time-varying CBFs to handle signal temporal logic specifications, that can be essentially translated to fulfilling piecewise constant time-varying constraints. Notably, these results necessitate precise model information. In contrast, a novel approach by \[115\] proposes a class of adaptive time-varying CBFs to ensure a user-defined prescribed performance of independent tracking errors for a class of uncertain nonlinear systems. This is achieved through the application of QPs for controller synthesis. Despite the capabilities of these methods in handling specific classes of output constraints, they all involve an optimization step for synthesizing controllers, which may not be suitable for certain applications. In this thesis, our emphasis shifts towards tractable closed-form control design methods tailored to address time-varying output constraints.

Building upon the existing literature in this subsection, this thesis endeavors to explore and provide insights into the following questions:
• What are the potential limitations and capabilities of funnel-based control methods in addressing coupled time-varying output constraints?

• Is it feasible to modify or directly leverage existing funnel-based control methods to handle time-varying coupled constraints in uncertain nonlinear control systems? If so, for what class of coupled constraints is this modification possible?

• How can one design closed-form robust feedback control laws for a broader class of time-varying output constraints in uncertain nonlinear systems? Furthermore, how can these control designs be adapted when time-varying constraints become infeasible for a certain duration?

1.3 Thesis Outline and Contributions

In this section, we present an overview of the thesis structure and highlight the key contributions of each chapter. In Chapter 2 we introduce essential notations and provide several mathematical concepts that play a vital role in this thesis. These include graph theory, with a particular emphasis on the intuitive exploration of graph rigidity and its relevance in formation control problems. Additionally, we delve into dynamical systems and stability analysis, discuss the Prescribed Performance Control (PPC) methodology, and present some mathematical lemmas and results that are instrumental in the thesis. At the end of this chapter, we briefly discuss the dynamical model of a nonholonomic underactuated mobile robot and a useful model transformation which gives an equivalent fullyactuated model, on which our developed control laws can be applied.

The thesis is divided into two main parts. Part I includes Chapter 3 focusing on decentralized robust coordinate-free formation control design for a directed leader-follower multi-agent system, ensuring guaranteed global convergence to the desired shape under a prescribed performance.

Chapter 3: Leader-Follower Directed Formation Control Based on Bipolar Coordinates with Global Convergence

Chapter 3 introduces an innovative 2-D formation control scheme for acyclic triangulated directed graphs, a class of minimally acyclic persistent graphs. This approach utilizes bipolar coordinates to achieve (almost) global convergence to the desired formation shape. To ensure robustness against external disturbances and avoid singularities, a decentralized control law is devised using the prescribed performance control method. This control law additionally guarantees predefined transient and steady-state performance for the closed-loop system. Furthermore, the proposed formation control scheme demonstrates its versatility by handling formation maneuvering, scaling, and orientation specifications concurrently with the contribution
of two leaders. Moreover, it is shown that the proposed decentralized control law is implementable in follower agents’ arbitrarily oriented local coordinate frames, requiring only bearing and ratio of distance measurements. These measurements can be easily obtained through low-cost onboard vision sensors, making this formation control strategy highly practical for various applications. This chapter is based on the following publication:


Part II, encompassing Chapters 4 and 5, delves into robust feedback control designs for uncertain nonlinear systems with coupled time-varying output constraints, drawing inspiration from funnel-based control approaches. In Chapter 4, we investigate constraints couplings involving time-varying hard and soft funnel constraints. Chapter 5 is dedicated to presenting a new feedback control design method tailored to address a broader spectrum of potentially coupled time-varying output constraints, which is beyond the scope of conventional funnel-based control methodologies.

**Chapter 4: Funnel Control Under Hard and Soft Output Constraints**

In Chapter 4, a new funnel control method is introduced, specifically addressing a class of time-varying hard and soft output constraints for uncertain nonlinear Euler-Lagragian systems. The approach begins with the design of an online funnel planning scheme that generates Constraint Consistent Funnel (CCF) constraints. These CCF constraints ensure the adherence to hard (safety) constraints, with soft (performance) constraints accommodated only when they do not conflict with the hard constraints. Subsequently, the prescribed performance control method is employed to craft a robust, low-complexity control law guaranteeing that the system’s outputs consistently stay within the online planned CCF constraints. This chapter is based on the following publication:


**Chapter 5: Feedback Control for Uncertain MIMO Nonlinear Systems under Generalized Time-Varying Output Constraints**

Chapter 5 presents a novel control framework for addressing the satisfaction of multiple time-varying output constraints in uncertain high-order MIMO nonlinear control systems. Unlike existing methods, which often assume that the constraints are always decoupled and feasible, in this chapter we tackle coupled time-varying constraints even in the presence of potential infeasibilities. First, it is shown that satisfying multiple constraints essentially boils down to ensuring the positivity of
a scalar variable, representing the signed distance from the boundary of the time-varying output-constrained set. To achieve this, a single consolidating constraint is designed that, when satisfied, guarantees convergence to and invariance of the time-varying output-constrained set within a user-defined finite time. Next, a novel robust and low-complexity feedback controller is proposed to ensure the satisfaction of the consolidating constraint. Additionally, a mechanism for online modification of the consolidating constraint is presented to secure a least-violating solution when potential constraint infeasibilities occur for an unknown time interval. From a different standpoint, this chapter takes a step towards designing feedback control laws to guarantee the invariance of a time-varying constrained set with an arbitrary shape, while earlier feedback control methods are limited to establishing invariance solely for time-varying box constraints. This chapter is mainly based on the following publications:


Chapter 5 Summary and Future Research Directions

In the final chapter, we provide a summary on the presented results and outline potential directions for future work.

Contributions not covered in this thesis:

The following works by the author are not covered in this thesis, but contain related material:


Contribution by the author

The order of authors reflects their contribution to each paper. The first author has the most important contribution, while the author Dimos V. Dimarogonas has taken
1.3. Thesis Outline and Contributions

the supervisory role. [C3] and [C4] stemmed from distinct master’s thesis projects under the supervision of the author Farhad Mehdifar. In all the listed publications, all the authors were actively involved in formulating the problems, developing the solutions, evaluating the results, and writing the paper.
Chapter 2

Preliminaries

In this chapter, we introduce notation and essential preliminaries that are used throughout this thesis.

2.1 Notation

The general notation used throughout the thesis is introduced in this section. Additional notation or conventions may be introduced in subsequent chapters if needed.

The set of natural and real numbers is denoted by \( \mathbb{N} \) and \( \mathbb{R} \), respectively. \( \mathbb{R}_{\geq 0} \) and \( \mathbb{R}_{>0} \) represent non-negative and positive real numbers. \( \mathbb{R}^n \) denotes the real \( n \)-dimensional vector space. For a column vector \( x \in \mathbb{R}^n \), its transpose is denoted by \( x^\top \in \mathbb{R}^{1\times n} \). The Euclidean norm of \( x \) is shown as \( \|x\| \). The concatenation operator is \( \text{col}(x_i) := [x_1^\top, \ldots, x_m^\top] \in \mathbb{R}^{mn} \), where \( x_i \in \mathbb{R}^n, i = 1, \ldots, m \). The space of real \( n \times m \) matrices is \( \mathbb{R}^{n\times m} \). For a matrix \( A \in \mathbb{R}^{n\times m} \), \( A^\top \) and \( \lambda_{\text{min}}(A) \) denote its transpose and minimum eigenvalue, respectively. The induced matrix norm is denoted by \( \|A\| \).

The operator \( \text{diag}(\cdot) \) constructs a diagonal matrix from its arguments. The absolute value of a real number is represented by \( |\cdot| \). For a set \( \Omega \), \( \partial \Omega \) is the boundary, and \( \text{cl}(\Omega) \) is the closure. \( \otimes \) represents the Kronecker product, and \( \mathbf{0}_n \in \mathbb{R}^n \) is the vector of zeros. The class \( C^1 \) functions encompass all differentiable functions with a continuous first-order derivative. \( C^n \) denotes the set of functions continuously differentiable up to order \( n \). For \( x \in \mathbb{R} \), we define \( \text{sign}(x > 0) = 1 \) and \( \text{sign}(x \leq 0) = -1 \).

The gradient of a scalar function \( f : \mathbb{R}^n \to \mathbb{R} \) with respect to its argument is denoted by \( \nabla f(x) = (\frac{\partial f(x)}{\partial x})^\top \), where \( \frac{\partial f(x)}{\partial x} \in \mathbb{R}^{1\times n} \) is the row vector of partial derivatives of \( f \) with respect to \( x \). Furthermore, the special orthogonal group \( SO(n) \) of dimension \( n \), corresponding to rotation matrices in an \( n \)-dimensional space, is defined as \( SO(n) := \{ \mathcal{R} \in \mathbb{R}^{n\times n} \mid \mathcal{R}\mathcal{R}^\top = I_n, \det(\mathcal{R}) = 1 \} \), where \( I_n \) is the identity matrix.
2.2 Graphs and Graph Rigidity

In this section, we briefly review some basic notions about graphs. Chapter 3 covers material closely related to the concepts of rigid and persistent graphs. However, these notions are not directly employed in developing the results presented in Chapter 3. Nevertheless, familiarity with graph rigidity and persistence can provide valuable insights for the reader when going through Chapter 3. Therefore, rather than a complete treatment, we provide a concise and intuitive background on graph rigidity and persistence. Interested readers can refer to the provided references for a detailed introduction to these concepts.

An undirected graph is denoted by the pair $G := (V, E)$, where $V = \{1, 2, \ldots, N\}$ represents the set of vertices (nodes), and $E \subseteq \{(i, j)|i, j \in V, i \neq j\} \subset V \times V$ is the set of undirected edges, in the sense that $(i, j)$ and $(j, i)$ correspond to the same edge. The cardinality of the edge set is indicated by $|E| = m$, where $m \in \{1, \ldots, N(N-1)/2\}$ represents the number of edges in $E$. Two vertices that share a common edge are called adjacent. An undirected graph in which every vertex is adjacent to every other vertex is a complete graph. For a directed graph $G$, expressed as the pair $(V, E)$, the edge set $E$ is directed, signifying that $(j, i) \in E$ denotes an edge originating from vertex $j$ (source) and sinking at vertex $i$. The out-degree of vertex $i$, denoted as $\text{out}(i)$, refers to the number of edges in $E$, where vertex $i$ is the source and the sinks are in $V \setminus \{i\}$. The neighbor set of vertex $i$ in a directed graph is represented by $N_i(E) = \{j \in V \mid (i, j) \in E\}$. A directed path in a graph is a finite sequence of vertices, such that from each vertex, there exists a directed edge to the next vertex in the sequence. A directed graph is said to be acyclic if there is no directed path within the graph that starts and ends with the same node.

Let $p_i \in \mathbb{R}^2$ denote the coordinates assigned to vertex $i \in V$. The stacked vector $p := \text{col}(p_i) \in \mathbb{R}^{2N}$ represents a realization of $G$ in $\mathbb{R}^2$ (i.e., the plane). The pair $(G, p)$ is referred to as a framework of $G$ in $\mathbb{R}^2$. The edge function $\Phi_g : \mathbb{R}^{2N} \to \mathbb{R}^m$ associated with the framework $(G, p)$ is defined as follows:

$$\Phi_g(p) = [\ldots, \|p_i - p_j\|^2, \ldots]^\top,$$  \hspace{1cm} (2.1)

where the $q$-th component ($q \leq m$), i.e., $\|p_i - p_j\|^2$, corresponds to the $q$-th edge of $G$ connecting the $i$-th and $j$-th vertices. Frameworks $(G, p)$ and $(G, p')$ are equivalent if $\Phi_g(p) = \Phi_g(p')$, and are congruent if $\|p_i - p_j\| = \|p'_i - p'_j\|$, $\forall i, j \in V$. In the following, we present the concept of graph rigidity in $\mathbb{R}^2$.

Rigid Graphs:

Consider the map $\mathcal{H} : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\mathcal{H}(x) = Rx + d_H$, where $R \in SO(2)$ and $d_H \in \mathbb{R}^2$. A framework $(G, p)$, defined by an undirected graph $G$, is considered as an isometric (shape preserving) transformation, which keeps the distance between any two points constant.
2.2. Graphs and Graph Rigidity

Figure 2.1: Graph (a) is a non-rigid (flexible). Graph (b) a minimally rigid. Graph (c) is a complete graph, hence it is globally rigid. Graph (d) is equivalent but not congruent to graph (b), and in particular, they are flip-ambiguous. (e) shows reflected frameworks (they are congruent but not strongly congruent).

A framework \((G, p)\) is rigid if it is not flexible. In other words, a framework is rigid if it can be continuously deformed while keeping all its edge lengths constant \([117, 118]\). In simpler terms, when we locally perturb the nodes while preserving all edge lengths of a framework, if all distances among the vertices of the graph \(G\) remain unaltered the framework is rigid, thus preserving its shape. Framework \((G, p)\) is flexible if it is not rigid. In other words, a framework is flexible if it can be continuously deformed while keeping all its edge lengths constant \([119]\) (see Fig. 2.1a). A framework is minimally rigid if it is rigid and if by removing any of its edges it loses rigidity (see Fig. 2.1b). In particular, a rigid framework \((G, p)\) in \(\mathbb{R}^2\) is minimally rigid in \(\mathbb{R}^2\) if and only if \(|E^u| = 2n - 3\) \([3]\). A minimally rigid graph in \(\mathbb{R}^2\) is also called a Laman graph \([120]\). Moreover, an undirected complete graph is always a globally rigid graph (see Fig. 2.1c) \([3]\).

If two rigid frameworks \((G, p)\) and \((G, p')\) are equivalent but not congruent, they fall into the category of flip- or flex-ambiguous configurations \([3]\) (see Fig. 2.1d). The concept of congruency distinguishes between two frameworks that exhibit flip- or flex-ambiguity. However, it does not account for a third type of ambiguity, which is the reflection of the entire framework (see Fig. 2.1e). Frameworks \((G, p)\) and \((G, p')\) are said to be strongly congruent if they are congruent and not mirrored versions of each other \([32, 33]\).

\(^2\)The terms rigid graph and rigid framework are used interchangeably in the literature.

\(^3\)Note that not every globally rigid graph is a complete graph. Global rigidity is a more demanding concept than rigidity, refer to \([3]\) for more information.

\(^4\)It is worth noting that flex-ambiguity differs from flip-ambiguity, and this scenario only occurs for non-minimally rigid graphs; refer to \([3]\) for in-depth information.
Henneberg Construction:

The Henneberg construction provides a systematic and iterative approach for constructing minimally rigid graphs in $\mathbb{R}^2$, encompassing two operations [3]: (I) vertex addition and (II) edge splitting. Vertex addition involves introducing a new vertex with two connecting edges to an existing graph. Edge splitting entails the addition of a new vertex with three edges, simultaneous with the removal of an existing edge. Importantly, the vertices originally incident to the removed edge must now be adjacent to the newly added vertex. It is a known fact that performing Henneberg operations on an existing minimally rigid graph guarantees that the resulting augmented graph remains minimally rigid.

Lemma 2.1. [24] A minimally rigid graph in $\mathbb{R}^2$, can be constructed using Henneberg construction (using both operations) starting from a complete graph of two vertices with one edge.

A minimally rigid graph in $\mathbb{R}^2$ is called a triangulated Laman graph if it is constructed exclusively by successive application of Henneberg Type I operations (i.e., vertex addition), provided that at each stage the new vertex is connected to two adjacent vertices [120].

Application of Graph Rigidity in Formation Control Problems:

Rigid graphs and rigidity theory are widely employed as a useful tool for the study of coordinate-free formations [3, 6, 14, 15]. In the realm of distance-based formation control problems, undirected distance rigid graphs play a crucial role in representing inter-agent interactions and characterizing the desired formation shape [3, 6, 22, 121, 122]. To elaborate, consider the nodes of a rigid graph as agents, the edges as inter-agent sensings, and the edge lengths as actual distances among neighboring agents. The objective is to control all agents in space so that each pair of neighboring agents achieves its desired edge length, thereby ensuring the desired relative distances. Due to practical consideration, when dealing with formation control applications, it is often advantageous to model the desired formation using minimally rigid graphs since it requires the minimum number of relations (sensing) among agents, facilitating the scalability of distance-based formation control solutions.

Note that, when only enough distance constraints are present to ensure minimal rigidity, the shape of the associated formation is not uniquely determined by those constraints (for example see Figs 2.1b and 2.1d). Rigidity, whether minimal or not, ensures that if a formation assumes one of the allowed shapes, it cannot smoothly deform from that shape. Minimal rigidity, therefore, enables the retention of shape only in the face of potential smooth deformations, although it does not inherently specify the retained shape. In other words, when distance constraints among agents adhere to a rigid graph that is not globally rigid, the achievement of desired inter-agent distances results in only a finite number of shape realizations. In
2.2. Graphs and Graph Rigidity

![Graphs and Graph Rigidity](image)

**Figure 2.2:** The underlying undirected graph in (a) is rigid, yet the shape of the corresponding 2D-formation may not be maintained at all times. For instance, agent 4 has only one distant constraint to fulfill and can move freely on a circle centered on agent 1. When this occurs, it becomes impossible for agent 3 to simultaneously satisfy the three constraints it is responsible for, while the other agents fulfill all their constraints and have no reason to move, as depicted in (b). Such a situation never arises with the formation depicted in (c). Formally, the graph in (a) is rigid but not persistent due to its lack of constraint consistency, whereas (c) is both rigid and constraint consistent, making it persistent.

In other words, the sets of edges and their corresponding desired distance constraints do not uniquely define the relative positions of the agents. Notice that, although global rigidity can ensure uniqueness of the shape, it is not desirable to characterize a desired formation by a globally rigid graph since such a choice might lead to scalability issues in formation control applications with a large number of agents.

**Persistent Graphs:**

Persistent graphs serve as the directed analog of undirected distance rigid graphs, playing a crucial role in studying directed distance-based formation problems, where the inter-agent sensing topology is directed [3, 23]. In scenarios involving directed formations, it turns out that the rigidity of the underlying undirected graph, obtained by replacing all directed edges with undirected ones, is not sufficient for feasibility of the formation [2]. This issue leads to the definition of persistent graphs. A directed graph is **persistent** if it is both constraint-consistent and exhibits rigidity in its underlying undirected graph [24]. Roughly speaking, a formation is constraint consistent if every agent is able to satisfy all its distance constraints while all other agents are trying to do so. In other words, if an agent’s responsibility for maintaining desired distances is too demanding to fulfill, the directed formation lacks constraint consistency [23]. Refer to Fig. 2.2 for an illustrative example of constraint-consistency in a directed graph. The following theorem provides a formal definition of persistence.

**Theorem 2.1.** [25] A directed graph $G := (V, E)$ is persistent if and only if it is constraint consistent and the underlying undirected graph is rigid.

The following lemma is useful for characterizing constraint consistency of directed graphs.
Lemma 2.2. [3] A directed graph in $n$-dimensional space is constraint consistent if none of its vertices has an out-degree greater than $n$.

Similar to minimal rigidity, minimal persistency for directed graphs is also defined. A directed graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ is minimally persistent if it is persistent and removing any of its edges results in a loss of persistency. The following theorem provides a simple way for checking minimally persistency of a directed graph in $\mathbb{R}^2$.

Theorem 2.2. [123] A directed graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ is minimally persistent in $\mathbb{R}^2$ if and only if its underlying undirected graph is minimally rigid and for all vertices we have $\text{out}(i) \leq 2$ for all $i \in \{1, \ldots, N\}$.

Henneberg construction is also defined for directed graphs. Particularly, Henneberg directed vertex addition (Henneberg directed type I operation) in a 2-dimensional space is defined as adding a vertex with exactly two directed edges to an existing directed graph provided that the added vertex is the source of the added edges. The following lemma provides a useful result.

Lemma 2.3. [127] A directed graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ in $\mathbb{R}^2$ obtained by applying Henneberg directed vertex addition to directed graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, is (resp. minimally) persistent if and only if $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ is (resp. minimally) persistent.

Note that not all minimally persistent directed graphs can be obtained by applying Henneberg directed Type I operations, as there are counterexamples [24]. The following lemma provides an important result about the structure of minimally persistent directed graphs, which can be constructed through successive applications of Henneberg directed Type I operations.

Lemma 2.4. [24] A minimally persistent directed graph which has no cycle (cycle-free or acyclic), always has a Leader-First-Follower (LFF) structure and hence can be obtained via Henneberg directed vertex insertion starting from a graph of two vertices connected with a single directed edge.

A minimally persistent graph in $\mathbb{R}^2$ obtained through Henneberg directed vertex addition, starting from a LFF structure, is also known as a (directed) Henneberg framework [32, 33]. Figure 2.3 illustrates the process of generating an acyclic minimally persistent graph using Henneberg Type I insertions. Similar to the undirected case, if at each stage of applying Henneberg directed vertex addition, the new vertex is connected to two adjacent vertices, the resulting acyclic minimally persistent graph is called a directed triangulated Laman graph [125].

Removing Ambiguous Configurations in Rigid and Persistent Graphs

Similar to undirected minimally rigid graphs, the characterization of the desired shape under minimally persistent graphs faces challenges related to flip-ambiguity and reflection issues (refer to Fig. 2.4a). Consequently, in directed formation control
2.2. Graphs and Graph Rigidity

Figure 2.3: Generating an acyclic minimally persistent graph under successive application of Henneberg directed type I operations starting from a graph of two vertices connected with a directed edge (a). (b): Primitive LFF structure (L: Leader, FF: First-Follower, F: Follower). (c) Henneberg directed type I operation at step \(i - 2\).

Figure 2.4: (a) Minimally persistent graphs suffer from ambiguous configurations. (b) Use of signed area to distinguish the position of vertex 4 with respect to its reflected version \(4'\). (c) Use of edge-angle for distinguishing the position of vertex 4 with respect to \(4'\).

problems, satisfying inter-agent distance constraints based on a minimally persistent graph does not guarantee convergence to the desired shape \([32]\). To address this, additional formation parameters such as signed area, edge-angle, etc., have been recently introduced alongside distances to uniquely characterize the desired formation \([28, 34]\). Fig. 2.4 illustrates how assigning signed areas (denoted by \(Z_{kij}\)) or edge-angles (denoted by \(\alpha_{kij} \in [0, 2\pi]\)) to a minimally persistent framework can address shape ambiguities (see \([29, 33]\) for more details). This concept is also applied to undirected distance rigid frameworks, leading to the development of sign rigidity theory for graphs \([28, 34]\).

As mentioned before, the notion of strong congruency is related to distinguishing the shape of congruent rigid frameworks and avoiding reflected frameworks. Originally, strong congruency is introduced for directed Henneberg frameworks by incorporating signed areas \([32]\) or edge-angles \([33]\) along with the edge distances. In the case of employing edge-angles, \([33]\) considers (directed) Henneberg frameworks, i.e., acyclic minimally persistent frameworks, with \(N\) vertices and a directed edge set \(\mathcal{E}\). Given that Henneberg frameworks can be decomposed into triangular sub-frameworks (graphs), an edge-angle can be defined for each sub-framework, leading
to the formulation of an edge-angle function $A : \mathbb{R}^{2N} \to [0, 2\pi)^{N-2}$ defined as:

$$A(p) = [\ldots, \alpha_{kij}, \ldots]^T, \quad \forall (k, i), (k, j) \in \mathcal{E} \setminus \{(2, 1)\},$$  
(2.2)

where $\alpha_{kij}$ is the edge-angle measured from edge $(k, i)$ to the edge $(k, j)$, for which $i < j$ (by convention). Moreover, the $m$-th component in (2.2) corresponds to the edge-angle of the $m$-th triangular sub framework with vertices $i < j < k$. With this definition, [33] extended the concept of congruency to strong congruency by incorporating the edge-angle function and provided the following result:

**Lemma 2.5.** [33] Henneberg frameworks (i.e., acyclic minimally persistent frameworks) $(\mathcal{G}, p)$ and $(\mathcal{G}, p')$ are strongly congruent if and only if they are equivalent, i.e., $\Phi_\mathcal{G}(p) = \Phi_\mathcal{G}(p')$, and $A(p) = A(p')$.

The above lemma, in essence, asserts that if two Henneberg frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, p')$ share identical edge lengths and all their edge-angles are the same, they are strongly congruent. This implies that their shapes are identical, and neither can be a flip-ambiguous nor a reflected version of the other. Consequently, a desired directed formation can be uniquely characterized when defined according to a Henneberg framework with a specified set of edge-angles.

### 2.3 Dynamical Systems

In this thesis, we deal with dynamical systems modeled by ordinary differential equations, which together with an initial condition of the system result in an Initial-Value Problem (IVP). In the following, we review the basic concepts of a solution to an IVP and the notions of maximal and complete solution.

Consider the following IVP:

$$\dot{x} = f(t, x), \quad x_0 := x(t_0) \in \mathcal{D},$$  
(2.3)

with $f : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D} \subset \mathbb{R}^n$ is a nonempty open set. Moreover, without loss of generality, assume that $t_0 = 0$.

**Definition 2.1** (Solution to an IVP). A solution to the IVP (2.3) is an absolutely continuous function $x : \mathcal{I} \to \mathcal{D}$, where $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$, that satisfies $x(t) = f(t, x(t))$ for all $t \in \mathcal{I}$.

In particular, it is known that if $f(t, x)$ is continuous in $x$ and piecewise continuous in $t$, the solution to (2.3) is continuously differentiable.

**Definition 2.2** (Maximal Solution). A solution $x : \mathcal{I} \to \mathcal{D}$ to the IVP (2.3) with $\mathcal{I} := [0, \tau_{\text{max}}] \in \mathbb{R}_{\geq 0}$, is maximal solution, if there is no other solution $\tilde{x} : [0, \tilde{\tau}) \to \mathcal{D}$ with $\tau_{\text{max}} < \tilde{\tau}$ such that $x(t) = \tilde{x}(t)$ for all $t \in [0, \tau_{\text{max}}]$.

**Definition 2.3** (Complete Solution). A solution $x : \mathcal{I} \to \mathcal{D}$ to the IVP (2.3) with $\mathcal{I} := [0, \tau_{\text{max}}] \in \mathbb{R}_{\geq 0}$, is said to be complete if $\tau_{\text{max}} = +\infty$. 


2.3. Dynamical Systems

Next, we review some standard result regarding the existence and uniqueness of a solution to an IVP from [126], which have been extensively used in this thesis.

**Theorem 2.3.** [126, Theorem 5.4] Consider the IVP in (2.3). Assume that \( f : \mathbb{R}^n \times \mathcal{D} \to \mathbb{R}^n \) is: (a) locally Lipschitz continuous in \( x \) for each \( t \in \mathbb{R}_0^+ \), (b) piecewise continuous in \( t \) for each fixed \( x \in \mathcal{D} \). Then there exists a unique and maximal solution \( x : \mathcal{I} \to \mathcal{D} \) to the IVP (2.3) with \( \mathcal{I} := [0, \tau_{\text{max}}] \subseteq \mathbb{R}_0^+ \).

The next result concerns the completeness of a solution an IVP.

**Lemma 2.6.** [126, Proposition C.3.6] Assume that the conditions of Theorem 2.3 holds. For a maximal solution \( x : [0, \tau_{\text{max}}) \to \mathcal{D} \) to the IVP (2.3) with \( \tau_{\text{max}} < +\infty \) (i.e., the solution is not complete), and for any compact set \( \mathcal{D}' \subseteq \mathcal{D} \), there exists a time instance \( t' \in [0, \tau_{\text{max}}) \) such that \( x(t') \notin \mathcal{D}' \).

Moreover, we review the concepts of boundedness and ultimate boundedness for nonautonomous dynamical systems, which play a crucial role in the presented stability analyses in this thesis. Consider the dynamical system (2.3) and assume \( f : \mathbb{R}^n \times \mathcal{D} \to \mathbb{R}^n \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \), where \( \mathcal{D} \subseteq \mathbb{R}^n \) is a domain that contains the origin.

**Definition 2.4** (Class \( K \) and \( K_\infty \) functions [127]). A scalar continuous function \( \alpha(r) \), defined for \( r \in [0, a) \), belongs to class \( K \) if it is strictly increasing and \( \alpha(0) = 0 \). Moreover, it belongs to class \( K_\infty \) if it is defined for all \( r \geq 0 \) and \( \alpha(r) \to +\infty \) as \( r \to +\infty \).

**Definition 2.5** (Class \( KL \) functions [127]). A scalar continuous function \( \beta : [0, a) \times [0, +\infty) \to [0, +\infty) \) is said to belong to class \( KL \) if, for each fixed \( s \), the mapping \( \beta(r, s) \) belongs to class \( K \) with respect to \( r \) and, for each fixed \( r \), the mapping \( \beta(r, s) \) is decreasing with respect to \( s \) and \( \beta(r, s) \to 0 \) as \( s \to +\infty \).

**Definition 2.6.** (Boundedness and Ultimate Boundedness [127]) The solutions of (2.3) are:

- **Uniformly bounded** if there exists a positive constant \( c \), independent of \( t_0 > 0 \), and for every \( a \in (0, c) \), there is a \( \beta = \beta(a) > 0 \), independent of \( t_0 \), such that:

  \[
  \|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \quad \forall t \geq t_0. \tag{2.4}
  \]

- **Globally uniformly bounded** if (2.4) holds for arbitrarily large \( a \).

- **Uniformly ultimately bounded with ultimate bound** \( b \) if there exist positive constants \( b \) and \( c \), independent of \( t_0 > 0 \), and for every \( a \in (0, c) \), there is \( T = T(a, b) > 0 \), independent of \( t_0 \), such that:

  \[
  \|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \quad \forall t \geq t_0 + T. \tag{2.5}
  \]

- **Globally uniformly ultimately bounded** if (2.5) holds for arbitrarily large \( a \).
**Theorem 2.4.** [128, Theorem 4.4] Let $D \subset \mathbb{R}^n$ be a domain containing $B_\mu := \|x\| \leq \mu$ (thus containing the origin) and $V(x)$ be a continuously differentiable function such that:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$  

(2.6a)

$$\frac{\partial V}{\partial x} f(t, x) \leq -W(\|x\|), \quad \forall x \in D \text{ with } \|x\| \geq \mu, \quad \forall t \geq 0,$$  

(2.6b)

where $\alpha_1$ and $\alpha_2$ are class $\mathcal{K}$ functions and $W(x)$ is a continuous positive definite function. Choose $c > 0$ such that $\Omega_c := \{V(x) \leq c\}$ is compact and contained in $D$ and suppose that $\mu < \alpha_2^{-1}(c)$. Then, $\Omega_c$ is positively invariant for the system (2.3) and there exists a class $\mathcal{KL}$ function $\beta$ such that for every initial state $x(t_0) \in \Omega_c$, the solution of (2.3) satisfies:

$$\|x\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \alpha_1^{-1}(\alpha_2(\mu))\}, \quad \forall t \geq t_0.$$  

(2.7)

If $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then (2.7) holds for any initial state $x(t_0)$, with no restriction on how large $\mu$ is.

Barrier Lyapunov functions (BLFs) share similarities with Lyapunov functions, as they are positive definite. However, they are intentionally designed to have a limited domain of definition, confined to open subsets of Euclidean space. Notably, these functions exhibit unbounded growth as the argument approaches the boundary of their specified domain.

**Definition 2.7.** (Barrier Lyapunov Function) [61] Consider the system $\dot{x} = f(x)$ and let $D \subset \mathbb{R}^n$ be an open set containing the origin. A Barrier Lyapunov Function is a scalar function $V : D \to \mathbb{R}_{\geq 0}$, that is continuous, positive definite, has continuous first-order partial derivatives at every point of $D$, has the property $V(x) \to +\infty$ as $x \to \partial D$, and satisfies $V(x(t)) \leq b, \forall t \geq 0$ along the solution of $\dot{x} = f(x)$ for all $x(0) \in D$ and some positive constant $b$.

### 2.4 Prescribed Performance Control

In this section, we start by introducing the Prescribed Performance Control (PPC) design method and its philosophy. Then, we explore extending PPC design to handle asymmetric funnel constraints, particularly relevant in the control design discussed in Chapter 4. Lastly, we draw explicit connections between PPC and the concepts of BLF and Time-Varying Barrier (Lyapunov) Functions, offering valuable insights for controller design presented in Chapter 5.

PPC is a funnel-based feedback control strategy originally proposed to address stabilization of tracking errors in nonlinear control systems under a user-defined transient and steady-state performance [53, 54]. In particular, PPC method belongs to a class of robust adaptive nonlinear controllers, which are also known as non-identifier based adaptive control methods [129, 131].
2.4. Prescribed Performance Control

2.4.1 Basic Problem Setup and Design Strategy

The main goal of conventional PPC design is to constrain the evolution of the tracking error signal $e(t) \in \mathbb{R}$, within a time-varying (symmetric) funnel constraint:

$$-\rho(t) < e(t) < \rho(t), \quad \forall t \geq 0,$$

where $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$ is a strictly positive, continuously differentiable, and decreasing function of time known as performance function, which is usually defined as [51]:

$$\rho(t) = (\rho_0 - \rho_\infty)e^{-lt} + \rho_\infty,$$  

in which $\rho_0, \rho_\infty \in \mathbb{R}_{>0}$, $\rho_0 \geq \rho_\infty$, and $l \in \mathbb{R}_{\geq 0}$. In particular, $\rho_0$ is selected such that $\rho_0 > |e(0)|$ to ensure $e(0) \in (-\rho(0), \rho(0))$ and parameters $l$ and $\rho_\infty$ characterize the desired transient and steady-state performance specifications on $e(t)$, respectively. More precisely, the decreasing rate of $\rho(t)$, affected by the constant $l$, introduces a lower bound on the speed of convergence of $e(t)$. Moreover, depending on the accuracy (resolution) of the sensors, the constant $\rho_\infty := \lim_{t \rightarrow \infty} \rho(t)$ can be set arbitrarily small to get practical convergence of $e(t)$ to zero at the steady-state. Fig. 2.5 illustrates the funnel constraint in (2.8) on the tracking error.

To design the PPC law, we progress as follows. First, the normalized error signal is defined as:

$$\hat{e}(t, e) = \frac{e(t)}{\rho(t)},$$

where $\hat{e}(t) \in (-1, 1)$ if and only if $e(t) \in (-\rho(t), \rho(t))$. Next, a nonlinear transformation is employed to map the constrained (normalized) error $\hat{e}(t)$ to an equivalent unconstrained one as follows:

$$\varepsilon(t, e) := T(\hat{e}) := \ln \left( \frac{1 + \hat{e}}{1 - \hat{e}} \right),$$

Figure 2.5: Evolution of the error inside the prescribed performance funnel.
where $\varepsilon(t)$ is the unconstrained transformed error corresponding to $e(t)$ and $\mathcal{T} : (-1, 1) \to (-\infty, +\infty)$ is a smooth strictly increasing bijective mapping, which satisfies $\mathcal{T}(0) = 0$. In general, the specific choice of $\mathcal{T}(\hat{e})$ in (2.11) can be replaced by any function that satisfies the mentioned properties. Note that enforcing the boundedness of $\varepsilon$ along the system trajectories ensures that $\varepsilon$ remains within the range of $(-1, 1)$, which leads to the satisfaction of (2.8). Hence, the key idea behind designing the PPC law is to come up with a feedback control law $u$ so that in the closed-loop system $\varepsilon(t)$ remains bounded for all time.

Similarly to the above, the mentioned control design strategy can be extended for dealing with funnel constraints over multiple tracking error signals, i.e., $-\rho_i(t) < e_i(t) < \rho_i(t)$, $i = 1, \ldots, n$, by defining a vector of normalized errors

$$\hat{e} := \left[ \frac{e_1(t)}{\rho_1(t)}, \ldots, \frac{e_n(t)}{\rho_n(t)} \right]^\top, \quad (2.12)$$

and considering a vector of unconstrained mapped error signals

$$\varepsilon := \left[ \ln \left( \frac{1 + \hat{e}_1}{1 - \hat{e}_1} \right), \ldots, \ln \left( \frac{1 + \hat{e}_n}{1 - \hat{e}_n} \right) \right]^\top, \quad (2.13)$$

in which $\hat{e}_i = \frac{e_i(t)}{\rho_i(t)}$. Again establishing the boundedness of $\|\varepsilon\|$ under an appropriate control law leads to the satisfaction of all funnel constraints. Often in the literature the tracking errors are considered as $e_i(t) = x_i - x_i^d(t)$, $i = 1, \ldots, n$, where $x_i$ is a state of the system and $x_i^d(t)$ is its associated (smooth) desired trajectory. It is not difficult to see that this choice will lead to having multiple independent (decoupled) funnel constraints since each funnel constraint is imposed on a distinct state variable (or tracking error). Combination of such funnel constraints resembles a time-varying box constraint in the state space.

### 2.4.2 Dealing with Asymmetric Funnel Constraints

The PPC method can be modified to address general asymmetric funnel constraints on the state variables. Let us consider that the control objective is to design a feedback control law $u$ such that the following time-varying asymmetric funnel constraint on $x \in \mathbb{R}$ is satisfied:

$$\underline{\rho}(t) < x(t) < \overline{\rho}(t), \quad \forall t \geq 0,$$

where $\underline{\rho}, \overline{\rho} : \mathbb{R}_\geq 0 \to \mathbb{R}$ are continuously differentiable functions of time. Note that, to ensure that the asymmetric funnel constraint $\underline{\rho}(t)$ is well-defined (feasible) $\overline{\rho}_d(t) := \overline{\rho}(t) - \underline{\rho}(t) > 0$ should be strictly positive for all time. Fig. 2.6 illustrates an example for asymmetric funnel constraint in (2.14) on $x(t)$. Now inspired from the PPC design philosophy, first, we define the normalized output (state) w.r.t. the asymmetric funnel constraint (2.14) as follows:

$$\hat{x}(t, x) := \frac{x - \frac{1}{2}(\overline{\rho}(t) + \underline{\rho}(t))}{\frac{1}{2}(\overline{\rho}(t) - \underline{\rho}(t))}, \quad (2.15)$$
2.4. Prescribed Performance Control

where $\hat{x} \in (-1,1)$ if and only if $x \in (\rho(t), \rho(t))$. Notice that, (2.15) reduces to (2.10) when $\rho(t) = \rho(t) = -\rho(t)$. Next, akin to the conventional PPC method we can employ the nonlinear transformation $T(\cdot)$ in (2.11) to get $\varepsilon(t, x) = \ln(1 + x)$. As a result, maintaining the boundedness of $\varepsilon$ along the system trajectories becomes equivalent to the satisfaction of (2.14). As mentioned in the previous subsection, this idea can be extended for dealing with multiple asymmetric funnel constraints on independent system states. We will utilize this approach for controller design in Chapter 4.

2.4.3 Connection with the Concept of BLF

Here, we emphasize that the PPC method can be interpreted as a control design approach based on the concept of Barrier Lyapunov Functions (BLFs). This insight proves valuable in the control design discussed in Chapter 5.

Specifically, consider the task of ensuring the asymmetric funnel constraint (2.14). As explained in the preceding subsection, this challenge boils down to establishing the boundedness of $\varepsilon(t, x) = T(\hat{x})$. Now, let us define $V(\varepsilon) := \frac{1}{2}\varepsilon^2$ as a positive definite and radially unbounded candidate Lyapunov function with respect to $\varepsilon$. This candidate Lyapunov function is usually used for designing PPC laws in the literature. It is important to note that one can regard this candidate Lyapunov function as $V(\hat{x})$, where $V : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$, and $\mathcal{D} := (-1, 1)$. In this context, since $V(\hat{x}) \rightarrow +\infty$ as $\hat{x} \rightarrow \partial \mathcal{D}$, $V(\hat{x})$ can be seen as a candidate BLF with respect to the dynamics of $\hat{x}$ (i.e., $\hat{x}$). Therefore, ensuring the boundedness of $\varepsilon$ through the design of $u$ is equivalent to demonstrating that $V(\hat{x})$ serves as a BLF for closed-loop normalized state space dynamics $\hat{x}$. Fig. 2.7a illustrates the mapping $\varepsilon(\hat{x}) = T(\hat{x})$ along with $V(\hat{x}) = \frac{1}{2}\varepsilon^2$ with respect to the normalized state $\hat{x}$.

Similarly, employing a slight abuse of notation, we can express $V(\varepsilon) := \frac{1}{2}\varepsilon^2$ as
V(\dot{x}), \text{ where } V: \mathbb{R}_{\geq 0} \times S \rightarrow \mathbb{R}_{\geq 0}, \text{ and } S(t) := \{ x \in \mathbb{R} \mid -\rho(t) < x < \varphi(t) \} \subset \mathbb{R}. \text{ As } V(t, x) \rightarrow +\infty \text{ when } x \rightarrow \partial S(t), \text{ it follows that } V(t, x) = \frac{1}{2} \varepsilon^2(t, x) \text{ can be considered a time-varying barrier function with respect to the system dynamics } \dot{x}. \text{ Note that a time-varying barrier function concerning a dynamical system is also termed a Time-Varying Barrier Lyapunov Function (TVBLF), as introduced in [62]. Once again, establishing the boundedness of } V(t, x) \text{ along the (closed-loop) system trajectories ensures the fulfillment of (2.14). Fig. 2.7b illustrates the mapping } \varepsilon = \mathcal{T}(t, x) \text{ along with } V(t, x) = \frac{1}{2} \varepsilon^2 \text{ in relation to the state variable } x. \text{ Finally, we highlight that the above interpretations can be extended when satisfaction of multiple funnel constraints are required by considering } V = \frac{1}{2} \varepsilon^2, \text{ which gives a candidate TVBLF with respect to system dynamics.}

## 2.5 Useful Mathematical Results

In this section, we review some key mathematical results essential for deriving the presented findings in this thesis.

**Definition 2.8** (Coercive Functions [132]). Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a continuous function defined over \( \mathbb{R}^n \). The function \( f \) is called coercive if

\[
\lim_{\|x\| \to +\infty} f(x) = +\infty. \tag{2.16}
\]

In control theory literature, coercive functions are also referred to as *radially unbounded* functions [133]. It is important to note that the limit condition in (2.16) implies that \( f(x) \) must tend towards infinity along any path where \( \|x\| \) approaches infinity. A continuous vector function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is *norm-coercive* under a similar condition, that is when \( \lim_{\|x\| \to +\infty} \|f(x)\| = +\infty \).
Theorem 2.5 (Compactness of all Level Sets). \[134, \text{Proposition 2.9}\] Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous function. Then all the level sets of \( f \) are compact if and only if \( f \) is coercive.

The following theorem provides a sufficient condition for existence of a global minimizer for a continuous function.

Theorem 2.6. \[132, \text{Theorem 1.4.4}\] Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous function. If \( f \) is coercive, then it has at least one global minimizer. If, in addition, the first partial derivatives of \( f \) exist on all of \( \mathbb{R}^n \), then these global minimizers can be found among the critical points of \( f \).

Theorem 2.7 (Global Inverse Function Theorem). \[135\] Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^k \) map. Then \( f \) is a \( C^k \) diffeomorphism (i.e., has a \( C^k \) continuously differentiable inverse) if and only if: (a) \( f \) has an invertible Jacobian matrix for all \( x \in \mathbb{R}^n \) (i.e., \( \det(Df(x)) \neq 0, \forall x \in \mathbb{R}^n \)), and (b) \( f \) is norm-coercive (i.e., \( \lim_{\|x\| \to +\infty} \|f(x)\| = +\infty \)).

Definition 2.9. (Invex Functions \[136\]) Assume \( X \subseteq \mathbb{R}^n \) be an open set. The differentiable function \( f : X \to \mathbb{R} \) is invex if there exists a vector function \( \eta : X \times X \to \mathbb{R}^n \) such that:

\[
f(x) - f(y) \geq \eta(x, y)^\top \nabla f(y), \quad \forall x, y \in X.
\]

Convexity is a stronger property than invexity. It is easy to see that the particular case of (differentiable) convex function is obtained from (2.17) by choosing \( \eta(x, y) = x - y \). The following theorem provides one characterization of differentiable invex functions.

Theorem 2.8. \[136\] Let \( f : X \to \mathbb{R} \) be differentiable. Then \( f \) is invex if and only if every stationary (critical) point of \( f \) is a global minimizer.

Lemma 2.7. \[137, 138\] Smooth over- and under-approximations of the max and min operators, respectively, can be expressed using the following Log-Sum-Exp functions:

\[
\max\{c_1, \ldots, c_n\} \leq \frac{1}{\nu} \ln \left( e^{\nu c_1} + \ldots + e^{\nu c_n} \right), \tag{2.18a}
\]

\[
-\frac{1}{\nu} \ln \left( e^{-\nu c_1} + \ldots + e^{-\nu c_n} \right) \leq \min\{c_1, \ldots, c_n\}, \tag{2.18b}
\]

Here, \( c_i \in \mathbb{R}, i = 1, \ldots, n \), and \( \nu > 0 \) is a tuning coefficient. Larger values of \( \nu \) result in a closer approximation. Moreover, the following inequality holds:

\[
\min\{c_1, \ldots, c_n\} \leq -\frac{1}{\nu} \ln \left( e^{-\nu c_1} + \ldots + e^{-\nu c_n} \right) + \frac{\ln(n)}{\nu}. \tag{2.19}
\]
2.6 Mobile Robot Dynamics

In Chapters 4 and 5, we consider an underactuated mobile robot as the case study in the simulation results. This section provides a concise overview of the mobile robot’s dynamics and introduces a model transformation that facilitates the application of our proposed control laws.

Consider a mobile robot operating in a 2-D plane (see Fig. 2.8) with kinematics and dynamics expressed by the following equations [139]:

\[
\begin{align*}
\dot{p}_c &= S(\theta) \zeta \\
\ddot{M} \dot{\zeta} + \ddot{D} \zeta &= \ddot{u} + \ddot{d}(t)
\end{align*}
\]

where \( p_c := [x_c, y_c, \theta]^T \) represents the position and orientation of the body frame \( C \) relative to the reference frame \( O \). The vector \( \zeta := [v_T, \dot{\theta}]^T \) includes the translational speed \( v_T \) along the direction of \( \theta \) and the angular speed \( \dot{\theta} \) about the vertical axis passing through \( C \). The matrices involved are defined as follows: \( M := \text{diag}(m_R, I_R) \), where \( m_R \) and \( I_R \) represent the mass and moment of inertia of the robot about the vertical axis, respectively. \( \ddot{u} \) denotes the force/torque-level control inputs, \( \ddot{D} = \text{diag}(\ddot{D}_1, \ddot{D}_2) \) is a constant damping matrix, and \( \ddot{d}(t) \) is the vector of bounded piecewise continuous external disturbances.

