Doctoral Thesis in Physics

Anomaly and Topology

On the axial anomaly, domain wall dynamics, and local topological markers in quantum matter

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Academic Dissertation which, with due permission of the KTH Royal Institute of Technology, is submitted for public defence for the Degree of Doctor of Philosophy on Wednesday the 21st of February 2024 at 9:00 a.m. in room FB53, AlbaNova Universitetscentrum, Roslagstullsbacken 21, Stockholm.
I am thinking about the aurora borealis. You can’t tell if it really does exist or if it just looks like existing. All things are so very uncertain, and that’s exactly what makes me feel reassured.

Too-ticki (Moominland Midwinter)
Chiral anomalies and topological phases of matter form the basis of the research presented in this dissertation. The chiral anomaly is considered both in the context of magnetic Weyl semimetals and in the context of non-Hermitian Dirac actions. Topological phases of matter play a role in this work through the research on Weyl semimetals and in the formulation of local topological markers.

The simplest example of magnetic Weyl semimetals consist of two Weyl cones separated in momentum space by a magnetisation vector which acts as an axial gauge field. We describe the emergence of axial electromagnetic fields by considering a magnetic field driven domain wall in this magnetisation. The parallel axial magnetic and axial electric fields give rise to the axial anomaly, and in turn to the chiral magnetic effect; a nonequilibrium current located at the domain wall. The chiral magnetic effect is a source of electromagnetic radiation, and a measurement of this radiation would provide evidence of the existence of the axial anomaly.

Electronic manipulation of domain walls is a central objective in spintronics. We describe how the axial anomaly, in terms of external electromagnetic fields, acts as a torque on the domain wall, and allows for electric control of the equilibrium configuration of the domain wall. We show how the axial anomaly is used to flip the chirality of the domain wall by tuning the electric field. Measuring the change in domain wall chirality constitutes a signal of the axial anomaly. We also describe how the Fermi arc boundary states of the Weyl semimetal at the domain wall result in an effective hard axis anisotropy which allows for large domain wall velocities irrespective of the intrinsic anisotropy of the material.

Our interest in non-Hermitian chiral anomalies stems from the existence of topological phases of matter in non-Hermitian models. We evaluate the chiral anomaly for a non-Hermitian Dirac theory with massless fermions with complex Fermi velocities coupled to non-Hermitian axial and vector gauge fields. The anomaly is compared with the corresponding anomaly of a Hermitianised and an anti-Hermitianised action derived from the non-Hermitian action. We find that the non-Hermitian anomaly does not correspond to the combined anomalous terms derived from the Hermitianised and anti-Hermitianised theory, as would be expected classically, resulting in new anomalous terms in the conservation laws for the chiral current.

Local topological markers are real space expressions of topological invariants
evaluated by local expectation values and are important for characterising topology in noncrystalline structures. We derive analytic expressions for local topological markers for strong topological phases of matter in odd dimensions, by generalising the formulation of the even dimensional local Chern marker. This is not a straightforward task since the topological invariants in odd dimensions are basis dependent. Our solution is to express the invariants in terms of a family of parameter dependent projectors interpolating between a trivial state and the topological state of interest. The odd dimensional invariant is therefore expressed as a Chern character integrated over the combined space of the odd dimensional Brillouin zone and the one dimensional parameter space. As a result, we provide an easy-to-use chiral marker for symmetry classes with a chiral constraint, and a Chern-Simons marker for symmetry classes with either time reversal symmetry (in three dimensions) or particle hole symmetry (in one dimension). These markers are readily extended to interacting systems by considering the topological equivalence between a gapped one-particle density matrix of the interacting state and a projector corresponding to a free fermion state.


Normalt karakteriseras topologiska faser av icke-lokala invarianter som beräknas i momentumrummet. Detta är praktiskt då systemen är translationsinvarianta,
List of Accompanying Papers

I Axial anomaly generation by domain wall motion in Weyl semimetals
Julia D. Hannukainen, Yago Ferreiros, Alberto Cortijo, and Jens H. Bardarson

II Electric manipulation of domain walls in magnetic Weyl semimetals via the axial anomaly
Julia D. Hannukainen, Alberto Cortijo, Jens H. Bardarson and, Yago Ferreiros,

III Non-Hermitian chiral anomalies
Research 4, L042004 (2022).

IV Local topological markers in odd spatial dimensions and their application to amorphous matter
Julia D. Hannukainen, Miguel F Martínez, Jens H. Bardarson, Thomas Klein

V Interacting Local Topological Markers: A one-particle density matrix approach for characterizing the topology of interacting and disordered states
Julia D. Hannukainen, Miguel F Martínez, Jens H. Bardarson, Thomas Klein

Scientific paper not included in this thesis

A Amorphous topological matter: Theory and experiment
Paul Corbae, Julia D. Hannukainen, Quentin Marsal, Daniel Muñoz-Segovia,
Adolfo G. Grushin, EPL 142 16001 (2023).
Preface

I am pleased to present the work that I have conducted during my graduate studies through this dissertation titled: Anomaly and Topology: On the axial anomaly, domain wall dynamics, and local topological markers in quantum matter. The subject matter of this dissertation concerns chiral anomalies and topology in quantum phases of matter. I began my research on the chiral anomaly by exploring Weyl semimetals. Weyl semimetals in turn opened up the world of topological phases of matter: of symmetry protected phases, topological invariants, and later amorphous topological matter and local topological markers.

Part one of this dissertation gives an introduction to the topics presented in the five accompanying research papers. Papers I and II explore the chiral anomaly in magnetic Weyl semimetals: how the chiral anomaly is generated when introducing a domain wall in the Weyl node separation, and how the chiral anomaly in turn may be used to manipulate the domain wall. Paper III is a mathematical enquiry into non-Hermitian chiral anomalies, exploring the anomalous terms resulting from a non-Hermitian action. Papers IV and V concern local topological markers in odd spatial dimensions, namely real space expressions of topological invariants that are important for characterising topology in strong topological phases of matter lacking translation invariance, such as amorphous matter. Research is not a solitary endeavour, and I have learned a lot by exploring physics together with my colleagues and collaborators. Below I have, as is customary, outlined my contributions to each research paper.

My contributions

I I took part in all the discussions forming the direction of the project. I performed all the calculations, including the ones in the appendix, and I created all the plots for the paper. I took part in writing the manuscript, and I wrote the first draft of the bulk text of the paper, including the abstract and the appendix.

II I took part in all the discussions forming the direction of the project. I performed all the calculations and made all the plots for the paper. I took part in writing the manuscript, and I wrote the first draft of the bulk text of the paper, including the abstract and the appendix.
Preface

III I was heavily involved in deciding the direction of this project, by analysing our intermediate results. I performed the calculations for the loop expansions and connected the resulting consistent anomaly with the covariant one given by the Fujikawa method. I took part in the writing of the paper and wrote the first draft on those parts connected to the calculations that I performed.

IV I took part in all the discussions leading to the final project. I wrote the code together with Miguel, and we produced all the plots. I took part in writing the manuscript, and I wrote the first draft of the bulk text, including the introduction, abstract, and parts of the appendix.

V I took part in all the discussions leading to the final project. I wrote the code together with Miguel. I took part in writing the manuscript, and I wrote the first draft of the abstract, and parts of the bulk text, and I co-wrote the introduction together with Miguel.

A I wrote the first draft of the sections on topological matter and topological markers. I took part in the polishing and rewriting of the rest of the review article.
Acknowledgements

My doctorate studies have like all endeavours in life been a journey in time and space. Along the way I have encountered a lot of people who all have affected the paths that I have treaded and the places that I have visited. I would like to acknowledge some of those who have helped me in making this journey a positive one.

Jens and Adolfo, your offices have been my favourite two places at work these past few years, places of inspiration and solace. Your research approach is similar in many ways, but you have your individual ways of communicating it. This has provided me with two different perspectives to consider as I develop my research practice and its place in the world. I have enjoyed browsing through the combined knowledge of your libraries; somehow you always found me even the most obscure requests behind the counter.

Jens, it has been a privilege to work with you, and I am so happy that you found such a suitable Master’s project for me back in the day. Through the years you have given me new opportunities and new responsibilities, and I have always felt that they came at a time when I was ready to take them on. Exploring physics with your guidance is fun and rewarding, and you most often seem to find a way to make sense of things. I appreciate the time you have taken to make sure that I can continue to find my place in academia; through our conversations, your advice, and through the proverbs from Hávamál.

Adolfo, your excitement for research is very contagious, and after our discussions, I feel full of energy to continue tinkering. My visits to Grenoble have been a highlight of my studies, and I have always felt at home in the creative environment in our group at Néel. Your particular skill of translating abstract maths to physical knowledge is very inspirational, and you often return my one line questions about some high energy concept with a paragraph explaining its physical consequences in simple terms. I am grateful for all your advice, encouragement, feedback, and pep talks through the years.

Jonas, you provided me with my first glimpse into research, which was when I knew that I wanted to pursue this path. You have always cheered me on through my studies, and I very much enjoy our encounters in the corridor. Thank you also for your help with the Swedish translation of the abstract.

Robert and Miguel, you have been my office family for the past two years,
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Yago, you taught me a lot about the world of high energy physics in a condensed matter setting, and I had fun collaborating with you. Thomas, thank you for taking the time to explain involved concepts from differential geometry and other physics to me when I needed it. This has included a lot of written communication, and sometimes also the odd phone call, which I appreciate a lot. Apoorv, your calm enthusiasm for explaining mathematical physics has been very inspirational and provided me with new interests. Mats, Andrea, and Ilaria, you were all part of the early days in the office, and I enjoyed our discussions of physics and beyond, thank you for creating such a pleasant atmosphere. Claudia, Daniel, David, and Matthias thank you for all the good conversations and group outings. I would like to acknowledge the faculty, students and postdocs of the condensed matter theory department that I have had the pleasure to spend time with through the years. Jack, thank you for reading through and giving feedback on my thesis.

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I spent a part of my graduate studies in Grenoble, and I met a lot of people who influenced me and my work in various ways. Katee, your presence and organisation of lunches and hikes made me feel instantly welcome at Néel. Your friendship made me and Peter feel at home in Grenoble, which in turn made it easier for me to take advantage of the new work environment. Thank you also for sustaining my obsession with alpine meadows. Quentin thank you for good discussions in Grenoble and San Sebastián, and for taking us on that spectacular hike to Saint-Pierre-de-Chartreuse. Selma, I take inspiration from your work and your talks, and I have enjoyed our conversations on work, life, and academia.
Dani, your enthusiasm and knowledge make for good discussions on physics and life; the image of you running up a mountain to jump into a patch of snow sums this up for me.

My research interests started to take root during my undergraduate studies, and I would like to thank Fawad and Ingemar in particular for their inspirational lectures. Eddy, thank you for introducing me to condensed matter field theory and conformal field theory, and for taking the time to discuss with me. Supriya, it is always a pleasure to talk with you, and I appreciate the advice that you have given me through the years.

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Chapter 1

Introduction

The research presented in this thesis takes place within the realm of topological quantum matter which is a subfield of condensed matter physics. Condensed matter physics is a large field containing a diverse range of research engaging a lot of physicists around the world. Yet, the name does not mean much to those asking me about my work. Maybe it is my vague introduction saying—I work with theoretical physics—that makes people immediately think about the Big Bang, or dark matter, and wonder whether I am researching the fundamental particles that make up our universe. The philosophy of condensed matter physics is quite the opposite. It is associated with the concept of emergence—how the collective behaviour of a large number of particles leads to phenomena that can not be explained by the individual particles alone [1]. If we want to explain a thunderstorm it is not very useful to study the individual molecules making up the atmosphere. We can not get a full picture of the storm by simply summing together what we know of the constituent air molecules; the thunderstorm resulting from the interacting molecules is something completely different. The same is true in quantum matter where a huge amount of particles coexist and interact with one another. The macroscopic phenomena that we observe in a laboratory are very different from the microscopic building blocks of the material. What is interesting is that structures that differ significantly on the microscopic scale may be qualitatively equivalent on the macroscopic scale. This means that we do not need to keep track of the microscopic details and we can instead characterize quantum matter through their universal properties, in terms of universality classes. These universality classes are referred to as quantum phases of matter [2].

My work concerns topological quantum phases of matter, where topological is the keyword [3]. Topology can be thought of as the study of the global properties of an object as opposite to its local properties [4]. Consider a window frame with no glass inside. We can for example measure the length of arbitrarily small parts of the frame and add up all local measurements to determine its height. So height can be understood as a local property. The fact that the frame has a hole in it is not a local property. We can not deduce the existence of the hole by considering individual parts of the frame separately—the hole is a global property of the window frame. Topology is about those properties of an object which can not be described by only considering arbitrarily small parts of the whole. The
window frame is topologically equivalent to other objects that have one hole, such as a door frame, a bagel or a swim ring. This brings us to the notions of stretching and squeezing, and tearing and gluing. Any two objects which can be transformed into each other through (figurative) stretching and squeezing without tearing and gluing are topologically equivalent. A ball is therefore not topologically equivalent to a bucket or swim ring, as depicted in Fig. 1.1. The same ideas of global properties and adiabatic transformations carry over to quantum matter. Topology seems to be prevalent in nature, and we can grow topological matter in a laboratory environment. This makes it a relevant task to both classify topological phases of matter and to understand the phenomena arising from the topological nature of these materials.

Topology relates to another main topic in my research: the chiral anomaly [5]. There are two new words here, chiral and anomaly. The word chiral is derived from the Greek word for hand, and the human hand is indeed a chiral object: an object which can not be transformed into its mirror image by using rotations and translations. The relevant objects in the context of the chiral anomaly are chiral fermions. Chiral fermions are massless fermions which are said to have a given handedness. The chirality of chiral fermions is defined through the relation between their momentum and spin: the momentum of right handed fermions is parallel with their spin, whilst the momentum of left handed fermions is antiparallel with their spin. The word anomaly in the context of the chiral anomaly is historical and refers to something out of the ordinary, something odd [5]. The oddity in this case is the nonconservation of chiral fermions upon quantisation, even though their number is conserved in the classical limit. Noether’s theorem relates symmetries of a theory with conserved quantities and forms a foundation for our understanding of classical mechanics. This notion carries over to quantum field theory in terms of classical conservation laws inside of correlation functions. But it is not a given that the classical conservation laws still hold in the quantum limit, and if they do not the theory is said to be anomalous.

The classical spacetime dynamics of massless fermions are given by the Dirac
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Figure 1.2: The integer quantum Hall effect. Left: the Hall bar in green connected to six leads in pink, with an applied magnetic field $B$, and current $I$. The Voltage $V$ across the sample is measured perpendicular to the direction of the current. Right: a schematic plot of the quantised Hall resistance as a function of the magnetic field. The plateaus in the resistance correspond to different values of the integer $\nu$.

The Dirac theory is invariant under rotations of the chiral fields, which in accordance with Noether’s theorem means that there is a corresponding conserved quantity—the chiral current [6]. This means that the number of left handed fermions and the number of right handed fermions are separately conserved, and are constant in time. The number of chiral fermions is no longer conserved in the quantum theory—this is the chiral anomaly. So how does topology and the chiral anomaly feature in my research? It can all to some degree be related to the discovery of the integer quantum Hall effect [7–9].

The corridors at the AlbaNova Physics Centre are covered with posters presenting the Physics of the Nobel prizes for each given year. As it happens, the poster hanging outside of my office highlights the discovery of the integer quantum Hall effect, a discovery that opened up the research on topological phases of matter. The poster describes a setup where a two dimensional electron gas is subject to a perpendicular magnetic field, depicted in Fig. 1.2. The bulk of the sample is insulating, but the sample is metallic on the boundaries where the fermions can only move in a given direction defined by the direction of the magnetic field. These fermions are chiral, and the edge states are a manifestation of the chiral anomaly.

The sample is connected to six leads with a current $I$ applied to two leads on the opposite side of the sample, as shown in Fig. 1.2. Measuring the voltage $V$ between two leads perpendicular to the direction of the current yields the Hall resistance $R_H = V/I$. The resulting value of the Hall resistance has a peculiar property—it is quantised. At large enough magnetic fields (of order 1 Tesla), the measured Hall resistance is quantised as $R_H = V/I = h/(e^2 \cdot \nu)$, where $h$ is Planck’s constant, and $e$ is the electron charge, and $\nu$ is a positive integer. The quantised Hall resistance as a function of the magnetic field is schematically depicted Fig. 1.2. The integer $\nu$ is a topological invariant characterising the topological phase of the quantum Hall insulator. The Hall current is carried by the boundary states at the edge of the sample, and the integer $\nu$ counts the
number of such metallic states.

The quantised resistance is said to be universal, it only depends on fundamental constants, and material properties do not play a role. What is even more important is that the quantisation is robust, it does not matter if the sample contains impurities and defects, the resistance remains the same. The poster outside of my office states that the precision of the quantisation to be of the order $10^{-7}$ and describes it as the difference between the diameter of a dust particle and the height of Kebnekaise, the highest mountain in Sweden. Contemporary measurements have improved this precision to the order of one part per billion [3], and today the constant $R_K = h/e^2$ defines the SI unit Ohm ($\Omega$).

The quantum Hall effect is an example of a topological insulator, a phase of matter which contains an insulating bulk but hosts metallic edge states [3]. Topological insulators together with topological superconductors are referred to as symmetry protected topological phases of matter [10, 11]. The symmetries in question are time reversal symmetry and particle hole symmetry, which are both antiunitary symmetries and their product, which is called chiral symmetry. Two topological $d$ dimensional insulators are said to be in the same topological phase if their Hamiltonians can be adiabatically deformed into one another without ever closing the bulk gap, and whilst preserving the symmetries of the Hamiltonian. The topology of these insulators and superconductors is characterised by a topological invariant; an integer valued number that depends on the global properties of the bulk, like the integer $\nu$ in the definition of the Hall resistance. The bulk topological invariants are related to the metallic boundary states through the bulk boundary correspondence [3, 10, 11]. The integer $\nu$ defining the Hall resistance is also equal to the number of metallic edge states at the boundary of the Hall bar. The integer nature of these invariants means that they can only change in integer steps, and can not be adiabatically deformed into one another. So two different values of $\nu$ in the Hall resistance describe two topologically different states, similarly to how the ball and the swim ring in Fig. 1.1 are two topologically different objects which can not be smoothly transformed into each other. This means that the metallic edge states are extremely robust against local impurities and perturbations.

Part of my work concerns amorphous topological matter, which is disordered matter that lacks long range order [12–14]. Topological insulators and superconductors do not rely on translational invariance, but translational invariance simplifies the formulation of their topological invariants. Crystalline structures allow us to describe our models in terms of periodic Bloch functions, describing the physics in terms of energy bands in the first Brillouin zone [15]. Topological invariants can therefore be formulated through momentum space expressions describing the global properties of the Bloch functions. This is no longer the case when translation invariance is lost, and amorphous materials require different tools for characterising their topology.
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Figure 1.3: Left: Two Weyl cones of opposite chirality in equilibrium where the Fermi level is at the nodes. Right: The application of parallel electric and magnetic fields creates an imbalance in the number of left handed and right handed fermions.

One way to characterise the topology in these structures is through local topological markers. Local topological markers are topological invariants expressed in real space, of which the local Chern marker is the most prominent example [16]. The local Chern marker is the Fourier transform of the Chern number; the topological invariant of the integer quantum Hall effect. The Chern number is a topological invariant which is defined in even spatial dimensions. The local Chern marker is evaluated for each point on the lattice; it is only quantised if the lattice is periodic in which case it assumes the value of the Chern number at each lattice point. To recover a quantised value for a nonperiodic lattice in practice, one must average the value of the local Chern marker over a large enough volume. In theory, this refers to a rescaling of the lattice—translation invariance is recovered in the long wavelength limit so averaging the local Chern marker over a larger and larger volume equates to evaluating it on a coarse grained translational invariant lattice. In papers IV and V in this dissertation, we extend the notion of the local Chern marker to odd spatial dimensions. This is not a straightforward task since the topological invariants in odd dimensions are basis dependent expressions [17] which can not be Fourier transformed into local expressions without some manipulation. The resulting local markers can be used for characterising the topology of several different classes of topological insulators and superconductors in odd spatial dimensions.

As part of my research, I explore Weyl semimetals which are topological structures without a bulk gap [15,18]. Weyl semimetals are described by emergent chiral fermions at low energies. Chiral fermions were theoretically predicted almost a hundred years ago, but they have not been found to exist as fundamental particles in the universe. But chiral fermions do emerge as quasiparticles in condensed matter systems due to the collective behaviour of fermions, describing the low energy behaviour of Weyl semimetals. Weyl semimetals are characterised by linearly dispersing, conical, band touching points close to the Fermi energy. Each band touching point, called a Weyl node, is described by a chiral fermion of a given handedness. The Weyl nodes are topological objects which are robust against perturbations; one can assign a Chern number to any two dimensional surface surrounding the Weyl node [15,18]. Weyl nodes always come in pairs of opposite
chirality on a crystalline lattice. Weyl semimetals have topologically protected metallic surface states called Fermi arcs [15, 18]. These surface states only exist for momenta between two Weyl nodes of opposite chirality, so the Fermi surface of the edge state forms an arc between the two nodes.

The existence of chiral quasiparticles makes Weyl semimetals a perfect stage in which to explore the physics of the chiral anomaly [19–24]. The chiral anomaly in Weyl semimetals is generated by applying parallel electric and magnetic fields to the material. The fermions move along the electric field, depleting fermions of one chirality, and increasing fermions of the other, schematically depicted in Fig. 1.3. The chiral anomaly leads to phenomena such as negative magnetoresistance: the decrease of the resistance as a function of the magnetic field [25], and the chiral magnetic effect: the emergence of a current parallel to the applied magnetic field [26, 27]. I research the chiral anomaly in models consisting of two Weyl cones with opposite chirality separated in momentum space. The separation of the Weyl nodes breaks time reversal symmetry and acts as a magnetisation vector. The vector separating two Weyl nodes couples to the fermions as an axial gauge field; it resembles an ordinary gauge field but it couples with an opposite sign to fermions with opposite chirality. Space and time dependent axial gauge fields generate axial electric and magnetic fields which couple with opposite sign to left and right handed fermions, analogously to how ordinary electromagnetic gauge fields give rise to electromagnetic fields. In papers I and II we assume a continuous domain wall in this magnetisation vector, which by construction is space dependent generating an axial magnetic field. By applying an external magnetic field the domain wall is put into motion making the axial gauge field time dependent and giving rise to external axial electromagnetic fields. This allows us to research the chiral anomaly in terms of these axial electromagnetic fields, and in reverse the effects of the chiral anomaly on the domain wall itself.

This dissertation aims to give the reader some context to the research presented in the five accompanying papers. Chapter two is dedicated to the chiral anomaly, illustrating what the chiral anomaly is, how it is derived in the quantised theory, and how it is connected to topology. The physics in chapter two relates to the underlying research in papers I and II on the chiral anomaly in Weyl semimetals, and to paper III which concerns non-Hermitian chiral anomalies. In chapter three we turn to topological phases of matter and give a general description of topological insulators and superconductors, with a special emphasis on the Chern insulator. The low energy physics of the Chern insulator is described in terms of Dirac theory, and some effort is spent on describing its bulk and edge properties, and the bulk boundary correspondence. The classification and characterisation of topological insulators and superconductors is relevant background knowledge for the physics presented in papers IV and V on local topological markers. The remainder of the chapter is focused on Weyl semimetals, starting with a general introduction to the subject. This is followed by an account of the low energy field
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theory description of magnetic Weyl semimetals; the bulk Chern-Simons theory and the chiral edge theory. Chapter four introduces magnetic domain walls, where the aim is to connect domain wall physics with magnetic Weyl semimetals. The latter part of chapter three and the entirety of chapter four prepare the reader for the research on Weyl semimetals presented in papers I and II. Chapter five provides an introduction to amorphous matter and the importance of characterising topology in matter lacking translation invariance. The chapter is dedicated to local topological markers, starting with a description of the local Chern marker, and why it is not a straightforward procedure to define equivalent markers in odd spatial dimensions. The latter part of chapter five introduces the chiral and Chern-Simons markers which are used to characterise the topology of insulators and superconductors in odd dimensions. The last chapter, chapter six, presents a summary of the results, ending with a reflection on future research directions.
Chapter 2

The chiral anomaly

Symmetries, conservation laws, and gauge invariance are essential for describing and understanding the physical world around us. In classical mechanics, symmetries of the Lagrangian lead to constants of motion—the conservation of energy, momentum, and angular momentum are a consequence of the invariance of the theory under time translations, space translations, and rotations. Noether’s theorem [28] tells us that all continuous symmetries of the classical Lagrangian are related to such conservation laws. The notion of symmetries and conservation laws extends to gauge symmetries, which form the basis of the theory of all known forces, from electromagnetism to gravitation. Gauge symmetries add constraints to our theories, bringing knowledge about the particles, and quasiparticles, describing the universe and the collective phenomena within it. The conservation of electric charge, for example, results from the global gauge symmetry of the free fermion Lagrangian. The same free fermion theory is only invariant under local gauge transformations if it is minimally coupled to an external gauge field, where the local gauge symmetry forces the photon to be massless.

There exist quantum field theories where the symmetries of the underlying classical theory are broken upon quantisation [5, 29–39]. These theories are said to contain a quantum anomaly, where the name anomaly likely refers to the oddity in discovering a broken conservation law [30]. The chiral anomaly was the first quantum anomaly to be discovered, by correctly describing the decay of a pion into two photons, which is a forbidden process if all symmetries of the underlying classical field theory are preserved [30–32]. The chiral anomaly has consequences beyond particle physics, and it plays the main part in this thesis due to its connection to Weyl semimetals [18, 27, 40, 41].

The chiral anomaly refers to the nonconservation of the chiral current, present in the corresponding classical Dirac theory, described by chiral fermions, massless fermions with a fixed chirality, or handedness [42]. In the massless limit chirality is equal to helicity, and the spin of the chiral fermion is either parallel or antiparallel to the momentum, depending on the chirality of the fermion. The massless Dirac equation in even spacetime dimensions decouples the Dirac fermions into chiral fermions, and the massless Dirac theory is invariant under both left and right handed chiral rotations, corresponding to two conservation laws for the right and left handed chiral currents [6, 42]. These symmetries of the classical theory are
The chiral anomaly

broken when the Dirac theory is quantised, and both chiral currents can not be simultaneously conserved \[5\], and the quantised Dirac theory in the massless limit is anomalous.

The chiral anomaly is often formulated in terms of the total vector current, \( j^\mu = j^\mu_L + j^\mu_R \), the sum of the individual currents of left and right handed fermions, and the axial vector current \( j_5^\mu = j^\mu_L - j^\mu_R \), which is the difference between the two chiral currents, such that

\[
\partial_\mu j^\mu = 0, \tag{2.1}
\]

\[
\partial_\mu j_5^\mu = \frac{1}{2\pi^2} \left( \mathbf{E} \cdot \mathbf{B} + \frac{1}{3} \mathbf{E}_5 \cdot \mathbf{B}_5 \right). \tag{2.2}
\]

In this form, the anomaly is referred to as the axial anomaly, where the total current is conserved, whilst the axial current is not. The nonconservation of axial current is proportional to the sum of parallel electric and magnetic fields, and parallel axial electric and magnetic fields, where the latter are fields which couple with opposite sign to fermions of opposite chirality.

Chiral fermions are not found to be fundamental particles in the universe, but they do exist as quasiparticles in condensed matter systems, in particular in Weyl semimetals. Weyl semimetals are three dimensional semimetals described by chiral (massless) fermions at low energies. The low energy theory of Weyl semimetals is defined by a massless Dirac theory which means that these materials provide a stage on which to explore and measure the chiral anomaly and its consequences \[18\]. Later, in chapter four we will see that Weyl semimetals also provide a physical setting for emergent axial electromagnetic fields. The axial anomaly in Weyl semimetals plays a large part in this dissertation and is the main topic of the research presented in papers I and II.

This chapter is dedicated to the exploration and derivation of the chiral anomaly in some detail. We will derive the chiral anomaly by using both a path integral method and a diagrammatic method. The derivations provide an understanding of why the chiral anomaly appears in the quantised field theory. Towards the end of the chapter, we will find a connection between the anomaly and topology, the other main theme in this dissertation. But before getting there we must start at the beginning, with classical symmetries.

### 2.1 Symmetries and conservation laws

To discuss the chiral anomaly, and how it breaks the underlying classical symmetries, we first need to know what these symmetries are. In classical mechanics they are defined by Noether’s theorem, and in quantum field theory by the Ward-Takahashi identities.
The chiral anomaly

2.1.1 Noether’s theorem

In 1918 Emmy Noether published a paper titled Invariante Variationsprobleme [28] in which she explained the connection between symmetries and conservation laws, a result referred to as Noether’s theorem in classical mechanics. Noether’s theorem states that each generator of a continuous symmetry of a theory is linked to a conserved current, where the symmetries of the Lagrangian give rise to constants of motion.

The proof of Noether’s theorem relies on the variation of the Lagrangian density $L$ described by a set of fields $\varphi_a$, under infinitesimal continuous transformations $\varphi_a \rightarrow \varphi_a + \delta \varphi_a$ [6]. The transformation $\delta \varphi_a$ defines a symmetry transformation if the lowest order variation of the Lagrangian density equals a total derivative, $\delta L = \partial_\mu F^\mu$, for some function $F^\mu(\phi_a)$. The variation of the corresponding action is zero for a symmetry transformation, and the theory contains a conserved current $\partial_\mu j^\mu = 0$. The variation of the Lagrangian density to lowest order $\delta \varphi_a$ is

$$\delta L = \frac{\delta L}{\delta \varphi_a} \delta \varphi_a + \frac{\delta L}{\partial_\mu \delta \varphi_a} \partial_\mu (\delta \varphi_a),$$

which is conveniently moulded into the expression

$$\delta L = \left( \frac{\delta L}{\delta \varphi_a} - \partial_\mu \frac{\delta L}{\partial_\mu \varphi_a} \right) \delta \varphi_a + \partial_\mu \left( \frac{\delta L}{\partial_\mu \varphi_a} \delta \varphi_a \right),$$

such that the terms in the first bracket vanish whenever the equations of motion are satisfied. The remaining term in Eq. (2.4) must be equal to $\partial_\mu F^\mu(\phi_a)$, which defines the conserved Noether current $j^\mu = \frac{\delta L}{\partial_\mu \varphi_a} \delta \varphi_a - F^\mu(\phi_a)$.

The connection between the laws of physics and symmetries is important for our understanding of the physical world, which is why quantum anomalies were considered to be suspicious when they were first discovered. The classical conservation laws are expressed in quantum field theory as conservation laws of Green’s functions and it is these quantised conservation laws that are broken by the quantum anomalies.

2.1.2 Ward identities—conservation laws in the quantum limit

The Ward-Takahashi identities define the quantum field theory equivalents of Noether’s theorem, where the classical conservation laws appear inside Green’s functions. We consider the path integral quantisation of the classical theory, which results in a compact derivation of the Ward-Takahashi identities [43]. The partition function

$$Z[K] = \int \mathcal{D}\varphi e^{-S[\varphi] + \int d^d y K_b(y) \varphi_b(y)},$$

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The chiral anomaly is expressed in Euclidean space, and the classical action $S[\varphi]$ is invariant under the continuous symmetry, $\varphi_a(x) \rightarrow \varphi'_a(x) = \varphi_a(x) + \delta \varphi_a(x)$, where the Roman subscripts label the fields in the specific theory. The second term in the action describes the coupling between $\varphi_a$ and a source field $K_a(y)$. Any correlation function can be obtained from the functional integral by taking derivatives with respect to the source field, and then setting the field to zero,

$$\langle 0 | T \varphi(x_1) ... \varphi(x_n) | 0 \rangle = \int \mathcal{D} \varphi \varphi(x_1) ... \varphi(x_n) e^{-S[\varphi] + \int d^4y K_a(y) \varphi_a(y)} = \frac{\delta^n Z[K]}{\delta K(x_1) ... \delta K(x_n)} \bigg|_{K=0} .$$

The partition function is invariant under the infinitesimal transformation of the fields, $\delta Z = 0$ since $\varphi'$ is an integration variable of the path integral,

$$Z[K] = \int \mathcal{D} \varphi' e^{-S[\varphi'] + \int d^4y K_a(y) \varphi'_a(y)} .$$

The variation of the path integral to first order in $\delta \varphi_a(x)$ is

$$\delta Z = \int \mathcal{D} \varphi e^{-S[\varphi] + \int d^4y K_a(y) \varphi_a(y)} \int d^4x (\partial_\mu j_\mu^a(x) - K_a(x)) \delta \varphi_a(x) \equiv 0 ,$$

where we assume that the path integral measure is invariant under the transformation, $\mathcal{D} \varphi' = \mathcal{D} \varphi$. By taking $n$ derivatives with respect to the source field, and setting it to zero in the end, we arrive at the expression

$$0 = \int \mathcal{D} \varphi e^{-S[\varphi]} \int d^4x (\partial_\mu j_\mu^a(x) \varphi_{a_1}(x_1) ... \varphi_{a_n}(x_n))$$

$$+ \sum_{j=1}^n \varphi_{a_1}(x_1) ... \varphi_{a_j} \delta(x_j) ... \varphi_{a_n}(x_n)) \delta \varphi_a(x) .$$

The variation, Eq. (2.9), is true for any choice of $\delta \varphi_a(x)$, so dropping the variation and the integral, results in the time ordered vacuum correlation function

$$0 = \partial_\mu \langle 0 | T j_\mu^a(x) \varphi_{a_1}(x_1) ... \varphi_{a_n}(x_n) | 0 \rangle$$

$$+ \sum_{j=1}^n \langle 0 | T \varphi_{a_1}(x_1) ... \delta_{a_j} \delta(x_j) ... \varphi_{a_n}(x_n) | 0 \rangle ,$$

defining the Ward-Takahashi identity. The Ward-Takahashi identity tells us that the conservation of the Noether current holds inside correlation functions in the quantised theory, defining a gauge invariant theory. The equivalence is true up to contact terms due to fields evaluated at the same spacetime points, where the form of the contact terms depends on the specifics of the infinitesimal transformations.
2.1.3 When the Jacobian determinant matters

The conservation of Noether currents inside correlation functions relies on the assumption that the Jacobian of the transformation of the path integral measure is unity. This assumption is not necessarily true—we would not expect the measure to be unchanged by default when transforming integrals in calculus. It is the path integral measure that gives rise to the quantum anomalies since the measure is the only object that contains information about the quantisation in the path integral formalism. The transformation of the path integral measure is \( D\varphi' = D\varphi J \), where the Jacobian matrix is \( J = e^{-A\delta\varphi_a(x)} \) for some function \( A \) of the fields [43]. The Jacobian provides an additional term, \(-A\delta\varphi_a(x)\), to the path integral variation in Eq. (2.8),

\[
\delta Z = \int D\varphi e^{-S[\varphi]+\int d^d y K_a(y)\varphi_a(y)} \int d^d x \left( -A + \partial_\mu j_\mu^a(x) - K_a(x) \right) \delta \varphi_a(x) = 0.
\]

(2.11)

This means that the corresponding Ward-Takahashi identities are modified such that the derivatives of the \( n \) point functions are no longer zero up to the contact term, but instead of the form

\[
A = \partial_\mu \langle 0 \mid T j_\mu^a(x) \varphi_{a_1}(x_1) \cdots \varphi_{a_n}(x_n) \mid 0 \rangle \\
+ \sum_{j=1}^n \langle 0 \mid T \varphi_{a_1}(x_1) \cdots \delta_{aa_j} \delta^n(x - x_j) \cdots \varphi_{a_n}(x_n) \mid 0 \rangle \delta \varphi_a(x),
\]

(2.12)

where \( A \) is the quantum anomaly.

The chiral anomaly was first discovered through perturbation theory [30,31], and it took some time before the result was confirmed through the invariance of the measure in the path integral language [44–46]. We will derive the chiral anomaly in some detail through both the diagrammatic and the path integral formalisms, to pinpoint how and why the classical conservation laws are broken in the quantum limit. But first, as a prelude, let us understand what the chiral anomaly is, and does, within a physical set up.

2.2 The nonconservation of chiral charge

The Dirac action in even spacetime dimensions forms the basis for our exploration of the chiral anomaly. The evolution of spin 1/2 particles in \( d \) spacetime dimensions is governed by the equations of motion derived from the Dirac action

\[
S_D = \int d^d x \bar{\Psi}(i\partial_\mu \gamma^\mu + m)\Psi,
\]

(2.13)

where \( \Psi(x) \) is a Dirac field with adjoint \( \bar{\Psi} = \Psi^\dagger \gamma^0 \), \( m \) is the Dirac mass, and repeated indices, \( \mu \in (0, \ldots, d-1) \), are summed over. The gamma matrices, \( \gamma^\mu \) are
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The chiral anomaly of the Clifford algebra $\mathcal{C}L_d$, with multiplication rule $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where $g^{\mu\nu}$ is the Minkowski metric of flat spacetime.

In $d = 4$ spacetime dimensions there exists a chiral element referred to as the fifth gamma matrix, which is defined as the product of the four generators, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The chiral matrix is Hermitian, squares to unity, and anti-commutes with $\gamma^\mu$. The matrix $\gamma^5$ defines the chirality of the chiral fermions—the chiral fermions are eigenvectors to $\gamma^5$ with eigenvalues $\chi = \pm 1$ defining the chirality. The equivalent of $\gamma^5$ only exists in even spacetime dimensions, which is a consequence of the matrix dimension of the representation of the gamma matrices; gamma matrices in both $d = 2n$ dimensions, and $2n+1$ dimensions are represented in terms of $d \times d$ matrices. The Clifford algebra in five spacetime dimensions is given by $\mathcal{C}L_5 = \{\gamma^0, \gamma^1, \gamma^2, \gamma^3, i\gamma^5\}$, where the product of the generators is proportional to the identity matrix. So there is no chiral matrix in five spacetime dimensions.

The two generators in two spacetime dimensions are represented by two of the Pauli matrices, for example in a basis $\gamma^0 = \sigma_x$, and $\gamma^1 = i\sigma_y$, which means that the chiral matrix is defined by $\gamma^5 = -\gamma^0\gamma^1 = \sigma_z$. In three spacetime dimensions, the generators constitute all three Pauli matrices, which square to unity, and no chiral matrix exists in three spacetime dimensions. The same argument extends to every odd spacetime dimension, and chiral elements with the same properties as $\gamma^5$ in four spacetime dimensions exist in every even spacetime dimension. Chiral fermions are eigenvectors to the chiral matrix, which means that the chiral anomaly only exists in even spacetime dimensions. We will explore the anomaly in two and four spacetime dimensions; the two dimensional anomaly appears as a chiral edge state of two dimensional topological insulators, and the three dimensional anomaly plays a role in the theory of Weyl semimetals.

2.2.1 The chiral anomaly in 1+1 dimensions

The massless Dirac theory in 1+1 spacetime dimensions is defined by the action

$$S_D = \int d^2x \bar{\Psi}i\partial\Psi,$$

(2.14)

where $\partial = \gamma^\mu \partial_\mu$, $\mu \in (0, 1)$, and the metric is $g^{\mu\nu} = \text{diag}(1, -1)$. The gamma matrices are represented by Pauli matrices where,

$$\gamma^0 = \sigma_x, \quad \gamma^1 = i\sigma_y, \quad \gamma^5 = -\gamma^0\gamma^1 = \sigma_z.$$

(2.15)

The Dirac spinors are composed by chiral fermions, $\Psi = \Psi_L \oplus \Psi_R$, where

$$\Psi_L = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}, \quad \Psi_R = \begin{pmatrix} 0 \\ \psi_- \end{pmatrix},$$

(2.16)

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The chiral anomaly

and $\gamma^5$ is used to construct the projectors $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$, such that $P_\pm \Psi = \Psi_{L/R}$. By using the projector together with the definition of the Dirac adjoint, we decouple the Dirac action in Eq. (2.14) into chiral fermions,

$$S_D = \int d^2 x (i \psi_+^\dagger \partial_- \psi_+ + i \psi_-^\dagger \partial_+ \psi_-),$$

(2.17)

where the partial derivative is defined as $\partial_\pm = \partial_0 \pm \partial_1$ [6]. The left, $\psi_+$, and right, $\psi_-$ moving fields $\psi_\pm = \psi_\pm(t \pm x)$ solve the equations of motions of the chiral fermions $\partial_\mp \psi_\pm = 0$.

