Symmetry in a free boundary problem

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1 Abstract

We consider a variational formulation of a Bernoulli-type free boundary problem for the Laplacian operator with discontinuous boundary data. We show the existence of a weak solution to the problem. Moreover, we show that the solution has symmetry properties inherited by symmetric data. These results are achieved through the use of comparison arguments, the celebrated method of moving planes, and several elaborated techniques from existing literature.

2 Sammanfattning

Vi studerar ett Bernoulli frirandsproblem för Laplaceoperatorn med diskontinuerliga randdata. Detta görs via en variationsformulering av problemet. Vi visar att en svag lösning existerar för problemet. Utöver det visar vi bland annat att den svaga lösningen har symmetriegenskaper. Dessa resultat uppnås genom jämförelseargument, den välkända ”moving-plane” metoden, samt flera utarbetade tekniker från befintlig litteratur

Contents

3 Introduction and statement of the problem

3.1 Background

In this thesis, we study a free boundary problem for a semilinear equation, with a so-called Bernoulli-type boundary condition.

Bernoulli free boundary problems are present in several applied sciences, ranging from electrostatics, fluid dynamics, and optimal design, However, the purpose of this thesis is to study this problem from a purely mathematical point of view. For readers interested in applications, we refer to the following works [10], [9] and [3].

The problem we are concerned with is a variation of the work by E. Lindgren and Y. Privat [15], where they showed, using the supersolutions technique of A. Beurling, that there exists a unique solution to the problem. The approach in this thesis is based on a variational method, through minimization of a certain energy functional, stemming from the celebrated work of H.W. Alt and L.A. Caffarelli [5], with methods inspired by the work of B. Gustafsson and H. Shahgholian [18].

3.2 Formulation of the problem

Problem 1. Let $\mathbb{R}^n_+ := \{x_1 > 0\}$, with $n \geq 2$, and suppose that $D \subset \{x_1 = 0\}$ is an open, bounded set. The goal is to find a function $u : \mathbb{R}^n_+ \to \mathbb{R}_+$, such that u solves the following equation

$$\begin{cases} \Delta u = f(u) & \text{in } \{u > 0\} \\ u = 0 & \text{on } \partial\{u > 0\} \setminus \overline{D} \\ u = 1 & \text{on } D \\ |\nabla u| = g(x) & \text{on } \partial\{u > 0\} \cap \{x_1 > 0\}, \end{cases} \tag{1}$$

where f, g are given functions with

$$f \geq 0, \quad g \geq c_0 > 0, \qquad f \in \mathrm{Lip}(\mathbb{R}^1_+), \qquad g \in C(\mathbb{R}^n_+)$$

and c_0 is a constant.

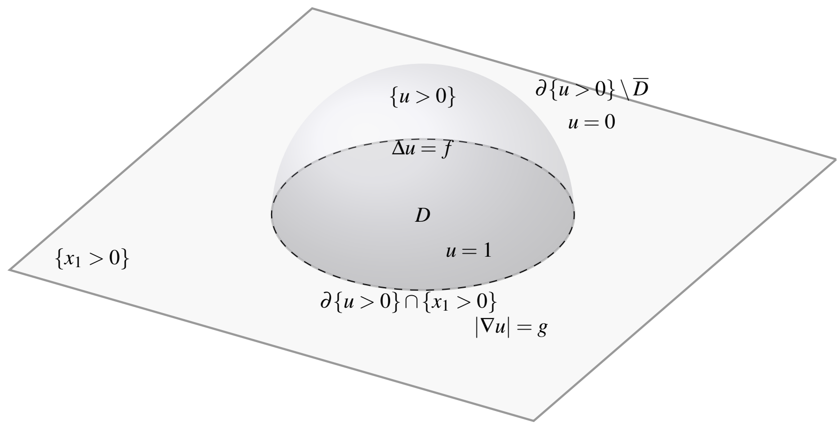
Here the gradient boundary condition $|\nabla u| = g$ is in a very weak sense,

$$\lim_{\varepsilon \to 0} \int_{\partial\{u > \varepsilon\}} (|\nabla u|^2 - g^2)\eta \nu = 0, \tag{2}$$

where ν denotes the outward normal of $\partial\{u > \varepsilon\}$ and any $\eta \in C_0^\infty(\mathbb{R}^n_+, \mathbb{R}^n_+)$, see Theorem 2.5 in [5] and Theorem 2.3 in [18].

Observe that we cannot assign boundary values at ∂D, as the solution function is expected to be discontinuous there.

We give the following illustration of the problem (1) when $n = 3$.



Equation (1) is a variation of the free boundary problem studied by E. Lindgren and Y. Privat [15] see also A. Laurin and Y. Privat [13]. Given a reasonably smooth function h, where $u=h$ on D, with $h \in W^{1,2}(\mathbb{R}^n_+)$, then the problem arises as a minimizer to the functional

$$J(u) = \int_{\mathbb{R}^n_+} |\nabla u|^2 + 2F(u) + g^2 \chi_{\{u>0\}}, \tag{3}$$

where χ denotes the characteristic function and $F'(u) = f(u) \geq 0$, with $f(t) = 0$ for $t<0$ and $F(0) = 0$.

For $h \in (W^{1,2} \cap C^1)(\mathbb{R}^n)$, we denote by $\mathbb{K}_h$ the class of admissible functions defined by

$$\mathbb{K}_h := \{v \in W^{1,2}(\mathbb{R}^n_+) : v \in h + W_0^{1,2}(\mathbb{R}^n_+)\}, \tag{4}$$

where $W_0^{1,2}(\mathbb{R}^n_+) = \{w : \ w, \nabla w \in L^2(\mathbb{R}^n_+), \text{ with compact support in } \mathbb{R}^n_+\}$. In proving the existence of solutions to our problem, later we shall let $h = h_\varepsilon \searrow \chi_D$; i.e. decreasing to χ_D. See the discussion preceding Theorem 4.0.2.

Similar types of problems have been studied in the literature, and we refer to two main articles of importance in our work [5], [18].

3.3 Main results

In this section, we present the main results of the thesis.

The methods we will use in the proofs of the statements are heavily inspired by the works of H.W. Alt and L.A. Caffarelli [5], B. Gustafsson and H. Shahgholian [18], L. El Hajj and H. Shahgholian [6].

In the previous section, we proposed that solutions to the problem (1) possibly arises as minimizers of the functional (3). Moreover, in the formulation of the problem, we proposed that any minimizer to the functional is an element of the set $\mathbb{K}_h$, where $\mathbb{K}_h$ is a subset of the Sobolev space $W^{1,2}(\mathbb{R}^n_+)$. See Appendix 6.3 for definitions and general results about Sobolev space.

In Section 4, we will see that this is indeed the case (see Theorem 4.0.1), at least in the weak sense (see Theorem 4.0.2). In Section 5, after establishing the existence of an absolute minimizer, we study the properties of such a minimizer. In particular, we show, under some additional conditions, that any minimizer is radially symmetric (see Theorem 5.0.1). Furthermore, considering the methods in [5] and [18], we are able to derive some geometric properties of the set $\{u > 0\}$.

4 Existence of global minimizers

To prove the existence of solutions to our main problem, we shall start by proving the existence of a global minimizer for the functional $J[u]$. For this, we use an array of classical techniques, such as the direct method in the calculus of variation, the comparison principles, and several other methods from functional analysis and geometric PDE.