Dynamical model (2.20) represents a category of underactuated robotic vehicles, encompassing wheeled mobile robots, marine vessels, underwater vehicles at a constant depth, and aircraft at a constant altitude. The system is underactuated in the sense that it has three degrees-of-freedom on the plane but only two control inputs. Especially in scenarios where the control objective is solely on controlling the position of the mobile robot (2.20), a transformed model can be derived to tackle the underactuated nature of the system and circumvent nonholonomic constraints [140, 141]. To this end, we transform (2.20) with respect to the hand position \( p_H \) (shown in Fig. 2.8), where:

\[
p_H := [x_c, y_c]^T + L[\cos \theta, \sin \theta]^T.
\]

From (2.20) and (2.21) we have:

\[
\zeta = T^T(\theta) \hat{p}_H,
\]
where
\[
\Upsilon = \begin{bmatrix}
\cos \theta & \sin \theta \\
-(\sin \theta)/L & (\cos \theta)/L
\end{bmatrix}.
\] (2.23)

After taking the time derivative of (2.22) and pre-multiplying the resulting equation first by \(\Upsilon^\top\) and then by \(\Upsilon p\), we arrive at the following fully actuated Euler-Lagrangian (EL) form [139]:
\[
M(x)\ddot{v} + C(x, v)v + D(x)v = u + d(t),
\] (2.24)

where \(x := p_H\) and \(v := \dot{p}_H\) represent the hand position and its velocity, respectively. Moreover, we have:
\[
M := \Upsilon^\top M \Upsilon, \quad C := \Upsilon^\top \dot{M} \Upsilon, \quad D := \Upsilon^\top \dot{D} \Upsilon, \quad d(t) := \Upsilon^\top \dot{d}(t), \quad u := \Upsilon^\top \dot{u},
\]
where the expressions of matrices \(M\) and \(C\) are given as follows:
\[
M = \begin{bmatrix}
m_R \cos^2 \theta + \frac{I_y^*}{L^2} \sin^2 \theta & (m_R - \frac{I_y^*}{L^2}) \cos \theta \sin \theta \\
(m_R - \frac{I_y^*}{L^2}) \cos \theta \sin \theta & m_R \sin^2 \theta + \frac{I_y^*}{L^2} \cos^2 \theta
\end{bmatrix},
\] (2.25a)
\[
C = \begin{bmatrix}
-(m_R - \frac{I_y^*}{L^2}) \dot{\theta} \cos \theta \sin \theta & m_R \dot{\theta} \cos^2 \theta + \frac{I_y^*}{L^2} \dot{\theta} \sin^2 \theta \\
-m_R \dot{\theta} \sin^2 \theta + \frac{I_y^*}{L^2} \dot{\theta} \cos^2 \theta & (m_R - \frac{I_y^*}{L^2}) \dot{\theta} \cos \theta \sin \theta
\end{bmatrix}.
\] (2.25b)

One can show that the transformed model in (2.24) satisfies the well-known properties of standard EL systems (see [139] for more details). In particular, \(M(x)\), \(C(x, v)\), and \(D(x)\) are locally Lipschitz continuous functions of their arguments and \(M(x)\) is positive definite for all \(x \in \mathbb{R}^2\).
Part I

Coordinate-Free Multi-Agent Formation Control
In this chapter we address the problem of robust 2-D leader-follower coordinate-free formation control with (almost) global convergence to the desired shape under directed inter-agent sensing topologies.

3.1 Introduction

In coordinate-free formation control problems, the desired formation shape is defined by a set of coordinate-free scalar variables, often involving distances or angles \[6, 7, 11–15\]. These scalar variables define the desired shape and, consequently, the formation errors for individual agents. To establish control laws for agents in coordinate-free formation control, they must possess measurements of vectorized relative information about their neighboring agents, such as relative positions or bearings, and actively control scalar variables, including inter-agent distances and/or angles, etc. Coordinate-free formation control methods are preferred over other approaches because they allow the design and implementation of formation control laws within each agent’s local coordinate frame, eliminating the need for global position measurements, like those provided by GPS systems, or assuming alignment of agents’ local coordinate frames. Another significant advantage of coordinate-free formation control strategies is their cost-effectiveness, primarily due to their simpler sensing and interaction mechanisms\[1\].

A common challenge in coordinate-free formation control methods is achieving local (rather than global) convergence to the desired shape. This implies that while agents may meet the desired formation criteria, they might not converge to the desired shape due to issues like reflection, flip, and flex ambiguities, as highlighted\[1\].

\[1\]Refer to Section 1.2.1 for a more comprehensive literature review on formation control.
in [2, 3]. To address this challenge, recent research has introduced additional types of formation constraints, such as signed area and edge/signed angle, in conjunction with inter-agent distances to uniquely characterize the desired formation and establish (almost) global shape convergence [17, 18, 28–35]. However, it is essential to note that introducing these additional (scalar) signed variables, while helpful in eliminating shape ambiguities, represents a necessary condition rather than a sufficient one for convergence to the desired shape. Specifically, incorporating extra formation constraints through controller designs can lead to undesired equilibria. For instance, when distances and auxiliary formation constraints like signed area or angle intersect at specific agent positions, new unwanted equilibria can emerge, thereby limiting the assurance of convergence to the desired shapes. Additionally, these approaches often require meticulous control gain tuning to ensure global convergence to the desired shape, complicating the controller design process, especially when applied to more complex formation control scenarios involving agents with intricate tasks or dynamics [28–33].

In contrast to previous works that employed signed variables in coordinate-free formation control, recently [36] introduced the use of formation error variables along orthogonal directions to characterize 2-D directed distance-based formations, achieving (almost) global shape convergence. An extension of this approach for 3-D directed distance-based formations is proposed in [37]. By characterizing the desired shape using orthogonal variables, this method circumvents the issues related to local convergence resulting from controller design, as observed when employing additional signed formation constraints.

### 3.2 Contributions

We introduce a robust 2-D directed coordinate-free formation control method utilizing bipolar coordinates, achieving (almost) global shape convergence under a user-defined transient and steady-state performance. The target formation and the sensing topology among agents are defined by a triangulated acyclic minimally persistent graph (constructed under Assumption 3.1 in Section 3.3). This configuration forms a distance-rigid, directed hierarchical leader-follower structure with the minimum possible number of edges. In this multi-agent setup, agent 1 (leader) handles formation translations, agent 2 (secondary-leader) only follows agent 1 and manages formation scaling and orientation adjustments. The remaining agents (followers), each of whom follows precisely two other agents, are responsible for generating and maintaining the desired shape. The followings describe key contributions of this chapter.

First and foremost, given a desired formation shape, we demonstrate that the desired position of each follower agent can be distinctly defined within a local bipolar coordinate system. More precisely, a bipolar coordinate system is locally assigned to each follower, with its two neighboring agents serving as the focal points [142, 143].

\(^2\)Global convergence, except for a zero-measure set of initial conditions, is guaranteed.
3.2. Contributions

Consequently, each follower’s formation errors can be precisely characterized by a unique pair of desired bipolar coordinate values: a desired angle and a desired (logarithm of) ratio of distances with respect to the two foci (neighbors). This characterization is illustrated in Fig. 3.2. Furthermore, by utilizing local bipolar coordinates to describe formation errors, our proposed control law only necessitates bearing and ratio of distances measurements for the followers. These measurements can be readily acquired through onboard vision sensing, as explained in more detail in Remark 3.3 in Subsection 3.4.1.

Secondly, leveraging the fact that each follower’s formation errors can be independently reduced by moving along the two orthogonal directions within its associated bipolar coordinate basis, we employ the Prescribed Performance Control (PPC) methodology to devise decentralized robust controllers. These controllers ensure practical stabilization of the formation errors, ensuring (almost) global (practical) convergence to the desired shape in the presence of external disturbances. In the control design process, we establish user-defined performance guarantees for the system’s response by imposing time-varying decreasing performance bounds, which can be regarded as spatiotemporal constraints on the formation errors. It is worth emphasizing that maintaining the formation errors within these desirable, diminishing performance bounds not only enhances robustness concerning external disturbances affecting the agents’ motion dynamics but also enables agent 2 and the followers to handle formation maneuvering task in the presence of a moving leader with time-varying scale and orientation adjustments.

The contributions of this chapter can be summarized as follows:

• Pioneering Use of Bipolar Coordinates: This work marks the first instance of employing bipolar coordinates to address 2-D coordinate-free formation control problems. It achieves (almost) global shape convergence while avoiding the introduction of undesirable equilibria and gain tuning issues reported in previous works [28–34]. Our formation control approach essentially falls under the category of new coordinate-free formation control methods that utilize orthogonal variables, akin to those introduced in [36, 37].

• Handling Formation Maneuvering and Sensing Advantages: In contrast to existing coordinate-free formation control results that achieve (almost) global shape convergence, such as [17, 28–33, 36–39], our approach excels in managing coordinate-free formation maneuvering with time-varying leader’s velocity, and accommodating scaling and orientation adjustments. Furthermore, while the previously mentioned results necessitate agents to measure relative positions, our approach only requires bearing and ratio of distance measurements (for the follower agents), which are easily obtainable through vision (camera) sensors, enhancing practical applicability. To our knowledge, this work is the first to offer such (almost) globally converging results for coordinate-free formation control.

• Prescribed Performance and Robustness to Uncertainties: To the best of our
knowledge, there are no previous results on (almost) globally converging coordinate-free formations with guaranteed transient and steady-state performance and robustness with respect to external disturbances/dynamical uncertainties.

- Practicality of Directed Inter-Agent Sensing: The choice of directed inter-agent sensing is often more practical due to two primary reasons. First, the sensing limitations of agents may necessitate such structures. Second, it inherently mitigates the undesired behaviors that can arise from measurement mismatches in undirected formation control setups, as evidenced in [21] [22].

### 3.3 Problem Formulation

Consider a multi-agent system comprised of $n$ mobile robots on a 2-D plane governed by the following dynamics:

$$\dot{p}_i = u_i + \delta_i(t), \quad i = 1, \ldots, n,$$

where $p_i \in \mathbb{R}^2$ and $u_i \in \mathbb{R}^2$ are the position and the velocity control input of agent $i$ expressed with respect to a global coordinate frame, respectively. Let $\delta_i(t) \in \mathbb{R}^2$ represent an unknown, bounded and piecewise continuous external disturbance on agent $i$ (e.g., wind gusts), which may also account for model uncertainties. Notice that the upper bound of the disturbances is not known a priori.

Let the sensing topology among agents be modeled by a directed graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of vertices representing the agents and $\mathcal{E} = \{(j, i) | j, i \in \mathcal{V}, j \neq i\}$ such that if $(j, i) \in \mathcal{E} \Rightarrow (i, j) \notin \mathcal{E}$, is the set of directed edges depicting the directed sensings among the agents. More precisely, $(j, i) \in \mathcal{E}$ denotes an edge that starts from vertex $j$ (source) and sinks at vertex $i$, and its direction is indicated by $j \rightarrow i$. For $(j, i)$ we say $i$ is the neighbor of $j$. The relative position vector corresponding to the directed edge $(j, i)$ is defined as:

$$p_{ji} = p_i - p_j, \quad (j, i) \in \mathcal{E},$$

and its associated relative bearing vector $z_{ji} \in \mathbb{R}^2$ is:

$$z_{ji} = \frac{p_{ji}}{\|p_{ji}\|}, \quad (j, i) \in \mathcal{E}. \quad (3.3)$$

In particular, here the physical meaning of the directed edge $(j, i) \in \mathcal{E}$ is that only agent $j$ can measure the relative bearing of agent $i$ with respect to itself, i.e., $z_{ji}$, and not vice versa. Furthermore, as we will elaborate in the subsequent sections, we make an additional assumption for the special case of agent 2, referred to as the secondary leader. Agent 2 not only possesses the ability to measure the relative bearing of agent 1 but also has the capability to determine the absolute distance from agent 1.
We also assume that the graph $G$ is triangulated and imposes a hierarchical structure, where agent 1 is the leader, agent 2 is the secondary leader with agent 1 acting as its only neighbour, and agents $i \geq 3$ are the followers with each one having exactly two neighbors to follow with smaller indices. Hence, we impose the following assumption for constructing $G$:

**Assumption 3.1.** The directed sensing graph $G$ is constructed such that:

1. $\text{out}(1) = 0$, $\text{out}(2) = 1$, and $\text{out}(i) = 2$, $\forall i \geq 3$;
2. If there is an edge between agents $i$ and $j$, where $i < j$, the direction must be $j \rightarrow i$;
3. If $(k, i), (k, j) \in \mathcal{E}$ then $(j, i) \in \mathcal{E}$,

where $\text{out}(i)$ denotes the out-degree of vertex $i$ that is the number of edges in $\mathcal{E}$ whose source is vertex $i$ and whose sinks are in $V \setminus \{i\}$.

We highlight that cases 1) and 2) in Assumption 3.1 impose $G$ to be a Leader-First-Follower type graph [26] constructed by applying Henneberg directed type I operations, hence, $G$ is acyclic minimally persistent with edge set cardinality $|\mathcal{E}| = 2n - 3$ [24, 33] (see Theorem 2.2 and Lemma 2.4). Moreover, case 3) establishes triangulation in $G$. Note that under Assumption 3.1, $G$ is composed of acyclic directed triangles (i.e., triangular sub-graphs as depicted in Fig. 3.1a). Fig. 3.1b shows an example of $G$ constructed under Assumption 3.1.

For each follower $k$ in $G$ with two neighbors $i$ and $j$, we can define an edge-angle as the angle $\alpha_{kij} \in [0, 2\pi)$ formed by the edges $(k, i), (k, j) \in \mathcal{E}$, measured by convention counterclockwise from edge $(k, i)$ to edge $(k, j)$ [33]. Fig. 3.1a shows the edge-angle $\alpha_{kij}$ assigned to the $k$-th follower in a (acyclic) directed triangular sub-graph of $G$, where Assumption 3.1 establishes the ordering of $i < j < k$ as well. Based on the bearing vectors $z_{ki}$ and $z_{kj}$, the edge-angle $\alpha_{kij}$ can be obtained by:

$$\alpha_{kij} = \begin{cases} \arccos(z_{ki}^T z_{kj}) & \text{if } (z_{ki}^T z_{kj}) \geq 0, \\ 2\pi - \arccos(z_{ki}^T z_{kj}) & \text{otherwise,} \end{cases}$$

in which $z_{ki}^\perp := Jz_{ki}$, where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

denotes the $\frac{\pi}{2}$-counterclockwise rotation matrix.

It is known that based on the directed sensing graph $G$ (respecting Assumption 3.1), we can uniquely define a desired formation characterized by [33]:

(i) A set of $2n - 3$ desired distances $d^*_{ji}$, appointed to the directed edges $(j, i) \in \mathcal{E}$.
(ii) A set of $n - 2$ desired edge-angles $\alpha^*_{kij}, (k, i), (k, j) \in \mathcal{E}\setminus\{(2, 1)\}, i < j < k$, (see Fig. 3.1b for an example).
Leader-Follower Directed Formation Control Based on Bipolar Coordinates with
Global Convergence

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_graph.png}
\caption{(a) edge-angle in a triangular subgraph. (b) example of a desired formation (note that \(d_{ij}^* = d_{ji}^*\) and \(d_{jk}^* = d_{kj}^*\)).}
\end{figure}

Given a desired formation characterized by a graph \(G\) (under Assumption 1) and the corresponding sets of desired distances and edge-angles, the objective is to design a decentralized robust control protocol for (3.1) such that:

\begin{align}
\|p_j(t) - p_i(t)\| &\to d_{ij}^* \quad \text{as } t \to \infty, \\
\alpha_{kij}(t) &\to \alpha_{kij}^* \quad \text{as } t \to \infty,
\end{align}

for all \((j, i) \in \mathcal{E}\) and \((k, i), (k, j) \in \mathcal{E}\backslash\{(2, 1)\}, i < j < k\), respectively, while avoiding zero distance among neighboring agents (i.e., \(\|p_{ji}\| \neq 0, \forall (j, i) \in \mathcal{E}, \forall t \geq 0\)) so that all edge-angles are well-defined. It is known that, satisfaction of (3.6) is equivalent to strong congruency \cite{32, 33} between the actual formation of the agents and the desired formation (see Lemma 2.5). This means that if (3.6) gets satisfied, the agents can achieve the desired formation only up to rotations and translations \cite{33}.

**Remark 3.1.** The notion of strong congruency defined in \cite{32, 33} is related to distinguishing the shape of congruent rigid frameworks and thus avoiding reflected frameworks, see \cite{33} Section II.c. In other words, in general, congruent rigid frameworks may have the issue of position reflections and are not shape-preserving, while strong congruency removes this issue by exploiting an additional parameter (i.e., edge-angle or signed area) to characterize the position of each vertex in a rigid framework. As an example, consider distances \(\|p_{ki}\|, \|p_{kj}\|\) and the edge-angle \(\alpha_{kij}\) in Fig. 3.1a where \(0 < \alpha_{kij} < \pi\). If the position of vertex \(k\) is reflected without altering its distances with respect to \(i\) and \(j\), then \(\pi < \alpha_{kij} < 2\pi\). This property will allow us to distinguish the position of agent \(k\) from its reflection with respect to the line passing through agents \(i\) and \(j\). This is further visually illustrated in Fig. 3.1b where we compare the desired edge-angles assigned to agents 4 and 5, both of whom share the same neighbors. Notably, the assigned edge angle for agent 4 exceeds \(\pi\), situating it in the half-plane opposite to agent 5’s position.

**Remark 3.2.** From Assumption 3.1 \(G\) is minimally persistent, meaning that it is a persistent graph with minimum number of edges. This is favorable in practice since
it requires minimum number of relations (sensing) among agents. This assumption on \( G \) is not restrictive since \( G \) can be easily adopted to any geometrical shape and scaled up to any number of agents through successive applications of Henneberg directed type I operation [122, 144], which eventually return a minimally persistent graph (see Subsection 2.2 for more information). In particular, \( G \) adhering to Assumption 3.1 is also referred to as a directed triangulated Laman graph [125].

### 3.4 Characterization of the Desired Formation Based on Bipolar Coordinates

Consider a triangular sub-graph of \( G \) as in Fig. 3.1a where \( i < j < k \). If \( \|p_{ji}\| \neq 0 \) one can define a virtual local Cartesian coordinate frame based on vertices \( i \) and \( j \), denoted by \( \{C_k\} \) as in Fig. 3.2a, with its origin located in the middle of the \( i-j \) line segment. Note that the position of node \( k \) can be uniquely determined in \( \{C_k\} \) w.r.t. its neighboring agents \( i \) and \( j \). It is also known that agent \( k \)'s position in \( \{C_k\} \) can be expressed by the bipolar coordinates \( (r_k, \alpha_{kij}) \in \mathbb{R}^2 \) associated with \( \{C_k\} \), where nodes \( i \) and \( j \) are the two foci of the bipolar coordinate system [142]. Recall that the bipolar coordinate variable \( \alpha_{kij} \) (edge-angle) was already introduced in Section 3.3 and it is given by (3.4). Moreover, \( r_k \) is the natural logarithm of the ratio of the distances \( r_{kij} := \|p_{ki}\|/\|p_{kj}\|, (k, i), (k, j) \in E \setminus \{(2, 1)\}, i < j < k \), between node \( k \) and the foci \( i \) and \( j \), expressed by:

\[
r_k := \ln r_{kij} = \ln \frac{\|p_{ki}\|}{\|p_{kj}\|}, \quad (k, i), (k, j) \in E \setminus \{(2, 1)\},
\]

(3.7)

where \( r_k \in \mathbb{R} \). Note that, when agent \( k \) approaches one of the foci \( i \) or \( j \) (i.e., either \( \|p_{ki}\| \to 0 \) or \( \|p_{kj}\| \to 0 \)), \( r_k \) tends to \( \pm \infty \). The bipolar coordinates are related to the \( \{C_k\} \) frame with the following (almost) one-to-one (except at the foci of the bipolar coordinates, \( i \) and \( j \)) transformation [143]:

\[
x_k^{[C_k]} = c_k \frac{\sinh r_k}{\cosh r_k - \cos \alpha_{kij}},
\]

(3.8a)

\[
y_k^{[C_k]} = c_k \frac{\sin \alpha_{kij}}{\cosh r_k - \cos \alpha_{kij}},
\]

(3.8b)

where \( p_k^{[C_k]} = [x_k^{[C_k]}, y_k^{[C_k]}]^\top \in \mathbb{R}^2 \) is the position of vertex \( k \) with respect to frame \( \{C_k\} \) and \( c_k = 0.5\|p_{ji}\| > 0, k \geq 3 \).

The bipolar coordinate system \( (r_k, \alpha_{kij}) \) is indeed a 2-D orthogonal curvilinear coordinate system [142, 143] (similar to the well-known polar coordinate system), therefore, one can define a local orthogonal basis at each point in the 2-D plane of \( \{C_k\} \) showing the directions of increase for \( \alpha_{kij} \) and \( r_k \). Fig. 3.2b shows orthogonal bipolar coordinates basis \( \hat{\alpha}_k \in \mathbb{R}^2 \) and \( \hat{r}_k \in \mathbb{R}^2 \) associated with \( \{C_k\} \) at some arbitrary points of interest as well as some \( \alpha_{kij} \) and \( r_k \) isoquant curves that create circles centered along the \( Y_k \) and \( X_k \) axis, respectively.
Given a target formation expressed by the graph $\mathcal{G}$ along with the desired edge-angles (i.e., $\alpha_{kij}^*$) and distances (i.e., $d_{ji}^*$), we can use the desired bipolar coordinates $(r_k^*, \alpha_{kij}^*) \in \mathbb{R}^2$ to uniquely determine the desired position of agent $k \geq 3$ with respect to its two neighbors $i$ and $j$ ($i < j < k$), where

$$r_k^* := \ln \frac{d_{ki}^*}{d_{kj}^*}, \quad (k, i), (k, j) \in \mathcal{E}\backslash\{(2, 1)\}. \quad (3.9)$$

In this regard, we propose the following lemma.
Lemma 3.1. Given a desired formation shape based on a specific directed sensing graph $G = (V, E)$ under Assumption 3.1 as well as $\alpha^{*}_{kij}$, $(k, i), (k, j) \in E\backslash\{(2, 1)\}$, $i < j < k$ and $d^{*}_{ji}$, $(j, i) \in E$, satisfying:

\[
\begin{align*}
\|p_2(t) - p_1(t)\| &\rightarrow d^{*}_{21}, & \text{as } t \rightarrow \infty, \\
\kappa_k(t) &\rightarrow \kappa^{*}_k, & k \geq 3, & \text{as } t \rightarrow \infty, \\
\alpha_{kij}(t) &\rightarrow \alpha^{*}_{kij}, & k \geq 3, & \text{as } t \rightarrow \infty,
\end{align*}
\]

is equivalent to the satisfaction of (3.6).

Proof. \((3.10) \Rightarrow (3.6)\): Recall that due to Assumption 3.1, $G$ is comprised of triangular sub-graphs. Every acyclic directed triangular sub graph of $G$ with vertices $i, j,$ and $k$ defines a triangle denoted by $\Delta_{ijk}$. Now consider the triangle (composed of agents 1, 2 and 3) of the desired formation and the actual triangle formed by the agents at time instance $t \geq 0$, which are denoted by $\Delta^{*}_{123}$ and $\Delta_{123}(t)$, respectively. Note that, owing to the side-angle-side similarity theorem between two triangles, if $\Delta_{123}$ is similar to $\Delta^{*}_{123}$. Therefore, satisfaction of (3.10a) and (3.10c) for agent 3 (i.e., $r_3 \rightarrow r^{*}_3$ and $\alpha_{312} \rightarrow \alpha^{*}_{312}$) ensures that $\Delta_{123}(t)$ becomes similar to $\Delta^{*}_{123}$ in the limit. In addition, satisfaction of (3.10a) further ensures that $\Delta_{123}(t)$ and $\Delta^{*}_{123}$ will have the same edge lengths in the limit, that is $\|p_3(t) - p_1(t)\| \rightarrow d^{*}_{31}$ and $\|p_3(t) - p_2(t)\| \rightarrow d^{*}_{32}$ as $t \rightarrow \infty$. Repeating these arguments for the rest of the triangular sub-graphs of $G$ in the desired and the actual formations, i.e., $\Delta^{*}_{ijk}$ and $\Delta_{ijk}(t)$, $(k, i), (k, j) \in E\backslash\{(2, 1)\}$, $i < j < k$, will result in satisfaction of (3.6) for all triangular sub-graphs of $G$. Therefore, (3.10) implies (3.6).

\((3.6) \Rightarrow (3.10)\): Again consider $\Delta_{123}(t)$ and $\Delta^{*}_{123}$. If (3.6) is satisfied for $\Delta_{123}(t)$, i.e., $\|p_3(t) - p_1(t)\| \rightarrow d^{*}_{31}$, $\|p_3(t) - p_2(t)\| \rightarrow d^{*}_{32}$, $\|p_2(t) - p_1(t)\| \rightarrow d^{*}_{21}$, and $\alpha_{312} \rightarrow \alpha^{*}_{312}$, then satisfaction of (3.10) for $\Delta_{123}(t)$ can be readily deduced. Similarly to the previous case, by repeating these arguments for the rest of the triangular sub-graphs in the desired and the actual formations, one can infer that, in general, (3.6) implies (3.10) and this completes the proof.

Recall that equations (3.6a) and (3.6b) indicate that the secondary leader (i.e., agent 2) is only required to keep a certain distance with respect to the leader (agent 1), whereas the rest of the agents (i.e., followers) are required to keep a certain edge-angle and two specific distances with respect to their neighbors. Therefore, a direct approach to achieve (3.6) for each follower agent is controlling three variables: two distances and an edge-angle, as it is studied in [33]. Alternatively, followers can use the signed area information (instead of the edge-angle) along with the distances to achieve the same objective, see [29–32]. However, using an extra shape constraint (i.e., signed area or edge-angle) for the followers to achieve the desired formation may introduce new undesirable equilibria as the distance and signed-area/edge-angle constraints interfere with each other at certain agent positions (see [28–33].
for more details and examples). Indeed, these variables do not always constitute an orthogonal space, in which each formation variable can be adjusted independently by moving along orthogonal directions. Lemma 3.1 overcomes this issue as it only requires the followers to control only two orthogonal (i.e., independent) formation variables (3.10b) and (3.10c). In the sequel, we will leverage this fact to design the formation controllers of the follower agents (see Section 3.5), which allows for (almost) global convergence to the desired shape.

The proof of Lemma 3.1 also reveals that by modifying the distance of agent 2 with respect to agent 1 (i.e., $\|p_2(t) - p_1(t)\|$) one can change the scale of the actual formation at the steady-state (formation scaling means maintaining all angles in the shape and increasing or decreasing all edge lengths with the same proportion). Therefore, if the secondary leader alters its desired distance with respect to the leader, by considering a time-varying desired distance $d^\ast_{21}(t)$, then it can control the formation’s scale, which is of high importance in practical formation control applications, e.g., passing through narrow passages, obstacle avoidance, etc.

### 3.4.1 Formation Errors

To quantify the control objective we define 3 types of error variables. First, the **squared distance error** between agents 2 and 1 is defined as:

$$e_d := \|p_{21}\|^2 - (d^\ast_{21}(t))^2,$$

where $d^\ast_{21}(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ is a strictly positive and continuously differentiable function of time with a bounded derivative representing the desired (in general, time-varying) distance between agents 2 and 1. Notice that $\|p_{21}\| := d^\ast_{21}(t)$ if and only if $e_d = 0$.

Secondly, the **logarithmic ratio of the distances error** is defined as:

$$e_{r_k} := r_k - r^\ast_k, \quad k = 3, \ldots, n,$$

where $r_k$ and $r^\ast_k$ are defined in (3.7) and (3.9), respectively.

Finally, the **edge-angle error** is defined as:

$$e_{\alpha_k} := \alpha_{kij} - \alpha^\ast_{kij}, \quad (k, i), (k, j) \in E \backslash \{(2, 1)\},$$

where $\alpha_{kij}$ is defined in (3.4) and $i < j < k$. Note that, (3.12) and (3.13) are independent (orthogonal) error variables defined only for the followers. More precisely, by moving along each bipolar coordinates basis, $\hat{r}_k$ and $\hat{\alpha}_k$, each follower can reduce (3.12) and (3.13), respectively, without affecting the other error variable. Due to the above discussion and Lemma 3.1 by adopting the bipolar coordinates approach, the control objective of (3.6) is met by zero stabilization of the errors defined in (3.11), (3.12), and (3.13) while maintaining $\|p_{ji}(t)\| \neq 0, \forall (j, i) \in E, \forall t \geq 0$. Recall that in a triangular subgraph of $G$ with agents $i, j$, and $k$, the edge-angle $\alpha_{kij}$ is not defined if $\|p_{ji}(t)\| = 0$.

---

3We have used the subscription of $e_{\alpha_k}$ instead of $e_{\alpha_{kij}}$ for better readability.
Remark 3.3. In order to meet (3.10b) and (3.10c) in Lemma 3.1, each follower is only required to sense and adjust its edge-angle formed by its neighbors, as well as the ratio of the distances with respect to them. It is known that, in general, onboard vision-based sensors (e.g., monocular cameras) give projective measurements that do not contain distance information. As a consequence, it is possible to obtain only bearing (direction) information, from which the angle information can be then retrieved [40] (e.g., by (3.4)). Moreover, as explained in [15, Section II.D], the ratio of the distances can also be extracted from a single image of a camera by comparing projections of two identical (yet unknown) sized (spherical or circular) objects/markers (i.e., two neighbors of a certain follower agent) on the image plane of a camera. In the absence of spherical (circular) shaped agents/markers, each robot may use a database of CAD models for obtaining the ratio of the distances [145]. Therefore, all followers are required to be equipped only with low-cost vision sensors to perceive the required information. This is in contrast to many related results in coordinate-free formation control with (almost) global shape convergence, where relative position measurements for all agents are assumed [28, 33, 36–38].

3.5 Controller Design

In this section, we will adopt the Prescribed Performance Control (PPC) method [53] for designing the formation control laws in order to:

(i) introduce robustness against external disturbances, which also allows us to deal with the formation maneuvering problem,

(ii) achieve predefined transient and steady state response for each formation error $e_h, h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\}$,

(iii) avoid singularities in the edge-angle definition when $\|p_{ji}\| \to 0$, for a pair $(j, i) \in \mathcal{E}$ or when either $\alpha_{kij} = 0$ or $\alpha_{kij} = 2\pi$.

Prescribed performance is achieved when the formation errors $e_h(t), h \in \{d, r_k, \alpha_k\}$, with $k = \{3, \ldots, n\}$ evolve strictly within the predefined regions that are bounded by absolutely decaying functions of time, called performance functions [53, 54]. The mathematical expression of prescribed performance is formulated by the following inequalities:

$$-b_h \rho_h(t) < e_h(t) < \tilde{b}_h \rho_h(t), \quad h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\},$$

where $b_h, \tilde{b}_h > 0$ are arbitrary positive scaling parameters selected properly to avoid singularities in the control problem, as presented in the sequel. Moreover, $\rho_h(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ are user-defined continuously decaying performance functions with bounded derivatives and strictly positive limit as $t \to \infty$ (i.e., $\lim_{t \to \infty} \rho_h(t) > 0$).

In this work, we will adopt the following performance functions:

$$\rho_h(t) = (1 - \rho_{\infty,h}) \exp(-l_h t) + \rho_{\infty,h}, \quad h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\},$$
where parameters \( l_h > 0 \) and \( 0 < \rho_{\infty,h} < 1 \) characterize the desired transient and steady-state performance specifications on \( e_h(t) \), respectively. In particular, the decreasing rate of \( \rho_h(t) \), affected by the constant \( l_h \), introduces a lower bound on the speed of convergence of \( e_h(t) \), \( h \in \{ d, r_k, \alpha_k \}, k = \{3, \ldots, n \} \). Furthermore, depending on the accuracy (resolution) of the sensors, the constants \( \rho_{\infty,h} \) can be set arbitrarily small, thus achieving practical convergence of \( e_h(t) \) to zero.

The task is to synthesize decentralized feedback control laws such that, given \(-b_h \rho_h(0) < e_h(0) < \bar{b}_h \rho_h(0), h \in \{ d, r_k, \alpha_k \}, k = \{3, \ldots, n \}\), the formation errors \( e_h(t) \) satisfy (3.14) for all \( t \geq 0 \) leading to (practical) stabilization of the errors in (3.11), (3.12), and (3.13).

### Selection of the Performance Bounds

We can incorporate the requirement of \( \|p_{ji}(t)\| \neq 0, \forall (j, i) \in \mathcal{E} \), \( \forall t \geq 0 \) by choosing the maximum (absolute) values of the performance bounds \(-b_h \rho_h(t), \bar{b}_h \rho_h(t)\) on \( e_h(t), h \in \{ d, r_k, \alpha_k \}, k = \{3, \ldots, n \}\) in (3.14), appropriately. In particular, from (3.11) and (3.14), choosing \( b_d \rho_d(t) \) such that \( \inf_{t \geq 0} \left( (d_{21}(t))^2 - b_d \rho_d(t) \right) > 0 \) is sufficient to ensure \( \|p_{21}(t)\| > 0 \) for all \( t \geq 0 \). On the other hand, \( \bar{b}_d \) can be chosen arbitrarily without affecting the positiveness of \( \|p_{21}(t)\| \). Furthermore, as boundedness of \( r_k \) in (3.7) implies \( \|p_{ki}(t)\|, \|p_{kj}(t)\| > 0 \), from boundedness of \( r_k^* \) in (3.12) and the fact that \( \max(\rho_{r_k}(t)) = \rho_{r_k}(0) = 1 \), setting any bounded arbitrary values for \( \bar{b}_{r_k} \) and \( \bar{b}_{r_k} \) in (3.14) ensures \( \|p_{ki}(t)\|, \|p_{kj}(t)\| > 0 \), for \( (k, i), (k, j) \in \mathcal{E} \backslash \{(2, 1)\} \) and \( \forall t \geq 0 \). Moreover, notice that since the edge-angles are defined over the domain \( \alpha_{kij} \in [0, 2\pi) \), from (3.13), by setting \( \bar{b}_{\alpha_k} \leq \alpha_{kij}^* \) and \( \bar{b}_{\alpha_k} \leq 2\pi - \alpha_{kij}^* \) in (3.14), we can enforce this domain for the edge-angles, which avoids sudden changes from \( 2\pi \) to \( 0 \). This further ensures continuous angle errors in (3.13) leading to a smooth control action. Finally, notice that each agent can observe its initial formation errors and select \(-b_h \) and \( \bar{b}_h \), \( h \in \{ d, r_k, \alpha_k \}, k = \{3, \ldots, n \}\) in agreement with the above conditions to further ensure the requirement of \(-b_h \rho_h(0) < e_h(0) < \bar{b}_h \rho_h(0)\).

### Transformed Errors

The problem of designing a controller that meets the error constraints in (3.14) can be transformed into establishing the boundedness of certain modulated error signals [53, 54]. More specifically, to handle the time-varying constraints in (3.14), a time-varying error transformation technique will be used to convert each of the original error dynamics \( \dot{e}_d, \dot{r}_{r_k}, \) and \( \dot{e}_{\alpha_k} \) (given in (3.36), (3.41), respectively) under the constraints (3.14) into equivalent unconstrained ones, whose stability merely ensure satisfaction of the constraints given in (3.14). First, we define the modulated formation errors as:

\[
\tilde{e}_h(t) := \frac{e_h(t)}{\rho_h(t)}, \quad h \in \{ d, r_k, \alpha_k \}, \quad k = \{3, \ldots, n \}.
\] (3.16)
To transform the constrained error dynamics (in the sense of (3.14)) into an equivalent unstrained one, we introduce the following error transformation:

\[
\sigma_h := T_h(\tilde{e}_h), \quad h \in \{d, r_k, \alpha_k\}, \quad k = \{3, \ldots, n\},
\]

where \(\sigma_h, h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\}\), are the transformed errors corresponding to \(e_h\). Moreover, \(T_h(\cdot) : (-\bar{b}_h, \bar{b}_h) \rightarrow (-\infty, +\infty)\), denote smooth, strictly increasing bijective mappings satisfying \(T_h(0) = 0\). Note that \(e_h = 0\) if and only if \(\sigma_h = 0\). Finally, notice that maintaining the boundedness of \(\sigma_h(t)\), enforces \(-\bar{b}_h < \tilde{e}_h(t) < \bar{b}_h\), and consequently the satisfaction of (3.14). Taking the time derivatives of (3.17), yields:

\[
\dot{\sigma}_h = \xi_h(\tilde{e}_h - \tilde{e}_h \dot{\rho}_h), \quad h \in \{d, r_k, \alpha_k\}, \quad k = \{3, \ldots, n\},
\]

where

\[
\xi_h := \frac{1}{\rho_h(t)} \frac{\partial T_h(\tilde{e}_h)}{\partial \tilde{e}_h} > 0, \quad h \in \{d, r_k, \alpha_k\}, \quad k = \{3, \ldots, n\}.
\]

In the sequel, we shall consider the following logarithmic function as a proper choice for the mapping functions in (3.17):

\[
\sigma_h = T_h(\tilde{e}_h) = \ln \left( \frac{\tilde{b}_h \tilde{e}_h + \tilde{b}_h \bar{b}_h}{\tilde{b}_h \bar{b}_h - \tilde{b}_h \tilde{e}_h} \right),
\]

where \(h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\}\). Note that, the specific form in (3.20) satisfies the aforementioned properties for \(T_h(\cdot)\) and \(\tilde{e}_h \in (-\bar{b}_h, \bar{b}_h)\) if and only if \(\sigma_h \in (-\infty, +\infty)\).

**Remark 3.4 (PPC Design Philosophy).** When \(-\bar{b}_h \rho_h(0) < e_h(0) < \bar{b}_h \rho_h(0), h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\}\), based on the properties of the error transformations (3.17), prescribed performance in the sense of (3.14) is achieved, if \(\sigma_h(t)\) are kept bounded. Notice that, although for \(\sigma_h \in \mathbb{R}\) the prescribed performance bounds in (3.14) are satisfied, the boundedness of \(\sigma_h\) is required to guarantee well-defined bounded control inputs. Moreover, it is important to note that the specific bounds of \(\sigma_h\) (no matter how large they are, which is the key property of the adopted error transformation) do not affect the achieved transient and steady-state performance on \(e_h(t)\), which is solely determined by (3.14) and thus by the selection of the performance functions \(\rho_h(t)\) as well as the scaling constants \(\bar{b}_h\) and \(\bar{b}_h\).

**Proposed Control Laws**

The following lemma is useful for the control design and stability analysis.

**Lemma 3.2.** For a given triangular directed sub-graph as in Fig. 3.1a, the bipolar coordinates basis \(\hat{\alpha}_k\) (see Fig. 3.2b) associated with the virtual Cartesian frame \(\{C_k\}\) in Fig. 3.2a can be expressed with respect to the global coordinate system as follows:

\[
\hat{\alpha}_k = -f_1(r_k, \alpha_{kij})z_{ji} + f_2(r_k, \alpha_{kij})J^T z_{ji},
\]

\[
\hat{r}_k = f_2(r_k, \alpha_{kij})z_{ji} + f_1(r_k, \alpha_{kij})J^T z_{ji},
\]

\[
\hat{\alpha}_k = \frac{1}{2} \left( \frac{\bar{b}_h \tilde{e}_h + \bar{b}_h \bar{b}_h}{\bar{b}_h \bar{b}_h - \bar{b}_h \tilde{e}_h} \right),
\]

where \(h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\}\). Note that, the specific form in (3.20) satisfies the aforementioned properties for \(T_h(\cdot)\) and \(\tilde{e}_h \in (-\bar{b}_h, \bar{b}_h)\) if and only if \(\sigma_h \in (-\infty, +\infty)\).

**Proposed Control Laws**

The following lemma is useful for the control design and stability analysis.

**Lemma 3.2.** For a given triangular directed sub-graph as in Fig. 3.1a, the bipolar coordinates basis \(\hat{\alpha}_k\) (see Fig. 3.2b) associated with the virtual Cartesian frame \(\{C_k\}\) in Fig. 3.2a can be expressed with respect to the global coordinate system as follows:

\[
\hat{\alpha}_k = -f_1(r_k, \alpha_{kij})z_{ji} + f_2(r_k, \alpha_{kij})J^T z_{ji},
\]

\[
\hat{r}_k = f_2(r_k, \alpha_{kij})z_{ji} + f_1(r_k, \alpha_{kij})J^T z_{ji},
\]

\[
\hat{\alpha}_k = \frac{1}{2} \left( \frac{\bar{b}_h \tilde{e}_h + \bar{b}_h \bar{b}_h}{\bar{b}_h \bar{b}_h - \bar{b}_h \tilde{e}_h} \right),
\]
for $k \geq 3$, $(k, i), (k, j) \in \mathcal{E}\backslash\{(2, 1)\}$, $i < j < k$, where $z_{ji}$ is the bearing vector associated with edge $(j, i)$, $J^\top$ is the $\frac{\pi}{2}$ clockwise rotation matrix (see (3.5)), and

$$f_1(r_k, \alpha_{kij}) = \frac{-\sinh r_k \sin \alpha_{kij}}{\cosh r_k - \cos \alpha_{kij}}, \quad (3.22a)$$

$$f_2(r_k, \alpha_{kij}) = \frac{\cos \alpha_{kij} \cosh r_k - 1}{\cosh r_k - \cos \alpha_{kij}}. \quad (3.22b)$$

**Proof.** See Subsection 3.10.1.

Notice that in the proposed formation control setup the leader (agent 1) does not participate in forming the desired shape, thus its behaviour is independent from the other agents. In this respect, the leader’s control law $u_L(t)$ is designed for objectives such as trajectory tracking, position stabilization, etc., in the presence of external disturbances/uncertainties $\delta_1(t)$. Note that, the response of the leader should be stable, therefore, in the following we will assume $u_L : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ is continuous and uniformly bounded in time and is designed for fulfilling a desirable high-level formation coordination task.

We propose the following formation control laws:

$$u_1 = u_L(t) \quad (3.23a)$$

$$u_2 = \xi_d \sigma_d p_{21} \quad (3.23b)$$

$$u_k = -\xi_{rk} \sigma_{rk} \hat{r}_k - \xi_{\alpha_k} \sigma_{\alpha_k} \hat{\alpha}_k, \quad k = 3, \ldots, n, \quad (3.23c)$$

where $\hat{\alpha}_k$ and $\hat{r}_k$ are the bipolar coordinates basis associated with agent $k \geq 3$ obtained in (3.21), and from (3.19) and (3.17), $\xi_h, h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\}$ is given by:

$$\xi_h = \frac{1}{\rho_h(t)} \left( \frac{1}{\hat{e}_h + \bar{b}_h} - \frac{1}{\hat{e}_h - \bar{b}_h} \right), \quad (3.24)$$

which is lower bounded by a positive constant over its domain $\hat{e}_h \in (-\bar{b}_h, \bar{b}_h)$ owing to strict positiveness of $\rho_h(t)$.

**Remark 3.5.** The control law (3.23c) indicates that the motion of the follower $k \geq 3$ results by the superposition of the motions along each of the orthogonal bipolar coordinate basis $\hat{r}_k, \hat{\alpha}_k \in \mathbb{R}^2$ to compensate the formation errors $e_{r_k}, e_{\alpha_k}$ with the given constraints in (3.14). In this respect, notice that one can select the performance bounds (error constraints) on $e_{r_k}, e_{\alpha_k}$ in (3.14) arbitrarily without any constraint infeasibility issues since these two error variables can vary independently along their respective bipolar coordinate basis, i.e., $e_{r_k}, e_{\alpha_k}$ are not interdependent/coupled.

**Remark 3.6.** Notice that for implementing (3.23c), from (3.21), agent $k$ should know $z_{ji}$, which is the relative bearing between its neighbours $i < j \in \mathbb{N}$. We argue that agent $k$ can obtain $z_{ji}$ by direct measurements of $z_{ki}, z_{kj}$, and the ratio of the
3.6. Stability Analysis

Example sentences from the document:

distances $r_{kij}, i < j < k$, which are available. First, notice that $p_{ji} = p_{ki} - p_{kj} = \|p_{ki}\|z_{ki} - \|p_{kj}\|z_{kj}$. Let $z_k := r_{kij}z_{ki} - z_{kj} \in \mathbb{R}^2$. One can verify that $z_k$ is parallel with $p_{ji}$; consequently, normalizing $z_k$ gives $z_{ji}$.