The Dirac action is invariant under two global $U(1)$ symmetries, $\psi_\pm \rightarrow \psi_\pm^{i\alpha\pm}$, where $\alpha_{\pm}$ is a constant depending on the chirality of the fermions according to Noether’s theorem, they separately conserve the number of left and right moving fermions, defining the chiral symmetry of the model. The chiral currents are defined as [47]

$$j^0 = \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-,$$

$$j^1 = \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-,$$

(2.18)

$$j^0_5 = \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-,$$

$$j^0_5 = - (\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-),$$

(2.19)

where $j^\mu = \varepsilon^{\mu\nu} j_\nu$, using the convention $\varepsilon^{01} = 1$ for the anti symmetric Levi-Civita symbol.

The chiral symmetry is often stated in terms of the Dirac fermions as the combination of a vector symmetry $\Psi \rightarrow e^{i\alpha} \Psi$, and an axial symmetry, $\Psi \rightarrow e^{i\alpha \gamma^5} \Psi$. The vector symmetry refers to the conservation of the total number of fermions, while the axial symmetry conserves the difference between right moving and left moving fermions, where the conserved vector current, $j^\mu$, and axial current, $j^\mu_5$, are defined as

$$j^\mu = \bar{\Psi} \gamma^\mu \Psi,$$

$$j^\mu_5 = \bar{\Psi} \gamma^\mu \gamma^5 \Psi.$$  

(2.20)

(2.21)

To understand what the chiral anomaly does, rather than how it comes about, we assume that the Dirac action, Eq. (2.14), is quantised. The corresponding momentum space Hamiltonian is

$$H = \Psi^\dagger H^0(k) \Psi,$$

$$H^0(k) = -\sigma_z k,$$

(2.22)

which disperses linearly in momentum, $E = \pm |k|$. The Dirac sea is depicted in Fig. 2.1, where all negative energy states are filled, and where chiral excitations with $k > 0$ are right moving fermions, whilst those with $k < 0$ are left moving. To induce the anomaly we introduce an external electric field by minimally coupling the fermions to a vector field; this model is called the massless Schwinger model [48]. By coupling the fermions to a temporal gauge field $A_0 = 0$, $A_1 = E$,
for a constant electric field $\mathcal{E}$, the equations of motion for the two chiral species are given by

$$i\partial_0 \psi_- = (-i\partial_1 - eA_1)\psi_-, \quad i\partial_0 \psi_+ = (i\partial_1 - eA_1)\psi_+. \quad (2.23)$$

where $e$ is the unit electron charge [47]. The electric field exerts a force on the charged fermions where $\dot{k} = e\mathcal{E}$, which due to the linear dispersion implies that also the energy changes linearly with the applied field, $\dot{E} = e\mathcal{E}$. By adding an electric field $\mathcal{E} > 0$ parallel to the momentum, the fermions start moving to the right creating right moving particles, and left moving holes, as shown in Fig. 2.1 [49]. The shift in momentum leads to a change of the Fermi surface, and the particle density $\rho_{\pm}$ of chiral fermions with charge $e$ subject to the electric field during a time $t$ is given by

$$\rho_{\pm} = \pm \frac{e\mathcal{E}t}{L} \frac{L}{2\pi} = \pm e\mathcal{E}t, \quad (2.24)$$

where $L/(2\pi)$ is the density of states for a system of length $L$ [49]. This means that the chiral symmetry is broken as the number of left moving fermions has increased while the number of right moving fermions has decreased. The chiral anomaly plays an important role in the field of topological phases of matter. In chapter three we show that the edge states of the integer quantum Hall effect and the Chern insulator are described by 1+1 dimensional chiral fermions, and are a manifestation of the 1+1 dimensional chiral anomaly.

The chiral anomaly is equivalently stated in terms of the vector and the axial vector currents; the vector symmetry is still intact since the total number of fermions is conserved, but the axial symmetry is broken. The 1+1 dimensional axial anomaly is expressed as

$$\partial_{\mu}j_5^\mu = \frac{e}{2\pi} \varepsilon_{\mu\nu} F^{\mu\nu}, \quad (2.25)$$
The chiral anomaly

where $F^{\mu\nu}$ is the field strength tensor.

The 1+1 dimensional chiral anomaly has physical relevance as an anomalous edge theory of topological matter in two spatial dimensions described by chiral fermions. These chiral edge fermions can only move in a given direction along the edge depending on their chirality.

2.2.2 The chiral anomaly in 3+1 dimensions

The classical Dirac action, and its symmetries in 3+1 dimensions, is described in complete analogy to the theory in 1+1 dimensions. To prepare for the derivation of the axial anomaly in the following sections, we will explore the 3+1 dimensional Dirac theory with some additional detail. The Lagrangian density for massive fermions in 3+1 dimensions is

$$L_D = \bar{\Psi} \left( i \gamma^\mu \partial_\mu - m \right) \Psi,$$

where the Greek indices $\mu \in (0, 1, 2, 3)$, and where the metric signature is $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The Dirac Lagrangian is associated with two currents, the vector current, $j^\mu = \bar{\Psi} \gamma^\mu \Psi$, and the axial vector current $j_5^\mu = \bar{\Psi} \gamma^\mu \gamma^5 \Psi$, where the quantity $\gamma^\mu \gamma^5$ is a pseudo vector, which changes sign under parity transformations. Any Dirac field satisfying the Dirac equations,

$$(i \gamma^\mu \partial_\mu - m) \Psi = 0, \quad (i \gamma^\mu \partial_\mu - m) \bar{\Psi} = 0,$$

leads to the current relations

$$\partial_\mu j^\mu = 0, \quad \partial_\mu j_5^\mu = 2im\bar{\Psi}\gamma^5\Psi.$$  

We will only consider the massless limit in which both the classical vector and the axial vector currents are conserved. The conserved currents imply two different global symmetries of the Lagrangian density Eq. (2.26) for $m = 0$; $\Psi \to e^{i\alpha(x)}$, and $\Psi \to e^{i\beta(x)} \bar{\Psi}$, for constant $\alpha$ and $\beta$.

The Lagrangian density for massless fermions coupled to external vector fields, $A_\mu$, and axial vector fields, $A_{5,\mu}$, is

$$L_D(A, A_5) = \bar{\Psi} \gamma^\mu iD_\mu \Psi,$$

where $D_\mu = \partial_\mu - iA_\mu - i\gamma^5 A_{5\mu}$.

in terms of the Dirac operator $D$. This minimally coupled Lagrangian density is invariant under both local vector gauge transformations,

$$\Psi \to \Psi e^{i\alpha(x)} , \quad \bar{\Psi} \to e^{-i\alpha(x)} \bar{\Psi} , \quad A_\mu \to A_\mu + \partial_\mu \alpha(x),$$

and local axial gauge transformations

$$\Psi \to e^{i\beta(x)} \gamma^5 \Psi , \quad \bar{\Psi} \to \bar{\Psi} e^{i\beta(x)} \gamma^5 , \quad A_{5,\mu} \to A_{5,\mu} + \partial_\mu \beta(x),$$

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so that the classical theory contains a combined $U_V(1) \times U_A(1)$ symmetry associated with the conserved vector and axial vector currents $[5]$. The Dirac Lagrangian density in terms of chiral fermions is

$$L_D(A, A_5) = \bar{\Psi}_L(i\partial + A_5^L)\Psi_L + \bar{\Psi}_R(i\partial + A_5^R)\Psi_R,$$

(2.33)

where the Dirac fermions are projected into chiral fermions with the projectors

$$P_{\pm} = \frac{1}{2}(\openone \pm \gamma^5).$$

The left and right handed gauge fields, $A_{\mu}^L/R$ are defined as

$$A_{\mu}^L = A_\mu + A_{5,\mu},$$

(2.34)

$$A_{\mu}^R = A_\mu - A_{5,\mu}.$$  

(2.35)

In terms of the left and right handed fermions and gauge fields, the Lagrangian density obeys a $U_L(1) \times U_R(1)$ symmetry, where the left/right handed gauge transformation, $U_{L/R}(1)$ is given by

$$\Psi_{L/R} \rightarrow \Psi_{L/R}e^{i\lambda_{L/R}(x)},$$

(2.36)

$$A_{\mu}^L/R \rightarrow A_{\mu}^L/R + \partial_\mu \lambda_{L/R}(x).$$

(2.37)

The corresponding conserved chiral currents are defined as

$$j_\mu^L = \bar{\Psi}_L\gamma^\mu\Psi_L,$$

(2.38)

$$j_\mu^R = \bar{\Psi}_R\gamma^\mu\Psi_R,$$

(2.39)

such that, (Eq. (2.28), $\partial_\mu j_\mu^L = \partial_\mu j_\mu^R = 0$.

The axial anomaly in 3+1 dimensions can be understood in terms of a linearly dispersing energy momentum spectrum in the same way as in the massless Schwinger model $[49]$. By applying a magnetic field, the Landau level spectrum of the chiral fermions displays a linearly dispersing zeroth Landau level in the direction of the magnetic field, as depicted in Fig. 2.2, where the sign of the linear dispersion depends on the chirality of the fermion. The gap between the lowest Landau level and the higher ones is proportional to the magnetic field, so the higher Landau levels can be ignored in a strong enough magnetic field. The low energy physics in strong magnetic fields is therefore equivalent to the 1+1 dimensional theory. By adding an electric field in the direction of the magnetic field the chiral fermions start to increase, or decrease depending on their chirality, as depicted in Fig. 2.2. The chiral anomaly in terms of the charge density is now

$$\dot{\rho}_{\pm} = \pm \frac{e\mathcal{E}}{2\pi} \frac{eB}{2\pi} = \pm \frac{e^2}{4\pi^2} \mathbf{E} \cdot \mathbf{B},$$

(2.42)
The chiral anomaly

Figure 2.2: Left: The Landau level spectrum of 3+1 dimensional chiral fermions subject to a magnetic field $B$. Right: The addition of an electric field $E$ parallel to the magnetic field creates an imbalance of right and left handed fermions.

which, up to the Landau level degeneracy $eB/2\pi$, corresponds to Eq. (2.24) describing the particle density in 1+1 dimensions.

The chiral anomaly in 3+1 dimensions is generated in Weyl semimetals subject to parallel electromagnetic fields. The Landau level picture provides an intuitive description of the chiral anomaly in Weyl semimetals, and we will explore it in more detail in chapter three. The coming sections will be contain more technical details as we go through the derivation of the anomaly. These details are important for understanding where the anomaly comes from, and how it is easy to miss when evaluating divergent integrals.

2.3 Deriving the anomaly diagrammatically

The chiral anomaly appears in the third order in perturbation theory in the three point functions of an odd number of axial currents described by triangle Feynman diagrams [5]. These diagrams are regularisation dependent, so the chiral anomaly appears as a surface term of diverging momentum space integrals in perturbation theory. The loop integrals of the triangle diagrams are finite when regularised, but they are not unique—the result depends on the labelling of the loop momenta. The loop integrals are linearly divergent, so the momentum variables can be chosen to be any arbitrary linear combination of each other. The form of the anomaly depends on the choice of labelling, which in turn is determined by the physical constraints of the theory in question.

2.3.1 Anomalous Ward identities—when surface terms matter

The axial anomaly in 3+1 dimensions stems from the Green’s function describing the correlation of two vector currents, and an axial vector current [5]

$$\tilde{\Gamma}_{\mu\nu\rho}(x, y, z) = \langle 0 | T_{\mu}(x) \tilde{j}_{\nu}(y) \tilde{j}_{\rho}^{5}(z) | 0 \rangle. \quad (2.43)$$
The chiral anomaly

The corresponding momentum space amplitude, schematically depicted in the triangle diagram Fig. 2.3, is

\[ \Gamma_{\mu
u\rho}(k_1, k_2, q) = \int d^4x d^4y d^4ze^{ik_1x+ik_2y-iz} \langle 0|Tj_\mu(x)j_\nu(y)j^5_\rho(z)|0 \rangle, \]  

(2.44)

where \( k_1 \) and \( k_2 \) labels the momenta of the vector currents, and \( q \) the momenta of the axial current. To obtain the classical conservation laws for the axial current we must differentiate the Green’s function with respect to \( z \),

\[ q^\rho \Gamma_{\mu
u\rho}(k_1, k_2, q) = \int d^4x d^4y d^4ze^{ik_1x+ik_2y-iz} \partial_\rho z \langle 0|Tj_\mu(x)j_\nu(y)j^5_\rho(z)|0 \rangle, \]  

(2.45)

where the total derivative is assumed to be zero. The notation \( \partial_\rho z \) means that the derivative is taken with respect to the \( \mu \) component of an (axial) vector with argument \( z \). By expanding the time ordered three point function in Eq. (2.43) in terms of Heaviside \( \theta \) functions, and using the product rule of differentiation, the resulting Green’s function takes the form

\[ \partial_\rho z \langle 0|Tj_\mu(x)j_\nu(y)j^5_\rho(z)|0 \rangle = \langle 0|Tj_\mu(x)j_\nu(y)[\partial_\rho j^5_\rho(z)]|0 \rangle. \]  

(2.46)

Since the axial current is conserved according to the classical conservation laws, the axial Ward-Takahashi identity

\[ q^\rho \Gamma_{\mu
u\rho}(k_1, k_2, q) = 0, \]  

(2.47)

is fulfilled. By instead differentiating the three point function with respect to the vector current, ignoring any surface terms, and using the classical conservation law of the vector current, results in the vector Ward-Takahashi identities

\[ k_1^\mu \Gamma_{\mu\nu\rho}(k_1, k_2, q) = k_2^\mu \Gamma_{\mu\nu\rho}(k_1, k_2, q) = 0. \]  

(2.48)

By ignoring the boundary term in Eq. (2.45) we seem to have generated a set of Ward-Takahashi identities, but these results are intentionally wrong. But
The chiral anomaly

Figure 2.4: The two triangle diagrams corresponding to a three point function of three chiral currents of the same chirality. The Green’s function in Eq. (2.44) is divergent, and the regularisation leads to a surface term due to the total derivative [5]. The surface term is finite and is responsible for the axial anomaly. We will evaluate this surface term in the next section.

2.3.2 The triangle diagrams—finite but not unique

We evaluate the three point function of the three currents of chiral fermions of the same chirality which leads to the chiral anomaly. The starting point is the massless Dirac action of a chiral fermion coupled to a vector gauge field,

\[ S_\pm = \int d^4x \bar{\Psi} \left( i \slashed{\partial} + A \right) P_\pm \Psi, \]  

for which the average chiral current is defined as

\[ \langle j_{L/R} \rangle = \frac{\int \mathcal{D}[\bar{\Psi}\Psi] e^{iS_\pm[\bar{\Psi},\Psi,A]} j_{L/R} \int \mathcal{D}[\bar{\Psi}\Psi] e^{iS_\pm[\bar{\Psi},\Psi,A]}}{\int \mathcal{D}[\bar{\Psi}\Psi] e^{iS_\pm[\bar{\Psi},\Psi,A]}}. \]

The anomalous contribution appears in second order in the gauge fields, given by the three point function \( \langle 0|T j_{L/R,\mu}(x) j_{L/R,\nu}(y) j_{L/R,\rho}(z) |0 \rangle \), of the chiral currents. The momentum space amplitude for the three point function is given by the sum of the triangle diagrams in Fig 2.4,

\[ i\Gamma^\mu_\chi(p,q,r) = -e^3 \int \frac{d^4k}{(2\pi)^4} \text{tr}[(-\slashed{k} + p)\gamma^\mu(-\slashed{k})\gamma^\nu(k + q)\gamma^\rho P_\pm] + \frac{\mu \rightarrow q}{\mu \leftarrow \nu}, \]

where the external momenta is conserved, \( \delta(p + q + r) = 0 \), \( \chi = \pm 1 \) labels the chirality of the fermions, and the propagators are rationalised, \( \frac{1}{k} = -\frac{\slashed{k}}{k^2}, \frac{\slashed{k}}{k^2} = -k^2 \). It is only the term proportional to \( \pm \gamma^5/2 \) in the projector \( P_\pm = (1 \pm \gamma^5)/2 \) that contributes; expectation values of an odd number of vector currents vanish.
The chiral anomaly
due to the charge conjugation symmetry of quantum electrodynamics, as stated
by Furry’s law§.
By evaluating the Feynman diagrams, we will find that the three Ward-
Takahashi identities
\[ p_\mu \Gamma^{\mu \nu \rho}(p, q, r) = 0, \quad (2.52) \]
\[ q_\nu \Gamma^{\mu \nu \rho}(p, q, r) = 0, \quad (2.53) \]
\[ r_\rho \Gamma^{\mu \nu \rho}(p, q, r) = 0, \quad (2.54) \]
can not be fulfilled simultaneously, which is the chiral anomaly. We start by
showing that Eq. (2.54) holds, by using the cyclic property of the trace, and by
exchanging \( p \rightarrow -(k + q) + (k - p) \). By using the trace relation
\[
\text{tr}\left[\gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\mu \gamma^5\right] = -4i\varepsilon^{\alpha \nu \beta \mu}, \quad (2.55)
\]
for metric signature \((-1, 1, 1, 1)\), the numerator of \( r_\rho \Gamma^{\mu \nu \rho}(p, q, r) \) splits into four
terms
\[
r_\rho \Gamma^{\mu \nu \rho}(p, q, r) = -2\varepsilon^3 \varepsilon^{\alpha \nu \beta \mu} \int \frac{d^4k}{(2\pi)^4} \left( \frac{k_\alpha p_\beta}{k^2(k - p)^2} + \frac{k_\alpha p_\beta}{k^2(k + q)^2} \right) + p^{\mu \leftarrow q \nu} \quad (2.56)
\]
These four momentum integrals are all linearly divergent, and vanish when using
dimensional regularisation,
\[
\varepsilon^{\alpha \nu \beta \mu} \int \frac{d^4k}{(2\pi)^4} \frac{k_\alpha p_\beta}{k^2(k - p)^2} \propto \varepsilon^{\alpha \nu \beta \mu} p_\alpha p_\beta = 0, \quad (2.57)
\]
generating the Ward-Takahashi identity
\[
r_\rho \Gamma^{\mu \nu \rho}(p, q, r) = 0. \quad (2.58)
\]
By expressing \( p = -(k + p) - k \), where the upper sign refers to the left dia-
gram in Fig. 2.4 and the lower sign to the right diagram in Fig. 2.4, we find the expression
\[
p_\mu \Gamma^{\mu \nu \rho}(p, q, r) = -2\varepsilon^3 \varepsilon^{\alpha \nu \beta \rho} \int \frac{d^4k}{(2\pi)^4} \left( \frac{(k - p)\alpha (p + q)\beta}{(k - p)^2(k + q)^2} + p^{\rho \leftarrow \nu} \right) \quad (2.59)
\]
It seems at first glance that we can reproduce the integrals in Eq. (2.56), by
shifting the integration variable in Eq. (2.59) by \( k \rightarrow k + p \), resulting in another
Ward-Takahashi identity. But this is not true, the integrals are linearly divergent,
so shifting the integration variable will lead to a surface term. This surface term
breaks the Ward-Takahashi identity, resulting in the chiral anomaly.

§The charge conjugation operator, \( C \), leaves the vacuum state invariant, \( C|\Omega\rangle = |\Omega\rangle \), and
anticommutes with the photon fields, \( \{ A_\mu, C \} = 0 \), so a correlation function of \( n \) number
of photon fields gives \( \langle \Omega | A_{\mu_1} A_{\mu_2} \ldots A_{\mu_n} | \Omega \rangle = \langle \Omega C^\dagger C A_{\mu_1} C^\dagger C A_{\mu_2} C^\dagger C \ldots C^\dagger C A_{\mu_n} C^\dagger C | \Omega \rangle = (-1)^n \langle \Omega | A_{\mu_1} A_{\mu_2} \ldots A_{\mu_n} | \Omega \rangle \), which is zero when \( n \) is odd.

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2.3.3 Linearly divergent integrals—the surface term matters II

The difference between two one dimensional integrals where the integration variable of one of the integrals is shifted by a constant \(a\),

\[
\Delta(a) = \int_{-\infty}^{\infty} dx [f(x + a) - f(x)]
\]

\[
= a[f(\infty) - f(-\infty)] + \frac{a^2}{2!} [f'(\infty) - f'(-\infty)] + ..., \tag{2.60}
\]

is only zero whenever the integral over \(f(x)\) converges, in which case the function \(f(x)\) and all its derivatives vanish at the infinities. A linearly divergent integral is instead constant at the infinities,

\[
\lim_{x \to \pm \infty} f(x) = c_{\pm},
\]

resulting in a constant surface term \(\Delta(a) = a(c_+ - c_-)\). By denoting the first term of the integrand in Eq. (2.59) by

\[
f_{\alpha\beta}(k - p) = \frac{(k - p)_\alpha(p + q)_\beta}{(k - p)^2(k + q)^2}, \tag{2.61}
\]

and expanding it to the lowest order in \(p\), we find an expression for the surface term

\[
\int \frac{d^4k}{(2\pi)^4} f_{\alpha\beta}(k - p) = \tilde{c}(p + q)_\alpha(p + q)_\beta - p^\lambda \int \frac{d^4k}{(2\pi)^4} \partial_\lambda f_{\alpha\beta}(k). \tag{2.62}
\]

The first term on the right hand of Eq. (2.62) vanishes when contracted with the epsilon tensor. By using Gauss’s law on the remaining integral,

\[
\Delta_{\alpha\beta}(p) = -p^\lambda \int \frac{d^4k}{(2\pi)^4} \partial_\lambda f_{\alpha\beta}(k) = -ip^\lambda \lim_{k \to \infty} \int \frac{dS_\lambda}{(2\pi)^4} f_{\alpha\beta}(k), \tag{2.63}
\]

where the surface element is \(dS_\lambda = k^2k_\lambda d\Omega_4\) in terms of the differential solid angle in four dimensions, \(d\Omega_4\), yields the result

\[
\Delta_{\alpha\beta}(p) = -ip^\lambda \frac{1}{32\pi^2} g_{\lambda\alpha}(p + q)_\beta. \tag{2.64}
\]

By inserting the surface term into Eq. (2.59), and combining it with the second term in the integrand obtained by simply changing the momentum and index labels, results in the expression

\[
p_\mu \Gamma^{\mu\nu\rho}_\chi(p, q, r) = \chi \frac{i e^3}{8\pi^2} \varepsilon^\alpha\nu\beta_\rho p_\alpha q_\beta. \tag{2.65}
\]

The evaluation of the remaining vertex \(q_\mu \Gamma^{\mu\nu\rho}_\chi(p, q, r)\) follows analogously, and results in

\[
q_\mu \Gamma^{\mu\nu\rho}_\chi(p, q, r) = \chi \frac{i e^3}{8\pi^2} \varepsilon^\alpha\rho\beta_\mu p_\alpha q_\beta. \tag{2.66}
\]
The Ward identities in Eqs. (2.52-2.54) are not all fulfilled, and the theory is anomalous. The triangle diagrams are finite, but since the surface term depends on the shift of the integration variables, the results themselves depend on how we label the loop integral momenta. At the beginning of this section, we made the statement that the chiral anomaly appears in the triangle diagram due to the lack of convergence. Diagrams with more than three vertices are convergent, so shifting integration variables is therefore allowed, and all Ward identities are fulfilled for these diagrams [37].

### 2.3.4 Labelling the loop momenta

The freedom in labelling the loop momenta means that the triangle diagrams are not unique. The most general form of the chiral anomaly corresponds to the choice \( k \rightarrow k + a \) of the loop momenta in Eq. (2.51), where \( a \) is an arbitrary combination of the three loop momenta. The limit \( a = 0 \) corresponds to the labelling in the diagrams in Fig 2.4. The relevant surface term is

\[
\delta \Gamma_{\mu\nu\rho}^{\chi} = \Gamma_{\chi}^{\mu\nu\rho}(p, q, r; a) - \Gamma_{\chi}^{\mu\nu\rho}(p, q, r; 0),
\]

Corresponding to a shift of variables, \( k \rightarrow k + a \), in Eq. (2.51). The dependence on \( a \) is denoted by the additional fourth argument in \( \Gamma_{\chi}^{\mu\nu\rho}(p, q, r; a) \), where

\[
\Gamma_{\chi}^{\mu\nu\rho}(p, q, r, a) = \frac{1}{2} i \chi e^3 I_{\alpha\beta\lambda}(a) \text{tr} [\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu \gamma^\lambda \gamma^5] + (p, \mu \leftrightarrow q, \nu),
\]

In the Taylor expansion of \( I_{\alpha\beta\lambda}(a) \) all derivatives of \( I_{\alpha\beta\lambda}(0) \) vanish at the infinities, and

\[
\Delta_{\alpha\beta\lambda}(a) = I_{\alpha\beta\lambda}(a) - I_{\alpha\beta\lambda}(0) = \frac{i}{192 \pi^2} (a_\alpha g_{\beta\lambda} + a_\beta g_{\lambda\alpha} + a_\lambda g_{\alpha\beta}),
\]

so that the surface term is

\[
\delta \Gamma_{\mu\nu\rho}^{\chi} = \frac{1}{2} i \chi e^3 \Delta_{\alpha\beta\lambda}(a) \text{tr} [\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu \gamma^\lambda \gamma^5] + (p, \mu \leftrightarrow q, \nu).
\]

The identity \( g_{\alpha\beta} \gamma^\alpha \gamma^\mu \gamma^\beta = 2 \gamma^\mu \) simplifies the trace over the gamma matrices, and

\[
\delta \Gamma_{\mu\nu\rho}^{\chi} = -i \chi e^3 \frac{1}{16 \pi^2} \varepsilon^{\mu\nu\rho\alpha} a_\alpha + (p, \mu \leftrightarrow q, \nu).
\]

The only terms in \( a_\alpha \) that give nonzero contributions are those antisymmetric in the momenta \( p, q \) since \( a_\alpha \) is contracted with the Levi-Civita tensor. So \( a_\alpha = c(p - q)_\alpha \) for some constant \( c \), which is regularisation dependent, and

\[
\delta \Gamma_{\mu\nu\rho}^{\chi} = -i \chi e^3 \frac{1}{8 \pi^2} \varepsilon^{\mu\nu\rho\alpha} c(p - q)_\alpha.
\]
The chiral anomaly

The sought after contractions of $\Gamma^{\mu\nu\rho}(p, q, r; a) = \delta \Gamma^{\mu\nu\rho} + \Gamma^{\mu\nu\rho}(p, q, r; 0)$ are obtained by combining $\delta \Gamma^{\mu\nu\rho}$ with the expressions for the momentum contractions of $\Gamma^{\mu\nu\rho}(p, q, r; 0)$ in Eqs. (2.58,2.65,2.66), together with the conservation of momentum $r + p + q = 0$:

$$r_\rho \Gamma^{\mu\nu\rho}(p, q, r, a) = -i\chi \frac{e^3}{8\pi^2} (2c) \varepsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta,$$

(2.72)

$$p_\mu \Gamma^{\mu\nu\rho}(p, q, r, a) = -i\chi \frac{e^3}{8\pi^2} (1 - c) \varepsilon^{\mu\nu\rho\alpha\beta} q_\alpha r_\beta,$$

(2.73)

$$q_\nu \Gamma^{\mu\nu\rho}(p, q, r, a) = -i\chi \frac{e^3}{8\pi^2} (1 - c) \varepsilon^{\rho\mu\alpha\beta} r_\alpha p_\beta.$$

(2.74)

This means that there exists no value of $c$ which simultaneously removes the anomaly from the right hand side of all three expressions. The theory is anomalous, and the chiral anomaly depends on the value of $c$. The allowed choices for $c$ result in different forms of the anomaly, where the correct choice depends on the physical constraints of a given theory.

2.4 The consistent and covariant anomaly

There are two demands that we can ask of the anomaly to constrain the value of the constant $c$ in Eq. (2.72-2.74). The first one is to conserve the vector current to preserve the total fermion number, and the second is to insist that the currents are gauge invariant [27]. It turns out that these two demands can not be fulfilled simultaneously, but correspond to different values for the constant $c$, and to two different forms of the anomaly. The form of the anomaly which preserves charge conservation is called the consistent anomaly§ [50]. The consistent current is by definition obtained through the functional derivative of an effective action with respect to the gauge fields, and is not gauge invariant [5]. We obtain the covariant gauge invariant current by adding a Chern-Simons current to the consistent current [50]. The covariant current is also anomalous, defining the covariant anomaly.

2.4.1 The consistent anomaly

The consistent anomaly corresponds to the value $c = 1/3$ in Eq. (2.72-2.74), for which the anomaly is equally spread between the three vertices [27]. This choice stems from the definition of the consistent currents as the variation of an effective action [5]:

$$j^\mu = \frac{\delta \Gamma[A, A_5]}{\delta A_\mu}, \quad j^\mu_5 = \frac{\delta \Gamma[A, A_5]}{\delta A_{5,\mu}}. \quad (2.75)$$

§The name consistent stems from the Wess-Zumino consistency condition which describes a condition which an anomaly has to fulfil [5]
The three point function of the chiral current is therefore obtained as the variation

$$\Gamma^\mu_{\nu\rho}(x, y, z) = \frac{\delta \Gamma[A, A_5]}{\delta A_x \delta A_y \delta A_z},$$

where the chiral currents are coupled to external gauge fields $$A^x = A^{L/R}$$. Since the result is independent of the order of the derivatives, the current couples in the same way to the gauge fields at each vertex, and $$c = 1/3$$.

The real space expression of the chiral anomaly is

$$\partial_\mu j_\mu^x = \chi \frac{1}{96\pi^2} \varepsilon^{\mu\nu\alpha\beta} F^x_{\mu\nu} F^\alpha_{\beta},$$

where the field strength tensor is defined with respect to the right and left gauge fields: $$F^x_{\mu\nu} = \partial_\mu A^x_\nu - \partial_\nu A^x_\mu$$ [27]. The vector, and axial vector anomalies are obtained by summing, and subtracting the expressions for the chiral anomaly for each chirality;

$$\partial_\mu j_\mu^5 = \partial_\mu (j_\mu^L - j_\mu^R) = \chi \frac{1}{48\pi^2} \varepsilon^{\mu\nu\alpha\beta} (F^5_{\mu\nu} F^\alpha_{\beta} + F^5_{\mu\nu} F^5_{\alpha\beta}),$$

where the field strengths are defined as, $$F^x_{\mu\nu} = \partial_\mu A^x_\nu - \partial_\nu A^x_\mu$$, and $$F^5_{\mu\nu} = \partial_\mu A^5_\nu - \partial_\nu A^5_\mu$$. We made the claim that the consistent anomaly conserves the vector current, but this does not seem to be the case, as Eq. (2.78) is anomalous. The reason is that the effective action $$\Gamma[A, A_5]$$ is regularisation dependent, and Eq. (2.78) is the result of a specific choice of regularisation. The effective action $$\Gamma[A, A_5]$$ is obtained by integrating out the fermionic degrees of freedom of a Dirac action coupled to both vector and axial gauge fields. By expanding the remaining functional determinant to second order in the gauge fields results in a sum of one loop expressions for the effective action. The loop integrals are divergent and the effective action is therefore regularisation dependent. The vector anomaly, Eq. (2.78), is a consequence of choosing a specific regularisation scheme that does not conserve the vector current. Choosing a dimensional, or Pauli-Villars regularisation of the effective action always conserves the vector current [27, 38]. However, no regulation scheme can cancel the anomaly, but only shuffle it between the vector and axial vector currents.

The effective action is not gauge invariant due to the anomaly—the nonconservation of the consistent current is the result of the nonconservation of the effective action under a gauge transformation. Since the effective action is regularisation dependent we can always add local counter terms to it by hand. By local we mean that the Lagrangian only depends on one spatial coordinate. These counter terms are general nongauge invariant functions of the gauge fields to second order which take different values corresponding to different regularisation schemes.
The chiral anomaly

These terms are called Bardeen counterterms, and the most general form of the effective action in terms of these counterterms is

$$\Gamma[A, A_5] \rightarrow \Gamma[A, A_5] + \int d^4x \varepsilon^{\mu\nu\sigma\rho} A_\mu A_5^\nu (c_1 F_{\sigma\rho} + c_2 F_5^{\sigma\rho}),$$  \hspace{1cm} (2.80)

where the mixed anomaly depends on the choice of the constants $c_1$, and $c_2$. The choice $c_1 = (12\pi^2)^{-1}$, and $c_2 = 0$ corresponds to the physically relevant form:

$$\partial_\mu j_5^\mu = 0,$$  \hspace{1cm} (2.81)

$$\partial_\mu j_5^\mu = \frac{1}{16\pi^2} \varepsilon^{\mu\nu\alpha\beta} (F_{\mu\nu} F_{\alpha\beta} + \frac{1}{3} F_5^{\mu\nu} F_5^{\alpha\beta}),$$  \hspace{1cm} (2.82)

which conserves the total fermion current. The axial anomaly in expressed in terms of electromagnetic, and axial electromagnetic fields is

$$\partial_\mu j_5^\mu = \frac{1}{2\pi^2} \left( E \cdot B + \frac{1}{3} E_5 \cdot B_5 \right),$$  \hspace{1cm} (2.83)

where bold$E$, bold$B$ are electric and magnetic fields, and bold$E_5$, bold$B_5$ are axial electric and axial magnetic fields.

2.4.2 The covariant anomaly

The covariant anomaly corresponds to the value $c = 1$ in Eqs. (2.72-2.74), where the anomaly only appears at one vertex [27]. This choice stems from the gauge invariance of the covariant current which couples without an anomaly to the two external gauge fields. All the anomaly must therefore be at the remaining vertex, and

$$\partial_\mu j_5^\mu = \frac{1}{2\pi^2} \varepsilon^{\mu\nu\alpha\beta} F_5^{\mu\nu} F_5^{\alpha\beta},$$  \hspace{1cm} (2.84)

in real space. The corresponding vector and axial anomalies are,

$$\partial_\mu j_5^\mu = \frac{1}{8\pi^2} \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_5^{\alpha\beta},$$  \hspace{1cm} (2.85)

$$\partial_\mu j_5^\mu = \frac{1}{16\pi^2} \varepsilon^{\mu\nu\alpha\beta} (F_{\mu\nu} F_{\alpha\beta} + F_5^{\mu\nu} F_5^{\alpha\beta}).$$  \hspace{1cm} (2.86)

They are a factor three larger than the consistent anomaly, Eqs. (2.78,2.79) [50]. In terms of electromagnetic vector, and axial vector fields, the anomaly is

$$\partial_\mu j_5^\mu = \frac{1}{2\pi^2} (E \cdot B_5 + E_5 \cdot B),$$  \hspace{1cm} (2.87)

$$\partial_\mu j_5^\mu = \frac{1}{2\pi^2} (E \cdot B + E_5 \cdot B_5).$$  \hspace{1cm} (2.88)
The covariant currents are by definition gauge invariant, which is not the case for the consistent current, but the two currents are connected through Chern-Simons currents, $j_{cS}, j_{cS,5}$, where

\[ J^\mu = j^\mu + j_{cS}, \]
\[ J_5^\mu = j_5^\mu + j_{cS,5}. \]

The Chern-Simons currents, which are also called Bardeen polynomials, precisely cancel the gauge invariance breaking contribution from the consistent current.

In the next section we use the massless Schwinger model as an example to derive the consistent current from the effective action to explore the consistent and covariant forms of the anomaly, as well as the Bardeen counter terms, and the Bardeen polynomials.

### 2.4.3 Consistent and covariant anomalies in the Schwinger model

We derive the consistent axial anomaly in $1+1$ dimensions by considering the quantised Schwinger model coupled to external vector, and axial vector fields,

\[ Z[\bar{\Psi}, \Psi] = \int D[\bar{\Psi}, \Psi] e^{iS_D[\bar{\Psi}, \Psi, A, A_5]}, \]
\[ S_D[\bar{\Psi}, \Psi, A, A_5] = \int d^2 x \bar{\Psi} i \not{D} \Psi, \]

\[ i \not{D} = i \not{\partial} + A - \gamma^5 A_5, \]

where $e = 1$, and where we work in Minkowski space with metric signature $\text{diag}(g) = (1, -1)$. After integrating out the fermionic degrees of freedom we are left with an effective action

\[ \Gamma[A, A_5] = -i \ln \det(i \not{D}) \sim -i \text{Tr} \ln[1 + G(A - \gamma^5 A_5)], \]

where $G^{-1} = i \not{\partial}$ is the inverse Green’s function, and $\sim$ means that we have ignored the vacuum contribution $-i \text{Tr} \ln G^{-1}$. The second order expansion of the fields results in the expression

\[ \Gamma^{(2)}[A, A_5] = \frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} [A_\mu(p)A_\nu(-p)\Pi^{\mu\nu} + A_{5\mu}(p)A_{5\nu}(-p)\Pi^{\mu\nu} + 2A_\mu(p)A_{5\nu}(-p)\Pi^{\mu\nu}_5]. \]

After rationalising the propagators, $\frac{1}{k} = -\frac{\not{k}}{k^2}, k\not{k} = -k^2$,

\[ \Pi^{\mu\nu} = i \int \frac{d^2 k}{(2\pi)^2} \frac{\text{tr}[\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu]}{k^2(k+p)^2} k_\alpha(k+p)_\beta. \]
The chiral anomaly

The only difference between the diagram coupling to two vector fields and the diagram coupling to two axial vector fields is that the latter couples with a $\gamma^5$ at each vertex. But $(\gamma^5)^2 = 1$, so the two polarisation functions describing the two different diagrams are equal. By using the trace relation $\text{tr} [\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu] = 2(g^\alpha_\mu g^\beta_\nu - g^\alpha_\beta g^\mu_\nu + g^\alpha_\nu g^\mu_\beta)$ in two spacetime dimensions, the polarisation function takes the form

$$\Pi^{\mu\nu} = 2i \int \frac{d^2k}{(2\pi)^2} \frac{k^\mu (k + p)^\nu - k_\alpha (k + p)^\alpha g^{\mu\nu} + k^\nu (k + p)^\mu}{k^2 (k + p)^2}. \quad (2.97)$$

The Feynman parametrisation:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}, \quad (2.98)$$

simplifies the denominator, where the loop integral is evaluated by using dimensional regularisation, resulting in

$$\Pi^{\mu\nu} = -\frac{1}{\pi} \frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{p^2}. \quad (2.99)$$

The remaining polarisation function couples to one vector field and one axial vector field, and contains one factor of $\gamma^5$:

$$\Pi_5^{\mu\nu} = i \int \frac{d^2k}{(2\pi)^2} \text{tr} \left[ \frac{1}{k} \gamma^\mu \frac{1}{k + p} (-\gamma^5) \gamma^\nu \right] \quad (2.100)$$
$$= -i \int \frac{d^2k}{(2\pi)^2} \varepsilon^{\nu\sigma} g_{\rho\sigma} \text{tr} [\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\rho] \frac{k_\alpha (k + p)_\beta}{k^2 (k + p)^2}, \quad (2.101)$$

where $\gamma^5 \gamma^\mu = \varepsilon^{\mu\nu} \gamma^\nu$. The relation $g_{\mu\sigma} g^{\nu\sigma} = \delta^\nu_\mu$ simplifies the trace expression $\varepsilon^{\nu\sigma} g_{\rho\sigma} \text{tr} [\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\rho] = 2(\varepsilon^\nu_\beta g^{\mu\alpha} - g^{\alpha\beta} \varepsilon^\nu_\mu + \varepsilon^\alpha_\nu g^\mu_\beta)$, such that

$$\Pi_5^{\mu\nu} = -2i \int \frac{d^2k}{(2\pi)^2} \varepsilon^{\nu\beta} k^\mu (k + p)_\beta - \varepsilon^{\nu\mu} k^\beta (k + p)_\beta + \varepsilon^{\nu\alpha} k_\alpha (k + p)^\mu}{k^2 (k + p)^2}. \quad (2.102)$$

Dimensional regularisation of the loop momenta results in

$$\Pi_5^{\mu\nu} = \frac{1}{\pi} \frac{\varepsilon^{\nu\beta} (p^\mu p_\beta - \delta^\mu_\beta p^2)}{p^2}. \quad (2.103)$$

The corresponding vector and axial currents are

$$j^\mu = \frac{\delta \Gamma^{(2)} [A, A_5]}{\delta A} = -\frac{1}{\pi} \frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{p^2} A_\nu + \frac{1}{\pi} \frac{\varepsilon^{\nu\beta} (p^\mu p_\beta - \delta^\mu_\beta p^2)}{p^2} A_5^\nu, \quad (2.104)$$
$$j_5^\mu = \frac{\delta \Gamma^{(2)} [A, A_5]}{\delta A_5} = -\frac{1}{\pi} \frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{p^2} A_5^\nu + \frac{1}{\pi} \frac{\varepsilon^{\mu\beta} (p^\mu p_\beta - \delta^\mu_\beta p^2)}{p^2} A_\nu, \quad (2.105)$$
The chiral anomaly

and the axial anomaly is

\[ p_\mu j^\mu = 0, \]  
\[ p_\mu j_5^\mu = \frac{1}{\pi} \varepsilon^{\mu \nu} p_\mu A_\nu. \]  

(2.106) \hspace{1cm} (2.107)

This is the consistent form of the anomaly. The vector anomaly is conserved due to the choice of the regulator, which equals a specific form of the Bardeen counter terms at the level of the effective action. By adding the Bardeen counter term \( \Gamma_B[A, A_5] = \frac{1}{2\pi} \int \frac{d^2 p}{(2\pi)^2} \varepsilon^{\mu \nu} A_5, \mu A_\nu \) to the effective action Eq. (2.95), the anomaly is equally spread over the vector and axial vector currents

\[ p_\mu j^\mu = \frac{1}{2\pi} \varepsilon^{\mu \nu} p_\mu A_{5, \nu}, \]  
\[ p_\mu j_5^\mu = \frac{1}{2\pi} \varepsilon^{\mu \nu} p_\mu A_\nu. \]  

(2.108) \hspace{1cm} (2.109)

The consistent currents, Eq. (2.104), and Eq. (2.105) are not gauge invariant, and change as

\[ \delta_\zeta j^\mu = \delta_\zeta j_5^\mu = \delta_\kappa j_5^\mu = 0, \quad \delta_\kappa j^\mu = -\frac{1}{\pi} \varepsilon^{\mu \nu} \delta_\nu \kappa. \]  

(2.110)

under the gauge transformations \( A_\mu \rightarrow A_\mu + \delta_\zeta A_\mu \), and \( A_{5, \mu} \rightarrow A_{5, \mu} + \delta_\kappa A_{5, \mu} \), where \( \delta_\zeta A_\mu = \partial_\mu \zeta \), and \( \delta_\kappa A_{5, \mu} = \partial_\mu \kappa \). The gauge invariance is restored by adding a Chern-Simons current \( j_{CS} = \frac{1}{\pi} \varepsilon^{\mu \nu} A_{5, \nu} \) to the vector current, where the corresponding consistent currents are defined as

\[ J^\mu = j^\mu + \frac{1}{\pi} \varepsilon^{\mu \nu} A_{5, \nu}, \]  
\[ J_5^\mu = j_5^\mu. \]  

(2.111) \hspace{1cm} (2.112)

The covariant anomaly in two spacetime dimensions is therefore

\[ p_\mu j^\mu = \frac{1}{\pi} \varepsilon^{\mu \nu} p_\mu A_{5, \nu}, \]  
\[ p_\mu j_5^\mu = \frac{1}{\pi} \varepsilon^{\mu \nu} p_\mu A_\nu, \]  

(2.113) \hspace{1cm} (2.114)

which is twice as large as the consistent mixed anomaly in Eqs. (2.108,2.109). The consistent anomaly in \( d \) dimensions can always be expressed as the covariant anomaly multiplied by a factor \( (1 + d/2)^{-1} \), which corresponds to a regularisation where the consistent vector current is not conserved.