Theorem 4.0.1. Let J, $\mathbb{K}_h$ and h be defined as earlier, see (3). Then, there exists an absolute minimizer u of J, in $\mathbb{K}_h$, with compact support.

Proof. We start by noticing that $h \in \mathbb{K}_h$. Hence, the admissible class is nonempty, and we thus can look for an absolute minimizer of the functional

$$J[v] = \int_{\mathbb{R}^n_+} \left\{ |\nabla v|^2 + 2F(v) + g^2 \chi_{\{v>0\}} \right\} dx.$$

for the class $\mathbb{K}_h$.

The functional is obviously bounded from below by zero $J[v] \geq 0$, since $F(v) \geq 0$, with $F(0) = 0$ (because $F' = f \geq 0$), and the other terms are also non-negative. In particular, $\inf_{v \in \mathbb{K}_h} J[v]$ exists.

Since $h \in \mathbb{K}_h$, and $0 < J[h] < \infty$, there exists a minimizing sequence $\{u^k\}_{k=1}^{\infty}$ in $\mathbb{K}_h$, such that

$$J[u^k] \to \inf_{v \in \mathbb{K}_h} J[v] \quad \text{as } k \to \infty. \tag{5}$$

Thus, it suffices to show that $\{u^k\}_{k=1}^{\infty}$ converges to an actual minimizer in $\mathbb{K}_h$ and that J is weakly lower semicontinuous. To do this, we consider a comparison argument similar to the one outlined in the proof of Theorem 1 in L. El Hajj and H. Shahgholian [6], with methods from the proof of Lemma 1.7 in H.W. Alt and D. Phillips [11], Lemma 1.1 in B. Gustafsson and H. Shahgholian [18], and the proof of Theorem 4 in [18].

Let us now consider the same energy functional over $\mathbb{R}^n$ and for the class of $h + W_0^{1,2}(\mathbb{R}^n)$ functions, and let $\tilde{u}$ be the minimizer of our functional in this class. Let now $\tilde{h}$ be the restriction of $\tilde{u}$ on $\{x_1 = 0\}$. Obviously, $\tilde{h} \geq h$ on $\{x_1 = 0\}$, and $\tilde{u}$ minimizes our functional in the set $\tilde{h} + W_0^{1,2}(\mathbb{R}^n_+)$. Hence, by Lemma 6.1.1 in the Appendix, we may consider the sequence $\{\min(u^k, \tilde{u})\}_{k=1}^{\infty}$. Therefore, without loss of generality, we may assume $u^k \leq \tilde{u}$, for every $k \geq 1$.

Next, suppose u_0 is an absolute minimizer of the energy functional in H.W. Alt and L.A. Caffarelli [5], defined by

$$J_0[u_0] = \int_{\mathbb{R}^n} |\nabla u_0|^2 + g^2 \chi_{\{u_0>0\}} \tag{6}$$

with g as in (3), and observe that $u_0 \leq 1$ in $\mathbb{R}^n$, since it is subharmonic in $\mathbb{R}^n$, see [5]. Applying the comparison between the functionals again, see Lemma 6.1.1 in Appendix 6.1, we see that $\tilde{u} \leq u_0$.

Now using $u^k \leq \tilde{u} \leq u_0$ for every $k \geq 1$, it suffices to show $\operatorname{supp}(u_0)$ is bounded. This follows from the non-degeneracy of u_0, in the following sense (see Lemma 3.4 in [5])

$$Cr \leq \sup_{\partial B_r(z)} u_0 \leq \sup_{\mathbb{R}^n} u_0 \leq 1, \tag{7}$$

where $B_r(z) \subset \mathbb{R}^n \setminus D$, with $z \in \partial\{u_0 > 0\}$ and $C > 0$ is a constant that depends only on n and r. Here in the last inequality we have used the comment following (6).

From the above argument it follows that the only possibility for balls $B_r(z) \subset \mathbb{R}^n$, with $z \in \partial\{u_0 > 0\}$ is that r is universally bounded. Hence, $z \in \partial\{u_0 > 0\}$ cannot have large distance to the set D.

The above argument, along with Lemma 6.1.1, implies that for any minimizing sequence $\{u^k\}_{k=1}^{\infty}$, also $\{\min(u_0, u^k)\}_{k=1}^{\infty}$ is a minimizing sequence of J. Thus, without loss of generality, we may assume $u_0 \geq u^k$, and $\operatorname{supp}(u^k)$ are uniformly bounded for every $k \geq 1$. Hence, $\{u^k\}_{k=1}^{\infty}$ is in $h + W_0^{1,2}(\mathbb{R}^n_+)$. Therefore, by Poincare's inequality, Theorem 6.3.20, we obtain

$$\|u^k\|_{L^2(\mathbb{R}^n_+)} \leq C\|\nabla u^k\|_{L^2(\mathbb{R}^n_+)} + \|h\|_{W^{1,2}(\mathbb{R}^n_+)}, \tag{8}$$

for some constant $C > 0$. From this it follows that $\{u^k\}_{k=1}^{\infty}$ is bounded in $W^{1,2}(\mathbb{R}^n_+)$.

In particular, the boundedness of $\{u^k\}_{k=1}^{\infty}$ in $W^{1,2}(\mathbb{R}^n_+)$, together with Theorem 6.3.26 guarantees the existence of a subsequence $\{u^{k_j}\}_{j=1}^{\infty} \subset \{u^k\}_{k=1}^{\infty}$ and a function u in $W^{1,2}(\mathbb{R}^n_+)$, such that

$$u^{k_j} \rightharpoonup u \quad \text{in } W^{1,2}(\mathbb{R}^n_+) \quad \text{as } j \to \infty, \tag{9}$$

i.e., u^{k_j} converges weakly to u in $W^{1,2}(\mathbb{R}^n_+)$.

We need to convince ourselves that u^{k_j} converges weakly to u in $h + W_0^{1,2}(\mathbb{R}^n_+)$, it suffices to note that h is finite, which implies that h is compact in $W_0^{1,2}(\mathbb{R}^n_+)$, thus $h + W_0^{1,2}(\mathbb{R}^n_+)$ is closed, by the Minkowski addition of a compact set and a closed set is a closed set.

Recall that J_0 is weakly lower semicontinuous; see Theorem 1.3 in [5]. Furthermore, the sum of two weakly lower semicontinuous functions is a weakly lower semicontinuous function.

Now, we claim that J is weakly lower semicontinuous, since F is continuous, it follows that the integral of F over $\mathbb{R}^n_+$ is weakly lower semicontinuous. Therefore, J is indeed weakly lower semicontinuous.

The above argument along with the weak convergences of u^{k_j} in $\mathbb{K}_h$ implies that

$$\inf_{v\in\mathbb{K}_h} J[v] \le J[u] \le \liminf_{j\to\infty} J[u^{k_j}] = \inf_{v\in\mathbb{K}_h} J[v]. \tag{10}$$

Therefore, we may conclude that $u \in \mathbb{K}_h$ is an absolute minimizer of J, and has compact support. □

To prove the existence of solutions for the free boundary problem (1), we shall now consider an appropriate choice of parameter-dependent h, for example

$$h = h_\varepsilon \in C^1(\mathbb{R}^n)$$

and with $h_\varepsilon \searrow \chi_D$ (decreasing as $\varepsilon \to 0$) and h_ε having compact support, we should obtain a solution to our problem. We assume also h_ε is decreasing in ε.