Although the proposed control laws (3.23) are given with respect to a global coordinate frame (only for the sake of analysis), we emphasize that the proposed formation controller can be implemented in any arbitrarily oriented local coordinate frame (i.e., in a coordinate-free fashion). First, notice that according to the leader’s objective (e.g., going to a specific position or following a trajectory, etc.), since it is not involved in the process of generating the desired shape, the leader can perform the required calculations for its control law with respect to its own local coordinate frame. Second, let $g_h(e_h) := \xi_h \sigma_h, h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\}$ in (3.23b), (3.23c), where all $g_h(e_h)$ are scalar functions of the formation errors. Now let the superscript $[k], k \geq 2$, indicates a quantity expressed in the local coordinate frame of the $k$-th agent. Furthermore, suppose that $R_k \in \text{SO}(2)$ is the transformation (rotation) matrix from the $k$-th local frame to the global frame. Notice that, we have $u_k = R_ku_k^{[k]}$, $p_{ki} = R_kp_{ki}^{[k]} = R_k(p_{i}^{[k]} - p_{k}^{[k]})$, and consequently from (3.3) we get $z_{ki} = R_kz_{ki}^{[k]}, i < k \in \mathbb{N}$. Considering (3.23c), we have:

$$u_k^{[k]} = R_k^{-1}u_k = -R_k^{-1}(-g_{r_k}(e_{r_k})\hat{r}_k - g_{\alpha_k}(e_{\alpha_k})\hat{\alpha}_k),$$

where from (3.21) we get:

$$\hat{\alpha}_k^{[k]} = R_k^{-1}\hat{\alpha}_k = f_2 R_k^{-1}z_{ji} + f_1 J^\top R_k^{-1}z_{ji} = f_2 z_{ji}^{[k]} + f_1 J^\top z_{ji}^{[k]},$$

and the permutation property between $R_k^{-1}$ and $J^\top$ is employed to achieve the right-hand sides (since both of them are rotation matrices). Note that, the values of the scalar functions $f_1(\alpha_{kij}, r_k), f_2(\alpha_{kij}, r_k)$ as well as $g_h(e_h) := \xi_h \sigma_h, h \in \{d, r_k, \alpha_k\}, k = \{3, \ldots, n\}$ do not depend on the coordinate systems since their arguments (i.e., edge-angles, logarithm of ratio of distances, and their errors) are the same in any coordinate system. One can verify that (3.26) and (3.25) have the same form as (3.21) and (3.23c), respectively, where all the quantities are expressed with respect to the $k$-th local coordinate frame. This indicates that the decentralized control law (3.23c) can be implemented in arbitrarily oriented local coordinate frame of agent $k$. The same claim can be verified in a similar manner for the control law of agent 2 (secondary leader) in (3.23b).

3.6 Stability Analysis

This chapter’s main results are summarized in the following theorems. Theorem 3.1 indicates that the secondary leader (i.e., agent 2) can keep a certain (in general time-


varying) distance with respect to the leader (which can move freely). Compensation of the formation errors assigned to the followers (agents $k \geq 3$) are derived in Theorem 3.2.

**Theorem 3.1.** Consider agents 1 and 2 with dynamics (3.1) and a desired formation given by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ under Assumption 3.1, where the desired (time-varying) distance $d_{21}^*(t)$ between agents 1 and 2 is such that $\inf_{t \geq 0} (d_{21}^*(t)) > 0$. Given that $-b_d \rho_d(0) < e_d(0) < \bar{b}_d \rho_d(0)$, where $b_d \rho_d(t)$ and $\bar{b}_d$ are chosen as explained in Section 3.5, the decentralized control protocols (3.23a), (3.23b) guarantee $-b_d \rho_d(t) < e_d(t) < \bar{b}_d \rho_d(t)$ and $\|p_{21}(t)\| > 0$, for all $t \geq 0$ as well as boundedness of all closed-loop signals.

**Proof.** See Subsection 3.10.2 for the proof. □

**Theorem 3.2.** Consider a group of $n$ agents with dynamics (3.1) in a 2-D plane. Let the desired formation be given by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ under Assumption 3.1 along with the sets of logarithms of the desired ratio of distances $r_k^*$ and desired edge-angles $\alpha_{kij}^*$, $(k, i), (k, j) \in \mathcal{E}\setminus\{(2, 1)\}$, $i < j < k$, assigned to agents $3 \leq k \leq n$. Assume that $-b_h \rho_h(0) < e_h(0) < \bar{b}_h \rho_h(0)$, where $b_h \rho_h(t)$ and $\bar{b}_h$ are chosen as in Section 3.5. Under the stability results of Theorem 3.1, the decentralized control protocol (3.23a) guarantees, $-b_h \rho_h(t) < e_h(t) < \bar{b}_h \rho_h(t)$ and $\|p_{k1}(t)\|, \|p_{ij}(t)\| > 0$, $(k, i), (k, j) \in \mathcal{E}\setminus\{(2, 1)\}$, for all $t \geq 0$, as well as boundedness of all closed-loop signals.

**Proof.** See Subsection 3.10.3 for the proof. □

**Remark 3.7.** We highlight that our results in Theorems 3.1 and 3.2 indicate almost global convergence to the desired formation. In particular, satisfaction of $-b_d \rho_d(0) < e_d(0) < \bar{b}_d \rho_d(0)$ in Theorem 3.1 as well as satisfaction of $-b_r \rho_r(0) < e_r(0) < \bar{b}_r \rho_r(0)$ in Theorem 3.2, along with choosing $b_d \rho_d(t)$, $\bar{b}_d$, $b_r$, and $\bar{b}_r$ according to Section 3.5 require agents not to be initially collocated with their neighbors (i.e., $\|p_{21}(0)\| \neq 0, \|p_{k1}(0)\| \neq 0, \|p_{kj}(0)\| \neq 0, (k, i), (k, j) \in \mathcal{E}\setminus\{(2, 1)\}$, $i < j < k$). Moreover, satisfaction of $-b_{\alpha_k} \rho_{\alpha_k}(0) < e_{\alpha_k}(0) < \bar{b}_{\alpha_k} \rho_{\alpha_k}(0)$ in Theorem 3.2 along with choosing $\bar{b}_{\alpha_k}$ and $b_{\alpha_k}$ according to Section 3.5 requires $0 < \alpha_{kij}(0) < 2\pi$, which affects the acceptable initial positions for agent $k \geq 3$ with respect to its neighbors, i.e., agent $k$ should not be initially collinear with agents $i$ and $j$ while locating on left- or right-hand side of them (being in the middle is feasible since in this case $\alpha_{kij}(0) = \pi$). Additionally, note that the target formation should respect the aforementioned conditions as well. Finally, we argue that the above restrictions only constitute a zero-measure set to avoid global convergence to the desired shape.

**Remark 3.8.** Recall that in Part 3 of Assumption 3.1, we ensure the triangulation of the formation graph, representing the sensing topology among agents. It is important to note that our proposed decentralized controllers guarantee the avoidance of zero distances between neighboring agents ($\|p_{ji}\| \neq 0, \forall (j, i) \in \mathcal{E}, \forall t \geq 0$), which is
crucial for all edge-angles \( \alpha_{kij}, (k, i), (k, j) \in E \setminus \{(2, 1)\} \) to remain well-defined. However, in practice, even if two non-neighboring agents collide, they cannot attain a zero distance due to their physical size. As a result, the edge-angles defined based on non-neighboring agents always remain well-defined. Therefore, in practice, the triangulation assumption of the formation graph in this work is not necessary for the proposed method to ensure global convergence to the desired formation shape and it mainly serves as a technical requirement.

### 3.7 Formation Control with Orientation Adjustment

Followed by (3.10), owing to the above results and the proposed control laws in (3.23), the leader determines the position of the formation (e.g., by tracking a reference velocity/trajectory that leads to formation maneuvering), the secondary leader determines the formation scale by tracking a time-varying desired distance \( d_{21}^{\ast}(t) \), and the followers contribute to obtain the desired shape based on the defined errors in their local bipolar coordinate system. In this section, we shall extend the aforementioned results to formation control with orientation adjustment, where by using an extended control law for the secondary leader, we can obtain a certain (in general, time-varying) desired orientation for the formation as well.

Let \( \beta := \arctan2(z_{21,y}, z_{21,x}) \in (-\pi, \pi] \) be the bearing angle between agents 2 and 1, where \( z_{21} = [z_{21,x}, z_{21,y}]^T \) denotes the corresponding bearing vector. Now consider the following new control requirement in addition to the ones in (3.10):

\[
\beta(t) \rightarrow \beta^\ast, \quad \text{as} \quad t \to \infty
\]  

where \( \beta^\ast \) is a desired bearing angle associated with a desired bearing vector \( z_{21}^\ast \) between agents 2 and 1. Note that \( \beta^\ast \) determines the desired orientation for the formation. In this regard, \( \beta^\ast \) and \( d_{21}^\ast \) determine a desired relative position vector \( p_{21}^\ast \) between agents 2 and 1, since \( p_{21}^\ast = d_{21}^\ast z_{21}^\ast \). Indeed satisfaction of (3.27) is equivalent to: \( p_2(t) - p_1(t) \rightarrow p_{21}^\ast \), as \( t \to \infty \), see Fig. 3.3 for further illustrations.

In this way the secondary leader not only controls the scale of the desired formation at the steady-state, but also alters the formation orientation by modifying its desired bearing angle with respect to the leader. In other words, satisfaction of (3.27) along with (3.10) achieves the desired formation only up to translations. Note that, given a \( p_{21}^\ast \) in a global coordinate system, one can always find \( d_{21}^\ast \) and \( \beta^\ast \). Having \( p_{21}^\ast \) defined in a global coordinate system is not restrictive since it is only required to be accessible to agent 2, hence, as it is shown in Fig. 3.3, one can always consider agent 2’s local coordinate frame as the reference frame in which the formation orientation is determined.

Let us define the bearing angle error between agent 2 and 1 as:

\[
e_\beta = \beta(t) - \beta^\ast(t),
\]

\( \arctan2 \) is the two argument arc tangent function, see [146, Appendix A].
Figure 3.3: Given a desired sensing graph $G$ as in Fig. 3.1b, in each case the desired formation is characterized by different desired relative positions between agents 2 and 1, whereas the sets of desired edge-angles and ratio of the distances for followers $(i \geq 3)$ are the same. The dashed arrows show the local coordinate frame of agent 2 in which the formation orientation can be characterized by the desired bearing angle $\beta^*$. $p^*_{21,a}$ and $p^*_{21,b}$ have the same orientation but different length while $p^*_{21,a}$ and $p^*_{21,c}$ have different orientations with the same length.

where $\beta^*(t) : \mathbb{R} \to (-\pi, \pi)$ is a continuously differentiable function of time with a bounded derivative representing the desired (in general, time-varying) orientation between agents 2 and 1 (which determines the desired formation orientation). Similarly to Section 3.5, the PPC method can be adopted to design a robust control law for practical stabilization of $e_\beta$ such that:

$$-b_\beta \rho_\beta(t) < e_\beta(t) < \bar{b}_\beta \rho_\beta(t)$$  \hspace{1cm} (3.29)

by utilizing an unconstrained transformed error $\sigma_\beta = \mathcal{T}_\beta(\bar{e}_\beta)$, where $b_\beta, \bar{b}_\beta > 0$, and $\rho_\beta, \bar{e}_\beta, \sigma_\beta, \xi_\beta$ are defined similarly to (3.15), (3.16), (3.20), and (3.24), respectively. Moreover, akin to Section 3.5, to ensure $\beta \in (-\pi, \pi)$ and avoid potential singularities (i.e., $\beta = \pi$ or $-\pi$) we should have $b_\beta \rho_\beta(t) \geq \pi + \beta^*(t)$ and $\bar{b}_\beta \rho_\beta(t) \leq \pi - \beta^*(t)$ for all $t \geq 0$. Hence, $\inf_{t \geq 0} (\pi + \beta^*(t) - b_\beta \rho_\beta(t)) \geq 0$ and $\inf_{t \geq 0} (\pi - \beta^*(t) - \bar{b}_\beta \rho_\beta(t)) \geq 0$ are sufficient to ensure $\beta \in (-\pi, \pi)$ for all $t \geq 0$. We thus propose the following extended control law for agent 2:

$$u_2 = \xi_d \sigma_d p_{21} + \xi_\beta \sigma_\beta Jz_{21},$$  \hspace{1cm} (3.30)

which indicates that the motion of secondary leader is resulted from the superposition of the motions along orthogonal directions $z_{21}$ and $Jz_{21} = z_{21}^T$ to compensate the distance error $e_d$ and the bearing angle error $e_\beta$, with the given constraints in (3.14) and (3.29), respectively.

The aforementioned results are summarized in Corollary 3.1 which along with Theorem 3.2 extends the results to formation control with scaling and orientation adjustment.

**Corollary 3.1.** Consider agents 1 and 2 with dynamics (3.1) and a desired formation given by a directed graph $G = (V, E)$ under Assumption 3.1 with a desired...
3.8. Simulations Results

(time-varying) distance $d_{21}^s(t)$ (such that $\inf_{t \geq 0} (d_{21}^s(t)) > 0$), and a desired bearing angle $-\pi < \beta^*(t) < \pi$ between agents 1 and 2. Given $-\dot{b}_h \rho_h(0) < \epsilon_h(0) < \dot{b}_h \rho_h(0)$, $h \in \{d, \beta\}$, where $\dot{b}_h \rho_h(t)$ and $b_h$ are chosen as explained in Sections 3.3 and 3.7 the decentralized control protocols (3.23a) and (3.30) guarantee $-\dot{b}_h \rho_h(t) < \epsilon_h(t) < \dot{b}_h \rho_h(t)$ and $\|p_{21}(t)\| > 0$, for all $t \geq 0$ as well as boundedness of all closed-loop signals.

Proof. The proof is similar to Theorem 3.1. Notice that based on the control law (3.30), $e_d$ and $e_\beta$ can also be independently treated (due to orthogonality of the control directions). \hfill \Box

3.8 Simulations Results

In this section, a simulation example of robust formation maneuvering with orientation and scaling control is presented to demonstrate the effectiveness of the proposed decentralized control protocols (3.23a), (3.30), and (3.23c)\footnote{A short video demonstrating the following simulation results can be found at: \url{https://youtu.be/jtsiU9DLp1k}}.

Consider a group of six agents modeled by (3.1) in a two-dimensional space. Suppose that the desired formation is an equilateral triangle composed of four equilateral sub-triangles (see Fig. 3.4), where its underlying sensing graph $G$ satisfies Assumption 3.1 with the following directed edge set:

$$
\mathcal{E} = \{(2, 1), (3, 1), (3, 2), (4, 2), (4, 3), (5, 2), (5, 4), (6, 3), (6, 5)\}.
$$

Let the desired formation be characterized by the following sets of desired log-arithmetic ratio of the distances and edge-angles: $r_3^* = r_4^* = r_6^* = 0$, and $\alpha_{312}^* = \alpha_{634}^* = \pi/3$, $\alpha_{423}^* = \alpha_{524}^* = 5\pi/3$, where $d_{31}^s = d_{32}^s = d_{42}^s = d_{43}^s = d_{52}^s = d_{53}^s = d_{63}^s = d_{64}^s = 1.875$. Moreover, assume that the local coordinate system of agent 2 (secondary leader), in which the formation orientation is defined, is aligned with the X-Y axis of the global coordinate system.

Fig. 3.5 shows the continuously differentiable time-varying reference signals $d_{21}^s(t)$ and $\beta^*(t)$ of agent 2 for adjusting the formation scale and orientation as times goes on. Note that $d_{21}(t)$ and $\beta^*(t)$ are initially constant, where $d_{21}^s(0 \leq t \leq 16) = 1.875$ and $\beta^*(0 \leq t \leq 13) = 0$. Without loss of generality, in the simulation we have assumed that the disturbance input to agent 1 (leader) is zero, $\delta_1(t) = 0$, and the leader follows a sinusoidal trajectory under the velocity control input of $u_L(t) = [1.25, \frac{\pi}{4} \cos(\frac{\pi}{4} t)]^\top$. The external disturbances of agents $\delta_k := [\delta_{kx}, \delta_{ky}]^\top$, $k = 2, \ldots, 6$, in the simulation are assumed to be: $\delta_{2x}(t) = \delta_{6y}(t) = 0.75 \sin(4t + \frac{\pi}{4}) + 0.5 \sin(2t + \frac{3\pi}{4})$, $\delta_{2y}(t) = \delta_{4y}(t) = \delta_{5x}(t) = 0.25 \cos(3t + \frac{\pi}{2}) + 0.75 \sin(2t - \frac{\pi}{2})$, $\delta_{3x}(t) = 0.75 \sin(t)$, $\delta_{3y}(t) = 0.25 \cos(t + \frac{\pi}{4}) + 0.25 \sin(2t + \frac{\pi}{4})$, $\delta_{4x}(t) = 0.5 \cos(5t + \frac{\pi}{8}) + 0.5 \sin(t + \frac{\pi}{4})$, $\delta_{5y}(t) = 0.5 \cos(t)$, $\delta_{6x}(t) = 0.5 \sin(2t + \frac{\pi}{4})$. Furthermore, the parameters of the performance functions (3.15) are considered as $l_h = 0.5$, $\rho_{x,k} = 0.04$, $h \in \{\beta, r_k, \alpha_k\}$, $k = \{3, \ldots, 6\}$, and $l_d = 0.5$, $\rho_{x,d} = 0.03$. \footnote{A short video demonstrating the following simulation results can be found at: \url{https://youtu.be/jtsiU9DLp1k}}
Leader-Follower Directed Formation Control Based on Bipolar Coordinates with Global Convergence

Figure 3.4: Starting from arbitrary initial positions, agents converge to the desired shape while following the leader’s (agent 1’s) motion. The scale and orientation of the formation is adjusted by agent 2 along the way. In particular, roughly around $t = 14$ agent 2 starts following a time-varying desired bearing and distance w.r.t. agent 1 that leads the formation to pass through a narrow passage (black curves).

Figure 3.5: Agent 2’s desired (time-varying) distance $d_{21}^*(t)$ and bearing angle $\beta^*(t)$.

Figure 3.6: Evolution of agent 2’s squared distance ($e_d$) and bearing angle ($e_\beta$) errors. The magnified subplots provide the details of error evolution in the steady-state.

Moreover, the positive constants $b_h, \overline{b}_h, h \in \{d, \beta, r_k, \alpha_k\}, k = \{3, \ldots, 6\}$ of the performance bounds are selected according to the guidelines in Sections 3.5 and 3.7.

Considering the aforementioned setting as well as a set of arbitrary initial positions for the agents, the results are summarized in Fig. 3.4, which depicts consecutive snapshots of the agents’ trajectories towards the desired formation as the leader follows its reference trajectory. Note that agent 2 starts tracking a time-varying bearing angle with respect to agent 1 from $t > 13$ (see Fig. 3.5) such that it follows the angle of the leader’s velocity direction with an offset of $\pi/6$ radians. In addition, the evolution of $d_{21}^*(t)$ during $16 < t < 26$ allows agents to pass through
3.9 Conclusions

In this chapter, we proposed a novel 2-D directed formation control approach with (almost) global convergence using bipolar coordinates for desired shapes that are modeled by acyclic triangulated directed graphs (also known as a class of minimally acyclic persistent graphs). Bipolar coordinates were used to characterize the desired formation to avoid undesired equilibria in 2-D coordinate-free directed formations. Then the prescribed performance control method was adopted for designing the formation control laws to introduce robustness against external disturbances/model uncertainties as well as ensuring user-defined transient and steady-state performance guarantees. We further showed that the proposed approach is capable of handling formation maneuvering with time-varying reference velocities along with scaling and orientation adjustment. Moreover, it was argued that the control ap-

Figure 3.7: Evolution of the edge-angle errors $\alpha_k, k \geq 3$. The magnified subplots provide the details of error evolution in the steady-state.

a narrow passage without colliding with obstacles when following the leader and maintaining the desired shape. The evolution of the edge-angle (3.13) and the logarithmic ratio of the distances (3.12) errors for agents $3 \leq k \leq 6$ are depicted in Fig. 3.7 and Fig. 3.8, respectively, where the dashed lines indicate the user-defined performance bounds. Moreover, Fig. 3.6 shows the evolution of bearing angle error (3.28) as well as the squared distance error (3.11) of agent 2. Notice that, the formation errors remain within the pre-defined performance bounds for all time. Hence, the results indicate that the proposed formation control scheme is capable of handling the problem of coordinate-free (stationary or maneuvering) formation control with adjustable scaling and orientation as well as global shape convergence under prescribed performance specifications that further introduce robustness to external disturbances.
proach can be easily implemented in arbitrarily oriented local coordinate frames of the (follower) agents by using onboard vision sensors, which are favorable for practical applications.

From a controller design perspective, it is crucial to recognize that using bipolar coordinates in this chapter not only facilitated the unique characterization of the desired formation shape but also enabled the generation of independent (orthogonal) formation error signals. This, in turn, paved the way for employing the PPC method in control design (refer to Remark 3.5). Specifically, due to the independence of formation error signals, the time-varying (performance) constraints defined in (3.14) are effectively decoupled, remaining feasible for any choice of time-varying bounds in (3.14). This property plays a pivotal role in the applicability of the conventional PPC method for control design. In the subsequent chapters of this thesis, we will focus on control design for uncertain nonlinear systems under potentially coupled time-varying (output) constraints, where it is not possible to directly apply the existing PPC designs.

3.10 Proofs of Lemmas, Theorems, and Some Technical Derivations

3.10.1 Proof of Lemma 3.2

From (3.8) recall that the position of node $k$ w.r.t. $\{C_k\}$ is $p_k^{[C_k]} = [x_k^{[C_k]}, y_k^{[C_k]}]$. Let $\hat{x}_k^{[C_k]}, \hat{y}_k^{[C_k]} \in \mathbb{R}^2$ be the (unit) orthogonal bases of $\{C_k\}$ that are expressed in $\{C_k\}$, hence:

$$p_k^{[C_k]} = x_k^{[C_k]} \hat{x}_k^{[C_k]} + y_k^{[C_k]} \hat{y}_k^{[C_k]} , \quad (3.31)$$
where, \( x_k^{[C_k]} \) and \( y_k^{[C_k]} \) are given in (3.8). It is known that for the bipolar coordinates (that is an orthogonal curvilinear coordinate system) associated with \( \{C_k\} \), the two vectors \( \alpha_k^{[C_k]} \), \( \beta_k^{[C_k]} \) form a local basis at any nonsingular point \( p_k^{[C_k]} \), where the following hold [142]:

\[
\frac{\partial p_k^{[C_k]}}{\partial \alpha_{kij}} = q_{\alpha_k} \hat{\alpha}_k^{[C_k]}, \quad (3.32a)
\]

\[
\frac{\partial p_k^{[C_k]}}{\partial r_k} = q_{r_k} \hat{r}_k^{[C_k]}, \quad (3.32b)
\]

in which

\[
q_{\alpha_k} = q_{r_k} = \frac{c_k}{\cosh r_k - \cos \alpha_{kij}}, \quad (3.33)
\]

are the scaling (metrical) factors [142, 143] and \( c_k \) was defined in (3.8). From (3.8), (3.31), (3.32), and (3.33) one can obtain:

\[
\hat{\alpha}_k^{[C_k]} = f_1(r_k, \alpha_{kij}) \hat{x}_k^{[C_k]} + f_2(r_k, \alpha_{kij}) \hat{y}_k^{[C_k]}, \quad (3.34a)
\]

\[
\hat{r}_k^{[C_k]} = -f_2(r_k, \alpha_{kij}) \hat{x}_k^{[C_k]} + f_1(r_k, \alpha_{kij}) \hat{y}_k^{[C_k]}, \quad (3.34b)
\]

where \( f_1, f_2 \) are given in (3.22). Notice that:

\[
\hat{x}_k^{[C_k]} = -\frac{p_{ji}}{\|p_{ji} \|} = -z_{ji}, \quad (3.35a)
\]

\[
\hat{y}_k^{[C_k]} = -J \frac{p_{ji}}{\|p_{ji} \|} = -J z_{ji} = J^T z_{ji}, \quad (3.35b)
\]

which provide the basis of \( \{C_k\} \) in the global coordinate frame. Finally, (3.34) and (3.35) yield (3.21).

### 3.10.2 Proof of Theorem 3.1

The proof proceeds in three phases: First, we show that \( \hat{e}_d(t) \) remains within \((-\hat{b}_d, \hat{b}_d)\) for a specific time interval \([0, \tau_{2, \text{max}}]\) (i.e., the existence and uniqueness of a maximal solution) by noticing to the fact that the closed-loop system of \( \hat{e}_d \) is locally Lipschitz continuous. Next, by establishing boundedness of the mapped error \( \sigma_d \) we prove that the proposed control scheme guarantees, for all \([0, \tau_{2, \text{max}}]\): a) the boundedness of all closed loop signals as well as b) that \( \hat{e}_d(t) \) remains strictly in a compact subset of \((-\hat{b}_2, \hat{b}_2)\), which leads to \( \tau_{2, \text{max}} = \infty \) (i.e., forward completeness), thus finalizing the proof.

**Phase I.** First notice that agent 2’s formation error dynamics can be obtained invoking (3.1), (3.11), (3.23a) as follows:

\[
\dot{e}_d = 2p_{21}^T \dot{p}_{21} - 2d_{21}^* \dot{d}_{21}^* = 2p_{21}^T (u_L - u_2 + \delta_{21}) - 2d_{21}^* \dot{d}_{21}^*
\]

\[
= 2p_{21}^T (A_2 - u_2) - \Gamma_d, \quad (3.36)
\]
where $\delta_{21} := \delta_1 - \delta_2 \in \mathbb{R}^2$, $\Lambda_2 := u_L + \delta_{21} \in \mathbb{R}^2$, $\Gamma_d := 2\dot{d}_{21}^* \dot{d}_{21}^* \in \mathbb{R}$ are uniformly bounded signals ($\delta_{21}, \Lambda_2, \Gamma_d \in \mathcal{L}_\infty$) by assumption. Now differentiating $\dot{e}_d(t)$ in (3.16) and employing (3.36), (3.23b), yields:

$$
\dot{e}_d := E\hat{e}_d(t, \hat{e}_d) = \rho_d^{-1}(t) (\hat{e}_d - \hat{e}_d \hat{\rho}_d(t))
= \rho_d^{-1}(2p_{21}^T \Lambda_2 - 2\xi_d \sigma_d \|p_{21}\|^2 - \Gamma_d - \hat{e}_d \hat{\rho}_d) .
$$

(3.37)

Let us also define the open set $\Omega_{\tilde{e}_d}$ as: $\Omega_{\tilde{e}_d} := (-b_d, b_d)$. Note that, $\Omega_{\tilde{e}_d}$ is nonempty and open by construction. Moreover, followed by the discussion in Section 3.5, agent 2 can always initially select $b_d, b_d > 0$ to ensure $\tilde{e}_d(0) \in \Omega_{\tilde{e}_d}$. Additionally, $E\hat{e}_d(t, \hat{e}_d)$ is continuous on $t$ and locally Lipschitz on $\hat{e}_d$ over the set $\Omega_{\tilde{e}_d}$. Therefore, the hypotheses of Theorem 2.3 hold and the existence and uniqueness of a maximal solution $\hat{e}_d(t)$ of (3.37) for a time interval $[0, \tau_{2,\text{max}}]$ such that $\hat{e}_d(t) \in \Omega_{\tilde{e}_d}, \forall t \in [0, \tau_{2,\text{max}}]$ is guaranteed. Based on this, we can further infer that $e_d(t)$ is bounded as in (3.14) for all $t \in [0, \tau_{2,\text{max}}]$.

Phase II. Owing to $\hat{e}_d(t) \in \Omega_{\tilde{e}_d}, \forall t \in [0, \tau_{2,\text{max}}]$, the error $\sigma_d$, as defined in (3.17), is well-defined for all $t \in [0, \tau_{2,\text{max}})$. Therefore, consider the following positive definite and radially unbounded Lyapunov function candidate: $V_2 = (1/4)\sigma_d^2$. Taking the time derivative of $V_2$, invoking (3.18), (3.36), (3.23b), and the positivity of $\xi_d$, we get:

$$
V_2 = -\xi_d^2 \sigma_d^2 \|p_{21}\|^2 + \xi_d \sigma_d p_{21}^T \Lambda_2 - \xi_d \sigma_d \frac{1}{2} (\Gamma_d + \hat{e}_d \hat{\rho}_d)
\leq -\xi_d^2 \sigma_d^2 \|p_{21}\|^2 + \xi_d \sigma_d \|p_{21}\| \|\Lambda_2\| + \xi_d \sigma_d \|\Psi_2\| ,
$$

(3.38)

where $\Psi_2 := 0.5(\Gamma_d + \hat{e}_d \hat{\rho}_d) \in \mathbb{R}$, which is bounded for all $t \in [0, \tau_{2,\text{max}})$ owing to the boundedness of $\hat{\rho}_d(t), \Gamma_d(t)$ and $\hat{e}_d(t)$ for $\forall t \in [0, \tau_{2,\text{max}})$ (as it was shown in Phase I). Let $0 < \theta_2 < 1$ be a constant; thus adding and subtracting $\theta_2 \xi_d^2 \sigma_d^2 \|p_{21}\|^2$ to the right-hand side of (3.38) yields:

$$
\begin{align*}
V_2 &\leq -(1 - \theta_2) \xi_d^2 \sigma_d^2 \|p_{21}\|^2 - \xi_d \sigma_d \left(\theta_2 \xi_d \sigma_d \|p_{21}\|^2 - \|p_{21}\| \|\Lambda_2\| - \|\Psi_2\|\right) \\
&\leq -(1 - \theta_2) \xi_d^2 \sigma_d^2 \|p_{21}\|^2 , \quad \forall \sigma_d \geq \frac{\|\Lambda_2\| \|p_{21}\| + \|\Psi_2\|}{\theta_2 \xi_d \|p_{21}\|^2} , \quad \forall t \in [0, \tau_{2,\text{max}}) .
\end{align*}
$$

(3.39)

Recall that $\Lambda_2(t)$, $\Psi_2(t), \theta_2 \in \mathcal{L}_\infty, \forall t \in [0, \tau_{2,\text{max}})$. Notice that $\xi_d$ is lower bounded by a positive constant. In addition, since $\hat{e}_d(t) \in \Omega_{\tilde{e}_d} = (-b_d, b_d), \forall t \in [0, \tau_{2,\text{max}})$, followed by (3.16), (3.11), (3.14), and Section 3.5, for all $t \in [0, \tau_{2,\text{max}})$ we have:

$$
\begin{align*}
\|p_{21}(t)\|^2 &> \inf_{t \in [0, \tau_{2,\text{max}})} \left((d_{21}^*(t))^2 - b_d \rho_d(t)\right) > 0 , \\
\|p_{21}(t)\|^2 &< \sup_{t \in [0, \tau_{2,\text{max}})} \left((d_{21}^*(t))^2 + b_d \rho_d(t)\right) .
\end{align*}
$$

Therefore, (3.39) indicates that $\sigma_d(t)$ is Uniformly Ultimately Bounded (UBB), see Theorem 2.4. Consequently, one can show that there exists an ultimate bound $\bar{\sigma}_d$ independent of $\tau_{2,\text{max}}$ such that $|\sigma_d(t)| \leq \bar{\sigma}_d$ for $\forall t \in [0, \tau_{2,\text{max}})$.
3.10. Proofs of Lemmas, Theorems, and Some Technical Derivations

Phase III. Owing to the properties of $T_d(\hat{e}_d)$ in (3.17), we have: $-\hat{b}_d < T_d^{-1}(\sigma_d) = \hat{e}_d < \bar{b}_d$. Furthermore, since $T_d^{-1}(\sigma_d)$ is strictly increasing and $|\sigma_d(t)| \leq \sigma_d$ there exist $-\hat{b}_d(\sigma_d), \bar{b}_d(\sigma_d)$ such that:

$$-\hat{b}_d < -\hat{b}_d(\sigma_d) \leq \hat{e}_d \leq \bar{b}_d(\sigma_d) < \bar{b}_d. \quad (3.40)$$

As a result $\hat{e}_d(t) \in \Omega_\tilde{\hat{e}}_d$, $\forall t \in [0, \tau_{\text{2, max}}]$ where $\Omega_\tilde{\hat{e}}_d = [-\hat{b}_d(\sigma_d), \bar{b}_d(\sigma_d)]$ is a nonempty compact subset of $\Omega_{\tilde{\hat{e}}_d}$. Hence, assuming a finite $\tau_{\text{2, max}} < \infty$ and since $\Omega^\prime_\tilde{\hat{e}}_d \subset \Omega_{\tilde{\hat{e}}_d}$, Lemma 2.6 dictates the existence of a time instant $t' \in [0, \tau_{\text{2, max}}]$ such that $\hat{e}_d(t') \notin \Omega_{\tilde{\hat{e}}_d}$, which is a contradiction. Therefore, $\tau_{\text{2, max}} = \infty$. Thus, all closed loop signals remain bounded and moreover $\hat{e}_d(t) \in \Omega^\prime_\tilde{\hat{e}}_d \subset \Omega_{\tilde{\hat{e}}_d}$, $\forall t \geq 0$. Multiplying (3.40) by $\rho_d(t)$ results in: $-\hat{b}_d(\rho_d(t)) < -\hat{b}_d^s(\rho_d(t)) \leq e_d(t) \leq \bar{b}_d(\rho_d(t)) < \bar{b}_d(\rho_d(t))$ for all $t \geq 0$, which further ensures (3.14) and thus $\|p_2(t)\| > 0$, for all $t \geq 0$, due to selection of $b_d, \bar{b}_d$ according to Section 3.5.

3.10.3 Proof of Theorem 3.2

The proof follows an organization similar to that of Theorem 3.1. In particular, first, we establish the results for agent 3’s formation errors (i.e., for the first triangular subgraph of the desired formation). Next, by leveraging the established results for agent 3 and Theorem 1, as well as exploiting the hierarchical leader-follower structure of the formation control system, we extend the results to all of the agents by induction.

First note that, using (3.1), (3.7), (3.12), and (3.13), the formation error dynamics of agent $k \geq 3$ is given by:

$$\dot{e}_{rk} = \frac{p_{ki}^T \dot{p}_{ki}}{\|k\|^2} - \frac{p_{kj}^T \dot{p}_{kj}}{\|p_{kj}\|^2} = \frac{z_{ki}^T}{\|p_{ki}\|} (u_i - u_k + \delta_{ki}) - \frac{z_{kj}^T}{\|p_{kj}\|} (u_j - u_k + \delta_{kj})$$

$$= \frac{z_{ki}^T}{\|p_{ki}\|} (\Lambda_{ki} - u_k) - \frac{z_{kj}^T}{\|p_{kj}\|} (\Lambda_{kj} - u_k), \quad (3.41a)$$

$$\dot{e}_{\alpha k} = \frac{z_{ki}^T}{\|p_{ki}\|} J(u_i - u_k - \delta_{ki}) - \frac{z_{kj}^T}{\|p_{kj}\|} J(u_j - u_k + \delta_{kj})$$

$$= \frac{z_{ki}^T}{\|p_{ki}\|} J(\Lambda_{ki} - u_k) - \frac{z_{kj}^T}{\|p_{kj}\|} J(\Lambda_{kj} - u_k), \quad (3.41b)$$

where $(k, i), (k, j) \in \mathcal{E} \setminus \{(2, 1)\}$, $i < j < k$, and $\delta_{ki} := \delta_i - \delta_k, \delta_{kj} := \delta_j - \delta_k$, $\Lambda_{ki} := u_i + \delta_{ki}, \Lambda_{kj} := u_j + \delta_{kj}$. Note that (3.41b) is obtained by using the arc length formula (see Subsection 3.10.4 for a detailed derivation). Define $e_k := [e_{r_k}, e_{\alpha k}]^T \in \mathbb{R}^2, k = 3, \ldots, n$, as the stacked formation errors for agent $k$. Based on (3.41) we have:

$$\dot{e}_k = H_k \Lambda_k + G_k u_k, \quad k = 3, \ldots, n, \quad (3.42)$$
where $\Lambda_k := [\Lambda_{k1}, \Lambda_{kj}]^T \in \mathbb{R}^{4 \times 1}$, and $H_k \in \mathbb{R}^{2 \times 4}, G_k \in \mathbb{R}^{2 \times 2}$ are as follows:

$$H_k := \begin{bmatrix} z_{k1}^T \|p_{k1}\| & - z_{k1}^T \|p_{kj}\| \\ - z_{kj}^T \|p_{kj}\| & z_{kj}^T \|p_{k1}\| \end{bmatrix}, \quad G_k := \begin{bmatrix} z_{kj}^T \|p_{kj}\| & - z_{kj}^T \|p_{k1}\| \\ - z_{k1}^T \|p_{kj}\| & z_{k1}^T \|p_{k1}\| \end{bmatrix}.$$  

Moreover, defining the stacked transferred formation errors as $\sigma_k := [\sigma_{rk}, \sigma_{ak}]^T$, $k = 3, \ldots, n$, and employing (3.18) gives:

$$\dot{\sigma}_k = \xi_k (\dot{e}_k - \hat{\rho}_k \tilde{e}_k),$$  \hspace{1cm} (3.43)

where $\xi_k := \text{diag}(\xi_{rk}, \xi_{ak}) \in \mathbb{R}^{2 \times 2}, \rho_k := \text{diag}(\rho_{rk}, \rho_{ak}) \in \mathbb{R}^{2 \times 2}$, and $\tilde{e}_k := [\tilde{e}_{rk}, \tilde{e}_{ak}]^T = \rho_k^{-1} e_k$. Finally notice that, the control law (3.23c) can be re-written as follows:

$$u_k = -B_k \xi_k \sigma_k, \quad k = 3, \ldots, n,$$  \hspace{1cm} (3.44)

where $B_k := [\hat{\varphi}_k | \hat{\vartheta}_k] \in \mathbb{R}^{2 \times 2}, k = 3, \ldots, n$, are matrices whose columns are the orthogonal bipolar basis associated with agent $k \geq 3$. In the following, we shall first establish the results for agent 3 and then extend the proof for all $3 < k \leq n$ by induction. Similarly to the proof of Theorem 3.1, we will proceed in three phases.

**Phase I.** Differentiating $\tilde{e}_{r3}$ and $\tilde{e}_{a3}$, gives:

$$\dot{\tilde{e}}_{r3} := E_{\tilde{e}_{r3}} (t, \tilde{e}_{r3}) = \rho_{r3}^{-1}(t) (\tilde{e}_{r3} - \tilde{\rho}_{r3}(t)), \hspace{1cm} (3.45a)$$

$$\dot{\tilde{e}}_{a3} := E_{\tilde{e}_{a3}} (t, \tilde{e}_{a3}) = \rho_{a3}^{-1}(t) (\tilde{e}_{a3} - \tilde{\rho}_{a3}(t)). \hspace{1cm} (3.45b)$$

Define $E_{\tilde{e}_3}(t, \tilde{e}_3) := [E_{\tilde{e}_{r3}}(t, \tilde{e}_{r3}), E_{\tilde{e}_{a3}}(t, \tilde{e}_{a3})]^T$. Using (3.42) and (3.44), the closed-loop dynamical system of $\tilde{e}_3 = [\tilde{e}_{r3}, \tilde{e}_{a3}]^T = \rho_3^{-1} e_3$ with $\rho_3 = \text{diag}(\rho_{r3}, \rho_{a3})$ may be written in compact form as:

$$\dot{\tilde{e}}_3 = E_{\tilde{e}_3}(t, \tilde{e}_3) = \rho_3^{-1}(t)(\tilde{e}_3 - \rho_3(t) \tilde{e}_3)$$

$$= \rho_3^{-1}(t)(H_k \Lambda_k - G_k B_k \xi_k \sigma_k - \hat{\rho}_3(t) \tilde{e}_3). \hspace{1cm} (3.46)$$

Let us also define the open set: $\Omega_{\tilde{e}_3} := \Omega_{\tilde{e}_{r3}} \times \Omega_{\tilde{e}_{a3}}$, where $\Omega_{\tilde{e}_{r3}} := (-\tilde{b}_{r3}, \tilde{b}_{r3})$, and $\Omega_{\tilde{e}_{a3}} := (-\tilde{b}_{a3}, \tilde{b}_{a3})$. Note that $\Omega_{\tilde{e}_3}$ is nonempty and open by construction. Followed by the discussion in Section 3.5, agent 3 can initially select $\tilde{b}_{r3}, \tilde{b}_{r3}, \tilde{b}_{a3}, \tilde{b}_{a3} > 0$ to ensure $\tilde{e}_3(0) \in \Omega_{\tilde{e}_3}$. Since $E_{\tilde{e}_3}(t, \tilde{e}_3)$ is continuous on $t$ and locally Lipschitz on $\tilde{e}_3$ over the set $\Omega_{\tilde{e}_3}$, the hypotheses of Theorem 2.3 dictates existence and uniqueness of a maximal solution $\tilde{e}_3(t)$ of (3.46) for a time interval $[0, \tau_{3,\text{max}})$ where $\tilde{e}_3(t) \in \Omega_{\tilde{e}_3}, \forall t \in [0, \tau_{3,\text{max}})$ is guaranteed. This further ensures that $\tilde{e}_{r3}(t)$ and $\tilde{e}_{a3}(t)$ are bounded as in (3.14) for all $t \in [0, \tau_{3,\text{max}})$.

**Phase II.** Owing to $\tilde{e}_3(t) \in \Omega_{\tilde{e}_3}, \forall t \in [0, \tau_{3,\text{max}})$, the stacked transformed errors $\sigma_3 := [\sigma_{r3}, \sigma_{a3}]^T$, where $\sigma_{r3}, \sigma_{a3}$ are defined in (3.17), are well-defined for all $t \in [0, \tau_{3,\text{max}})$. Therefore, consider the following positive definite and radially
unbounded Lyapunov function candidate $V_3 = (1/2)\sigma_3^\top \sigma_3$. Differentiating $V_3$ with respect to time, using (3.42), (3.43), and (3.44), gives:

$$
\dot{V}_3 = -\sigma_3^\top \xi_3 (G_3 B_3) \xi_3 \sigma_3 + \sigma_3^\top \xi_3 H_3 \Lambda_3 - \sigma_3^\top \xi_3 \dot{p}_3 \dot{e}_3,
$$

$$
\leq -m_3 \|\sigma_3^\top \xi_3\|^2 + \|\sigma_3^\top \xi_3\| H_3 \|\Lambda_3\| + \|\sigma_3^\top \xi_3\| \|\Psi_3\|,
$$

(3.47)

where $m_3 > 0$ is a positive constant related to $M_3 := G_3 B_3$ (see Subsection 3.10.5 for details), and $\Psi_3 := \dot{p}_3 \dot{e}_3 \in \mathbb{R}^2$, which is bounded for $\forall t \in [0, \tau_{3,\text{max}}]$ owing to the boundedness of $\dot{p}_3(t)$ for all $t \geq 0$ and the boundedness of $\dot{e}_3(t)$ for $\forall t \in [0, \tau_{3,\text{max}}]$ (as it is shown in Phase I). Moreover, note that due to the boundedness of $\delta_{31}, \delta_{32}, u_1 = u_L(t) \in \mathcal{L}_\infty$ as well as boundedness of $u_2(t) \in \mathcal{L}_\infty$ (owing to Theorem 3.1), we have that $\Lambda_{31}, \Lambda_{32} \in \mathcal{L}_\infty$, which leads to the boundedness of $\Lambda_3$. Let $0 < \theta_3 < m_3$ be a constant, adding and subtracting $\theta_3 \|\sigma_3^\top \xi_3\|^2$ to the right-hand side of (3.47), and invoking diagonal and positive definiteness of $\xi_3$, yields:

$$
\dot{V}_3 \leq -(m_3 - \theta_3) \|\sigma_3^\top \xi_3\|^2 - \|\sigma_3^\top \xi_3\| \left(\theta_3 \|\sigma_3^\top \xi_3\| - \|H_3\| \|\Lambda_3\| - \|\Psi_3\|\right) \leq 0,
$$

(3.48)

where $\lambda_{\text{min}}(\xi_3^2)$ is the minimum eigenvalue of the diagonal positive definite matrix $\xi_3^2 \in \mathbb{R}^{2 \times 2}$. Note that, $\dot{e}_3(t) \in \Omega_{\dot{e}_3} = (-b_{r_3}, b_{r_3}) \times (-b_{\alpha_3}, b_{\alpha_3}), \forall t \in [0, \tau_{3,\text{max}})$, hence followed by (3.7), (3.12), (3.16), (3.14), and Section 3.5 for the selection of $b_{r_3}, b_{\alpha_3}$, we can infer that $\|p_{31}(t)\|, \|p_{32}(t)\|$ are bounded away from zero for $\forall t \in [0, \tau_{3,\text{max}}]$. Moreover, since $J, z_{31}, z_{32} \in \mathcal{L}_\infty$, the elements of matrix $H_3$ are all bounded for $\forall t \in [0, \tau_{3,\text{max}}]$, thus $\|H_3\| \in \mathcal{L}_\infty, \forall t \in [0, \tau_{3,\text{max}}]$. Finally, as $\|H_3(t)\|, \|\Lambda_3(t)\|, \|\Psi_3(t)\|, \theta_3 \in \mathcal{L}_\infty, \forall t \in [0, \tau_{3,\text{max}}]$, and $\xi_3$ is a diagonal positive definite matrix, (3.48) implies that $\sigma_3$ is uniformly ultimately bounded (Theorem 2.4). Therefore, one can show that there exists an ultimate bound $\bar{\sigma}_3$ independent of $\tau_{3,\text{max}}$ such that $\|\sigma_3(t)\| \leq \bar{\sigma}_3$ for $\forall t \in [0, \tau_{3,\text{max}}]$. 