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2.5 The anomaly is in the measure

We are now ready to derive the anomaly again, but this time using Fujikawa’s path integral method. The path integral method is all about evaluating the path integral measure due to the rotational symmetries of the classical action. It is the Jacobian of the transformation that contains the chiral anomaly, as we learnt in section 2.1.3.

The path integral derivation of the axial anomaly is relatively simple compared to perturbation theory, and it also highlights the connection between the chiral anomaly and topology. Fujikawa’s path integral method [5, 44, 45, 51] evaluates the path integral measure under infinitesimal rotations. We consider a 3+1 dimensional Dirac theory coupled to an external vector field

\[ Z[A] = \int \mathcal{D}[\bar{\Psi}, \Psi] e^{S_D[\bar{\Psi}, \Psi, A]}, \]  

(2.115)

\[ S_D[\bar{\Psi}, \Psi, A] = \int d^4x \bar{\Psi} i \gamma \partial \Psi, \]  

(2.116)

\[ \gamma \partial = \partial - i A, \]  

(2.117)

expressed in Euclidean space with metric \( g^{\mu\nu} = -\delta^{\mu\nu} \), where the time components in Minkowski spacetime are connected to the fourth space component in flat space through a Wick rotation: \( ix^0 = x^4, \partial_0 = i \partial_4, i \gamma^0 = \gamma^4, \) and \( A_0 = i A_4. \) The \( \gamma \) matrices are chosen to be anti-Hermitian, \( \gamma^{\mu\dagger} = -\gamma^{\mu}, \) but the chiral matrix is still Hermitian \( \gamma^{5\dagger} = \gamma^5. \) The Dirac operator is therefore Hermitian,

\[ \gamma \partial^{\dagger} = \gamma \partial. \]  

(2.118)

The classical action transforms as

\[ S_D[\bar{\Psi}_rot^5, \Psi^5, A] = S_D[\bar{\Psi}, \Psi, A] + \int d^4x \beta(x) \partial_\mu j^\mu_5, \]  

(2.119)

to lowest order in \( \beta \) under infinitesimal local transformations,

\[ \Psi^5 = (1 + i \beta \gamma^5) \Psi(x), \]  

(2.120)

\[ \bar{\Psi}_rot^5 = \bar{\Psi}(x) \left(1 + i \beta \gamma^5\right), \]  

(2.121)

The path integral measure transforms as \( \mathcal{D}[\bar{\Psi}_rot^5, \Psi^5] = \mathcal{D}[\bar{\Psi}, \Psi] \mathcal{J}, \) where \( \mathcal{J} \) is the Jacobian determinant of the transformation to first order in \( \beta(x). \) Since the transformed fields are integration variables of the path integral the variation of the path integral is zero, \( Z[A, \beta] - Z[A] = \delta Z, \) and

\[ Z[A, \beta] = \int \mathcal{D}[\bar{\Psi}, \Psi] \mathcal{J} e^{S_D[\bar{\Psi}_rot^5, \Psi^5, A]}, \]  

(2.122)

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The Jacobian equals \[ J = e^{-\int d^4x \beta(x) A_5}, \] where \( A_5 \) is the axial anomaly. To first order in \( \beta(x) \) the path integral is

\[
Z[A, \beta] = \int D[\bar{\Psi}, \Psi] e^{S_D[1 + \beta(x)[\partial_\mu j_5^\mu - A_5]}.
\]

(2.123)

Since the functional variation with respect to \( \beta(x) \) must vanish,

\[
\delta Z[A, \beta] \Big|_{\beta=0} = \int D[\bar{\Psi}, \Psi] e^{S_D[\partial_\mu j_5^\mu - A_5]} \equiv 0,
\]

(2.124)

which breaks the classical conservation law for the axial vector current,

\[
\partial_\mu j_5^\mu = A_5.
\]

(2.125)

Our task is to evaluate \( A_5 \).

We consider the complete set of orthonormal eigenfunctions of the Dirac operator,

\[
\mathcal{D} \varphi_n(x) = \lambda_n \varphi_n(x),
\]

(2.126)

for real eigenvalues \( \lambda_n \), where

\[
\int d^4x \varphi_1^\dagger(x) \varphi_m(x) = \delta_{nm}, \quad \sum_n \varphi_1^\dagger(y) \varphi_n(x) = \delta(x-y).
\]

(2.127)

The Dirac spinors are expanded in terms of the eigenfunctions, \( \varphi_n(x) \),

\[
\Psi(x) = \sum_n a_n \varphi_n(x) = \sum_n \langle x | \varphi_n \rangle a_n,
\]

\[
\bar{\Psi}(x) = \sum_n \bar{\varphi}_n^\dagger(x) \bar{b}_n = \sum_n \langle \varphi_n | x \rangle \bar{b}_n,
\]

(2.128)

(2.129)

where \( a_n \) and \( \bar{b}_n \) are independent anticommuting Grassmann coefficients. In terms of the Grassmann coefficients the path integral measure becomes \([44]\)

\[
D\bar{\Psi} D\Psi \propto \prod_n da_n \bar{d}b_n.
\]

(2.130)

The proportionality constant is \([\det(|m|) \det(|n|)]^{-1}\), which does not play a role in the derivation of the anomaly. The chiral transformation of the fields,

\[
\Psi_5^{\text{rot}} = e^{i\gamma^5 \beta(x)} \Psi(x) \sim (1 + i\beta \gamma^5) \sum_n a_n \varphi_n(x) = \sum_n \varphi_n a_n^{\text{rot},5},
\]

\[
\bar{\Psi}_5^{\text{rot}} = \bar{\Psi}(x) e^{i\gamma^5 \beta(x)} \sim \sum_n \bar{b}_n \bar{\varphi}_n^\dagger(x) (1 + i\beta \gamma^5) = \sum_n \bar{b}_n^{\text{rot},5} \bar{\varphi}_m^\dagger(x),
\]

(2.131)

(2.132)
The chiral anomaly

leads to the transformation of the Grassman variables

\[ a_{n}^{\text{rot}} = C_{nm}^{5} a_{m}, \quad C_{nm}^{5} = \delta_{nm} + i \int d^{4} x \beta(x) \varphi_{n}^{\dagger}(x) \gamma^{5} \varphi_{m}(x), \quad (2.133) \]

\[ \bar{b}_{n}^{\text{rot}} = \bar{b}_{m} C_{nm}^{5}, \quad (2.134) \]

where we have used the orthonormality of the field \( \varphi_{n}(x) \). The path integral measure therefore transforms with Jacobian determinant \( J[\beta] = \det(C)^{-1}, \)

\[ \mathcal{D}\bar{\Psi}_{\text{rot}}^{5} \mathcal{D}\Psi_{\text{rot}}^{5} = J[\beta] \mathcal{D}\bar{\Psi} \mathcal{D}\Psi. \quad (2.135) \]

By using the identity \( \det C = \exp(\text{tr} \ln C) \) to expand the logarithm to first order in \( \beta(x) \), the Jacobian determinant becomes,

\[ J[\beta] = \exp\left[ -\int d^{4} x \beta(x) A_{5} \right], \quad (2.136) \]

\[ A_{5} = 2i \sum_{n} \varphi_{n}^{\dagger}(x) \gamma^{5} \varphi_{n}(x). \quad (2.137) \]

The sum \( A_{5} = \delta(0) \text{tr} \gamma^{5} \) is not well defined and is regularised by using a Gaussian cut off \( e^{-\frac{\lambda^{2}}{M^{2}}} \),

\[ A_{5} = \lim_{M \to \infty} 2i \sum_{n} \varphi_{n}^{\dagger}(x) \gamma^{5} e^{-\frac{\mu^{2}}{M^{2}}} \varphi_{n}(x), \quad (2.138) \]

favouring contributions from small eigenvalues \( \lambda \) of the Dirac operator. By using the completeness of the eigenfunctions to express \( \sum_{n} \xi_{n}^{\dagger}(\ell) \Gamma \xi_{n}(k) = \text{tr} \delta(\ell - k) \), for a diagonal operator \( \Gamma \), The Fourier transform of \( A_{5} \) is

\[ A_{5} = \lim_{M \to \infty} 2i \text{tr} \int \frac{d^{4} k}{(2\pi)^{4}} e^{-ikx} \gamma^{5} e^{-\frac{\mu^{2}}{M^{2}}} e^{ikx}, \quad (2.139) \]

where \( \varphi_{n}(x) = \int \frac{d^{4} k}{(2\pi)^{4}} e^{ikx} \varphi_{n}(k) \). To simplify Eq. (2.139) we rescale the momenta \( k_{\mu} \to k_{\mu}/M^{2} \), and use the relation

\[ D^{2} = D_{\mu} D^{\mu} + \frac{1}{4} [\gamma^{\mu}, \gamma^{\nu}] F_{\mu\nu}, \quad (2.140) \]

together with the differential operator shift \( e^{-ikx} f(\partial_{\mu}) e^{ikx} = f(\partial_{\mu} + ik_{\mu}) \), so that

\[ A_{5} = \lim_{M \to \infty} 2M^{4} i \text{tr} \int \frac{d^{4} k}{(2\pi)^{4}} e^{k_{\mu} k^{\mu}} \text{tr} \gamma^{5} e^{-\frac{2ik_{\mu} D^{\mu}_{\mu}}{M^{2}}} - \frac{D_{\mu} D^{\mu}}{M^{2}} - \frac{\gamma^{\mu} \gamma^{\nu}}{2M^{2}} F_{\mu\nu}. \quad (2.141) \]

The Taylor expansion of the exponential leaves only one nonzero term in the limit \( M \to \infty \), which is the term quadratic in \( \gamma^{\mu} \gamma^{\nu} F_{\mu\nu} \). This follows from the
trace identities $\text{tr} \gamma^5 = \text{tr}[\gamma^5 \gamma^\mu \gamma^\nu] = 0$, and $\text{tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = -4 \varepsilon^{\mu\nu\rho\sigma}$, where $\varepsilon^{1234} = 1$, and

$$A_5 = \frac{1}{2!} \frac{2i}{4} \int \frac{d^4k}{(2\pi)^4} e^{k\mu k\nu} \text{tr}[\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] F_{\mu\nu} F_{\rho\sigma}. \quad (2.142)$$

The momentum space integral is a Gaussian integral equal to $\pi^2$, where $k\mu k\nu = -k\mu k\nu \delta^{\mu\nu}$, and

$$A_5 = -i \frac{1}{16 \pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad (2.143)$$

which is the axial anomaly in the Euclidean signature. The Minkowski space equivalent is obtained by removing the factor $-i$, due to the Minkowski space expression for the trace, $\text{tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4i \varepsilon^{\mu\nu\rho\sigma}$. The vector current is still conserved; since the Jacobian for infinitesimal vector rotations of the fields is unity—without axial fields the vector current is always conserved.

### 2.5.1 The covariant anomaly—including the axial vector field

The axial vector field, $A_{5,\mu}$, breaks the Hermiticity of the Dirac operator due to the anticommutation relation $\{\gamma^\mu, \gamma^5\} = 0$, where

$$\slashed{D} = \gamma^\mu (\partial_\mu - iA_\mu - i\gamma^5 A_{5,\mu}),$$

$$\slashed{D}^\dagger = -\gamma^\mu (\partial_\mu + iA_\mu - i\gamma^5 A_{5,\mu}),$$

such that $\slashed{D}^\dagger [A, A_5] = \slashed{D} [A, -A_5]$. The physical theory is Hermitian as the full action remains Hermitian. By defining the Hermitian Laplace operator $\slashed{D} \slashed{D}^\dagger$, the axial anomaly is derived analogously to the $A_{5,\mu} = 0$ case. The orthonormal eigenbasis of the Laplace operator is

$$\slashed{D} \slashed{D}^\dagger |\phi_n\rangle = \lambda_n^2 |\phi_n\rangle,$$

$$\slashed{D}^\dagger |\phi_n\rangle = \lambda_n^* |\phi_n\rangle,$$

where

$$\int d^4x \eta_n^\dagger(x) \eta_n(x) = \delta_{nm}, \quad \sum_n \eta_n^\dagger(y) \eta_n(x) = \delta(x-y), \quad (2.148)$$

for $\{|\eta\rangle\} \subset \{|\varphi\rangle\}, \{|\phi\rangle\}$. The fermionic fields in terms of the eigenbases are

$$\Psi(x) = \sum_n a_n \varphi_n(x) = \sum_n a_n \langle x | \varphi_n\rangle,$$

$$\bar{\Psi}(x) = \sum_n \bar{\phi}_n^\dagger(x) \bar{b}_n = \sum_n \langle \phi_n | x \rangle \bar{b}_n,$$
The chiral anomaly

where \( a_n \) and \( \bar{b}_n \) are independent Grassmann coefficients. The path integral measure is again expressed as a product of the Grassmann variables, \( D\bar{\Psi}D\Psi = \prod_n a_n\bar{b}_n \). The rotated Grassman fields are \( a_n^{rot, 5} = C_{nm}^5 a_m \), and \( \bar{b}_n^{rot, 5} = \bar{b}_m D_{nm}^5 \), analogously to Eqs. (2.131, 2.132), where

\[
C_{nm}^5 = \delta_{nm} + i \int d^4x \beta(x) \varphi_n^\dagger(x) \gamma^5 \varphi_m(x),
\]

\[
D_{nm}^5 = \delta_{nm} + i \int d^4x \beta(x) \phi_m^\dagger(x) \gamma^5 \phi_n(x).
\]

The rotated path integral measure is now \( D\bar{\Psi}^{5, rot} D\Psi^{5, rot} = J[\beta] D\bar{\Psi} D\Psi \), where the Jacobian is \( J[\beta] = (\det(C) \det(D))^{-1} \). To first order in \( \beta(x) \)

\[
J[\beta] = \exp \left[ -\int d^4x \beta(x) A_5 \right],
\]

\[
A_5 = i \sum_n \varphi_n^\dagger(x) \gamma^5 \varphi_n(x) + i \sum_n \phi_n^\dagger(x) \gamma^5 \phi_n(x).
\]

The sum is regularised with a Gaussian cut off \( \exp(-\lambda^2/M^2) \), in the limit \( M \to \infty \) where \( \lambda \), and \( \lambda^* \) are the eigenvalues of the Laplacian operators in Eq. (2.146),

\[
A_5 = \lim_{M \to \infty} i \sum_n \left[ \varphi_n^\dagger(x) \gamma^5 e^{\frac{-\beta^\dagger \beta}{M^2}} \varphi_n(x) + \phi_n^\dagger(x) \gamma^5 e^{\frac{-\beta^\dagger \beta}{M^2}} \phi_n(x) \right],
\]

\[
= \lim_{M \to \infty} i \int \frac{d^4k}{(2\pi)^4} \text{tr} e^{-i k x} \gamma^5 \left[ e^{\frac{-\beta^\dagger \beta}{M^2}} + e^{\frac{-\beta^\dagger \beta}{M^2}} \right] e^{i k x}.
\]

Rather than continuing in analogy to the previous section, we rewrite the Dirac operators in terms of the chiral operators

\[
\slashed{D}_\pm = \slashed{\partial} - iA_\mu^{L/R},
\]

where the chiral fields are \( A_\mu^{L/R} = A_\mu \pm A_{5,\mu} \), as defined in Eq.(2.34), and Eq.(2.35). By using the chiral projector \( P_\pm = (1 \pm \gamma^5)/2 \), together with the identities

\[
P_+(\slashed{A} + \slashed{A}_5 \gamma^5) = \slashed{A}^R P_-, \quad P_-(\slashed{A} + \slashed{A}_5 \gamma^5) = \slashed{A}^L P_+,
\]

the Dirac operator, and its Hermitian conjugate are separated into chiral vector fields:

\[
\slashed{D} = \slashed{\partial} - i(P_+ + P_-)(\slashed{A} + \slashed{A}_5 \gamma^5) = \slashed{\partial} - i\slashed{A}_L P_- - i\slashed{A}_R P_+,
\]

\[
\slashed{D}^\dagger = \slashed{\partial} - i(P_+ + P_-)(\slashed{A} - \slashed{A}_5 \gamma^5) = \slashed{\partial} - i\slashed{A}_L P_- - i\slashed{A}_R P_+.
\]
The chiral anomaly

The Laplacians in Eq.(2.164) are decomposed into the chiral Dirac fields,

\[ \mathcal{D}_+ \mathcal{D}_+^\dagger = \mathcal{D}_+^2 P_+ + \mathcal{D}_-^2 P_+ , \quad (2.161) \]

\[ \mathcal{D}_+ \mathcal{D}_- = \mathcal{D}_+^2 P_+ + \mathcal{D}_-^2 P_- , \quad (2.162) \]

\[ (P_+ + P_-) \mathcal{D}_+ \mathcal{D}_+^\dagger + (P_+ + P_-) \mathcal{D}_- \mathcal{D}_-^\dagger = \mathcal{D}_+^2 + \mathcal{D}_-^2 , \quad (2.163) \]

and the axial anomaly is

\[ \mathcal{A}_5 = \lim_{M \to \infty} \frac{i}{16} \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \gamma^5 \left[ e^{-\frac{\mathcal{D}_+^2}{M^2}} + e^{-\frac{\mathcal{D}_-^2}{M^2}} \right] e^{ikx} . \quad (2.164) \]

Eq. (2.164) is of the same form as Eq. (2.139), and results in the axial anomaly

\[ \mathcal{A}_5 = -i \frac{1}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} (F_{\mu\nu}^L F_{\rho\sigma}^L + F_{\mu\nu}^R F_{\rho\sigma}^R) . \quad (2.165) \]

The Minkowski space expression in terms of vector and axial vector fields is

\[ \partial_\mu j_5^\mu = \frac{1}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} (F_{\mu\nu} F_{\rho\sigma} + F_5^{\mu\nu} F_5^{\rho\sigma}) . \quad (2.166) \]

This is the covariant form of the axial anomaly.

### 2.5.2 The covariant vector anomaly

The covariant anomaly also breaks the classical conservation laws of the vector current under local rotations of the spinor fields:

\[ \Psi_{\text{rot}}(x) = e^{-ik(x)} \Psi(x) , \quad \bar{\Psi}_{\text{rot}}(x) = \bar{\Psi}(x) e^{ik(x)} , \quad (2.167) \]

where the Jacobian matrix of the transformation of the path integral measure is equal to \( J = [\det C \det D]^{-1} = \exp \left[ - \int d^4 x k(x) A \right] \). \( C_{nm} \) and \( D_{nm} \) are the transformation matrices of the Grassmann fields:

\[ a_{n}^{\text{rot}} = C_{nm} a_{m} , \quad C_{nm} = \delta_{nm} + i \int d^4 x k(x) \varphi_n^\dagger(x) \varphi_m(x) , \quad (2.168) \]

\[ \bar{b}_{n}^{\text{rot}} = \bar{b}_{m} D_{nm} , \quad D_{nm} = \delta_{nm} - i \int d^4 x k(x) \phi_n^\dagger(x) \phi_m(x) , \quad (2.169) \]

such that the vector anomaly is

\[ A = i \sum_n \varphi_n^\dagger(x) \varphi_n(x) + i \sum_n \phi_n^\dagger(x) \phi_n(x) \]

\[ = \lim_{M \to \infty} \frac{i}{16} \int \frac{d^4 k}{(2\pi)^4} \text{tr} e^{-ikx} \left[ e^{-\frac{\varphi_n^\dagger \varphi_n}{M^2}} - e^{-\frac{\phi_n^\dagger \phi_n}{M^2}} \right] e^{ikx} . \quad (2.170) \]
The chiral anomaly

The only difference between $A$ in Eq. (2.171), and $A_5$ in Eq. (2.156), is the sign difference between the exponential operators, and that the $\gamma^5$ matrix only appears in $A_5$. By expressing the Dirac operators in terms of chiral Dirac operators,

$$\Psi \Psi^\dagger - \Psi^\dagger \Psi = (P_+ - P_-) \slashed{D}_+^2 + (P_+ - P_-) \slashed{D}_-^2 = \gamma^5 (\slashed{D}_+^2 - \slashed{D}_-^2),$$

(2.172)

and following the derivation of $A_5$, leads to the vector anomaly

$$A = \lim_{M \to \infty} \int \frac{d^4 k}{(2\pi)^4} \text{tr} e^{-ikx} \gamma^5 \left[ e^{-\frac{\bar{\psi}^2}{2M^2}} - e^{-\frac{\psi^2}{2M^2}} \right] e^{ikx},$$

(2.173)

$$= -i \frac{1}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} (F^{L\mu}_\nu F^{L\rho}_\sigma - F^{R\mu}_\nu F^{R\rho}_\sigma),$$

(2.174)

The vector anomaly in Minkowski space is

$$\partial_\mu j^\mu = \frac{1}{8\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}^5,$$

(2.175)

which is the covariant form of the vector anomaly.

We have derived the chiral anomaly in terms of vector and axial vector fields using both a diagrammatic approach, and a path integral approach. Both methods require some care to not miss the existence of the anomaly, and both methods rely on perturbation theory. But both methods yield the same expression for the anomaly, even though we have relied on different perturbative methods. The reason is that the chiral anomaly is exact to one loop in perturbation theory, which as we will learn next, is a consequence of its topological nature.

2.6 The chiral anomaly and topology

The chiral anomaly is one loop exact [37,52,53]. By coupling the free theory to external fields one needs to evaluate the anomaly for higher order loops, which could give corrections to the anomaly. But all higher order loops are zero, which is known as the nonrenormalisation of the anomaly [37,52,53]. The idea of the diagrammatic proof [52] is that the higher order loops contain more fermion propagators and are highly convergent. This means that we can freely shift the integration variables of the fermion loop momenta to ensure that the Ward-Takahashi identities are fulfilled. Another proof of the nonrenormalisation of the anomaly is due to topology. The axial anomaly is a topological object determined by a quantity called the index [4,5], this means that it can not be changed by adiabatic corrections.

2.6.1 The index of the Dirac operator

If $\varphi_n(x)$ is an eigenvector of the Hermitian Dirac operator, Eq. (2.117), with eigenvalue $\lambda_n$, then $\gamma^5 \varphi_n$ is also an eigenvector of the Dirac operator with eigenvalue
The chiral anomaly

\(-\lambda_n\), since

\[ \mathcal{D} \varphi_n = \lambda_n \varphi_n \implies \mathcal{D} \gamma^5 \varphi_n = -\lambda_n \gamma^5 \varphi_n, \]

(2.176)

where the anticommutation relation \( \{\gamma^5, \gamma^\mu\} = 0 \) is responsible for the minus sign. The eigenvectors \( \varphi_n \) and \( \gamma^5 \varphi_n \) are orthogonal for \( \lambda_n \neq 0 \):

\[ (\varphi_n, \gamma^5 \varphi_n) = \int d^4x \varphi_n^\dagger(x) \gamma^5 \varphi_n(x) = 0, \]

(2.177)

which follows from the Hermiticity of the Dirac operator where,

\[ \lambda_n(\gamma^5 \varphi_n, \varphi_n) = (\gamma^5 \varphi_n, \mathcal{D} \varphi_n) = (\mathcal{D}^\dagger \gamma^5 \varphi_n, \varphi_n) = -\lambda_n(\gamma^5 \varphi_n, \varphi_n). \]

(2.178)

The zero modes are degenerate as \( \varphi_n \) and \( \gamma^5 \varphi_n \) both have eigenvalue zero, which is also the case for the chiral eigenvectors \( \varphi_{0_n}^\pm = P_\pm \varphi_{0_n} \), where \( P_\pm = (\mathbb{1} \pm \gamma^5)/2 \).

The Jacobian of the path integral measure, Eq. (2.136), contains the axial anomaly \( A_5 \),

\[ \mathcal{J}[\beta] = \exp \left[ -\int d^4x \beta(x) A_5 \right], \]

(2.179)

\[ A_5 = 2i \sum_n \varphi_n^\dagger(x) \gamma^5 \varphi_n(x) = -\frac{2i}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \]

(2.180)

The integral over the anomaly for a constant variation \( \beta \) is proportional to an integer, where only the zero modes contribute due to the orthogonality of all other eigenvalues:

\[ 2i \int d^4x \sum_n \varphi_n^\dagger(x) \gamma^5 \varphi_n(x) = 2i \sum_n \varphi_n^{0\dagger}(x) \gamma^5 \varphi_n^0(x) \]

\[ = 2i \sum_n \varphi_n^{0\dagger}(x)(P_+ + P_-) \gamma^5(P_+ + P_-) \varphi_n^0(x). \]

(2.181)

By using that \( P_\pm \gamma^5 = \pm P_\pm \), \( P_+ P_- = 0 \) we are left with

\[ \int d^4x \sum_n \varphi_n^\dagger(x) \gamma^5 \varphi_n(x) = \sum_n \int d^4x (\varphi_n^{0\dagger} \varphi_n^0 - \varphi_n^{0\dagger} \varphi_n^{0-}) = n_+ - n_-, \]

(2.182)

which counts the number of zero modes of positive and negative chirality. The integral over the axial anomaly is therefore proportional to an integer; \( \int d^4x \partial_\mu j_5^\mu = 2\pi(n_+ - n_-) \). We have considered four spacetime dimensions, but this argument is the same in any even spacetime dimension.

The quantity \( n_+ - n_- \) is equal to the index of the chiral Dirac operator \( D_+ = \mathcal{D} P_+ \), where index \( D_+ = n_+ - n_- \) [4,5]. We will define the index of an operator with as few definitions as possible, a complete discussion is given in [4,5].
**Bounded** A linear operator $T : E \to F$, where $E, F$ are Hilbert spaces with inner product $\|x\| = \sqrt{(x, x)}$, is bounded if
\[
\|T\| = \sup \frac{\|Tx\|}{\|x\|} < \infty \quad x \neq 0.
\]

**Kernel** The Kernel of $T$ is
\[
\ker T = \{ e \in E | Te = 0 \}.
\]

**Adjoint operator** The adjoint operator $T^* : E \leftarrow F$ is adjoint with respect to the inner product
\[
(f, Te) = (T^*f, e) \quad \text{for} \quad e \in E, \; f \in F.
\]

**Fredholm operator** A linear operator $T \in \text{Fred}(E, F)$ is a Fredholm operator:
\[
\text{if} \; T : E \to F \; \text{where} \; T \text{is bounded}
\quad \dim \ker T < \infty, \text{and} \quad \dim \ker T^* < \infty.
\]

**Index** The index of $T \in \text{Fred}(E, F)$ is defined as
\[
\text{index } T = \dim \ker T - \dim \ker T^*.
\]

The relevant operator for us is the Dirac operator, which is a differential operator. The Dirac operator is a Fredholm operator [4,5], but since it is self-adjoint it has a zero index. Instead, we consider the chiral Dirac operators:
\[
D_+ = \slashed{D} P_+, \quad D_- = \slashed{D} P_-;
\]

with adjoints
\[
D_+^\dagger = \slashed{D} P_- = D_-, \quad D_- = \slashed{D} P_+ = D_+.
\]

We define the space of chiral spinors $S_+ = \{ \varphi_+ \}$, and $S_- = \{ \varphi_- \}$ such that the chiral Dirac operators act as
\[
S_+ \xleftrightarrow{D_+} D_+ S_-, \quad (2.185)
\]
which is true since
\[
D_+ \varphi_+ = \slashed{D} P_+ P_+ \varphi = \slashed{D} P_+ \varphi = P_- \slashed{D} \varphi = \lambda P_- \varphi = \lambda \varphi_- \in S_-,
\]
\[
D_- \varphi_- = \slashed{D} P_- P_- \varphi = \slashed{D} P_- \varphi = P_+ \slashed{D} \varphi = \lambda P_+ \varphi = \lambda \varphi_+ \in S_+.
\]

This means that the index of $D_+$ is precisely what we have already defined it to be, namely the difference between zero modes of positive and negative chirality:
\[
\text{index } D_+ = \dim \ker D_+ - \dim \ker D_+^\dagger
= \dim \ker D_+ - \dim \ker D_-
= n_+ - n_-.
\]

The chiral anomaly

\[\text{index } D_+ = \dim \ker D_+ - \dim \ker D_+^\dagger
= \dim \ker D_+ - \dim \ker D_-
= n_+ - n_-.
\]

(2.188)
The chiral anomaly

The above argument carries over to the case of a non-Hermitian Dirac operator, like the one considered in section 2.5.1. The definition of the index remains the same, but our Hermitian operators are the Laplacians $\mathcal{D}^\dagger \mathcal{D}$, and $\mathcal{D}\mathcal{D}^\dagger$, with the following eigenvalue relations:

$$\begin{align*}
\mathcal{D}\mathcal{D}^\dagger |\phi_n⟩ &= \lambda^2_n |\varphi_n⟩, \\
\mathcal{D}^\dagger |\phi_n⟩ &= \lambda^*_{-n} |\varphi_n⟩, \\
\mathcal{D}^\dagger |\varphi_n⟩ &= \lambda_{-n} |\phi_n⟩.
\end{align*}$$

The integral over the axial vector anomaly is:

$$\int d^4 x \, A_5 = \int d^4 x \, \sum_n (\varphi_n(x)\gamma^5 \varphi_n(x) + \phi_n(x)\gamma^5 \phi_n(x)).$$

(2.191)

It is only the zero modes which contribute to the sums since the two inner products in Eq. (2.191) cancel for all higher order eigenvalues:

$$\lambda^2_n (\gamma^5 \varphi_n, \varphi_n) = (\gamma^5 \varphi_n, \mathcal{D}^\dagger \mathcal{D} \varphi_n) = -(\gamma^5 \mathcal{D} \varphi_n, \mathcal{D} \varphi_n) = -\lambda^2 (\gamma^5 \varphi_n, \phi_n).$$

(2.192)

By projecting the remaining zero modes into chiral zero modes we again find that the integral over the anomaly results in an integer,

$$\int d^4 x \, A_5 = \sum_n \int d^4 x \, \left[ (\varphi^0_+ \varphi^0_+ - \varphi^0_- \varphi^0_-) + (\phi^0_+ \phi^0_+ - \phi^0_- \phi^0_-) \right]$$

$$= (n^\varphi_+ - n^\varphi_-) + (n^\phi_+ - n^\phi_-).$$

(2.193)

The zero modes of the Laplacian operators are equal to the zero modes of the non-Hermitian Dirac operator and its adjoint. Since $\ker \mathcal{D}^\dagger \mathcal{D} = \{\varphi | \mathcal{D}^\dagger \mathcal{D} \varphi = 0\}$, and $\ker \mathcal{D} = \{\varphi | \mathcal{D} \varphi = 0\}$ we have that $\mathcal{D} \varphi = 0 \implies \mathcal{D}^\dagger \mathcal{D} \varphi = 0$. And if $\mathcal{D}^\dagger \mathcal{D} \varphi = 0$ then $0 = (\mathcal{D}^\dagger \mathcal{D} \varphi, \varphi) = (\mathcal{D} \varphi, \mathcal{D} \varphi) \implies \mathcal{D} \varphi = 0$. The same argument holds for the Laplacian $\mathcal{D}^\dagger \mathcal{D}^\dagger$, and

$$\begin{align*}
\ker \mathcal{D}^\dagger \mathcal{D} &= \ker \mathcal{D}, \\
\ker \mathcal{D}^\dagger \mathcal{D}^\dagger &= \ker \mathcal{D}^\dagger.
\end{align*}$$

(2.194)

(2.195)

We can therefore define the chiral operators $D_\pm = \mathcal{D} P_\pm$, and $D^\dagger_\pm = \mathcal{D}^\dagger P_\pm$, and use the same argument as for the Hermitian Dirac operator to find that

$$\int d^4 x \, A_5 = i \left( \text{index } D_+ + \text{index } D^\dagger_+ \right).$$

(2.196)

The index corresponds to the integral over an even $d = 2n$ dimensional manifold of the Chern character:

$$\text{index } D_+ = \int_{\mathcal{M}_{2n}} \text{ch}(F) = \frac{1}{n!} \left( \frac{i}{2\pi} \right)^n \int_{\mathcal{M}_{2n}} \text{tr } F^n,$$

(2.197)

42
where $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$. This result is known as the Atiyah-Singer index theorem. The index is a topological invariant, the $n$th Chern number. We will encounter the first Chern number, often denoted simply the Chern number in the next chapter, and discuss the Chern character in some more detail in chapter four.

2.7 Summary

We have derived the chiral anomaly in two different ways by using both the diagrammatic method, and the path integral method. The anomaly appears as a consequence of divergent loop integrals in the evaluation of triangle Feynman diagrams. The diagrams are finite but undetermined due to the freedom in labelling the loop momenta. This freedom means that there is a choice in how to distribute the anomaly between the three vertices in the triangle diagram, which defines the covariant and consistent form of the anomaly. The path integral derivation of the chiral anomaly is a derivation of the Jacobian matrix of the transformation of the path integral measure. This method reproduces the results from the diagrammatic evaluation of the chiral anomaly, but it also provides a language through which to connect the chiral anomaly with the topological index. Topology is a global property, and so is the axial anomaly, and this is the reason why we cannot get rid of it through the addition of local counter terms. In chapter three we will describe how the chiral anomaly in 1+1 dimensions appears in the edge theory of the integer quantum Hall effect described by chiral fermions. The left handed and the right handed fermions live on opposite edges of the two dimensional plane, which demonstrates the nonlocality of the chiral anomaly in a physical setting.

The reason that we derived the chiral anomaly in detail is because it gives us an understanding of what the anomaly is, and how it breaks the classical conservation laws. The derivation also serves a purpose in explaining how the chiral anomaly was overlooked historically due to the subtleties in evaluating divergent integrals. The theory of chiral anomalies forms the basis for the research in papers I-III, concerning both Weyl semimetals and a non-Hermitian Dirac action. The detailed calculations are especially important in paper III which explores the non-Hermitian chiral anomaly for fermions with complex velocities coupled to non-Hermitian gauge fields. The formulation of the Fujikawa method is based on the Hermitian Dirac operator or the Hermitian Laplacian, so considering a non-Hermitian action complicates the derivation of the anomaly. In paper III we show that the Non-Hermitian action leads to new terms in the chiral anomaly in both 1+1, and 3+1 dimensions.

The Axial anomaly plays an important role in the theory of topological matter where it appears as an even (spacetime) dimensional edge theory connected to a bulk theory in one higher dimension. The integer quantum Hall effect is a prominent example of this; it consists of an anomalous edge theory described by chiral fermions. We will describe the integer quantum Hall effect in chapter three. The
Axial anomaly is responsible for the response theory of Weyl semimetals, which we will explore in chapters three and four. We saw that the chiral anomaly may depend on axial gauge fields, which appear as emergent fields in Weyl semimetals. This means that one can consider both the consistent and the covariant forms of the anomaly in Weyl semimetals, which we will discuss in chapter four. This concludes our exploration of the chiral, and axial anomalies for now, but we will return to it in the following chapters.
Chapter 3

**Topological phases of matter**

The discovery of the quantum Hall effect \([7–9]\) in the early 1980s initiated the field of topological quantum matter. Since that first measurement of the quantised Hall resistance, topological physics has become a broad field of research, involving researchers from many different disciplines \([3]\). The focus of this thesis is on symmetry protected topological insulators and superconductors which are characterised by their metallic edge states \([10,11,54,55]\), and on Weyl semimetals \([18,56–58]\), three dimensional structures described by chiral fermions at low energies. Symmetry protected topological phases and Weyl semimetals are both defined in terms of noninteracting fermions and can be understood through topological band theory in crystalline structures. These phases of matter have been explored both in theory and experiment, and it is predicted that more than 50 per cent of all nonmagnetic crystalline materials are topological, catalogued in databases of topological materials \([59–63]\).

Topology relies on the notion of adiabatic change, where two objects are topologically equivalent if they can be transformed into one another without the use of scissors or glue \([4]\). Topological equivalence of strong symmetry protected topological phases of matter combines the notion of adiabaticity \([64]\) with local symmetries. Two gapped Hamiltonians are topologically equivalent if their band structures can be smoothly connected without ever closing the band gap, whilst preserving the symmetries of the Hamiltonian \([65]\). The relevant symmetries are time reversal symmetry, particle hole symmetry, and their combination, the chiral symmetry. These three symmetries can be combined in ten different ways forming the tenfold classification of free fermion Hamiltonians \([66–68]\). The topological classification of free fermion Hamiltonians further describes which symmetry classes in a given dimension can host a topological phase \([17,65,69–71]\). The topology is characterised by a quantised topological invariant which depends on the symmetry class and the dimension of the respective Hamiltonian \([17,54,65,69,72]\). Topological invariants are readily defined in translation invariant structures described by Bloch functions, but topology itself does not rely on translation invariance. Strong topological phases of matter play a role in papers IV and V which explore the characterisation of strong topological phases beyond crystalline symmetries in terms of local topological markers.