Theorem 4.0.2. Let $u_\varepsilon \in \mathbb{K}_{h_\varepsilon}$ be a minimizer of J_ε, then $\lim_{\varepsilon\to 0} u_\varepsilon$ solves (1), in the sense of equation (2), with boundary data χ_D on D.

Proof. By Theorem 4.0.1, there is a global minimizer $u_\varepsilon \in \mathbb{K}_{h_\varepsilon}$ to the functional J_ε.

We claim that $u := \lim_{\varepsilon\to 0} u_\varepsilon$ solves (1), with boundary data $u = \chi_D$ on D. Since $h_\varepsilon \searrow \chi_D$ as $\varepsilon \to 0$ (decreasing pointwise), it suffices to show that $u_\varepsilon \searrow u$, which is a minimizer of J over the admissible class $\mathbb{K}_h$.

By the comparison of functionals, Lemma 6.1.1, a smaller boundary value gives a smaller solution for global minimizers, we must have u_ε is a decreasing sequence. Hence it has a limit, which we call u. By lower semicontinuity (discussed in the proof of Theorem 4.0.1) we have u is a minimizer of our functional, with boundary values $h = \chi_D$. This depends also on the fact that the boundary and the boundary value is smooth at points $z \in D$, but not on ∂D. The latter follows from standard elliptic regularity theory, which states that regular boundary and boundary data imply continuity up to the boundary of solutions.

One can in a standard way make variations (see [5] and [18]) that the weak free boundary condition (2) for the gradient holds.

Now, we show that u is a weak solution to the problem $\Delta u = f(u)$ in $\{u > 0\}$.

Given any $\varphi \in C_0^\infty(\mathbb{R}^n_+)$, with $\varphi \ge 0$ and $\operatorname{supp}(\varphi) \Subset \{u > 0\} \cap \mathbb{R}^n_+$. We define $v_\delta = u \pm \delta\varphi$ and choose $\delta > 0$ sufficiently small, so that $v_\delta > 0$ in $\{u > 0\}$. This is possible, since $u > 0$ in $\{u > 0\}$ and $\operatorname{supp}(\varphi) \Subset \{u > 0\} \cap \mathbb{R}^n_+$, with $\varphi \ge 0$, it follows that $v_\delta \ge u > 0$, for any $\delta > 0$ sufficiently small, thus $v_\delta > 0$ in $\{u > 0\}$. Therefore, we have $v_\delta \in \mathbb{K}_h$. In particular, we obtain $J[u] \le J[v_\delta]$.

From this along with the definition of v_δ it follows that

$$0 \le J[v]-J[u] = \int_\Omega |\nabla v|^2 - |\nabla u|^2 + 2(F(v)-F(u)) \tag{11}$$

$$= \int_\Omega |\nabla(u+\delta\varphi)|^2 - |\nabla u|^2 + 2(F(u+\delta\varphi)-F(u)) \tag{12}$$

$$= \int_\Omega \delta^2|\nabla\varphi|^2 + 2\delta\nabla u\nabla\varphi + 2(F(u+\delta\varphi)-F(u)), \tag{13}$$

where $v = u+\delta\varphi$ and $\Omega := \operatorname{supp}(\varphi)$. Furthermore, if we divide by δ on both sides of (13) and let $\delta \to 0$, then

$$0 \le \lim_{\delta\to 0}\Big(\int_\Omega \delta|\nabla\varphi|^2 + 2\nabla u\nabla\varphi + 2\int_\Omega \frac{F((u+\delta\varphi))-F(u))}{\delta}\Big) = \int_\Omega 2\nabla u\nabla v + 2\int_\Omega F'(u)\varphi,$$

thus

$$0 \le \int_\Omega \nabla u\nabla\varphi + F'(u)\varphi. \tag{14}$$

This along with the analogues argument with respect to $v = u-\delta\varphi$, implies that

$$\int_{\mathbb{R}^n_+} \nabla u\nabla\varphi + F'(u)\varphi = 0, \tag{15}$$

Therefore, we conclude that u is a weak solution of the problem $\Delta u = f$ in $\{u>0\}$, with $u = \chi_D$ on D.

□

5 Geometric properties of the free boundary

In this section, we show under some additional assumptions that solutions to problem (3.2) are symmetric about coordinate axes. To do this, we use the method of moving plane, introduced by J. Serrin in [17] and further developed by H. Berestycki and L. Nirenberg in [2]. However, [17] and [2] do not consider a variational formulation of their respective problem, and both make assumptions on the regularity of their respective solution. Instead, we use an approach similar to that in the proof of Theorem 3.9, in [18]. We also would like to mention that Theorem 5.0.2 is inspired by Theorem 3.12, along with Remark 2.7 in [18], respectively. Now, we introduce relevant notation.

Setting 1. Given a fixed unit vector $a \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$, with $\lambda \ge 0$, we define the hyperplane

$$\Gamma_\lambda = \Gamma_{\lambda,a} := \{x\cdot a = \lambda\} \subset \mathbb{R}^n$$

and the half-spaces

$$\Gamma_\lambda^+ := \{x\cdot a > \lambda\}, \quad \Gamma_\lambda^- := \{x\cdot a < \lambda\}.$$

For $x \in \mathbb{R}^n$, we denote by x_λ the reflection of x in Γ_λ, and for a function ϕ, we set $\phi_\lambda(x) = \phi(x_\lambda)$. Given any subset $\Sigma \subset \mathbb{R}^n$, then we denote by

$$\Sigma_\lambda := \Sigma \cap \Gamma_\lambda^+, \qquad \Sigma'_\lambda := \{x_\lambda : x \in \Sigma_\lambda\}. \tag{16}$$

These represent a cap of Σ, cut by Γ_λ^+, respectively its reflection.

Remark. Given the above setting, we will, in Theorem 5.0.1, apply the method of moving plane. The idea is to continuously move a hyperplane in its normal direction facing the domain $\{u > 0\}$, then there exists a "moment", more specifically a λ for which the intersection of the hyperplane and $\{u > 0\}$ is empty, as illustrated in the method of moving planes 6.1.2. However, if we continuously decrease λ, then there exists a "moment" for which the intersection of the hyperplane and $\{u > 0\}$ is nonempty, as illustrated in the Method of moving planes 6.1.2, which implies that we can reflect the points in the cut of part of $\{u > 0\}$, thus, by studying these types of reflections, we are, under additional assumptions, able to derive properties of u.

We will now formulate one of the main results of this section, Theorem 2, using the "moving-plane methods" argument in the proof of Theorem 3.9 in [18]. From this we can derive results concerning the geometric and topological properties of the set $\{u > 0\}$. Moreover, under some additional assumptions, we show that u is symmetric in the x_n direction.

Theorem 5.0.1. Let u minimize J over $\mathbb{K}_h$, and suppose that $D = B_1$, the $(n-1)$-dimensional unit ball. Then the following hold:

(i) If $g(x)$ is non-decreasing in x_1-direction, then $\partial_1 u \leq 0$.

(ii) If $g(x_1, x') = g(x_1, |x'|), \quad \forall\, x \in \mathbb{R}^n$, then u is axially symmetric around the x_1-axis, and $u(x) = u(x_1, |x'|)$.