**Phase III.** Owing to $\|\sigma_3(t)\| \leq \bar{\sigma}_3$ we have $|\sigma_{r_3}(t)| \leq \bar{\sigma}_3$ and $|\sigma_{\alpha_3}(t)| \leq \bar{\sigma}_3$. Similarly to Phase III in the proof of Theorem 3.1 due to properties of $T_k(\tilde{e}_h)$ in (3.17) and its inverse, there exist $-b_{r_3}^*(\bar{\sigma}_3), b_{r_3}^*(\bar{\sigma}_3), -b_{\alpha_3}^*(\bar{\sigma}_3), b_{\alpha_3}^*(\bar{\sigma}_3)$ such that:

$$
-b_{r_3} < -b_{r_3}^*(\bar{\sigma}_3) \leq \dot{e}_{r_3} \leq b_{r_3}^*(\bar{\sigma}_3) < b_{r_3},
$$

(3.49a)

$$
-b_{\alpha_3} < -b_{\alpha_3}^*(\bar{\sigma}_3) \leq \dot{e}_{\alpha_3} \leq b_{\alpha_3}^*(\bar{\sigma}_3) < b_{\alpha_3}.
$$

(3.49b)

As a result $\tilde{e}_3(t) \in \Omega_{\tilde{e}_3} := \Omega'_{\tilde{e}_3} \times \Omega'_{\tilde{e}_{\alpha_3}}, \forall t \in [0, \tau_{3,\text{max}}]$ where $\Omega'_{\tilde{e}_3} = [-b_{r_3}^*, b_{r_3}^*]$ and $\Omega'_{\tilde{e}_{\alpha_3}} = [-b_{\alpha_3}^*, b_{\alpha_3}^*]$ are nonempty compact subset of $\Omega_{\tilde{e}_3}$ and $\Omega_{\tilde{e}_{\alpha_3}}$, respectively. Hence, assuming a finite $\tau_{3,\text{max}} < \infty$, since $\Omega'_{\tilde{e}_3} \subset \Omega_{\tilde{e}_3}$, Lemma 2.5 leads to the existence of a time instant $t' \in [0, \tau_{3,\text{max}}]$ such that $\tilde{e}_3(t') \notin \Omega'_{\tilde{e}_3}$, which is a contradiction. Therefore, $\tau_{3,\text{max}} = \infty$. Thus, all closed loop signals remain bounded, and $\tilde{e}_3(t) \in \Omega'_{\tilde{e}_3} \subset \Omega_{\tilde{e}_3}, \forall t \geq 0$. Multiplying (3.49a) and (3.49b) by $\rho_{r_3}(t)$ and $\rho_{\alpha_3}(t)$, respectively, gives: $-b_{r_3} \rho_{r_3}(t) \leq -b_{r_3}^* \rho_{r_3}(t) \leq \dot{e}_{r_3}(t) \leq b_{r_3}^* \rho_{r_3}(t) < b_{r_3} \rho_{r_3}(t)$ and
Leader-Follower Directed Formation Control Based on Bipolar Coordinates with
Global Convergence

\begin{align*}
-\bar{b}_{\alpha_3}\alpha_3(t) \leq e_{\alpha_3}(t) \leq \bar{b}_{\alpha_3}\alpha_3(t) \leq b_{\alpha_3}\alpha_3(t) \quad \text{for } t \geq 0,
\end{align*}
which ensure (3.14) for \(e_{\alpha_3}(t)\) and \(e_{\alpha_3}(t)\). This also leads to \(\|p_{31}(t)\|, \|p_{32}(t)\| > 0\), for all \(t \geq 0\) due to the selection of \(\bar{b}_{\alpha_3}, \bar{b}_{\alpha_3}\) according to Section 3.5.

**Induction Step:** Now let us assume that the stability results of Theorem 2 holds for agents \(3, \ldots, k-1\) (i.e., boundedness of all signals and satisfaction of (3.14) for all agents \(3, \ldots, k-1\)). Hence, one can verify that the results of Phase I for agent \(k\) still holds. Moreover, by employing the radially unbounded Lyapunov function candidate \(V_k = \left(1/2\right)\sigma_k^{-1} \sigma_k\), and since agent \(k\) has its arbitrary two neighbors from the set \(1, \ldots, k-1\), we can establish existence of an ultimate bound \(\bar{\sigma}_k\) for \(\sigma_k(t)\) in the same way as in Phase II. Finally it is straightforward to repeat Phase III and establish satisfaction of (3.14) for \(e_{\alpha_k}, e_{\alpha_k}\) along with \(\|p_{ki}(t)\|, \|p_{kj}(t)\| > 0\), \((k, i), (k, j) \in \mathcal{E}\setminus\{(2, 1)\}\) for all \(t \geq 0\), which finalizes the proof.

### 3.10.4 Derivation of \(\dot{e}_{\alpha_k}\) in (3.41b)

Consider a triangular sub-graph of \(\mathcal{G}\), where \(i < j < k\). An alternative way of calculating the edge-angle \(\alpha_{kij}\) is given by

\begin{align*}
\alpha_{kij} = \text{mod}\{\alpha_{kj} - \alpha_{ki}, 2\pi\}, \quad (k, i), (k, j) \in \mathcal{E}\setminus\{(2, 1)\},
\end{align*}

where \(\alpha_{ki}\) and \(\alpha_{kj}\) are the angles of the edges \((k, i)\) and \((k, j)\) measured counterclockwise from the \(x\)-axis of the global coordinate frame (see Fig. 3.9a). Taking the time derivative of (3.13) based on (3.50) yields:

\begin{align*}
\dot{e}_{\alpha_k} = \dot{\alpha}_{kj} - \dot{\alpha}_{ki}.
\end{align*}

where \(\dot{\alpha}_{ki}\) and \(\dot{\alpha}_{kj}\) should be calculated explicitly. In this regard, consider \(p_{ki}^{+} = p_{ki} + dp_{ki}\), where \(p_{ki}^{+}\) represents the new relative position vector associated with edge \((k, i)\) subject to the infinitesimal changes in the positions of agents \(k\) and \(i\) that are captured by \(dp_{ki}\). Notice that the infinitesimal motions of agents \(k\) and \(i\)
modeled by $dp_{ki}$ can be seen as if only agent $i$ is moving. Therefore, for a better geometric representation, without loss of generality we assume that only agent $i$ has an infinitesimal motion as illustrated in Fig. 3.96. Moreover, assume that $d\alpha_{ki}$ represents the infinitesimal variation of $\alpha_{ki}$, and $ds_{ki}$ shows the infinitesimal variation of its corresponding curve with radius of $\|p_{ki}\|$. Since $\alpha_{ki}$ is in radians, from the arc length formula\footnote{It holds $ds = r d\theta$, where $r$ is the radii, $d\theta$ is the variation of the angle, and $ds$ is the variation of its corresponding arc length.} we get:

$$ds_{ki} = \|p_{ki}\| d\alpha_{ki}. \quad (3.52)$$

For the infinitesimal right triangle $\triangle_{ii' i^+}$ we also have:

$$ds_{ki} = (J z_{ki})^T dp_{ki}, \quad (3.53)$$

that is the projection of $dp_{ki}$ on the infinitesimal arc $ds_{ki}$. Invoking (3.52) and (3.53), $\dot{\alpha}_{ki}$ is given by:

$$\dot{\alpha}_{ki} = \frac{d\alpha_{ki}}{dt} = \frac{z_{ki}^T}{\|p_{ki}\|} J^T p_{ki}. \quad (3.54)$$

A similar expression can also be obtained for $\dot{\alpha}_{kj}$. Therefore, using (3.1), (3.51), and (3.54) followed by the fact that $J^T = -J$, yields (3.41b).

### 3.10.5 Quadratic form for $M_k := G_k B_k \in \mathbb{R}^{2 \times 2}$, $k \geq 3$ in (3.47)

Consider matrices $B_k = [\hat{r}_k | \hat{\alpha}_k] \in \mathbb{R}^{2 \times 2}$ and $G_k \in \mathbb{R}^{2 \times 2}$, $k = 3, \ldots, n$, as defined in the proof of Theorem 3.2. Let $\eta_k := (z_{kj}/\|p_{kj}\|) - (z_{ki}/\|p_{ki}\|) \in \mathbb{R}^2$ then $G_k = [\eta_k^T | \eta_k^T J]^T$. Now $M_k := G_k B_k$ gives:

$$M_k = \begin{bmatrix} \eta_k^T \hat{r}_k & \eta_k^T \hat{\alpha}_k \\ \eta_k^T J \hat{r}_k & \eta_k^T J \hat{\alpha}_k \end{bmatrix} = \begin{bmatrix} \eta_k^T \hat{r}_k & -\eta_k^T J \hat{r}_k \\ \eta_k^T J \hat{r}_k & \eta_k^T \hat{\alpha}_k \end{bmatrix} \quad (3.55)$$

where orthogonality of the bipolar basis is employed to obtain the right-hand side, that is: $\hat{r}_k = J \hat{\alpha}_k$ along with the fact that $J^{-1} = J^T = -J$. Let $x = [x_1 \ x_2]^T \in \mathbb{R}^2$, then it one can verify that:

$$x^T M_k x = m_k (x_1^2 + x_2^2) = m_k \|x\|^2 \quad (3.56)$$

where $m_k := \eta_k^T \hat{r}_k$. Since $\hat{r}_k, z_{ki}, z_{kj}$ are unit vectors, from the (geometric) inner product formula we get:

$$m_k = \eta_k^T \hat{r}_k = \frac{\cos \gamma_{kj}}{\|p_{kj}\|} + \frac{-\cos \gamma_{ki}}{\|p_{ki}\|}, \quad (3.57)$$

where $\gamma_{ki}$ represents the (smallest) angle formed between $z_{ki}$ and $\hat{r}_k$, and $\gamma_{kj}$ shows the (smallest) angle formed between $z_{kj}$ and $\hat{r}_k$. In the sequel we will prove that
Global Convergence

$m_k > 0$, which ensures positiveness of (3.56). Consider three cases for agent $k$’s position with respect to its neighbors in the virtual Cartesian coordinate frame \{\(C_k\)\} that are: (a) left half-plane, (b) right half-plane, and (c) on the \(Y_k\) axis, as illustrated in Fig. 3.10. First, note that \(\gamma_{kj} \leq \gamma_{ki}\) always holds. Without loss of generality, let us also assume that \(\|p_{ki}\|, \|p_{kj}\|\) are bounded.

**Case (a):** Note that in this case \(\hat{r}_k\) is always directed outwards the isocount curves \(r_k\) = constant. Moreover it always holds that \(\pi/2 < \gamma_{ki} \leq \pi\), \(0 \leq \gamma_{kj} \leq \pi\), and \(\|p_{ki}\| < \|p_{kj}\|\). In this regard, whenever \(0 \leq \gamma_{kj} \leq \pi/2\) then the first term in the right-hand-side of (3.57) is always positive or zero whereas the second term is always positive, thus \(m_k > 0\) is ensured. Now consider when \(\pi/2 < \gamma_{kj} \leq \pi\). In this case the first term in the right-hand-side of (3.57) is always negative whereas the second term is always positive, however, due to \(\gamma_{kj} \leq \gamma_{ki}\) and \(\|p_{ki}\| < \|p_{kj}\|\) the second term always dominates the first term, thus we will always have \(m_k > 0\). As a result we can conclude that for bounded \(\|p_{ki}\|, \|p_{kj}\|\), we always have \(m_k > 0\), whenever agent \(k\) is in the left-half plane of \{\(C_k\)\}.

**Cases (b) and (c):** Using similar arguments, we can also show that \(m_k > 0\) whenever agent \(k\) is on the \(Y_k\) axis or in the right-half plane of \{\(C_k\)\} for bounded \(\|p_{ki}\|, \|p_{kj}\|\).

Note that when \(\|p_{ki}\|, \|p_{kj}\|\) become unbounded, \(m_k\) may approach to zero in all three cases (a), (b), and (c). However, unboundedness of \(\|p_{ki}\|, \|p_{kj}\|\) is avoided in the proof of Theorem 3.2. In particular, for \(k = 3\), from Phase I of Theorem 3.2 we have \(\hat{e}_3(t) \in \Omega_{\hat{e}_3} = (-\hat{b}_{r_3}, \hat{b}_{r_3}) \times (-\hat{b}_{\alpha_3}, \hat{b}_{\alpha_3}), \forall t \in [0, \tau_{3,\text{max}}]\), hence, followed by (3.13), (3.16), (3.14), and Section 3.5 for the selection of \(\hat{b}_{\alpha_3}, \hat{b}_{\alpha_3}\), we ensure that the edge-angle \(\alpha_{312}\) is positively lower bounded away from zero and its upper bound is less than \(2\pi\), \(\forall t \in [0, \tau_{3,\text{max}}]\), which is sufficient to have \(\|p_{31}\|, \|p_{32}\|\) bounded \(\forall t \in [0, \tau_{3,\text{max}}]\). Note that similarly we can show that this property also holds for \(k > 3\), as explained in the induction step in the proof of Theorem 3.2. Therefore, there always exists a positive lower bound \(\overline{m}_k\) (depending on the choice of \(\hat{b}_{\alpha_k}, \hat{b}_{\alpha_k}\)) such that \(0 < \overline{m}_k < m_k\).

**Figure 3.10:** Configuration of \(z_{ki}, z_{kj}\), and \(\hat{r}_k\) in three arbitrary positions of agent \(k\) with respect to its neighbors.
Part II

Feedback Control under Coupled Spatiotemporal Constraints
This chapter studies time-varying hard and soft constraints in the context of funnel control for uncertain Euler-Lagrangian systems. The existence of hard and soft constraints leads to a class of coupled constraints that goes beyond the conventional capabilities of funnel-based controller designs. In this chapter, we introduce a new approach that extends the applicability of funnel-based control methods, especially prescribed performance control, to address this problem.

4.1 Introduction

Funnel-based control methods provide a low-complexity and robust (model-free) feedback controllers to address (time-varying) output constraints in uncertain non-linear systems. In recent years, these methods have been primarily used to guarantee user-defined transient and steady-state performance in terms of output tracking and error stabilization. This is achieved by constraining the evolution of output error signals within predefined time-varying funnel constraints. Two prominent control strategies for achieving this goal are Funnel Control (FC) [64, 70, 96] and Prescribed Performance Control (PPC) [54, 68, 147]. Time-Varying Barrier Lyapunov Functions (TVBLFs) have also been applied to tackle similar challenges [62].

In addition to addressing performance specifications, FC, PPC, and TVBLF have also proven valuable in managing constraints that resemble safety requirements in specific applications. However, these methods often necessitate modeling safety constraints as constant upper and lower bounds on tracking/stabilization errors, allowing their integration into the traditional performance bounds used in FC and PPC design methods, see [9, 78, 148] for some examples. This approach implicitly assumes that safety constraints should align with performance requirements. Nevertheless, it is important to note that performance constraints on tracking and stabilization may not always align with safety specifications in practice.

In recent years, Control Barrier Functions (CBFs) have been also used as a
promising approach for addressing tracking and stabilization while incorporating safety specifications. This is achieved through the utilization of Quadratic Programs (QPs), where safety requirements are treated as hard constraints, while the stabilization and tracking criteria are regarded as soft constraints within the optimization problem [66, 67]. Although CBFs are traditionally applied to constant constraints, a recent development by [113] introduced the use of time-varying CBFs to handle time-varying output constraints. However, it is worth noting that conventional CBF-based control synthesis assumes precise knowledge of the system dynamics. Additionally, it is important to highlight that, unlike funnel-based control methods, CBF schemes do not directly predefine (constrain) the quality of the tracking and stabilization task.

4.2 Contributions

In this chapter, we introduce a novel funnel-based control scheme aimed at addressing time-varying hard and soft (funnel-like) output constraints. The key concept revolves around dynamically generating Constraint Consistent Funnel (CCF) constraints to ensure adherence to hard (safety) constraints while accommodating soft (performance) constraints, as long as they do not conflict with the hard constraints. To achieve this, we first present a novel online funnel planning scheme that constructs a CCF for each system output. Subsequently, we design a model-free robust controller under a low-complexity PPC scheme to maintain each system’s output within its corresponding (online) planned CCF. The controller design is proposed for uncertain nonlinear Euler-Lagrangian systems, a broad category encompassing various practical physical systems such as mobile robotic vehicles and robot manipulators. Notably, our work stands as the first to consider the integration of hard and soft constraints in funnel-based control designs. Furthermore, in comparison to existing time-varying CBF-based control synthesis methods, our approach offers a closed form feedback control law that is both computationally tractable (optimization-free) and robust to uncertainties.

4.3 Problem Formulation

Consider the following Euler-Lagrange (EL) system:

\[
\begin{align*}
M(x)\ddot{v} + C(x,v)\dot{v} + g(x) + D(x)v &= u + d(t) \\
y &= x
\end{align*}
\]

(4.1)

where \(x := \text{col}(x_i) := [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n\) and \(v := \text{col}(v_i) = \dot{x}\) are the generalized coordinates and their velocities, respectively, \(M(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}\) is the inertia matrix, \(C(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}\) is the centrifugal and Coriolis forces matrix, \(g(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is the vector of gravitational forces, \(D(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}\) is the matrix of friction like terms. Moreover, \(d(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n\) is the vector of unknown
bounded piecewise continuous external disturbances and \( u \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) denote the control input and the output, respectively. Within this chapter, we refer to \( x(t) \) as output trajectory of the closed-loop dynamical system \( 4.1 \) under the control input \( u \) and initial conditions \( x(0) \) and \( v(0) \). In particular, \( x(t) \) resembles the spatial position vector of the mechanical EL system \( 4.1 \).

**Assumption 4.1.** \( M(x), C(x,v), g(x), D(x), \) and the upper bound of \( d(t) \) are unknown for the controller design.

The following property holds for the EL system \( 4.1 \):

**Property 4.1.** \( M(x) \) is symmetric and positive definite for all \( x \in \mathbb{R}^n \). Moreover, \( M(x), g(x), D(x) \) are continuous over \( x \) and \( C(x,v) \) is continuous over \( x \) and \( v \).

Suppose that the outputs of \( 4.1 \) are required to satisfy the following time-varying constraints:

\[
\begin{align*}
\underline{\rho}_i^h(t) &< x_i < \overline{\rho}_i^h(t), \quad i = \{1, \ldots, n\}, \quad (4.2a) \\
\underline{\rho}_i^s(t) &< x_i < \overline{\rho}_i^s(t), \quad i = \{1, \ldots, n\}, \quad (4.2b)
\end{align*}
\]

where \( \underline{\rho}_i^h(\cdot), \overline{\rho}_i^h(\cdot), \underline{\rho}_i^s(\cdot), \overline{\rho}_i^s(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}, i = \{1, \ldots, n\}, \) are bounded continuously differentiable functions of time with bounded derivatives. In particular, let \( 4.2a \) and \( 4.2b \) represent time-varying hard and soft constraints on \( x_i(t) \), respectively. Note that, inequalities \( 4.2a \) and \( 4.2b \) denote separate hard and soft constrained (time-varying) funnels for the evolution of \( x_i(t) \).

**Definition 4.1** (Feasibility of a funnel constraint). A funnel constraint, as in \( 4.2a \) or \( 4.2b \), is said to be feasible when it has a nonzero distance between its upper and lower boundaries for all times. In particular, \( 4.2a \) and \( 4.2b \) are feasible funnels when \( \overline{\rho}_i^h(t) - \underline{\rho}_i^h(t) \geq \epsilon_i^h > 0 \) and \( \overline{\rho}_i^s(t) - \underline{\rho}_i^s(t) \geq \epsilon_i^s > 0, i = \{1, \ldots, m\} \) hold \( \forall t \geq 0 \), respectively.

**Assumption 4.2.** Each hard and soft funnel constraint on \( x_i \) in \( 4.2a \) and \( 4.2b \) is feasible for all times.

**Definition 4.2** (Compatibility of time-varying constraints). A number of time-varying constraints on \( x_i \) are said to be compatible, whenever all of them can be satisfied at the same time. In particular, the hard and soft funnel constraints in \( 4.2a \) and \( 4.2b \) on \( x_i \) are compatible at time \( t \) if \( \overline{\rho}_i^h(t) > \underline{\rho}_i^s(t) \) and \( \overline{\rho}_i^s(t) > \underline{\rho}_i^h(t) \) hold simultaneously.

Fig. 4.1a illustrates a case when both hard and soft constrained funnels \( 4.2a \) and \( 4.2b \) on \( x_i \) are compatible \( \forall t \geq 0 \), while Fig. 4.1b shows hard and soft constrained funnels that are incompatible during \( t \in [4.8, 9] \).

The following assumption is proposed for simplicity in the presentation and will be relaxed later.
**Assumption 4.3.** The hard \(4.2a\) and soft \(4.2b\) funnel constraints on \(x_i\) are compatible at \(t = 0\) and the initial outputs \(x_i(0), i = \{1, \ldots, n\}\), satisfy both \(4.2a\) and \(4.2b\) at \(t = 0\).

**Problem (Hard and Soft Constrained Funnel Control):** Given the hard and soft output (funnel) constraints in \(4.2a\) and \(4.2b\), design under Assumptions 4.1, 4.2, and 4.3:

1. a continuous time-varying Constraint Consistent Funnel (CCF) constraint with boundary functions \(\rho^U_i(t), \rho^L_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}\) for each output \(x_i, i = \{1, \ldots, n\}\), such that \(\rho^U_i(t) - \rho^L_i(t) \geq \epsilon_i > 0, \forall t \geq 0\) (i.e., CCF should be feasible for all times);

2. a robust control law \(u(t, x)\) for \(4.1\) to ensure:

\[
\rho^L_i(t) < x_i(t) < \rho^U_i(t), \quad \forall t \geq 0, \quad i = \{1, \ldots, n\}.
\]  

By **constraint consistency** of \(\rho^U_i(t), \rho^L_i(t)\) we mean: (i) satisfaction of \(4.3\) always implies satisfaction of \(4.2a\), i.e., \(4.3\) always respects the hard constraints \(4.2a\), and (ii) whenever hard and soft constraints \(4.2a\) and \(4.2b\) are compatible, \(4.3\) ensures satisfaction of \(4.2b\) (or **exponentially fast recovery** of \(4.2b\), which will be clarified later in Section 4.4), i.e., \(4.3\) respects (resp. recovers) the soft constraints \(4.2b\) only when its satisfaction is not conflicting with the hard constraints \(4.2a\).

Fig. 4.1 shows examples of Constraint Consistent Regions (CCRs) for \(x_i\), in which hard constraints are always satisfied and the soft constraints are met only when they are compatible with the hard constraints. As can be seen in Fig. 4.1a, if hard and soft constraints are compatible for all \(t \geq 0\) then the boundaries of the CCR determines the boundary functions of a CCF in \(4.3\). However, this is not the case if hard and soft constraints become incompatible for a time interval, since the upper or the lower boundary of the CCR becomes discontinuous and

---

**Figure 4.1:** (a) compatible \((\forall t \geq 0)\), and (b) incompatible hard and soft funnel constraints.
thus cannot be used to construct a feasible (well-defined) continuous CCF. For example, in Fig. 4.1b the upper bound of the CCR is discontinuous. Hence, to have a continuous transition region for the evolution of \( x_i(t) \), \( \rho^U_i(t) \) in (4.3) needs to be designed (planned), as depicted in Fig. 4.1b (dashed curve), while a continuous \( \rho^L_i(t) \) can be directly determined by the lower boundary of the CCR.

**Remark 4.1.** In practical applications soft constraints (4.2b) can be considered as the required performance for reference tracking or stabilization, while hard constraints (4.2a) can be considered as safety requirements. For example, consider a mobile robot whose motion is governed by (4.1), where \( y = [x_1,x_2]^T \) indicates its position on the plane. Assume that the robot requires to: (i) always remain in a box shaped region indicated by \( |x_i| < s_i, s_i > 0, i = \{1, 2\} \), for safety consideration, and (ii) track a desired time-varying continuously differentiable reference trajectory \( x_d(t) = [x_{d_1}(t), x_{d_2}(t)]^T \), such that \( |x_i - x_{d_i}(t)| < \gamma_i(t), i = \{1, 2\} \), where \( \gamma_i(t) \) represent user-defined positive (performance) functions decaying to a sufficiently small neighborhood of zero. For example one can take \( \gamma_i(t) = (\rho_{0_i} - \rho_{x_i}) \exp(-l_it) + \rho_{x_i} \), in which \( l_i, \rho_{x_i}, > 0 \) determine the convergence rate and ultimate bound of the tracking errors, respectively, and \( \rho_{0_i} > |x_i(0) - x_{d_i}(0)|, i = \{1, 2\} \). Note that, in general, the desired trajectory \( x_d(t) \) may not always be inside of the safe region. In this example, (4.2a) and (4.2b) can be written as \(-s_i < x_i < s_i \) and \( x_{d_i}(t) - \gamma_i(t) < x_i - x_{d_i}(t) + \gamma_i(t), i = \{1, 2\} \), respectively.

In the next section, given the hard and soft constraints in (4.2a) and (4.2b), we will first propose an online funnel planning method to construct the constraint consistent feasible funnel boundary functions in (4.3). Then, in Section 4.5 we will design a robust low-complexity funnel-based control law using the prescribed performance control method to ensure (4.3).

### 4.4 Online Constraint Consistent Funnel Planning

Consider the hard and soft constraints in (4.2a) and (4.2b). Note that, whenever hard and soft constraints are compatible one can simply choose\(^1\)

\[
\begin{align*}
\rho^U_i(t) &= \max\{\rho^s_i(t), \rho^h_i(t)\}, \\
\rho^L_i(t) &= \min\{\rho^s_i(t), \rho^h_i(t)\},
\end{align*}
\]

for \( i = \{1, \ldots, n\} \) as proper candidates for CCF’s boundary functions within which \( x_i(t) \) is allowed to evolve (see Fig. 4.1a). Nevertheless, when hard and soft constraints temporarily become incompatible (as illustrated in Fig. 4.1b), the aforementioned selection results in \( \min\{\rho^s_i(t), \rho^h_i(t)\} \leq \max\{\rho^s_i(t), \rho^h_i(t)\} \) for a specific time point. This implies that \( \rho^U_i(t) \leq \rho^L_i(t) \) at that moment, thereby causing an

---

\(^1\)In this chapter, the notation \( \max \) (or \( \min \)) refers to taking the maximum (or minimum) of their respective arguments at each time instance.
infeasible funnel constraint in \((4.3)\). To avoid this issue, for each \(x_i, i = \{1, \ldots, n\}\), we propose an online dynamic CCF planning scheme, by design \(\rho^L_i(t)\) and \(\rho^U_i(t)\) in \((4.3)\) as follows:

\[
\begin{align*}
\rho^L_i(t) &:= \max\{\rho^*_i(t) - \varphi^L_i(t), \rho^h_i(t)\}, \\
\rho^U_i(t) &:= \min\{\rho^*_i(t) + \varphi^U_i(t), \rho^h_i(t)\},
\end{align*}
\]

\((4.5a)\)

\((4.5b)\)

where \(\varphi^L_i(t), \varphi^U_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, i = \{1, \ldots, m\}\) are continuous nonnegative modification signals that are governed by the following dynamics:

\[
\begin{align*}
\dot{\varphi}^L_i &= \frac{1}{2} \left(1 - \text{sign}(\eta^L_i - \mu)\right) \frac{1}{\eta^L_i + \varphi^L_i} - k_c \varphi^L_i, \\
\dot{\varphi}^U_i &= \frac{1}{2} \left(1 - \text{sign}(\eta^U_i - \mu)\right) \frac{1}{\eta^U_i + \varphi^U_i} - k_c \varphi^U_i,
\end{align*}
\]

\((4.6a)\)

\((4.6b)\)

in which \(\mu, k_c > 0\) are user-defined arbitrary positive constants and \(\eta^L_i(t) := \widetilde{\rho}^h_i(t) - \rho^*_i(t), \eta^U_i(t) := \widetilde{\rho}^*_i(t) - \rho^h_i(t), i = \{1, \ldots, n\}\). For simplicity of the explanations in the sequel, let the initial conditions of \((4.6)\) be \(\varphi^L_i(0) = \varphi^U_i(0) = 0\). Later we show that by refining the choice of \(\varphi^L_i(0)\) and \(\varphi^U_i(0)\) one can relax Assumption \(4.3\). The modification signals \(\varphi^L_i(t), \varphi^U_i(t)\) introduced in \((4.5)\), adjust \(\rho^L_i(t)\) and \(\rho^U_i(t)\) whenever hard and soft constraints become conflicting so that the soft constraints \((4.2b)\) are violated in favor of satisfying the hard constraints \((4.2a)\). Moreover, \(\mu > 0\) is considered as a user-defined minimal distance between potentially conflicting hard and soft constraint bounds (i.e., \(\rho^h_i(t)\) and \(\rho^*_i(t)\) or \(\rho^h_i(t)\) and \(\rho^*_i(t)\)) and it will be used to trigger the process of disregarding the soft constraints.

In the following, we summarize the philosophy behind adopting \((4.5)\) and \((4.6)\), as well as the impact of choosing \(\mu\) and \(k_c\). Recall that when hard and soft constraints are compatible and \(\varphi^L_i(0) = \varphi^U_i(0) = 0\), \((4.5)\) becomes identical to \((4.4)\). Therefore, under Assumption \(4.3\) and \(\varphi^L_i(0) = \varphi^U_i(0) = 0\), \((4.5)\) gives a feasible CCF for \(x_i(t)\) at \(t = 0\). Now, without loss of generality, consider the conflicting case illustrated in Fig. \(4.1b\) (i.e., \(3t > 0\) such that \(\widetilde{\rho}^h_i(t) - \rho^*_i(t) < 0\) and \(\widetilde{\rho}^*_i(t) - \rho^h_i(t) < 0\), \(\forall t \geq 0\)). Based on a user-defined minimal distance \(\mu > 0\) between the conflicting constraints \(\rho^h_i(t)\) and \(\rho^*_i(t)\), the idea is to design a triggering process by which \(\rho^U_i(t)\) in \((4.5b)\) starts disregarding the soft constraint’s boundary \(\rho^*_i(t)\), thus allowing the output \(x_i(t)\) to enter the region \(\rho^*_i(t) < x_i(t) < \rho^h_i(t) + \varphi^U_i(t)\). In this respect, as soon as \(\rho^*_i(t) - \rho^h_i(t) \leq \mu\), the term \(1/(\eta^U_i + \varphi^U_i) = 1/((\rho^*_i + \varphi^U_i) - \rho^h_i)\) in \((4.6b)\) becomes active and sufficiently increases \(\varphi^U_i(t)\). On the other hand, whenever the conflict between hard and soft constraints is resolved (i.e., \(\rho^*_i(t) - \rho^h_i(t) > \mu\), \((4.6b)\) reduces to \(\dot{\varphi}^U_i = -k_c \varphi^U_i\) and ensures exponential convergence of \(\varphi^U_i(t)\) to zero. Owing to \((4.6b)\), this allows \(\rho^U_i(t)\) to converge exponentially towards \(\rho^*_i(t)\), i.e., the violated soft constraint gets recovered exponentially fast. In this case, the rate of convergence to \(\rho^*_i(t)\) can be adjusted by tuning the constant \(k_c\). Moreover, a larger \(k_c\) can impede the growth of \(\varphi^U_i(t)\) (and also \(\varphi^L_i(t)\)), leading to a less conservative violation of the soft constraints. Finally, notice that in the described case if at some
time $\varphi_i^L(t)$ increases such that $\tilde{p}_i^h(t) + \varphi_i^U(t) > \beta_i^h(t)$, according to (4.5b), $\rho_i^U(t)$ will become equal to $\rho_i^h(t)$ to respect the hard constraint and due to Assumption 4.2 the CCF’s boundaries $\rho_i^U(t) = \rho_i^h(t)$ and $\rho_i^L(t) = \rho_i^l(t)$ will remain feasible. In a similar fashion, one can justifi the proposed dynamic modification of $\rho_i^U(t)$ in (4.5a).

Note that, $\rho_i^L(t)$ and $\rho_i^U(t)$, $i = \{1, \ldots, n\}$ obtained from (4.5) are continuous (but in general nonsmooth) functions of time. Moreover, one can verify from (4.6) that $\dot{\varphi}_i^L(t)$ and $\dot{\varphi}_i^U(t)$ are piecewise continuous functions of time.

**Lemma 4.1.** Under Assumptions 4.2 and 4.3, the online dynamic CCF planning scheme in (4.5), (4.6) constructs $\rho_i^U(t)$ and $\rho_i^L(t)$ such that: (i) $\dot{\rho}_i^L(t), \dot{\rho}_i^U(t) \in L_\infty$ (are bounded signals), and (ii) $\rho_i^L(t) \rho_i^L(t)$, $\rho_i^U(t)$, $\rho_i^L(t)$, $\rho_i^U(t)$ are piecewise continuous functions of time.

**Proof.** First, we establish $\varphi_i^L, \varphi_i^L \in L_\infty, i = \{1, \ldots, n\}$. Consider $\dot{\varphi}_i^L$ given by (4.6a), which operates in two modes:

- **Mode I.** When $\eta_i^L(t) > \mu$, (4.6a) reduces to $\dot{\varphi}_i^L = -k_\varphi \varphi_i^L$, thus $\varphi_i^L(t)$ becomes exponentially stable and $\varphi_i^L, \dot{\varphi}_i^L \in L_\infty$.

- **Mode II.** When $\eta_i^L(t) \mu$, the first term on the right hand-side of (4.6a) is active. It holds that if $\eta_i^L(t) + \varphi_i^L(t) \rightarrow 0$ (does not converge to zero), then $\varphi_i^L \in L_\infty$. Note that, by assumption we have $\tilde{p}_i^h, \tilde{p}_i^h, \tilde{p}_i^h, \tilde{p}_i^h \in L_\infty$, which leads to $\eta_i^L, \eta_i^L \in L_\infty$. Now, let $\eta_i^L(t) + \varphi_i^L(t) \rightarrow 0$, which also indicates $\eta_i^L + \varphi_i^L < 0$ or $\eta_i^L + \varphi_i^L > 0$ depending on the sign of $\eta_i^L(t) + \varphi_i^L(t)$ at Mode II’s activation time. Owing to (4.6a), $\eta_i^L(t) + \varphi_i^L(t) \rightarrow 0$ leads to $\dot{\varphi}_i^L \rightarrow +\infty$, which also requires $\dot{\eta}_i^L \rightarrow -\infty$ or $\dot{\eta}_i^L \rightarrow +\infty$, however, this is a contradiction, since we had assumed $\eta_i^L \in L_\infty$. Therefore, $\eta_i^L(t) + \varphi_i^L(t) \rightarrow 0$ and $\dot{\varphi}_i^L \in L_\infty$. Now let $\dot{\varphi}_i^L \rightarrow +\infty$ (resp. $\dot{\varphi}_i^L \rightarrow -\infty$), since $\eta_i^L \in L_\infty$, from (4.6a) we get $\dot{\varphi}_i^L \rightarrow +\infty$ (resp. $\dot{\varphi}_i^L \rightarrow -\infty$), which contradicts the infinite growth of $\varphi_i^L$, thus $\varphi_i^L \in L_\infty$.

Next we prove that if $\varphi_i^L(0) \geq 0$, indeed we will have $\eta_i^L(t) + \varphi_i^L(t) \geq \epsilon_i^L > 0$ ($\epsilon_i^L$ is a positive constant) and $\dot{\varphi}_i^L(t) \geq 0$, $\forall t \geq 0$, $i = \{1, \ldots, n\}$. Consider a sequence of switching times between Modes I and II in (4.6a): $\{t_1, \ldots, t_j\}, j \in \mathbb{N}$, where $0 < t_1 < \ldots < t_j$. Notice that Zeno behavior is excluded as $\eta_i^L(t)$ and $\eta_i^L(t)$ in (4.6) are continuous time-varying signals independent of $\varphi_i^L$ and $\varphi_i^L$. Now consider two cases:

- **Case I.** Suppose that at $t = 0$ (4.6a) starts evolving under Mode I (that means $\eta_i^L(0) > \mu$). In this case, since (4.6a) in Mode I is an exponentially stable system if $\varphi_i^L(0) \geq 0$, then $\varphi_i^L(t) \geq 0$ for all $t \leq t_1$. As soon as Mode II becomes active at time $t = t_1$, we have $\eta_i^L(t_1) = \mu > 0$ and $\varphi_i^L(t_1) \geq 0$. Since it is proved that $\eta_i^L(t) + \varphi_i^L(t) \rightarrow 0$, for $t_1 \leq t \leq t_2$ and we also have $\eta_i^L(t_1) + \varphi_i^L(t_1) > 0$, we can infer that there exists a $\epsilon_i^L > 0$ such that $\eta_i^L(t) + \varphi_i^L(t) \geq \epsilon_i^L > 0$, for $t_1 \leq t \leq t_2$. Moreover, as $\eta_i^L(t) + \varphi_i^L(t) > 0$, for $t_1 \leq t \leq t_2$, if $\varphi_i^L(t) = 0$ for $t \in [t_1, t_2]$, $\varphi_i^L(t)$ in (4.6a) (operating in Mode II) will remain positive and prevents $\varphi_i^L(t)$ from getting negative, thus $\varphi_i^L(t) \geq 0$, for $t_1 \leq t \leq t_2$. Followed by this, when (4.6a) switches to Mode I at $t = t_3$, again we will have $\varphi_i^L(t_3) \geq 0$, and $\eta_i^L(t_3) > \mu > 0$ so the above
results can be repeatedly extended for any $t \geq t_3$. Therefore, $\eta^L_i(t) + \phi^L_i(t) \geq \epsilon^L_i > 0$ and $\varphi^L_i(t) \geq 0$, $\forall t \geq 0$.

**Case II.** Now this time suppose that at $t = 0$ (4.6a) starts running under Mode II (that means $\eta^U_i(0) \leq \mu$). Note that due to Assumption 4.3 we have $\eta^L_i(0) > 0$. Therefore, similarly to Case I, we can prove that $\eta^L_i(t) + \phi^L_i(t) \geq \epsilon^L_i > 0$ and $\varphi^L_i(t) \geq 0$, $\forall t \geq 0$.

In a similar fashion, based on (4.6b) one can show that: (i) $\eta^U_i(t) + \phi^U_i(t) \to 0$ and $\varphi^U_i(t) \geq 0$, $\forall t \geq 0$, i.e., $\varphi^U_i(t) \geq 0$, $\forall t \geq 0$, $i = \{1, \ldots, n\}$.

Owing to (4.5), we know that $\hat{\rho}^L_i(t) \in \{\hat{\rho}_1^L(t) - \hat{\varphi}^L_i(t), \hat{\varphi}^L_i(t)\}$ and $\hat{\rho}^U_i(t) \in \{\hat{\rho}_1^U(t) + \varphi^U_i(t), \hat{\varphi}^U_i(t)\}$. Since $\hat{\rho}_1^L(t), \hat{\rho}_1^U(t), \hat{\varphi}_1^L(t), \hat{\varphi}_1^U(t) \in \mathcal{L}_\infty$ and $\varphi^L_i(t), \varphi^U_i(t) \in \mathcal{L}_\infty$ hold, we get $\rho^L_i, \rho^U_i \in \mathcal{L}_\infty$, $\forall t \geq 0$, $i = \{1, \ldots, n\}$. Moreover, followed by $\hat{\rho}_1^L(t), \hat{\rho}_1^U(t), \hat{\varphi}_1^L(t), \hat{\varphi}_1^U(t) \in \mathcal{L}_\infty$ and boundedness of $\varphi^L_i(t), \varphi^U_i(t)$, (4.5) establishes $\rho^L_i, \rho^U_i \in \mathcal{L}_\infty$, $\forall t \geq 0$, $i = \{1, \ldots, n\}$.

Finally, from (4.5) we get $\rho^U_i(t) - \rho^L_i(t) \in \{\rho^1_i - \rho^1_i, \rho^1_i - \rho^1_i + \varphi^L_i(t), \rho^1_i - \rho^1_i + \varphi^L_i(t), \rho^1_i + \varphi^U_i(t)\}$, where due to Assumption 4.2 and the nonnegativity of $\varphi^L_i(t), \varphi^U_i(t)$, the first and the fourth elements are lower bounded by a positive constant. Moreover, the second and third elements of the above set are always lower bounded by a positive constant, owing to $\eta^L_i(t) + \phi^L_i(t) \geq \epsilon^L_i > 0$ and $\eta^L_i(t) + \phi^L_i(t) \geq \epsilon^L_i > 0$, respectively. Therefore, there exists a positive constant $\epsilon^L_i = \min\{\epsilon^L_i, \epsilon^L_i, \epsilon^L_i, \epsilon^L_i\}$ such that $\rho^U_i(t) - \rho^L_i(t) \geq \epsilon^L_i > 0$, $\forall t \geq 0$, $i = \{1, \ldots, n\}$.

**Remark 4.2** (Smooth CCF). The online funnel planning scheme given by (4.5) and (4.6) provides a continuous but (in general) nonsmooth CCF, which can lead to continuous but nonsmooth control inputs under funnel-based control design methods. There are two sources for nonsmoothness in $\rho^L_i(t), \rho^U_i(t)$: (i) the nonsmooth switch in (4.6) and (ii) the nonsmooth max and min operators in (4.5). To generate smooth $\rho^L_i(t), \rho^U_i(t)$ we can use smooth switching mechanisms instead of the nonsmooth ones in (4.6). In this respect, $\text{sign}(\eta^L_i - \mu)$ and $\text{sign}(\eta^U_i - \mu)$ in (4.6) can be replaced by $\tanh(\kappa (\eta^L_i - \mu))$ and $\tanh(\kappa (\eta^U_i - \mu))$, respectively, where a larger $\kappa > 0$ captures the sign($\cdot$) function behavior better. Moreover, instead of nonsmooth max and min operators in (4.5), one can utilize their smooth over- and under-approximations using (2.18), to obtain:

$$
\rho^L_i(t) := \frac{1}{\nu} \ln \left[ e^{\nu (\varphi^L_i(t) - \varphi^L_i(t))} + e^{\nu \phi^L_i(t)} \right] > \rho^L_i(t), \quad (4.7a)
$$

$$
\rho^U_i(t) := -\frac{1}{\nu} \ln \left[ e^{-\nu (\varphi^L_i(t) + \varphi^L_i(t))} + e^{-\nu \phi^L_i(t)} \right] < \rho^U_i(t). \quad (4.7b)
$$

Note that, $\rho^L_i(t)$ and $\rho^U_i(t)$ determine a smooth inner-approximation of the CCF defined by (4.5). It is important to note that increasing the value of $\nu$ enhances the quality of this inner-approximation. However, using an excessively small $\nu$ in (4.7) may result in an overly conservative inner-approximation of the original (nonsmooth) CCF, potentially jeopardizing its feasibility.
4.4. Online Constraint Consistent Funnel Planning

On Relaxation of Assumption 4.3:

Before proposing the controller design, let us describe how the choice of \( \varphi_i^L(0) \) and \( \varphi_i^U(0) \) in (4.6) can lead to relaxation of Assumption 4.3. In this respect, we introduce the following assumption instead of Assumption 4.3.

**Assumption 4.4.** The initial outputs of (4.1), \( x_i(0), i = \{1, \ldots, n\} \), satisfy the hard funnel constraints (4.2a) at \( t = 0 \), i.e., \( x_i(0) \in (\rho_i^h(0), \overline{\rho}_i^h(0)), i = \{1, \ldots, n\} \).

Clearly, Assumption 4.4 only requires that the initial values of the system’s outputs satisfy the hard constraints without any necessity for the compatibility condition of hard and soft constraints at \( t = 0 \) (Assumption 4.3). To ensure that the proposed funnel planning approach is still useful under Assumption 4.4 we propose choosing \( \varphi_i^L(0) \) and \( \varphi_i^U(0) \) in (4.6) as follows:

\[
\begin{align*}
\varphi_i^L(0) &= \begin{cases} 
0 & x_i(0) - \rho_i^s(0) > 0 \\
\rho_i^s(0) - x_i(0) + \varsigma_i^L & \text{otherwise}
\end{cases}, \\
\varphi_i^U(0) &= \begin{cases} 
0 & \overline{\rho}_i^s(0) - x_i(0) > 0 \\
x_i(0) - \overline{\rho}_i^s(0) + \varsigma_i^U & \text{otherwise}
\end{cases},
\end{align*}
\]

(4.8a) (4.8b)

where \( \varsigma_i^L \) and \( \varsigma_i^U \) are some user-defined sufficiently small positive constants.

First, note that (4.8) always ensures that \( \varphi_i^L(0), \varphi_i^U(0) \geq 0 \). Moreover, since \( x_i(0) \in (\rho_i^h(0), \overline{\rho}_i^h(0)) \) holds by assumption, satisfaction of both \( x_i(0) - \rho_i^s(0) > 0 \) and \( \overline{\rho}_i^s(0) - x_i(0) > 0 \) indicates that the soft constrained funnel (4.2b) is compatible with the hard constrained funnel (4.2a) at \( t = 0 \). Indeed, for this particular case since Assumption 4.3 holds (4.8) gives \( \varphi_i^L(0) = \varphi_i^U(0) = 0 \). On the other hand, if either \( x_i(0) - \rho_i^s(0) > 0 \) or \( \overline{\rho}_i^s(0) - x_i(0) > 0 \) does not hold (note that only one of these two conditions can be violated since \( \overline{\rho}_i^s(0) > \rho_i^s(0) \)), then \( x_i(0) \) does not lie within the soft constrained funnel (i.e., \( x_i(0) \notin (\rho_i^s(0), \overline{\rho}_i^s(0)) \)). Notice that in this scenario it is undetermined whether the soft constrained funnel (4.2b) is compatible with the hard constrained funnel (4.2a) at \( t = 0 \). For example, consider (4.8b) and let \( \overline{\rho}_i^s(0) - x_i(0) \leq 0 \), employing (4.8b) in (4.5b) yields \( \rho_i^U(0) := \min\{x_i(0) + \varsigma_i^U, \overline{\rho}_i^h(0)\} \), which gives a valid upper bound for \( x_i(t) \) at \( t = 0 \) (i.e., \( x_i(0) < \rho_i^U(0) \)). Note that even if \( \varsigma_i^U \) is chosen to be large enough such that \( x_i(0) + \varsigma_i^U > \overline{\rho}_i^h(0) \), \( \rho_i^U(0) \) will become equal to \( \overline{\rho}_i^h(0) \), which is still a valid upper bound according to Assumption 4.4. Now if the soft constrained funnel is compatible with the hard constrained funnel under the given user defined minimal distance \( \mu \) (i.e., \( \eta_i^U = \rho_i^U(t) - \rho_i^h(t) > \mu \)) for \( t \geq 0 \), then as discussed before according to (4.6b), \( \varphi_i^U(t) \) tends to decrease exponentially towards zero, thus based on (4.5b) the compatible soft constraint boundary \( \overline{\rho}_i^h(t) \) gets recovered exponentially fast. Conversely, if the soft constrained funnel is incompatible with the hard constrained funnel under the given user defined minimal distance (i.e., \( \eta_i^U = \rho_i^U(t) - \rho_i^h(t) \leq \mu \)) for \( t \geq 0 \), then (4.6b) ensures evolution of \( \varphi_i^U(t) \) such that the online planned CCF boundary \( \rho_i^U(t) \) gets as much as possible closer to the incompatible soft constraint bound \( \overline{\rho}_i^h(t) \) (depending on
the tuning parameter $k_c$) while respecting the hard constrained funnel. Similarly, one can justify the application of (4.8a) in choosing $\varphi_i^L(0)$ under Assumption 4.4.

Finally, we emphasize that, if $\varphi_i^L(0)$ and $\varphi_i^U(0)$ are selected according to (4.8), then it is not difficult to show that the results stated in Lemma 4.1 remain valid by replacing Assumption 4.3 with Assumption 4.4.