Weyl semimetals are three dimensional topological semimetals where the band
touching points, referred to as Weyl nodes, are robust against perturbations [18, 56]. The Weyl nodes are not symmetry protected, but they are topological in that they are sources and sinks of Berry curvature, acting as magnetic monopoles in reciprocal space [15]. The low energy description of Weyl semimetals is given by the Weyl Hamiltonian [42], from which the semimetal receives its name [56]. The solutions to the Weyl Hamiltonian are conically dispersing chiral fermions, and although chiral fermions do not exist as particles in the standard model, they do appear as emergent quasiparticles in Weyl semimetals. Weyl nodes always appear in pairs of opposite chirality on the lattice [73, 74], and their location on the lattice is dictated by symmetry. For the Weyl semimetal phase to exist, either time reversal symmetry, inversion symmetry, or both symmetries must be broken. If time reversal symmetry is broken the Weyl nodes are separated in momentum, while a broken parity symmetry separates the nodes in energy. Weyl semimetals have metallic Fermi arcs surface states. By projecting the two Weyl nodes of opposite chirality onto the two dimensional surface of the sample it turns out that the Fermi surface on this surface forms an arc, a Fermi arc for momenta between the nodes. The existence of chiral fermions in a condensed matter setting makes Weyl semimetals a platform in which to explore the chiral anomaly [30, 31, 49], which is the main object of study in papers I and II.

This chapter is intended as a general introduction to strong topological phases of matter and their classification, with a specific focus on the two dimensional Chern insulator characterised by the Chern number. The Chern insulator is described by the massless Dirac equation in 2+1 dimensions, and many results carry over to the theory of Weyl semimetals, which can be understood as a stacking of Chern insulators in momentum space. This chapter delves into the field theory description of Weyl semimetals and their electromagnetic response properties, building the foundation for the research on domain walls in Weyl semimetals presented in papers I and II.

Figure 3.1: A two dimensional electron gas subject to a magnetic field out of the plane. The electrons move in cyclotron orbits. The electrons on the edge can not complete the full circle as they bounce on the boundary, and instead form skipping orbits along the edge.
Topological phases of matter

3.1 Topological insulators and superconductors

The integer Quantum Hall effect [7–9] provided the first measurement of an exactly quantised observable. The set up consists of electrons confined in a two dimensional plane in a perpendicular magnetic field. In the semiclassical picture, the electrons move in cyclotron orbits and are localised in the bulk. These orbits are disrupted by collisions with the boundaries, forming a string of skipping orbits on the edges of the sample moving in opposite directions along the opposing edges—the edge states are chiral, as depicted in Fig. 3.1. We consider a two dimensional plane in the $xy$ directions with a magnetic field in the $z$ direction. By sending a current $I_x$ in the $x$ direction a voltage $V_y$ is built up, giving rise to the Hall conductance

$$\sigma_{xy} \equiv \frac{I_x}{V_y} = \frac{e^2}{h} \nu. \quad (3.1)$$

$\nu$ is an integer corresponding to the number of filled bands in the Landau level picture [3]. The physical set up is depicted in Fig. 3.2. The degenerate energy levels are given by $E_\nu = \hbar \omega_c (|\nu| + 1/2)$, where $\omega_c$ is the classical cyclotron frequency. The Landau levels in the bulk are flat bands, and the spectrum is gapped, with $|\nu|$ filled Landau levels. By introducing boundaries to the sample modelled by a potential wall, the energy spectrum is modified by bending up across the Fermi level, resulting in $|\nu|$ number of chiral edge modes of opposite chirality at the two boundaries. The induced Hall current is carried by these chiral edge states. The integer $\nu$, is a topological bulk invariant, a Chern number, and the connection between the Hall conductivity and the chiral edge states is a manifestation of the

Figure 3.2: Integer quantum Hall effect: A magnetic field in the $z$ direction, and a current in the $x$ direction leads to a voltage drop in the $y$ direction.
bulk boundary correspondence in topological insulators.

The Chern insulator is another topological insulator in two dimensions with chiral edge states and a bulk topological phase characterised by the Chern number \[ \left. \right| 75 \right]. The Chern insulator realises the quantum Hall effect in the absence of a magnetic field and is an example of the anomalous quantum Hall effect. To understand the topological properties of the Chern insulator, and topological insulators and superconductors in general, it is useful to first introduce the adiabatic theorem and the geometric phase [64].

3.1.1 Adiabatic transformations and the geometric phase

The description of topological band structures depends on the adiabatic theorem. The theorem states that an instantaneous eigenstate \( |\psi\rangle \) with energy \( E \) stays in the same eigenstate at any later time \( t \), as long as the eigenenergy is separated from any other eigenenergy by a gap \( \Delta \), for a slow enough time evolution [64]. The Hamiltonian \( H(t) = H[R(t)] \) depends on a \( D \) dimensional time dependent parameter \( R \), with instantaneous eigenstates \( |n(R(t_0))\rangle \) and eigenenergies \( E_n(R(t_0)) \) at time \( t = t_0 \). At a later time \( t \) the time evolved state is in a superposition, \( |\psi_n(t)\rangle = \sum_n c_n(t)|n(R(t))\rangle e^{-i\phi_n^D(t)} \), where \( \phi_n^D(t) = -i \int_{t_0}^{t} dt' E_n(t')/\hbar \) is the dynamic phase. The dynamics of the coefficients, \( \dot{c}_n = -c_n \langle n| \frac{d}{dt} |n\rangle - \sum_{m \neq n} c_m \frac{\langle n| \frac{dH}{dt} |n\rangle}{E_m - E_n} e^{i\phi_n^D(t)} \), (3.2)

result from the Schrödinger equation of the time evolved state. At this point we evoke the adiabatic theorem: In the adiabatic limit, we assume that \( E_m - E_n \geq \Delta \), and that \( \frac{dH}{dt} \) can be arbitrarily small [76]. This implies that the second term in Eq. (3.2) vanishes, so that the time evolved state is proportional to the initial state \( c_n(t) = c_n(t_0) \exp[i\gamma_n(t)] \). The phase \( \gamma_n(t) \) is called a geometric phase since it only depends on the path traversed in the parameter space:

\[ \gamma_n(t) = \frac{i}{\hbar} \int_{t_0}^{t} \langle n(R(t'))| \frac{d}{dt'} \langle n(R(t'))|dt' = \int_{R(t_0)}^{R(t)} A_n(R) \cdot dR. \] (3.3)

The integrand of the rightmost integral,

\( A_n(R) = i \langle n(R)| \frac{d}{dR} |n(R)\rangle \),

is the Berry connection which is a gauge dependent quantity. A transformation \( |n(R(t))\rangle \rightarrow e^{i\lambda(R)} |n(R(t))\rangle \) of the state alters the Berry connection to \( A_n(R) \rightarrow A_n(R) - \frac{d}{dR} \lambda(R) \). The geometric phase therefore transforms as

\[ \gamma_n(t) \rightarrow \gamma_n(t) + \lambda[R(t_0)] - \lambda[R(t)], \] (3.5)
Topological phases of matter

and is only gauge invariant for closed paths where \( \mathbf{R}(t = t_0) = \mathbf{R}(t = t_0 + T) \). When the geometric phase factor is unity \( |n(\mathbf{R}(t_0))| = |n(\mathbf{R}(t_0 + T))| \), leaving the geometric phase invariant modulo \( 2\pi \), where \( \lambda(\mathbf{R}(t_0 + T)) = \lambda(\mathbf{R}(t_0)) + 2\pi c \) for an integer \( c \in \mathbb{Z} \). The geometric phase is commonly referred to as the Berry phase after Michael Berry’s contribution to the physics of the geometrical phase [64]. But the geometric phase, as most theories and discoveries, cannot be attributed to one single person. Michael Berry made a point of this during a lecture and asked the audience to either call the phase using a descriptive nomenclature suggesting the name the geometric phase or name it after everyone who in some way have contributed to the knowledge of the phase, suggesting the open ended name: Hamilton, Lloyd, Bortolotti, Rytov, Vladimirskii, Pancharatnam, Longuet-Higgins-Pryce-Öpik-Sack, Aharonov-Bohm, Budden-Smith, Mead-Truhlar, Berry, Simon, Hannay, Wilczek-Zee, Aharonov-Anandan...phase [77]. Berry’s name is also associated with the Berry connection, and the Berry curvature which we will introduce next, and we will use these names to comply with the contemporary literature on the subject.

For a closed path \( \partial S \) we can use Stokes theorem to define the geometric phase as the integral over any two dimensional surface \( S \),

\[
\gamma_n(t) = i \oint_{\partial S} \mathbf{A}_n(\mathbf{R}) \cdot d\mathbf{R} = i \int_S \Omega_{\mu\nu}^n dS^\mu \wedge dS^\nu. 
\]

The quantity

\[
\Omega_{\mu\nu}^n = \partial_{\mu} A_{\nu}^n(\mathbf{R}) - \partial_{\nu} A_{\mu}^n(\mathbf{R}),
\]

is called the Berry curvature, and \( dS^\mu \wedge dS^\nu \) defines an area element on \( S \). In a three dimensional parameter space, the geometric phase is

\[
\gamma_n = \int_S \Omega_n \cdot d\mathbf{S},
\]

with Berry curvature \( \Omega_n = \nabla \times \mathbf{A}_n \), where \( d\mathbf{S} = \mathbf{\hat{n}} dS \) for a normal vector \( \mathbf{\hat{n}} \) of the surface element \( dS \). The integral of the Berry curvature over a closed two dimensional manifold equals an integer multiple of \( 2\pi \),

\[
\int_S \Omega_n \cdot d\mathbf{S} = 2\pi c.
\]

This result is called the Chern theorem, where \( c \) is the Chern number [15]. The Chern number defines the topology of both the two dimensional Chern insulator and the three dimensional Weyl semimetal. We will also encounter the Chern number in chapter five when we introduce the local Chern marker, which is interpreted as the Fourier transform of the Berry curvature.

Previously, in the introduction, we described topologically equivalent objects through stretching and squeezing, with no scissors or glue allowed, which relates
to the notion of adiabatic change. The adiabatic theorem is important beyond
the geometric phase and the Chern number and forms a part of the definition of
topologically equivalent insulators and superconductors. Topological insulators
and superconductors are symmetry protected topological phases, and adiabaticity
must be considered alongside the symmetries protecting the topology.

### 3.1.2 Three important symmetries

Wigner’s theorem states that there are only two types of symmetries, unitary
symmetries, and antiunitary symmetries [78, 79]. Topological insulators and su-
perconductors are described by noninteracting fermion Hamiltonians constrained
by antiunitary symmetries and their product. The relevant symmetries are time
reversal symmetry and particle hole symmetry, which are defined by antiunitary
operators, and chiral symmetry, defined as the combination of the time rever-
sal and the particle hole symmetries. All strong topological phases are classified
according to these symmetry constraints in each given dimension.

We consider the classification of free fermion Hamiltonians in the language
of first quantised Hamiltonians. The second quantised Hamiltonian is defined in
terms of the first quantised Hamiltonian as

\[ \hat{H} = \sum_{\alpha, \beta} \Psi_\alpha^\dagger H_{\alpha \beta} \Psi_\beta, \]

where \{\Psi^\dagger\} are either fermion creation operators (nonsuperconducting systems), or Bogolobov-
de-Gennes creation operators (superconducting systems), and \( H \) is the first quanti-
sed, single particle, Hamiltonian. A unitary symmetry \( U \) leaves the first quanti-
sed Hamiltonian \( H \) invariant under the transformation \( U^\dagger H U = H \). Unitary
operators do not enter the topological classification since one can always choose a
basis which block diagonalises the Hamiltonian reducing the problem to the study
of each separate block. Repeating the procedure for each unitary symmetry on
each block results in an irreducible set of blocks of the Hamiltonian. After using
up all unitary symmetry of the Hamiltonian we are left with a classification of
antiunitary symmetries of the irreducible blocks. The properties of the remaining
symmetries are summarised as follows:

- **Time reversal symmetry** \( T \):

  The time reversal operator is defined as \( T = U_T K \), where \( U_T \) is a unitary
  operator and \( K \) is complex conjugation. The square of the time reversal
  operator is

  \[ T^2 = \pm 1. \]  

  (3.10)

  The first quantised Hamiltonian \( H \) is invariant under time reversal symme-
  try if

  \[ T H T^{-1} = H. \]  

  (3.11)
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- **Particle hole symmetry** $P$: The particle hole operator is defined as $P = U_P K$, where $U_P$ is a unitary operator and $K$ is complex conjugation. The square of the particle hole operator is

$$P^2 = \pm 1. \quad (3.12)$$

The first quantised Hamiltonian $H$ is invariant under particle hole symmetry if

$$PHP^{-1} = -P. \quad (3.13)$$

- **Chiral symmetry** $C$

The chiral symmetry operator is defined as the product of the time-reversal and the particle-hole symmetries, $C = TP$, such that $C = U_T K U_P K = U_T U_P^* U_C$. The chiral symmetry squares to

$$C^2 = 1. \quad (3.14)$$

The first quantised Hamiltonian $H$ is invariant under chiral symmetry if

$$CHC^{-1} = -C. \quad (3.15)$$

Although the chiral symmetry is unitary in the first quantised language, it anticommutes with the Hamiltonian and does not behave like an ordinary unitary symmetry.

The three symmetries, $T$, $P$, and $C$, can be combined in ten different ways forming ten different symmetry classes. There are three possibilities for time reversal symmetry: The Hamiltonian either lack time reversal symmetry, or is invariant under time reversal symmetry squaring to either one or minus one. There are similarly three different possibilities for the particle hole symmetry, which combined with the three possibilities for the time reversal symmetry leads to $3 \cdot 3 = 9$ possible combinations. The chiral symmetry is fully determined whenever the Hamiltonian is invariant under either time reversal symmetry or particle hole symmetry, or both, which covers eight of the nine possible combinations of these two symmetries. But if the Hamiltonian is not invariant under time reversal symmetry nor particle hole symmetry, it might still be invariant under chiral symmetry, such that the total number of combinations of the three symmetries is $(3 \cdot 3 - 1) + 2 = 10$.

These ten possible combinations of $T$, $P$, and $C$ form the Altland-Zirnbauer symmetry classification of first quantised Hamiltonians [66–68] shown in Table 3.1. The symmetry classes are labelled with respect to Cartan’s historic labels for the classification of symmetric spaces. The first quantised Hamiltonian is represented by a finite square matrix $H$ on the lattice. It turns out that the time evolution
Topological phases of matter

Table 3.1: The Altland-Zirnbauer \[67,68\] classification of noninteracting fermionic Hamiltonians with regards to time reversal symmetry $T$, particle hole symmetry $P$, and chiral symmetry $S$ labelled by the Cartan labels of symmetric spaces \[66\]. The absence of a symmetry is indicated by $0$. $\pm 1$ means that the symmetry is present, and the sign represents the value of the operator squared. The last column lists the symmetric spaces to which the time evolution operator obtained from the free fermion Hamiltonian in the corresponding symmetry class belongs to \[72\].

<table>
<thead>
<tr>
<th>Cartan label</th>
<th>T</th>
<th>P</th>
<th>S</th>
<th>Time evolution operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$U(N)$</td>
</tr>
<tr>
<td>AIII</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$U(N + M)/U(N) \times U(M)$</td>
</tr>
<tr>
<td>AⅠ</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>$U(N)/O(N)$</td>
</tr>
<tr>
<td>BDI</td>
<td>+1</td>
<td>+1</td>
<td>1</td>
<td>$O(N + M)/O(N) \times O(M)$</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$SO(2N)$</td>
</tr>
<tr>
<td>DIII</td>
<td>−1</td>
<td>+1</td>
<td>1</td>
<td>$SO(2N)/U(N)$</td>
</tr>
<tr>
<td>AⅡ</td>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>$U(2N)/sp(2N)$</td>
</tr>
<tr>
<td>CI</td>
<td>−1</td>
<td>−1</td>
<td>0</td>
<td>$sp(N + M)/sp(N) \times sp(M)$</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>−1</td>
<td>0</td>
<td>$sp(2N)$</td>
</tr>
<tr>
<td>CI</td>
<td>+1</td>
<td>−1</td>
<td>0</td>
<td>$sp(2N)/U(N)$</td>
</tr>
</tbody>
</table>

operator $U = \exp(itH)$ is an element of a group which corresponds to one of the ten symmetric spaces depending on the symmetry constraints of the Hamiltonian. These symmetric spaces are expressed in the column labelled Hamiltonian in Table 3.1.

3.1.3 Classification of topological insulators and superconductors

Free fermion topological insulators and superconductors are described by gapped Hamiltonians which belong to one of the ten Altland-Zirnbauer symmetry classes. The classification of topological insulators and superconductors can be understood through seemingly different means, through K-theory \[65\], Anderson localisation \[17,69,70\], and quantum anomalies \[71\]. The topological classification is presented in Table 3.2 for one, two, and three spatial dimensions. The nonzero entries in Table 3.2 mean that the free fermion Hamiltonian supports topological phases where the topology is characterised by topological invariants which are either $\mathbb{Z}$, $2\mathbb{Z}$, or $\mathbb{Z}_2$ valued.

The K-theory classification relies on translation invariance where the eigenvalue equation of the bulk Hamiltonian can be expressed in momentum space. By adiabatically transforming the energy spectrum the Hamiltonian can be expressed in a band flattened form $Q(k) = 1 - 2P(k)$ with eigenvalues $E = \pm 1$, where $P(k)$ is projector onto the occupied bands \[54,69\]. We assume that all negative eigenvalue states are filled, which represents the ground state of the Hamiltonian. We consider symmetry class A where $Q(k)$ does not have any symmetries, and diag-
Table 3.2: The Altland-Zirnbauer classification of free fermion Hamiltonians with regards to time reversal symmetry $T$, particle hole symmetry $P$, and chiral symmetry $S$ labelled by the Cartan labels of symmetric spaces [66]. The absence of a symmetry is indicated by 0, and $\pm 1$ indicates that the symmetry is present. The sign represents the value of the operator squared. $d$ is the spatial dimension of the Hamiltonian. A nonzero entry in the columns labelled by dimension implies the presence of a topological phase defined by a $\mathbb{Z}$, $\mathbb{Z}_2$, or $\mathbb{Z}_2$ topological invariant.

<table>
<thead>
<tr>
<th>Cartan label</th>
<th>T</th>
<th>P</th>
<th>S</th>
<th>d=1</th>
<th>d=2</th>
<th>d=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AI</td>
<td>$\pm 1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BDI</td>
<td>$\pm 1$</td>
<td>$\pm 1$</td>
<td>1</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>$\pm 1$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
</tr>
<tr>
<td>DIII</td>
<td>$-1$</td>
<td>$\pm 1$</td>
<td>1</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>AII</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>CII</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
<td>$2\mathbb{Z}$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>$2\mathbb{Z}$</td>
<td>0</td>
</tr>
<tr>
<td>CI</td>
<td>$\pm 1$</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$2\mathbb{Z}$</td>
</tr>
</tbody>
</table>

Table 3.2: The Altland-Zirnbauer classification of free fermion Hamiltonians with regards to time reversal symmetry $T$, particle hole symmetry $P$, and chiral symmetry $S$ labelled by the Cartan labels of symmetric spaces [66]. The absence of a symmetry is indicated by 0, and $\pm 1$ indicates that the symmetry is present. The sign represents the value of the operator squared. $d$ is the spatial dimension of the Hamiltonian. A nonzero entry in the columns labelled by dimension implies the presence of a topological phase defined by a $\mathbb{Z}$, $\mathbb{Z}_2$, or $\mathbb{Z}_2$ topological invariant.

To diagonalise the band flattened Hamiltonian,

$$Q(k) = U(k) \Lambda U(k)^\dagger,$$

and

$$\Lambda = \text{diag}(1_{m \times m}, -1_{n \times n}).$$  \hspace{1cm} (3.16)\n
$\Lambda$ is a diagonal matrix with $m$ eigenvalues equal to 1, and $n$ eigenvalues equal to $-1$, and $U(k) \in U(m+n)$ is an $m+n$ dimensional unitary matrix where the columns are formed by the eigenvectors of $Q(k)$. The band flattened Hamiltonian is diagonal, $Q(k) = \Lambda$ when $U(k) = 1_{m+n \times m+n}$, and for the block diagonal matrix $U(k) = \text{diag}[U_m(k), U_n(k)]$, where $U_m(k) \in U(m)$, and $U_n(k) \in U(n)$. Conversely $Q(k)$ is not diagonal when $U(k)$ is a nontrivial element in the coset space $G_{m,m+n}(\mathbb{C}) = U(m+n)/U(m) \times U(n)$. $G_{m,m+n}(\mathbb{C})$ is called the complex Grassmannian [4,72,76]. Every ground state of the band flattened Hamiltonian is described by a map $k \rightarrow Q(k)$ from the Brillouin zone to the complex Grassmannian. The topological classification counts the number of different ground states which can not be adiabatically deformed to each other. This number of topologically different ground states is given by the Homotopy group, $g = \pi_d(G_{m,m+n}(\mathbb{C}))$ in the continuous limit, where $g$ is the group of inequivalent maps between the sphere $S^d$ and $G_{m,m+n}(\mathbb{C})$ [4,72,76]. The homotopy group $\pi_2(G_{m,m+n}(\mathbb{C}))$ in two dimensions is the group of integers $\mathbb{Z}$, as depicted in Table. 3.2 for symmetry class A in two dimensions. This means that the topological invariants in class A in two spatial dimensions are integer valued—each integer is connected to a set of topologically equivalent ground states, so ground states defined by different integers are topologically distinct.
This procedure can be extended to all symmetry classes, but the symmetry constraints on the band flattened Hamiltonian complicate the analysis [72]. K-theory solves this problem and provides a classification for all ten symmetry classes [65]. The K-theory classification deals with topologically equivalent vector bundles which have a large enough fibre dimension [65]. A vector bundle consists of a base manifold with a fibre at each point on the manifold. We consider a translation invariant structure where the base manifold is the periodic momentum space. The relevant quantity is the image of the projector onto occupied bands which constitutes a vector space at each momenta—this vector space is the fibre [4,72,76]. Since the addition of any topologically trivial bands does not affect the topology of a model, the classification should remain invariant as the bundle dimension changes due to trivial bands. K-theory takes care of this, and this is exactly what the clause large enough fibre dimension refers to [4,65,72,76].

The classification with respect to Anderson localisation classifies the $d-1$ dimensional edge of $d$ dimensional topological insulators and superconductors [17,69,70]. Anderson localisation means that spatially extended eigenstates of the Hamiltonian become localised under sufficiently large disorder [80]. The metallic edge states of topological insulators and superconductors are topologically protected by local symmetries. Disorder does not affect the topology of the edge states: The topological phase is not protected by translation invariance so the topology remains unaltered when translation symmetry is broken. The fact that topological edge states evade Anderson localisation contrary to the edge states of topologically trivial insulators implies that the phenomena acts as a signature of topology. The starting point is the boundary of a Hamiltonian in one of the ten symmetry classes subject to a disorder potential. The effective long wavelength description of Anderson localisation physics of these boundaries is given by non-linear sigma-models at length scales much larger than the mean free path [81–84]. These non-linear sigma models are scalar field theories, where the values of the scalar fields belong to one of the ten symmetric spaces. The field theory evades Anderson localisation if there exists a topological term that belongs to the relevant target space and boundary dimension that can be added to the non-linear sigma model [17,69,70]. The topological insulators and superconductors are therefore classified in terms of these topological terms, where the topology is defined by the homotopy group of the target space [17,69,70,72].

The boundary classification through non-linear sigma models is related to the classification in terms of quantum anomalies [71]. All topological insulators and superconductors are associated with a massive Dirac Hamiltonian in the same symmetry class [17,85]. The topological phase of all insulators and superconductors can therefore be classified through the corresponding Dirac Hamiltonian. The classification is achieved by coupling the Dirac Hamiltonian to a classical background field preserving the continuous symmetries of the Hamiltonian, or if
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the Hamiltonian does not contain any continuous symmetries\(^8\), by coupling the
theory to gravitational fields so that it lives on a curved spacetime [71, 72]. The
topology appears in the quantised response function of the effective field theory af-
fter integrating out the fermionic degrees of freedom. In odd spacetime dimensions
the response functions are given by the topological Chern-Simons terms, which
are proportional to \(\mathbb{Z}\) valued topological invariants [71, 72]. In even spacetime
dimensions they are given by the \(\mathbb{Z}_2\) invariant topological \(\theta\) terms [71, 72].

The presence of these topological bulk terms in \(d\) dimensions implies the exis-
tence of an anomaly associated with the \(d-1\) dimensional edge. The Chern-Simons
terms are only gauge invariant on manifolds without a boundary [4]. This means
that the gauge invariance of the full theory is restored by a contribution from the
boundary. The nonzero variation of the boundary indicates that the boundary is
anomalous, and can not exist in isolation of the bulk. In the next section, we show
that the topological response of the 2+1 dimensional bulk in class A is a Chern-
Simons term which contains a parity anomaly cancelled by the contribution from
its 1+1 dimensional chiral edge. The connection between quantum anomalies in
\(d-1\) dimensions and Chern-Simons terms in \(d\) dimensions [86–88] is referred to
as anomaly inflow [89].

The \(\theta\) term in even spacetime dimensions is constrained by either time reversal
or particle hole symmetry, where the corresponding boundary theory is defined
by a Chern-Simons term [54]. This boundary Chern-Simons term is not a \(\mathbb{Z}\)
invariant but takes on noninteger values not allowed by the odd dimensional bulk
theories [4,72]. This again suggests that the boundary theory can not exist as a
theory of its own, but must be associated with a bulk.

We have not discussed the specific form of the topological invariants in the
different symmetry classes for different dimensions. The important classes for
our purposes are class A in two spatial dimensions, and class AII and DIII in
three spatial dimensions. Class AII in three spatial dimensions is a topological
insulator phase characterised by the topological \(\theta\) term, whilst class DIII in three
spatial dimensions is a topological superconductor phase characterised by an in-
teger winding number. We will explore these two symmetry classes in more detail
in chapter five in the context of local topological markers. The next section is
dedicated to the topology of the Chern insulator phase in class A. The focus on
the Chern insulator, especially the field theory description is useful beyond the
theory of Chern insulators. Our main purpose in the coming sections is to intro-
duce Weyl semimetals which can be thought of as a stacking of Chern insulators
in momentum space. We will see that this stacking is apparent in both the bulk
and the edge theory of the Weyl semimetal.

\(^8\)This is the case for certain superconducting phases of matter [71, 72].
3.2 The Chern insulator

Chern insulators are two dimensional bulk insulators hosting chiral edge states. The low energy two band model of the Chern insulator is described by the Dirac action in 2+1 dimensions [3]

$$S = \int d^3x \overline{\Psi} [i \partial - m(x)] \Psi,$$

(3.17)

with metric signature $(1,-1,-1)$, where $\partial = \gamma^\mu \partial_\mu$, $m(x)$ is the Dirac mass, and where the partial derivative is scaled to include the Fermi velocity $\nu_F$, $\partial_\mu = (\partial_0, \nu_F \partial_i)$. The $\gamma$-matrices are represented by Pauli matrices in the basis

$$\gamma^0 = \sigma_z, \quad \gamma^1 = -i\sigma_x, \quad \gamma^2 = -i\sigma_y,$$

(3.18)

In this basis the momentum space Hamiltonian with energy $E(k)$ is

$$H_D = k_x \sigma_y - k_y \sigma_x + m \sigma_z, \quad E(k) = \pm \sqrt{|k|^2 + m^2}.$$  

(3.19)

The Dirac mass breaks the time reversal invariance of the Hamiltonian, where the time reversal operator is defined as $T = \sigma_y K$. A zero mass term $m = 0$ closes the spectral gap and defines a boundary between a topological insulator phase and a trivial insulator phase characterised by a mass term of the opposite sign. Our aim is to derive the topological edge states which are located at the boundary. We will find that these edge states are chiral, which means that the electrons can only move in one given direction along the wall, just as the chiral edge states of the integer quantum Hall effect. To derive the edge states we consider a space dependent mass term of the form $m(x) = \sigma m_0 \tanh(x/\lambda)$, which acts as a continuous domain wall between the trivial and the topological phase. Here $m_0 > 0$ is a positive constant and $\lambda$ is the width of the domain wall. $\sigma = \pm 1$ is called the chiral charge, and we will find that $\sigma$ is directly related to the chirality of the edge states. The solution of the eigenvalue equation $H_D \psi = E \psi$ for the states bound to the domain wall with energies in the mass gap $-m_0 < E < m_0$, is a chiral edge state

$$\psi_\sigma(x,y) = \left( \begin{array}{c} 1 \\ -\sigma \end{array} \right) \frac{1}{N \sqrt{2}} e^{-\sigma \int_0^x m(x') dx'} e^{ik_y y},$$

(3.20)

where $N$ is the norm of $e^{-\sigma \int_0^x m(x') dx'}$. The chiral edge state disperses linearly in momentum where $E_\sigma = \sigma k_y$ [90]. The chiral states are confined to the one dimensional edge of the insulators and are proportional to the eigenvectors of $\sigma_x$ with eigenvalues $\pm 1$. The Pauli matrix $\sigma_x$ represents the chiral element $\gamma^5$ in a given basis of the 1+1 dimensional edge theory. The chiral edge states are therefore a manifestation of the chiral anomaly in 1+1 dimensions [91]. The chirality of the domain wall determines the chirality of the edge states, analogously
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to how the magnetic field direction in the integer quantum Hall effect decides the chirality.

The number of chiral edge states on the boundary of a topological insulator is a topological invariant \( n = |n_+ - n_-| \) counting the number of chiral edge states with chirality \( \pm 1 \). \( n \) is also equal to the bulk Chern number, through the bulk boundary correspondence \([3]\). The Chern number Eq. (3.9) is evaluated for the negative energy eigenstates of the Dirac Hamiltonian with a constant mass \( m \) deep in the bulk, for which the Berry curvature, Eq. (3.4), is found to be \( \Omega = (m/2E^3)e_z \).

The Berry curvature is integrated over the manifold \( \mathbb{R}^2 \), resulting in the Chern number

\[
\begin{align*}
    c &= \frac{m}{4\pi} \int_0^{2\pi} d\theta \int_0^{\infty} dk \frac{k}{(k^2 + m^2)^{3/2}} = \frac{\text{sgn}(m)}{2}.
\end{align*}
\]

We know that the Chern number is an integer, but Eq. (3.21) is equal to \( \pm 1/2 \). By focusing on the low energy model we have lost information about the band structure at higher energies \([47,49]\). The contribution to the Chern number at higher energies, would shift the Chern numbers of the two signs of the mass term to two consecutive integer values corresponding to a topological and a trivial phase.

3.2.1 Coupling to electromagnetism—the quantum Hall effect

The Chern insulator is an example of the anomalous quantum Hall effect with a quantised Hall conductivity proportional to the Chern number. The Hall current is the linear response of the Chern insulator subject to an external electric field. We will derive the Hall conductivity using a field theory description of the effective electromagnetic theory. In the field theory description, the external fields enter as electromagnetic gauge fields, and the response theory is obtained from the effective theory in terms of these gauge fields. The effective response theory of the Chern insulator is described by the Chern-Simons action

\[
\Gamma_{\text{CS}} = \frac{k}{2\pi} \int \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho,
\]

where \( k \) is a model dependent constant, \( \varepsilon^{\mu\nu\rho} \) is the completely antisymmetric Levi-Civita symbol, and \( A_\mu \) are external electromagnetic gauge fields \([3,92]\). The Chern-Simons action is topological, all the indices are contracted with the Levi-Civita symbol so it does not depend on a metric, and is independent of the geometry of spacetime. Parity in two spatial dimensions is defined as the coordinate transformation \( x \rightarrow -x, y \rightarrow y \), (changing the sign on both coordinates defines a rotation). This convention leads to the parity transformation of the gauge fields \( A_1 \rightarrow -A_1, A_2 \rightarrow A_2, \) and \( A_0 \rightarrow -A_0 \). So the Chern-Simons term is odd under parity transforms: \( \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \rightarrow -\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \).
The Chern-Simons action is an effective theory obtained from the Dirac action in 2+1 dimensions,

\[
S_D = \int d^3x \bar{\Psi} (\slashed{D} + m) \Psi,
\]

(3.23)

minimally coupled to an external gauge field, \( D_\mu = \partial_\mu - iA_\mu \). Here \( e = 1 \), and \( \partial_\mu = (\partial_0, \nu F \partial_1) \). \( S_D \) is expressed in Euclidean space§ with metric signature \(-\delta_{\mu\nu}\) for \( \mu \in (0, 1, 2) \). The \( \gamma \) matrices are represented by the Pauli matrices, \( \gamma^\mu = \sigma^\mu \), where \( \sigma_0 \) is the identity matrix. After integrating out the fermionic degrees of freedom the remaining effective action \( \Gamma(A) = -\ln \det (\slashed{D} + m) \) is, by use of the operator identity \( \ln \det \hat{O} = \text{tr} \ln \hat{O} \), expanded to second order in the gauge fields,

\[
\Gamma^{(2)}[A] = -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} A_\mu (-p) \Pi^{\mu\nu} A_\nu(p),
\]

(3.24)

\[
\Pi^{\mu\nu} = \int \frac{d^3k}{(2\pi)^3} \text{tr} (G(k)\gamma^\mu G(k + p)\gamma^\nu).
\]

(3.25)

Here the vacuum polarisation function \( \Pi^{\mu\nu} \) is expressed in terms of the fermion propagator \( G(k) = \frac{1}{i(k + m)} = \frac{i_k - m}{k^2 + m^2} \). The only terms that contribute to the Chern-Simons action are those proportional to the trace of three gamma matrices \( \text{tr}[\gamma^\mu \gamma^\nu \gamma^\rho] = 2i\varepsilon^{\mu\nu\rho} \). These are the only terms that are both parity odd and proportional to the Levi-Civita symbol. By using a Feynman parameterisation (Eq. (2.98)) and changing variables to \( k_\mu = \ell_\mu - (1 - x)p_\mu \) the parity odd part of \( \Pi^{\mu\nu} \) is evaluated as

\[
\Pi^{\mu\nu}_{\text{odd}} = 2m \int_0^1 dx \int \frac{d^3\ell}{(2\pi)^3} \frac{1}{(\ell^2 + x(1 - x)p^2 + m^2)^2}
\]

(3.26)

\[
= \frac{m}{4\pi} \int_0^1 dx \frac{1}{\sqrt{x(1 - x)p^2 + m^2}}
\]

(3.27)

\[
= \frac{m}{2|p|\pi} \arctan \left( \frac{|p|}{2|m|} \right).
\]

(3.28)

The Dirac theory of the Chern insulator is valid in the low energy limit, so the relevant theory is obtained in the limit of a large mass, and small momenta \( p \), since \( m \) regulates the size of the bulk energy gap. \( \Pi^{\mu\nu}_{\text{odd}} \approx \frac{1}{4\pi} \text{sgn}(m) \) in this limit. The Minkowski space Chern-Simons action Fourier transformed to real space is

\[
\Gamma_{CS} = -\frac{1}{4\pi} \int d^3x \frac{\text{sgn}(m(x))}{2} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho,
\]

(3.29)

§The Dirac Hamiltonian, Eq. 3.19 is expressed in Minkowski space. The operators are connected by a Wick rotation between Euclidean, labelled by E, and Minkowski, labelled by M, space: \( x_0 \to -i\tau, D_0^M \to iD_0^E, \gamma_M = -i\gamma_0^E \), and the other operators remain the same.
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where the $x$ dependence is restored, and the partial derivatives have been rescaled not to include the Fermi velocity. The constant mass approximation, where the $x$ dependence is restored at the end of the derivation, is only valid deep in the bulk of the material. The approximation breaks down in the vicinity of the edge, which means that the bound states must be added by hand. The constant mass approximation is more accurate the steeper the domain wall is since the position dependent region is smaller.

The current is defined as the variation of the Chern-Simons action with respect to the gauge fields,

$$ j^\mu = \frac{\delta \Gamma_{CS}}{\delta A^\mu} = -\frac{c}{4\pi} \varepsilon^{\mu\nu\rho} F_{\nu\rho}, $$

where $c = \text{sgn}(m)/2$ is the Chern number. In terms of the electric fields the current is given by

$$ j^x = \frac{e^2}{\hbar} E_y, \quad j^y = -\frac{e^2}{\hbar} E_x, $$

where we have reintroduced $\hbar = h/2\pi$, and the charge $e$. We recognise the currents as the Hall current presented in section 3.1, with a quantised conductivity proportional to the Chern number. We previously explained that the Chern number in the Hall conductivity of the integer quantum Hall effect corresponds to the number of chiral edge states. The same is true for the Chern insulator, and in the next section, we will describe this bulk boundary correspondence in terms of gauge invariance.

3.2.2 Bulk boundary correspondence

The Chern-Simons action is anomalous in that it is only gauge invariant up to a boundary term. The anomaly is cancelled by the contribution from the chiral edge states through the bulk boundary correspondence, and it is only the total theory of bulk and edge combined which is gauge invariant. The variation of the Chern-Simons action under the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu f(x)$ is

$$ \delta \Gamma_{CS} = -\frac{1}{8\pi^2} \int d^3x \, \varepsilon^{\mu\nu\rho} [\partial_\mu (\text{sgn}(m(x)) f(x) \partial_\nu A_\rho) - f(\partial_\mu \text{sgn}(m))(\partial_\nu A_\rho)], $$

where the total derivative vanishes since $f(x)$ goes to zero at $x \rightarrow \pm \infty$. The theory is therefore gauge invariant as long as the mass term is constant. Since the mass changes sign at the boundary $x = 0$ we can express the sign of the mass term as $\text{sgn}(m) = \sigma \text{sgn}(x)$. The derivative of the sign-functions is $\partial_\mu (\text{sgn}(m(x)) = 2\delta(x)$, and the variation of the Chern-Simons action is

$$ \delta \Gamma_{CS} = \frac{\sigma}{4\pi^2} \int d^2x \, f(x) \varepsilon^{\mu\nu} \partial_\mu A_\nu, $$

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where the index $\mu \in (0, 2)$.

The effective 1+1 dimensional edge action is obtained by inserting the chiral edge states Eq. (3.20) into the Dirac action, Eq. (3.17). The edge action coupled to external gauge fields is therefore

$$S_{b, \sigma}[\bar{\psi}, \psi] = \int d^2x \varphi^\dagger(t, y)[iD_0 + \sigma iD_2]\varphi(t, y), \quad (3.34)$$

where the $x$ dependence is integrated out as $\int dx N^{-2} e^{\sigma^2 \int_0^L m(x') dx'} = 1$, and where $\varphi(x_0, x_2) = e^{ik_y x_y} \phi(t)$ for some unspecified time dependence of the system. The gauge is partially fixed to depend only on the spacetime coordinates on the edge; $A_\mu = A_\mu(t, y), \mu \in (0, 2)$. By using the Wick rotation $t \rightarrow -i\tau, D_0 \rightarrow -i\tau_0, D \rightarrow D$, the partition function is expressed as

$$Z_b = \int D\varphi D\bar{\varphi} e^{-S_{b, \sigma}[\phi, A]}, \quad (3.35)$$

$$S_{b, \sigma}[\phi, A] = \int d^2x \varphi^\dagger(x_0, x_2) (D_0 + \sigma iD_2) \varphi(x_0, x_2). \quad (3.36)$$

The effective edge action to second order in the external fields is

$$\Gamma^{(2)}[A] = -\frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} i k_0 + \sigma k_2 . \frac{i(k_0 + p_0) + \sigma (k_2 + p_2)}{(k + p)^2} (A_0(p) + \sigma iA_2(p))(A_0(-p) + \sigma iA_2(-p)), \quad (3.37)$$

where $p = k' - k$. The loop integrals are simplified by expressing the denominator in terms of a Feynman parameter,

$$\text{denominator} = \int_0^1 \frac{dx}{[\ell^2 + p^2 x(1 - x)]^2}, \quad (3.38)$$

where, $\ell = k + p(1 - x)$. All terms odd in $\ell$ are zero when integrating over all momenta since the denominator is even in $\ell$ so

$$\Gamma^{(2)}[A] = -\frac{1}{2} \int dx \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2\ell}{(2\pi)^2} \left( A_\mu(p) \frac{\ell^2 \delta_{\mu\nu} - 2\ell_\mu \ell_\nu + \sigma (i\varepsilon^{\nu\alpha}\ell_\alpha - i \ell_\mu \ell_\beta \varepsilon^{\beta\nu})}{[\ell^2 + p^2 x(1 - x)]^2} A_\nu(-p) + 2p_\mu p_\nu + \sigma (i\varepsilon^{\mu\alpha}p_\alpha p_\nu + i p_\mu p_\beta \varepsilon^{\beta\nu}) ] A_\nu(-p)x[1 - x] \right), \quad (3.39)$$

which is obtained by expressing everything in terms of the Greek indices using

$$[A_0(p) + \sigma iA_2(p)][-\ell_0^2 + \ell_2^2 + \sigma i2\ell_0\ell_2][A_0(-p) + \sigma iA_2(-p)] = A_\mu(p)[\ell^2 \delta_{\mu\nu} - 2\ell_\mu \ell_\nu + \sigma (i\varepsilon^{\nu\alpha}\ell_\alpha - i \ell_\mu \ell_\beta \varepsilon^{\beta\nu})] A_\nu(-p). \quad (3.40)$$
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Integrating over the loop momenta and Fourier transforming the remaining expression into real space, leaves the Minkowski spacetime effective action of the edge

$$\Gamma^\sigma_b[A] = \frac{1}{4\pi\nu} \int d^2 x A_\mu \left[ a g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2} + \frac{\sigma}{2} \left( -\varepsilon^{\mu\alpha} \partial_\alpha \partial^\nu + \partial^\mu \partial_\beta \varepsilon^{\beta\nu} \right) \right] A_\nu. \quad (3.41)$$

Here $a$ is a regularisation dependent constant where $a = 1$ when using dimensional regularisation, and $a = \frac{1}{2}$ when using a hard cut off. The regularisation dependence of the field theory means that the observables are model dependent, and can not be predicted by the field theory. The Chern insulator is a gauge invariant model, so the regularisation constant is fixed by demanding that the full theory, including the bulk and the edge is gauge invariant.