(iii) If $g(x_1, |x'|)$ is non-decreasing in $|x'|$-direction, then $\partial_{|x'|} u(x) \leq 0$.

Proof. We first prove ii) as it is more straightforward, and simpler. Suppose that u minimizes J over $\mathbb{K}_h$, and let σ be any rotation of $\mathbb{R}^{n-1}$ around the origin, then $u_\sigma = u \circ \sigma$ is in $\mathbb{K}_h$; see Theorem 6.3.15. Since $g(x) = g(x_1, |x'|)$ and F is independent of x, we have u_σ is a minimizer of J over $\mathbb{K}_h$, as the value of the functional does not change. Applying Lemma 6.1.1 in the Appendix, to $\max(u, u_\sigma)$, and $\min(u, u_\sigma)$ and observing that

$$J(\max(u, u_\sigma)) + J(\min(u, u_\sigma)) = J(u) + J(u_\sigma),$$

we can conclude they both are absolute minimizers, and consequently $u = u_\sigma$. This proves u has axial symmetry with respect to x_1-axis.

Now to prove i) and iii) we use the moving plane technique, see Appendix 6.1.2.

Since the proof of Case, i) and iii) are similar, we shall only prove case iii), and then explain how case i) follows in the same manner.

Indeed, our proof below also shows that u has spherical symmetry, already proven above in a slightly simpler way.

Given the setting (1), with a and $\lambda_0 \in \mathbb{R}$. Fix some direction d, and consider $\Gamma_\lambda = \Gamma_{\lambda,a} := \{x_d \cdot a = \lambda\}$, defined in (1). We denote by x_λ the reflection of x about Γ_λ in the x_d-direction. Furthermore, we may assume, without loss of generality, that the normal of Γ_λ is orthogonal to the normal of D. Since the converse would imply that there is $x \in \{u > 0\}_\lambda$, such that $x_\lambda \notin \{u > 0\}$.

Consider $u_\lambda = u(x_\lambda)$, we claim that $u_\lambda \geq u$ in $\mathbb{R}^n_+$. Indeed, for any $\lambda \geq \lambda_0$, then $u_\lambda = u(x_\lambda)$ is in $\mathbb{K}_h$, since x_λ is in $\mathbb{R}^n_+$, for any $x \in \mathbb{R}^n_+$, it follows that there exists $y \in \mathbb{R}^n_+$, such that $y = x_\lambda$, thus $u_\lambda = u(x_\lambda) = u(y)$. Hence, $u_\lambda \in \mathbb{K}_h$, for any $\lambda \geq \lambda_0$. This implies that $\min(u, u_\lambda)$ and ,$\max(u, u_\lambda)$ are both in $\mathbb{K}_h$, see Corollary 6.3.5 together with (4). Therefore, we obtain $J[\min(u, u_\lambda)] \leq J[u]$, whenever $\min(u, u_\lambda) \not\equiv u$, see the proof of Lemma 6.1.1. Hence, contradicting the assumption that u is a minimizer of J over $\mathbb{K}_h$. Therefore, we get $u_\lambda \geq u$ in $\mathbb{R}^n_+$.

Observe that $u = u_\lambda$ in all of $\mathbb{R}^n_+ \setminus \{u > 0\}$. Indeed, since $u = 0$ in $\mathbb{R}^n_+ \setminus \{u > 0\}$, it follows that $u_\lambda(x) = u(x_\lambda) = 0$ for any $x_\lambda \in \mathbb{R}^n_+ \setminus \{u > 0\}$. This, along with the definition of x_λ, implies that $u = u_\lambda$ in all of $\mathbb{R}^n_+ \setminus \{u > 0\}$. Therefore, the observation holds. Thus, it suffices to convince ourselves that $u_\lambda \geq u$, whenever $\{u > 0\}_\lambda \subset \{u > 0\}$.

Given any $\lambda \geq \lambda_0$, consider

$$v_\lambda = \begin{cases} \min(u, u_\lambda) & \text{in } \Gamma^+_\lambda \\ \max(u, u_\lambda) & \text{in } \Gamma^-_\lambda, \end{cases} \tag{17}$$

then $v_\lambda = u$ and $\{u > 0\}'_\lambda \subset \{u > 0\}$, whenever $\{u > 0\} \subset \Gamma^-_\lambda$, see the previous observation together with $\{u > 0\} \cap \Gamma = \emptyset$. Therefore, we only need to consider the case where $\{u > 0\} \cap \Gamma_\lambda \neq \emptyset$.

Note that for any $\lambda \geq \lambda_0$, so that $\{u > 0\} \cap \Gamma_\lambda \neq \emptyset$, then $u_\lambda \geq u$, whenever $v_\lambda = u$, and $\{u > 0\}'_\lambda \subset \{u > 0\}$. In particular, for any $\lambda_1 > \lambda_0$, such that $\{u > 0\} \cap \Gamma_\lambda \neq \emptyset$, if

$$v_\lambda = u \tag{18}$$

and

$$\{u > 0\}'_\lambda \subset \{u > 0\} \tag{19}$$

then (18) and (19) hold for any $\lambda \geq \lambda_0$.

Given $\lambda_1 > \lambda_0$, suppose that $\{u > 0\} \cap \Gamma_{\lambda_1} \neq \emptyset$, we need to verify that (18) and (19) hold, respectively.

Let $w = \max(u, v_{\lambda_1}) - u$, then

$$w = \begin{cases} 0 & \text{in } \Gamma^+_{\lambda_1} \cup \Gamma_{\lambda_1} \\ (u_{\lambda_1} - u)_+ & \text{in } \Gamma^-_{\lambda_1}. \end{cases} \tag{20}$$

Observe that $u_{\lambda_1} = u$ on Γ_{λ_1}, therefore, it suffices to consider w over $\Gamma^+_{\lambda_1}$ and $\Gamma^-_{\lambda_1}$, respectively. Recall that $v_{\lambda_1} = \min(u, u_{\lambda_1})$ in $\Gamma^+_{\lambda_1}$. Therefore, we have $w = 0$ over $\Gamma^+_{\lambda_1} \cup \Gamma_{\lambda_1}$. Similarly, for u, v_{λ_1} over $\Gamma^-_{\lambda_1}$, then $\max(u, u_{\lambda_1})$ is equal to u or u_{λ_1}. From this it follows that $w = (u_{\lambda_1} - u)^+$ and we may conclude that w satisfies (20). In particular, this, along with the definition of u_{λ_1}, implies that $w = 0$ in $\mathbb{R}^n_+ \setminus \{u > 0\}'_{\lambda_1}$.

Observe that if $\{u > 0\}'_{\lambda_1} \subset \{u > 0\}$, then $w = 0$ in $\{u > 0\}$. Indeed, since $w = 0$ in $\mathbb{R}^n_+ \setminus \{u > 0\}'_{\lambda_1}$, it suffices to show that $w = 0$ in $\{u > 0\}'_{\lambda_1}$.

Given $\{u > 0\}'_{\lambda_1} \subset \{u > 0\}$, consider w over $\{u > 0\}'_{\lambda_1}$, then $\Delta w = f(u_{\lambda_1}) - f(u) \geq 0$ or $\Delta w = 0$. Indeed, since $u_\lambda \geq u$ in $\mathbb{R}^n_+$, for any $\lambda \geq \lambda_0$, it follows that $F(u) \leq F(u_\lambda)$, thus $f(u) \leq f(u_\lambda)$. Hence, we get $f(u_{\lambda_1}) - f(u) \geq 0$. Therefore, we obtain $w = 0$ in $\{u > 0\}'_{\lambda_1}$, by the maximum principle. From this it follows that $v_{\lambda_1} = u$ in $\{u > 0\}'_{\lambda_1} \subset \{u > 0\}$ whenever (19) holds. In particular, (18) and (19) are equivalent, whenever $\Delta w \geq 0$ in $\{u > 0\}$.