4.5 Funnel-Based Controller Design

In this section, inspired by [54], we design a robust low-complexity funnel-based controller using the Prescribed Performance Control (PPC) to ensure that the output signals $x_i(t), i = \{1, \ldots, n\}$ will always remain within the online planned (in general asymmetric) CCF (4.3).

The EL system (4.1) can be re-written in state-space form as follows:

\[
\begin{aligned}
\dot{x} &= v, \\
\dot{v} &= M^{-1}(x)(-C(x,v)v - g(x) - D(x)v + u + d(t)).
\end{aligned}
\] (4.9)

The controller design is two-fold:

(I) velocity-level control design,

(II) acceleration-level control design.

Step I-a. Given the initial output values $x(0)$ and hard constraints (4.2a) at $t = 0$, determine boundary functions $\rho_i^s(t), \overline{\rho}_i^s(t)$ of the soft constraints in (4.2b) (i.e., user defined performance constraints) according to the control application (e.g., see Remark 4.1) such that conditions in Assumption 4.3 are met.

Step I-b. Define the normalized system outputs, w.r.t. the asymmetric funnel given by (4.3), as $\hat{x}(t,x) := \text{col}(\hat{x}_i(t,x_i)) \in \mathbb{R}^n$, where:

\[
\hat{x}_i(t,x_i) := \frac{x_i - \frac{1}{2}(\rho_i^U(t) + \rho_i^L(t))}{\frac{1}{2}(\rho_i^U(t) - \rho_i^L(t))}, \quad i = \{1, \ldots, n\},
\] (4.10)

in which $\hat{x}_i \in (-1,1)$ if and only if $x_i \in (\rho_i^L(t), \rho_i^U(t))$. Moreover, define control related signals $\xi_x := \text{diag}(\xi_{x_i}(t,\hat{x}_i)) \in \mathbb{R}^{n \times n}$ and $\varepsilon_x := \text{col}(\varepsilon_{x_i}(\hat{x}_i)) \in \mathbb{R}^n$, where:

\[
\begin{aligned}
\xi_{x_i}(t,\hat{x}_i) &:= 4 \frac{1}{(\rho_i^U(t) - \rho_i^L(t))(1 - \hat{x}_i^2)}, \\
\varepsilon_{x_i}(\hat{x}_i) &:= T(\hat{x}_i) := \ln \left( \frac{1 + \hat{x}_i}{1 - \hat{x}_i} \right).
\end{aligned}
\] (4.11a, 4.11b)

Finally, design the desired reference velocity vector as:

\[
v_d(t,\hat{x}) := -k_x \xi_x \varepsilon_x,
\] (4.12)

\[\text{When Assumption 4.4 holds, by utilizing (4.8) this step can be skipped.}\]
with \( k > 0 \), where \( v_d(t, \hat{x}) = \text{col}(v_d(t, \hat{x}_i)) \in \mathbb{R}^n \).

**Step II-a.** Define the vector of velocity errors \( e_v := \text{col}(e_{v_i}) = v - v_d \in \mathbb{R}^n \). Now the objective is to design the acceleration level controller \( u \) in (4.9) to compensate the velocity errors by enforcing an (optionally symmetric) exponentially narrowing funnel on \( e_{v_i}(t) \) indicated by:

\[
-\gamma_i^v(t) < e_{v_i}(t) < \gamma_i^v(t), \quad \forall t \geq 0, \quad i = \{1, \ldots, n\},
\]

where \( \gamma_i^v(\cdot) : \mathbb{R} \to \mathbb{R}_{>0}, i = \{1, \ldots, n\} \), are continuously differentiable positive performance functions for the velocity errors, which decay to a small neighborhood of zero. A possible choice for \( \gamma_i^v(t) \) is \((\rho_{v_0}^i - \rho_{v_{x_i}}^i)\exp(-l_i^v t) + \rho_{v_{x_i}}^i\), where \( l_i^v, \rho_{v_{x_i}}^i \), are user-defined positive constants and \( \rho_{v_0}^i > |e_{v_i}(0)| \), ensuring \( e_{v_i}(0) \in (-\gamma_i^v(0), \gamma_i^v(0)) \), \( i = \{1, \ldots, n\} \).

**Step II-b.** Similarly to the first step, define the normalized velocity errors, w.r.t. the symmetric funnel constraints in (4.13), as \( \hat{e}_{v_i}(t, e_v) := \text{col}(\hat{e}_{v_i}(t, e_{v_i})) \in \mathbb{R}^n \), where:

\[
\hat{e}_{v_i}(t, e_{v_i}) := \frac{e_{v_i}}{\gamma_i^v(t)}, \quad i = \{1, \ldots, n\},
\]

in which \( \hat{e}_{v_i} \in (-1, 1) \) i and only if \( e_{v_i} \in (-\gamma_i^v(t), \gamma_i^v(t)) \). In addition, define control related signals \( \xi_v := \text{diag}(\xi_{v_i}(t, \hat{e}_{v_i})) \in \mathbb{R}^{n \times n} \) and \( \varepsilon_v := \text{col}(\varepsilon_{v_i}(\hat{e}_{v_i})) \in \mathbb{R}^n \), where:

\[
\xi_{v_i}(t, \hat{e}_{v_i}) := \frac{2}{\gamma_i^v(t)(1 - \hat{e}_{v_i}^2)}, \quad (4.15a)
\]

\[
\varepsilon_{v_i}(\hat{e}_{v_i}) = T(\hat{e}_{v_i}) := \ln\left(\frac{1 + \hat{e}_{v_i}}{1 - \hat{e}_{v_i}}\right), \quad (4.15b)
\]

Finally, design the control input \( u \) for (4.9) as:

\[
u(t, \hat{e}_v) := -k_v \xi_v \varepsilon_v,
\]

with \( k_v > 0 \), where \( u(t, \hat{e}_v) = \text{col}(u_i(t, \hat{e}_{v_i})) \in \mathbb{R}^n \).

**Remark 4.3.** The prescribed performance control technique guarantees prescribed transient and steady-state performance specifications that are encapsulated by a time-varying funnel condition as in (4.3) (or (4.13)). The basic idea is to enforce the normalized state in (4.10) (resp. normalized error in (4.14)) to remain strictly within the set \((-1, 1)\). This is achieved through exploiting a smooth, strictly increasing, bijective nonlinear mapping function \( T(\cdot) : (-1, 1) \to (-\infty, +\infty) \), as in (4.12) (resp. (4.15b)), which transforms the normalized state (resp. error) from constrained space \((-1, 1)\) to an unconstrained one over \( \mathbb{R} \). In particular when \( x_i(0) \in (\rho_{v_0}^i(0), \rho_{v_{x_i}}^i(0)) \) (resp. \( e_{v_i}(0) \in (-\gamma_i^v(0), \gamma_i^v(0)) \)), the mapped signal \( \varepsilon_{x_i}(\hat{x}_i) \) (resp. \( \varepsilon_{v_i}(\hat{e}_{v_i}) \)) is initially well-defined. In this case, one can verify that maintaining boundedness of the mapped signal \( \varepsilon_{x_i}(\hat{x}_i) \) (resp. \( \varepsilon_{v_i}(\hat{e}_{v_i}) \)) for all \( t \geq 0 \) is equivalent to ensuring \( \hat{x}_i(t) \in (-1, 1) \) (resp. \( \hat{e}_{v_i}(t) \in (-1, 1) \)).
4.6 Stability Analysis

The following theorem summarizes the main result of this chapter, where it is proven that the proposed feedback control law \([4.16]\) is capable of maintaining \(x_i(t), i = \{1, \ldots, n\}\) within the online planned CCRs in Section 4.4 thus solving the robust hard and soft output constrained funnel control problem for uncertain EL systems.

**Theorem 4.1.** Consider the Euler-Lagrange system \([4.1]\) with hard and soft output constraints \([4.2a]\) and \([4.2b]\) under Assumptions \([4.1]\) and \([4.3]\). Given \(\rho_i^L(t), \rho_i^U(t), i = \{1, \ldots, n\}\) obtained from the constraint consistent online funnel planning scheme in \([4.5]\) and \([4.6]\), the feedback control law \([4.16]\) guarantees satisfaction of \(\rho_i^L(t) < x_i(t) < \rho_i^U(t), \forall t \geq 0, i = \{1, \ldots, n\}\), as well as boundedness of all closed-loop signals.

**Proof.** The proof comprises of three phases. First, we show that \(\hat{x}_i(t, x_i), \hat{e}_{v_i}(t, e_{v_i}), i = \{1, \ldots, n\}\) remain within \((-1, 1)\) for a specific time interval \([0, \tau_{max}]\) (i.e., the existence and uniqueness of maximal solutions). Next, we prove that the proposed control scheme guarantees, for all \([0, \tau_{max}]\): (i) the boundedness of all closed loop signals as well as (ii) that \(\hat{x}_i(t, x_i), \hat{e}_{v_i}(t, e_{v_i})\) remain strictly in a compact subset of \((-1, 1)\), which leads to \(\tau_{max} = \infty\) (i.e., forward completeness), thus finalizing the proof.

Differentiating \(\hat{x}_i \) \([4.10]\) and \(\hat{e}_{v_i} \) \([4.14]\) yields:

\[
\begin{align*}
\dot{x}_i &= \frac{2}{\rho_i^L - \rho_i^U} \left[ v_i - \frac{1}{2} \left( \rho_i^U + \rho_i^L \right) + \hat{x}_i (\rho_i^U - \rho_i^L) \right], \quad i = \{1, \ldots, n\} \quad (4.17a) \\
\dot{e}_{v_i} &= (\gamma^v)^{-1} [\hat{e}_{v_i} - \hat{e}_{v_i} (\gamma^v)'], \quad i = \{1, \ldots, n\}. \quad (4.17b)
\end{align*}
\]

Define \(\rho_m := \text{diag}(\rho_i^U - \rho_i^L) \in \mathbb{R}^{n \times n}, \rho_a := \text{col}(\rho_i^L + \rho_i^L) \in \mathbb{R}^n, \gamma^v := \text{diag}(\gamma_i^v) \in \mathbb{R}^{n \times n}\), then the above equations can be written in stacked form as:

\[
\begin{align*}
\dot{x} &= 2(\rho_m(t))^{-1} \left( v - \frac{1}{2} (\rho_a(t) - \rho_m(t) \dot{x}) \right), \quad (4.18a) \\
\dot{e}_v &= (\gamma^v(t))^{-1} (\dot{e}_v - \gamma^v(t) \dot{e}_v). \quad (4.18b)
\end{align*}
\]

Now employing \([4.9]\), as well as the fact that \(v = v_d + \gamma^v \dot{e}_v, \dot{e}_v = \dot{v} - \dot{v}_d, \) and substituting \([4.12]\) and \([4.16]\) into \([4.18]\) gives:

\[
\begin{align*}
\dot{x} := h_x(t, \dot{x}, \dot{e}_v) &= 2(\rho_s(t))^{-1} \left[ v_d(t, \dot{x}) + \gamma^v(t) \dot{e}_v - \frac{1}{2} (\rho_a(t) - \rho_m(t) \dot{x}) \right], \quad (4.19a) \\
\dot{e}_v := h_{e_v}(t, \dot{x}, \dot{e}_v) &= (\gamma^v(t))^{-1} \left[ M^{-1}(x) \left( -C(x, v) v - g(x) - D(x) v + u(t, \dot{e}_v) + d(t) \right) - \dot{v}_d(t, \dot{x}) - \dot{\gamma}^v(t) \dot{e}_v \right]. \quad (4.19b)
\end{align*}
\]

The results of this theorem remains valid if Assumption \([4.3]\) is replaced with Assumption \([4.4]\). Moreover, in this case the proof does not change.
Note that in the above equations, based on (4.10), \( x \) can be rewritten as \( x(t, \hat{x}) \). Moreover, from \( v = v_d + \gamma^v \hat{e}_v \), and (4.12), we have \( v = v(t, \hat{x}, \hat{e}_v) \). Thus, the closed-loop dynamical system of \( [\hat{x}, \hat{e}_v]^T \) can be written in compact form as:

\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{e}}_v
\end{bmatrix} =
\begin{bmatrix}
h_x(t, \hat{x}, \hat{e}_v) \\
h_{\hat{e}_v}(t, \hat{x}, \hat{e}_v)
\end{bmatrix} = h(t, \hat{x}, \hat{e}_v). \tag{4.20}
\]

Let us also define an open set \( \Omega_h = \Omega_{\hat{x}} \times \Omega_{\hat{e}_v} \subset \mathbb{R}^{2n} \) with:

\[
\Omega_{\hat{x}} = \Omega_{\hat{e}_v} = (-1, 1) \times \ldots \times (-1, 1).
\]

**Phase I.** It is clear that the set \( \Omega_h \) is nonempty and open. Followed by Assumption 4.3 and usage of (4.5), \( x_i(0), i = 1, \ldots, n \) satisfy (4.3) at \( t = 0 \), which based on (4.10) ensures \( \hat{x}(0) \in \Omega_{\hat{x}} \). Moreover, followed by (4.14), selecting \( \rho^0_0 > |e_{v_i}(0)| \) in \( \gamma^v_i(t), i = 1, \ldots, n \) guarantees \( e_{v_i}(0) \) to satisfy (4.13) at \( t = 0 \), which leads to \( \hat{e}_v(0) \in \Omega_{\hat{e}_v} \) based on (4.14). Additionally, \( h(t, \hat{x}, \hat{e}_v) \) is piecewise continuous on \( t \) and locally Lipschitz on \( [\hat{x}, \hat{e}_v]^T \) over the set \( \Omega_h \), Therefore, the Theorem 2.3 hold and the existence and uniqueness of a maximal solution \( [\hat{x}(t), \hat{e}_v(t)]^T \) of (4.20) for a time interval \([0, \tau_{\text{max}}]\) such that \( [\hat{x}(t), \hat{e}_v(t)]^T \in \Omega_h, \forall t \in [0, \tau_{\text{max}}] \) is guaranteed. Accordingly, \( \hat{x}(t) \in \Omega_{\hat{x}} \) and \( \hat{e}_v(t) \in \Omega_{\hat{e}_v}, \forall t \in [0, \tau_{\text{max}}] \). As a result, we can further infer that \( x_i(t) \) and \( e_{v_i}(t), i = 1, \ldots, n \) are bounded for all \( t \in [0, \tau_{\text{max}}] \) as in (4.3) and (4.13), respectively.

**Phase II.** Owing to \( \hat{x}(t) \in \Omega_{\hat{x}} \) and \( \hat{e}_v(t) \in \Omega_{\hat{e}_v}, \forall t \in [0, \tau_{\text{max}}] \), which reveal \( \hat{x}_i(t) \in (-1, 1) \) and \( \hat{e}_{v_i}(t) \in (-1, 1), \forall t \in [0, \tau_{\text{max}}], \xi_{x_i}, \xi_{e_{v_i}}, i = 1, \ldots, n \) in (4.11a) and (4.15a) are lower bounded by a positive constant \( \forall t \in [0, \tau_{\text{max}}] \). Moreover, the signals \( \varepsilon_{x_i} \) and \( \varepsilon_{e_{v_i}} \), defined in (4.11b) and (4.15b), are well-defined for all \( t \in [0, \tau_{\text{max}}] \).

**Step 1:** Consider the following positive definite and radially unbounded Lyapunov function candidate: \( V_1 = \frac{1}{2} \xi^T \xi_x \). Differentiating \( V_1 \) with respect to time, substituting (4.18a), (4.9), (4.12), and exploiting (4.11a) and diagonality of \( \xi_x \) gives:

\[
\dot{V}_1 = \xi^T \dot{\xi}_x \left[ -k_x \xi_x \varepsilon_x + \gamma_{v} \hat{e}_v - \frac{1}{2} (\hat{\rho}_a + \hat{\rho}_m \hat{x}) \right]
\leq -k_x \| \xi_x \| \| \xi_x \| \| \varepsilon_x \| + \| \xi_x \| \| \varepsilon_x \| \| \Psi \|, \tag{4.21}
\]

where \( \Psi = \gamma_{v} \hat{e}_v - \frac{1}{2} (\hat{\rho}_a + \hat{\rho}_m \hat{x}) \in \mathbb{R}^n \). From Phase I we have \( \hat{e}_v, \hat{x} \in \mathcal{L}_x, \forall t \in [0, \tau_{\text{max}}] \). Moreover, Lemma 4.1 indicates \( \hat{\rho}_a, \hat{\rho}_m \in \mathcal{L}_x, \forall t \geq 0 \), and \( \gamma_{v} \in \mathcal{L}_x \) by construction. Hence, \( \Psi \in \mathcal{L}_x, \forall t \in [0, \tau_{\text{max}}] \). Let \( 0 < \theta_x < k_x \) and \( \sigma_x := k_x - \theta_x > 0 \) be constants; thus adding and subtracting \( \theta_x \| \xi_x \| \| \varepsilon_x \| \) to the right-hand side of (4.21) yields:

\[
\dot{V}_1 \leq -\sigma_x \| \xi_x \| \| \varepsilon_x \| - \| \xi_x \| \| \varepsilon_x \| \left( \theta_x \| \xi_x \| \| \varepsilon_x \| - \| \Psi \| \right)
= -\sigma_x \| \xi_x \| \| \varepsilon_x \|, \quad \forall \| \varepsilon_x \| \geq \frac{\| \Psi \|}{\theta_x \| \xi_x \|}, \quad \forall t \in [0, \tau_{\text{max}}], \tag{4.22}
\]
which, indicates that $\varepsilon_x$ is Uniformly Ultimately Bounded (UBB) (see Theorem 2.4), thus there exists an ultimate bound $\bar{\varepsilon}_x \in \mathbb{R}_{>0}$ independent of $\tau_{\text{max}}$ such that $\|\varepsilon_x\| \leq \bar{\varepsilon}_x$ for all $t \in [0, \tau_{\text{max}}]$. Moreover, by taking the inverse logarithmic function of (4.11b) and exploiting $\bar{\varepsilon}_x$, we obtain:

$$-1 < \frac{e^{-\bar{\varepsilon}_x} - 1}{e^{-\bar{\varepsilon}_x} + 1} := -b_{\bar{\varepsilon}_x} \leq \hat{\varepsilon}_i(t) \leq b_{\bar{\varepsilon}_x} := \frac{e^{\bar{\varepsilon}_x} - 1}{e^{\bar{\varepsilon}_x} + 1} < 1,$$

(4.23)

for all $t \in [0, \tau_{\text{max}}], i = 1, \ldots, n$. Thus, based on (4.11a), $\xi_x \in \mathcal{L}_\infty, \forall t \in [0, \tau_{\text{max}})$, hence, the designed desired reference velocity vector $v_d$ in (4.12), remains bounded for all $t \in [0, \tau_{\text{max}})$. Moreover, invoking $v = v_d + \gamma^v \hat{e}_v$, we also conclude that $v(t) \in \mathcal{L}_\infty, \forall t \in [0, \tau_{\text{max}})$. Finally, by taking the time derivative of $v_d(t, \hat{x})$, substituting (4.18a) and utilizing (4.23) and Lemma 4.1 it is straightforward to deduce $v_d \in \mathcal{L}_\infty$ for all $t \in [0, \tau_{\text{max}})$ as well.

**Step 2:** Consider the following positive definite and radially unbounded Lyapunov function candidate: $V_2 = \frac{1}{2} \varepsilon_v^T \Gamma \varepsilon_v$. Differentiating $V_2$ with respect to time, substituting (4.18a), (4.19), (4.16), and exploiting (4.18b) and diagonality of $\xi_v$ yields:

$$\dot{V}_2 = \varepsilon_v^T \xi_v \left[ -M^{-1} k_v \varepsilon_v v - M^{-1} ((C + D)(v_d + \gamma^v \hat{e}_v) + g - d) - \dot{v}_d - \gamma^v \hat{e}_v \right]$$

$$\leq -k_v \lambda \|\xi_v\|^2 \|\varepsilon_v\|^2 + \|\xi_v\| \|\varepsilon_v\| \|\Gamma\|,$$

(4.24)

where $\lambda$ is the minimum eigenvalue of the positive definite matrix $M^{-1}$ and $\Gamma := M^{-1}((C + D)(v_d + \gamma^v \hat{e}_v) + g - d) - \dot{v}_d - \gamma^v \hat{e}_v$. Note that we have already showed $x, v \in \mathcal{L}_\infty, \forall t \in [0, \tau_{\text{max}})$ in Phase I and Step 1 of Phase II, thus, Property 4.1 for EL system (4.1) indicates boundlessness of the EL system’s matrices and vectors; that is $\|C(x, v)\|, \|g(x)\|, \|D(x)\|, \|M^{-1}(x)\| \in \mathcal{L}_\infty$ for all $t \in [0, \tau_{\text{max}})$. As a result, owing to $x, v \in \mathcal{L}_\infty, \forall t \in [0, \tau_{\text{max}})$ and Property 4.1 of (4.1), boundedness of $d(t), \hat{v} = \gamma^v v, \forall t \geq 0$, as well as boundedness of $\hat{v}_d, \hat{e}_v, \forall t \in [0, \tau_{\text{max}})$ established in Phase I and Phase II-Step 1, we have $\Gamma \in \mathcal{L}_\infty, \forall t \in [0, \tau_{\text{max}})$. Now let $0 < \theta_v < k_v \lambda$ and $\sigma_v := k_v \lambda - \theta_v > 0$ be constants; thus adding and subtracting $\theta_v \|\xi_v\|^2 \|\varepsilon_v\|^2$ in the right-hand side of (4.24) yields:

$$\dot{V}_2 \leq -\sigma_v \|\xi_v\|^2 \|\varepsilon_v\|^2 - \|\xi_v\| \|\varepsilon_v\| (\theta_v \|\xi_v\| \|\varepsilon_v\| - \|\Gamma\|)$$

$$= -\sigma_v \|\xi_v\|^2 \|\varepsilon_v\|^2, \forall \|\varepsilon_v\| \geq \frac{\|\Gamma\|}{\theta_v \|\xi_v\|}, \forall t \in [0, \tau_{\text{max}}),$$

(4.25)

which concludes that $\varepsilon_v$ is UBB from Theorem 2.4, hence, there exists an ultimate bound $\bar{\varepsilon}_v \in \mathbb{R}_{>0}$ independent of $\tau_{\text{max}}$ such that $\|\varepsilon_v\| \leq \bar{\varepsilon}_v$ for all $t \in [0, \tau_{\text{max}})$. Similarly to Step 1, taking the inverse of (4.15b), and using $\bar{\varepsilon}_v$ leads to:

$$-1 < \frac{e^{-\bar{\varepsilon}_v} - 1}{e^{-\bar{\varepsilon}_v} + 1} := -b_{\bar{\varepsilon}_v} \leq \hat{\varepsilon}_v(t) \leq b_{\bar{\varepsilon}_v} := \frac{e^{\bar{\varepsilon}_v} - 1}{e^{\bar{\varepsilon}_v} + 1} < 1,$$

(4.26)

for all $t \in [0, \tau_{\text{max}}), i = 1, \ldots, n$. From (4.15a), we can deduce that $\xi_v \in \mathcal{L}_\infty, \forall t \in [0, \tau_{\text{max}})$, thus the designed control input $u$ in (4.16) is bounded for all $t \in [0, \tau_{\text{max}})$. Therefore, $\xi_v \in \mathcal{L}_\infty, \forall t \in [0, \tau_{\text{max}})$.
4.7 Simulation Results

Phase III. Now we shall establish that $\tau_{\text{max}} = \infty$. In this direction, notice by (4.23) and (4.26) that $[\hat{x}(t), \hat{e}_v(t)]^T \in \Omega_h \triangleq \Omega_\hat{x} \times \Omega_\hat{e}_v$, where $\Omega_\hat{x} := [-b_{\hat{x}_1}, b_{\hat{x}_1}] \times \ldots \times [-b_{\hat{x}_n}, b_{\hat{x}_n}]$ and $\Omega_\hat{e}_v := [-b_{\hat{e}_v}, b_{\hat{e}_v}] \times \ldots \times [-b_{\hat{e}_v}, b_{\hat{e}_v}]$ are nonempty and compact subsets of $\Omega_\hat{x}$ and $\Omega_\hat{e}_v$, respectively. Hence, assuming a finite $\tau_{\text{max}} < \infty$, since $\Omega'_h \subset \Omega_h$, Lemma 2.6 dictates the existence of a time instant $t' \in [0, \tau_{\text{max}})$ such that $[\hat{x}(t'), \hat{e}_v(t')]^T \notin \Omega'_h$, which is a contradiction. Therefore, $\tau_{\text{max}} = \infty$. As a result, all closed-loop control signals remain bounded $\forall t \geq 0$, and moreover $[\hat{x}(t), \hat{e}_v(t)]^T \in \Omega'_h \subset \Omega_h, \forall t \geq 0$. Finally, recall from (4.23) that $\hat{x}_i(t) \in [-b_{\hat{x}_i}, b_{\hat{x}_i}] \subset (-1, 1), \forall t \geq 0, i = \{1, \ldots, n\}$, thus invoking (4.10), it can be deduced that $\rho_{t}^l(t) < x_i(t) < \rho_{t}^u(t)$ for all $t \geq 0, i = \{1, \ldots, n\}$.

4.7 Simulation Results

For the simulation example we consider the case study of a mobile robot operating on a 2-D plane with kinematics and dynamics given by (2.20). As described in Chapter 2 Section 2.6 to avoid the nonholonomic constraints, we can transform the model (2.20) w.r.t the hand position $p_H$ (see Fig. 2.8) and obtain an equivalent EL form as in (4.1) (with $x = p_H$ being the output of (4.1)) that meets Property 4.1. The numerical values of the robot’s parameters are assumed to be $m_R = 3.6$, $I_R = 0.0405$, $D_1 = 0.3$, $D_2 = 0.04$, and $L = 0.2$. Moreover, the bounded external disturbances affecting of the mobile robot’s motion in (2.20) are considered to be $\dot{d}(t) = [0.75 \sin(2t + \frac{\pi}{3}) + 1.5 \cos(3t + \frac{2\pi}{3}), 0.25 \cos(3t + \frac{\pi}{3}) + 0.75 \sin(5t - \frac{\pi}{3})]^T$.

Now consider the scenario described in Remark 4.1 and let $x_d(t) = [-1.5 + 5.8 \cos(0.24t + 1.5), 5.8 \sin(0.24t + 1.5)]^T$ be the trajectory of a moving object (reference trajectory). Let $\rho_{1}^h = -6.58$, $\rho_{1}^h = 6.58$ and $\rho_{2}^h = -4.63$, $\rho_{2}^h = 4.63$ represent the (time-invariant) box shaped hard constraints (4.2a) on robot’s hand position $p_H = x = [x_1, x_2]^T$. Moreover, let $x(0) = [-3.19, 1.70]^T$. In addition, let $\gamma_i(t), i = \{1, 2\}$ be the user-defined performance function (given in Remark 4.1) for trajectory tracking errors with $l_1 = l_2 = 0.7, \rho_{x_1} = \rho_{x_2} = 0.2$ and $\rho_{e_1}, \rho_{e_2}$ are selected such that $\rho_{e_1} > |x_i(0) - x_d_i(0)|, i = \{1, 2\}$. Recall that, in this scenario the soft constraints (4.2b) on $x_i$ (accounting for trajectory tracking performance) are given by $\rho_{x_1}^v(t) = x_{d_1}(t) - \gamma_i(t)$ and $\rho_{x_2}^v(t) = x_{d_2}(t) + \gamma_i(t), i = \{1, 2\}$. In the simulation, the proposed control law (4.16) designed for the EL system (4.1), is applied to (2.20) through the inverse transformation between $\ddot{u}$ and $u$ (see Section 2.6). The numerical values of the parameters used for $\gamma_i(t), i = \{1, 2\}$, in (4.13), are considered as: $\rho_{x_1}^v = \rho_{x_2}^v = 0.1, l_1^0 = l_2^0 = 0.3$, and $\rho_{e_1}^v, i = \{1, 2\}$ are selected such that $\rho_{e_1}^v > |e_{v_i}(0)|$. Moreover, $k_x = 0.2, k_v = 3$ are considered for (4.12) and (4.16), respectively. Finally, in the simulations we employed smooth CCF planning proposed in Remark 4.2 with $\epsilon = 0.01, k_x = 3, \kappa = 4$ and $\nu = 10$.

Under the proposed control scheme, Fig. 4.2 shows snapshots of the mobile robot’s hand position trajectory (in solid blue) when tracking the moving object (depicted by *) whose trajectory and initial position are depicted by the dashed black line and $\bigcirc$, respectively. The red lines depict the hard (box-shaped) con-
straints on the robot’s hand position, and it can be seen that the robot respects the hard constraints and tracks the object whenever it is possible. Figs. 4.3 and 4.4 (on the left) depict the evolution of the mobile robot’s hand position in $X_0$ and $Y_0$ directions with time, i.e., $x_1(t)$ and $x_2(t)$, respectively. In Figs. 4.3 and 4.4 (left), the online planned constraint consistent funnels for $x_1(t)$ and $x_2(t)$ are illustrated by the green regions, that satisfy the hard (safety) and soft constraints (tracking performance) together. Finally, Figs. 4.3 and 4.4 (on the right) show the evolution of the nonnegative modification signals $\varphi_L^i(t), \varphi_U^i(t), i = \{1, 2\}$ that contribute in generating the CCFs for $x_1(t)$ and $x_2(t)$.

Followed by the discussion in Section 4.4, a smaller $k_c$ in (4.6) results in a slower recovery of the soft constraints and a more conservative behavior when the planned CCF violate these constraints (for respecting the hard constraints). The simulation results with $k_c = 0.3$, illustrated in Figs. 4.5, 4.6, and 4.7 support this observation. In this scenario, the mobile robot shows a tendency to violate the soft constraints more frequently, and the recovery rate of the soft constraints is notably slower. Moreover, as expected, in this scenario $\varphi_U^i(t), \varphi_L^i(t), i = \{1, 2\}$ show a larger increase with $k_c = 0.3$.

### 4.8 Conclusions

In this chapter, we introduced a funnel-based control scheme designed for uncertain Euler Lagrange nonlinear systems operating under both hard and soft time-varying output constraints. The hard constraints represent safety specifications, while the soft constraints define performance criteria for the outputs. To address these constraints, we presented an online funnel planning scheme aimed at crafting a constraint-consistent funnel (CCF). This CCF intentionally violates the soft constraints when they conflict with the hard constraints. To ensure effective control, we employed a prescribed performance control approach, devising low-complexity robust control law to keep the system’s outputs within the planned CCF.

It is worth noting that in this chapter, for each $i \in 1, \ldots, n$, although the pairs of hard and soft funnel constraints in (4.2) are interconnected, still there are no couplings between distinct hard (or soft) constraints. This lack of coupling arises because each constraint is applied to an independent system output, aligning with the traditional funnel-based control methods where hard constraints operate independently. In the upcoming chapter, we will present a novel feedback control method tailored for uncertain MIMO nonlinear systems, where it allows for couplings among time-varying (hard) constraints.
Figure 4.2: Mobile robot’s trajectory (blue line) tracking a moving object (dashed line) under hard constraints (red lines) with $k_c = 3$.

Figure 4.3: Evolution of $x_1(t)$ in its respective CCF under hard and soft constraints with $k_c = 3$ (left). The evolution of the associated modification signals (right).

Figure 4.4: Evolution of $x_2(t)$ in its respective CCF under hard and soft constraints with $k_c = 3$ (left). The evolution of the associated modification signals (right).
Figure 4.5: Mobile robot’s trajectory (blue line) tracking a moving object (dashed line) under hard constraints (red lines) with $k_c = 0.3$. 

Figure 4.6: Evolution of $x_1(t)$ in its respective CCF under hard and soft constraints with $k_c = 0.3$ (left). The evolution of the associated modification signals (right). 

Figure 4.7: Evolution of $x_2(t)$ in its respective CCF under hard and soft constraints with $k_c = 0.3$ (left). The evolution of the associated modification signals (right).
This chapter addresses the problem of closed-form feedback control for high-order uncertain MIMO nonlinear systems under generalized time-varying output constraints. Unlike the preceding chapter, this discussion omits the consideration of soft constraints, but focuses instead on effectively addressing potentially coupled time-varying (hard) constraints. The key emphasis lies on managing the invariance problem of a time-varying output-constrained set. The analysis begins by establishing that the satisfaction of multiple time-varying constraints essentially hinges on ensuring the positivity of a scalar variable, representing the signed distance from the boundary of the time-varying output-constrained set.

5.1 Introduction

Existing feedback control approaches for addressing time-varying output constraints fall into three primary categories: Funnel Control (FC) [63, 64], Prescribed Performance Control (PPC) [53, 54], and Time-Varying Barrier Lyapunov Function (TVBLF) methods [62]. Typically, control designs based on FC, PPC, and TVBLF are commonly employed to achieve user-defined transient and steady-state performance for tracking and stabilization errors. These designs restrict errors evolution within user-defined time-varying funnels, serving as the sole output constraints. For example, in these studies, constraints of the form $-\rho_i(t) < e_i = x_i - x_i^*(t) < \rho_i(t)$ are frequently used for independent tracking errors, where $x_i$ represents independent state variables, $x_i^*(t)$ denotes desired trajectories, and $\rho_i(t)$ signifies bounded, strictly positive time-varying functions that model the evolving behavior of these constraints. To ensure the desired transient and steady-state performance of tracking errors, $\rho_i(t)$ is often selected as a strictly positive exponentially decaying function, approaching a small neighborhood of zero [53, 63].
While FC, PPC, and TVBLF approaches have demonstrated success in various applications, they still face limitations when it comes to handling couplings between multiple time-varying constraints. These methods primarily focus on time-varying funnel constraints applied to independent states or error signals, which inherently remain decoupled from each other. In other words, the satisfaction of one funnel constraint does not impact the satisfaction of the others. To be more precise, the funnel constraints considered in FC, PPC, and TVBLF methods can be likened to time-varying box constraints in the system’s output or error space \cite{54, 70, 107}. In this respect, it is also known that standard control methods for dealing with time-varying output constraints (mentioned above) are restricted to systems with the same number of inputs and outputs. However, in various practical applications, such as those involving general safety considerations \cite{149} and general spatiotemporal specifications \cite{101}, there is a need to address arbitrary and potentially coupled multiple time-varying output constraints. Consequently, it becomes crucial to develop control methodologies for uncertain nonlinear systems that can handle a more general class of time-varying output constraints.

5.2 Contributions

In this chapter, we present a novel feedback control law aimed at satisfying potentially coupled, time-varying output constraints for uncertain high-order MIMO nonlinear systems. Drawing inspiration from the approach introduced in \cite{101}, our control design revolves around consolidating all time-varying constraints into a carefully crafted single constraint. To ensure the satisfaction of this consolidating constraint, we introduce a new low-complexity robust control strategy inspired by \cite{54}. Notably, our control method does not rely on approximations or parameter estimation schemes to handle system uncertainties. Additionally, we demonstrate that by adaptively adjusting the consolidating constraint online, we can achieve a least violating solution for the closed-loop system when the constraints become infeasible during an unknown time interval.

In contrast to existing FC, PPC, TVBLF methods, which primarily impose symmetric funnel constraints on system outputs, our approach not only incorporates generic asymmetric funnel constraints but also integrates one-sided (time-varying) constraints on system outputs. This enables the consideration of a broader range of spatiotemporal specifications. Moreover, while the aforementioned control methods require the fulfillment of all output constraints at the initial time, our proposed method allows for achieving convergence to the time-varying output-constrained set within a user-defined finite time, even when the initial constraints are not met. More precisely, our proposed control method guarantees convergence to and invariance of the time-varying output-constrained set within an appointed finite time. Overall, our results extend the scope of feedback control designs for nonlinear systems, accommodating a broader range of time-varying output constraints, and simplifying controller synthesis and stability analysis. Notably, reference tracking under
prescribed transient and steady-state specifications and time-invariant output constraints for nonlinear systems fit into our results as special cases. From a different standpoint, this chapter takes a step towards designing feedback control laws to guarantee the invariance of time-varying constrained sets with an arbitrary shape, while FC, PPC, TVBLF methods are limited to establishing invariance solely for time-varying box constrained sets.

In connection with our methodology presented in this chapter, related works in [149][151] share a common approach of constructing a single time-invariant Control Barrier Function (CBF) to satisfy multiple time-invariant constraints. Although it might be possible to extend the mentioned method for time-varying constraints, traditional control synthesis using the CBF concept typically necessitates precise knowledge of the system dynamics and involves solving an online Quadratic Programming (QP) problem, which may not be favorable in certain applications. In contrast, this chapter offers a computationally tractable (optimization-free) and robust (model-free) feedback control law.

5.3 Problem Formulation

**Notation:** In this chapter, we adopt $I_i^j = \{i, \ldots, j\}$ as the index set notation, where $i, j \in \mathbb{N}$ and $i < j$.

Consider a class of general high-order MIMO nonlinear systems described by the following dynamics:

\[
\begin{aligned}
\dot{x}_i &= f_i(t, \bar{x}_i) + G_i(t, \bar{x}_i)x_{i+1} + w_i(t), \quad i \in I_1^{r-1}, \\
\dot{\bar{x}}_r &= f_r(t, \bar{x}_r) + G_r(t, \bar{x}_r)u + w_r(t), \\
y &= h(t, x_1),
\end{aligned}
\]

where $x_i := [x_{i,1}, x_{i,2}, \ldots, x_{i,n}]^T \in \mathbb{R}^n$, $\bar{x}_i := [x_1^T, \ldots, x_i^T]^T \in \mathbb{R}^{ni}$, $i \in I_1^r$, $r \in \mathbb{N}$, and $x := \bar{x}_r \in \mathbb{R}^{nr}$ is the state vector. Moreover, $u \in \mathbb{R}^n$ and $y = [y_1, y_2, \ldots, y_m]^T \in \mathbb{R}^m$ denote the control input and output vectors, respectively, and $w_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^n, i \in I_1^r$ represent external disturbance vectors. In addition, $f_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^{ni} \to \mathbb{R}^n, i \in I_1^r$ denote the vectors of nonlinear functions that are locally Lipschitz in $\bar{x}_i$ and piecewise continuous in $t$. Moreover, $G_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^{ni} \to \mathbb{R}^{n \times n}, i \in I_1^r$ stand for the control coefficient matrices whose elements are locally Lipschitz in $\bar{x}_i$ and piecewise continuous in $t$. Finally, $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ is a $C^2$ map in $x_1$ and $C^1$ in $t$. In particular, let $h(t, x_1) = [h_1(t, x_1), h_2(t, x_1), \ldots, h_m(t, x_1)]^T$, so that $y_i = h_i(t, x_1), i \in I_1^m$. Let $x(t; x(0), u)$ denote the solution of the closed-loop system (5.1) under the control law $u$ and the initial condition $x(0)$. For brevity in the notation, from now on we will use $x(t; x(0))$ instead of $x(t; x(0), u)$. Moreover, consider $x_1(t; x(0))$ as the partial solution of the closed-loop system (5.1) with respect to states $x_1$ under the initial condition $x(0)$ and the control input $u$.

In this chapter, we pose the following technical assumptions for (5.1). Note that these assumptions do not restrict the applicability of our results, as they are relevant to the high-order practical mechanical systems under consideration.
Assumption 5.1. The functions $f_i(t, \bar{x}_i), i \in T_1^r$, are unknown and there exist locally Lipschitz functions $\tilde{f}_i : \mathbb{R}^{ni} \rightarrow \mathbb{R}^n, i \in T_1^r$, with unknown analytical expressions such that $\|\tilde{f}_i(t, \bar{x}_i)\| \leq \|f_i(\bar{x}_i)\|$, for all $t \geq 0$, and all $\bar{x}_i \in \mathbb{R}^{ni}$.

Assumption 5.2. The matrices $G_i(t, \bar{x}_i), i \in T_1^r$ are unknown and (A) there exist locally Lipschitz functions $\tilde{g}_i : \mathbb{R}^{ni} \rightarrow \mathbb{R}, i \in T_1^r$, with an unknown analytical expression such that $\|G_i(t, \bar{x}_i)\| \leq \tilde{g}_i(\bar{x}_i), \forall t \geq 0$; (B) the symmetric components denoted by $J^*_i(t, \bar{x}_i) := \frac{1}{2} \left( G_i^T(t, \bar{x}_i) + G_i(t, \bar{x}_i) \right), i \in T_1^r$, are uniformly sign-definite with known signs. Without loss of generality, we assume that all $J^*_i(t, \bar{x}_i)$ are uniformly positive definite, i.e., $z^T J^*_i(t, \bar{x}_i) z > 0, \forall t \in \mathbb{R}_{\geq 0}, \forall z \in \mathbb{R}^n$, and $\forall \bar{x}_i \in \mathbb{R}^{ni}$, with $z \neq 0$.

Assumption 5.3. The functions $w_i(t), i \in T_1^r$ are unknown, bounded and piecewise continuous, with $|w_i(t)| \leq \tilde{w}_i, \forall t \geq 0, i \in T_1^r$, in which the constant upper bounds $\tilde{w}_i$ are unknown.

Part (B) of Assumption 5.2 establishes a global controllability condition for (5.1). It implies the existence of strictly positive constants $\lambda_i > 0, i \in T_1^r$, such that $\lambda_{\min}(G_i^*$) $\geq \lambda_i > 0$, for all $\bar{x}_i \in \mathbb{R}^{ni}$ and all $t \geq 0$. Furthermore, Assumptions 5.1 and 5.2 suggest that while the elements of $f_i(t, \bar{x}_i)$ and $G_i(t, \bar{x}_i), i \in T_1^r$, can grow arbitrarily large due to variations in $\bar{x}_i$, they cannot do so as a result of increase in $t$. The following assumptions solely pertain to the output map of (5.1).

Assumption 5.4. There exists a continuous function $\kappa_0 : \mathbb{R}^n \rightarrow \mathbb{R}$, such that $\|J(t, x_1)\| \leq \kappa_0(x_1)$, where $J(t, x_1) := \frac{\partial h_i(t, x_1)}{\partial x_1}$ is the Jacobian of the output map.

Assumption 5.5. There exist continuous functions $\kappa_i, \bar{h}_i : \mathbb{R}^n \rightarrow \mathbb{R}, i \in T_1^m$, such that $|h_i(t, x_1)| \leq \bar{h}_i(x_1)$ and $\left|\frac{\partial h_i(t, x_1)}{\partial x_1}\right| \leq \kappa_i(x_1)$, respectively.

Assumptions 5.4 and 5.5 ensure that the elements of the Jacobian matrix $J(t, x_1)$ and the functions $h_i(t, x_1)$ and $\frac{\partial h_i(t, x_1)}{\partial x_1}, i \in T_1^m$, can grow arbitrarily large only as a result of changes in $x_1$, and not due to increase in $t$. Recall that, $h_i(t, x_1)$ are the elements of the output map $h(t, x)$ in (5.1).

Remark 5.1. Assumptions 5.4 and 5.5 can be omitted if $h(t, x_1)$ in (5.1) does not explicitly depend on time, i.e., $h(x_1)$, with $h(x_1)$ only requiring to be a $C^2$ function. Additionally, if $f_i(t, \bar{x}_i)$ and $G_i(t, \bar{x}_i), i \in T_1^r$ in (5.1), are replaced by $f_i(\bar{x}_i)$ and $G_i(\bar{x}_i)$, then Assumptions 5.1 and 5.2 simplify to standard requirements. These requirements then only necessitate $f_i$ and $G_i$ to be locally Lipschitz and $G_i^*(\bar{x}_i) = \frac{1}{2} \left( G_i^T(\bar{x}_i) + G_i(\bar{x}_i) \right)$ to be positive definite for all $\bar{x}_i \in \mathbb{R}^{ni}$.

Let the outputs of (5.1) be subject to the following class of time-varying constraints:

$$\rho_i(t) < h_i(t, x_1) < \bar{h}_i(t), \quad i \in T_1^m, \quad \forall t \geq 0,$$

where $\rho_i, \bar{h}_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \cup \{ \pm \infty \}, i \in T_1^m$. We assume for each $i \in T_1^m$, that at least one of $\bar{h}_i(t)$ and $\rho_i(t)$ is a bounded $C^1$ function of time with a bounded derivative.
In other words, we allow \( \rho_i(t) = -\infty \) (resp. \( \bar{\rho}_i(t) = +\infty \)) when \( \rho_i(t) \) (resp. \( \bar{\rho}_i(t) \)) is bounded for all \( t \geq 0 \). In this respect, \( \text{(5.2)} \) can either represent \textit{Lower Bounded One-sided} (LBO) time-varying constraints in the form of \( \rho_i(t) < h_i(t,x_1) \), \textit{Upper Bounded One-sided} (UBO) time-varying constraints in the form of \( h_i(t,x_1) < \bar{\rho}_i(t) \), as well as (time-varying) \textit{funnel constraints} in the form of \( \rho_i(t) < h_i(t,x_1) < \bar{\rho}_i(t) \), for which both \( \rho_i(t) \) and \( \bar{\rho}_i(t) \) are bounded.

Without loss of generality, we assume that the first \( p \) constraints in \( \text{(5.2)} \), i.e., for \( i \in \mathcal{I}_p^0, 0 \leq p \leq m \), are funnel constraints, \( q \) LBO constraints are indexed by \( i \in \mathcal{I}_{p+1}^q, 0 \leq q \leq m - p \) in \( \text{(5.2)} \), and the remaining \( m - p - q \) constraints represent UBO constraints for which \( i \in \mathcal{I}_{p+q+1}^m \) in \( \text{(5.2)} \). We make the assumption that each funnel constraint in \( \text{(5.2)} \) is well-defined, i.e., for every \( i \in \mathcal{I}_1^p \), there exists a positive constant \( \epsilon_i \) such that \( \bar{\rho}_i(t) - \rho_i(t) \geq \epsilon_i \) for all \( t \geq 0 \). This condition guarantees that the \( p \) funnel constraints are \textit{independently feasible}.