The variation of the edge action, Eq. (3.41), under a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu f(x)$ is

$$\delta \Gamma^\alpha_b = \delta \Gamma^\alpha_b + \delta \Gamma^\sigma_b,$$

where

$$\delta \Gamma^\alpha_b = \frac{1}{4\pi\nu} \int d^2 x \left[ (A_\mu + \partial_\mu f) (a \partial^\mu f - \partial^\mu f) + a \partial_\mu f A^\mu - \partial_\mu f A^\mu \right], \quad (3.42)$$

$$\delta \Gamma^\sigma_b = -\frac{\sigma}{4\pi\nu} \int d^2 x \varepsilon^{\mu\nu} f(x) \partial_\mu A_\nu, \quad (3.43)$$

after discarding any total derivatives, and using the antisymmetry of the Levi-Civita symbol. The variation proportional to $\sigma$ cancels the variation of the Chern-Simons action for the bulk, Eq. (3.33), which implies that $\delta \Gamma^\alpha_b \equiv 0$, which fixes the regularisation dependent constant for this theory to be $a = 1$.

The chiral edge of the Chern insulator realises the chiral anomaly in 1+1 dimensions [91]. The total current is conserved since $\delta \Gamma^\alpha_b$ vanishes when summed over both chiralities, $\sigma = \pm 1$. The difference $\delta \Gamma^+ - \delta \Gamma^-$ is nonzero so the axial current is not conserved; the edge is anomalous. The bulk boundary correspondence between the bulk Chern-Simons term and the chiral anomaly on the edge is a manifestation of anomaly inflow, also referred to as the Callan-Harvey mechanism [89].

This concludes our introduction to the Chern insulator characterised by the integer valued Chern number. We have found that the boundary between the topological and trivial state host chiral edge states. The net number of edge states correspond to the value of the Chern number in the bulk. This bulk boundary correspondence manifests in the low energy field theory through gauge invariance—the bulk and boundary theories are only gauge invariant when considered together as a whole. In the next sections we will introduce Weyl semimetals, which can be understood as a stacking of Chern insulators in momentum space. This means that we get to reuse some of the expressions derived in this section.
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Figure 3.3: From left to right: A Weyl node, yellow, emits Berry flux through a spherical surface surrounding it, corresponding to a Chern number $c$ of the surface. The surface is smoothly deformed until it breaks into two surfaces $a$ with Chern number $c_a$, and $b$ with Chern number $c_b$, such that $c = c_a - c_b$.

3.3 Weyl semimetals

Weyl semimetals are three dimensional topological semimetals hosting Weyl nodes; points in the Brillouin zone where nondegenerate valence and conduction bands coincide in energy [18]. At energies close to the band touching points the relevant physics is described by a generic two band Hamiltonian $H(k) = \sum_{\mu\nu} f_{\mu}(k) \sigma_{\nu}$, where $\mu \in (0,1,2,3)$, $\sigma_i$ for $i \in (1,2,3)$ are the Pauli matrices, and $\sigma_0$ is the identity matrix. The effective Hamiltonian to lowest order at small momenta $p_i = k_i - b_i$ around a Weyl node at momentum $b$ is,

$$H(k) = (f_0(b_i) + v_0 \cdot p) \sigma_0 + \sum_i v_i \cdot p \sigma_i + \mathcal{O}(p^2),$$

(3.44)

where $v_\mu = \partial_k f_\mu(k)|_{k=b}$. The continuum description of the Weyl semimetal is given by the Weyl Hamiltonian

$$H_\chi(p) = \chi v_F \mathbf{p} \cdot \mathbf{\sigma},$$

(3.45)

in the limit $v_0 = 0$, and $v_i = \chi v_F \hat{n}_i$, where $\chi = \text{sgn}[v_1 \cdot v_2 \times v_3]$ is the chirality, and $v_F$ is the Fermi velocity [18]. The linearly dispersing energy spectrum of the Weyl Hamiltonian, $\varepsilon_{\pm} = \pm \chi |k|$, is called a Weyl cone. The Weyl nodes are stable to any local perturbation—since all three Pauli matrices are used in the Weyl Hamiltonian there is no freedom left for adding a mass term which could open up a gap.

The Weyl nodes are sources and sinks of Berry curvature, Eq. (3.7), and are assigned a Chern number, calculated with respect to either the positive or negative energy eigenstates of the Weyl Hamiltonian. The Berry curvature has only a radial component $\Omega_{\chi}^\pm = \pm \chi/(2k)^2 \hat{k}$. For $\chi = +1$ the Berry curvature acts as a source for positive energy states and as a sink for negative energy eigenstates [15]. When $\chi = -1$, it instead acts as a sink for positive eigenstates and as a source for positive eigenstates [15]. The Berry flux is obtained by integrating the Berry
Topological phases of matter

Figure 3.4: Two Weyl cones of opposite chirality which acts as a source (left cone), and a sink (right cone) of Berry curvature $\Omega$.

curvature over a two dimensional surface surrounding the Weyl point

$$\Phi_B^\chi = \int_{S^2} \Omega \cdot \hat{v}_F dS = -2\pi \chi.$$  \hfill (3.46)

$\hat{v}_F$ is a unit vector along the direction of the Fermi velocity normal to the surface $S^2$, and has a positive sign when considering eigenstates with positive energy eigenstates, and a negative sign when considering negative energy eigenstates [15]. The Chern number of the surface around a Weyl cone of chirality $\chi$ is therefore $c = -\chi$.

There is an alternative way of calculating the Chern number of the surface around the Weyl cone. This method includes the adiabatic expansion and deformation of the sphere surrounding the Weyl node until it breaks into two planes on each side of the Weyl node, as depicted in Fig. 3.3 [15]. The total Chern number $c = -\chi$ of the sphere is now divided between the two planes, where plane $a$ has Chern number $c_a$, and plane $b$ has Chern number $c_b$, Fig. 3.3. Since the normal of the two planes points in opposite directions, the total Chern number is given by the difference $c = c_a - c_b$. In this way, we can define the Chern number for the two dimensional planes on each side of the Weyl node. These planes have an energy bulk gap. Every two dimensional plane away from a Weyl node is gapped and can be assigned a Chern number. The planes are either trivial or topological insulators depending on the value of the Chern number, and the Chern number can only change if the gap closes. The gap only closes at the Weyl point where the Chern number changes by $\chi$.

Weyl nodes come in pairs of opposite chirality on a crystalline lattice; the periodicity of the Brillouin zone requires that the Berry curvature emanating from
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A source ends up in a sink [73, 73], as depicted in Fig. 3.4. By viewing the Weyl semimetal as a stack of Chern insulators in momentum space, the Chern number of such slices can only be periodic across the Brillouin zone boundary if there are an equal number of nodes with positive and negative chirality. This means that Weyl semimetals consist of an even number of Weyl cones of opposite chirality. The existence of Weyl semimetals depends on the inversion symmetry and the time reversal symmetry of a given model. Inversion symmetry takes $k \rightarrow -k$, and $\sigma \rightarrow \sigma$, such that the Weyl Hamiltonian transforms as

$$\mathcal{H}_\chi(k - b) = \chi \nu_F(k - b) \cdot \sigma \rightarrow \chi \nu_F(-k - b) \cdot \sigma = \mathcal{H}_\chi(k + b).$$  \hspace{1cm} (3.47)

If there exists a Weyl node with positive chirality at momentum $b$, there must be another Weyl node with negative chirality at momentum $-b$. Time reversal symmetry defined by the operator $T = \sigma_2 \mathcal{K}$, where $\mathcal{K}$ is complex conjugation, transforms the Weyl Hamiltonian

$$T\mathcal{H}_\chi(k - b)T^{-1} = \sigma_2 (\mathcal{H}_\chi(k - b))^* \sigma_2 = \chi \nu_F(k + b) \cdot \sigma = \mathcal{H}_\chi(k + b),$$  \hspace{1cm} (3.48)

so a time reversal symmetric Weyl semimetals must contain at least four Weyl nodes. Weyl points do not require symmetry protection, but symmetries can prevent the Weyl semimetal from existing.

The simplest model of a time reversal breaking Weyl semimetal consists of two Weyl nodes of opposite chirality separated in momentum and can be thought of as a stack of two dimensional insulators in the direction of the Weyl node separation. Without loss of generality, we choose the Weyl nodes to be at $k = \pm b = bk_z$ with Chern number $c = 1$ for $|k| < b$, and Chern number $c = 0$, in the rest of the Brillouin zone. Two dimensional insulators with nontrivial topology are Chern insulators with metallic edge states, which means that the four $xz$ and $yz$ surfaces of the three dimensional Weyl semimetal are metallic for momenta $|k| < b$. The edge states form Fermi arcs in momentum space—the edge states project an arc between the two Weyl nodes onto the two dimensional surface so that the Fermi surfaces on each side of the material are disconnected, Fig. 3.5. Fermi arcs always exist between two Weyl nodes of opposite chirality, where material properties of multi node Weyl semimetals dictate how the nodes pair up.

The solutions of the Weyl Hamiltonian are massless fermions with a fixed chirality, which means that their momentum is either parallel or antiparallel to their spin. The low energy description of Weyl semimetals consists of dressed quasiparticles which are chiral fermions. The two Weyl Hamiltonians of opposite chirality make up the diagonal blocks of the four band massless Dirac Hamiltonian, which means that they are readily described through quantum electrodynamics. In the following sections we build up the field theory vocabulary of magnetic time reversal breaking Weyl semimetals. Weyl semimetals, and the axial anomaly, are characterised by massless fermions and are therefore described by Dirac theory. Our aim is to explore the axial anomaly in Weyl semimetals, which is why it is useful to introduce the field theory description.
3.4 Field theory description of Weyl semimetals with two Weyl nodes

The effective low energy description of a Weyl semimetal phase of two Weyl nodes is contained in the action

\[ S = \int d^4x \left( \bar{\Psi} i \gamma^0 \partial_0 \Psi - H \right), \]

\[ H = \Psi^\dagger \left( -i \nu_F \gamma^0 \gamma^i \partial_i - e \nu_F \gamma^0 b_\mu \gamma^\mu \gamma^5 + \gamma^0 m \right) \Psi, \]

with metric signature \((1, -1, -1, -1)\), in units where \(c = \hbar = 1\) [19–21, 24, 25]. \(\Psi = \Psi_L \otimes \Psi_R\) is a Dirac spinor composed of two chiral spinors of opposite handedness, where the Dirac adjoint is defined as \(\bar{\Psi} = \Psi^\dagger \gamma^0\), and \(m\) is the Dirac mass. Greek indices run over the four spacetime coordinates, \(\mu \in (0, 1, 2, 3)\), whilst Latin indices correspond to the three space coordinates \(i \in (1, 2, 3)\), and all repeated indices are summed over. In condensed matter settings it is more common to use the notation \(\mu \in (t, x, y, z)\), and we will switch between the two notations, choosing the appropriate one for each given situation. \(b_\mu\) is a constant four vector which is related to the separation of the Weyl nodes in energy-momentum space, and as it comes with \(\gamma^5\) it couples with opposite sign to fermions of opposite chirality.

The action \(S\) corresponds to a Weyl semimetal phase when \(-b^2 = -b_0^2 + b^2 > m^2\). This condition is obtained from the spectrum of the Hamiltonian. By representing the gamma matrices in the chiral basis,

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[(3.51)\]
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where \( \mathbf{1} \) is the identity matrix in two dimensions, and \( \sigma^i \) are the Pauli matrices, the momentum space Hamiltonian, \( H = \Psi^\dagger H^0 \Psi \) is represented by

\[
H^0 = \begin{pmatrix}
  b_0 + \sigma \cdot (k + b) & \frac{m}{m} \\
-\frac{b_0}{m} - \sigma \cdot (k - b) & m
\end{pmatrix}.
\]

(3.52)

The spectrum takes a simple analytic form in certain parameter limits, and analysing the spectrum in these limits provides an understanding of the different phases of the model. \( H^0 \) represents the massless Dirac Hamiltonian in the limit \( b_\mu = m = 0 \), consisting of two copies of the Weyl Hamiltonian of opposite chirality. The spectrum is a doubly Kramers degenerate cone which can be gapped out by adding a finite mass term. The phase defines a stable Dirac semimetal phase when it is protected by additional point group symmetries. The time reversal symmetry is broken by the addition of a space like axial vector acting like a magnetisation vector, separating the Weyl nodes in momentum with a distance \( 2b \).

The energy spectrum of \( H^0 \) in the limit of a finite mass term, \( m \neq 0 \), and a finite space like axial vector \( b \neq 0 \), is given by the closed form expression \[21\]

\[
E_{k,s} = \pm \sqrt{\left| k \times \hat{b}\right|^2 + \left| b\right| + s\sqrt{m^2 + \left| k \cdot \hat{b}\right|^2}}, \quad s = \pm 1,
\]

(3.53)

where \( \hat{b} \) is a unit vector in the direction of \( b \). The two bands corresponding to \( s = 1 \) are always gapped, whilst the bands defined by \( s = -1 \) are insulating only for parameter values \( m^2 > b^2 \). The spectrum is semimetallic when \( m^2 < b^2 \), where the two bands cross at momenta \( k = \pm b \sqrt{b^2 - m^2} \), and the Weyl node separation is given by \( \delta k = 2b \sqrt{1 - m^2/|b|^2} \). By adding a small, in relation to the mass term, time like axial vector component \( b_0 \), the Weyl nodes are separated in energy. The Weyl semimetal phase exists for parameter values \(-b_0^2 = -b_0^2 + b^2 > m^2\), and the separation of the Weyl nodes given by \( \delta k_\mu = 2b_\mu \sqrt{1 - m^2/|b|^2} \) \[24\]. This means that the spectrum is always gapped for a purely time like axial vector.

The purpose of this section was to figure out the conditions for when Eq. 3.49 describes a Weyl semimetal phase. The Weyl node separation is constant in the bulk of the semimetal but vanishes at the boundary with another phase. To describe the edge physics of the Weyl semimetal phase we will consider a space dependent Weyl node separation which vanishes at the edge.

3.4.1 Fermi arc boundary states

The axial vector defining the Weyl node separation in energy-momentum space vanishes at the boundary of a Weyl semimetal, giving rise to Fermi arc states bound to the edge. We consider a model with a boundary on the \( yz \) plane, and a purely space like Weyl node separation \( b_\mu = b \hat{z} \) along the \( k_z \) direction. The Weyl
node separation is assumed to abruptly go to zero at the boundary, such that $b(x) = b\theta(x)$, where $\theta(x)$ is the Heaviside step function. The sample is infinite in the $y$ and $z$ directions so that $k_y$ and $k_z$ are good quantum numbers. This leads to the eigenvalue equation, $H^0\psi(x) = E\psi(x)$, where

$$H^0 = \left( \begin{array}{cc} i\sigma_x \partial_x - k_y\sigma_y - k_z\sigma_z + b\theta(x)\sigma_z & m \\ -i\sigma_x \partial_x + k_y\sigma_y + k_z\sigma_z + b\theta(x) & \end{array} \right).$$

(3.54)

The only normalisable solution is given by the ansatz

$$\psi(x) = u(x)(0, 0, 1, i)^T + v(x)(1, -i, 0, 0)^T,$$

(3.55)

which is an eigenstate of $\gamma^0\gamma^2$ with eigenvalue $+1$. To see why this is the case we will use the transfer matrix $T(x_i, x_f)$ evolving a state at initial position $x_i$ to a state in final position $x_f$ through $\psi(x_f) = T(x_i, x_f)\psi(x_i)$. The normalisation condition for the final state translates to the condition $\lim_{x_f \to \infty} T(x_i, x_f) = 0$. We rearrange the eigenvalue equation to the form $\partial_x \psi(x) = \hat{A}\psi(x)$, where

$$\hat{A} = -iE\sigma_x \otimes \tau_z + k_y\sigma_z \otimes \tau_0 - k_3\sigma_y \otimes \tau_0 - b\theta(x)\sigma_y \otimes \tau_3 - m\sigma_x \otimes \tau_2,$$

(3.56)

and $\tau_i$ is a second set of Pauli matrices with $\tau_0$ being the identity. This connects the eigenvalue equation with the transfer matrix through $\hat{A}\psi(L) = \partial_x \psi(x)|_{x=L} = \partial_x T(x, 0)|_{x=L}\psi(0)$ for initial position $x_i = 0$, and final position $x_f = L$. This implies that $\hat{A}T(L, 0) = \partial_x T(x, 0)|_{x=L}\psi(0)$. By solving for the transfer matrix, the final state is expressed as

$$\psi(L) = T(L, 0)\psi(0) = e^{\int_0^L \hat{A}dx} \psi(x) = e^{\lambda L}\psi(0),$$

(3.57)

where $\lambda$ is a negative eigenvalue of $\hat{A}$, such that $\psi(L)$ is a normalisable state. Since the eigenvalue equation holds for all energies, we can choose $E = 0$, which means that also $k_y = k_z = 0$ due to the linear dispersion of the fermions. To simplify the transfer matrix further we assume a boundary between a Weyl phase and a gapped Dirac phase at $x = 0$ so that $m = m\theta(-x)$. The transfer matrix is therefore $T(L, 0) = \exp\left[\int_0^L (-b\sigma_y \otimes \sigma_z)\right]$, where the normalisation of $\psi(L)$ requires $\lambda$ to be the positive eigenvalues of the matrix $\sigma_y \otimes \sigma_z = \gamma^0\gamma^y$. So the eigenstate $\psi(x)$ is indeed given by the ansatz in Eq. (3.55).

The ansatz in Eq. (3.55) leads to the dispersion relation $E = \nu_F k_y$, where the coefficients are solved to be

$$u(x) = Ae^{\sqrt{k_y^2 + m^2} x}, \quad x < 0,$$

(3.58)

$$u(x) = Ae^{(-b + \sqrt{k_y^2 + m^2}) x}, \quad x > 0,$$

(3.59)

$$v(x) = -\frac{A}{m} \left( k_z + \sqrt{k_y^2 + m^2} \right) u(x),$$

(3.60)
where $A$ is a normalisation constant. The Weyl phase $u(x > 0)$ is only normalisable for $k_z$ in between the Weyl nodes, $-k_\Delta < k_z < k_\Delta$, $k_\Delta = \sqrt{b^2 - m^2}$. The eigenstates are summed over all $k_z$ so that the Fermi arc bound states are

$$
\Psi(x) = N \int \frac{dk_z}{2\pi} V(k_z) e^{(-b + \sqrt{k_z^2 + m^2})x} e^{k_y y} e^{k_z z} \theta(k_\Delta - k_z),
$$

(3.61)

where $N$ is the normalisation constant, and $V(k_z) = (-f(k_z), i f(k_z), 1, i)^T$, $f(k_z) = \frac{A}{m} \left(k_z + \sqrt{k_z^2 + m^2}\right)$. The linear dispersion $E = \nu_F k_y$ forms a two dimensional plane in between the two Weyl nodes, forming a Fermi arc when projected onto the surface, depicted in Fig. 3.5.

We have derived the Fermi arc bound states at the edge of a Weyl semimetal phase. In the next section, we will consider the low energy response theory of Weyl semimetals with respect to external electromagnetic fields. The low energy theory is very similar to the Chern insulator with a bulk Chern-Simons theory and chiral edge theory which are not gauge invariant when considered separately. We will again need to consider both the bulk and edge to restore gauge invariance.

### 3.4.2 Lorentz breaking quantum electrodynamics

The electromagnetic response of Weyl semimetals includes the chiral anomaly, the related chiral magnetic effect, and quantum Hall effect [19–25,93]. The response theory is obtained by minimally coupling the fermionic degrees of freedom to external gauge fields, resulting in the action:

$$
S[A] = \int d^4x \bar{\Psi} \left(i \partial^\mu + A^\mu + b^5 \gamma^5 + m\right) \Psi,
$$

(3.62)

reducing to the action of quantum electrodynamics in the limit where $b^\mu = 0$.

To simplify notation we have set $c = \nu_F = e = 1$, which assumes that the Fermi velocity is equal in all spatial directions. To account for a nonuniform Fermi velocity one may introduce the tensor $M^\nu_\mu = \text{diag}(c, \nu_F^x, \nu_F^y, \nu_F^z)$ in the action Eq. (3.62) [24].

A nonzero $b^\mu$ couples like an axial gauge field with opposite sign to fermions of opposite chirality, but $b^\mu$ differs from an axial gauge field since $b^\mu$ is a parameter of the theory rather than an external gauge field. The action $S[A]$ with a nonvanishing $b^\mu$ has been extensively explored in high energy physics and is referred to as the action for Lorentz breaking quantum electrodynamics [94–98]. It is the constant vector $b^\mu$ that breaks particle Lorentz symmetry.

There are two types of Lorentz transformations: observer transformations, and particle transformations [24,99]. To understand the difference we consider an electron in a background magnetic field where the electron momenta is perpendicular to the magnetic field so that it moves in a circle with radius $r$. An observer transformation takes us to another inertial frame in which the electron
would be subject to both magnetic and electric fields and no longer move in a circle. A particle transformation changes the momenta of the particle, but leaves the external magnetic field invariant; the particle still moves in a circle but with a different radius [99]. Another way to view the difference is by assuming a particle in a box under the influence of a constant gravitational field [24]. The observer transformation rotates both the box and the external gravitational field, leaving the model invariant. But a particle transformation only rotates the box leaving the external field unchanged. So the particle in a box subject to an external gravitational field is not invariant under particle transformations.

The quantity $\bar{\Psi} \gamma^5 \gamma^5 \Psi$ in the Weyl semimetal phase breaks particle Lorentz transformations. Here $b_\mu$ is a physical parameter of the model; $b$ couples like a magnetisation vector in the Hamiltonian, and $b_0$ couples like an axial chemical potential, a chemical potential which couples with opposite sign to fermions of opposite chirality. $b_\mu$ can therefore be viewed as an external field that does not change under particle transformations. The axial vector $\bar{\Psi} \gamma^\mu \gamma^5 \gamma^5 \Psi$ transforms as a vector under both types of Lorentz transformations. $b_\mu$ transforms as a vector under observer transformations leaving the theory invariant. But under particle transformations, $b_\mu$ transforms as four scalars so that $\bar{\Psi} \gamma^5 \gamma^5 \Psi$ as a whole is not invariant.

The physics of Lorentz breaking quantum electrodynamics plays a role in the effective bulk theory of the gauge fields in Weyl semimetals, which is described by a Chern-Simons like term multiplied by $b_\mu$. Chern-Simons terms only exist in odd spacetime dimensions, which is why the bulk theory in Weyls is described as a Chern-Simons like theory. This term is effectively three dimensional since $b_\mu$ is constant, and the bulk theory can be thought of as a momentum space stacking of the Chern-Simons terms along the direction of the Weyl node separation.

### 3.4.3 The bulk Chern-Simons action

The effective bulk theory responsible for the linear response of a Weyl semimetal phase is given by the Chern Simons like effective action [19–21,24]

$$
\Gamma^{(2)}[A] = \int \varepsilon^{\mu\nu\rho\sigma} c_\mu A_\nu \partial_\rho A_\sigma.
$$

(3.63)

Chern-Simons field theories only exist in odd space time dimensions, which is why $\Gamma^{(2)}[A]$ is often described as a Chern-Simons like action, allowed due to the constant $c_\mu$ four vector. Here $c_\mu$ depends on the parameters $m$, and $b_\mu$, so $\Gamma^{(2)}[A]$ is viewed effectively as an action of Chern-Simons terms stacked in the direction of $c_\mu$ [20]. The constant $c_\mu$ is finite but undetermined and must be fixed by observables, or equivalently through gauge invariance, of any physical theory [20,95].
The effective Chern-Simons action, $\Gamma^{(2)}[A]$ is obtained by integrating out the fermionic degrees of freedom from the path integral of the action Eq. (3.62)

$$Z[A] = \int D\bar{\Psi} D\Psi e^{iS} = e^{i\Gamma} = \det (i\partial + A + b\gamma^5 + m). \tag{3.64}$$

By using the operator identity $\ln \det \hat{O} = \text{tr} \ln \hat{O}$, and expanding the logarithm to second order in the fields leads to the effective action

$$\Gamma^{(2)}[A] = -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} A_\mu(p)\Pi^{\mu\nu} A_\nu(-p), \tag{3.65}$$

$$\Pi^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \text{tr}[\gamma^\mu S_b(k)\gamma^\nu S_b(k-p)]. \tag{3.66}$$

The vacuum polarisation $\Pi^{\mu\nu}$ is expressed in terms of the modified fermion propagator

$$S_b(k) = \frac{i}{\not{k} - m - \not{b}\gamma^5}. \tag{3.67}$$

The Chern-Simons action Eq. (3.63) is odd under parity and proportional to both the Levi-Civita tensor and the external momenta $p_\mu$. So we are looking to derive a parity odd expression of the form $\Pi^{\mu\nu}_{\text{odd}} \sim \varepsilon^{\mu\nu\rho\sigma}c_\mu p_\nu$. Since $c_\mu$ is a constant four vector it must be proportional to $b_\mu K(p^2, b^2, m)$, where $K(p = 0, b^2, m)$ is a scalar function with respect to the parameters $m$, and $b_\mu$ so that

$$\Pi^{\mu\nu}_{\text{odd}} \sim \varepsilon^{\mu\nu\rho\sigma}b_\mu p_\nu K(p = 0, b^2, m). \tag{3.68}$$

The correct value for $K(p = 0, b^2, m)$ is obtained by evaluating the vacuum polarisation function in Eq. (3.66) by taking the trace of the gamma-matrices and performing the loop integral over $k$. To obtain Eq. (3.68) we only keep terms which are odd in $b_\mu$ and set $p_\mu = 0$ in the loop integral. This derivation is performed in detail in Ref. [95] and leads to the result

$$\Pi^{\mu\nu}_{\text{odd}} \sim \varepsilon^{\mu\nu\rho\sigma}b_\mu p_\nu K(p = 0, b^2, m). \tag{3.69}$$

where $C$ is a regularisation dependent constant, which stems from the divergent loop integral over $k$. The Fourier transformed Chern-Simons term in real space is

$$\Gamma_{\text{cs}}[A] = -\frac{e^2}{2\nu_F} \int \varepsilon^{\mu\nu\rho\sigma}c_\mu A_\rho \partial_\sigma A_\nu, \tag{3.70}$$

where we have reintroduced $e$, and $\nu_F$. In the Weyl semimetal phase $c_\mu = b_\mu[C - (2\pi^2)^{-1/2}(1 - m^2/b^2)^{1/2}]$. 

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The constant $C$ depends on the microscopic model, or equally on the high energy theory, and is determined by observables, such as the conductivity \cite{20}. We will instead use the bulk boundary correspondence to fix $C$. The Chern-Simons action is not gauge invariant on a closed manifold, but the nonzero variation is cancelled by the anomalous contribution from the boundary states. The procedure mimics that of the Chern insulator in section 3.2.2. The variation of the Chern-Simons action for $b = b\theta(x)\hat{z}$ under $A_\mu \rightarrow \partial_\mu A_\mu + f(x)$ is

$$\delta \Gamma_{cs}[A] = \left(C - \frac{1}{2\pi^2} \sqrt{1 - \frac{m^2}{b^2}}\right) \frac{b}{2\nu_F} \int d^2x \varepsilon^{\mu\nu} f(x) \partial_\mu A_\nu. \quad (3.71)$$

The chiral boundary state Eq. (3.61):

$$\Psi(x) = N \int \frac{dk_z}{2\pi} V(k_z) e^{-b+\sqrt{k_z^2+m^2}x} e^{k_y y} e^{k_z z} \theta(k_\Delta - k_z), \quad (3.72)$$

corresponds to the boundary action

$$S_b = \int \frac{dk_z}{2\pi} \varphi^\dagger(t,y) [iD_0 + \sigma iD_2] \varphi(t,y) \theta(k_\Delta - k_z), \quad (3.73)$$

coupled to external electromagnetic gauge fields $A_\mu = A_\mu(t,y)$, $\mu \in (0,2)$ where $\varphi(t,y) = e^{ik_y y} \phi(t)$ includes an unspecified time dependence. The boundary action Eq. (3.73) is equal to the boundary action for the Chern insulator Eq. (3.34), but summed over all momenta between the Weyl cones. The effective boundary action of the Weyl semimetal is therefore equal to the effective action of the Chern insulator, Eq. (3.41) but summed over $k_z$ between the Weyl nodes:

$$\Gamma_\sigma^\sigma_b[A] = \frac{k_\Delta}{4\pi \nu_F} \int d^2x A_\mu \left[ g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2} + \frac{\sigma}{2} \left(-\varepsilon^{\mu\alpha} \partial_\alpha \partial^\nu + \partial^\mu \partial_\beta \varepsilon^{\beta\nu}\right)\right] A_\nu. \quad (3.74)$$

Here $\Gamma_\sigma^\sigma_b[A]$ has been regularised by using dimensional regularisation. The variation of the effective edge action under $A_\mu \rightarrow \partial_\mu A_\mu + f(x)$ is

$$\delta \Gamma_\sigma^\sigma_b[A] = \frac{1}{2\pi^2} \sqrt{1 - \frac{m^2}{b^2}} \frac{b}{2\nu_F} \int d^2x \varepsilon^{\mu\nu} f(x) \partial_\mu A_\nu, \quad (3.75)$$

and the condition $\delta \Gamma_{cs}[A] + \delta \Gamma_\sigma^\sigma_b[A] = 0$ fixes the constant $C$ for this model to be equal to zero.

The physics in this section emulates that of the Chern insulator, where both the bulk and edge terms can be realised by summing the 2+1 dimensional theory along the Weyl node separation. We have previously emphasised that Weyl semimetals provide a condensed matter realisation for the axial anomaly and axial

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emergent electromagnetic fields. Now we can also add Lorentz breaking quantum electrodynamics to this list, and the notion of the finite but undetermined effective bulk theory which is fixed through knowledge of the microscopic theory.

In the next section we will explore the chiral anomaly through the Landau level formalism briefly introduced in chapter two, and see how it generates a bulk current called the chiral magnetic effect.

3.5 The chiral anomaly and Landau levels

The bulk action of the Weyl semimetal phase, Eq. (3.49) with $-b^2 > m^2$ with a constant Weyl node separation $b_\mu$ is anomalous. The chiral anomaly may be calculated from the action by using the methods explored in chapter two to yield the expression

$$\partial_t n_5 = \frac{e^2}{2\pi^2\hbar^2} E \cdot B.$$ (3.76)

We will instead focus on the Landau level derivation of the chiral anomaly.

The Landau level spectrum of the magnetic Weyl semimetal in a strong magnetic field paints an intuitive picture of the chiral anomaly in the bulk. The Weyl Hamiltonian of chirality $\chi$ for a Weyl node at momentum $p$ coupled to an external gauge field is

$$H = \chi \nu_F (p - eA) \cdot \sigma,$$ (3.77)

where the Landau gauge $A = B_z \hat{x} e_y$ corresponds to a magnetic field in the z direction, $B = B_z e_z$ [49]. We introduce the dimensionless creation and annihilation
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operators

\[ \hat{a}_{py} = \frac{1}{\sqrt{2}} \left( \frac{\hat{x} - p_y \ell_B^2}{\ell_B} + i\hat{p}_x \ell_B \right), \] (3.78)

\[ \hat{a}^\dagger_{py} = \frac{1}{\sqrt{2}} \left( \frac{\hat{x} + p_y \ell_B^2}{\ell_B} + i\hat{p}_x \ell_B \right), \] (3.79)

which obey the commutation relation \([a_{py}, a_{py}^\dagger] = 1\), where \(eB_z > 0\), and \(\ell_B = (eB_z)^{-\frac{1}{2}}\) is the magnetic length \([100]\). In the basis of eigenvectors to \(\hat{p}_y\) the Hamiltonian is

\[ H = \chi \nu_F \begin{pmatrix} p_z & i\sqrt{2} \hat{a}^\dagger_{py} / (\ell_B) \\ -i\sqrt{2} \hat{a}_{py} / (\ell_B) & -p_z \end{pmatrix}. \] (3.80)

The spectrum is obtained through the ansatz

\[ \psi(x) = u_n \begin{pmatrix} 1 \\ 0 \end{pmatrix} |n\rangle + v_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} |n-1\rangle, \] (3.81)

where the states obey the relations \(\hat{a}_{py}|n\rangle = \sqrt{n}|n-1\rangle\), and \(\hat{a}_{py}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\). The lowest Landau level for \(n = 0\) is \(\epsilon^\chi_0(p_y) = \chi \nu_F p_z\), which is chiral and dispersing linearly in the direction of the magnetic field. All higher order Landau levels \(n \geq 1\) are given by \(\epsilon^\chi_{n>0}(p_y) = \pm \chi \nu_F (p_z^2 + 2n \ell_B^2)^{1/2}\), see Fig. 3.6. The Landau levels are independent of \(p_y\), and their degeneracy is given by \(N_{LL} = \frac{L_x L_y B_z}{2\pi/e}\), where \(L_i\) is the length of the system in the \(i\) direction.

The gap between the two Landau levels with \(n = 1\) scales as \(\Delta \epsilon^\chi_N \sim \nu_F / \ell_B\). The low energy physics of the Fermi energies within the gap is only determined by the zeroth linearly dispersing chiral Landau level. In the presence of an electric field \(\mathcal{E} = E_z e_z\) obtained from the potential \(A_{\mu} = (0, 0, 0, E_z t)\), the fermions in the lowest Landau level start moving along the electric field, Fig. 3.6, and states are created or destroyed at a rate \(\frac{dp_z}{dt} = \chi e \mathcal{E}\) \([49]\). The lowest Landau level subject to an electric field yields the chiral anomaly in 1+1 dimensions, but multiplied with the Landau level degeneracy so that the rate of change is

\[ \partial_t n_\chi = \frac{1}{V} \frac{L_z}{2\pi} \frac{dp_z}{dt} N_{LL} = \chi e^3 4\pi^2 \hbar^2 \mathcal{E} \cdot B, \] (3.82)

where the total number of states is divided by the volume \(V\), and \(\frac{2\pi}{L_z}\) is the density of states per quantisation length. Since the Weyl nodes come in pairs of opposite chirality, the total particle density is conserved, but the axial particle density, \(n_5 = n_L - n_R\) is not and leads to the axial anomaly

\[ \partial_t n_5 = \frac{e^2}{2\pi^2 \hbar^2} \mathcal{E} \cdot B. \] (3.83)
The Weyl cones of opposite chirality on a lattice are connected at higher energies, resulting in an inter valley scattering between the two cones, which is taken into account by including a relaxation term in the anomaly equation,

$$\partial_t n_5 = \frac{e^2}{2\pi^2\hbar^2} E \cdot B - \frac{n_5}{\tau},$$  \hfill (3.84)

where $\tau$ is the inter valley scattering time [101, 102].

Even though a large magnetic field is necessary for describing the chiral anomaly in the Landau level picture, this is not necessary for the anomaly to exist. The chiral anomaly at low magnetic fields is obtained through the Boltzmann kinetic equation assuming a finite chemical potential [25].

3.5.1 The chiral magnetic effect

The chiral magnetic effect is the emergence of a current due to the chiral anomaly which is proportional to the external magnetic field and the difference in chemical potential between the two Weyl cones of opposite chirality [26]. The current is only carried by the lowest Landau level defining the chiral edge state. The total edge current is defined as the sum of the chiral currents of left and right handed fermions. The current in momentum space for a given chirality $\chi$ is given by the velocity $\nu = \partial_{p_z} E_0^\chi = \chi (\nu_F = 1)$, multiplied with the charge density and the Landau level degeneracy per unit volume [103],

$$J_3^\chi = \chi e B_z \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \left( \frac{1}{1 - e^{\beta(\epsilon - \mu_\chi)}} - \theta(-\epsilon) \right) = \chi \frac{e^2 \mu_\chi}{4\pi^2} B_z. \hfill (3.85)$$

Here $\theta$ is the Heaviside step function which subtracts all negative energy states, and $\mu_\chi \in (\mu_L, \mu_R)$ is the chemical potential for the left and right handed chiral fermions, Fig. 3.7. The chemical potential of the cones are defined as $\mu_L = \mu + \mu_5$, and $\mu_R = \mu - \mu_5$, so that the total chemical potential of both cones is $\mu = (\mu_L + \mu_R)/2$, and the axial chemical potential is given by the difference $\mu_5 = (\mu_L - \mu_R)/2$. The total current is therefore proportional to the axial chemical potential:

$$J^3 = J^3_L + J^3_R = \frac{e^2}{4\pi^2} B_z (\mu_L - \mu_R) = \frac{e^2}{2\pi^2} \mu_5 B_z. \hfill (3.86)$$
This is the chiral magnetic effect. The chiral magnetic effect is a consequence of the chiral anomaly, and can only exist out of equilibrium where the axial chemical potential is nonzero. For inversion breaking Weyl semimetals $b_0 \neq 0$, and would contribute to the chiral magnetic effect which is modified to $J = (\mu_5 - b_0) \frac{e^2}{2\pi^2} B$ [27]. The chiral magnetic effect plays an important role in paper I as a suggested signature of the anomaly.

3.6 Summary

We have provided a general introduction to strong symmetry protected phases of matter and their classification. Topological insulators and superconductors are described by free fermion Hamiltonians which belong to one of the ten Altland-Zirnbauer classes. The topological classification of these Hamiltonians specifies which symmetry classes can host topological phases in a given dimension. The topology is characterised by quantised integers referred to as topological invariants. The focus on topological phases of matter in this chapter serves as a preparation for the research on local topological markers in papers IV and V. Topological phases of matter are not restricted to crystalline structures, and we will pay more attention to topological invariants of topological insulators and superconductors in chapter five, where we will focus on amorphous matter and topology in structures lacking translation invariance.

We have also introduced the Chern insulator in this chapter by focusing on the field theory description of the topological edge states and the Chern-Simons theory of the bulk. The purpose of the detailed derivations is twofold: To describe the bulk boundary correspondence between the Chern number of the bulk and the chiral edge states, and to introduce the field theory which is also used for describing Weyl semimetals. Weyl semimetals are the main object of study in papers I and II, where we consider the chiral anomaly in magnetic Weyl semimetals by using a field theory description. We have introduced Weyl semimetals and their connection to the chiral anomaly, focusing on Weyl semimetals consisting of two Weyl cones of opposite chirality. Such Weyl semimetals can be described as a stacking of Chern insulators in momentum space in the direction of the Weyl node separation, highlighting our reason for deriving the Chern insulator in such detail.