Now, if $\{u > 0\}$ is connected, then $v_\lambda = u$ and $\{u > 0\}'_\lambda \subset \{u > 0\}$, for any $\lambda \geq \lambda_0$.

Suppose that $\lambda_1 > \lambda_0$, and assume that (18) and (19) holds with respect to λ_1. We claim that for any $\varepsilon > 0$ and every λ, such that $\operatorname{dist}(\lambda, \lambda_1) < \varepsilon$, then $\Delta w = 0$ in $\{u > 0\}$. Indeed, if $\{u > 0\} \subset \Gamma^-_\lambda$, then $\Delta w = 0$, see the above argument. Conversely, assume that $\{u > 0\} \cap \Gamma_\lambda \neq \emptyset$, then $w = 0$ in $\mathbb{R}^n_+ \setminus \{u > 0\}'_\lambda$. This, along with the previous argument, implies that $\Delta w \geq 0$. Therefore, $v_\lambda = u$ and $\{u > 0\}'_\lambda \subset \{u > 0\}$, for any $\lambda \geq \lambda_0$. Moreover, we have $\{u > 0\} \not\subset \Gamma^+_\lambda$, whenever $\operatorname{dist}(\lambda, \lambda_1) < \varepsilon$. Indeed, since

$$\operatorname{dist}(x_\lambda, x) = \operatorname{dist}(2\lambda_1 - x_d, x_d) \leq \operatorname{diam}(\{u > 0\}),$$

for any $x_{\lambda_1} \in \{u > 0\}_{\lambda_1}$ and any $x \in \{u > 0\}$, it follows that

$$\operatorname{dist}(\lambda_1, x_n) \leq \frac{1}{2} \operatorname{diam}(\{u > 0\})$$

thus $\operatorname{diam}(\{u > 0\}_{\lambda_1}) \leq \frac{1}{2} \operatorname{diam}(\{u > 0\})$. Hence, we get $\{u > 0\} \not\subset \Gamma^+_\lambda$, whenever $\operatorname{dist}(\lambda, \lambda_1) < \varepsilon$. Therefore, (18) and (19) hold for any $\lambda \geq \lambda_0$, whenever $\{u > 0\}$ is connected. If $\{u > 0\}$ is not connected, then the above argument holds for each component, see Theorem 6.1.2.

Therefore, the claim that $u_\lambda \geq u$ and $\{u > 0\}'_\lambda \subset \{u > 0\}$, for x_d is verified. Hence, for any direction d, he previous argument holds, by arbitrariness of d. This, along with the image of $x_d \mapsto -x_d$, implies that $u_\lambda = u$. From this, it follows that $u = u_\sigma$. Hence, u is radially symmetric about the x_1-axis.

From this it follows that u is nondecreasing in the $|x'|$-direction. Indeed, since $u_\lambda \geq u$ for any $\lambda \geq \lambda_0$, it follows that $u(x, |x'_\lambda|) \geq u(x, |x'|)$ for all $\lambda \geq \lambda_0$, where $u_\lambda \to u$ as $\lambda \to 0$, thus u is decreasing in $|x'|$. Therefore, we obtain that $\partial_{|'x|} u \leq 0$ in the sense of weak derivatives, see Lemma 6.1.5.

For case i), the moving plane technique is done in direction of the negative x_1-axis. Since the method is similar to the $|x'|$-direction, it suffices to consider the steps outlined in case (iii). From this it follows that $u_\lambda \geq u$ for any $\lambda \geq \lambda_0$. In particular, we see that u

is decreasing in the x_1-direction; see the analogue argument for the $|x'|$ direction. This, along with Lemma 6.3.17, implies that $\partial_1 u \leq 0$.

□

Theorem 5.0.2. Suppose that $u \in \mathbb{K}_h$ minimizes J, and assume that

$$F(u_t) \leq tF(u) \quad \text{in } \mathbb{R}^n_+, \quad \text{and} \quad g(tx) \geq g(x) \quad \text{in } \mathbb{R}, \tag{21}$$

for all $t \in (0,1]$, with f, g as in (1). If D is starshaped with respect to the origin, then

$$tu_t \leq u, \tag{22}$$

and $\{u > 0\} \subset \mathbb{R}^n_+$ is starshaped with respect to the origin, with $u_t = u(x/t)$.

Proof. To verify the claim, we consider a method similar to the one of Theorem 3.12 in [18], along with Lemma 6.1.3. Suppose that $u \in \mathbb{K}_h$ is a minimizer of J, and set $g_t = g(tx)$, with $x \in \mathbb{R}$. Now recall the definition of J_t

$$J_t[tu_t] = \int_{\mathbb{R}^n_+} |\nabla tu_t|^2 + \frac{2F(u_t)}{t} + g_t^2 \chi_{\{u_t>0\}}.$$

Then tu_t minimizes J_t over $\mathbb{K}_h$, for any $t \in (0,1]$ (see Lemma 6.1.3). Note that $tu_t \leq u$ for any $x \in D$ and $t \in [0,1]$, since D is starshaped with respect to the origin, it follows that $y = tx$ for any $y \in D$, where $t \in [0,1]$ and $x \in D$, therefore $tu_t \leq u$ in D. We claim that $tu_t \leq u$, for any $t \in (0,1]$. From this, along with the assumption that D is starshaped with respect to the origin, it follows that $\{u > 0\}$ is starshaped with respect to the origin.

To verify the first claim, we consider an argument similar to that in the proof of Theorem 5.0.1. Note that $tu_t = u$ when $t = 1$. Therefore, it suffices to show that $tu_t \leq u$ for any $t \in (0,1)$.

Let $v = \min(tu_t, u)$ and $w = \max(tu_t, u)$, and suppose that $v \not\equiv tu_t$. The comparison principle (see Lemma 6.1.1), along with (21) and the decreasing property of F, produces the following contradiction

$$J_t[v] < J_t[tu_t] \leq J_t[v]. \tag{23}$$

Therefore, it follows that $tu_t \leq u$ in $\mathbb{R}^n_+$. In particular, the support of tu_t is contained in u. Hence, we obtain $tu_t = 0$ whenever $u = 0$.

Let $w_t = \max(tu_t, u)$ and consider $\varphi = \max(w_t, u) - u$, then $\varphi = 0$ in $\mathbb{R}^n_+ \setminus \{u > 0\}$. This, along with $\Delta \max(w_t, u) = f$ in $\{u > 0\}$, implies that $\Delta\varphi = 0$ in $\{u > 0\}$. Therefore, φ is harmonic in $\{u > 0\}$. The the maximum principle, along with $\varphi = 0$ in $\mathbb{R}^n_+ \setminus \{u > 0\}$, entails $\varphi = 0$ in $\{u > 0\}$, which implies that $w_t = u$, for any $t \in (0,1)$.