**Remark 5.2.** Note that the output constraints specified in \( \text{(5.2)} \) depend on \( x_1 \), which signifies the spatial coordinates (positions) of mechanical systems. Furthermore, while previous works, namely \[53, 54, 61, 62, 68, 82, 92, 106, 107\], particularly only deal with \( n \) decoupled funnel constraints by considering \( y = x_1 \), we account for \( m \geq n \) generalized system outputs denoted as \( y = h(t,x_1) \) in \( \text{(5.1)} \), which allows for possible couplings between different constraints presented in \( \text{(5.2)} \).

We emphasize that, the output map \( h(t,x_1) \) is primarily employed to incorporate various types of constraints into the nonlinear dynamics described by \( \text{(5.1)} \). Specifically, we utilize \( h(t,x_1) \) in conjunction with the time-varying functions \( \rho_i(t) \) and \( \bar{\rho}_i(t) \) in \( \text{(5.2)} \) to represent various forms of spatiotemporal constraints for \( \text{(5.1)} \). Note that, \( h(t,x_1) \) is not necessarily related to the available measurements of the system. As we will further elaborate, we assume that the states of \( \text{(5.1)} \) are available for the measurement and will be utilized for designing the control input \( u(t,x) \).

**Definition 5.1.** An output of \( \text{(5.1)} \), \( y_i = h_i(t,x_1) \), is called \textit{separable} if it can be expressed as the sum of a component solely varying with time and another component dependent solely on \( x_1 \), i.e., \( h_i(t,x_1) = h_i^{x_1}(x_1) + h_i^t(t) \). Otherwise, it is called \textit{inseparable}.

Note that, if \( y_i = h_i(t,x_1) \) is separable, the time-varying term \( h_i^t(t) \) can be incorporated into the time-varying bounds \( \bar{\rho}_i(t) \) and \( \rho_i(t) \) in \( \text{(5.2)} \). Thus, one can consider the time-independent output \( h_i^{x_1}(x_1) \) instead of \( h_i(t,x_1) \) in \( \text{(5.1)} \). For example, let \( \text{(5.1)} \) model a moving vehicle with position \( [x_{1,1}, x_{1,2}]^T \), and the objective is to track a \( C^1 \) reference trajectory characterized by \( x^d_i(t) = [x^d_{1,1}(t), x^d_{1,2}(t)]^T \) under the funnel constraints \( -\rho_i(t) < h_i(t,x_1) = x_{1,i} - x^d_{1,i}(t) < \rho_i(t), i \in \mathcal{I}_1^2 \), where \( \rho_i(t) \) are positive functions decaying to a small neighborhood of zero (tracking under a prescribed performance, see \[53\]). Here \( h_i(t,x_1), i \in \mathcal{I}_1^2 \) represent the tracking errors and the constraints can be written as \( x^d_{1,i}(t) - \rho_i(t) < x_{1,i} < \rho_i(t) + x^d_{1,i}(t), i \in \mathcal{I}_1^2 \).

On the other hand, if the vehicle is tasked with reaching a moving target defined by its position \( x^d(t) \), then a single constraint can be imposed: \( -\rho(t) < h(t,x_1) = \)
\[(x_{1,1} - x_{1,1}^d(t))^2 + (x_{1,2} - x_{1,2}^d(t))^2 < \rho(t), \text{ where } \rho(t) \text{ is a positive function decaying to a neighborhood of zero and } h(t, x_1) \text{ represents the squared distance error, which has inseparable time-varying terms.}\]

Finally, let us define the output constrained set \(\bar{\Omega}(t)\) based on (5.2) as:
\[
\bar{\Omega}(t) := \{ x_1 \in \mathbb{R}^n \mid \rho_i(t) < h_i(t, x_1) < \bar{\rho}_i(t), i \in I^m_1 \}. \tag{5.3}
\]

Control Objective:

In this chapter, our primary goal is to design a low-complexity continuous robust feedback control law \(u(t, x)\) for (5.1) such that \(x_1(t; x(0))\) satisfies the time-varying output constraints \(5.2\) \(\forall t > T \geq 0\), where \(T\) is a user-defined finite time after which the output constraints are satisfied for all time (i.e., \(x_1(t; x(0)) \in \bar{\Omega}(t), \forall t > T \geq 0\)). Note that this problem reduces to establishing only invariance of \(\bar{\Omega}(t)\) for all \(t \geq 0\), if \(x_1(0) \in \bar{\Omega}(0)\) \((T = 0)\). On the other hand, having \(x_1(0) \notin \bar{\Omega}(0)\) indicates establishing: (i) finite time convergence to \(\bar{\Omega}(t)\) at \(t = T\), and (ii) ensuring invariance of \(\bar{\Omega}(t)\), for all \(t > T\). Furthermore, when \(\bar{\Omega}(t)\) becomes infeasible (empty) for an unknown time interval, we aim at enhancing the control scheme such that \(u(t, x)\) drives the closed-loop system trajectory towards a least violating solution for (5.1) under constraints (5.2) (see Section 5.7 for more details).

5.4 Methodology

In this section, inspired from [101], we first introduce a novel scalar variable, which is the signed distance from the boundary of the time-varying output constrained set (5.3). This variable serves as a metric of both feasibility and satisfaction of the output constraints. Next, we propose a robust and low-complexity controller design for (5.1), which ensures the ultimate positivity of the aforementioned variable. This, in turn, leads to the satisfaction of the output constraints.

5.4.1 Satisfying Constraints using a Scalar Variable

Notice that the \(m\) output constraints in (5.2) can be re-written in the following format:
\[
\begin{align*}
\psi_{2i-1}(t, x_1) &= h_i(t, x_1) - \rho_i(t) > 0, \quad i \in I^p_1, \quad \text{(funnel constraints)} \tag{5.4a} \\
\psi_{2i}(t, x_1) &= \bar{\rho}_i(t) - h_i(t, x_1) > 0, \quad i \in I^p_1, \\
\psi_i(t, x_1) &= h_j(t, x_1) - \rho_j(t) > 0, \quad i \in I^{2p+q}_{2p+1}, \quad j \in I^{p+q}_{p+1}, \quad \text{(LBO constraints)} \\
\psi_i(t, x_1) &= \bar{\rho}_j(t) - h_j(t, x_1) > 0, \quad i \in I^{m+p}_{2p+q+1}, \quad j \in I^{m}_{p+q+1}, \quad \text{(UBO constraints)} \tag{5.4b}
\end{align*}
\]

Now, without loss of generality, consider all these \(m + p\) constraints in (5.4) as:
\[
\psi_i(t, x_1) > 0, \quad i \in I^{m+p}_1, \tag{5.5}
\]
where $\psi_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ are $C^2$ in $x_1$ and $C^1$ in $t$. As a result, one can re-write (5.3) as:

$$\bar{\Omega}(t) = \{ x_1 \in \mathbb{R}^n \mid \psi_i(t, x_1) > 0, \forall i \in I_1^{m+p} \}. \quad (5.6)$$

Now let us define the scalar function $\bar{\alpha} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$, as:

$$\bar{\alpha}(t, x_1) := \min \{ \psi_1(t, x_1), \ldots, \psi_{m+p}(t, x_1) \}, \quad (5.7)$$

where $\bar{\alpha}(t, x_1)$ represents the signed (minimum) distance from $\partial \text{cl}(\bar{\Omega}(t))$. In this respect, one can re-write (5.6) as the zero super level set of $\bar{\alpha}(t, x_1)$:

$$\Omega(t) = \{ x_1 \in \mathbb{R}^n \mid \bar{\alpha}(t, x_1) > 0 \}. \quad (5.8)$$

Note that if $\bar{\alpha}(t', x_1) < 0$, then at least one constraint is not satisfied at $t = t'$, while $\bar{\alpha}(t, x_1) > 0, \forall t \geq 0$ means that all constraints are satisfied for all time. Owing to the min operator in (5.7), in general, $\bar{\alpha}(t, x_1)$ is a continuous but nonsmooth function, therefore, to facilitate the controller design and stability analysis we will consider the smooth under-approximation of $\bar{\alpha}(t, x_1)$ given by the following Log-Sum-Exp function (see Lemma 2.7):

$$\alpha(t, x_1) := -\frac{1}{\nu} \ln \left( \sum_{i=1}^{m+p} e^{-\nu \psi_i(t, x_1)} \right) \quad (5.9a)$$

$$\leq \bar{\alpha}(t, x_1) \leq \alpha(t, x_1) + \frac{1}{\nu} \ln(m + p), \quad (5.9b)$$

where $\nu > 0$ is a tuning coefficient whose larger values gives a closer (under) approximation (i.e, $\alpha(t, x_1) \to \bar{\alpha}(t, x_1)$ as $\nu \to \infty$). Note that, $\alpha(t, x_1)$ provides the signed distance from the boundary of a smooth inner-approximation of $\text{cl}(\Omega(t))$. Therefore, ensuring $\alpha(t, x_1) > 0, \forall t \geq 0$ guarantees $\bar{\alpha}(t, x_1) > 0, \forall t \geq 0$ and thus the satisfaction of (5.5) (equivalently (5.2)). Define $\Omega(t) \subset \bar{\Omega}(t)$ as the smooth inner-approximation of the set $\bar{\Omega}(t)$, given by:

$$\Omega(t) := \{ x_1 \in \mathbb{R}^n \mid \alpha(t, x_1) > 0 \}. \quad (5.10)$$

Note that we have $x_1 \in \Omega(t) \Rightarrow x_1 \in \bar{\Omega}(t)$, and when $\bar{\Omega}(t)$ is bounded, then $\Omega(t)$ is also bounded. Fig. 5.1 depicts snapshots of $\Omega(t)$ and $\Omega(t)$ with $\nu = 2$ in (5.9) for the following examples:

**Example 5.1.** Consider $h(x_1) = [h_1(x_1), h_2(x_1), h_3(x_1)]^T$, where $h_1(x_1) = x_{1,1}$, $h_2(x_1) = -x_{1,1} + x_{1,2}$, and $h_3(x_1) = 0.3x_{1,1}^2 + x_{1,2}$ and let the output constraints be $\rho_1(t) < h_1(x_1) < \bar{\rho}_1(t)$ (funnel constraint), $\rho_2(t) < h_2(x_1)$ (LBO constraint), and $h_3(x_1) < \bar{\rho}_3(t)$ (UBO constraint), respectively. Fig. 5.1a depicts a snapshot of the time-varying output constrained set and its smooth inner approximation, for which $-\rho_1(t) = \bar{\rho}_1(t) = 2$, $\rho_2(t) = -2$, and $\bar{\rho}_3(t) = 4$. 
Feedback Control for Uncertain MIMO Nonlinear Systems under Generalized Time-Varying Output Constraints

Figure 5.1: Snapshots of $\bar{\Omega}(t)$ and its corresponding inner-approximation under (5.9) for three different examples.

Example 5.2. Consider $h(x_1) = [h_1(x_1), h_2(x_1)]^T$, with $h_1(x_1) = x_{1,1}$ and $h_2(x_1) = 0.3x_{1,1}^2 - x_{1,2}$ and let the output constraints be $\rho_1(t) < h_1(x_1) < \bar{\rho}_1(t)$ and $\rho_2(t) < h_2(x_1) < \bar{\rho}_2(t)$ (two funnel constraints), respectively. Fig. 5.1b depicts a snapshot of the time-varying output-constrained set and its smooth inner-approximation, for which $\rho_1(t) = -3, \bar{\rho}_1(t) = 2$, and $-\rho_2(t) = \bar{\rho}_2(t) = 2$.

Example 5.3. Consider the constraints of Example 2, however, this time we modify the second output such that $h_2(t, x_1) = c_1(t)(x_{1,1} - o_1(t))^2 - x_{1,2}$, where $c_1(t)$ and $o_1(t)$ are bounded continuously differentiable time-varying functions. Fig. 5.1c depicts three snapshots of the time-varying output-constrained set and its smooth inner-approximations, for which $\rho_1(t_1) = -3, \bar{\rho}_1(t_1) = 2, \rho_1(t_2) = 4, \rho_1(t_3) = 9, \rho_1(t_3) = 11, \bar{\rho}_1(t_3) = 16$, and $-\rho_2(t) = \bar{\rho}_2(t) = 2, \forall t \in \{t_1, t_2, t_3\}$, where $t_1 < t_2 < t_3$. Moreover, $c_1(t_1) = 0.3, c_1(t_2) = 0, c_1(t_3) = -0.3$ and $o_1(t_1) = 0, o_1(t_2) = 6, o_1(t_3) = 13$. Note that different from Example 5.2, $o_1(t)$ and $c_1(t)$ in $h_2(t, x_1)$ can contribute in shifting and changing the boundaries of the time-varying constrained set simultaneously at different time instances.

Assumption 5.6. The function $-\bar{\alpha}(t, x_1)$ is coercive (radially unbounded) in $x_1$ and uniformly in $t$, i.e., $-\bar{\alpha}(t, x_1) \to +\infty$ as $\|x_1\| \to +\infty, \forall t \geq 0$.

Note that, the focus of this work is the satisfaction of (5.2). On the other hand, it is also essential to design $u(t, x)$ such that $\|x_i(t)\|, i = 1, \ldots, n$ remain bounded.
∀t ≥ 0. In this respect, if the output-constrained set \( \bar{\Omega}(t) \) is well-posed (i.e., it is bounded), the satisfaction of the constraints inherently leads to the boundedness of \( \|x_1(t)\| \). Assumption 5.6 serves as a necessary and sufficient condition for ensuring the boundedness of \( \Omega(t) \) (and \( \Omega(t) \)) for all \( t ≥ 0 \). The following lemma establishes this.

**Lemma 5.1.** Under Assumption 5.6, \( \bar{\Omega}(t) \) (resp. \( \Omega(t) \)) is a bounded set for all \( t ≥ 0 \).

**Proof.** See Subsection 5.10.1.

Note that, Assumption 5.6 implies that for any time instant \( -\bar{\alpha}(t, x_1) \) should approach \( +\infty \) along any path within \( \mathbb{R}^n \) on which \( \|x_1\| \) tends to infinity. Define \( h_f(t, x_1) := \text{col}(h_i(t, x_1)) \in \mathbb{R}^p, i \in I^m_r \), \( h_L(t, x_1) := \text{col}(h_i(t, x_1)) \in \mathbb{R}^q, i \in I^{m+1}_{p+1} \), and \( h_U(t, x_1) := \text{col}(h_i(t, x_1)) \in \mathbb{R}^{m-p-q}, i \in I^n_{p+q+1} \), as the stacked vectors of system outputs associated with funnel, LBO, and UBO constraints in (5.2), respectively. The following lemma provides explicit conditions on \( h_i(t, x_1), i \in I^m_r \), to ensure that \( -\bar{\alpha}(t, x_1) \) (resp. \( -\alpha(t, x_1) \)) is coercive.

**Lemma 5.2.** The function \( -\bar{\alpha}(t, x_1) \) (resp. \( -\alpha(t, x_1) \)) is coercive in \( x_1 \) for all \( t ≥ 0 \) if and only if, for each time instant \( t \), at least one of the following conditions holds:

(I) \( \|h_f(t, x_1)\| → +\infty \);

(II) one or more elements of \( h_L(t, x_1) \) approaches \(-\infty\);

(III) one or more elements of \( h_U(t, x_1) \) approaches \(+\infty\);

along any path in \( \mathbb{R}^n \) as \( \|x_1\| → +\infty \).

**Proof.** See Subsection 5.10.2 for the proof.

In Example 5.1 we observe that \( h_f(t, x_1) = h_1(x_1), h_L(t, x_1) = h_2(x_1), h_U(t, x_1) = h_3(x_1) \). It can be verified that the condition in Lemma 5.2 is satisfied along any path in \( \mathbb{R}^2 \) such that \( \|x_1\| → +\infty \). Hence, \( -\bar{\alpha}(t, x_1) \) (resp. \( -\alpha(t, x_1) \)) is coercive, implying that \( \Omega(t) \) (resp. \( \bar{\Omega}(t) \)) is bounded based on Lemma 5.1 (See Fig. 5.1a). However, if we remove the LBO constraint \( \rho_2(t) < h_2(x_1) = x_{1,2} - x_{1,1} \), \( \bar{\Omega}(t) \) (and \( \Omega(t) \)) will not be bounded since along the path where \( x_{1,1} = 0 \) and \( x_{1,2} → -\infty \) we get \( h_f(x_1) = 0, h_U(x_1) → -\infty \), which do not satisfy any of the conditions in Lemma 5.2. In Example 5.2, we have \( h_f(t, x_1) = [h_1(x_1), h_2(x_1)]^T \). One can verify that \( \|h_f(t, x_1)\| → +\infty \) along any path in \( \mathbb{R}^2 \) such that \( \|x_1\| → +\infty \). In Example 5.3 we observe that \( h_f(t, x_1) = [h_1(x_1), h_2(x_1), h_3(x_1)]^T \). It is not difficult to see that Condition I in Lemma 5.2 holds for all time instances and as depicted in Fig. 5.1c \( \bar{\Omega}(t) \) (resp. \( \Omega(t) \)) remains bounded \( \forall t ≥ 0 \).

Verifying the boundedness of \( \bar{\Omega}(t) \) as per Lemma 5.1 can be challenging, especially when dealing with time-dependent outputs (i.e., \( h_i(t, x_1) \) instead of \( h_i(x_1) \)). In such cases, one must check the condition in Lemma 5.2 for all time instances.
However, note that ensuring the boundedness of $\bar{\Omega}(t)$ is merely a technical requirement. To meet this requirement, one approach is to introduce an auxiliary output, denoted as $h_{aux}(x_1)$, under the UBO constraint: $h_{aux}(x_1) := \|x_1\| < c_{aux}$, where $c_{aux} > 0$ is a sufficiently large constant. This constraint represents a large ball around the origin in the $x_1$ space, encompassing all other time-varying constraints in (5.2). Notably, this constraint guarantees the satisfaction of Lemma 5.2's condition at all times, regardless of the choice of other system outputs, $h_i(t, x_1), i \in I^n$.

Note that Assumption 5.6 also guarantees the existence of at least one global maximizer for $\tilde{\alpha}(t, x_1)$ (resp. $\alpha(t, x_1)$) $\forall t \geq 0$ (see Theorem 2.6). In this regard, for each time instant $t$, we define:

$$\tilde{\alpha}^*(t) := \max_{x_1 \in \mathbb{R}^n} \tilde{\alpha}(t, x_1),$$

(5.11)

where $\tilde{\alpha}^*(t)$ is bounded and denotes the maximum value of $\tilde{\alpha}(t, x_1)$ at time $t$. It is clear that if $\tilde{\alpha}^*(t') \geq 0$ then the time-varying output constraints are feasible at time $t = t'$, whereas $\tilde{\alpha}^*(t') < 0$ indicates that the constraints are infeasible at time $t = t'$, thus impossible to be satisfied. Similarly, for a given $\nu$ in (5.9) we can define:

$$\alpha^*(t) := \max_{x_1 \in \mathbb{R}^n} \alpha(t, x_1) \leq \tilde{\alpha}^*(t).$$

(5.12)

From (5.12) and (5.9), one can conclude that having $\alpha^*(t') > 0$ is sufficient for the feasibility of the time-varying output constraints (5.2) at time $t = t'$. In addition, notice that $\alpha^*(t') < 0$, does not necessarily imply that the actual output constrained set $\bar{\Omega}(t')$ in (5.8) is empty, i.e, $\tilde{\alpha}^*(t') < 0$ in (5.11). In fact, from (5.9b) and the fact that $\alpha(t, x_1) \leq \alpha^*(t)$ for all $t \geq 0$ and all $x_1 \in \mathbb{R}^n$, we can deduce that $\alpha^*(t') < -\frac{1}{t} \ln(m + p)$ provides a sufficient condition for the infeasibility of $\bar{\Omega}(t')$.

### 5.4.2 Consolidating Multiple Output Constraints into a Single Constraint

As discussed in Subsection 5.4.1 satisfying (5.2) can be achieved by maintaining the positivity of $\alpha(t, x_1(t; x(0)))$. Therefore, the main challenge in designing the control law outlined in Section 5.3 is to determine $u(t, x)$ for (5.1) such that if $\alpha(0, x_1(0)) > 0$, then $\alpha(t, x_1(t; x(0))) > 0$ for all $t \geq 0$, and if $\alpha(0, x_1(0)) \leq 0$, then $\alpha(t, x_1(t; x(0))) > 0$ for all $t \geq T$. To achieve this objective, we propose ensuring the following single consolidating constraint for (5.1):

$$\rho_\alpha(t) < \alpha(t, x_1(t; x(0))), \quad \forall t \geq 0,$$

(5.13)

where $\rho_\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a properly designed bounded and continuously differentiable function of time with a bounded derivative. Before proposing a design for $\rho_\alpha(t)$ we emphasize that, in general, any appropriate $\rho_\alpha(t)$ in (5.13) has to satisfy the following two Properties:

i) $\alpha^*(t) - \rho_\alpha(t) \geq \varsigma > 0, \forall t \geq 0$, where $\varsigma$ is a positive constant that may be unknown;
Recall that owing to (5.11) and Assumption 5.6 we have $\alpha(t, x_1) \leq \alpha^*(t), \forall t \geq 0, \forall x_1 \in \mathbb{R}^n$. Therefore, $\alpha(t, x_1(t;x(0)))$ in (5.13) is implicitly upper bounded by $\alpha^*(t)$ for all time. In this respect, (5.13) is a feasible (valid) constraint when Property (i) holds. Furthermore, it is imperative to design $\alpha(t)$ in such a way that Property (ii) holds, i.e., (5.13) should be satisfied at the initial time $t = 0$. Notably, this requirement is due to the controller design, which will be detailed later in Section 5.5 and it does not pose significant restrictions since the initial condition of system (5.1) is readily available.

5.4.3 Design of $\alpha(t)$ under Feasibility of the Constraints

In order to guarantee the fulfillment of the time-varying output constraints specified in (5.2) through enforcing (5.13), one needs to properly design $\alpha(t)$ to enforce the positivity of $\alpha(t, x_1(t;x(0)))$ while respecting Properties (i) and (ii) mentioned in Subsection 5.4.2. It turns out that it is straightforward to design $\alpha(t)$ in accordance with the following assumption:

**Assumption 5.7.** There exists $\epsilon_f > 0$ such that $\alpha^*(t) \geq \epsilon_f > 0, \forall t \geq 0$, i.e., $\Omega(t)$ is non-empty (feasible) for all time.

This assumption implies that all output constraints in (5.2) are mutually satisfiable for all time. Under Assumption 5.7 one can design $\alpha(t)$ through the following strategy:

(a) If $\alpha(0, x_1(0)) > 0$ (i.e., the constraints are initially satisfied), set $\alpha(t) = 0, \forall t \geq 0$;

(b) If $\alpha(0, x_1(0)) \leq 0$, design $\alpha(t)$ such that $\alpha(t) < \alpha(0, x_1(0)) \leq 0$ and $\alpha(t \geq T) = 0$.

Note that in the second case, the lower bound in equation (5.13) needs to be increased over time to ensure that $\alpha(t, x(t;x(0)))$ becomes and remains positive for all $t \geq T > 0$. To achieve this, inspired by [106, Remark 4], we can design:

$$
\alpha(t) = \begin{cases} 
(\frac{T-t}{T})^{\beta} (\rho_0 - \rho_x) + \rho_x, & 0 \leq t < T, \\
\rho_x, & t \geq T, 
\end{cases}
$$

(5.14)

where $\beta \in (0, 1)$ is a constant, $\rho_0, \rho_x$, are constants such that $\rho_0 \leq \rho_x$, and $T > 0$ is the user-defined appointed finite time for constraints satisfaction. Note that (5.14) is an increasing function and we have $\alpha(0) = \rho_0$ and $\alpha(t \geq T) = \rho_x$. Moreover, for case (a) above, we set $\rho_0 = \rho_x = 0$, while for case (b), we set $\rho_0$ such that $\rho_0 \leq \alpha(0, x_1(0)) < 0$ and $\rho_x = 0$. Finally, note that, the proposed design of $\alpha(t)$ ensures feasibility of (5.13) since owing to $\rho_x = 0$ and Assumption 5.7, we get $\alpha^*(t) - \alpha(t) \geq \zeta = \epsilon_f > 0, \forall t \geq 0$. 

ii) $\rho_\alpha(0) < \alpha(0, x_1(0))$. 


Remark 5.3. We highlight that, when Assumption 5.7 holds and \( \rho_x = 0 \) in (5.14), no information about the solution of the time-varying optimization problem (5.11) is required for designing \( \rho_\alpha(t) \) in (5.13). However, taking \( \rho_x > 0 \) requires \( \rho_x < \inf_t (\alpha^*(t)) \) to hold for ensuring the feasibility of (5.13).

Remark 5.4. Under Assumption 5.7, the choice of a larger \( \rho_x \), with the condition \( 0 < \rho_x < \inf_t (\alpha^*(t)) \), dictates the extent to which the time-varying output constraints are satisfied. Specifically, when \( \rho_x > 0 \) is increased, it leads to a more robust enforcement of a positive \( \alpha(t, x_1(t; x(0))) \) for all \( t \geq T \). As a result, for all \( t \geq T \), the trajectory of \( x_1(t; x(0)) \) is confined further away from the boundary of \( \text{cl}(\Omega(t)) \), effectively pushing it deeper inside \( \Omega(t) \).

5.5 Low-Complexity Controller Design

Now similarly to the PPC method in [54], we design a model-free low-complexity robust state feedback controller for (5.1) to ensure the satisfaction of the consolidating constraint (5.13). Due to the lower triangular structure of (5.1), we employ a backstepping-like design scheme. The process begins by creating an intermediate (virtual) control input \( s_1(t, x_1) \) for the dynamics of \( x_1 \) in (5.1), ensuring the fulfillment of (5.13). Subsequently, we design a second intermediate control \( s_2(t, \bar{x}_2) \) for the dynamics of \( x_2 \), making certain that \( x_2 \) follows the trajectory set by \( s_1(t, x_1) \). This iterative top-down approach to design intermediate control laws \( s_i(t, x_i), i \in I_1^i \), continues until we obtain the actual control input of the system, \( u(t, x) \). The controller design is summarized in the following steps:

**Step 1-a.** Given \( x_1(0) \) obtain \( \alpha(0, x_1(0)) \) and design \( \rho_\alpha(t) \) such that \( \rho_\alpha(0) < \alpha(0, x(0)) \), i.e., Property (ii) in Subsection 5.4.2 is satisfied (for the particular design of \( \rho_\alpha(t) \) in Subsection 5.4.3), this leads to \( \rho_0 < \alpha(0, x(0)) \).

**Step 1-b.** Define:

\[
e_\alpha(t, x_1) := \alpha(t, x_1) - \rho_\alpha(t),
\]

and consider the following nonlinear transformation:

\[
\varepsilon_\alpha(t, x_1) = \mathcal{T}_\alpha(e_\alpha) := \ln \left( \frac{e_\alpha}{v} \right),
\]

where \( v > 0 \) is a constant and \( \mathcal{T}_\alpha : (0, +\infty) \to (-\infty, +\infty) \) is a smooth strictly increasing bijective mapping, which satisfies \( \mathcal{T}_\alpha(v) = 0 \). Note that maintaining boundedness of \( \varepsilon_\alpha \) enforces \( e_\alpha \in (0, \infty) \), and thus the satisfaction of (5.13). We call \( \varepsilon_\alpha \in (-\infty, +\infty) \) as the unconstrained transformed signal corresponding to \( e_\alpha \).

**Step 1-c.** To design the first intermediate (virtual) control law we proceed as follows: first, define \( V_1(\varepsilon_\alpha) := \frac{1}{2} \varepsilon_\alpha^2 \), which is a positive definite and radially unbounded (implicitly time-varying) barrier function associated with the consolidating constraint in (5.13). Note that \( V_1(0) = 0 \) and as \( \alpha(t, x_1) \) approaches \( \rho_\alpha(t) \) (i.e., as \( e_\alpha \) approaches zero) we get \( V_1(\varepsilon_\alpha) \to +\infty \). Next, from (5.16), with a slight abuse of notation one may consider \( V_1(t, x_1) \), and design the first intermediate
(gradient-based) control law as:

$$s_1(t, x_1) := -k_1 \nabla_{x_1} V_1(t, x_1),$$  \hspace{1cm} (5.17) \]

where $k_1 > 0$ is a control gain and $\nabla_{x_1}$ denotes the gradient with respect to $x_1$. Applying the chain rule in (5.17) gives $s_1(t, x_1)$ more explicitly as:

$$s_1(t, x_1) = -k_1 \left( \partial V_1(\varepsilon_\alpha) \frac{\partial \varepsilon_\alpha}{\partial e_\alpha} \frac{\partial e_\alpha}{\partial \alpha} \frac{\partial \alpha(t, x_1)}{\partial x_1} \right)^T$$

$$= -k_1 \nabla_{x_1} \alpha(t, x_1) \varepsilon_\alpha.$$  \hspace{1cm} (5.18)

**Step i-a ($2 \leq i \leq r$).** Define the $i$-th intermediate error vector as:

$$e_i = \text{col}(e_{i,j}) := x_i - s_{i-1}(t, \bar{x}_{i-1}),$$  \hspace{1cm} (5.19)

where $e_i \in \mathbb{R}^n$. Now the objective is to design the $i$-th intermediate (virtual) control law $s_i(t, e_i)$ for (5.1) to compensate $e_{i,j}(t, \bar{x}_i), \ j \in \mathcal{I}_1^n$, by enforcing the following narrowing intermediate funnel constraints:

$$-\theta_i(t) < e_{i,j}(t, \bar{x}_i) < \theta_i(t), \ j \in \mathcal{I}_1^n,$$  \hspace{1cm} (5.20)

for all $t \geq 0$, where $\theta_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$, are continuously differentiable strictly positive performance functions that are decaying to a neighborhood of zero. One choice for $\theta_i(t)$ is:

$$\theta_i(t) := (\theta_{i,j}^0 - \theta_{i,j}^\infty) \exp(-l_{i,j} t) + \theta_{i,j}^\infty,$$  \hspace{1cm} (5.21)

where $l_{i,j}, \theta_{i,j}^0, \theta_{i,j}^\infty$ are user-defined positive constants. Moreover, one should choose $\theta_{i,j}^0 > |e_{i,j}(0, \bar{x}_i(0))|$ to ensure $e_{i,j}(0, \bar{x}_i(0)) \in (-\theta_{i,j}(0), \theta_{i,j}(0)), \ j \in \mathcal{I}_1^n$.

**Step i-b ($2 \leq i \leq r$).** Now define the diagonal matrix $\Theta_i(t) := \text{diag}(\theta_{i,j}(t)) \in \mathbb{R}^{n \times n}$, and consider

$$\hat{e}_i(t, e_i) = \text{col}(\hat{e}_{i,j}) := \Theta_i^{-1}(t) e_i,$$  \hspace{1cm} (5.22)

as the vector of normalized errors, whose elements are:

$$\hat{e}_{i,j}(t, e_i) = \frac{e_{i,j}}{\theta_{i,j}(t)}, \ j \in \mathcal{I}_1^n.$$  \hspace{1cm} (5.23)

Note that, $\hat{e}_{i,j} \in (-1, 1)$ if and only if $e_{i,j} \in (-\theta_{i,j}(t), \theta_{i,j}(t))$. Next we introduce the following nonlinear transformations:

$$\varepsilon_{i,j}(t, e_i) = \mathcal{T}(\hat{e}_{i,j}) := \ln \left( \frac{1 + \hat{e}_{i,j}}{1 - \hat{e}_{i,j}} \right), \ j \in \mathcal{I}_1^n,$$  \hspace{1cm} (5.24)

where $\varepsilon_{i,j}$ represents the unconstrained transformed signal that corresponds to $e_{i,j}(t, \bar{x}_i)$ and $\mathcal{T} : (-1, 1) \rightarrow (-\infty, +\infty)$ is a smooth strictly increasing bijective mapping, which satisfies $\mathcal{T}(0) = 0$. Note that enforcing the boundedness of $\varepsilon_{i,j}$
ensures that $\hat{e}_{i,j}$ remains within the range of $(-1, 1)$, which leads to the satisfaction of (5.20).

**Step i-c ($2 \leq i \leq r$).** Finally, similarly to Step 1-c we can design $s_i(t, e_i)$. In particular, define $\varepsilon_i := \text{col}(\varepsilon_{i,j}) \in \mathbb{R}^n$ and let $V_i(\varepsilon_i) := \frac{1}{2} \varepsilon_i^T \varepsilon_i$, which is a positive definite and radially unbounded (implicitly time-varying) composite barrier function associated with the intermediate funnel constraints in (5.20). Note that $V_i(0_n) = 0$ and for all $j \in I_1^n$ if any $e_{i,j}(t, \bar{x}_i)$ approaches $\pm \theta_{i,j}(t)$ (i.e., as $\hat{e}_{i,j}$ approaches $\pm 1$) we get $V_i(\varepsilon_i) \to +\infty$. From (5.24), with a slight abuse of notation, one can consider $V_i(t, e_i)$, and design the $i$-th intermediate control as:

$$s_i(t, e_i) := -k_i \nabla_{e_i} V_i(t, e_i),$$  
\hspace{1cm} (5.25)

where $k_i > 0$ is a control gain and $\nabla_{e_i}$ denotes the gradient with respect to $e_i$. Consequently, one can obtain $s_i(t, e_i)$ more explicitly by applying the chain rule:

$$s_i(t, e_i) = -k_i \left( \frac{\partial V_i(\varepsilon_i)}{\partial \varepsilon_i} \frac{\partial \varepsilon_i(\hat{e}_i)}{\partial \hat{e}_i} \frac{\partial \hat{e}_i(t, e_i)}{\partial e_i} \right)^T$$

$$= -k_i \Theta_i \varepsilon_i,$$  
\hspace{1cm} (5.26)

where $\Theta_i := \text{diag}(\xi_{i,j}) := \frac{\partial \varepsilon_i(\hat{e}_i)}{\partial \hat{e}_i} \frac{\partial \hat{e}_i(t, e_i)}{\partial e_i} \in \mathbb{R}^{n \times n}$ is a diagonal matrix whose diagonal entries are:

$$\xi_{i,j}(t, e_i, j) := \frac{2}{\vartheta_{i,j}(t)(1 - \hat{e}_{i,j}^2)}, \quad j \in I_1^n.$$  
\hspace{1cm} (5.27)

Notice that $s_i(t, e_i)$ can be considered as a function of $t$ and $\bar{x}_i$, as $e_i$ itself depends on $\bar{x}_i$ (see (5.19)) so with a slight abuse of notation one can write $s_i(t, \bar{x}_i)$.

**Step r + 1.** Finally we design the control input $u(t, x)$ as:

$$u(t, x) := s_r(t, x).$$  
\hspace{1cm} (5.28)

**Remark 5.5.** The proposed control method, similarly to backstepping, aims to make $x_2$ closely track $s_1(t, x_1)$ in the dynamics (5.1), where $s_1(t, x_1)$ is designed to satisfy (5.13). We design the second intermediate control $s_2(t, e_1)$ to ensure that all components of the error $e_2 = x_2 - s_1(t, x_1)$, denoted as $e_{2,j}, j \in I_1^n$, become sufficiently small through the satisfaction of (5.20). This iterative design process continues until we obtain $u(t, x)$ for (5.1). Importantly, unlike the classical backstepping method, we do not use derivatives of $e_i, i \in I_2^n$, or any filtering scheme in the design of intermediate control laws $s_i(t, e_i), i \in I_2^n$. Furthermore, we do not rely on prior knowledge of the system’s nonlinearities or any upper/lower bounds on uncertainties in the design of (5.28).

**Remark 5.6.** It is important to note that the satisfaction of the proposed consolidating constraint (5.13), as well as (5.20) for the intermediate error signals $e_{i,j}, i \in I_2^n, j \in I_1^n$, are ensured merely by keeping $\varepsilon_0$ and $\varepsilon_i$ bounded, respectively. This is achieved by applying the designed control input (5.28) in (5.1). This key observation will be leveraged in the stability analysis of the closed-loop system.
5.6 Stability Analysis

It is worth noting that \( \nabla_{x_1} \alpha(t, x_1) \) in (5.18) represents the control direction of the first intermediate control law \( s_1(t, x_1) \) for satisfying the output constraints (5.4) (or equivalently (5.2)). The expression for \( \alpha(t, x_1) \) in (5.9a) directly originates from the constraints in (5.4). However, a crucial point to consider is that \( \nabla_{x_1} \alpha(t, x_1) \) may become zero at certain undesirable critical points, leading to \( s_1(t, x_1) = 0 \). As \( s_1(t, x_1) \) is designed to ensure the fulfillment of the output constraints (5.2) when \( s_1(t, x_1) = 0 \), the control law (5.28) might no longer be capable of satisfying the output constraints unless \( \nabla_{x_1} \alpha(t, x_1) = 0 \) occurs solely at points where the constraints are already satisfied. Consequently, it becomes crucial to prevent the closed-loop system from encountering such undesired critical points of \( \alpha(t, x_1) \), which can include saddle points and local minima. To address this concern, we introduce the following technical assumption:

**Assumption 5.8.** For all \( t \geq 0 \) the function \( -\alpha(t, x_1) \) is invex, i.e., every critical point of \( \alpha(t, x_1) \) is a (time-varying) global maximizer (see Definition 2.9 and Theorem 2.8).

The following lemma gives some sufficient conditions for ensuring Assumption 5.8:

**Lemma 5.3.** The function \( -\alpha(t, x_1) \) is invex \( \forall t \geq 0 \), if at each time instant \( t \) one of the following conditions holds:

1. \( \psi_i(t, x_1), \forall i \in \mathcal{I}_1^{m+p} \) in (5.5) are concave in \( x_1 \).
2. Having only \( n \) funnel constraints (i.e., \( n = m = p \) in (5.2)) such that: (i) the output map \( y = h(t, x_1) \) in (5.1) is norm-coercive (i.e., \( \| h(t, x_1) \| \to +\infty \) as \( \| x_1 \| \to +\infty \)), and (ii) the Jacobian matrix \( J(t, x_1) := \frac{\partial h(t, x_1)}{\partial x_1} \in \mathbb{R}^{n \times n} \) is full rank for all \( x_1 \in \mathbb{R}^n \).

**Proof.** See Subsection 5.10.3.

The concavity of \( \psi_i(t, x_1), i = \mathcal{I}_1^{m+p} \) at time \( t \) in Lemma 5.3 can be understood by examining (5.4) in terms of system outputs \( h_i(t, x_1), i \in \mathcal{I}_1^m \). Specifically, for funnel constraints, the functions \( h_i(t, x_1), i \in \mathcal{I}_1^p \), should be an affine function of \( x_1 \) at time \( t \), as \( \psi_{2i}(t, x_1) \) and \( \psi_{2i-1}(t, x_1) \) are concave only when \( h_i(t, x_1) \) and \( -h_i(t, x_1) \) are concave, see (5.4a). On the other hand, for LBO constraints, \( h_i(t, x_1), i \in \mathcal{I}_1^{p+q} \), should be concave, and for UBO constraints, \( h_i(t, x_1), i \in \mathcal{I}_1^m \), should be convex at time \( t \), see (5.4b).

It is straightforward to see that Example 5.1 (illustrated in Fig. 5.1a) satisfies Condition I of Lemma 5.3 at all times. Additionally, Example 5.2 (shown in Fig. 5.1b) meets Condition II of Lemma 5.3 at all times. Specifically, in Example 5.2, we have \( n = m = p = 2 \) and the Jacobian matrix of \( h(x_1) \), denoted as \( J(x_1) = \begin{bmatrix} 1 & 0 \\ 0.6x_{1,1} & -1 \end{bmatrix} \), has full rank for all \( x_1 \in \mathbb{R}^2 \). Additionally, \( h(x_1) = h_f(x_1) \),
is norm-coercive. Note that Example 5.2 fails to satisfy Condition I of Lemma 5.3 because \( h_2(x_1) \) is not an affine function. Likewise, we can easily verify that Example 5.3 also meets Condition II of Lemma 5.3 at all times.

It is worth emphasizing that Condition II in Lemma 5.3 accurately captures the notion of independence between \( n \) funnel constraints in \( \mathbb{R}^n \). This means that the satisfaction of individual feasible funnel constraints does not interfere with each other, meaning that the funnel constraints are decoupled.

**Remark 5.7.** If \( \Omega(t) \) is the interior of a time-varying bounded convex polytope in \( \mathbb{R}^n \), then \( \alpha(t, x_1) \) satisfies Assumptions 5.6 and 5.8. The former holds as a consequence of the polytope’s boundedness assumption and the latter is true because in a convex polytope, all \( h_i(t, x_1) \) are affine in \( x_1 \) for all time (which satisfies Condition I of Lemma 5.3 for all \( t \geq 0 \)).

**Remark 5.8.** The invexity of \(-\alpha(t, x_1)\) is ensured even if conditions I and II of Lemma 5.3 interchange at different time instances. Unlike Examples 1-3, this case allows for a modified version of Example 5.1 with \( h(t, x_1) = [h_1(x_1), h_2(t, x_1), h_3(t, x_1)]^\top \). Here, \( h_1(x_1) = x_{1,1}, h_2(t, x_1) = c_1(t)x_{1,1}^2 + c_2(t)x_{1,2} + c_3(t)x_{1,1}, h_3(x_1) = 0.3x_{1,1}^2 + c_4(t)x_{1,2} \), and \( c_i(t), i \in T_1 \), are bounded continuous time-varying functions. Initially, at \( t = t_1 \), with \( c_1(t_1) = 0, c_2(t_1) = 1, c_3(t_1) = -1 \), and \( c_4(t_1) = 1 \), the constrained set mirrors Fig. 5.1a satisfying Condition I in Lemma 5.3. Then, as \( c_i(t), i \in T_1 \), continuously vary over time, at \( t = t_2 \), where \( c_1(t_2) = 0.3, c_2(t_2) = -1, c_3(t_2) = 0 \), and \( c_4(t_2) = -1 \), we observe \( h_2(t_2, x_1) = h_3(t_2, x_1) \). The LBO and UBO constraints for \( h_2(t_2, x_1) \) and \( h_3(t_2, x_1) \) combine into a single funnel constraint, resulting in a constrained set resembling the one in Example 5.2 depicted in Fig. 5.1b. Hence, if, for \( t \in (t_1, t_2) \), the functions \( c_i(t), i \in T_1 \), vary in such a way that any condition in Lemma 5.3 holds (requiring the constrained set in Fig. 5.1a to transform into a box and then into the one in Fig. 5.1b), the invexity of \(-\alpha(t, x_1)\) is guaranteed for all \( t \in [t_1, t_2] \).

**Remark 5.9.** Notice that satisfying Condition I of Lemma 5.3 alone is not enough to ensure the boundedness of \( \Omega(t) \). To guarantee that \( \Omega(t) \) is bounded, the functions \( \psi_i(t, x_1), i \in T_m^m \) should also meet the condition of Lemma 5.2 (see Lemma 5.2’s proof). However, for Condition II of Lemma 5.3 it is worth noting that since \( h(t, x_1) \) is norm-coercive and only funnel-type constraints are considered (i.e., \( h(t, x_1) = h_f(t, x_1) \)), one can verify that Condition I in Lemma 5.2 is already satisfied. This, in turn, ensures the boundedness of \( \Omega(t) \).

The following theorem summarizes our main result:

**Theorem 5.1.** Consider the MIMO nonlinear system (5.1) subject to time-varying output constraints (5.2). Let the design of \( \rho(t) \) satisfy conditions (i) and (ii) in Subsection 5.4.2. Additionally, select constants \( \psi_{i,j}^0, i \in T_2, j \in T_1 \) in (5.21) such that \( \psi_{i,j}^0 > |e_{i,j}(0, \bar{x}(0))| \) (as explained in Step i-a in Section 5.5). Under Assumptions 5.1-5.6 and 5.8, the feedback control law (5.28) ensures the satisfaction of the
consolidating constraint \(5.13\), as well as the boundedness of all closed-loop signals for all time.

**Proof.** See Subsection 5.10.4.

**Remark 5.10.** The results in Theorem 5.1 are independent of Assumption 5.7. Specifically, Theorem 5.1 remains valid when \(\rho_\alpha(t)\) satisfies Properties (i) and (ii) outlined in Subsection 5.4.2, and it is bounded along with its derivative \(\dot{\rho}_\alpha(t)\). As discussed in Subsection 5.4.3 Assumption 5.7 primarily aids in the design of \(\rho_\alpha(t)\), ensuring the fulfillment of Property (i). In Section 5.7, we will introduce an adaptive \(\rho_\alpha(t)\) design that does not rely on Assumption 5.7.

**Remark 5.11.** Control law \(5.28\) ensures that \(5.13\) is met for all time, but the parameter \(\nu > 0\) in \(5.16\) significantly shapes \(\alpha(t, x_1(t; x(0)))\) concerning \(\rho_\alpha(t)\) in \(5.13\). Specifically, with a very small \(\nu\), even a slight increase in \(e_\alpha > 0\) strongly influences \(e_\alpha\) growth. Consequently, the intermediate control law \(s_1(t, x_1)\) \(5.18\) restricts \(e_\alpha\) growth, keeping \(\alpha(t, x_1(t; x(0)))\) closer to \(\rho_\alpha(t)\). Conversely, a larger \(\nu\) relaxes this restriction, allowing \(\alpha(t, x_1(t; x(0)))\) more freedom to approach \(\alpha^*(t)\).

The validity of this statement can also be intuitively confirmed by observing that \(s_1(t, x_1)\) in \(5.17\) is formulated as the negative gradient of \(V_1(e_\alpha)\). As depicted in Fig. 5.2, it becomes apparent that for smaller values of \(\nu\), \(s_1(t, x_1)\) tends to maintain \(e_\alpha\) closer to zero. Consequently, it imposes constraints on the evolution of \(\alpha(t, x_1(t; x(0)))\), forcing it to stay in the proximity of \(\rho_\alpha(t)\).