We described the chiral anomaly in Weyl semimetals through the Landau level spectrum since it gives an intuitive picture of the nonconservation of chiral fermions as a response to parallel electric and magnetic fields. This imbalance of fermions of opposite chirality can be viewed as a difference of chemical potential between the two Weyl cones, and we described how this leads to the chiral magnetic effect—a nonequilibrium current in the direction of the magnetic field. The focus on the chiral anomaly and the chiral magnetic effect provides a foun-
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dation for the physics of domain wall induced axial electromagnetic fields in Weyl semimetals that we will explore in chapter four.
Weyl semimetals are closely related to the axial anomaly due to the emergence of Weyl fermions as quasiparticles in their low energy description. The axial anomaly in Weyl semimetals describes the nonconservation of chiral charge due to the influence of external parallel electric and magnetic fields. The axial anomaly can in theory also depend on parallel axial electromagnetic fields, resulting from a minimal coupling between fermions and axial gauge fields in the Dirac Lagrangian. Axial electromagnetic fields do not exist in nature, but they do appear as emergent fields in Weyl semimetals, giving us the possibility to explore these fields and their consequences in real materials.

A space and time dependent Weyl node separation in Weyl semimetals generates emergent axial electromagnetic fields [104]. The Weyl node separation enters the action similarly to an electromagnetic gauge field, the only difference is that the Weyl node separation couples with a chiral $\gamma^5$ matrix [19–21]. It can therefore be interpreted as an axial gauge field which couples with opposite sign to fermions of opposite chirality. The space and time variation of the axial gauge fields leads to axial electromagnetic fields, in analogy to the relation between vector gauge fields and ordinary electromagnetic fields [27]. The presence of axial electromagnetic fields provides another approach through which to explore chiral anomaly physics in Weyl semimetals [104].

Mechanical strain can be used to induce a spatially dependent Weyl node separation [104–107]. Applying strain to a sample of Weyl semimetal leads to lattice deformations which couple to the Weyl fermions as chiral elastic gauge fields. These emergent chiral elastic gauge fields act as axial gauge fields on the Weyl fermions and produce axial magnetic fields. By sending sound waves through the sample the strain becomes time dependent, causing axial electric fields to appear [106]. The strain induced axial electromagnetic fields provide insight into the covariant form of the chiral anomaly in a lattice model, and how to understand the connection between the locally nonconserved covariant current and the conserved covariant current. We provide an alternative approach for axial electromagnetic field generation in paper I, based on the notion of domain wall motion in Weyl semimetals [108].

The simplest model of a time reversal symmetry breaking Weyl semimetal consists of two Weyl nodes of opposite chirality separated in momentum space.
Such a model is described by the Hamiltonian, $H = (\tau_z \otimes \mathbf{k} \cdot \mathbf{\sigma} + \mathbf{1} \otimes \mathbf{b} \cdot \mathbf{\sigma})$, where $\tau_i$ and $\sigma_i$, $i \in (x, y, z)$ are Pauli matrices [18]. Choosing the coordinate system in such a way that the Weyl nodes are located at momenta $\mathbf{k} = \pm \mathbf{b}$, results in a Weyl node separation given by a vector $2\mathbf{b}$. The Weyl node separation enters the Hamiltonian as a Zeeman coupling, $\mathbf{b} \cdot \mathbf{\sigma}$, so the Weyl node separation $\mathbf{b}$ acts as a magnetisation vector breaking time reversal symmetry [18]. This is why time reversal symmetry breaking Weyl semimetals are referred to as magnetic Weyl semimetals. Magnetic Weyl semimetals have been observed experimentally, and the research into material realisations of magnetic Weyl semimetals is a continuing process [109–114]. It was recently discovered that the proposed magnetic Weyl semimetal EuCd$_2$As$_2$ [115,116], is not at all a Weyl semimetal but a semiconductor [117,118]. This demonstrates how research is an ongoing endeavour, where new and different methods can overturn previous results.

Our concern is on magnetic domain wall dynamics in Weyl semimetals, which requires materials with magnetic domains. There are experimental realisations of magnetic Weyl semimetals where the Weyl node separation points in different directions in momentum space in different domains [119–121]. A spatially dependent Weyl node separation is therefore obtained through a domain wall in the magnetisation vector separating the two Weyl cones in momentum space. The existence of such domain walls give rise to emergent axial magnetic fields generated due to the spatial variation of $\mathbf{b}$. Axial electromagnetic fields depend on a spatial variation of the Weyl node separation and require dynamic domain walls.

The generation of axial electromagnetic fields due to dynamical domain walls is the main focus of papers I and II. Paper I focuses on the emergence of the axial anomaly in terms of the axial electromagnetic fields, and provides a suggestion on how to measure the anomaly in such a setup. Electronic domain wall control is a relevant objective in the field of spintronics, for developing new magnetic technologies for example for memory storage. Paper II describes how the axial anomaly with respect to external electromagnetic fields acts as an emergent spin torque on the domain wall, mediating domain wall manipulation through an external electric field.

The first part of this chapter is dedicated to the description of magnetic domain walls and their dynamics under external magnetic fields. In the latter part of this chapter, we explore the consequences of axial electromagnetic fields in Weyl semimetals and bring together the concepts of magnetic domain walls, Weyl semimetals and the axial anomaly.

4.1 Magnetic domain walls

Magnetic domain walls in magnetic materials form boundaries between magnetic domains with magnetisation vectors of opposite orientations. A magnetic domain wall is described by a spacetime dependent field $\mathbf{m}(x, t)$ with three components
m = (m₁, m₂, m₃) satisfying the constraint m² = 1, so that m(x, t) is a unit magnetisation vector [122]. We consider two domains separated by a domain wall along the x direction with boundary conditions limₓ→±∞ m(x) = ±ẑ depicted in Fig. 4.1. The degrees of freedom of the dynamics of such a domain wall are very large, making it hard to evaluate the equations of motions analytically. The collective coordinate method reduces the number of degrees of freedom to two: the position X describing the centre of the domain wall along the x axis, and the spatially averaged internal angle φ out of the xz plane [122]. The value φ = 0 describes a Néel wall [123] where all the magnetisation vectors lie in the xz plane, whilst the angle φ = ±π/2 defines a Bloch wall [124]. The sign of the angle φ = ±π/2 is referred to as the chirality of the Bloch wall; φ = π/2 is called a right handed wall, and φ = π/2 is called a left handed wall [125]. A Néel wall is said to have no chirality. The domain wall depicted in Fig 4.1 is a right handed Bloch wall.

The existence of a domain wall in the magnetisation vector of a magnetic Weyl semimetal gives rise to axial magnetic fields confined to the domain wall. To generate axial electromagnetic fields the domain wall must also be time dependent. We will focus on field driven dynamics through an external magnetic field, where the domain wall dynamics depend on the magnitude of the magnetic field. A small enough magnetic field leads to rigid domain wall motion along the domain wall direction, and the solutions to the equations of motion in this region are referred to as Walker solutions [126]. The Walker solutions break down at some critical value of the magnetic field, at which the averaged angle φ starts rotating [126]. We will derive the equations of motions in both regions to get an understanding of field driven domain wall dynamics.

We aim to combine domain wall dynamics with the low energy theory of magnetic Weyl semimetals. We will therefore develop a field theory description of the dynamics of the domain wall in terms of two collective coordinates. The first step is to describe the motion of a rigid domain wall with a constant spatially averaged angle out of the easy plane [127,128]. The full dynamics, beyond Walker
breakdown, is obtained by considering low energy quantum fluctuations around the rigid solution, where the collective coordinates emerge as zero modes of the quantum fluctuations [127,128]. The field theory description of domain wall motion allows us to explore the axial anomaly in terms of emergent axial fields in magnetic Weyl semimetals, where the axial electromagnetic fields are a direct consequence of the domain wall dynamics. We will also show how the coupling between the electrons and the magnetisation in the Weyl semimetal contributes to the domain wall dynamics.

4.1.1 The domain wall Lagrangian and rigid dynamics

Field theories are often defined through their Lagrangian, so also the field theory of a magnetic domain wall. The magnetic domain wall coupled to an external magnetic field is described by the Lagrangian

\[ L_{dw} = L_B - H_H - H_Z, \]

where the \( L_B \) is the Berry phase term of a single magnetisation vector, \( H_H \) is the Heisenberg Hamiltonian in the continuous limit taking into account the interaction between neighbouring spins, and \( H_Z \) is the Zeeman term coupling the magnetisation with the magnetic field [127,128]. The Berry phase Lagrangian can be written in terms of a fictitious gauge field \( A(m) \) containing a singularity for any gauge choice [83,129], where

\[ L_B = \frac{1}{a^3} \int d^3x \partial_t m \cdot A(m). \] (4.2)

The corresponding action is a geometric phase: the magnetisation vector draws a positively orientated curve on a unit sphere which does not depend on the velocity, only on the path:

\[ S_B = \frac{1}{a^3} \int d^3x \int dt \partial_t m \cdot A(m) = \int d^3x \int dm \cdot A(m). \] (4.3)

The Berry phase action therefore describes the path of a single magnetisation vector as it carves out a solid angle on the unit sphere. The path of the magnetisation vector on the sphere can be viewed as the motion of a charged particle on the unit sphere \( |m| \), subject to a magnetic monopole, \( \nabla_m \times A(m) = B(m) = -\tilde{J}m \), with magnetic charge \( \tilde{J} \), at the centre of the sphere [83,129,130]. This becomes clearer when we use the common gauge choice

\[ A(m) = \frac{(m_2, -m_1, 0)}{1 - m_3}, \] (4.4)
with the singularity in the south pole of the unit sphere \([83,129,130]\). In spherical polar coordinates, the same gauge choice yields the expression
\[
A(m) = \frac{1 - \cos \theta}{\sin \theta} \hat{\phi},
\]
for the fictitious gauge field, where \(\theta\) is the azimuthal angle and \(\phi\) is the polar angle. This gauge choice corresponds to the magnetic field \(B_m = \hat{e}_r\), defining a magnetic monopole emanating a magnetic field in the radial direction \([83]\).

The second term in the domain wall Lagrangian in Eq. (4.1) is the Heisenberg Hamiltonian in the continuous limit:\[131\]:
\[
H_H = \frac{1}{a^3} \int d^3x \left[ \frac{Ja^2}{2} (\nabla m)^2 - \frac{K}{2} (m_z)^2 + \frac{K}{2} (m_y)^2 \right].
\]
\(J\) is a positive coupling energy between neighbouring vectors, \(K\) is the easy axis energy gain along the \(z\) direction, and \(K_\perp\) is the hard axis energy loss along the \(y\) direction. Both \(K\) and \(K_\perp\) are positive and have units of energy, making the \(xz\) plane the easy plane. The last term in in Eq. (4.1) is the Zeeman term which couples the magnetisation with an external magnetic field \(B\),
\[
L_Z = \frac{1}{a^3} \int d^3x \ m \cdot \gamma B,
\]
where \(\gamma\) is the electron gyromagnetic ratio.

The Euler-Lagrange equations of motion are expressed in a generalised form,
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} + \nabla \cdot \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = -\frac{\partial W_s}{\partial \dot{q}_\alpha},
\]
for a set of coordinates \(q_\alpha\) \([128]\). Here the damping of the magnetisation vector is included as a classical force term \([132]\) at the level of the Lagrangian through the dissipation term
\[
W_s = \frac{\alpha}{2a^3} \int d^3x (\partial_t m)^2,
\]
where \(\alpha\) is the Gilbert damping constant \([133]\). We aim to derive the domain wall dynamics in terms of collective coordinates by first considering the dynamics of a rigid domain wall. We describe the rigid domain wall in terms of spherical polar coordinates defined by the unit magnetisation vector
\[
m(x, t) = [\sin \theta(x, t) \cos \phi(x), \sin \theta(x, t) \sin \phi(x), \cos \theta(x, t)],
\]
\[\text{§}\]The continuum limit Heisenberg Hamiltonian, excluding anisotropies, with exchange coupling \(J\) follows by Taylor expanding the discrete Hamiltonian \(H = -J \sum_{i, \hat{a}} (m_i \cdot m_{i+\hat{a}})\), where the sum over \(i\) is over all lattice sites, and \(\hat{a} \in \{a_x, a_y, a_z\}\) is the lattice constant which we take to be the same in all spatial directions, \(a = a_x = a_y = a_z\).
Domain wall dynamics in Weyl semimetals

where the polar angle is a constant in time. The equations of motion for the rigid domain wall described by $m(x, t)$ are obtained by setting $q_a \in (\theta, \phi)$ in Eq. (4.8), which yields the equations

$$\dot{\alpha} \theta = K \left( \frac{\lambda}{\lambda_c} \right)^2 \left[ \lambda_c^2 \partial_x^2 \theta - \sin \theta \cos \theta \right] + \gamma B \sin \theta \quad (4.11)$$

$$\dot{\theta} = \frac{K_\perp}{2} \sin \theta \sin 2\phi, \quad (4.12)$$

where $\lambda = \sqrt{Ja^2/K}$. These two equations of motion combine into a single equation

$$\gamma (B_c \sin 2\phi - B) = K \left( \frac{\lambda}{\lambda_c} \right)^2 \left[ \lambda_c^2 \partial_x^2 \theta - \sin \theta \cos \theta \right][\sin \theta]^{-1}, \quad (4.13)$$

where $\lambda_c = \lambda/\sqrt{1 + \kappa \sin^2 \phi}$ has the unit of length, $\kappa = K_\perp/K$, and the critical magnetic field is defined as $B_c = \alpha \nu_\perp/\gamma \lambda$, where $\nu_\perp = \lambda K_\perp/2$ has the unit of velocity. The left hand side of Eq. (4.13) is a function of $\phi$, while the right hand side of Eq. (4.13) is a function of $\theta$, so both sides of Eq. (4.13) must be equal to a constant with the unit of frequency $\gamma \tilde{B}$. This constant is fixed by the initial condition of the domain wall, and the choice $\tilde{B} = 0$ gives the equations

$$\sin 2\phi = \frac{B}{B_c}, \quad (4.14)$$

$$\lambda_c^2 \partial_x^2 \theta - \sin \theta \cos \theta = 0. \quad (4.15)$$

The solution for $\phi$ in Eq. (4.14) is only valid for $|B| \leq |B_c|$, which is the Walker regime of solutions in which $\phi = \frac{1}{2} \arcsin \frac{B}{B_c}$ is constant [128]. To solve for $\theta$ we multiply Eq. (4.15) with $\partial_x \theta$ and integrate the resulting equation with respect to $x$, resulting in the second order equation

$$\lambda_c^2 (\partial_x \theta)^2 - \sin^2 \theta + C = 0. \quad (4.16)$$

The boundary condition $\lim_{x \to \pm \infty} m(x) = \pm \hat{z}$ sets the constant of integration $C$ to zero. The nontrivial solutions for the $\theta$ angle in Eq. (4.16) are

$$\theta(x) = 2 \tan^{-1} \left( e^{Q \frac{x - X(t)}{\lambda_c}} \right), \quad (4.17)$$

which describes two different domain walls depending on $Q = \pm 1$ and $X(t)$ is a constant of integration that may be time dependent. By expressing the $\theta$ dependence as the $z$ component of the unit magnetisation vector

$$\cos(\theta) = -Q \tanh \left( \frac{x - X(t)}{\lambda_c} \right), \quad (4.18)$$
we identify $X(t)$ as the position at the centre of the domain wall at time $t$. $Q$ is called the chiral charge of the domain wall [122]. $Q = 1$ defines a domain wall where the direction of $m_z$ changes from $\hat{z}$ at $x \to -\infty$ to $-\hat{z}$ at $x \to \infty$, while $Q = -1$ defines a domain wall where the direction of $m_z$ changes from $-\hat{z}$ at $x \to -\infty$ to $\hat{z}$ at $x \to \infty$. The constant $\lambda_c = \lambda/\sqrt{1 + \kappa\sin^2 \phi}$ is the width of the domain wall, which depends on the hard axis anisotropy through $\kappa = K_\perp/K$.

The result of the detailed maths in the previous paragraph is that we now can express the domain wall solution in the Walker regime as

$$\mathbf{m} = \begin{bmatrix} \cos \phi \\ \cosh \left( \frac{x-X(t)}{\lambda_c} \right) \end{bmatrix}, \begin{bmatrix} \sin \phi \\ \cosh \left( \frac{x-X(t)}{\lambda_c} \right) \end{bmatrix}, -Q \tanh \left( \frac{x-X(t)}{\lambda_c} \right) \right], \quad (4.19)$$

for a centre of the wall coordinate $X(t) = -Q\gamma B\lambda_c t/\alpha$, and a constant polar angle $\phi = \frac{1}{2} \arcsin B/\gamma$. The domain wall moves rigidly along the $x$ axis with a velocity $\dot{X} = -Q\gamma B\lambda_c /\alpha$, obtained by inserting the expression for $\theta(x)$ in Eq. (4.17) into Eq. (4.15). So far we can only describe the dynamics of the magnetic domain wall in response to magnetic fields $|B| \leq B_c = \alpha\nu_\perp / (\gamma\lambda)$. To obtain the full dynamics for all magnetic fields, we must consider the effect of quantum fluctuations around the solutions for the rigid domain wall. This requires a little bit more algebra which we will go through in the next section.

### 4.1.2 The collective coordinate description

The Walker breakdown means that the constant solution $\phi = \frac{1}{2} \arcsin B/\gamma$ is no longer valid, and the equations of motion become complicated to solve analytically due to the large degree of freedom of the domain wall coordinates. This is why we aim to describe the domain wall in terms of only two dynamical coordinates encapsulating the collective behaviour of the domain wall. The two collective coordinates are the position at the centre of the domain wall $X(t)$, and the spatially averaged angle $\phi(t)$ out of the easy plane [122]. These coordinates stem from the translation invariance along the domain wall direction, and the rotational invariance along the easy axis of the domain wall Lagrangian, Eq. (4.1). The hard axis anisotropy breaks the rotational invariance, so the description in terms of two collective coordinates is only exact in the limit of zero hard axis anisotropy. We will obtain the collective coordinate description in this limit, and describe how the zero modes of the quantum fluctuations promote the constants $X$ and $\phi$ to dynamical variables $X(t)$ and $\phi(t)$, where $\phi(t) = \langle \phi(t,x) \rangle_x$ is the spatial average of $\phi(t,x)$ [127, 128].

A finite hard axis anisotropy breaks the rotational invariance of the Lagrangian, and couples the zero modes to higher order fluctuations. This means that additional collective coordinates are needed to accurately describe the dynamics of the domain wall. We will show that the interaction between the zero modes and
higher order quantum fluctuations is negligible in the limit $K_\perp \ll K$ and that $X(t)$ and $\phi(t)$ remain the only relevant collective coordinates in this limit.

We consider a static domain wall in a wire of length $\ell$ in the $x$ direction with a cross section area $A$ in the $yz$ plane. The low energy dynamics of the domain wall are evaluated by using the stereographic projection of the sphere onto the complex plane, Fig. 4.2. We introduce the complex variable $\xi = e^{i\phi}e^{\tan \frac{\theta}{2}}$, such that the static solution of the domain wall in Eq. (4.17) for $Q = -1$, is given by

$$\xi = e^{-\frac{x-X}{\lambda}} + i\phi.$$  

(4.20)

The domain wall Lagrangian for the static wall with zero hard axis anisotropy, and a zero magnetic field, as a function of $\xi$, is

$$L = \int \frac{d^3x}{a^3} \left[ \frac{i}{1 + |\xi|^2} (\xi \dot{\xi} - \dot{\xi} \xi) - \frac{2K}{(1 + |\xi|^2)^2} (\lambda^2 |\partial_x \xi|^2 + |\xi|^2) \right],$$

(4.21)

where the bar in $\bar{\xi}$ represents complex conjugation. The low energy dynamics of the domain wall are incorporated by introducing the fluctuations through a complex function $\eta(t, x - X)$,

$$\xi = \exp[-u(t, x) + i\phi(t) + \eta(t, x - X)],$$

(4.22)

where $u = -\frac{x-X}{\lambda}$. By expanding the domain wall Lagrangian, Eq. (4.21) to second order in $\eta(t, x - X)$ it takes the form

$$L = i \int \frac{d^3x}{a^3} \left( \bar{\eta} \dot{\eta} - \dot{\eta} \bar{\eta} \right) - 2K \int \frac{d^3x}{a^3} \left[ \lambda^2 |\partial_x \eta|^2 + \left( 1 - \frac{2}{\cosh^2 u} \right) |\eta|^2 \right],$$

(4.23)

where $\eta \equiv 2 \cosh(u)\bar{\eta}$. The expansion is performed in detail in Ref. [127].

We aim to express the fluctuations in a basis where we can separate the zero modes from higher order fluctuations and identify the zero modes as collective
coordinates for position and rotation angle. The relevant basis for our purpose is the basis of orthonormal eigenfunctions $\varphi(x - X)$ which are solutions to the eigenequation

$$
\left( -\lambda^2 \nabla_x^2 + 1 - \frac{2}{\cosh^2 \frac{x}{\lambda}} \right) \varphi_k = \omega_k \varphi_k,
$$

where the eigenstates $\varphi_k(x)$ and corresponding eigenvalues $\omega_k$ are

$$
\varphi_k(x) = \frac{1}{\sqrt{\ell \omega_k}} \left( -ik\lambda + \tanh \frac{x}{\lambda} \right) e^{ikx}, \quad \omega_k = 1 + k^2 \lambda^2, \quad k > 0,
$$

$$
\varphi_0(x) = \frac{1}{\sqrt{2\lambda \cosh \frac{x}{\lambda}}}, \quad \omega_0 = 0,
$$

for a wire of length $\ell$. We expand the fluctuations in the basis of the eigenfunctions $\varphi_k$ separating the zero modes from the higher order fluctuations,

$$
\tilde{\eta}(t, x - X) = \frac{2}{\sqrt{2\lambda}} \eta_0(t) + 2 \cosh(x - X) \sum_k \eta_k(t) \varphi_k(x - X),
$$

such that the space dependence of the zero mode $\varphi_0$ is cancelled by the factor $\cosh(x - X)$. This allows us to express the state Eq. (4.22) as

$$
\xi = \exp \left[ -\frac{x - X(t)}{\lambda} + i\phi(t) + 2 \cosh(x - X) \sum_{k \neq 0} \eta_k(t) \varphi_k(x - X) \right],
$$

where we have combined the constant position $X$, and angle $\phi$ with the zero modes to define the collective coordinates $[127,128]$

$$
X(t) = X + \sqrt{2\lambda} \text{Re} [\eta_0(t)],
$$

$$
\phi_0(t) = \phi_0 + \sqrt{\frac{2}{\lambda}} \text{Im} [\eta_0(t)].
$$

This means that the low energy motion of the domain wall is described in terms of these two collective coordinates.

The time dependence of the spherical polar angles is defined as

$$
\cos \theta \equiv \tanh \frac{x - X(t)}{\lambda},
$$

$$
\sin \theta \equiv \text{sech} \frac{x - X(t)}{\lambda},
$$

$$
\phi \equiv \phi(t),
$$
in terms of the two collective coordinates. We should highlight that we use the same notation for the collective coordinate $X(t)$ and for the time dependent integration variable in section 4.1.1, even though these two functions are not related. For the rest of this chapter, we use the function $X(t)$ to refer to the collective coordinate defined by Eq. (4.29).

By using Eqs. (4.31-4.33), the unit magnetisation in terms of the collective coordinates is

$$m(t) = \begin{bmatrix} \frac{\cos \phi(t)}{\cosh \left( \frac{x-X(t)}{\lambda} \right)} & \frac{\sin \phi(t)}{\cosh \left( \frac{x-X(t)}{\lambda} \right)} & -QT \tanh \left( \frac{x-X(t)}{\lambda} \right) \end{bmatrix}$$  \hspace{1cm} (4.34)

where the domain wall is depicted in Fig. 4.3. To be able to make use of the collective coordinate formalism to analyse the dynamics of the domain wall we must express the Lagrangian in terms of Eq. (4.34). Introducing the collective coordinates allows us to explore the dynamics of the domain wall for magnetic fields in the Walker regime and beyond.

### 4.1.3 Equations of motion in terms of collective coordinates

The collective coordinate description in terms of $X(t)$ and $\phi(t)$ is exact in the absence of hard axis anisotropy. To understand the effect of a finite hard axis anisotropy it is useful to express the full domain wall Lagrangian, Eq. (4.1), and the dissipation term, Eq. (4.9), in terms of the collective coordinates and the higher order fluctuations. The expansion of the Lagrangian and the dissipation term to second order in the fluctuations, ignoring terms beyond second order which
Domain wall dynamics in Weyl semimetals couple fluctuations and collective coordinates, leads to the Lagrangian terms:

\[
L = -\frac{2A}{a^3} \left[ \dot{\phi} X + \nu_\perp \sin^2 \phi + \frac{\pi \lambda \gamma}{2} (B_x \cos \phi - B_y \sin \phi) - \gamma B_x X \right],
\]

(4.35)

\[
W = \frac{A \alpha \lambda}{a^3} \left[ \left( \frac{\dot{X}}{X} \right)^2 + \phi^2 \right],
\]

(4.36)

\[
L_{sw} = \frac{A}{a^3} \sum_k \left[ i (\eta^*_k \dot{\eta}_k - \dot{\eta}^*_k \eta_k) - 2 \Omega_k \eta^*_k \eta_k + \frac{\nu_\perp}{\lambda} (\eta_{-k} \eta_k + \eta^*_{-k} \eta^*_k) \right],
\]

(4.37)

\[
L_{int} = -\frac{A}{a^3} \frac{1}{\sqrt{L}} \sum_k \nu_\perp \sin^2 \phi A_k (\eta_k + \eta^*_k),
\]

(4.38)

where \( A \) is the cross section of the sample in the \( yz \) plane, \( \nu_\perp = K_\perp \lambda / 2 \), and \( A_k = \pi \sqrt{\omega_k / \cosh(\pi k \lambda / 2)} \) [128]. \( L_{sw} \) is the contribution to the Lagrangian from higher order fluctuations, which we will ignore in the low energy limit. A finite hard axis anisotropy generates an interaction term \( L_{int} \) between the collective coordinates and the higher order fluctuations. This means that the dynamics are not described by the collective coordinates alone, and more coordinates are needed for an accurate description of the domain wall dynamics. The interaction Lagrangian can be ignored for a small enough hard axis anisotropy. To understand why this is the case, we find the stable solutions for the fluctuations by taking the derivative of \( L_{sw} + L_{int} \) with respect to the modes \( \eta_k \) and \( \eta^*_k \) and set the derivative to zero [127,128]. The stable solutions are

\[
\eta_{sk} = \eta^*_{sk} = -\frac{\pi \lambda}{4\sqrt{L}} \frac{\kappa \sin^2 \phi}{\sqrt{\omega_k} \cosh(\pi k \lambda / 2)}.
\]

(4.39)

By inserting \( \eta_{sk} \) and \( \eta^*_{sk} \) into the expression for the domain wall solution \( \xi \) in Eq: (4.28), and evaluating the sum over \( k \) as an integral, we find that

\[
\xi = e^{-x - X (1 + K_\perp \sin^2 \phi)} \left( \frac{\kappa \lambda}{\omega_c} \right) e^{-x - X \lambda c / \omega_c},
\]

(4.40)

where \( \lambda_c \simeq \lambda (1 + K_\perp \sin^2 \phi)^{-1/2} \). This shows us that a small hard axis anisotropy only contracts the domain wall slightly and that the collective coordinate approximation remains valid in the limit \( K_\perp \ll K \).

We consider the full dynamics of the domain wall in terms of the two collective coordinates \( X(t) \) and \( \phi(t) \) in the limit \( K_\perp \ll K \), ignoring the interaction Lagrangian and the higher order fluctuations. This leads to the equations of motion

\[
\dot{\phi} + \frac{\alpha}{\lambda} \dot{X} = \gamma B,
\]

(4.41)

\[
\dot{X} - \alpha \lambda \dot{\phi} = \nu_\perp \sin 2 \phi,
\]

(4.42)
Domain wall dynamics in Weyl semimetals

which combine into a single equation for \( \phi \):

\[
\dot{\phi} = a_1 - a_2 \sin(2\phi),
\]

where the constants are defined as

\[
a_1 = \gamma B/(1 + \alpha^2)
\]

and

\[
a_2 = \alpha \nu_\perp/[(\alpha^2 + 1)\lambda].
\]

The long time limit solution for constant \( \phi \) in the Walker regime is

\[
\phi = \frac{1}{2} \arcsin \left( \frac{B}{B_c} \right),
\]

reproducing the result in Eq. (4.14). The long time limit velocity of the domain wall position \( X(t) \) in the Walker regime is \( \dot{X} = \gamma B \lambda / \alpha \), obtained from Eq. (4.41) for a constant angle \( \phi \). The Walker solutions break down at the critical magnetic field, \( B_c = \alpha \nu_\perp / (\gamma \lambda) \), and the internal domain wall angle starts rotating, oscillating with frequency \( \omega = \sqrt{a_1^2 - a_2^2} \) in time:

\[
\phi(t) = \arctan \left[ a_1 \tan(\omega t) \right] / \omega + a_2 \tan(\omega t).
\]

This means that the domain wall centre moves with an oscillatory motion along the \( x \) axis in the region where \( B > B_c \). By evaluating the time averaged velocity of \( \sin 2\phi \) over the period time \( T = 2\pi / \omega \), and inserting the expression into Eqs (4.42,4.43) we obtain the time averaged velocity of the domain wall over the period time \( T = 2\pi / \omega \),

\[
\langle \dot{X} \rangle = \frac{\lambda \gamma}{\alpha} B \left[ 1 - \frac{1}{1 + \alpha^2} \sqrt{1 - \left( \frac{B_c}{B} \right)^2} \right], \quad B \geq B_c,
\]

valid for magnetic fields larger than the critical field \( B = B_c \) \[127,128\]. The velocity of the domain wall grows linearly with the magnetic field in the Walker regime, with a maximum at \( \dot{X} = \gamma B_c \lambda / \alpha \) after which it starts decreasing, see Fig 4.4 for a schematic depiction.

The equations of motion of the domain wall subject to an external magnetic field are sometimes more conveniently expressed in terms of the unit magnetisation:

\[
\frac{dm}{dt} = \gamma B \times m + \alpha m \times \frac{dm}{dt} + T_e.
\]

These are the Landau-Lifshitz-Gilbert equations \[127,133,134\]. The last term on the right hand side is a spin torque \( T_e \sim m \times S \) where \( S \) is the spin density operator. We have not considered the spin torque in our derivation of the equations of motion, but mention it here as it plays a role as an emergent torque in the context of Weyl semimetals. In paper II we show that the axial anomaly generates a spin torque on the domain wall, and can be used to change the equilibrium configuration of the wall.
Domain wall dynamics in Weyl semimetals

Figure 4.4: A schematic depiction of the time averaged domain wall velocity $\langle \dot{X} \rangle$ along the $x$ direction as a function of the magnetic field $B$ in the $z$ direction.

The aim of this section was to obtain the Lagrangian formalism of the domain wall in terms of collective coordinates and understand the dynamics of the domain wall under the influence of an external magnetic field. We will consider these results in the context of domain walls in magnetic Weyl semimetals as a means to induce a spatiotemporal variation of the magnetisation to generate axial electromagnetic fields. The appearance of axial electromagnetic fields allows us to understand the covariant anomaly in a condensed matter setting. The axial electromagnetic fields result in the nonconservation of the vector current in the covariant form of the anomaly, which does not make sense in the physical setting of a Weyl semimetal. The solution is that the current is not locally conserved, but it is globally conserved, which we will explain in the next section.

4.2 A spacetime dependent Weyl node separation

The Weyl node separation between two Weyl cones of opposite chirality can be thought of as a magnetisation vector which breaks the time reversal symmetry of the Hamiltonian describing the magnetic Weyl semimetal. In this section, we will focus on the emergence of axial electromagnetic fields due to a spatiotemporal magnetisation vector and their consequences on the axial anomaly. The presence of axial fields in the anomaly equations means that the covariant vector current is not locally conserved. To describe a physical system in which the total charge must be conserved it is important to understand the nonconservation of the covariant current. We will use the Landau level picture to explain the physical meaning of the covariant anomaly in a finite Weyl semimetal.

4.2.1 Axial electromagnetic fields and the axial anomaly

The Weyl node separation couples to fermions like an axial vector field with opposite sign to fermions of opposite chirality. This means that a space and time dependent Weyl node separation $b = b(x, t)$ generates the axial electromagnetic
Domain wall dynamics in Weyl semimetals

Figure 4.5: Left: The Landau level dispersion of a Weyl semimetal with two cones of opposite chirality due to a magnetic field $B$. Right: The Landau level dispersion of a Weyl semimetal with two cones of opposite chirality due to a magnetic field $B_5$.

Fields

$$E_5 = -\partial_t b, \quad B_5 = \nabla \times b.$$  \hspace{1cm} (4.48)

The emergence of parallel axial electromagnetic fields contributes to the axial anomaly in Weyl semimetals,

$$\partial_t n_5 = \frac{1}{2\pi^2} \left( E \cdot B + \frac{1}{3} E_5 \cdot B_5 \right) - \frac{n_5}{\tau},$$  \hspace{1cm} (4.49)

where $\tau$ is the intervalley scattering rate, and provide condensed matter realisation of the axial anomaly in terms of the axial fields.

It is intuitive to think about the axial anomaly in the framework of Landau levels in strong magnetic fields. The lowest Landau level of a magnetic Weyl semimetal in a magnetic field is linearly dispersing, where the sign depends on the chirality of the fermions, the left image in Fig. 4.5. By adding a parallel electric field the fermions start moving from one cone to the other through higher energy bands, creating an imbalance in left and right handed fermions. The inclusion of axial electromagnetic fields complicates this picture. The dispersion relation of a Weyl semimetal subject to an axial magnetic field forms pseudo Landau levels, where the lowest Landau level is linearly dispersing, but with the same sign for both fermion chiralities, since the axial fields couple with opposite sign to fermions of opposite chirality, as depicted in the right image in Fig. 4.5. The Landau level picture of the anomaly is no longer intuitive; by adding an ordinary parallel electric field it seems that fermions of both chiralities are depleted, breaking the conservation of the total charge. It is not clear from Fig. 4.5 how the two lowest Landau levels connect.

The presence of axial fields means that the covariant form of the vector anomaly

$$\partial_\mu j_{\text{cov}}^\mu = (2\pi^2)^{-1} (E \cdot B_5 + E_5 \cdot B),$$ \hspace{1cm} (4.50)

$$\partial_\mu j_{5\text{cov}}^\mu = (2\pi^2)^{-1} (E \cdot B + E_5 \cdot B_5),$$ \hspace{1cm} (4.51)
differs from the consistent form of the anomaly. The connection between the
covariant and consistent anomalies plays a role in explaining the apparent non-
conservation of charge. We know that the total charge in a Weyl semimetal sample
must be conserved, but Eq. (4.50) and Fig. 4.5 seem to suggest that this is not
the case. The answer to this conundrum is that the charge is mediated by the
surface states: If the bulk states say, are losing charge due to an applied electric
field, then this charge loss is compensated by the gaining of charge of the surface
states. The details of this argument are presented in Ref. [102] where the authors
consider a magnetic Weyl semimetal with a constant Weyl node separation in the
bulk, for a sample which is finite in one direction. An axial magnetic field
described by pseudo Landau levels is therefore generated at each surface where the
Weyl node separation vanishes. By applying a parallel electric field, the charge
starts to decrease at one of the surfaces and increase at the other, and charge
conservation is broken locally at each surface.

The nonconservation of the total charge is the manifestation of the covariant
anomaly at the surfaces, \( \partial_\mu j_\mu^{\text{cov}} = (2\pi^2)^{-1} E \cdot B_5 \). However the surfaces do not exist
without the bulk, and the charge is pumped from one edge to the other through
the bulk states generating a net current across the sample. The covariant anomaly
only tracks the flow of charge at the Fermi level and is anomalous since it does
not take into account what is happening at energies away from the Fermi level.
The vector current is conserved in the consistent form of the anomaly, and the
consistent current is equal to the sum of the covariant current and a Bardeen
polynomial. The Bardeen polynomial in this example is precisely the current
mediated by the bulk states. The same reasoning is used to explain the covariant
anomaly in the general case of axial electromagnetic fields in the bulk [102]. The
pseudo Landau levels provide an explanation of the meaning of the covariant
current and the Bardeen polynomials in a finite Weyl semimetal.

4.3 Domain walls and emergent axial electromagnetic fields

The purpose of this section is to unite what we know about magnetic domain
wall dynamics, Weyl semimetals, and the axial anomaly [108,135–138]. The two
main conclusions are that domain wall dynamics in Weyl semimetals give rise to
emergent axial electromagnetic fields and that the resulting axial anomaly can
be measured through the chiral magnetic effect [108]. We consider a magnetic
Weyl semimetal with a domain wall in the Weyl node separation which acts as
a source of an emergent axial magnetic field. We are interested in the axial
anomaly and require the existence of parallel axial electric and magnetic fields, so
to generate the axial anomaly the domain wall also needs to be time dependent.
We will use the theory of magnetic field driven domain wall dynamics that we
introduced in section 4.1 to induce an axial electric field. The emergent axial
electromagnetic fields, and therefore also the axial anomaly, are located at the
moving domain wall. This means that the chiral magnetic effect yields a current along the magnetisation direction acting as a source of electromagnetic radiation, and measuring this radiation would constitute a signature of the chiral anomaly.

Consider a magnetic Weyl semimetal as a wire in the $x$ direction with two domains where the magnetisation vector separating the Weyl nodes points in opposite directions. Each domain is described by a minimal magnetic Weyl semimetal consisting of two Weyl cones of opposite chirality separated by a vector $2b$ in the $z$ direction, Fig. 4.6. $b$ becomes space dependent at the domain boundaries and is modelled by a continuous domain wall $b(x,t) = \Delta/(e\nu_F)m(x,t)$, where $\Delta$ is an effective exchange coupling between the fermions and the magnetisation, $e$ is the elementary charge, $\nu_F$ is the Fermi velocity, and $m(x,t)$ is the unit magnetisation in Eq. (4.34). The domain wall is described by the Lagrangian in Eq. (4.35) subject to an external magnetic field $B = B\hat{z}$.

The unit magnetisation allows us to use the physics of domain wall dynamics in an external magnetic field to generate both the axial electric, and axial magnetic fields, $E_5 = -\partial_t b$, and $B_5 = \nabla \times b = (0, -\partial_x m_z, \partial_x m_y)$. The axial anomaly in the absence of an electric field is given only in terms of the axial electromagnetic fields,

$$\partial_t n_5 = \frac{e^2}{6\hbar^2 \pi^2} (E_5 \cdot B_5) - \frac{n_5}{\tau}, \quad (4.52)$$

where $n_5$ is the axial number density, $\tau$ is the intervalley scattering time, and

$$E_5 \cdot B_5 = \frac{\Delta^2}{e^2 \nu_F \lambda} \frac{\dot{\phi} \cos \phi}{\cosh^3 \left(\frac{x - X(t)}{\lambda}\right)}. \quad (4.53)$$

The parallel axial fields in Eq. (4.53) are proportional to $\dot{\phi}$, which means that it only exists for magnetic fields beyond the Walker breakdown, for $|B| > B_c$. The axial electric field has nonzero components in the Walker regime, but all terms independent of $\dot{\phi}$ are orthogonal to the axial magnetic field. To realise the axial
Domain wall dynamics in Weyl semimetals

anomaly in practice it is important that the required magnetic field strength can be obtained experimentally. The critical magnetic field \( B_c = \alpha \nu_\perp / (\gamma \lambda) \) is of order 30 \( \mu \text{T} \) for typical parameter values \([108]\), which is of the same order of magnitude as the Earth’s magnetic field. This means that field driven domain wall dynamics can be used to generate the axial anomaly in a laboratory environment. In section 4.1.3 we learnt that the internal angle

\[
\phi(t) = \frac{\arctan [a_1 \tan (\omega t)]}{\omega + a_2 \tan (\omega t)},
\]

oscillates with a frequency \( \omega = \sqrt{a_1^2 - a_2^2} \) where \( a_1 = \gamma B / (1 + \alpha^2) \) and \( a_2 = \alpha \nu_\perp / (\alpha^2 + 1) \lambda \) in the Walker breakdown region. So the anomaly equation contains two relevant time scales: the period time \( \tau_\phi = 2\pi / \omega \) of the fields \( E_5 \cdot B_5 \), and the inter valley scattering time \( \tau \). The anomaly equation is solved adiabatically in the limit \( \tau_\phi \gg \tau \), where

\[
n_5(x) = \frac{e^2 \tau}{6\pi^2} (E_5(x) \cdot B_5(x)), \quad \tau_\phi \gg \tau,
\]

for large times \( t \gg \tau \). In the opposite limit, \( \tau_\phi \ll \tau \), where the axial electromagnetic fields oscillate much faster than the scattering between the Weyl cones, the axial number density is

\[
n_5(x,t) = \frac{e^2 \tau}{6\pi^2} \int_0^t E_5(x,s) \cdot B_5(x,s), \quad \tau_\phi \ll \tau.
\]

The period time \( \tau_\phi \) depends on the size of the external magnetic field through \( a_1 \), so the relevant solutions to the anomaly equation depend on the size of the magnetic field. For typical parameter values with an intervalley scattering time of order \( \tau \sim 1 \text{ ns} \) \([108]\), \( \tau_\phi \ll \tau \) for magnetic fields larger than 0.1 T.