We claim that $\{u > 0\}$ is star convex with respect to the origin, see Definition 6.1.4. We need to convince ourselves that $tx \in \{u > 0\}$, for any $x \in \{u > 0\}$ and for all $t \in [0,1]$, it suffices to show that $u(tx) > 0$, for any $t \in (0,1)$ and every $x \in \{u > 0\}$.

Suppose not, then there exist $x \in \{u > 0\}$ and $t \in (0,1)$, such that $y = tx \notin \{u > 0\}$, which implies that $u(y) = 0$. This, along with the previous argument, implies that $tu_t(y) \leq u(y) = 0$. However, since $tu_t(y) = tu(tx/t) = tu(x) > 0$, we obtain a contradiction. Therefore, we may conclude that $u(tx) > 0$, for any $t \in (0,1)$.

□

6 Appendix

This section contains the tools and Sobolev space theory used in Sections 4 and 5. In Sections 6.1.1 and 6.1.2, the reader will find the comparison principle lemma used in the previous sections, respectively, illustrations for the moving plane argument in Theorem 5.0.1. Section 6.2 contains general results on Mollifiers, using the results of L.C Evans in [7] and L.C. Evans and R.F. Garzepy in [8]. We also consult the lecture notes of J. Kinnunen in [12]. In Section 6.3, the reader will find general results and definitions of Sobolev space. There is no shortage of literature concerning Sobolev space, the ones used in this thesis are the following, L.C Evans in [7], R. Adams and J. Fournier in [1]. We also consult the lecture notes of J. Kinnunen in [12].

6.1 Tools

The tools used in the previous two sections are presented here. In particular, the reader will find the comparison principle lemma 6.1.1 used in Section 4.

6.1.1 Comparison principle

Lemma 6.1.1. Suppose that $u, u_0 \in \mathbb{K}_h$ minimizes J and J_0, respectively, with J_0 as in (6), and that $u_0 \geq u$ on $\{x_1 = 0\}$. Then $u_0 \geq u$ in $\mathbb{R}^n_+$.

Proof. We want to show that $u_0 \geq u$ in all of $\mathbb{R}^n_+$. To show this, we use a comparison argument similar to the one in Lemma 1.1 [18] and Lemma 1.7 in [16].

We denote by $v := \min(u_0, u)$ and $w := \max(u_0, u)$, which are in the respective admissible class, since $u_0 \geq u$ on $\{x_1 = 0\}$. Suppose $v \not\equiv u$, then

$$\int_{\mathbb{R}^n_+} F(v) < \int_{\mathbb{R}^n_+} F(u). \tag{24}$$

From this it follows that

$$\begin{aligned} J[v] + J_0[w] = J_0[v] + 2\int_{\mathbb{R}^n_+} F(v) + J_0[w] = J_0[u_0] + 2\int_{\mathbb{R}^n_+} F(v) + J_0[u] \\ < J_0[u] + 2\int_{\mathbb{R}^n_+} F(u) + J_0[u_0] = J[u] + J_0[u_0]. \end{aligned} \tag{25}$$

Indeed, we first recall that F is by definition an increasing function. In particular, if $f > 0$ in some neighborhood $V \subset \mathbb{R}^n_+$, then F is strictly increasing in V, thus (24) holds.

Now if $v \not\equiv u$, then there exists $x \in \mathbb{R}^n_+$, such that $v < u$ in some neighborhood U of x, which implies that $u_0 < w$ in U, whenever $v < u$ in U. Since we have

$$J[u] = J_0[u] + 2\int_{\mathbb{R}^n_+} F(u),$$

it follows that

$$J[v] + J_0[w] = J_0[v] + 2\int_{\mathbb{R}^n_+} F(v) + J_0[w] = J_0[u_0] + 2\int_{\mathbb{R}^n_+} F(v) + J_0[u],$$

thus

$$J_0[u_0] + 2\int_{\mathbb{R}^n_+} F(v) + J_0[u] < J_0[u_0] + 2\int_{\mathbb{R}^n_+} F(u) + J_0[u] = J[u] + J_0[u_0],$$

by the inequality in (24). Therefore, the inequality in (25) is verified.

Now, using that u_0, and u minimize J_0 and J, respectively, we have

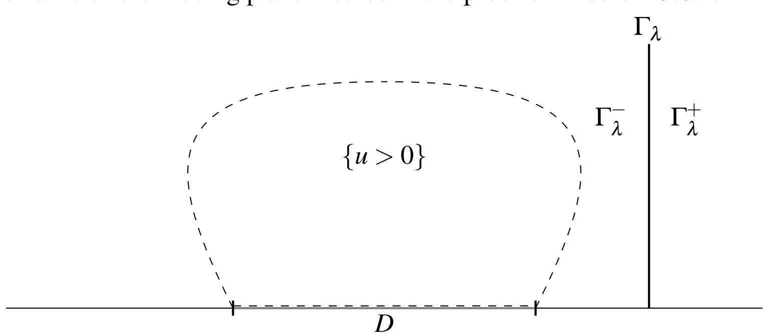
$$J[v] < J[u] + J_0[u_0] - J_0[w] \leq J[u] \leq J[v] \tag{26}$$

which is a contradiction. Therefore, $v = u$, and consequently $u \leq u_0$. In particular, the support of u, denoted by $\mathrm{supp}(u)$, is contained in $\mathrm{supp}(u_0)$.

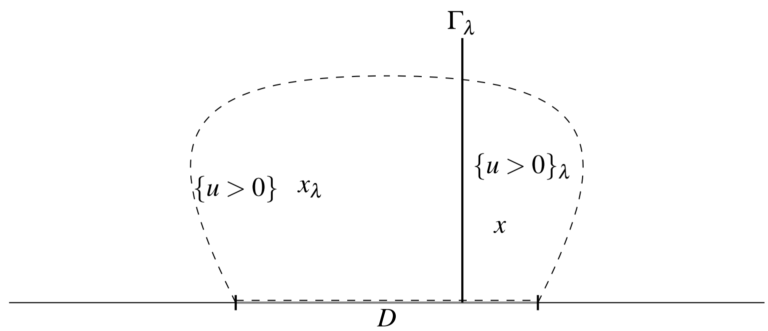
□

6.1.2 Method of moving planes

In this section, we give illustrative examples followed by explanatory text in the form of remarks on the moving plane method in the proof of Theorem 5.0.1.



Remark. The figure above illustrates the moment λ for which the intersection of the hyperplane Γ and $\{u > 0\}$ is empty. In particular, if λ is sufficiently large, the above picture illustrates the argument in the previous paragraph (18).



Remark. The figure above illustrates the moment $\lambda > 0$ for which the intersection of the hyperplane Γ_λ and $\{u > 0\}$ is nonempty. We recall the notation introduced in Setting 1. The left-hand side of the hyperplane contains the cut-off part of $\{u > 0\}$ by the hyperplane Γ_λ. The point $x \in \{u > 0\}_\lambda$ is reflected through the hyperplane Γ_λ. The reflection, denoted by x_λ, is contained on the right-hand side of the hyperplane in the figure above.

6.1.3 Disconnected discussion of Theorem 5.0.1

Theorem 6.1.2. If the assumptions in 5.0.1 are satisfied, then the conclusion of Theorem 5.0.1 holds, whenever $\{u > 0\}$ is not connected.

Proof. From the proof of the properties (i), (ii) and (iii), it suffices to verify the case when $\{u > 0\}$ is not connected, with respect to the claim (iii).