Assumption 5.8 is crucial for Theorem 5.1 and ensures the effectiveness of the proposed control law \(5.28\). However, it places certain limitations on the class of time-varying output constraints suitable for \(5.1\). Nevertheless, it is worth noting that there are scenarios where \(5.28\) can still work effectively without satisfying Assumption 5.8. For instance, let \(y = h(x_1) = x_{1,1}^2 + x_{1,2}^2\) be the sole output of \(5.1\). Implicitly, \(h(x_1) \geq 0\), and if we choose \(h(x_1) < \rho(t)\) as the output constraint,
5.7 Dealing with Potential Infeasibilities in the Constraints

In this section, we propose an adaptive design for $\rho_\alpha(t)$ in (5.13) to address the potential infeasibility of the inner-approximated output constrained set $\Omega(t)$ within an unknown time interval $I$, which is captured by having $\alpha^*(t) < 0$ in (5.12) for all $t \in I$. Our objective is to address conflicts that may arise from the couplings between multiple time-varying output constraints, leading to a possible violation of Assumption 5.7, which renders the proposed design of $\rho_\alpha(t)$ in Subsection 5.4.3 inapplicable. To resolve this issue, first, we introduce the concept of least violating solution for (5.1) to cope with the case when it is not possible to satisfy the output constraints.
Let us recall that according to (5.12), \( \alpha^*(t) < 0 \) for an unknown time interval \( I \) implies that the inner-approximated output constrained set \( \Omega(t) \) in (5.10) is empty (infeasible) for all \( t \in I \).

**Definition 5.2.** When \( \alpha^*(t) < 0 \), \( x(t; x(0)) \) is a least violating solution for (5.1) with a given gap of \( \mu^* > 0 \) if:

\[
\alpha^*(t) - \mu^* < \alpha(t, x_1(t; x(0))), \quad \forall t \in I.
\]

In other words, a least violating solution for (5.1) under the constraints in (5.2) is obtained by maintaining \( \alpha(t, x_1(t; x(0))) \) in a sufficiently small neighborhood below \( \alpha^*(t) \) whenever \( \alpha^*(t) < 0 \).

### 5.7.1 Estimating \( \alpha^*(t) \) via Online Continuous Time Optimization

Upon examining (5.13) and (5.29), it becomes evident that having knowledge of \( \alpha^*(t) \) is crucial for effective design of \( \rho_\alpha(t) \), ensuring the attainment of a least violating solution when \( \alpha^*(t) < 0 \). However, direct access to \( \alpha^*(t) \) can be limiting in various applications. To overcome this limitation, we introduce \( \hat{\alpha}(t) \) as an online estimate of \( \alpha^*(t) \) and propose an online continuous-time optimization scheme to estimate \( \alpha^*(t) \). Recall that \( \alpha^*(t) \) in (5.12) does not depend on the dynamical system (5.1) but the behavior of the output constraints in (5.2). To prevent any potential ambiguity in the notations, henceforth, we distinguish between the state vector \( x_1 \) of the dynamical system (5.1) and the optimization variable \( x_1 \) in (5.12). Thus, we denote the optimization variable in (5.12) as \( \tilde{x}_1 \in \mathbb{R}^n \), yielding:

\[
\alpha^*(t) := \max_{\tilde{x}_1 \in \mathbb{R}^n} \alpha(t, \tilde{x}_1).
\]

To estimate \( \hat{\alpha}(t) \), we propose the following first-order continuous-time optimization scheme:

\[
\begin{align*}
\dot{\tilde{x}}_1 &= k_{\alpha} \nabla_{\tilde{x}_1} \alpha(t, \tilde{x}_1) \quad \text{(5.31a)} \\
\dot{\hat{\alpha}}(t) &= \alpha(t, \tilde{x}_1) \quad \text{(5.31b)}
\end{align*}
\]

where \( k_{\alpha} > 0 \) and \( \tilde{x}_1(0) \) is an arbitrarily value. In equation (5.31), \( \hat{\alpha}(t) \) represents the continuous-time evaluation of the time-varying cost function \( \alpha(t, \tilde{x}_1) \) at each time instant \( t \) using the gradient ascent optimization scheme in (5.31a). Since \( \alpha^*(t) \) denotes the maximum value of \( \alpha(t, \tilde{x}_1) \) for any \( \tilde{x}_1 \) at each time instant, it is evident that \( \alpha(t, \tilde{x}_1) \leq \alpha^*(t) \) holds for all \( t \geq 0 \). Consequently, in (5.31), \( \hat{\alpha}(t) \) can only approach \( \alpha^*(t) \) from below, meaning that \( \hat{\alpha}(t) \leq \alpha^*(t) \) for all \( t \geq 0 \).

Note that, according to Assumption 5.8 every critical point of \( \alpha(t, \tilde{x}_1) \) is a global maximizer. If \( \alpha^*(t) \) varies slowly over time, it is anticipated that following the gradient of \( \alpha(t, \tilde{x}_1) \) with respect to \( \tilde{x}_1 \) can approximate \( \alpha^*(t) \) effectively. While this approach may not guarantee precise convergence to \( \alpha^*(t) \), it is well-suited for our needs. Notably, enhancing the parameter \( k_{\alpha} \) in (5.31) can significantly...
improve the estimation of $\alpha^*(t)$, especially when $\alpha^*(t)$ exhibits rapid variations over time. Additionally, initializing $\hat{x}_1(0)$ in (5.31) such that $\hat{\alpha}(0) = \alpha(t, \hat{x}_1(0)) \approx \alpha^*(0)$ can further enhance the performance of (5.31). To achieve this, obtaining an approximate solution for (5.30) at $t = 0$ through an offline procedure can aid in selecting an appropriate $\hat{x}_1(0)$ that closely approximates $\alpha^*(0)$.

**Remark 5.12.** In the realm of continuous-time optimization for time-varying cost functions, second-order gradient flows under a prediction-correction scheme have been proposed for achieving asymptotic convergence to the optimal point [52]. However, this method relies on the Hessian inverse of the time-varying cost function, necessitating the cost function to be $m$-strong concave (or convex). It is important to note that in our work $\alpha(t, \hat{x}_1)$ does not always satisfy this condition. Since this second-order approach is akin to continuous-time variant of Newton’s method, using it in our context does not guarantee convergence to the global optimum of $\alpha(t, \hat{x}_1)$ and, in the worst-case scenario, could result in divergence. Consequently, we have chosen to employ a first-order optimization scheme (5.31), which offers practical convergence to the time-varying optimum of $\alpha(t, \hat{x}_1)$, provided an appropriate choice of $k_\alpha$.

In Subsection 5.4.3, we proposed a method to design $\rho_\alpha(t)$ effectively, achieving the fulfillment of (5.2), which, however, relied on Assumption 5.7. In the following subsection, we aim to present an adaptive design for $\rho_\alpha(t)$ that does not require Assumption 5.7. Instead, we will utilize the available information on $\alpha^*(t)$ through the estimation scheme (5.31) to handle potentially infeasible time-varying output constraints. Our objective is to design $\rho_\alpha(t)$ in such a way that, whenever $\alpha^*(t) < 0$, it ensures a least violating solution, while still preserving Properties (i) and (ii) described in Subsection 5.4.2.

### 5.7.2 Design of $\rho_\alpha(t)$ for Potentially Infeasible Constraints

Let us first introduce $\varrho(t)$ as a nominal lower bound for $\alpha(t, x_1(t; x(0)))$, which determines the nominal behavior of the lower bound in (5.13). Specifically, $\varrho(t)$ is designed to ensure the satisfaction of output constraints by enforcing $\alpha(t, x_1(t; x(0)))$ to become and remain positive within a user-defined finite time $T$. In this regard, similar to the design of $\rho_\alpha(t)$ in Subsection 5.4.3, we can design $\varrho(t)$ as:

$$
\varrho(t) := \begin{cases} 
\left(\frac{T-t}{T}\right)^\beta (\varrho_0 - \varrho_\infty) + \varrho_\infty, & 0 \leq t < T, \\
\varrho_\infty, & t \geq T,
\end{cases}
$$

(5.32)

where $\beta \in (0, 1)$, $\varrho(0) = \varrho_0 < \alpha(0, x_1(0))$, and $\varrho_\infty \geq 0$ is a user-defined arbitrary non-negative constant. Recall that a larger $\varrho_\infty$ enforces how well the output constraints should be satisfied (in the nominal case) after finite time $t = T$ (see Remark 5.4). In this respect, we refer to $\varrho_\infty$ as the nominal constraint satisfaction margin. We now propose an alternative design for $\rho_\alpha(t)$ as follows:

$$
\rho_\alpha(t) = \iota(t) \varrho(t) + (1 - \iota(t))(\hat{\alpha}(t) - \mu),
$$

(5.33)
Figure 5.4: The evolution of $\alpha(t, x_1(t; x_0))$ under the consolidating constraint (5.13), where $\rho_\alpha(t)$ is determined by (5.33). The adaptation of $\rho_\alpha(t)$ (dashed line) based on the evolution of $\hat{\alpha}(t)$ in (5.31) (dotted line) allows for deviations of $\rho_\alpha(t)$ from the nominal lower bound function $\varrho(t)$ in (5.32). Consequently, satisfaction of (5.13) during the time intervals when $\alpha^*(t) < 0$ (shaded intervals) results in a least violating solution. In this illustrative example, $\hat{\alpha}(t)$ provides a reliable estimate of $\alpha^*(t)$, and the parameters used for this illustration are $\mu = 0.45$, $\varrho_x = 0.5$, and $T = 4$.

where $\mu > 0$ is a user-defined small positive constant, and $\iota : \mathbb{R}_{\geq 0} \to [0, 1]$ is a $C^1$ switch function given by:

$$
\iota(t) = \begin{cases} 
1 & \varphi(t) > \mu \\
-\frac{2}{\mu^3} \varphi^3(t) + \frac{3}{\mu^2} \varphi^2(t) & 0 \leq \varphi(t) \leq \mu \\
0 & \varphi(t) < 0
\end{cases}, \quad (5.34)
$$

in which $\varphi(t) := \hat{\alpha}(t) - \varrho(t)$. It is important to note that the third-order polynomial in (5.34) is deliberately designed to ensure that $\iota(t)$ varies smoothly between 1 and 0.

The logic behind the design in (5.32), (5.33) and (5.34) is summarized as follows: first, $\varrho(t)$ in (5.32) is designed as the nominal lower bound on $\alpha(t, x_1(t; x(0)))$ to address the user’s desired specifications regarding the satisfaction of the time-varying output constraints while ignoring whether these constraints are feasible or not for all time. Next, $\rho_\alpha(t)$ in (5.33) is designed as a convex combination of two terms such that when $\hat{\alpha}(t) - \varrho(t) > \mu$, we obtain the nominal lower bound behavior $\rho_\alpha(t) = \varrho(t)$. Otherwise, when $\hat{\alpha}(t) - \varrho(t) < 0$, we get $\rho_\alpha(t) = \hat{\alpha}(t) - \mu$. The transition between these two modes is achieved through the smooth switch (5.34). Note that since (5.33) is a convex combination, $\rho_\alpha(t)$ always takes a value between $\varrho(t)$ and $\hat{\alpha}(t) - \mu$ during the transition phase where $0 \leq \varphi(t) \leq \mu$. In particular, by employing (5.33), we allow the lower bound $\rho_\alpha(t)$ in (5.13) to deviate from its nominal behavior $\varrho(t)$ in order to achieve a minimum user-defined gap of $\mu$ with respect to $\hat{\alpha}(t)$. Fig. 5.4 illustrates the behavior of $\rho_\alpha(t)$ in (5.33).
Lemma 5.4. Let \( \tilde{e} := \alpha^*(t) - \hat{\alpha}(t) \). If \( \alpha^*(t) < 0 \) and \( \varphi(t) \geq 0 \) for all \( t \in I \), where \( I \) is some unknown time interval, then the satisfaction of \eqref{eq:5.13} under \( \rho_\alpha(t) \) given by \eqref{eq:5.33} guarantees a least violating solution with the gap of \( \mu^* = \tilde{e} + \mu \).

**Proof.** First, note that since \( \dot{\hat{\alpha}}(t) = \alpha(t, \bar{x}_1) \leq \alpha^*(t), \forall t \geq 0 \), we always have \( \tilde{e} \geq 0 \). Given the conditions in the lemma it is easy to verify that \( \varphi(t) = \dot{\hat{\alpha}}(t) - \varphi(t) < 0 \) for all \( t \in I \). Hence, from \eqref{eq:5.33} and \eqref{eq:5.34} we get \( \rho_\alpha(t) = \dot{\hat{\alpha}}(t) - \mu \). Consequently, the satisfaction of \eqref{eq:5.13} leads to \( \dot{\hat{\alpha}}(t) - \mu < \alpha(t, x_1(t; x(0))), \forall t \in I \), which is equivalent to \( \alpha^*(t) - \mu^* < \alpha(t, x_1(t; x(0))), \forall t \in I \), with \( \mu^* = \tilde{e} + \mu \).

From Lemma 5.4, it is evident that the gap of the obtained least violating solution through utilizing \eqref{eq:5.33} in \eqref{eq:5.13} is affected by \( \tilde{e} \) and the tunable constant \( \mu > 0 \) in \eqref{eq:5.33}. In particular, the better \( \dot{\hat{\alpha}}(t) \) estimates \( \alpha^*(t) \), the smaller the gap \( \mu^* \) becomes.

Consider the case where \( \alpha^*(t) > 0, \forall t \in I \), indicating that the time-varying constrained set \( \Omega(t) \) is feasible for all \( t \in I \), and further assume that \( \alpha^*(t) - \varphi(t) > \mu, \forall t \in I \). In this scenario, when \( \dot{\hat{\alpha}}(t) \) poorly estimates \( \alpha^*(t) \), a situation may arise where \( \varphi = \dot{\hat{\alpha}}(t) - \varphi(t) \leq \mu \), or particularly, \( \varphi < 0 \) in \eqref{eq:5.34}. Consequently, \( \rho_\alpha(t) \) in \eqref{eq:5.33} might not effectively follow the intended behavior designed by \( \varphi(t) \) for all \( t \in I \). This discrepancy can lead to a certain degree of conservativeness in fulfilling the output constraints. To clarify, even when the constraints are feasible at all time, enforcing \eqref{eq:5.13} under \eqref{eq:5.33} might not guarantee constraint satisfaction if \( \dot{\hat{\alpha}}(t) \) has a very poor performance in estimating \( \alpha^*(t) \). Therefore, it is crucial to properly tune the parameter \( k_\alpha \) in \eqref{eq:5.31} to enhance the performance of \eqref{eq:5.31} especially when \( \alpha^*(t) \) does not vary slowly enough with time. It is also important to emphasize that although \( \tilde{x}_1(0) \) in \eqref{eq:5.31} can be set arbitrarily, it is recommended to select \( \tilde{x}_1(0) \) such that \( \alpha(0, \tilde{x}_1(0)) = \alpha(0, x_1(0; x(0))) \), which helps in reducing the conservatism in satisfying the output constraints. As described in Subsection 5.7.1, it is better to select \( \tilde{x}_1(0) \) such that \( \dot{\hat{\alpha}}(0) = \alpha(0, \tilde{x}_1(0)) \) becomes closer to \( \alpha^*(0) \).

Finally, note that the dynamical system \eqref{eq:5.31}, given an initialization \( \tilde{x}_1(0) \), runs in parallel with the closed-loop system dynamics \eqref{eq:5.1}. It generates \( \dot{\hat{\alpha}}(t) \) at each time instant \( t \), which is then used in the (online) computation of \( \rho_\alpha(t) \) in \eqref{eq:5.33}. Recall that \( \rho_\alpha(t) \) is utilized in the control law \( u(t, x) \), specifically in the first intermediate control \eqref{eq:5.18}. Therefore, the gradient flow dynamic \eqref{eq:5.31} is connected to the closed-loop system in a cascaded form (see Fig. 5.5), and thus is independent of \eqref{eq:5.1}.

To conclude this section we show that the results stated in Theorem 5.1 still hold under \( \rho_\alpha(t) \) given in \eqref{eq:5.33}. In this regard, to ensure the boundedness of \( \dot{\hat{\alpha}}(t) \) in \eqref{eq:5.31}, we require the following technical assumption:

**Assumption 5.9.** There exist \( \pi_\alpha > 0 \) such that \( |\dot{\hat{\alpha}}(t)| \leq \pi_\alpha \) and \( k_c > 0 \) such that \( \Lambda(t, \tilde{x}_1) := \frac{\dot{\hat{\alpha}}(t, \tilde{x}_1)}{\dot{\hat{\alpha}}(t)} + k_\alpha \| \nabla \tilde{x}_1 \alpha(t, \tilde{x}_1) \|^2 \) is coercive (radially unbounded) in \( \tilde{x}_1 \) for all \( k_\alpha \geq k_c \).
5.8. Simulation Results

In this section, we present two simulation examples to validate the proposed control approach. The first example demonstrates our method’s effectiveness in addressing problems that are already solvable using existing methods, such as the conventional PPC method, where time-varying output constraints are decoupled. Subsequently, we offer an example involving coupled time-varying constraints, which cannot be accommodated by previous approaches.

In the upcoming simulation examples, we will consider a mobile robot operating in a 2-D plane with kinematics and dynamics expressed by (2.20). To implement the proposed control law of this chapter, we transform the mobile robot dynamics with respect to the hand position $p_H$ (refer to Fig. 2.8) as described in Chapter 2 Section 2.6. This transformation leads to the equivalent (fully actuated) Euler-Lagrangian dynamics (2.24), which can be written in the state-space form as follows:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= M(x_1)^{-1}(-C(x_1,x_2)x_2 - D(x_1)x_2 + u + d(t)),
\end{align*}
\] (5.35)

where $x_1$ corresponds to the hand position of the mobile robot ($x_1 = p_H$), and $x_2$ represents its velocity. The elements of matrices $M(x_1)$, $C(x_1, x_2)$, and $D(x_1)$ are locally Lipschitz continuous functions of their arguments. Note that, (5.35) can...
be viewed as a specific form of (5.1) with $n = 2$ and $r = 2$ and it is not difficult to verify that Assumptions 5.1, 5.2, and 5.3 hold for (5.35). In the simulations, we set $m_R = 3.6$, $I_R = 0.0405$, $D_1 = 0.3$, $D_2 = 0.04$, $L = 0.2$, and $d(t) = [0.75 \sin (3t + \frac{\pi}{2}) + 1.5 \cos (t + \frac{3\pi}{2}), -2.4 \exp (\cos (t + \frac{\pi}{3}) + 1) \sin (t)]^T$ in (2.20).

### 5.8.1 Decoupled Time-Varying Constraints

In our first simulation example, we will focus on trajectory tracking for the mobile robot described by (5.35), where the desired trajectory is $x_1(t) = [4.2 \cos (0.47t), 4.2 \sin (0.47t)]^T$. Our goal is to enforce specific performance funnel constraints on tracking errors, defined as follows: $-\rho_i(t) < x_{1,i} - x_{1,i}^d(t) < \rho_i(t)$ for $i \in I_1^2$ and all $t \geq 0$. Here, $\rho_1(t)$ and $\rho_2(t)$ are strictly positive, time-varying performance bounds. Without loss of generality, we assume $\rho_1(t) = \rho_2(t) = (1.75 - 0.3) \exp (-0.35t) + 0.3$.

To express these requirements analogously to the problem formulation in Section 5.3, we consider $y = h(x_1) = x_1$ for the dynamics (5.35) under the following funnel constraints: $\underline{P}_i(t) := -\rho_i(t) + x_{1,i}^d(t) < x_{1,i} < \rho_i(t) + x_{1,i}^d(t) := \overline{P}_i(t)$ for $i \in I_1^2$ and all $t \geq 0$. Note that $h(x_1)$ readily satisfies Assumptions 5.4 and 5.5. Moreover, the above funnel constraints resemble a time-varying box constraint in the $x_1$ space (i.e., Assumption 5.6 holds). As $\rho_1(t)$ and $\rho_2(t)$ are strictly positive, both funnel constraints are well-defined and feasible. Furthermore, these two funnel constraints are decoupled, as each one imposes time-varying upper and lower bounds on independent state variables, namely $x_{1,1}$ and $x_{1,2}$. This feature is also evident by verifying that Condition II of Lemma 5.3 holds. Satisfaction of Condition II of Lemma 5.3 also indicates that Assumption 5.8 is valid. Now since the aforementioned funnel constraints are well-defined and decoupled, they are mutually satisfiable for all time. Therefore, the constrained set $\Omega(t)$ defined in (5.8) is guaranteed to be feasible for all $t \geq 0$.

It is important to note that the feasibility of $\tilde{\Omega}(t)$ for all time does not necessarily imply the feasibility of its corresponding smooth under-approximation $\Omega(t)$ defined in (5.10). However, in this specific example, one can reasonably assume that Assumption 5.7 holds, given that $\tilde{\Omega}(t)$ remains feasible for all time and does not become excessively tight over certain time intervals (i.e., having overly stringent time-varying constraints).

This assumption allows us to use the suggested design of $\rho_\alpha(t)$ in (5.14) for consolidating constraint (5.13).

The numerical values of the parameters used to implement control law (5.28) are provided in Table 5.1. It is important to note that, as per the guidelines outlined in Subsection 5.4.3 and Section 5.5, the values of $\rho_0$ and $\vartheta^0_{2,j}$, $j \in I_1^2$ in Table 5.1 are determined based on the initial condition $x(0)$ of the transformed mobile robot dynamics in (5.35). Henceforth, we use $x_1(t)$ instead of $x_1(t; x(0))$ for brevity. In what follows, we consider two scenarios, where the mobile robot’s tracking errors

\[\text{Recall that, selecting a sufficiently large } \nu \text{ in (5.9) gives a closer under-approximation of } \tilde{\Omega}(t) \text{ and thus can provide more confidence on feasibility of } \Omega(t).\]
Table 5.1: Numerical values of the parameters involved in control law (5.28).

<table>
<thead>
<tr>
<th>Eq. no</th>
<th>Parameter(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5.9a)</td>
<td>$\nu = 10$</td>
</tr>
<tr>
<td>(5.14)</td>
<td>$T = 3$, $\beta = 0.3$, $\rho_{\infty} = 0$, $\rho_0 &lt; \alpha(0, x_1(0))$</td>
</tr>
<tr>
<td>(5.16)</td>
<td>$v = 8$</td>
</tr>
<tr>
<td>(5.18)</td>
<td>$k_1 = 1$</td>
</tr>
<tr>
<td>(5.21)</td>
<td>$\vartheta_{2,j}^0 = 0.1$, $l_{2,j} = 1$, $\vartheta_{2,j}^0 &gt;</td>
</tr>
<tr>
<td>(5.26)</td>
<td>$k_2 = 1$</td>
</tr>
</tbody>
</table>

performance specifications are not initially satisfied (Scenario 1), and the situation in which the performance specifications are initially met (Scenario 2).

The simulation results of Scenario 1 under the proposed control method are illustrated in Figs. 5.6, 5.7, and 5.8 respectively. In particular, Figs. 5.6 and 5.9 provide snapshots of the mobile robot’s hand position trajectory $x_1(t)$ along with the time-varying constrained set $\Omega(t)$ for both scenarios. Recall that $\bar{\text{cl}}(\Omega(t)) = \{x_1 \in \mathbb{R}^2 \mid \alpha(t, x_1) = 0\}$.

In Scenario 1, since $\alpha(0, x_1(0)) < 0$ (see Fig. 5.7) the robot’s initial position does not initially satisfy the prescribed performance bounds on the tracking errors $(x_1(0) \notin \bar{\Omega}(0))$ as depicted in Fig. 5.8. However, by enforcing (5.13) under $\rho_{\alpha}(t)$ in (5.14) and applying (5.28), we observe that $\alpha(t, x_1(t))$ becomes and remains positive within the user-defined finite time limit of $T = 3$ seconds. This signifies the achievement of the tracking error performance specifications within 3 seconds.

In Scenario 2, where $\alpha(0, x_1(0)) > 0$ (see Fig. 5.10), the performance bounds on the tracking errors are initially satisfied ($x_1(0) \in \Omega(0)$). By maintaining $\alpha(0, x_1(t))$ positive, we ensure the continuous fulfillment of the tracking errors’ specifications throughout the simulation as depicted in Fig. 5.11.

It is worth noting that, as anticipated, $\alpha^*(t)$ remains positive for all time, which indicates the feasibility of $\Omega(t)$ for all time. Note that, $\alpha^*(t)$ is unknown to the control system and is included in the figures solely for the purpose of verifying the simulation results. The value of $\alpha^*(t)$ at each time step is obtained through solving optimization problem (5.12) offline for a dense set of time instances.

The simulation results presented above highlight a key advantage of our proposed control methodology. Unlike conventional PPC and TVBLF-based control design approaches, which necessitate the initial satisfaction of the (output) constraints for their effective implementation, our approach operates without such restrictions.

5.8.2 Coupled Time-Varying Constraints

For our second simulation example we consider coupled time-varying (output) constraints, for which previous approaches (FC, PPC, TVBLF-based control) are not applicable. Moreover, previous approaches cannot ensure a least violating solution
Figure 5.6: Snapshots of the mobile robot’s hand position trajectory $x_1(t)$ along with the time-varying constrained set $\Omega(t)$ in Scenario 1.

Figure 5.7: Evolution of $\alpha(t, x_1(t))$ in Scenario 1.

Figure 5.8: Tracking errors and performance bounds evolution in Scenario 1.
5.8. Simulation Results

Figure 5.9: Snapshots of the mobile robot’s hand position trajectory $x_1(t)$ along with the time-varying constrained set $\Omega(t)$ in Scenario 2.

Figure 5.10: Evolution of $\alpha(t, x_1(t))$ in Scenario 2.

Figure 5.11: Tracking errors and performance bounds evolution in Scenario 2.
Feedback Control for Uncertain MIMO Nonlinear Systems under Generalized Time-Varying Output Constraints

Consider the transformed mobile robot dynamics in (5.35) with the output map $y = h(t, x_1) = [h_1(t, x_1), h_2(t, x_1), h_3(t, x_1)]^T$, for which we assume the following (coupled) time-varying constraints:

$$
\rho_1(t) < h_1(t, x_1) < \overline{\rho}_1(t), \quad \text{(funnel constraint)},
$$
$$
\rho_2(t) < h_2(t, x_1), \quad \text{(LBO constraint)},
$$
$$
h_3(t, x_1) < \overline{\rho}_3(t), \quad \text{(UBO constraint)},
$$

with

$$
\rho_1(t) = -0.7 - \sin(0.4t), \quad \overline{\rho}_1(t) = 1.1 + 3 \sin(0.45t),
$$
$$
\rho_2(t) = -1 - 0.5 \cos(0.3t),
$$
$$
\overline{\rho}_3(t) = 0.5 + \sin(0.4t).
$$

Moreover, let:

$$
h_1(t, x_1) = x_{1,1} - o_1(t)
$$
$$
h_2(t, x_1) = c_1(t)(x_{1,1} - o_1(t))^2 + c_2(t)(x_{1,2} - o_2(t)) + c_3(t)(x_{1,1} - o_1(t)),
$$
$$
h_3(t, x_1) = c_4(t)(x_{1,1} - o_1(t))^2 + (x_{1,2} - o_2(t)),
$$

in which $o_1(t) = 5 \cos(0.28t), o_2(t) = 5 \sin(0.28t), c_1(t) = -2 + 2 \cos(t), c_2(t) = 1 + 0.5 \sin(0.7t), c_3(t) = \sin(0.4t), \text{ and } c_4(t) = 1 - \cos(0.5t)$ are all bounded continuously differentiable functions of time. The time-varying output map $h(t, x_1)$ satisfies Assumptions 5.4 and 5.5. Furthermore, due to the way the constraints are designed, the set $\Omega(t)$ (and consequently $\Omega(t)$) remains bounded for all time (Assumption 5.6). Moreover, it can be verified that the constraints fulfill Condition I of Lemma 5.3, thus confirming the validity of Assumption 5.8. In simple terms, these constraints define a bounded time-varying region that the mobile robot’s (hand) position should enter and remain within for all time (i.e., a time-varying region tracking problem). Nevertheless, we did not assume that the constrained region is always feasible. Therefore, we utilize the proposed estimation scheme (5.31) along with $\rho_\alpha(t)$ given by (5.33) for the consolidating constraint (5.13).

For the simulations of this subsection, all numerical values used for the control law (5.28) match those in Table 5.1 with the only difference being that $\rho_\alpha(t)$ follows (5.33). Specifically, we set $\mu = 0.2$ in (5.33), and the parameters for $\rho(t)$ in (5.32) are set to $\rho_0 < \alpha(0, x_1(0)), \rho_\infty = 0.5, T = 3, \text{ and } \beta = 0.3$. Building on the discussion in Subsection 5.7.1, we conduct two simulations to highlight how the performance of the estimation scheme (5.31) impacts constraint satisfaction under control law (5.28) in these simulations, we explore two cases: setting $k_\alpha = 4$ and $k_\alpha = 1$ in (5.31a). Both simulations assume that the initial condition for the estimator (5.31a) is the same as the initial hand position of the mobile robot, i.e., $\hat{x}_1(0) = x_1(0)$, leading to $\hat{\alpha}(0) = \alpha(t, x_1(0))$. 

(as per (5.29)) when constraint infeasibilities arise.
Fig. 5.12 (bottom) shows the evolution of $\alpha(t, x_1(t))$ under control law (5.28) with $k_\alpha = 4$ in the estimation scheme (5.31a). After a brief transient period, the estimator’s output ($\hat{\alpha}(t)$) closely follows $\alpha^*(t)$, such that the estimation error remains small for all time. Additionally, thanks to the satisfaction of consolidating constraint (5.13), time-varying output constraints are guaranteed to be met with a margin of $q_\infty = 0.5$ after a user-defined $T = 3$ seconds. However, roughly between $t = 8$ and $t = 14$ (shaded interval), $\alpha^*(t) < 0$, indicating that the constraints become temporarily infeasible. During this time, the proposed $\rho_\alpha(t)$ (5.33) diverts from the nominal lower bound $\varrho(t)$ to ensure a least violating solution. When the constraints become feasible again ($\alpha^*(t) > 0$), $\rho_\alpha(t)$ quickly returns to the nominal constraint satisfaction requirement i.e., $\rho_\alpha(t) = q_\infty = 0.5$. Finally, in Fig. 5.12 (top), snapshots of the mobile robot’s hand position are shown along with the constrained region. Note that, the constrained region is shown only when it is feasible (nonempty).

The same simulation scenario is repeated with a lower gain, $k_\alpha = 1$ in (5.31a), and the results are presented in Fig. 5.13. This reduced gain leads to a poorer performance in estimating $\alpha^*(t)$. In Fig. 5.13 (bottom), we can see that the evolution of $\rho_\alpha(t)$ is influenced by $\hat{\alpha}(t)$, initially deviating from the nominal lower bound $\varrho(t)$ until $t = 4$. However, as this deviation is not significant it turns out that the controller is still capable of meeting the user-defined specifications for constraints satisfaction by maintaining $\alpha(t, x_1(t))$ above the nominal lower bound $\varrho(t)$ for over 7 seconds (although only $\rho_\alpha(t) < \alpha(t, x_1(t))$ is guaranteed by the proposed controller). As the time-varying constraints tend to become infeasible (shaded interval), $\alpha^*(t)$ rapidly decreases, which induces a significant divergence between $\rho_\alpha(t)$ and $\varrho(t)$ due to a large estimation error. As a result, the controller can only ensure a least violating solution with a considerably large gap. Recall that, as per Lemma 5.4, the gap for the least violating solution is given by $\mu^* = \hat{\epsilon} + \mu$, where $\hat{\epsilon} = \alpha^*(t) - \hat{\alpha}(t) \geq 0$ represents the estimation error. From Fig. 5.13 (bottom), it is evident that, even when the constraints become feasible again, owing to a rapid increase of $\alpha^*(t)$ a large estimation error continues to persist for some time, which hinders $\rho_\alpha(t)$ from approaching $\varrho(t)$. This phenomenon makes the controller to present a more conservative constraint satisfaction behavior. This is more evident in Fig. 5.13 (top), where the mobile robot is still out of the (feasible) constrained region at $t = 15.57$. This simulation underscores the direct impact of the estimator’s performance on the control law. Therefore, if one expects that $\alpha^*(t)$ can change rapidly, careful tuning of the gain for estimator (5.31a) becomes essential.

### 5.9 Conclusions

In this chapter, we introduced a novel low-complexity feedback control design for high-order uncertain MIMO nonlinear systems with multiple (potentially coupled) time-varying output constraints. Our method addresses these constraints by interpreting their satisfaction as the fulfillment of a single consolidating constraint.
Figure 5.12: Time-varying region tracking of the mobile robot (case 1). With an estimator gain of $k_\alpha = 4$ in (5.31a), minimal conservative behavior is achieved in satisfaction of the time-varying constraints (or region tracking), attributed to the estimator’s good performance in estimating (unknown) $\alpha^*(t)$. Moreover, a least violating solution is ensured with a small gap whenever the constraints become infeasible (empty region).
Figure 5.13: Time-varying region tracking of the mobile robot (case 2). A reduced estimator gain of $k_\alpha = 1$ in (5.31a) leads to a least violating solution with a large gap, which adversely impacts the controller’s performance, resulting in a more conservative satisfaction of time-varying constraints.
related to the signed distance with respect to the boundary of the time-varying constrained set. We have shown that by dynamically adjusting the lower bound of the consolidating constraint, our method ensures a least violating solution when the time-varying constraints become infeasible for an unknown time interval. Moreover, it overcomes the limitations of existing feedback control design approaches (like FC, PPC, and TVBLFs) when dealing with coupled time-varying constraints. As a result, the proposed method can be applied to a broader range of applications.

From an alternative perspective, the control methodology proposed in this chapter provides a means to guarantee the forward invariance of time-varying sets through feedback control design. Unlike preceding chapters and existing methods limited to handling box-constrained time-varying sets, our approach advances by addressing a broader class of constrained time-varying sets.

### 5.10 Proofs of Lemmas and Theorems

#### 5.10.1 Proof of Lemma 5.1

To start with, notice that all $p_i(t), p_{\perp}(t)$ in (5.4) are bounded, and based on Assumption 5.5 $|h_i(t,x_1)| \leq \bar{h}_i(x_1), i \in I_1^m$ holds. Therefore, $\psi_i(t,x_1), i \in I_1^m$ are bounded for all $t \geq 0$ and any fixed $x_1$. As a result, $\bar{\alpha} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ in (5.7) is bounded for all $t \geq 0$ and any fixed $x_1$. Let $\bar{\alpha}_t(x_1) := \bar{\alpha}(t,x_1)$. According to Assumption 5.6, $-\bar{\alpha}_t(x_1)$ is coercive in $x_1$ for each $t$. Therefore, by Theorem 2.5, all super-level sets $\bar{\alpha}_t(x_1) \geq c$, where $c \in \mathbb{R}$, are bounded for each $t$. Furthermore, based on Assumption 5.6 and Theorem 2.6, we can infer that there exists a time-dependent constant $\bar{c}(t) \in \mathbb{R}$ such that the super-level sets $\bar{\alpha}_t(x_1) > \bar{c}(t)$ are empty. This implies that $\Omega(t)$ in (5.8) is bounded, and in particular, it is empty if $\bar{c}(t) < 0$ at time $t$. Moreover, from (5.9b) we can verify that $-\bar{\alpha}(t,x_1)$ is coercive if and only if $-\alpha(t,x_1)$ is coercive. As a result, we can apply similar arguments as above to establish the boundedness of $\Omega(t)$.

#### 5.10.2 Proof of Lemma 5.2

Using (5.7), we can determine whether $-\bar{\alpha}(t,x_1)$ is coercive by verifying that, for each time instant $t$, at least one of the functions $\psi_i(t,x_1)$ in (5.4) approaches $-\infty$ as $\|x_1\| \to +\infty$ (along any path on $\mathbb{R}^n$). Note that $-\alpha(t,x_1)$ in (5.9) is also coercive under the same condition. Since the functions $\psi_i(t,x_1), i \in I_1^p$ in (5.4) are bounded for all $t \geq 0$ and any fixed $x_1$ (see proof of Lemma 5.1), we can interpret this requirement in terms of $h_i(t,x_1)$. Specifically, if there exists an $i \in I_1^p$ such that $h_i(t,x_1) \to \pm \infty$ for each time instant $t$, then it ensures that $\psi_{2i-1}(t,x_1) \to -\infty$ or $\psi_{2i}(t,x_1) \to -\infty$ in (5.4a) and vice versa. To simplify the verification process, we only need to check whether $\|h_f(t,x_1)\| \to +\infty$ for each time instant $t$ and along a path on $\mathbb{R}^n$ as $\|x_1\| \to +\infty$. From (5.4b), we can also see that if there exists a $j \in I_{p+1}^{p+q}$ such that $h_j(t,x_1) \to -\infty$ for each time instant $t$, then there exists
an \( i \in I_{2p+1}^{2p+q} \) such that \( \psi_i(t, x_1) \to -\infty \) and vice versa. Similarly, if there exists a \( j \in I_{m+q+1}^m \) such that \( h_j(t, x_1) \to +\infty \) for each time instant \( t \), then there exists an \( i \in I_{2p+1}^{2p+m} \) such that \( \psi_i(t, x_1) \to -\infty \) and vice versa. In summary, if along any path on \( \mathbb{R}^n \) as \( \|x_1\| \to +\infty \) at least one of the conditions I-III in the lemma holds, then \( -\bar{a}(t, x_1) \) (resp. \( -a(t, x_1) \)) is coercive and vice versa.

5.10.3 Proof of Lemma 5.3

Case I: Consider \( \alpha(t, x_1) \) in (5.9). First, note that since \( \psi_i(t, x_1), i \in I_{m+p}^m \) are concave functions in \( x_1 \in \mathbb{R}^n \) at time \( t \) then as \( \nu > 0 \), \( -\nu \psi_i(t, x_1), i \in I_{m+p}^m \) are convex at time \( t \). Moreover, from [137, Section 3.5] it is known that \( e^{-\nu \psi_i(t, x_1)}, i \in I_{m+p}^m \) are log-convex functions. Hence, \( \sum_{i=1}^{m+p} e^{-\nu \psi_i(t, x_1)} \) is log-convex. Consequently, \( \alpha(t, x_1) \) in (5.9) is a concave function at time \( t \). Furthermore, since \( \alpha(t, x_1) \) has bounded level sets, from Assumption 5.6 it attains a well-defined global maximum (i.e., the global maximum exists). Therefore, one can conclude that every critical point of \( \alpha(t, x_1) \) is a global maximizer at time \( t \).

Case II: Here, we first establish that under the given conditions \( \alpha(t, x_1) \) attains only one critical point and then we show that the critical point is the (unique) global maximizer of \( \alpha(t, x_1) \). Recall that the critical points of \( \alpha(t, x_1) \) are obtained by solving \( \nabla_x \alpha(t, x_1) = 0 \). Given the assumed ordering of constraint types in (5.4), one can write \( \alpha(t, x_1) \) in (5.9) as follows:

\[
\alpha(t, x_1) = -\frac{1}{\nu} \ln \left( \sum_{i=1}^{p} e^{-\nu (h_i(t, x_1) - \mu_i(t))} + \sum_{i=p+1}^{p+q} e^{-\nu (h_i(t, x_1) - \mu_i(t))} + \sum_{i=p+1}^{m} e^{-\nu (\bar{\mu}_i(t) - h_i(t, x_1))} \right). \tag{5.36}
\]

Using (5.36) and (5.9), and after some calculations, we can obtain \( \nabla_x \alpha(t, x_1) \) in a compact form as:

\[
\nabla_x \alpha(t, x_1) = J^T(t, x_1) \gamma(t, x_1) e^{\nu \alpha(t, x_1)}, \tag{5.37}
\]

where \( J(t, x_1) = \frac{\partial h(t, x_1)}{\partial x_1} \in \mathbb{R}^{m \times n} \) is the Jacobian of \( y = h(t, x_1) \), and \( \gamma(t, x_1) := \text{col}(\gamma_i(t, x_1)) \in \mathbb{R}^m \), in which \( \gamma_i(t, x_1), i \in I_1^m \) are given by:

\[
\begin{align*}
\{ e^{-\nu (h_i(t, x_1) - \mu_i(t))} - e^{-\nu (\bar{\mu}_i(t) - h_i(t, x_1))} \}, & \quad i \in I_1^p \tag{5.38a} \\
\{ e^{-\nu (h_i(t, x_1) - \mu_i(t))} \}, & \quad i \in I_{p+q}^{p+q} \tag{5.38b} \\
\{ -e^{-\nu (\bar{\mu}_i(t) - h_i(t, x_1))} \}, & \quad i \in I_{p+q+1}^m \tag{5.38c}
\end{align*}
\]

Notice that in (5.37) \( e^{\nu \alpha(t, x_1)} > 0 \), therefore, \( \nabla_x \alpha(t, x_1) = 0 \) if and only if \( J^T(t, x_1) \gamma(t, x_1) = 0 \). If \( m = n \) and each output constraint is a funnel constraint.
Feedback Control for Uncertain MIMO Nonlinear Systems under Generalized Time-Varying Output Constraints

(i.e., \( p = m = n \)) then all \( \gamma_i(t, x_1) \) will be given by (5.38a). In this case, \( J(t, x_1) \in \mathbb{R}^{n \times n} \) in (5.37) is a square matrix. If at time instance \( t \) we have \( \text{rank}(J) = n \) for all \( x \in \mathbb{R}^n \), then \( J^\top(t, x_1) \gamma(t, x_1) = 0 \) holds if and only if \( \gamma(t, x_1) = 0 \) at time \( t \). Therefore, under the above conditions, at time instant \( t \), we get \( \nabla_x \alpha(t, x_1) = 0 \) if and only if \( \gamma_i(t, x_1) = 0, \forall i \in I_t^p \). Owing to (5.38a) this leads to having \( h_i(t, x_1) = 0.5(\rho_i(t) + \rho(t)), \forall i \in I_t^p \), hence, we get the following system of nonlinear equations:

\[
F(t, x_1) := h(t, x_1) - 0.5 \left( \overline{\rho}(t) + \rho(t) \right) = 0,
\]

with \( \overline{\rho}(t) := \text{col}(\overline{\rho}_i(t)) \in \mathbb{R}^n, \rho(t) := \text{col}(\rho_i(t)) \in \mathbb{R}^n \), where \( F(t, x_1) \) is \( C^1 \) in \( t \) and \( C^2 \) in \( x_1 \) owing to the properties of \( h(t, x_1) \), \( \overline{\rho}(t) \), and \( \rho(t) \). Now for each time instant \( t \) define \( F_t(x_1) := F(t, x_1) \) and recall that in (5.39) \( 0.5 \left( \overline{\rho}(t) + \rho(t) \right) \) is bounded for all time and the elements of \( h(t, x_1) \) do not grow unbounded by the variation of \( t \) (Assumption 5.5). We are interested in checking the existence and uniqueness of the solution to (5.39) at each time instant \( t \), which boils down to checking the existence and uniqueness of the solution to \( F_t(x_1) = 0 \) for each \( t \). Since \( h(t, x_1) \) is norm-coercive (i.e., \( \|h(t, x_1)\| \rightarrow +\infty \) as \( \|x_1\| \rightarrow +\infty \)) then \( F_t(x_1) \) is norm-coercive as well. Moreover, from (5.39) \( F_t(x_1) \) has the same Jacobian matrix as \( h(t, x_1) \), which is invertible by assumption. Consequently, all conditions of the global inverse function theorem (Theorem 2.7) are met, and thus \( F_t(x_1) \) is a diffeomorphism at each time instant \( t \). Therefore, \( F_t(x_1) = 0 \) or equivalently \( F(t, x_1) = 0 \) has a (single) unique solution \( x_1^*(t) \) for each \( t \), which is unique critical point of \( \alpha(t, x_1) \) at time \( t \). Note that, since \( F(t, x_1) \) is continuous, \( x_1^*(t) \) depends continuously on time.

Next, we will show that the unique critical point of \( \alpha(t, x_1) \) at time \( t \), i.e., \( x_1^*(t) \), is indeed the global maximum point of \( \alpha(t, x_1) \) at time \( t \). In this regard, we consider the second derivative test on the critical point’s trajectory of \( \alpha(t, x_1) \). From (5.37) and followed by matrix differentiation rules [153], we obtain the Hessian matrix of \( \alpha(t, x_1) \), i.e., \( \mathcal{H}(t, x_1) := \frac{\partial}{\partial x_1} (\nabla_x \alpha(t, x_1)) \) as:

\[
\mathcal{H}(t, x_1) = \frac{\partial}{\partial x_1} \left( J^\top(t, x_1) \left[ \gamma(t, x_1) e^{\nu(t,x_1)} \right] \otimes I_n \right) + J^\top(t, x_1) \frac{\partial \gamma(t, x_1)}{\partial x_1} e^{\nu(t,x_1)} \\
+ J^\top(t, x_1) \gamma(t, x_1) \frac{\partial e^{\nu(t,x_1)}}{\partial x_1}.
\]

Recall that on the critical point’s trajectory we have \( \gamma(t, x_1^*(t)) = 0 \). Hence, evaluating (5.40) on \( x_1^*(t) \) gives:

\[
\mathcal{H}(t, x_1^*(t)) = J^\top(t, x_1^*(t)) \left. \frac{\partial}{\partial x_1} \left( \gamma(t, x_1) \right) \right|_{x_1=x_1^*(t)} e^{\nu(t,x_1^*(t)).}
\]

From (5.38a), one can get:

\[
\frac{\partial}{\partial x_1} \left( \gamma(t, x_1) \right) \bigg|_{x_1=x_1^*(t)} = \Gamma(t, x_1^*(t)) J(t, x_1^*(t)),
\]
where \( \Gamma(t, x^*(t)) \in \mathbb{R}^{n \times n} \) is a negative definite diagonal matrix whose diagonal entries are given by:
\[
-\nu \left[ e^{-\nu(h_i(t, x_i^*(t)) - L_i(t))} + e^{-\nu(\varphi_i(t) - h_i(t, x_i^*(t)))} \right]. \tag{5.43}
\]

Therefore:
\[
\mathcal{H}(t, x_i^*(t)) = J^T(t, x^*_i(t)) \Gamma(t, x^*_i(t)) J(t, x^*_i(t)) e^{\nu \alpha(t, x^*_i(t))}. \tag{5.44}
\]

Notice that \( e^{\nu \alpha(t, x^*_i(t))} > 0 \) for all time. Since \( J(t, x^*_i(t)) \) is a full rank square matrix and \( \Gamma(t, x^*_i(t)) \) is a negative definite matrix, one can infer that the Hessian matrix \( \mathcal{H}(t, x^*_i(t)) \) is negative definite \[54]. Therefore, the critical point \( x^*_i(t) \) is a local maximizer. Since \( x^*_i(t) \) is the unique critical point of \( \alpha(t, x_i) \) at time \( t \), we conclude that \( x^*_i(t) \) is indeed the (unique) global maximizer of \( \alpha(t, x_i) \) at time \( t \).