The imbalance of left and right handed fermions generates a chemical potential difference \( \mu_5 = (\mu_L - \mu_R) / 2 \) between the two Weyl cones with chemical potential \( \mu_{L/R} \). The relation between the axial number density and the axial chemical potential takes a simple form in the limit \( \hbar eB \ll \mu_5^2 / \nu_F \), where \( 3\pi^2 \hbar^2 \nu_F \mu_5^3 = \mu_5^3 + \mu_5(\pi^2 k_B^2 T^2 + 3\mu^2) \). Here \( T \) is the temperature, \( k_B \) is the Boltzmann constant and \( \mu = (\mu_L + \mu_R) / 2 \) is the total chemical potential.

The difference in the chemical potential in the two Weyl cones gives rise to the chiral magnetic effect:

\[
J_z = \frac{e^2}{2\pi^2 \hbar^2} \mu_5 B_z,
\]

a current proportional to the axial chemical potential \( \mu_5 \) in the direction of the applied magnetic field. This current is located at the moving domain wall and acts as a source of electromagnetic radiation, which if measured constitutes a
domain wall dynamics in Weyl semimetals

direct signal of the axial anomaly. The strength of the radiation depends on the magnitude of the current which in turn depends on the magnitude of the external magnetic field. The chiral magnetic effect is not the only current that appears at the domain wall, the curl of the magnetisation also gives rise to an effective current which contributes to the radiation. The research presented in paper I delves into the details of the measured radiation. We find that the measured power of the radiation due to the chiral magnetic effect dominates in both the near and far field limits, making it possible to distinguish the two sources [108]. The measured power therefore acts as an indication of the emergent anomaly due to domain wall motion in Weyl semimetals.

The manipulation of magnetic domain walls and other magnetic structures is a field of its own, and it is interesting to ask what role Weyl semimetals and the axial anomaly play on domain wall dynamics. We have explained how the presence of a domain wall in a Weyl semimetal leads to the axial anomaly. In the next section, we instead explore how the axial anomaly in terms of external electromagnetic fields affects the domain wall. The axial anomaly acts as a mediator which couples the electric field with the magnetisation. The electric field can therefore be used to manipulate the equilibrium configuration of the domain wall.

4.4 Axial anomaly manipulation of the domain wall

The previous section was dedicated to the emergence of axial electromagnetic fields and the axial anomaly in Weyl semimetals with a domain wall. In this section we are instead interested to learn how Weyl physics affects the domain wall—how the coupling between Weyl fermions and the magnetisation in the Weyl semimetal contributes to the magnetic domain wall theory presented in section 4.1. The fact that the domain wall is considered a part of the Weyl semimetal means that properties such as the Fermi arc boundary states and the axial anomaly will affect the domain wall and its dynamics. The physical consequences are all described by the total Lagrangian combining the bare domain wall Lagrangian (Eq. (4.35)), and the effective magnetisation Lagrangian derived from the Weyl semimetal. The derivation of this effective Lagrangian is presented in paper II [138], and we will just summarise the results below.

4.4.1 Effective domain wall Lagrangian

The total effective domain wall Lagrangian, \( L_{\text{tot}} = L_{\text{FM}} + L_{\text{eff}} \) consists of the domain wall Lagrangian in Eq. (4.35),

\[
L_{\text{FM}} = -\frac{2A}{a^3} \left[ \dot{\phi}X + \nu_\perp \sin^2 \phi + \frac{\pi \lambda \gamma}{2} (B_x \cos \phi - B_y \sin \phi) - \gamma B_z X \right],
\]

(4.58)
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and the effective Lagrangian, $L_{\text{eff}}$, derived from the coupling between fermions and
the magnetisation in the Lagrangian of the Weyl semimetal. The Weyl semimetal
Lagrangian is defined as

$$L_{\text{Weyl}} = \int d^3x \left( \bar{\Psi} i\gamma^0 \partial_0 \Psi - H \right),$$

(4.59)

$$H = \bar{\Psi} (-i\nu_F \gamma^i \partial_i + e\gamma^\mu A_\mu - e\nu_F b_\mu \gamma^\mu \gamma^5) \Psi,$$

(4.60)

with a metric signature $g^{\mu\nu} = \text{diag}[1, -1, -1, -1]$, Greek indices $\mu \in (0, 1, 2, 3)$, and Roman indices $i \in (1, 2, 3)$. $A_\mu$ is an external electromagnetic gauge field
defined as $A_\mu = (A_t, \nu_F A_i)$, where $\nu_F$ is the Fermi velocity and $e$ is the
elementary charge. The magnetisation separates the Weyl nodes in momentum
$b(x, t) = \Delta/(e\nu_F) m(x, t)$, where $\Delta$ is an exchange coupling between the electrons and the magnetisation.
The effective theory of the magnetisation is obtained by integrating out the fermionic degrees of freedom. In chapter three
we derived an effective theory for bound states at the edge of a domain wall
where the magnetisation vector vanishes. The domain wall similarly constitutes a
boundary between two domains where the magnetisation changes sign, and hosts
bound states. The effective theory is therefore obtained by considering the bound
states at the domain wall, and the extended plane wave bulk states away from
the domain wall separately. To do this in practice we consider the magnetisation
to consist of a background magnetisation and fluctuations, $b_i = \tilde{b}_i + \delta b_i$, where $\tilde{b}_i = \Delta/(e\nu_F)(0, 0, \tanh[x/\lambda])$ is centred at the domain wall. The back-
ground magnetisation $\tilde{b}_z$ is the zeroth order term in an expansion of $b_z$ around $X(t) = 0$ expressed in the frame of reference of the domain wall, and the fluctuations are $\delta b_i = \Delta/(e\nu_F)(m_x, m_y, \delta b_z)$, where $m_x = \cos \phi / \cosh([x - X(t)]/\lambda)$ and $m_y = \sin \phi / \cosh([x - X(t)]/\lambda)$ are components of the unit magnetisation, and $\delta b_z$ consists of all higher order terms in $X(t)$.

The derivation of the effective Lagrangian for the bound states is done analogously to that of the bound states in chapter three and is found in Ref. [138].
Integrating out the fermionic degrees of freedom from the bound state action, and
expanding the remaining action to second order in both electromagnetic gauge
fields, leaves the effective action for the magnetisation

$$\Gamma_{\text{bound}} = -\frac{\Delta^2}{2\pi\nu_F \hbar} \int dt dy m_0^0(t)m_0^0(t).$$

(4.61)

$m_0^0 = \sin \phi$ is the zeroth order term in the expansion of $m_0$ around the domain
wall centre, this expansion is implemented already at the level of the bound states
action since $m_0^0$ is independent of position and simplifies the analytic evaluation
of the effective action. $\Gamma_{\text{bound}}$ is proportional to $m^2 = \sin^2 \phi$, which is also how
the hard axis anisotropy enters the domain Lagrangian, Eq. (4.58). So $\Gamma_{\text{bound}}$ is
a Fermi-arc induced effective hard-axis anisotropy,

\[
K_{\text{eff}}^\perp = \frac{\Delta^2}{L_z \lambda \nu_F}, \quad (4.62)
\]

where \(L_z\) is the thickness of the domain wall in the \(z\) direction. In terms of the effective hard axis anisotropy the effective Lagrangian corresponding to Eq. (4.61) is

\[
L_{\text{bound}} = -L_y L_z \lambda K_{\text{eff}}^\perp \sin^2 \phi, \quad (4.63)
\]

where \(L_y\) is the width of the sample in the \(y\) direction.

The maximal domain wall velocity \(\dot{X} = \gamma B_c \lambda / \alpha\) is proportional to the hard axis anisotropy since \(B_c = \alpha K_{\text{eff}}^\perp / (2\gamma)\). So the emergent hard axis anisotropy results in a critical magnetic field of order \(\gamma B_c \sim 15 \text{ mT}\), which leads to domain wall velocities of about 2.6 km per second even when the intrinsic hard axis anisotropy of the material is small [138]. In comparison, the maximum domain wall velocity in nanowires is of order \(10 - 100 \text{ m/s}\) [139, 140]. Achieving high domain wall velocities in Weyl semimetals is relevant in the field of spintronics, where the search for ways to maximise the velocity of domain walls is an ongoing exploration.

The effective bulk states are calculated by integrating out the fermionic degrees of freedom from the Weyl action, the temporal integral over the Weyl Lagrangian in Eq. (4.59), and expanding the remaining action to second order in the external electromagnetic field and the fluctuations of the magnetisation

\[
\Gamma^{(2)}[A,b] = \frac{i}{2} \text{Tr} \left( \frac{eA - ev_F \delta b \gamma^5}{i\partial - ev_F \tilde{b} \gamma^5} \cdot \frac{eA - ev_F \delta b \gamma^5}{i\partial - ev_F \tilde{b} \gamma^5} \right), \quad (4.64)
\]

This derivation is analogous to that for the electromagnetic field in chapter three, and leads to a Chern-Simons like term for the magnetisation [138],

\[
\Gamma^5_{\text{CS}} = \frac{\Delta^3}{12\pi^2 \nu_F^3 \hbar^2} \int d^4 x e^{\mu\nu\rho\sigma} m_\mu \tilde{m}_\nu \partial_\rho m_\sigma \approx -\frac{A \Delta^3}{9\pi^2 \nu_F^3 \hbar^2} \int dt \dot{\phi} X, \quad (4.65)
\]

to first order in the domain wall position \(X(t)\). \(\Gamma^5_{\text{CS}}\) does not add anything new to the domain wall Lagrangian, it simply renormalises the Berry phase term in Eq. (4.58) [138], and as \(\Gamma^5_{\text{CS}}\) is much smaller than the intrinsic Berry phase term in Eq. (4.58), it can be ignored. There exist two more bulk contributions which are not derived from the effective action: the chiral separation effect \(j_5^i \propto \mu B^i\), and the axial separation effect \(j_5^i \propto \mu_5 B^i_5\) [138]. These two currents are derived by taking the functional derivative of their underlying actions with respect to the axial gauge field. This means that the chiral separation effect stems from an effective action of the form \(\Gamma_{\text{cse}} \propto \int dt \mu A_{5,i} B_i\), where \(J^i_5 = \delta \Gamma_{\text{cse}} / \delta A^i_5\). The axial gauge field acts as a magnetisation, \(A_5 = \Delta m / e \nu_F\) so \(\Gamma_{\text{cse}} \propto m \cdot B\). The
chiral separation effect therefore renormalises the Zeeman term in the domain wall Lagrangian.

The axial separation effect is obtained by differentiating the action

\[ \Gamma_{\mu_5} \propto \mu_5 \int d^4x \, \varepsilon^{ijk} m_i \partial_j m_k, \quad (4.66) \]

with respect to an axial gauge field, where \( A_5 = \Delta m / e \nu_F \). The axial separation effect is

\[ J_5^i = \frac{e^2}{6 \pi^2 \hbar^2} \mu_5 B^i, \quad (4.67) \]

where the coefficients are obtained by comparing it to the chiral magnetic effect,

\[ J_i^5 = \frac{e^2}{2 \pi^2 \hbar^2} \mu_5 B_5^i. \]

Both of these currents are obtained from a triangular Feynman diagram. The chiral magnetic effect stems from a Feynman diagram with one axial gauge field and two vector gauge fields, whilst the axial separation effect is obtained from a Feynman diagram with three insertions of axial gauge fields, where \( \mu_5 \) is a component of \( A_{5,0} \). Each axial gauge field couples with a \( \gamma^5 \), so there is one \( \gamma^5 \) corresponding to the chiral magnetic effect, and three \( \gamma^5 \) corresponding to the axial magnetic effect. But \((\gamma^5)^2 = 1\) which means that the triangle diagrams corresponding to the two currents are equivalent up to a factor of proportionality of one third. In the chiral magnetic effect, one can permute the two vector gauge fields to obtain a factor \( 1/2! \), while the permutation of the three axial gauge fields in the axial separation effect leads to a factor \( 1/3! \). This means that the corresponding Lagrangian is

\[ L_{\mu_5} = \frac{\Delta^2}{12 \pi^2 \nu_F^2 \hbar^2} \mu_5 \int d^3x \, \varepsilon^{ijk} m_i \partial_j m_k. \quad (4.68) \]

The axial chemical potential is only nonzero in the presence of the axial anomaly. Consider the axial anomaly in terms of constant parallel electromagnetic fields:

\[ \partial_t n_5 = \frac{e^2}{2 \hbar^2 \pi^2} \mathbf{E} \cdot \mathbf{B} - \frac{n_5}{\tau}, \quad (4.69) \]

which has the steady state solution \( n_5 = \frac{e^2}{2 \hbar^2 \pi^2} \mathbf{E} \cdot \mathbf{B} \). The axial number density is proportional to the axial chemical potential in the limit where \( \hbar e B \ll \mu_5^2 / \nu_F^2 \) and \( \mu_5 \ll \mu, k_B T \) [26],

\[ \mu_5 = \frac{3 \hbar v_F^2 e^2 \tau}{2 (\pi^2 k_B^2 T^2 + 3 \mu^2)} \mathbf{E} \cdot \mathbf{B}. \quad (4.70) \]

Inserting Eq. (4.70) into the expression for the Lagrangian Eq. (4.68), and expressing the magnetisation in terms of the collective coordinates yields the expression [138]

\[ L_{\mu_5} = \frac{A \Delta^2 \nu_F e^2 \tau}{8 \pi \hbar (\pi^2 k_B^2 T^2 + 3 \mu^2)} (\mathbf{E} \cdot \mathbf{B}) \sin \phi. \quad (4.71) \]
This means that the axial anomaly mediates a coupling between the electric field and the magnetisation.

The final effective Lagrangian due to the coupling between fermions and magnetisation is $L_{\text{eff}} = L_{\text{bound}} + L_{\mu_5}$ where we ignore the terms renormalising the domain wall Lagrangian as these only provide very small corrections for typical parameter values [138]. $L_{\text{bound}}$ contributes as an effective hard axis anisotropy and $L_{\mu_5}$ acts as a spin torque on the domain wall. The spin torque is defined as $T_e \propto m \times S$, where $S = \Psi^\dagger \sigma \Psi$ is the spin density operator. The axial gauge field couples to an axial current, $(e\nu_F b_\mu j_5^\mu)$, in the Weyl semimetal Hamiltonian,

$$H = \bar{\Psi} \left( -i\nu_F \gamma^i \partial_i + e\gamma^\mu A_\mu - e\nu_F b_\mu \gamma^\mu \gamma^5 \right) \Psi,$$  

which written in a chiral basis leads to the expression $\Psi^\dagger (\Delta^1 \otimes \sigma \cdot m) \Psi$ for the magnetisation part. The spin density is therefore $S = \Psi^\dagger \sigma \Psi = j_5^5/e\nu_F$ in terms of the axial current. This means that the spin torque is $T_e \propto m \times J_5 \propto m \times (\mu_5 B_5)$, where we evoked the axial separation effect in the last step. The axial separation effect is the cause of $L_{\mu_5}$, which is a spin torque term. The contribution of $L_{\mu_5}$ to the equations of motion is therefore a spin torque

$$T_e = -\frac{A\Delta^2 \nu_F \epsilon^2 \tau}{8\pi \hbar (\pi^2 k_B^2 T^2 + 3\mu^2)} (E \cdot B) \cos \phi.$$ 

The anomaly induced spin torque term is used to manipulate the equilibrium configuration of the domain wall as shown in paper II. By tuning the electric field the domain wall changes smoothly between a left handed Bloch wall and a right handed Bloch wall. By observing such a chirality change in the domain wall by changing the electric field would constitute a signature of the chiral anomaly.

### 4.5 Summary

We have derived a Lagrangian formalism for domain wall dynamics in terms of two collective coordinates describing the position of the centre of the domain wall and the spatially averaged internal angle out of the easy plane of the domain wall. The domain wall is driven by a magnetic field, resulting in two different kinds of dynamics. The domain wall moves rigidly, with a fixed internal angle, along the $x$ axis for magnetic fields below a critical value. The velocity of the domain wall grows linearly with the magnetic field in this regime, reaching a maximum at the critical value of the magnetic field, which is called the Walker breakdown. The internal angle starts rotating at magnetic fields larger than the critical magnetic field, and the velocity decreases.

Domain wall dynamics play a role in the generation of axial electromagnetic fields in magnetic Weyl semimetals, which is why we have derived the domain wall formalism to such detail. We consider a domain wall in the magnetisation vector...
Domain wall dynamics in Weyl semimetals

separating two Weyl nodes of opposite chirality. The Weyl node separation acts as an axial gauge field which is made space time dependent by the moving domain wall, giving rise to axial electromagnetic fields. Our interest in domain wall dynamics is twofold: we have discussed how the axial anomaly is induced by parallel axial electromagnetic fields and how it can be measured via the chiral magnetic effect. We have also described how the axial anomaly due to external electromagnetic fields acts as a spin torque on the domain wall mediating a coupling between the magnetisation and the electric field. The equilibrium configuration of the domain wall can be continuously changed between a right handed and a left handed Bloch wall by tuning the electric field. The manipulation of domain walls with electric fields plays an important role in the field of spintronics. The change of domain wall chirality is a direct consequence of the axial anomaly, and a measurement of this chirality change would be a direct signature of the anomaly in the Weyl semimetal.
Chapter 5

Local topological markers

When we think about topological matter we often think about crystalline materi-
als. Translation invariance simplifies the description and classification of topo-
logical matter, but it is not a requirement for its existence. Symmetry protected
Topological phases are protected by time reversal symmetry, particle hole sym-
metry, and chiral symmetry. These three symmetries are local, and their role in
topological protection does not rely on lattice details. Topological insulators and
superconductors have been modelled on many types of nontranslational invari-
ant lattice structures including amorphous [141–145], hyperbolic [146,147], and
quasicrystalline [148,149] lattices. There are also examples of amorphous matter
protected by spatial symmetries [150–153].

As topology exists in noncrystalline matter it becomes relevant to understand
how it is characterised in structures far from a translation invariant limit. Cryst-
talline structures have the advantage that they can be described by periodic func-
tions using Bloch’s theorem. The K-theory classification of topological insulators
and superconductors [65] relies on the existence of the Brillouin zone, and these
classes are characterised by topological invariants expressed in closed form us-
ing the periodicity of momentum space. These closed form expressions are no
longer accessible when translation invariance is broken, and one must instead rely
on alternative ways of characterising the topological phases of noncrystalline in-
sulators and superconductors. Many such characteristics fall under the diverse
category of topological markers, which for example include the local topological
markers [16,154–159], the nonlocal Bott index and spin Bott index [160–169], and
the spectral localisers [170–173].

The focus of this chapter is on the local Chern marker [16,154] and the local
chiral, and Chern-Simons markers [157]. The local Chern marker is the Fourier
transform of the Berry curvature and is a local value that can be evaluated for
each point on a lattice. This value is quantised and equals the Chern number
at each site if the lattice is translation invariant. For a disordered lattice, the
local Chern marker varies from site to site, fluctuating around the quantised
value of the topological phase in a translation invariant limit. The quantisation is
only obtained by averaging the local Chern marker over a large enough volume,
where the number of averaged sites needed is model dependent [16,157]. The
theory behind this averaging corresponds to a rescaling of the lattice. In the long
wavelength limit the lattice becomes translation invariant, so the volume required to quantise the average Chern marker corresponds to a single lattice site on the coarse grained, now translation invariant, lattice. This rescaling is also the reason why the local Chern marker remains a well defined object for nontranslation invariant lattices; we can only evaluate the Berry curvature and derive the Chern marker through a Fourier transform on the coarse grained lattice [16,157].

The local Chern marker is a $\mathbb{Z}$ invariant which for example characterises the topological phases of the Chern insulator in class A in the Altland-Zirnbauer classification [16]. The Chern number, and in turn the local Chern marker, only exists in even spatial dimensions. The chiral winding number is a $\mathbb{Z}$ valued topological invariant characterising the topological phase of Hamiltonians in odd dimensional symmetry classes which obey the chiral constraint. It is not a straightforward procedure to obtain a local topological marker in odd spatial dimensions through a Fourier transform of the chiral winding number. This chapter aims to explain why this is the case, and how we overcome this problem, which is also the topic of the research presented in paper IV. The resulting odd dimensional markers are a $\mathbb{Z}$ invariant local topological chiral marker characterising topology in odd dimensions protected by chiral symmetry, and a $\mathbb{Z}_2$ invariant local Chern-Simons marker for phases protected by either time reversal symmetry or particle hole symmetry, depending on the odd spatial dimension. These local markers can be used to define the topological phase on any lattice, but since the examples provided in paper IV focus on topological amorphous matter [13,14], we will start with a brief introduction to amorphous matter.

5.1 Amorphous matter

An amorphous lattice is a disordered lattice which lacks long range order but with a well defined local environment defined by the preferred bonding between atoms in amorphous materials [12]. The atomic bonding corresponds to preferred bond lengths and bond angles, resulting in a lattice with a fixed coordination number where each lattice site has the same number of vertices [12,174]. Take graphene for example; crystalline graphene consists of a honeycomb lattice made up of hexagons with threefold coordination. Amorphous graphene, whilst still defined by a threefold coordination, consists of polygons of any size. The atomic structure of crystalline lattices is deciphered from their diffraction spectra characterised by sharp Bragg peaks. The lack of long range order in amorphous matter means that the Bragg peaks are replaced by diffuse rings, and it is this diffraction pattern that determines whether a material is amorphous [12]. The width of the diffraction rings is associated with the distribution of varying bond lengths, and more structural disorder leads to broader and blurrier rings.

Amorphous matter is abundant in physical applications, and amorphous structures can be grown more flexibly in a laboratory compared to crystalline mate-
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There are several theoretical examples of topological phases in amorphous matter, including strong symmetry protected phases of matter \cite{141, 145, 175–182}, topological phases protected by spatial symmetries \cite{150–152}, and amorphous Weyl semimetals \cite{160}. Topology in amorphous structures has been realised experimentally in mechanical Chern insulators \cite{143} and through the observation of localised edge modes in photonic models \cite{142, 144}. There is also an experimental realisation of thin films of amorphous Bi$_2$Se$_3$ displaying metallic edge modes \cite{183}.

Amorphous lattices can be modelled in different ways. One option is to generate a random point set and define a radius of connectivity where a lattice site is connected to all other lattice sites within the volume defined by the radius \cite{141}. Another option is to start with a crystalline lattice and draw each lattice point from a Gaussian distribution centred at the crystalline lattice sites. The disorder strength of the lattice is therefore controlled by the width of the Gaussian distribution \cite{160}. One can construct more realistic models by imposing a fixed coordination number when defining the lattice \cite{179}.

The examples in paper IV explore the topology of two different three dimensional amorphous models: an amorphous topological superconductor, and an amorphous time reversal invariant topological insulator. The chosen Hamiltonians are known to host topological phases for certain parameter values in the crystalline limit, and the aim is to use the local chiral, and Chern-Simons markers to define the topological phase and explore how the strength of the structural disorder affects the topology. These topological markers, including the Chern marker, are functions of the projection operator projecting onto the occupied bands. Before we define the odd dimensional local chiral and Chern-Simons markers, we will derive the expression for the Chern character in terms of projectors. We have already met the Chern character very briefly in chapter one when we explored the topological index. In the remainder of this chapter, we will focus on this object and explain how we can use it to define local markers in odd dimensions. Since the projectors are the defining property of the local topological markers we must make a connection between the topological classification of free fermion Hamiltonians and these projectors.

5.2 Topological classification of free fermion ground states

In chapter three we described how all topological insulators and superconductors belong to one of the ten Altland-Zirnbauer classes of free fermion Hamiltonians. The symmetries of the Hamiltonians in the topological classification are also symmetries of the projectors onto the occupied states. So we can equally define a classification in terms of ground states, or the corresponding projectors onto the occupied states.
Topological insulators and superconductors are examples of a class of objects referred to as symmetry protected topological phases. These are topological phases of matter where the states are short range entangled. Short range entangled states are defined by local unitary transformations—any short range entangled state can be transformed into a product state through a local unitary transformation [72, 184].

A free fermion state is a Slater determinant state and can be transformed into any other free fermion state through a unitary transformation. Two different Slater determinant states are defined to be topologically equivalent if they are connected by a local unitary transformation which preserves the local symmetries of the state in a given Altland-Zirnbauer class. The topological phase of a Slater determinant state
\[ |\psi\rangle \propto \prod_\alpha \psi_\alpha^\dagger |0\rangle \]
is characterised by the Bogoliubov-de Gennes single-particle density matrix
\[ \rho_{\alpha\beta} = \langle \psi | \psi_\beta^\dagger \psi_\alpha | \psi \rangle , \] (5.1)
where \( \{ \psi_\alpha^\dagger \} \) are Bogoliubov-de Gennes creation operators, and \( |0\rangle \) denotes the state with fermion number zero [185–187]. The index \( \alpha = (a, s) \) defines the single-particle space \( \psi_{a,s}^\dagger = \delta_{s,1}c_a^\dagger + \delta_{s,-1}c_a \), where \( c^\dagger \) and \( c \) are fermion creation and annihilation operators, and \( s = \pm 1 \) is a particle hole index. The diagonalised single-particle density matrix \( \rho_{\alpha\alpha} = n_\alpha |\phi_\alpha\rangle \) is described in a basis of natural orbitals \( |\phi_\alpha\rangle \), where the corresponding eigenvalues \( n_\alpha \in [0, 1] \) are interpreted as the occupation of the natural orbitals [185, 186, 188, 189]. The Slater determinant state \( |\psi\rangle \) is a product state of \( N \) single particle states, so the corresponding eigenvalues \( n_\alpha \) of \( \rho \) are either one or zero, where the eigenstates with eigenvalue \( n_\alpha = 1 \) span the space of the \( N \) single particles states in \( |\psi\rangle \) [185, 186]. This means that the single-particle density matrix of a free fermion state is a projector onto the occupied states.

The single-particle density matrix is block diagonal for states \( |\psi\rangle \) that are constrained by a unitary symmetry. It is the blocks of the single-particle density matrix that will enter the topological classification, equivalently to the classification of symmetry reduced blocks of the free fermion Hamiltonian. For the rest of this chapter, we will refer to these blocks as the single-particle density matrix, and use the term projector and single-particle density matrix interchangeably. States which have a conserved fermion number are described by the block diagonal density matrix
\[ \rho = \begin{pmatrix} \rho^{U(1)} & 0 \\ 0 & 1 - (\rho^{U(1)})^* \end{pmatrix} , \] (5.2)
where \( \rho^{U(1)} \) is a matrix with elements \( \rho^{U(1)}_{ab} = \langle \phi | c_b^\dagger c_a | \phi \rangle \), in terms of a free fermion state \( |\phi\rangle \propto \prod_i c_i^\dagger |0\rangle \). So \( \rho^{U(1)} \) is the single-particle density matrix in this example. The single-particle density matrix is basis independent: connecting two states \( |\phi\rangle \) and \( |\psi\rangle \) through a unitary operator \( U \) such that \( |\phi\rangle = U |\psi\rangle \) leads to the relation,
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c_i = U^d_i U, such that \( \rho_{ij} = \langle \phi|c_i^\dagger c_j|\phi \rangle = \langle \psi|d_i^\dagger d_j|\psi \rangle \). In the language of the single-particle density matrix two states are defined to be topologically equivalent if their projectors can be adiabatically connected without closing the spectral gap whilst remaining localised and preserving the symmetries of the symmetry class in question.

The Chern number can be expressed as a bundle invariant defined as an integral over the Brillouin zone of the Chern character. The odd dimensional markers that we introduce in paper IV can also be expressed through the Chern character, but only after some manipulation which requires that the Chern character is defined in terms of the single-particle density matrix projecting onto the filled bands. Our first task on our quest to formulate odd dimensional local markers is therefore to define the Chern character in terms of projectors.

5.3 The Chern character in terms of projectors

The Chern marker expressed in terms of projectors is a bundle invariant, a topological invariant which depends on a given fibre bundle. To derive the desired expression of the Chern character we will need some concepts from differential geometry, including the directional derivative, parallel transport, and fibre bundles. In chapter three we learnt that the Chern number depends on the adiabatic theorem, and on traversing states along closed loops in momentum space. In the language of fibre bundles we are again concerned with closed loops in parameter space, so we need to understand what it means to compare states at different points on the base space of the fibre bundle.

A fibre bundle is a structure defined by the onto map \( \pi : E \rightarrow M \), where the manifold \( E \) is called the total space, the manifold \( M \) is called the base space, and \( \pi \) is called a projection map. The space \( E_p = \{ q \in E : \pi(q) = p \} \) for \( p \in M \) is called a fibre over \( p \), where the total space is made up by the union of all fibres, \( E = \bigcup_{p \in M} E_p \) [4, 190]. The fibre bundles that we consider are vector bundles where the total space is defined as a subset of \( \mathcal{H} \times \Lambda \), where \( \mathcal{H} \) is a Hilbert space and \( \Lambda \) is a parameter space, such that

\[
E = \bigcup_{\lambda \in \Lambda} \{ h_\lambda \times \{ \lambda \} \}. \tag{5.3}
\]

Here \( h_\lambda \) is a sub-Hilbert-space of \( \mathcal{H} \), and \( \Lambda \) is the first Brillouin zone. The projection map is a projector onto the filled bands of a given Hamiltonian and the image of this projector constitutes a vector space—the fibres consist of a vector space at each momenta.

We are interested in an adiabatic time evolution of the states in the filled bands and want to know how constant they are as they traverse along a curve in momentum space. A state is constant with respect to a parameter if its derivative is zero such that \( \partial_t |\psi(\lambda(t))\rangle = 0 \). But we need to compare states at different momenta on the base space, which corresponds to different vector spaces. The
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relevant condition for a constant state is instead the projected derivative [76]

\[ P_{\lambda(t)} \partial_t |\psi(\lambda(t))\rangle = 0, \tag{5.4} \]

where \( P_{\lambda(t)} \) is a projector onto \( h_{\lambda(t)} \). Let us consider an example of tangent vectors on a curved manifold to understand why this is the case.

We define a two dimensional vector field \( V(x) \equiv V^\mu(x)e_\mu(x) \) which lives on a tangent plane to a curved surface embedded in a three dimensional Euclidean space. Since the basis vectors are space dependent they contribute to the derivative of the vector field:

\[ \partial_\nu V(x) = \partial_\nu [V^\mu(x)e_\mu(x)] = [\partial_\nu V^\mu(x)]e_\mu(x) + V^\mu(x)\partial_\nu e_\mu(x). \tag{5.5} \]

The expression \( \partial_\nu e_\mu(x) \) defines the change of the basis vector in the direction \( \nu \) which will in general contain a component in the direction normal to the plane, \( \hat{n} = e_1 \times e_2 / |e_1 \times e_2| \). So we split the derivative into normal and tangential components \( \partial_\nu e_\mu(x) = \Gamma^\mu_{\sigma\nu} e_\mu(x) + K^\mu_{\nu} \hat{n}(x) \). But the vectors live on the tangential plane of the two dimensional curved surface and do not care for any vectors sticking out of this plane. We are therefore only interested in the covariant derivative which ignores the component which is normal to the plane:

\[ P_{\hat{n}\perp} \partial_\nu V(x) = [\partial_\nu V^\mu(x)]e_\mu(x) + V^\mu(x)\Gamma^\mu_{\sigma\nu} e_\mu(x), \tag{5.6} \]

where \( P_{\hat{n}\perp} \) is the projector onto the tangent plane, and \( \Gamma^\mu_{\sigma\nu} \) is called a connection; this specific connection is the Christoffel symbol [191]. This procedure is closely connected with the concept of parallel transport of a vector, of moving the vector along a curve whilst keeping it pointing in the same direction. In flat space, we can simply move the vector along the curve to compare two tangent vectors. In curved space, we must throw away the normal component of the tangent vector after moving it along the curve. So the covariant derivative compares the vector \( V(x + \delta x) \) with the vector \( V(x) \) parallel transported to the point \( x + \delta x \) [191].

A fibre bundle is a more complicated structure than the tangent vectors of a curved manifold, but the principle of comparing vectors at different points on the manifold is the same [76]. The state \( |\psi(\lambda(t))\rangle \) depends on its location in the base space and therefore changes in time internally because of its time dependence, but also externally because \( h_{\lambda(t)} \) is changing. To evaluate the change of a state \( |\psi(\lambda(t + \delta t))\rangle \) we compare it with the state \( |\psi(\lambda(t))\rangle \) by projecting away any states that are orthogonal to the vectors that belong to the vector space at \( t + \delta t \) [76]. This is in complete analogy to the case of the tangent space of the curved surface where we threw away the normal components to the plane since we were only interested in the world of the two dimensional surface of the curve. Now we are only interested in those states which belong to the vector space that constitutes the fibres, and must discard any other states. So the projective
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derivative \( P_{\lambda(t)} \partial_t |\psi(\lambda(t))\rangle \) (Eq. 5.4) is the relevant derivative, which defined in terms of the connection \( \mathcal{A}(\partial_t) \) is [76]

\[
P_{\lambda(t)} \partial_t = \partial_t + i\mathcal{A}(\partial_t).
\]

(5.7)

The connection tells us how much a state has to change as \( h_{\lambda(t)} \) changes. From now on we suppress the subscript \( \lambda(t) \) of \( P \) for notational convenience. The connection as a function of the projector is given by the commutator

\[
\mathcal{A}(\partial_t) = i[(\partial_t P), P],
\]

(5.8)

which follows from the fact that for all states in the Hilbert space \( P_{\lambda} |\psi(\lambda)\rangle = |\psi(\lambda)\rangle \), and \( P^2 = P \). So \( \partial_t = \partial_t P = (\partial_t P) + P\partial_t \), and \( P\partial_t = P\partial_t P + P\partial_t \implies P\partial_t P = 0 \).

The \( n \)’th Chern character is defined in terms of a field strength \( \mathcal{F} \) of the connection \( \mathcal{A} \) as [4]:

\[
ch_n = \frac{1}{(2\pi)^n} \text{tr} \left( \mathcal{F} \wedge \mathcal{F} \ldots \wedge \mathcal{F} \right),
\]

(5.9)

where \( n \) relates to the dimension \( d = 2n \) of the base space manifold. To express the Chern character in terms of projectors we need to define a relation between the field strength and the projectors. The minimal change of a state \( |\psi(\lambda(t))\rangle \) along \( \lambda(t) \) is

\[
|\psi(\lambda(t + \delta t))\rangle = |\psi(\lambda(t))\rangle + i\delta t \mathcal{A}(\partial_t) |\psi(\lambda(t))\rangle + \mathcal{O}(\delta t^2).
\]

(5.10)

The field strength is similarly defined as the minimal change of a state taken around an infinitesimal loop. By introducing the coordinate system \( \{\lambda^\mu\} \) on \( \Lambda \), we define the field strength through the minimal change of \( |\psi(\lambda(t))\rangle \) traversed around a parallelogram spanned by \( d\lambda^\mu \) and \( d\lambda^\nu \) as

\[
|\psi(\lambda)\rangle \rightarrow |\psi(\lambda)\rangle + \mathcal{F}_{\mu\nu} d\lambda^\mu d\lambda^\nu |\psi(\lambda)\rangle + \mathcal{O}((d\lambda)^3).
\]

(5.11)

By using Eq (5.10) we can express the field strength in terms of the connection by traversing along the same infinitesimal parallelogram in steps of \( d\lambda^\mu \) resulting in a change of the state:

\[
|\psi(\lambda)\rangle \rightarrow (\mathbb{1} + i\lambda^\mu \mathcal{A}_\mu + \mathcal{O}((d\lambda^\mu)^2))(\mathbb{1} + i\lambda^\nu \mathcal{A}_\nu + \mathcal{O}((d\lambda^\nu)^2))\times

(\mathbb{1} - i\lambda^\mu \mathcal{A}_\mu + \mathcal{O}((d\lambda^\mu)^2))(\mathbb{1} - i\lambda^\nu \mathcal{A}_\nu + \mathcal{O}((d\lambda^\nu)^2)) |\psi(\lambda)\rangle.
\]

(5.12)

This expression is modified by taking the commutator between the middle two factors resulting in

\[
|\psi(\lambda)\rangle \rightarrow |\psi(\lambda)\rangle + (\mathbb{1} + i\lambda^\mu \mathcal{A}_\mu + \mathcal{O}((d\lambda^\mu)^2))\times

[\mathbb{1} + i\lambda^\nu \mathcal{A}_\nu + \mathcal{O}((d\lambda^\nu)^2), \mathbb{1} - i\lambda^\mu \mathcal{A}_\mu + \mathcal{O}((d\lambda^\mu)^2)]\times

(\mathbb{1} - i\lambda^\nu \mathcal{A}_\nu + \mathcal{O}((d\lambda^\nu)^2)) |\psi(\lambda)\rangle.
\]

(5.13)
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To the lowest order in $d\lambda$ the state changes as

$$|\psi(\lambda)\rangle \rightarrow |\psi(\lambda)\rangle + [A_\mu, A_\nu]d\lambda^\mu d\lambda^\nu + O((d\lambda)^3),$$  \hspace{1cm} (5.14)

along the parallelogram. Comparing Eq. (5.14) with the definition of the field strength in Eq. (5.11) gives a relation between the field strength and the projector:

$$F_{\mu\nu} = [A_\mu, A_\nu] = P((\partial_\mu P), (\partial_\nu P))P,$$  \hspace{1cm} (5.15)

which written in terms of differentials is

$$F = P(dP \wedge dP).$$  \hspace{1cm} (5.16)

This expression is precisely what we need as it allows us to write the Chern character in terms of projectors. The desired expression for the Chern character (Eq. (5.9)) in terms of the projector is

$$ch_n = \frac{1}{(2\pi i)^n} \text{Tr} \left( P dP \wedge dP \wedge \cdots \right)$$

$$= \frac{1}{(2\pi i)^n} \varepsilon^{i_1 \cdots i_{2n}} \text{Tr} \left( P \partial_{i_1} P \partial_{i_2} P \cdots \partial_{i_{2n}} P \right).$$  \hspace{1cm} (5.17)

This expression has been simplified by using the cyclic property of the trace, the antisymmetry of the Levi-Civita tensor, and the relation $(\partial_\alpha P)P = \partial_\alpha P - P(\partial_\alpha P) \implies P(\partial_\alpha P)P = 0$, which translates to the condition $P(dP \wedge dP) = P(dP \wedge dP)P = P(dP \wedge dP) = P(dP \wedge dP)$ in terms of differential forms.

The integral of the Chern character over the base manifold, the first Brillouin zone, results in the Chern number, the topological invariant of the Brillouin zone bundle. The remaining task on our journey towards defining the Chern marker is to Fourier transform the Chern character into real space.

5.4 The local Chern marker

The Chern marker is in a sense a glorified, but relevant, Fourier transform of the Chern character. The Chern marker can be used even though translation invariance is lost, extending the use of the Chern number to all those models beyond crystalline structures. This section aims to explain the Fourier transform of the Chern character in detail to emphasise the importance of a topological invariant expressed in real space.