Consider $\{u > 0\} = \sqcup_{\alpha=1}^{n} C_\alpha$, where C_α denotes a connected component of $\{u > 0\}$. Then the conclusion of Theorem 5.0.1 holds for each of the connected components. Therefore, (18) and (19) hold for any $\lambda \geq \lambda_0$, that is, $v_\lambda = u$ and $\{u > 0\}'_\lambda \subset \{u > 0\}$.

Indeed, since Γ_λ can intersect at most one of the C_α, for any λ, it follows that the components of $\{u > 0\} \setminus C_\alpha$ are contained in Γ_λ^- or Γ_λ^+. Thus, $v_\lambda = u$ in any $C_\beta \subset \Gamma_\lambda^-$, see the proof of Theorem 5.0.1. Furthermore, $w = 0$ in any $C_\beta \subset \Gamma_\lambda^+$, according to the definition of w in Γ_λ^+ (see (20)). Hence (18) and (19) hold for any $\lambda \geq \lambda_0$. Therefore, the conclusion of Theorem 5.0.1 is verified. □

6.1.4 Homogeneity

Lemma 6.1.3. If u minimizes J over $\mathbb{K}_h$, then $tu_t \in \mathbb{K}_h$ is a minimizer of

$$J_t[tu_t] = \int_{\mathbb{R}^n_+} |\nabla tu_t|^2 + \frac{2F(u_t)}{t} + g_t^2 \chi_{\{u_t>0\}}, \tag{27}$$

for any $t > 0$, where $u_t = u(x/t)$ and $g_t = g(tx)$, with f and g as in (1).

Proof. Given u as above, suppose that $t > 0$. Now, consider $S_t(x) = x/t$ defined on $\mathbb{R}^n_+$, then S_t is bi-Lipschitz, which implies that $u_t = u \circ S_t \in W^{1,2}(\mathbb{R}^n_+)$, see Theorem 6.3.15. This, along with a comparison argument analogous to the one in the proof of Theorem

4.0.1, implies that $u_t \in \mathbb{K}_h$. In particular, $tu_t \in \mathbb{K}_h$, as $\mathbb{K}_h$ is a vector space. Therefore, we obtain $tu_t \in \mathbb{K}_h$

We claim that tu_t is a minimizer of J_t, defined above. To verify the claim, it suffices to show that $J_t[tu_t] = t^n J[u]$.

Recall that

$$\int_{\mathbb{R}^n_+} |\nabla tu_t|^2 = t^2 \int_{\mathbb{R}^n_+} |\nabla u_t|^2 = t^2 t^{-2} \int_{\mathbb{R}^n_+} |\nabla u \circ S_t|^2 |J_{S_t}| = \int_{\mathbb{R}^n_+} |\nabla u \circ S_t|^2 |J_{S_t}|, \quad (28)$$

where J_{S_t} denotes the Jacobian of S_t, see Theorem 6.3.15. Furthermore, we have

$$\frac{2}{t} \int_{\mathbb{R}^n_+} F(u_t) = \frac{2}{t} \int_{\mathbb{R}^n_+} tF(u)|J_{S_t}| = 2 \int_{\mathbb{R}^n_+} F(u)|J_{S_t}|. \quad (29)$$

This, along with the free boundary condition of u, implies that

$$\begin{aligned} J_t[tu_t] &= \int_{\mathbb{R}^n_+} |\nabla tu_t|^2 + \frac{2F(u_t)}{t} + g_t^2 \chi_{\{u_t>0\}} \\ &= \int_{\mathbb{R}^n_+} (|\nabla u \circ S_t|^2 + 2F(u) + g^2 \chi_{\{u>0\}})|J_{S_t}| \\ &= \frac{1}{t^n} \int_{\mathbb{R}^n_+} |\nabla u|^2 + 2F(u) + g^2 \chi_{\{u>0\}}. \end{aligned}$$

From this it follows that $J_t[tu_t] = t^n J[u]$. □

6.1.5 Starshaped

Definition 6.1.4. Given a nonempty set $A \subset \mathbb{R}^n$, then A is starshaped with respect to a point $a_0 \in A$, if for any $a \in A$, the closed interval $\{ta_0 + (1-t)a : 0 \leq t \leq\}$ is a subset of A.

6.1.6 Weak derivatives and monotonic functions

The following statement is present for the sake of completeness, and the result is an analog to the general one in the theory of distributions.

Lemma 6.1.5. Suppose that $u \in W^{1,2}(U)$, with $U \subset \mathbb{R}^n$, then u is decreasing (increasing) in x_i, for any $i = 1, \dots, n$ if and only if $\partial_i u \leq 0 (\geq 0)$.

Proof. It suffices to verify the claim in in the x_i-direction, for any $i = 1, \dots n$. Given $v \in C^\infty(U)$, then the weak derivative of v is the strong derivative. Fix any $i = 1, \dots, n$, and suppose that v is decreasing (increasing) in the x_i-direction, then $\partial_i v \leq 0$ (≥ 0).

Assume that $u \in W^{1,2}(U)$ we claim that u is decreasing (increasing) in the x_i-direction if and only if $\partial_i u \leq 0 (\geq 0)$. If u is decreasing then $-u$ is increasing, thus it suffices to verify the claim for u decreasing and $\partial_i u \leq 0$.

Let $\eta \in C^\infty(\mathbb{R}^n)$ denote the standard mollifier, with $\eta = (\eta_1, \dots, \eta_n)$, see Definition 6.2.2. Suppose that u is decreasing in x_i, we claim that $\partial_i u \leq 0$. Given any $\varepsilon > 0$, we consider the mollification of u by η_ε, then $\eta_\varepsilon \star u$ is decreasing in x_i. From this, it follows that

$$0 \geq \partial_i(\eta_\varepsilon \star u) = \eta_\varepsilon \star \partial_i u, \tag{30}$$

see the above argument along with property (i) of Theorem 6.3.5. In particular, letting $\varepsilon \to 0$ in (30) we obtain $\partial_i u \leq 0$, see property (ii) of Theorem 6.3.5. Therefore it suffices to show that if $\partial_i u \leq 0$, then u is decreasing in the x_i-direction.

We consider the translated mollifier defined by $\eta^+(x) = \eta(x-1)$, for any $x \in \mathbb{R}^n$, then η^+ satisfies the conditions outlined in Definition 6.2.2, with $\operatorname{supp} \eta^+ \subset B_2(0)$, see Definition 6.2.3.

Recall that any function $u \in W^{1,2}(U)$ can be approximated locally and globally by smooth functions; see Theorem 6.3.12 and Theorem 6.3.13, respectively. From this, it follows that $u_\varepsilon \in C^\infty(U_\varepsilon)$, where $u_\varepsilon = \eta_\varepsilon \star u$, for any $\varepsilon > 0$.

Suppose that $\partial_i u \leq 0$, then $\partial_i(u_\varepsilon) = (\eta_\varepsilon \star \partial_i u) \leq 0$, which implies that u_ε is decreasing in x_i, see the previous argument. Now given $\delta > 0$, consider $u_{\varepsilon,\delta} := u_\varepsilon \star (-\eta_\delta^+)$, then $u_{\varepsilon,\delta} \to u \star (-\eta_\delta^+)$ a.e as $\varepsilon \to 0$, see (ii) of Theorem 6.2.5. From this, along with the definition of η^+, it follows that

$$u \star (-\eta_\delta^+)(x) = -\int_{B_2(0)} u(x - \delta y)\eta^+(y) dy$$

and $u \star (-\eta_\delta^+)$ is decreasing in x and $\delta > 0$, respectively. The previous argument, along with the existence of a weak partial derivative $\partial_i u \in L^2(U)$ and Lebesgue's dominant convergence theorem, implies that $\inf_{\delta > 0} u \star (-\eta_\delta^+)(x) > -\infty$. Therefore, u is the limit of a bounded decreasing sequence. Hence u is decreasing in direction e_i.