### 5.10.4 Proof of Theorem 5.1

First, note that from \[5.19\] we have \( x_2 = e_2 + s_1(t, x_1), \) \( x_3 = e_3 + s_2(t, \hat{x}_2) \) and \( x_i = e_i + s_{i-1}(t, \hat{x}_{i-1}), i \in \mathcal{I}_4 \). Therefore, from \[5.22\] and with a slight abuse of notation, we can recursively obtain:
\[
x_2 = \Theta_2^{-1}(t) \hat{e}_2 + s_1(t, x_1), \tag{5.45a}
\]
\[
x_3 = \Theta_3^{-1}(t) \hat{e}_3 + s_2(t, x_1, \hat{e}_2), \tag{5.45b}
\]
\[
x_i = \Theta_i^{-1}(t) \hat{e}_i + s_{i-1}(t, x_1, \hat{e}_2, \ldots, \hat{e}_{i-1}), \quad i \in \mathcal{I}_4. \tag{5.45c}
\]

From the system dynamics \[5.1\] and \[5.45a\] one can write:
\[
\dot{x}_1 := \phi_1(t, x_1, \hat{e}_2) = f_1(t, x_1) + G_1(t, x_1) \left( \Theta_2^{-1}(t) \hat{e}_2 + s_1(t, x_1) \right) + w_1(t). \tag{5.46}
\]

Taking the time derivative of \[5.15\] and utilizing \[5.1\] and \[5.45a\] yields:
\[
\dot{e}_\alpha := \phi_\alpha(t, x_1, \hat{e}_2) \tag{5.47}
\]
\[
= \frac{\partial \alpha(t, x_1)}{\partial x_1} \dot{x}_1 + \frac{\partial \alpha(t, x_1)}{\partial t} - \dot{\rho}_\alpha(t)
\]
\[
= \frac{\partial \alpha(t, x_1)}{\partial x_1} \left[ f_1(t, x_1) + G_1(t, x_1) \left( \Theta_2^{-1}(t) \hat{e}_2 + s_1(t, x_1) \right) + w_1(t) \right]
\]
\[
+ \frac{\partial \alpha(t, x_1)}{\partial t} - \dot{\rho}_\alpha(t).
\]

Moreover, differentiating \[5.22\] with respect to time and substituting equations \[5.1\], \[5.45\], and \[5.28\] results in:
\[
\dot{\hat{e}}_2 := \phi_2(t, x_1, \hat{e}_2, \hat{e}_3) \tag{5.48a}
\]
\[
= \Theta_2^{-1}(t) \left[ f_2(t, x_1, \hat{e}_2) + G_2(t, x_1, \hat{e}_2) \left( \Theta_3^{-1}(t) \hat{e}_3 + s_2(t, x_1, \hat{e}_2) \right) + w_2(t) - \hat{s}_1(t, x_1) - \dot{\Theta}_2(t) \hat{e}_2 \right].
\]
\[ \dot{e}_i := \phi_i(t, x_1, \dot{e}_2, \ldots, \dot{e}_i) \]  
\[ = \Theta^{-1}_i(t) \left[ f_i(t, x_1, \dot{e}_2, \ldots, \dot{e}_i) + G_i(t, x_1, \dot{e}_2, \ldots, \dot{e}_i) \right. \]  
\[ \times \left( \Theta^{-1}_{i+1}(t) \dot{e}_{i+1} + s_i(t, x_1, \dot{e}_2, \ldots, \dot{e}_i) \right) + w_i(t) \]  
\[ - \dot{s}_{i-1}(t, x_1, \dot{e}_2, \ldots, \dot{e}_{i-1}) - \dot{\Theta}_i(t) \dot{e}_i \right], \quad i \in I_3^{-1}, \]  
\[ \dot{e}_r := \phi_r(t, x_1, \dot{e}_2, \ldots, \dot{e}_r) \]  
\[ = \Theta^{-1}_r(t) \left[ f_r(t, x_1, \dot{e}_2, \ldots, \dot{e}_r) + G_r(t, x_1, \dot{e}_2, \ldots, \dot{e}_r) \times u(t, x_1, \dot{e}_2, \ldots, \dot{e}_r) \right. \]  
\[ + w_r(t) - \dot{s}_{r-1}(t, x_1, \dot{e}_2, \ldots, \dot{e}_{r-1}) - \dot{\Theta}_r(t) \dot{e}_r \right]. \]  

Next, define the following time-varying set:
\[ \Omega_{x_1}(t) := \{ x_1 \in \mathbb{R}^n | \alpha(t, x_1) > \rho_\alpha(t) \}. \]  
\[ (5.49) \]

Owing to Assumption 5.6, \(-\alpha(t, x_1)\) is coercive (see the proof of Lemma 5.1), and then from Theorem 2.5, \(\Omega_{x_1}(t)\) is bounded at each time instance \(t\) (\(\Omega_{x_1}(t)\) is a time-dependent super level-set of \(\alpha(t, x_1)\)). Therefore, one can infer that \(\Omega_{x_1}(t)\) is bounded and open for all \(t \geq 0\). In addition, notice that \(\Omega_{x_1}(t)\) is nonempty for all \(t \geq 0\), since by Property (i) of \(\rho_\alpha(t)\) in Subsection 5.4.2, \(\rho_\alpha(t) < \alpha^*(t)\) holds for all time. Now define:
\[ \Omega_{x_1}^s := \bigcup_{t=0}^{+\infty} \Omega_{x_1}(t) \subset \mathbb{R}^n, \]  
\[ (5.50) \]
which is the (time-invariant) super set containing \(\Omega_{x_1}(t), \forall t \geq 0\). Owing to the properties of \(\Omega_{x_1}(t)\) established above, \(\Omega_{x_1}^s\) is nonempty, bounded, and open.

Now, let us define \(z := [x_1^T, e_\alpha, \dot{e}_2, \ldots, \dot{e}_r]^T \in \mathbb{R}^{nr+1}\) and consider the dynamical system:
\[ \dot{z} = \phi(t, z) = \begin{bmatrix} \phi_1(t, x_1, \dot{e}_2) \\ \phi_\alpha(t, x_1, \dot{e}_2) \\ \phi_2(t, x_1, \dot{e}_2, \dot{e}_3) \\ \vdots \\ \phi_r(t, x_1, \dot{e}_2, \ldots, \dot{e}_r) \end{bmatrix}, \]  
\[ (5.51) \]
as well as the (nonempty) open set:
\[ \Omega_z := \Omega_{x_1}^s \times (0, +\infty) \times (-1, 1)^n \times \ldots \times (-1, 1)^n. \]  
\[ (r-1) \text{ times} \]  
\[ (5.52) \]
In the sequel, we proceed in three phases. First, we show that there exists a unique and maximal solution \(z : [0, \tau_{\max}) \rightarrow \mathbb{R}^{nr+1}\) for (5.51) over the set \(\Omega_z\) (i.e., \(z(t; z(0)) \in \Omega_z, \forall t \in [0, \tau_{\max})\)). Next, we prove that the proposed control scheme
5.10. Proofs of Lemmas and Theorems

guarantees, for all \( t \in [0, \tau_{\text{max}}) \): (a) the boundedness of all closed loop signals in (5.51) as well as that (b) \( z(t; z(0)) \) remains strictly within a compact subset of \( \Omega_z \) for all \( t \in [0, \tau_{\text{max}}) \), which leads by contradiction to \( \tau_{\text{max}} = + \infty \) (i.e., forward completeness) in the last phase. Recall that, the latter means the signals \( e_\alpha \) and \( \bar{e}_i, i \in \mathcal{I}_2 \), remain within some strict subsets of \( (0, +\infty) \) and \( (-1, 1)^n \), respectively, which in turn leads to the satisfaction of (5.13) and (5.20).

**Phase I.** The set \( \Omega_z \) is nonempty, open and independent of time. In addition, note that for a given initial condition \( x(0) \) in (5.1) we know \( \rho_\alpha(0) < \alpha(0, x_1(0)) \) holds by construction of \( \rho_\alpha(t) \). Consequently, we have \( x(0) \in \Omega^*_{x_1} \), and from (5.15) one can verify that \( e_\alpha(0, x_1(0)) \in (0, +\infty) \). Moreover, as mentioned at Step i-a in Section 5.5, \( \vartheta_{i,j}^0 \) in (5.21) are selected such that \( \vartheta_{i,j}^0 > |e_{i,j}(0, \bar{x}_i(0))| \), which ensures \( \dot{e}_{i,j}(0, \bar{x}_i(0)) \) is locally Lipschitz in \( \bar{x}_i \) and piecewise continuous in \( t \), \( w_i(t) \) are piecewise continuous in \( t \) and the intermediate control laws \( s_i(t, \bar{x}_i) \) and \( u(t, x) \) are smooth. Consequently, one can verify that \( \phi(t, z) \) on the right hand side of (5.51) is locally Lipschitz in \( z \) over the set \( \Omega_z \) and is piecewise continuous in \( t \). Therefore, the hypotheses of Theorem 2.3 hold and the existence and uniqueness of a maximal solution \( z(t; z(0)) \in \Omega_z \) for a time interval \( t \in [0, \tau_{\text{max}}) \) is guaranteed.

**Phase II.** We have proven in Phase I that \( z(t; z(0)) \in \Omega_z, \forall t \in [0, \tau_{\text{max}}) \), which implies:

\[
\begin{cases}
  x_1(t; z(0)) \in \Omega^*_{x_1}, \\
  e_\alpha(t; z(0)) \in (0, +\infty), \\
  \dot{e}_i(t; z(0)) \in (-1, 1)^n, \quad \forall i \in \mathcal{I}_2,
\end{cases}
\]

for all \( t \in [0, \tau_{\text{max}}) \).

Therefore, \( e_\alpha \) in (5.16) and \( e_{i,j} \) in (5.24) (i.e., \( e_i \in \mathbb{R}^n, i \in \mathcal{I}_2 \)) are well-defined for all \( t \in [0, \tau_{\text{max}}) \).

Taking the time derivative of (5.16) and using (5.47) gives:

\[
\dot{e}_\alpha = \frac{\partial e_\alpha}{\partial e_\alpha} \dot{e}_\alpha = \frac{1}{e_\alpha} \left[ \frac{\partial \alpha(t, x_1)}{\partial x_1} \left( f_1(t, x_1) + G_1(t, x_1) \times \left( \Theta^{-1}_{i+1} \hat{e}_i + s_i(t, x_1, \hat{e}_2, \ldots, \hat{e}_i) \right) \right) + \frac{\partial \alpha(t, x_1)}{\partial t} - \dot{\rho}_\alpha \right].
\]

(5.53)

Moreover, differentiating \( e_i \in \text{col}(e_{i,j}) \) with respect to time and using (5.24), (5.19), (5.1), and (5.45) results in:

\[
\dot{e}_i = \Xi_i \left[ f_i(t, x_1, \hat{e}_2, \ldots, \hat{e}_i) + G_i(t, x_1, \hat{e}_2, \ldots, \hat{e}_i) \times \left( \Theta^{-1}_{i+1} \hat{e}_{i+1} + s_i(t, x_1, \hat{e}_2, \ldots, \hat{e}_i) \right) \right] + w_i - \dot{s}_{i-1}(t, x_1, \hat{e}_2, \ldots, \hat{e}_{i-1}) - \dot{\Theta}_i \hat{e}_i, \quad i \in \mathcal{I}_2, \quad (5.54)
\]

where for \( i = r \), the term \( \Theta^{-1}_{i+1} \hat{e}_{i+1} + s_i \) should be replaced by \( u \) in (5.28). Recall that, \( \Theta_i := \text{diag}(\vartheta_{i,j}) \) and \( \Xi_i := \text{diag}(\xi_{i,j}) \), in which \( \xi_{i,j} \) are given in (5.27).
Step 1. To ensure the satisfaction of (5.13), we are interested in establishing the boundedness of $|\varepsilon_\alpha|$. We begin by considering the implicit upper bound property $\alpha(t, x_1) \leq \alpha^*(t)$ stated in (5.13). Combining this property with (5.15) and Phase I, we obtain:

$$e_\alpha(t) \in (0, b), \quad \forall t \in [0, \tau_{\text{max}}),$$

(5.55)

where $b := \sup_{t \geq 0}(\alpha^*(t) - \rho_\alpha(t)) > 0$. It is important to note that although $b$ can be arbitrarily large, it remains bounded due to the boundedness of $\alpha^*(t)$ and $\rho_\alpha(t)$. Next, by examining (5.16) and (5.55), we observe that $|\varepsilon_\alpha|$ can only grow unbounded when $e_\alpha(t) \to 0$ or equivalently when $\alpha(t, x_1(t; z(0))) \to \rho_\alpha(t)$. Note that Property (ii) of $\rho_\alpha(t)$ in Subsection 5.4.2 ensures $\alpha^*(t) - \rho_\alpha(t) \geq \varsigma > 0$ for all $t \geq 0$. Now, let us consider the following two cases:

Case (a): When $e_\alpha \in \left[\frac{\varsigma}{2}, b\right)$ holds, from (5.15) and (5.16), it is evident that $|\varepsilon_\alpha|$ is bounded by a positive constant $\tilde{\varepsilon}_{\alpha, 1} > 0$, which is given by:

$$\tilde{\varepsilon}_{\alpha, 1} := \max \left\{|\ln \left(\frac{\varsigma}{2\upsilon}\right)|, |\ln \left(\frac{b}{\upsilon}\right)|\right\},$$

(5.56)

Recall that according to Assumption 5.8, $\|\nabla_{x_1} \alpha(t, x_1)\| = 0$ if and only if $\alpha(t, x_1) = \alpha^*(t)$. Therefore, $\|\nabla_{x_1} \alpha(t, x_1)\| = 0$ can only occur for values of $e_\alpha$ within the interval $[\varsigma, b)$. Consequently, even when $\|\nabla_{x_1} \alpha(t, x_1)\| = 0$, the bound in (5.56) still holds, which ensures the boundedness of $|\varepsilon_\alpha|$.

Case (b): When $e_\alpha \in (0, \frac{\varsigma}{2})$ holds, due to the continuity of $\nabla_{x_1} \alpha(t, x_1)$, there exists a positive constant $\varepsilon_\alpha$ such that $\|\nabla_{x_1} \alpha(t, x_1)\| \geq \varepsilon_\alpha$. Now consider the barrier function $V_1(\varepsilon_\alpha) = \frac{1}{2}\varepsilon_\alpha^2$ (introduced in Section 5.5) as a positive definite and radially unbounded Lyapunov function candidate with respect to $\varepsilon_\alpha$. Taking the time derivative of $V_1$, substituting (5.53) and (5.18), and exploiting the fact that $G^*_1(t, x_1)$ is uniformly positive definite (see Assumption 5.2), we obtain:

$$\dot{V}_1 = \frac{\varepsilon_\alpha}{e_\alpha} \left[\eta_1 + \frac{\partial\varepsilon_\alpha}{\partial x_1} G_1(t, x_1) s_1(t, x_1)\right]$$

$$= -k_1 \varepsilon_\alpha^2 \nabla_{x_1, \alpha^1} (t, x_1) G_1^s(t, x_1) \nabla_{x_1} \alpha(t, x_1) + \frac{\varepsilon_\alpha}{e_\alpha} \eta_1$$

$$\leq -k_1 \Delta_1 \left[\varepsilon_\alpha^2 \right] \|\nabla_{x_1} \alpha(t, x_1)\|^2 + \frac{|\varepsilon_\alpha|}{e_\alpha} |\eta_1|$$

$$= -\frac{|\varepsilon_\alpha|}{e_\alpha} \left[k_1 \Delta_1 \|\nabla_{x_1} \alpha(t, x_1)\|^2 \frac{|\varepsilon_\alpha|}{e_\alpha} - |\eta_1|\right],$$

(5.57)

where

$$\eta_1 := \frac{\partial\varepsilon_\alpha}{\partial x_1} (f_1(t, x_1) + G_1(t, x_1) \Theta_2^{-1} \varepsilon_2 + w_1) + \frac{\partial\varepsilon_\alpha}{\partial t} - \dot{\rho}_\alpha.$$

In the following, we show that $|\eta_1|$ is bounded for all $t \in [0, \tau_{\text{max}}]$. Firstly, it is important to note that $|\dot{\rho}_\alpha(t)|$ and $\|\Theta_2^{-1}(t)\|$ are bounded by construction for all time. Additionally, Assumption 5.3 ensures that $\|w_1(t)\|$ is bounded for all $t \geq 0$. 
Moreover, due to Assumptions 5.1 and 5.2, we know that }f1(t,x1)\leq \bar{g}_1(x_1). Owing to the continuity of \bar{f}_1(x_1) and \bar{g}_1(x_1), and the fact that \(x_1(t)\in \Omega_{x_1}\) for all \(t\in [0,\tau_{\text{max}}]\), by employing the Extreme Value Theorem, we conclude that \(\|f_1(t,x_1)\|\) and \(\|G_1(t,x_1)\|\) are bounded for all \(t\in [0,\tau_{\text{max}}]\). Similarly, under Assumptions 5.4 and 5.5, and by considering (5.37) while acknowledging the smoothness of \(\alpha(t,x)\), and the boundedness of \(\partial_1(t)\) and \(\eta_i(t)\), we conclude that \(\|\tilde{\alpha}(t,x)\|\) is bounded for all \(t\in [0,\tau_{\text{max}}]\) using (5.47). Furthermore, due to (5.60) and (5.61) as well as the boundedness of \(\tilde{\alpha}(t,x)\), the first intermediate control signal \(s_1\) in (5.18) is well-defined (since \(\tilde{e}_2\) remains strictly positive) and bounded for all \(t\in [0,\tau_{\text{max}}]\). Additionally, using (5.45) we also conclude the boundedness of \(x_2\) for all \(t\in [0,\tau_{\text{max}}]\). Finally, differentiating \(s_1(t,x_1)\) with respect to time and substituting (5.46), (5.47), and (5.53) yields:

\[
\dot{s}_1 = -k_1 \tilde{\varepsilon}_\alpha e_\alpha \mathcal{H}(t,x_1) \left[ f_1(t,x_1) + G_1(t,x_1) \left( \Theta^{-1}_2(t) \tilde{e}_2 + s_1 \right) + w_1(t) \right] - k_1 \nabla_x \alpha(t,x_1) \left( \frac{1 - \tilde{\varepsilon}_\alpha}{e_\alpha} \tilde{\varepsilon}_\alpha \right),
\]

where \(\mathcal{H}(t,x_1)\) denotes the Hessian of \(\alpha(t,x_1)\). It is straightforward to deduce the boundedness of \(\tilde{e}_\alpha\) for all \(t\in [0,\tau_{\text{max}}]\) using (5.47). Furthermore, due to

\[|\tilde{e}_\alpha| > \frac{\bar{\eta}_1 \epsilon_\alpha}{2k_1 \lambda_1 \epsilon_\alpha^2}; \tag{5.58}\]

and consequently:

\[|\varepsilon_\alpha(t)| \leq \bar{\varepsilon}_\alpha, \quad \forall t \in [0,\tau_{\text{max}}]. \tag{5.59}\]

Now based on the results of Case (a) and Case (b), combining (5.56) and (5.59) leads to:

\[|\varepsilon_\alpha(t)| \leq \bar{\varepsilon}_\alpha := \max \{ \bar{\varepsilon}_\alpha, \varepsilon_\alpha, \varepsilon_\alpha, \}, \quad \forall t \in [0,\tau_{\text{max}}], \quad \forall e_\alpha \in (0,b), \tag{5.60}\]

where \(\bar{\varepsilon}_\alpha\) is independent of \(\tau_{\text{max}}\). Furthermore, by taking the inverse logarithmic function in (5.16) and utilizing (5.60), we obtain:

\[e^{-\bar{\varepsilon}_\alpha} =: \bar{e}_\alpha \leq e_\alpha(t) \leq \bar{e}_\alpha := e^{\bar{\varepsilon}_\alpha}, \quad \forall t \in [0,\tau_{\text{max}}]. \tag{5.61}\]

As a result, considering (5.60) and (5.61) as well as the boundedness of \(\nabla_x \alpha(t,x_1)\) for all \(t\in [0,\tau_{\text{max}}]\), the first intermediate control signal \(s_1\) in (5.18) is well-defined (since \(e_\alpha\) remains strictly positive) and bounded for all \(t\in [0,\tau_{\text{max}}]\). Additionally, using (5.45) we also conclude the boundedness of \(x_2\) for all \(t\in [0,\tau_{\text{max}}]\).
the smoothness of $\alpha(t, x_1)$, we can establish that $\|H(t, x_1)\|$ is bounded for all $t \in [0, \tau_{\text{max}}]$. Consequently, since the boundedness of all other terms on the right-hand side of (5.62) has already been proved for all $t \in [0, \tau_{\text{max}})$, it can be concluded that $\dot{s}_1$ remains bounded for all $t \in [0, \tau_{\text{max}})$.

**Step 2.** Similarly to Step 1, we can consider the barrier function $V_2(\epsilon_2) = \frac{1}{2} \epsilon_2^T \Xi_2$ as a positive definite and radially unbounded Lyapunov function candidate with respect to $\epsilon_2$. By taking the time derivative of $V_2$ and substituting (5.54) and (5.26), while also incorporating the fact that $G^*_2(t, x_1, \hat{e}_2)$ is uniformly positive definite, we obtain the following expression:

$$
\dot{V}_2 = \epsilon_2^T \Xi_2 (\eta_2 + G_2(t, x_1, \hat{e}_2)) s_2(t, x_1, \hat{e}_2) \\
= -k_2 \epsilon_2^T \Xi_2 G_2^*(t, x_1, \hat{e}_2) \epsilon_2 + \epsilon_2^T \Xi_2 \eta_2 \\
\leq -k_2 \lambda_2 \|\epsilon_2\| \|\Xi_2\| \|\eta_2\| \\
\leq -\|\epsilon_2\| \|\Xi_2\| (k_2 \lambda_2 \|\epsilon_2\| \|\Xi_2\| - \|\eta_2\|),
$$

(5.63)

where:

$$
\eta_2 := f_2(t, x_1, \hat{e}_2) + G_2(t, x_1, \hat{e}_2) \Theta_3^{-1} \tilde{e}_3 + w_2 - \dot{s}_1 - \dot{\Theta}_2 \hat{e}_2.
$$

Akin to the analysis provided in Step 1, Assumptions 1-3, and the application of the Extreme Value Theorem, it is straightforward to establish the existence of a positive (unknown) constant $\bar{\eta}_2$ such that $\|\eta_2\| \leq \bar{\eta}_2$ for all $t \in [0, \tau_{\text{max}})$. Furthermore, it was previously shown in Phase I that $\hat{e}_2 \in (-1, 1)^n$, $\forall t \in [0, \tau_{\text{max}})$, which implies $\hat{e}_{2,j} \in (-1, 1)$, $\forall t \in [0, \tau_{\text{max}})$, $\forall j \in I_1^n$. Consequently, from (5.27) and (5.21) we deduce $\xi_{2,j} \geq \frac{2}{s_{2,j}} > 0$ for all $j \in I_1^n$ and all $t \in [0, \tau_{\text{max}})$. As a result, since $\Xi_2 = \text{diag}(\xi_{2,j})$, there exists a positive constant $\epsilon_{\bar{\xi}_2} := \max_j \left| \frac{2}{s_{2,j}} \right|$ such that $\|\Xi_2\| \geq \epsilon_{\bar{\xi}_2}$, $\forall t \in [0, \tau_{\text{max}})$.

Now, considering (5.63) and the aforementioned facts, it is evident that $\dot{V}_2$ is negative under the condition:

$$
\|\epsilon_2\| > \frac{\bar{\eta}_2}{k_2 \lambda_2 \epsilon_{\bar{\xi}_2}},
$$

(5.64)

which implies an upper bound on $\|\epsilon_2\|$ as follows:

$$
\|\epsilon_2(t)\| \leq \bar{\epsilon}_2 := \max \left\{ \|\epsilon_2(0)\|, \frac{\bar{\eta}_2}{k_2 \lambda_2 \epsilon_{\bar{\xi}_2}} \right\}, \quad \forall t \in [0, \tau_{\text{max}}),
$$

(5.65)

where $\bar{\epsilon}_2 > 0$ is independent of $\tau_{\text{max}}$. Moreover, taking the inverse of (5.24) and using the upper bound in (5.65) reveals that:

$$
-1 < \frac{e^{-\bar{\epsilon}_2} - 1}{e^{-\bar{\epsilon}_2} + 1} =: -\sigma_{2,j} \leq \hat{e}_{2,j}(t) \leq \sigma_{2,j} := \frac{e^{\bar{\epsilon}_2} - 1}{e^{\bar{\epsilon}_2} + 1} < 1,
$$

(5.66)

for all $t \in [0, \tau_{\text{max}})$ and all $j \in I_1^n$. By (5.66) and (5.27), it becomes evident that $\xi_{2,j}, j \in I_1^n$ remain bounded for all $t \in [0, \tau_{\text{max}})$. Consequently, considering (5.65),
we can establish that the second intermediate control signal \( s_2(t, x_1, \hat{e}_2) \) in (5.26) remains bounded for all \( t \in [0, \tau_{\text{max}}] \). Moreover, invoking (5.45b) we also conclude the boundedness of \( x_3 \) for all \( t \in [0, \tau_{\text{max}}] \).

Finally, differentiating \( s_2(t, x_1, \hat{e}_2) \) with respect to time and substituting (5.54) gives:

\[
\dot{s}_2 = -k_2 \hat{\Sigma}_2 \hat{e}_2 - k_2 \Sigma_2 \hat{e}_2 \\
= -k_2 \hat{\Sigma}_2 \hat{e}_2 - k_2 \left[ f_2(t, x_1, \hat{e}_2) + G_2(t, x_1, \hat{e}_2) \times \left( \Theta_3^{-1} \hat{e}_3 + s_2(t, x_1, \hat{e}_2) \right) + w_2 - \dot{s}_1 - \dot{\Theta}_2 \hat{e}_2 \right].
\]

(5.67)

Note that, by taking the time derivative of (5.27) one can obtain the diagonal elements of \( \hat{\Sigma}_i = \text{diag}(\hat{\xi}_{i,j}), i \in I_2^0 \), as follows:

\[
\dot{\xi}_{i,j} = -0.5 \xi_{i,j}^2 \dot{\theta}_{i,j} \left( 1 - 2 \hat{e}_{i,j} \dot{\hat{e}}_{i,j} \right), \quad j = I_1^n.
\]

(5.68)

In particular, from (5.68) and (5.48a) and using the aforementioned results, it is straightforward to infer the boundedness of \( \xi_{2,j}, j = I_1^n \). Accordingly, since the boundedness of all terms on the right-hand side of (5.67) are already established for all \( t \in [0, \tau_{\text{max}}] \), we conclude that \( \dot{s}_2 \) remains bounded for all \( t \in [0, \tau_{\text{max}}] \).

Step \( i (3 \leq i \leq r) \). Applying the same analysis described in Step 2 iteratively to the subsequent steps, while considering \( V_i(\xi_i) = \frac{1}{2} \xi_i^T \xi_i \), we can draw the following conclusion:

\[
\|\xi_i(t)\| \leq \bar{\xi}_i := \max \left\{ \|\xi_i(0)\|, \frac{\bar{\eta}_i}{k_i \lambda_i \epsilon_{\xi_i}} \right\}, \quad \forall t \in [0, \tau_{\text{max}}],
\]

(5.69)

in which \( \bar{\xi}_i > 0 \) is independent of \( \tau_{\text{max}} \) and \( \epsilon_{\xi_i} := \max_j |\frac{2}{\bar{\eta}_i}||\eta_i| > 0 \), and there exist (unknown) constants \( \bar{\eta}_i > 0, i \in I_3^0 \), which satisfy \( \|\eta_i\| < \bar{\eta}_i, \forall t \in [0, \tau_{\text{max}}], \) where:

\[
\begin{align*}
\eta_i := & f_i(t, x_1, \hat{e}_2, \ldots, \hat{e}_i) + G_i(t, x_1, \hat{e}_2, \ldots, \hat{e}_i) \Theta_{i+1}^{-1} \hat{e}_{i+1} + w_i \\
- \dot{s}_{i-1} - \dot{\Theta}_i \hat{e}_i, \quad i = I_3^{i-1}, \\
\eta_r := & f_r(t, x_1, \hat{e}_2, \ldots, \hat{e}_r) + w_r - \dot{s}_{r-1} - \dot{\Theta}_r \hat{e}_r.
\end{align*}
\]

(5.70a)

Correspondingly, (5.24) and (5.69) lead also to:

\[
-1 < \frac{e^{-\bar{\xi}_i} - 1}{e^{-\bar{\xi}_i} + 1} =: -\sigma_{i,j} \leq \hat{e}_{i,j}(t) \leq \sigma_{i,j} := \frac{e^{\bar{\xi}_i} - 1}{e^{\bar{\xi}_i} + 1} < 1,
\]

(5.71)

for \( i \in I_3^0, j \in I_1^n \), and all \( t \in [0, \tau_{\text{max}}] \). As a result, we can show that all intermediate control signals \( s_i \) and system states \( x_{i+1}, i = I_3^{i-1} \), as well as the control law \( u \) remain bounded for all \( t \in [0, \tau_{\text{max}}] \).
Phase III. Now we shall establish that $\tau_{\text{max}} = \infty$. In this direction, firstly, consider inequalities (5.61), (5.66), and (5.71), and accordingly define:

$$
\Omega'_{x_{\alpha}} := [\mathcal{E}_{\alpha}, \bar{\mathcal{E}}_{\alpha}],
\Omega'_{e_{i}} := [-\sigma_{i,1}, \sigma_{i,1}] \times \ldots \times [-\sigma_{i,n}, \sigma_{i,n}], \quad i \in \mathcal{I}',
\Omega'_{e} := \Omega'_{e_{1}} \times \ldots \times \Omega'_{e_{n}} \subseteq (-1,1)^{n} \times \ldots \times (-1,1)^{n}.
$$

In addition, owing to (5.61), from (5.49) it is straightforward to infer that $x_1(t) \in \Omega'_{x_{1}}(t) \subset \Omega_{x_{1}}(t)$ for all $t \in [0, \tau_{\text{max}})$, where:

$$
\Omega'_{x_{1}}(t) := \{ x_{1} \in \mathbb{R}^{n} | \mathcal{E}_{\alpha} \leq \alpha(t, x_{1}) - \rho_{\alpha}(t) \leq \bar{\mathcal{E}}_{\alpha} \},
$$

from which we can define $\Omega'_{x} := \bigcup_{t=0}^{\infty} \Omega'_{x_{1}}(t) \subset \Omega_{x}$ and claim that $x_1(t) \in \Omega'_{x_{1}}$, $\forall t \in [0, \tau_{\text{max}})$. Secondly, define $\Omega'_{z} = \Omega'_{x_{1}} \times \Omega'_{e_{1}} \times \Omega'_{e}$, which is a nonempty and compact subset of $\Omega_{z}$ given in (5.52). Note that, from (5.61), (5.66), and (5.71) we have $z(t; z(0)) \in \Omega'_{z}$, $\forall t \in [0, \tau_{\text{max}})$. Now assuming a finite $\tau_{\text{max}} < \infty$, since $\Omega'_{z} \subset \Omega_{z}$, Lemma 2.6 dictates the existence of a time instant $t' \in [0, \tau_{\text{max}})$ such that $z(t', z(0)) \notin \Omega'_{z}$, which is a contradiction. Therefore, $\tau_{\text{max}} = \infty$. As a result, all closed-loop control signals remain bounded $\forall t \geq 0$. Finally, recall that since $e_{\alpha}(t) \in \mathcal{E}_{\alpha} \subset (0, +\infty)$ for all $t \geq 0$, invoking (5.15) ensures the satisfaction of the consolidating constraint in (5.13) for all time, which completes the proof.

5.10.5 Proof of Theorem 5.2

We begin by establishing that $\rho_{\alpha}(t)$ given by (5.33), along with its derivative, remain bounded for all time. Next, we further show that $\rho_{\alpha}(t)$ attains Properties (i) and (ii) outlined in Subsection 5.4.2 which allows us to conclude that the specific design of $\rho_{\alpha}(t)$ in (5.33) fulfills the prerequisites stipulated in Theorem 5.1, see Remark 5.10. Consequently, the proposed control law in (5.28) effectively ensures the satisfaction of the consolidating constraint (5.13), as well as guaranteeing the boundedness of all closed-loop signals for all time.

Firstly, consider the estimation error $\hat{e} = \alpha^{*}(t) - \hat{\alpha}(t) \geq 0$, $\forall t \geq 0$, and from (5.12) and Assumption 5.6 recall that $\alpha^{*}(t)$ is bounded for all time. Taking the time-derivative of $\hat{e}$ and substituting (5.31a) yields:

$$
\dot{\hat{e}} = \dot{\alpha}^{*}(t) - \frac{\partial \alpha(t, \tilde{x}_{1})}{\partial t} - k_{\alpha} \| \nabla_{x_{1}} \alpha(t, \tilde{x}_{1}) \|^{2} \leq |\dot{\alpha}^{*}(t)| - \Lambda(t, \tilde{x}_{1})
\leq \pi_{\alpha} - \Lambda(t, \tilde{x}_{1}),
$$

where $\Lambda(t, \tilde{x}_{1}) := \frac{\partial \alpha(t, \tilde{x}_{1})}{\partial t} + k_{\alpha} \| \nabla_{x_{1}} \alpha(t, \tilde{x}_{1}) \|^{2}$. Assuming $\hat{e} \to +\infty$, owing to the boundedness of $\alpha^{*}(t), -\hat{\alpha}(t) = -\alpha(t, \tilde{x}_{1})$ should tend toward $+\infty$. Since $-\alpha(t, \tilde{x}_{1})$ is continuous and coercive (as per Assumption 5.6), the only way for $-\alpha(t, \tilde{x}_{1})$ to approach $+\infty$ is if $\| \tilde{x}_{1} \| \to +\infty$. Now, based on Assumption 5.5 it is evident that the right-hand side of (5.74) becomes negative as $\| \tilde{x}_{1} \| \to +\infty$. Consequently, $\hat{e}$ cannot
tend to $+\infty$, leading to a contradiction. Thus, we conclude that $\tilde{c}$ remains bounded from above, ensuring the boundedness of $\dot{\hat{\alpha}}(t)$, and consequently, the boundedness of $\rho_\alpha(t)$ in (5.33).

Taking the time-derivative of (5.33) gives:

$$\dot{\rho}_\alpha(t) = \dot{\alpha}(t) + \mu \dot{g}(t) + (1 - \alpha(t)) \dot{\hat{\alpha}}(t)$$  \hspace{1cm} (5.75)

Note that, $\alpha(t), g(t), \dot{g}(t)$ and $\dot{\hat{\alpha}}(t)$ are bounded. Moreover, from (5.34) one can see that $\dot{\alpha}(t)$ is bounded if $\dot{\hat{\alpha}}(t)$ is bounded. Consequently, the boundedness of $\dot{\rho}_\alpha(t)$ is ensured by establishing the boundedness of $\dot{\hat{\alpha}}(t) \leq \hat{\alpha}(t, \tilde{x}_1) + k_\alpha \| \nabla \hat{\alpha}(t, \tilde{x}_1) \|^2$. Owing to Assumptions 5.4 and 5.5 and the boundedness of $\overline{p}_1(t)$, $\rho_\alpha(t), \rho_\alpha^1(t), \rho_\alpha^2(t)$ in (5.4) one can deduce that for any fixed $\tilde{x}_1$, $\Lambda(t, \tilde{x}_1)$ bounded. Moreover, since we have already established that $\hat{c} \leq \hat{c}$ remains bounded for all time, one can infer that there is some constant $C > 0$ for which $\| \tilde{x}_1(1) \| \leq C$ holds for all time. Accordingly, since $\Lambda(t, \tilde{x}_1)$ continuous and radially unbounded in $\tilde{x}_1$ we can deduce that $\dot{\hat{\alpha}}(t)$ remains bounded for all time.

Secondly, recall that $\dot{\hat{\alpha}}(t) = \alpha(t, \tilde{x}_1) \leq \alpha^*(t)$ always holds. Now for the case that $\nu(t) = 0$ from (5.33) we have $\rho_\alpha(t) = \dot{\alpha}(t) - \mu$, and thus $\alpha^*(t) - \rho_\alpha(t) \geq \mu$. In addition, when $\nu(t) = 1$ from (5.33) we get $\rho_\alpha(t) = g(t)$ and from (5.34) it also holds that $\varphi = \dot{\alpha} - g(t) > \mu$. Hence, one can verify that $\alpha^*(t) - \rho_\alpha(t) = \varphi(t) - g(t) > \alpha^*(t) + \mu - \dot{\alpha}(t) \geq \mu$. When $\nu(t) \in (0, 1)$, from (5.34) we know that $0 \leq \varphi(t) \leq \mu$, from which we get $0 \leq \dot{\alpha} - g(t) \leq \mu$. Now from (5.33) and under the worst case scenario that is $\alpha^*(t) = \dot{\alpha}(t)$ we obtain:

$$\alpha^*(t) - \rho_\alpha(t) = \alpha^*(t) - \nu(t)g(t) - (1 - \nu(t))(\dot{\alpha}(t) - \mu)$$

$$\geq \nu(t)(\alpha^*(t) - g(t)) + (1 - \nu(t))\mu$$

$$> (1 - \nu(t))\mu.$$

Consequently one can infer that there exist a positive constant $\zeta$ such that $\alpha^*(t) - \rho_\alpha(t) \geq \zeta > 0$ holds for all $t \geq 0$. Hence, Property (i) in Subsection 5.4.2 holds for $\rho_\alpha(t)$ given by (5.33).

Finally, if $g_0 < \alpha(0, x_1(0))$ in (5.32), one can ensure that $\rho_\alpha(0) < \alpha(0, x_1(0))$ holds (i.e., Property (ii) in Subsection 5.4.2 holds) for any initialization $\tilde{x}_1(0)$ in (5.31). To this end, assume $g(0) = g_0 < \alpha(0, x_1(0))$ and consider $\tilde{x}_1(0)$ is such that:

(a) $\varphi(0) = \dot{\alpha}(0) - g(0) > \mu$, (b) $0 \leq \varphi(0) \leq \mu$, and (c) $\varphi(0) < 0$. For case (a), from (5.33) and (5.34) it is obvious that $\rho_\alpha(0) = g(0) < \alpha(0, x_1(0))$. Considering case (b) since $g(0) - \mu \leq \dot{\alpha}(0) - \mu \leq g(0)$ and $0 \leq \nu(0) \leq 1$ one can infer that the convex combination $\rho_\alpha(0) = \nu(0)g(0) + (1 - \nu(0))(\dot{\alpha}(0) - \mu)$ can only take a value less than or equal to $g(0)$, hence, we get $\rho_\alpha(0) < \alpha(0, x_1(0))$. For case (c) it is straightforward to verify that $\rho_\alpha(0) = \dot{\alpha}(0) - \mu < g(0) - \mu < \alpha(0, x_1(0))$. Therefore, Property (ii) in Subsection 5.4.2 holds for $\rho_\alpha(t)$ given by (5.33).

Overall, owing to the above analysis $\rho_\alpha(t)$ in (5.33) satisfies the conditions of Theorem 5.1, thereby, applying the control law (5.28) in (5.1) leads to the satisfaction of $\rho_\alpha(t) < \alpha(t, x_1(t); x(0))$, as well as boundedness of all closed-loop signals for all time.
Chapter 6

Summary and Future Research Directions

This chapter summarizes the content of the thesis and provide potential future research directions.

6.1 Summary

In this thesis, we investigated the robust coordinate-free formation control problem of a multi-agent system, aiming for global convergence. Additionally, we addressed the feedback control design for uncertain nonlinear systems under coupled spatiotemporal constraints. These constraints were treated as time-varying output constraints in nonlinear systems, commonly imposed to ensure desirable transient and steady-state performance in tracking errors, ensuring safety, or accomplishing complex tasks within deadlines. Our emphasis was on leveraging and extending the Prescribed Performance Control (PPC) method as a funnel-based control approach to satisfy these constraints.

In the first part of the thesis, detailed in Chapter 3, we presented a novel 2-D coordinate-free formation control scheme tailored for multi-agent systems operating under directed leader-follower interactions. Robust decentralized formation controllers were crafted using the PPC control method, imposing user-defined funnel constraints on the formation errors to ensure a desirable transient and steady-state performance. Our approach utilized bipolar coordinates to characterize the desired relative edge angles and the ratio of distances among follower agents. We highlighted that this characterization not only guarantees a unique formation shape but also results in independent (orthogonal) formation errors, facilitating both global convergence of the multi-agent system to the desired formation shape and employing the PPC method for controller design. Additionally, we demonstrated that the proposed decentralized control laws are implementable in follower agents’ arbitrarily oriented local coordinate frames, requiring only bearing and ratio of distance measurements. These values can be readily obtained using low-cost onboard vision sensors. Furthermore, we discussed how the proposed formation control scheme can adeptly handle time-varying formation maneuvering, scaling, and orientation
adjustments with the assistance of two leader agents. In the second part of the thesis we focused on feedback control design for uncertain nonlinear systems with coupled time-varying output constraints.

In Chapter 4, we expanded the conventional PPC method to address time-varying hard and soft funnel constraints for uncertain nonlinear Euler-Lagrangian systems. Specifically, we introduced a soft funnel constraint corresponding to each hard funnel constraint and addressed the couplings between hard and soft constraints by dynamically planning a Constraint Consistent Funnel (CCF) for each pair of soft and hard constraints. These CCFs were crafted to ensure adherence to hard (safety) constraints, accommodating soft (performance) constraints only when they do not conflict with the hard constraints. We then directly applied the PPC design to guarantee the satisfaction of all online planned CCFs. The key advantage of this control approach lies in its tractability and robustness to system uncertainties when dealing with time-varying hard and soft constraints.

Motivated by the limitations inherent in conventional funnel-based control methods, such as PPC, which are restricted to enforcing decoupled funnel constraints resembling feasible time-varying box constraints on system outputs, we introduced a novel feedback control method in Chapter 5. In this chapter, we considered uncertain high-order MIMO nonlinear control systems subject to potentially coupled time-varying constraints in the form of asymmetric funnel constraints and/or one-sided inequality constraints. Recognizing that the satisfaction of multiple constraints can be encoded as the positivity of the signed distance from the boundary of the time-varying constrained set, we formulated a single consolidating constraint. The satisfaction of this consolidating constraint ensures convergence to and invariance of the time-varying output-constrained set within a user-defined finite time. To achieve this, we proposed a novel robust and low-complexity feedback controller to ensure the fulfillment of the consolidating constraint. Furthermore, we presented an adaptive modification for the consolidating constraint, aimed at securing a least-violating solution for the closed-loop system in case of potential constraint infeasibilities occurring over an unknown time interval.

6.2 Potential Future Research Directions

There are several interesting and challenging research directions that could be investigated in the future.

In Chapter 3, we established the absence of collocation between neighboring agents, modeled as points, for all time. However, in practical scenarios, ensuring collision avoidance becomes crucial, which depends on the actual size of the agents. Additionally, extending the proposed formation controller design to address collision avoidance not only among neighboring agents but also with non-neighboring

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1 The findings of Chapter 3 are reported in [77].
2 The findings of Chapter 4 are reported in [155].
3 The findings of Chapter 5 are partially reported in [156].
agents is vital for safety compliance. Another future research direction could be on ensuring connectivity maintenance between neighboring agents, which is essential to ensure the perpetuity of inter-agent sensing links throughout the operation. This aspect plays a pivotal role in the applicability of our proposed formation controllers. Exploring the extension of the control laws to accommodate more realistic agent dynamics, such as nonholonomic mobile robots is another aspect for future research. Additionally, given the inherent limitation of the bipolar coordinate system to two-dimensional spaces, a logical next step would be to explore the extension of the proposed formation control scheme to three-dimensional spaces. Lastly, in Chapter 3, we used edge-angles and the ratio of distances to characterize unique 2D rigid formations under acyclic minimally persistent graphs. Exploring whether these formation parameters can contribute to the development of a new graph rigidity theory applicable to undirected formations is an intriguing avenue for further investigation.

In Chapter 4, the presented results primarily address hard and soft constraints at the system output level. A compelling direction for future research involves addressing compatibility issues that may arise between output and state constraints in high-order nonlinear systems. This is particularly pertinent when state constraints directly impact the satisfaction of output constraints. Exploring the concept of treating state constraints as hard constraints while managing output constraints as soft constraints could offer a promising solution to this problem. Additionally, a noteworthy research area is the investigation of hard and soft funnel constraints in the context of multi-agent applications.

Concerning Chapter 5, an important direction for future work involves a deeper analysis of the proposed control method when Assumption 5.8 is not satisfied. As illustrated in the example at the end of Section 5.6, the control law introduced in Chapter 5 remains effective for time-varying output constraints that deviate from Assumption 5.8. Aligned with this direction, utilizing the proposed method for handling time-constrained navigation in obstacles cluttered environments could be another future work. In recent years, a model-based approach employing Control Barrier Functions (CBFs) along with quadratic programs has been utilized to synthesize minimally invasive control laws, acting as safety filters. This involves an optimization step to generate a modified control input while respecting safety requirements. The control method proposed in Chapter 4 offers a robust and optimization-free solution to ensure invariance of time-varying constrained sets. Investigating the integration of this method with a nominal control law to serve as a model-free safety filter is also another interesting direction of research. Furthermore, incorporating the concepts of hard and soft constraints into the control method presented in Chapter 5 is another promising research direction. Finally, it is worth noting that the control scheme in Chapter 5 can be applied in multi-agent setup for time-varying constraints satisfaction. However, the current design idea leads to centralized solution due to the consolidation of all agents’ constraints into a single constraint used for designing the feedback control law. Extending this control design for distributed implementation in multi-agent systems remains an open problem.