We express the nth Chern character in Eq. (5.17) in terms of the single-particle density matrix

$$ch_n = \frac{1}{(2\pi i)^n} \frac{1}{n!} \varepsilon^{i_1 \cdots i_{2n}} \text{Tr} \left( \tilde{\rho}(k) \partial_{k_{i_1}} \tilde{\rho}(k) \cdots \tilde{\rho}(k) \partial_{k_{i_{2n}}} \tilde{\rho}(k) \right).$$  \hspace{1cm} (5.18)
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with a corresponding Chern number

\[ C = \frac{1}{(2\pi)^{D/2}} \frac{1}{(D/2)!} \int_{BZ} d^D k \varepsilon^i_1 \cdots \varepsilon^i_D \text{Tr} \left( \tilde{\rho}(k) \partial_{k_i} \tilde{\rho}(k) \cdots \tilde{\rho}(k) \partial_{k_{i_D}} \tilde{\rho}(k) \right). \]  

(5.19)

The matrices \( \tilde{\rho}(k) \) are the Fourier components of the single-particle density matrix

\[ \rho = (2\pi)^D \int_{BZ} d^D k \tilde{\rho}(k) |k\rangle \langle k|. \]  

(5.20)

The states in Hilbert space are similarly decomposed as

\[ |\psi\rangle = \frac{1}{(2\pi)^D} \int_{BZ} d^D k \tilde{\psi}(k) |k\rangle, \]  

(5.21)

where \( \tilde{\psi}(k) \) is a vector in the local Hilbert space. The single-particle density matrix acting on a state \( |\psi\rangle \) is therefore expressed through the integral

\[ \rho |\psi\rangle = \frac{1}{(2\pi)^D} \int_{BZ} d^D k \tilde{\rho}(k) \tilde{\psi}(k) |k\rangle, \]  

(5.22)

where we have used the norm \( \langle k'|k\rangle = (2\pi)^{-D} \delta^D(k-k') \). To Fourier transform the Chern character we need to express \( X_i |r\rangle \) in momentum space, where \( X_i \) is the position operator. By defining the eigenvalues of the position operator through \( X_i |r\rangle = x_i |r\rangle \) and using that \( |k\rangle = \sum_r e^{-ik \cdot r} |r\rangle \), we arrive at the expression

\[ X_i |\psi\rangle = \int_{BZ} d^D k \tilde{\psi}(k) X_i |k\rangle = \sum_r \int_{BZ} d^D k \tilde{\psi}(k) i \partial_{k_i} e^{-ik \cdot r} |r\rangle \]

\[ = -i \sum_r \int_{BZ} d^D k \partial_{k_i} \tilde{\psi}(k) e^{-ik \cdot r} |r\rangle \]

(5.23)

\[ = -i \int_{BZ} d^D k (\partial_{k_i} \tilde{\psi}(k)) |k\rangle. \]

We now work backwards to show that the Chern number becomes a local expectation value in state space. The key point is that a localised wave function \( \tilde{\psi}(k) = \alpha e^{ik \cdot r} \) is proportional to a constant vector \( \alpha \) in the local Hilbert space. By introducing the notation \( |\psi\rangle = |r; \alpha\rangle \), the local expectation value in momentum space is

\[ \langle r; \alpha| \rho X_{i_1} \rho X_{i_2} \cdots X_{i_D} \rho |r; \alpha\rangle = \]

\[ = (-i)^D \int_{BZ} d^D k' \int_{BZ} d^D k \alpha^\dagger \tilde{\rho}(k) \partial_{k_{i_1}} \tilde{\rho}(k) \cdots \partial_{k_{i_D}} \tilde{\rho}(k) \alpha e^{i(k-k') \cdot r} \langle k'|k\rangle \]

\[ = (2\pi)^{-D} \int_{BZ} d^D k [\tilde{\rho}(k) \partial_{k_{i_1}} \tilde{\rho}(k) \partial_{k_{i_2}} \cdots \partial_{k_{i_D}} \tilde{\rho}(k)]_{\alpha, \alpha}, \]

(5.24)

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where $[...]_{\alpha,\alpha}$ denotes an expectation value with respect to the vector $\alpha$. Inserting Eq. (5.24), into the expression for the Chern number in Eq. (5.19) results in the real space expression

$$C = \frac{1}{(2\pi)^{D/2}} \frac{1}{(D/2)!} \int_{BZ} d^D k \varepsilon^{i_1 \cdots i_D} \text{Tr} \left( \tilde{\rho}(k) \partial_{k_{i_1}} \tilde{\rho}(k) \cdots \tilde{\rho}(k) \partial_{k_{i_D}} \tilde{\rho}(k) \right)$$

$$= \frac{(2\pi)^{D/2}}{(D/2)!} \sum_{\alpha} \varepsilon^{i_1 \cdots i_D} \langle \mathbf{r}; \alpha | \rho X_{i_1} \rho X_{i_2} \cdots X_{i_D} \rho | \mathbf{r}; \alpha \rangle.$$ (5.25)

We have reached our goal: Eq. (5.25) is the local Chern marker [16,157]. By using the notation $[...]_{(\mathbf{r},\alpha),(\mathbf{r},\alpha)}$ for the expectation value, the Chern character is equivalently expressed as [157]

$$C(\mathbf{r}) = \sum_{\alpha} \varepsilon^{i_1 \cdots i_D} [\rho X_{i_1} \rho X_{i_2} \cdots X_{i_D} \rho]_{(\mathbf{r},\alpha),(\mathbf{r},\alpha)} / (D/2)!/(2\pi i)^{D/2}.$$ (5.26)

The local Chern marker is as anticipated a local expectation value. The trace is taken with respect to any internal degrees of freedom, such as spin.

In formulating Eq. (5.25) we are halfway in our task of defining local markers in odd dimensions. We want to use the Chern character to do so, but the problem is that the Chern character is only defined in even dimensions. In the next section, we will show how the relevant topological invariant in odd dimensions can be expressed as a Chern character in even dimensions. The trick is to introduce one extra dimension.

### 5.5 The chiral winding number and the chiral marker

The chiral winding number is a $\mathbb{Z}$ topological invariant that characterises the topology of states which are protected by chiral symmetry in odd dimensions [17]. If we go back and take a look at Table 3.2 in chapter three depicting the Altland-Zirnbauer classification, we find that there are three such classes in each odd dimension. The chiral winding number is derived from the Chern-Simons invariant which is a $\mathbb{Z}_2$ valued half integer [17]. The Chern-Simons invariant is defined as the integral of the $n$th Chern-Simons form $\text{cs}_n$ over a $d = 2n - 1$ dimensional closed manifold [4]:

$$\mathcal{CS} = \int_{S_{2n-1}} \text{cs}_n.$$ (5.27)

The integrand is basis dependent so this integral is only defined modulo one, and $\mathcal{CS}$ is not gauge invariant [4,17,76]. The $n$th Chern Simons form is locally an exterior derivative of the $n$th Chern character, $\text{ch}_n = d(\text{cs}_n)$, and by using Stokes
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\[ \Lambda = \mathbb{BZ} \times V \]

Figure 5.1: The extended space \( \Lambda = \mathbb{BZ} \times V \) for the range of parameter values \( \nu \in V \).

Theorem

\[ \int_S ch_n = \int_S d(cs_n) = \int_{\partial S} cs_n. \] (5.28)

This is what we are after, a way to connect the chiral winding number with the Chern character. But there is a subtlety here; the fact that the integral \( \int_{\partial S} cs_n \) is basis dependent means that the integral of the Chern character in one higher dimension also is basis dependent \([4,17,76]\). The reason is that the Brillouin zone, denoted by \( \partial S \) in Eq. (5.28), is not a boundary of a higher dimensional space \( S \), so there is always a choice of how to define the extended space. The ambiguity of the Chern-Simons invariant stems from the ambiguity of choosing a basis of the Chern-Simons form, or equally from the ambiguity of choosing the extended space. These two ways of defining the Chern Simons invariant are equivalent; by extending the space we implicitly impose restrictions on the choice of basis of the Chern Simons form.

The chiral winding number is defined as \( \nu = 2CS \) modulo 2; \( CS \) modulo 1 is half integer valued which implies that \( \nu \) is integer valued \([17,76]\). The relation \( \nu = 2CS \) is only exact for a specific gauge choice \([17,76]\). But the single-particle density matrix is a basis invariant object, which means that it is invariant under a basis change of the filled bands. Because of the basis independence of the projector, it is unclear how the Chern-Simons invariant could be written in terms of a projector with a fixed gauge.

So what do we need to do to be able to express the Chern-Simons invariant in terms of a Chern character? The answer is to introduce a family of projectors \( P_\vartheta \) where the parameter \( \vartheta \) acts as an additional dimension. The extended base space depicted in Fig. 5.1 is defined as \( \mathbb{BZ} \times V \) where \( \mathbb{BZ} \) is the odd dimensional first Brillouin zone, and \( (\vartheta_l \leq \vartheta \leq \vartheta_f) \in V \). The projector at \( \vartheta_l \) is defined to be a projector of a topologically trivial phase characterised by a zero chiral winding number, and the projector at \( \vartheta_f \) is defined to be a projector \( \rho \) of the topological phase of interest in a given situation. This means that the difference in the Chern-Simons invariant can be expressed as the integral over the Chern character defined
Figure 5.2: A schematic picture of the path of the projector $P_\vartheta$, Eq. (5.33), interpolating between a topologically trivial state and a state with a chiral constraint.

in terms of the projector $P_\vartheta$ integrated over the extended base space $BZ \times V$:

$$
\Delta CS = \int_{BZ} d^Dk \, cs_n = \int_{\vartheta_i}^{\vartheta_f} d\vartheta \int_{BZ} d^Dk \, ch_n. \tag{5.29}
$$

This chosen construction means that the integral over $\vartheta$ picks out the difference in the Chern-Simons invariant between the two boundaries of the extended base manifold where the Chern character is a well defined function of $P_{\vartheta_i}$ and $P_{\vartheta_f}$. Since the projector at $\vartheta = \vartheta_i$ is defined to be a projector of a trivial state it has per definition a zero Chern Simons invariant, so that

$$
\Delta CS = CS_{\vartheta_f} - CS_{\vartheta_i} = CS_{\vartheta_f}. \tag{5.30}
$$

This is the desired result: $\Delta CS = CS_{\vartheta_f}$ is the Chern-Simons invariant of the topological phase of interest characterised by a chiral winding number $\nu = 2CS_{\vartheta_f}$.

The Chern character in terms of the projector $P_\vartheta$ is

$$
ch_n = \frac{(2\pi i)^{-(D+1)/2}}{[(D+1)/2]!} \varepsilon^{\vartheta,i_1 \ldots i_D} \text{Tr}[P(k)\partial_\vartheta P(k)\partial_{k_{i_1}} \cdots P(k)\partial_{k_{i_D}} P(k)], \tag{5.31}
$$

where $P(k)$ are the Fourier components of $P_\vartheta$, and $D$ is the odd spatial dimension. We can now Fourier transform the Chern character to real space using the same methods as for the Chern character in section 5.4 resulting in the integer valued local chiral marker

$$
\nu(\mathbf{r}) = 2i \sum_\alpha \int_0^{\pi/2} d\vartheta \varepsilon^{\vartheta,0,i_1 \ldots i_D} P_{\vartheta_0} X_{\vartheta_0} P_{\vartheta} \cdots X_{i_D} P_{\vartheta} |(\mathbf{r},\alpha),(\mathbf{r},\alpha)| / [(D+1)/2]!(2\pi i)^{(D-1)/2}, \tag{5.32}
$$

where $\nu(\mathbf{r}) \mod 2 = 2\Delta CS$. The position operator with subscript zero is defined as $X_0 = i\partial_\vartheta$. In this formalism, the chiral winding number in $D$ dimensions is derived from the $n = [(D+1)/2]$th Chern number through dimensional reduction.

To evaluate the Chern-Simons invariant in practice we need to define a path in the parameter space where the projectors take on the desired form at the endpoints. In theory, we could choose any localised path between the trivial states and $\rho$ to evaluate the Chern-Simons invariant. But the resulting expression
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will depend on the localisation length $\xi$ of $P_\vartheta$, resulting in finite size corrections to the quantised integer values scaling as $\sim e^{-\xi/L}$ for linear system size $L$. The closed form expression for the family of projectors,

$$P_\vartheta = \frac{1}{2} \left[1 - \sin(\vartheta) (1 - 2\rho) - \cos(\vartheta)S\right], \quad (5.33)$$

interpolates between the trivial state at $\vartheta_i = 0$, $P_0 = 1/2(1 - S)$, and the topological state in question at $\vartheta_f = \pi/2$, see Fig. 5.2. Here $S$ is the chiral constraint \{S, $\rho$\} = S where $S^2 = 1$. The projector in Eq. (5.33) depends explicitly on $\rho$ and has the same localisation length as $\rho$. $\rho$ is a property of the model with a fixed localisation length, so the projector defined by Eq. 5.33 yields a Chern-Simons invariant with minimal possible corrections due to the localisation length. The closed form expression Eq. (5.33) corresponds to the anticommutation relation \{S, $P_\vartheta$\} = S - cos $\vartheta$, and $P_\vartheta$ only obeys the chiral constraint at $\vartheta = \pi/2$, at $P_{\pi/2} = \rho$. $P_\vartheta$ is, therefore, a topologically trivial projector for all $\vartheta \neq \pi/2$ and undergoes a topological phase transition into a topological phase only at the final point $\vartheta = \pi/2$ in parameter space.

The closed form expression for the family of projectors, Eq. (5.33), results in a closed form expression for the chiral winding number:

$$\nu(r) = \gamma_D \varepsilon^{i_1, \ldots, i_D} \sum_\alpha [\rho SX_i \rho \ldots \rho X_i \rho]|(r\alpha), (r\alpha)|, \quad (5.34)$$

where the dimension dependent coefficient is

$$\gamma_D = -\frac{4(8\pi i)^{D-1} [(D + 1)/2]!}{(D + 1)!}. \quad (5.35)$$

Deriving this general form involves using the antisymmetric Levi-Civita tensor to show that every product between two position operators $X_iX_j$ vanishes. The $\vartheta$ dependence in Eq. (5.32) factorises, and after integrating over $\vartheta$ only terms of the form

$$\sum_\alpha \varepsilon^{i_1, \ldots, i_D} \langle r, \alpha| \rho X_i \rho \ldots X_i \rho |(r\alpha), (r\alpha)|, \quad (5.36)$$

$$\sum_\alpha \varepsilon^{i_1, \ldots, i_D} \langle r, \alpha| \rho SX_i \rho \ldots X_i \rho |(r\alpha), (r\alpha)|, \quad (5.37)$$

$$\sum_\alpha \varepsilon^{i_1, \ldots, i_D} \langle r, \alpha| \rho X_i \rho SX_i \rho \ldots X_i \rho |(r\alpha), (r\alpha)|, \quad (5.38)$$

etc.,

remain, where the etc refers to all permutations of Eq. (5.37) with respect to the placement of the factor $\rho S$. The chiral operator can be moved around inside
the expectation value by using the relations \( \{S, \rho\} = S \) and \([S, X_i] = 0\), and the identity \(S^2 = 1\), which implies that only two terms in the above sums remain, namely Eq. (5.36) and Eq. (5.37). The expectation values without any chiral operators vanish in the translation invariant limit where they can be expressed as a trace in momentum space, (Eq. (5.25)). The reader might recollect that the local Chern marker in Eq. (5.26) is of the form of Eq. (5.36), but it does not vanish. The reason is that the local Chern marker is defined in even spatial dimensions.

The operator \(\varepsilon^{i_1 \cdots i_D} x_{i_1} \rho \cdots x_{i_D} \rho\) is expressed in terms of \(D\) commutators by using the commutator \([x_i, x_j] = 0\), where we now take \(D\) to be any dimension, even or odd:

\[
\varepsilon^{i_1 \cdots i_D} \langle r, \alpha | \rho x_{i_1} \rho \cdots x_{i_D} \rho | r, \alpha \rangle = \varepsilon^{i_1 \cdots i_D} \langle r, \alpha | \rho [x_{i_1}, \rho] \cdots [x_{i_D}, \rho] | r, \alpha \rangle,
\]

which in the long wavelength limit is proportional to a trace over the operator (Eq. (5.25)). By using the cyclic property of the trace and the definition of a projector, \(\rho^2 = \rho\), the operator inside the trace can be expressed as

\[
\varepsilon^{i_1 \cdots i_D} \mathrm{tr}(\rho [x_{i_1}, \rho] \cdots [x_{i_D}, \rho]),
\]

which is always zero if \(D\) is odd, but not if \(D\) is even. The proof follows from induction.

The proportionality constant \(\gamma_D\) is derived similarly, but here one must leave the integration over \(\vartheta\) until the end and keep track of the contributions from the trigonometric functions in \(\vartheta\). After some algebra, it can be shown that the proportionality factor of each expectation value with one factor of \(\rho S\) appearing (for example Eq. (5.37)) is proportional to the integral \(I_\vartheta = 1/2 \int \sin^D \vartheta d\vartheta\)· There are \(D + 1\) permutations of the factor \(\rho S\) appearing between any two position operators, so combining \(I_\vartheta \cdot (D + 1)\) with the overall prefactor in the definition of the chiral marker in Eq. (5.32) results in the coefficient \(\gamma_D\) defined in Eq. (5.35).

The chiral marker Eq. (5.32), provides an easy-to-use expression for evaluating topological invariants in matter that lack translation invariance. The relevant models are one or three dimensional phases constrained by chiral symmetry characterised by a \(\mathbb{Z}\) invariant. The explicit form of the chiral marker in one spatial dimension is

\[
\nu(r) = -2 \sum_\alpha [\rho S X \rho X \rho]_{(r, \alpha), (r, \alpha)},
\]

and the chiral marker in three spatial dimensions is

\[
\nu(r) = -\frac{8\pi i}{3} \sum_\alpha \varepsilon^{ijk} [\rho S x_i \rho x_j \rho x_k \rho]_{(r, \alpha), (r, \alpha)}.
\]

All we need to characterise the topology of a given free fermion state is the position operator \(X\), the chiral operator \(S\), and the projector \(\rho\). These matrices
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are $\alpha N$-dimensional square matrices where $N$ is the number of lattice sites, and $\alpha$ is the orbital degree of freedom. We can obtain $\rho$ in two different ways. If we are given a Hamiltonian matrix we can evaluate $\rho$ as the projector onto all occupied bands through $\rho = \sum_{i\in \text{occ}} \phi_i \phi_i^\dagger$, where $\phi_i$ are the eigenvectors of the Hamiltonian. Constructing the projector over the filled bands requires a full diagonalisation of the Hamiltonian, which makes it costly to increase the system size in three dimensions. We can also obtain $\rho$ by considering the single-particle density matrix, $\rho_{\alpha\beta} = \langle \phi | c_{\alpha}^\dagger c_{\beta} | \phi \rangle$, which only requires knowledge about the free fermion ground state. The chiral marker and the local Chern marker have the advantage that they can characterise the topology of a state without requiring knowledge about the parent Hamiltonian.

The chiral marker characterises all $\mathbb{Z}$ topological phases in odd dimensions with a chiral constraint. But we can do even better. The Chern-Simons invariant is also responsible for the $\mathbb{Z}_2$ invariant characterising topological phases in odd dimensions protected by either particle hole symmetry or time reversal symmetry in three spatial dimensions. In one dimension it characterizes the topological phases of superconductors in class $D$ protected by particle hole symmetry. In three dimensions the $\mathbb{Z}_2$ invariant is proportional to the $\theta$-angle which is constrained to be either 0 or $\pi$ due to the time reversal symmetry which protects the topology of the topological insulators in class AII. In the next section, we will expand the notion of the chiral marker to define the Chern-Simons marker; the local topological marker for the $\mathbb{Z}_2$ invariant in odd spatial dimensions.

5.6 The local Chern-Simons marker

The local chiral marker and the local Chern-Simons marker are the results of a collaboration that started after reading the paper on the Chern marker by Ref. [16]. We wanted to find out if there was a simple way to characterise the topology of an amorphous time reversal invariant topological insulator in three dimensions. Three dimensional topological insulators have been explored extensively in translation invariant structures in both theory and experiment, so it is of interest to understand these phases in a noncrystalline setting. So we set out to find local markers for three dimensional topological insulators. The local markers that we ended up with turned out to be simple to use expressions generalising to other odd dimensions and to more symmetry classes [157]. Before exploring the Chern-Simons marker, we will introduce the $\theta$ term since it was the main inspiration for constructing this marker.

5.6.1 The time reversal invariant topological insulator

The time reversal symmetric topological insulators in three dimensions are a class of topological insulators [84,192–194], with several proposed and experimentally
observed material realisations [84,195,196]. Crystalline Bi$_2$Se$_3$ is a three dimensional topological insulator which exists at room temperature without an external magnetic field making it an interesting topological insulator to explore [196]. The amorphous Bi$_2$Se$_3$ has also been shown to host topological boundary states, which highlights the importance of understanding topology and its characterisation in amorphous matter [183]. The topology is detected in experiment through the existence of an odd number of chiral surface states, which are a direct consequence of the time reversal symmetry. Time reversal symmetry demands that the energy eigenstates of the Hamiltonian are even functions of momentum; $E(k) = E(-k)$ since momentum changes sign under time reversal [84]. Spin also changes sign under time reversal, so the states at $k$ and $-k$ must have opposite spin. The corresponding eigenstates are Kramers pairs. Spin-orbit coupling breaks the degeneracy for energies at general momenta, but the degeneracy remains at time reversal invariant momenta where $k = -k$ up to a lattice vector. The simplest Hamiltonian that we can write down respecting time reversal symmetry on the two dimensional surface is a gapless Dirac Hamiltonian [84]

\[
H = \nu_F k_x \sigma_x + \nu_F k_y \sigma_y + \mu \sigma_0,
\]

where $\nu_F$ is the Fermi velocity, $k_x$, and $k_y$ are the momenta parallel with the surface, and $\mu$ is the chemical potential. The $\sigma_i$, $i \in (x,y)$ are Pauli matrices, and $\sigma_0$ is the identity matrix in two dimensions. The massless Dirac Hamiltonian has a linear, conical dispersion. The Hamiltonian in Eq. (5.43) is time reversal symmetric, where we define the time reversal operator as $T = i\sigma_y K$, where $K$ defines complex conjugation. The only way to produce a gap is to introduce a mass term $m \sigma_z$ to the Hamiltonian in Eq. (5.43). But this is not allowed as the mass term breaks time reversal symmetry—the metallic surface states are protected by time reversal symmetry. A topological insulator has an odd number of symmetry protected Dirac cones on the surface. A pair of Dirac Hamiltonians on the surface are not symmetry protected. They can be gapped when the combined mass terms respect time reversal symmetry [197,198]. There are only two possible phases, a topological phase defined by an odd number of Dirac cones on the surface, and a trivial phase defined by an even number of Dirac cones on the surface.

The bulk invariant characterising the topology of the three dimensional time reversal invariant insulator is called the $\theta$ term and is expressed as [54,199]

\[
\theta = \frac{1}{2\pi} \int_{BZ} d^3k \varepsilon^{ijk} \text{tr} (A_i \partial_j A_k + i A_i A_j A_k),
\]

where the trace is taken over the occupied bands. Eq. (5.44) is derived through dimensional reduction from a four dimensional theory with the Berry connection $A_i$. The $\theta$ angle is a constant axial field in the theory of axion electrodynamics
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which explores the consequences of adding a topological term

\[ S_\theta = \frac{\theta e^2}{4\pi^2 \hbar c} \mathbf{E} \cdot \mathbf{B}, \]  

(5.45)
to the Maxwell theory of electrodynamics, where \( \theta \in (0, 2\pi) \) [200]. The electric field remains invariant under time reversal while the magnetic field changes sign, so \( S_\theta \) changes sign under time reversal and is only time reversal invariant for the \( \theta = 0 \) and \( \theta = \pi \). Time reversal takes \( \theta \) to \(-\theta\), but \(-\pi = \pi\) since \( \pi \in (0, 2\pi) \), which defines the two possible phases of a three dimensional time reversal invariant insulator. \( S_\theta \) for \( \theta = \pi \) is the topological response term for topological insulators subject to external electromagnetic fields defining a polarisation [84,199].

As a bundle invariant, the \( \theta \) term is defined through the Chern-Simons invariant taken modulo two [17,76]. We can therefore define the local topological marker of the \( \mathbb{Z}_2 \) invariant \( \theta \) term analogously to the local chiral marker. We call this new marker the local Chern-Simons marker, and it characterises the \( \mathbb{Z}_2 \) invariant in topological classes protected by either particle hole symmetry or time reversal symmetry in every odd spatial dimension [157].

5.6.2 The family of projectors of the local Chern-Simons marker

The purpose of this section is to derive an analytic form of the family of projectors used to define the Chern-Simons marker introduced in paper IV [157]. The Chern-Simons marker is defined as \( \nu_{cs}(\mathbf{r}) = \nu(\mathbf{r}) \mod 2 \), where \( \nu(\mathbf{r}) \) is defined by Eq. (5.32) with the important difference that \( \rho \) is a projector which is protected either by time reversal symmetry or particle hole symmetry. \( \rho \) does not obey the chiral constraint so the closed form projector in Eq. (5.33) is of no use any more. This statement is not quite true; to obtain a closed form projector where \( P_{\nu_i} \) is the projector of a topologically trivial state and \( P_{\nu_i} = \rho \), we can introduce an intermediate point on our path which has chiral symmetry, see Fig. 5.3. By denoting the projector of this intermediate state by \( Q \) with a chiral constraint \( \{Q,S_R\} = S_R \), we can define the two projectors

\[ P^{\rho}_{\vartheta} = \frac{1}{2} \left[ 1 - \sin(\vartheta) (1 - 2Q) - \cos(\vartheta)S_R \right], \]  

(5.46)

\[ P^{\rho,Q}_{\vartheta} = \frac{1}{2} \left[ 1 - \sin(\vartheta) (1 - 2\rho) - \cos(\vartheta)(1 - 2Q) \right]. \]  

(5.47)

The Chern-Simons invariant is then equal to the sum of the two integrals of the Chern characters defined with respect to the projectors \( P^{\vartheta}_{\rho} \) and \( P^{\rho,Q}_{\vartheta} \). This procedure is well defined, but it is a bit cumbersome to analytically evaluate the \( \vartheta \) integral over the Chern character with respect to the projector \( P^{\rho,Q}_{\vartheta} \); the \( \vartheta \) integral in three dimensions for example results in an expression with sixteen terms. To avoid the algebra, we introduce a trick which will allow us to ignore the
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\[ P^R_{\theta=0} = P^R_{\theta=\pi/2} = Q(\rho) \]

Figure 5.3: A schematic picture of the paths of the two projectors \( P^R_\theta \) Eq. (5.46), and \( P^\rho_{\theta} Q \) Eq. (5.47). \( P^R_\theta \) interpolates between a topologically trivial bundle \( P^R_0 \) and a bundle \( P^R_{\pi/2} = Q(\rho) \) with a chiral constraint. In addition \( Q(\rho) \) is either time reversal or particle hole symmetric. \( P^\rho_{\theta} Q \) interpolates between the bundle \( P^\rho_{\theta} = Q(\rho) \) and the bundle \( P^\rho_{\pi/2} = \rho \) protected by either time reversal symmetry or particle hole symmetry. \( Q(\rho) \) and \( \rho \) belong to the same topological phase by construction.

contribution from the path described by \( P^\rho_{\theta} Q \). We define the intermediate point in such a way that \( Q \) is also invariant under both chiral symmetry and the symmetry of \( \rho \), see Fig. 5.3. This means that \( Q \) and \( \rho \) belong to the same symmetry class and are defined by the same Chern-Simons invariant. The difference in Chern-Simons invariant is therefore \( \Delta CS = 0 \) for the projector \( P^\rho_{\theta} Q \). Even though \( Q \) is invariant under the chiral symmetry \( S_R \), it is not topologically protected by this symmetry. There exists no symmetry class in the Altland-Zirnbauer classification which is protected by either a combination of chiral symmetry and particle hole symmetry or by a combination of chiral symmetry and time reversal symmetry. So \( Q \) and \( \rho \) are in the same symmetry class and no topological phase transition occurs along the path defined by the projector \( P^\rho_{\theta} Q \).

Let us consider the three dimensional case as an example. The time reversal invariant topological insulator belongs to class AII and obeys the time reversal constraint, \( T \rho^* T^\dagger = \rho \), where \( T = T_K \) is the time reversal operator in real space and \( K \) is complex conjugation. We construct the operator \( Q \) to obey a chiral constraint \( S_R = 1 - 2R \), such that \( \{ Q, S_R \} = S_R \):

\[ Q = \frac{1}{2} \left( 1 + i[\rho, S_R]^{-1}[\rho, S_R] \right), \tag{5.48} \]

where \( ||[\rho, S_R]|| \) is the matrix absolute value. The matrix absolute value \( |M| \) for a matrix \( M \) is defined to be diagonal in the eigenbasis of \( M \), where the diagonal elements consist of the absolute value of the eigenvalues of \( M \). By rearranging Eq. (5.48) we see that \( i[\rho, S_R]^{-1}[\rho, S_R] = 1 - 2Q \) is a band flattened Hamiltonian with projector \( Q \). We need the projector \( Q \) to be localised since the topological classification is defined with respect to localised states. The operator \( R \) can be chosen to be any trivial projector for which \( i[\rho, S_R]^{-1}[\rho, S_R] \propto i[\rho, R] \) is gapped, which implies that \( Q \) is a localised operator. \( R \) depends on the model and is constructed as a tensor product of local operators. In the simplest case, \( R \) is an operator which only acts on a single site, but if no such operator exists for
a given model one can construct it to act on more sites. Since $Q$ is localised by construction, and $\rho$ is localised by definition, $P_{\vartheta}^{\rho,Q}$ is also a localised. We demand that the projector $P_{\vartheta}^{\rho,Q}$ should be time reversal invariant, which forces $Q$ to be time reversal invariant: $TQ^*T^\dagger = Q$ or equally (Eq. (5.48)) $TS_R^*T^\dagger = -S_R$. By constructing a local operator $R$ which is both local and obeys the relation $TS_R^*T^\dagger = -S_R$ we have succeeded in keeping $P_{\vartheta}^{\rho,Q}$ in a single topological phase for all values of $\vartheta$. The Chern-Simons invariant is solely given by the Chern character defined in terms of the path $P_R^\vartheta$ in Eq. (5.46). This is a path between a trivial projector and a projector $Q$ which obeys the chiral constraint, which is the same path as we introduced in Eq. (5.33) for the chiral marker. The Chern-Simons marker is defined $\nu_{cs} = \nu \mod 2$ is therefore evaluated by using the expression for the local chiral marker in Eq. (5.34).

The same approach translates to all odd dimensions where the operator $R$ is enforced to have either time reversal symmetry or particle hole symmetry depending on the symmetry class. The Chern-Simons marker is therefore as easy to use as the chiral marker and provides a method to determine the topological phase of matter lacking translational symmetry in odd dimensions. Paper IV provides two examples where the chiral and Chern-Simons markers are used to determine the topological phase in an amorphous superconductor and an amorphous topological insulator in three dimensions.

The generalisation of the single-particle density matrix to the many-body one-particle density matrix extends the use of the local topological markers to include interacting systems. The key point is that the many-body one-particle density matrix is no longer a projector, but as long as it is still gapped it can be adiabatically connected, without closing the gap, to a projector of a free fermion state. The many-body state therefore belongs to the same topological class as the free fermion state.

5.7 The local topological markers in interacting systems

Earlier in this chapter we defined the single-particle density matrix of free fermion states to be a projector onto the filled orbitals \([185,186,188,189]\). The single-particle density matrix for interacting states is referred to as the one-particle density matrix. The eigenvalues of the one-particle density matrix are no longer restricted to be either one or zero but can take any value between in the interval $n_\alpha \in [0,1]$ \([185,186]\). This means that the one-particle density matrix is not a projector, but as long as the one-particle density matrix is gapped, it is still a relevant indicator of the topological phase.

The definition for short range entangled states still stands—any short range entangled states can be transformed into a product state through a local unitary transformation \([184]\). This means that any interacting state which belongs to one of the ten symmetry classes can be connected to a Slater determinant state.
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through a local unitary transformation. The interacting state and the Slater determinant state belong to the same topological class if the unitary transformation preserves the symmetries of the given symmetry class. This definition provides a theoretical tool to determine the topological phase of interacting topological insulators and superconductors, but it can be very hard to find such a unitary operator in practice.

The equivalence of states translates to an equivalence of projectors. The definition of topological equivalence of two projectors is based on adiabaticity—two projectors are topologically equivalent if they are adiabatically connected, remain localised and preserve the relevant symmetries of the topological class. If the one-particle density matrix is gapped it can be adiabatically band flattened into a projector while preserving the spectral gap determining the topological phases of the interacting state. The parameter range where the one-particle density matrix remains gapped is physically relevant and includes both many-body localised states, and ground states of weekly interacting structures.

The local Chern, chiral, and Chern-Simons markers therefore provide a versatile tool that can be used to characterise the topology of interacting short ranged entangled fermionic states.

5.8 Summary

We have considered the Chern number as an integral over the Chern character over the Brillouin zone and derived the expression for the Chern character in terms of projectors onto the filled states. The Fourier transform of the Chern character yields the local Chern marker, which is a local expectation value that can be used to identify the topological phase of topological phases in even dimensions that are not constrained by any local symmetries. This closed form expression of the local Chern marker is easy to implement as it only requires knowledge about the single-particle density matrix of a given ground state.

Our purpose was to use the Chern character to derive local markers in odd spatial dimensions, which required some extra consideration as the Chern character is only defined in even dimensions. We have described how the odd dimensional Chern-Simons invariant can be expressed as an integral over the Chern character if we extend the integration manifold to be even dimensional. Introducing a family of projectors depending on a parameter \( \vartheta \) acting as an extra dimension, allows us to express the Chern character in terms of this family of projectors. We define the projector to interpolate between a projector onto a topologically trivial state, and a projector onto the topological state of interest. The integral is over the extended space consisting of the odd dimensional Brillouin zone, and the parameter space therefore defines the Chern-Simons invariant of the state that we are interested in. The Chern-Simons invariant gives rise to both the \( \mathbb{Z} \) invariant chiral winding number for symmetry classes with a chiral constraint, and the \( \mathbb{Z}_2 \)
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invariant characterising the topology in odd dimensions protected by either time reversal symmetry, or particle hole symmetry. We introduced the corresponding local chiral marker, and the local Chern-Simons marker by Fourier transforming the Chern character expressed in terms of the family of projectors.

Topology does not rely on translation invariance, but it is harder to write down general closed form expressions for topological invariants when the Brillouin zone does not exist. The local chiral marker and the local Chern-Simons are important, as they do provide closed form expressions for characterising topology in structures that lack translation invariance, such as amorphous matter. These markers are expressed in terms of the single-particle density matrix which allows us to extend their use to interacting systems. The Chern, chiral, and Chern-Simons markers can therefore be used to characterise the topological phase of topological insulators and superconductors with or without translation invariance, for both interacting and noninteracting fermions.
In this dissertation, I have introduced my research into the axial anomaly, domain wall dynamics, and local topological markers in quantum matter. My research on the anomaly is focused on non-Hermitian chiral anomalies, axial anomaly generation through domain wall dynamics in Weyl semimetals, and domain wall manipulation mediated by the axial anomaly. The theory of the axial anomaly is well established in the high energy community, and a lot of what I have included in the introduction to the topic is found in textbooks on quantum field theory. But textbooks tend to omit a discussion on axial gauge fields, which of course play an important role in the realisation of the axial anomaly in Weyl semimetals. I, therefore, took some care to derive the axial anomaly in the presence of axial gauge fields through both a diagrammatic and a path integral approach, to provide a foundation for the theory of domain wall induced axial gauge fields and axial anomaly in Weyl semimetals. The detailed derivations also highlight the subtleties of divergent integrals, and the tricks involved in evaluating the path integral measure, especially when the Dirac operator is non-Hermitian due to the presence of an axial gauge field. These details become even more important when working with a non-Hermitian action, where one needs to consider complex integration and the consequences that arise from the fact that all operators are non-Hermitian. Our research on non-Hermitian chiral anomalies is of theoretical importance, where the main purpose was to establish the classical symmetries of the action and understand how these fare in the quantum limit.

The introduction to topological insulators and superconductors serves as an introduction to the research on local topological markers. It highlights how established the classification and characterisation of symmetry protected topological phases of matter is for crystalline structures. There are several different ways of classifying topology and evaluating topological invariants through general closed form expressions in translation invariant systems. The same is not true when translation invariance is lost, and one have to use different direct and indirect ways of characterising topological phases in such structures. In this dissertation I focus on the local topological markers, and derive the odd dimensional local topological markers by extending the formalism of the local Chern marker to odd dimensions. The derivation of odd dimensional markers relies on the classification of fibre bundles in terms of projectors onto the filled bands, and I explained
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how Chern character is expressed in terms of these projectors. The local Chern marker is defined as the Fourier transform of the Chern character, and equals the Chern number at each point on a translation invariant lattice. For noncrystalline structures the Chern number is quantised on a coarse grained translation invariant lattice, and is in practise obtained by integrating the Chern marker over a volume. I described how the odd dimensional markers are obtained through dimensional reduction of the local Chern marker.

6.1 Discussion

The combination of magnetic domain walls and Weyl semimetals offer a playground in which to explore both the axial anomaly and domain wall physics, as they influence one another. The magnetic domain wall in the Weyl node separation provides a straightforward method to generate axial electromagnetic fields in Weyl semimetals containing magnetic domains. The domain wall structure guarantees the existence of axial magnetic fields, which are accompanied by axial electromagnetic fields when the domain wall is put into motion. This framework provides a theoretical setting for exploring the axial anomaly in terms of axial electromagnetic fields and a suggestion for how to measure the presence of the anomaly through experiments. The axial anomaly gives rise to the chiral magnetic field, an oscillating current located at the moving domain wall, which acts as a source of electromagnetic radiation. Measuring this radiation would confirm the presence of the axial anomaly in the Weyl semimetal.

Anomaly physics and Weyl semimetals also play a role in manipulating the domain wall, by delaying the Walker breakdown and as a way to tune the domain wall chirality with an electric field. The axial anomaly acts as an emergent torque on the domain wall coupling the magnetisation with an external electric field. The equilibrium configuration of the domain wall changes due to the magnetic field, and the domain wall chirality can be changed continuously by tuning the electric field. Such a chirality flip would be another signature of the axial anomaly in Weyl semimetals.

The local topological markers that we have introduced in odd dimensions characterise topological invariants in symmetry classes defined by a chiral winding number or a $\mathbb{Z}_2$ invariant protected by either time reversal symmetry or particle hole symmetry depending on the dimension. The local topological markers are significant as they extend the formalism of the well established local Chern marker to more symmetry classes including topological insulators in three dimensions. These markers are useful as they provide closed form expressions for topological invariants which are easy to use in practice. The local markers are defined in terms of the single-particle density matrix which means that they can be evaluated even if we are only given a ground state and not a full Hamiltonian. So all we need to characterise the topology of a given state is the state itself. The single-particle
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density matrix allows us to use the local markers for interacting systems, as long as the one-particle density matrix, as it is called in the interacting case, has a spectral gap. A spectral gap means that the one-particle density matrix can be adiabatically transformed into a projector whilst preserving the symmetries of the relevant symmetry class;—the markers evaluated in terms of the one-particle density matrix define the topological phase of the given many-body state.

6.2 Outlook

There are different future directions that directly build on the research presented in this dissertation. One can for example consider magnetic skyrmion structures in Weyl semimetals and explore how skyrmion dynamics affect Weyl semimetals, and in turn how and if the axial anomaly affects the skyrmion. Skyrmions are more complicated structures than domain walls, and the equations of motion for a magnetic skyrmion can not, in general, be solved analytically. The skyrmion motion in condensed matter settings is often assumed to be rigid, described by the position of the centre of the skyrmion. Since the rotational angle plays a role in generating the axial anomaly in the domain wall case, it is reasonable to ask if this is also the case for skyrmions.

Another direction could be to focus on amorphous matter from a field theory point of view and develop a field theory for electrons on an amorphous lattice. Topological matter is abundant in nature, and as amorphous matter is easier to grow in experiments than perfect crystals, it is of interest to understand its topological properties, making a field theory description relevant. One approach could be to formulate the amorphous field theory through a geometric description of curved spacetime in terms of a random metric. Questions that arise are how to incorporate the local order, and how to perform the average over the random strain fields to end up with a continuum theory that still contains information of the underlying microscopic amorphous lattice. The structure of the local environment has been shown to be important for amorphous structures to be topological through both experiment and in lattice models This would suggest that the microscopic structure does matter, but does this information survive in the long wavelength limit?
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Part B

Accompanying Papers