□

6.2 Mollifiers

Definition 6.2.1. Given measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}$, the convolution of f and g, denoted by $f \star g : \mathbb{R}^n \to \mathbb{R}$, is the measurable function defined by

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy \tag{31}$$

Definition 6.2.2. Given $\varphi \in C^\infty(\mathbb{R}^n)$, if φ satisfies the following:

(i) The support of φ is compact, that is, $\operatorname{supp}(\varphi)$ is compact.

(ii) The measure of φ is 1, that is, $\int_{\mathbb{R}^n} \varphi dx = 1$

(iii) For any $\varepsilon > 0$, the function $\varphi_\varepsilon := \varepsilon^{-n} \varphi(x/\varepsilon)$ is C^∞, with $\operatorname{supp} \varphi_\varepsilon \subset B_\varepsilon(0)$ and $\int_{\mathbb{R}^n} \varphi_\varepsilon dx = 1$.

Then φ is a mollifier.

Definition 6.2.3. Given $\eta \in C^\infty(\mathbb{R}^n)$, we define the standard mollifier as the function given by

$$\eta := \begin{cases} Ce^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases} \tag{32}$$

with a constant $C > 0$, such that $\int_{\mathbb{R}^n} \eta dx = 1$, and for any $\varepsilon > 0$, η_ε satisfies (iii) in Definition 6.2.2.

Definition 6.2.4. Given any $U \subset \mathbb{R}^n$ open, and for any $\varepsilon > 0$, denote by $U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}$. If $f \in L^1_{loc}(U)$, then the mollification of f is given by $f_\varepsilon := \eta_\varepsilon \star f$ in U_ε, where $\eta_\varepsilon \star f$ is the convolution of f and η_ε in U_ε.

Remark. From the above definition it follows that for any locally integrable function defined in an open subset of $\mathbb{R}^n$, there exists a mollification of, by the standard mollifier in Definition 6.2.3.

Theorem 6.2.5. Given $U \subset \mathbb{R}^n$ open and $\varepsilon > 0$, let $U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}$.

(i) Suppose that $f : U \to \mathbb{R}$ is locally integrable, then $f_\varepsilon \in C^\infty(U_\varepsilon)$.

(ii) Given f_ε, then f_ε converges to f a.e. as $\varepsilon \to 0$.

(iii) If $f \in C(U)$, then f_ε converges to f uniformly on compact subsets of U.

(iv) If $1 \leq p < \infty$ and $f \in L^p_{loc}(U)$, then f_ε converges to f in the norm, that is, $f_\varepsilon \to f$ in $L^p_{loc}(U)$.

Proof. Given $U \subset \mathbb{R}^n$ open, and $\varepsilon > 0$, denote by $U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}$. We prove each of the claims separately.

(i) Given a locally integrable function $f : U \to \mathbb{R}$, let f_ε denote the mollification of f by the standard mollifier. We claim that f_ε is in $C^\infty(U_\varepsilon)$.

Let $e_i \in \mathbb{R}^n$ denote the unit vector, with 1 in the i-entry and zero everywhere else. Given $x \in U_\varepsilon$, choose $h > 0$ small, so that $x + he_i \in U_\varepsilon$. Now, consider

$$\frac{f_\varepsilon(x+he_i) - f_\varepsilon(x)}{h} = \int_U \Big(\frac{\eta_\varepsilon(x+he_i-y) - \eta_\varepsilon(x-y)}{h}\Big) f(y)dy,$$

then, since $\operatorname{supp}(\eta_\varepsilon)$ is compactly contained in U, there exists an open, compactly contained $V \subset U$, such that

$$\int_U \Big(\frac{\eta_\varepsilon(x+he_i-y) - \eta_\varepsilon(x-y)}{h}\Big) f(y)dy = \int_V \Big(\frac{\eta_\varepsilon(x+he_i-y) - \eta_\varepsilon(x-y)}{h}\Big) f(y)dy,$$

which implies that

$$\frac{\eta_\varepsilon(x+he_i-y) - \eta_\varepsilon(x-y)}{h} \to \eta_{\varepsilon,x_i}(x-y) \quad \text{as} \quad h \to 0,$$

uniformly, thus

$$f_{\varepsilon_{x_i}}(x) = \int_U \eta_{\varepsilon,x_i}(x-y)f(y)dy.$$

Hence, by the analogous argument, we find that the partial derivatives $D^\alpha f_\varepsilon$ exist and are equal to

$$D^\alpha f_\varepsilon(x) = \int_U D^\alpha \eta_\varepsilon(x-y)f(y)dy,$$

for $x \in U_\varepsilon$ and multiindex α, therefore, it suffices to prove the following general claim.

Given $\varphi \in C_0^\infty(\mathbb{R}^n)$, denote by $\varphi_{h_i} = \frac{\varphi(x+e_ih)-\varphi(x)}{h}$, the differential quotient of φ, with e_i defined above. Then φ_{h_i} converges uniformly to φ' in $\mathbb{R}^n$ as $h \to 0$. Indeed, since $\varphi \in C_0^\infty(\mathbb{R}^n)$, it follows that φ_{h_i} is continuous, thus, by the fundamental theorem of calculus, we have

$$\int_{\mathbb{R}^n} \varphi_{h_i} dx = \int_U \frac{\varphi(x+e_ih)-\varphi(x)}{h}dx \tag{33}$$

is uniformly continuous over $\mathbb{R}^n$, by the support of φ is compact, and $\varphi \equiv 0$ in $\mathbb{R}^n \setminus \operatorname{supp}(\varphi)$, which implies that

$$\int_U \varphi_{h_i} dx = \int_U \frac{\varphi(x+e_ih)-\varphi(x)}{h}dx, \tag{34}$$

is uniformly continuous in any open set $U \subset \operatorname{supp}(\varphi)$, thus by the mean value theorem, φ_{h_i} converges uniformly to φ' as h tends to 0 in any open set $U \subset \operatorname{supp}(\varphi)$, that is,

$$\lim_{h\to 0}\sup_{x\in U}\Big|\int_U \frac{\varphi(x+e_ih)-\varphi(x)}{h}dx - \int_U \varphi'(x)dx\Big| = 0.$$

Hence, by $\varphi \equiv 0$ in the complement of $\operatorname{supp}(\varphi)$, we conclude that φ_{h_i} converges uniformly to φ' as $h \to 0$ in $\mathbb{R}^n$. Therefore, the claim is verified. In particular, claim (i) is verified.

(ii) Given f_ε, we claim that $f_\varepsilon \to f$ a.e. as $\varepsilon \to 0$.

We recall Lebesgue's Differential Theorem. If $f \in L^1_{loc}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$

$$\int_{B_r(x)} |f(y)-f(x)|dy \to 0 \qquad \text{as } r \to 0. \tag{35}$$

Now, fix $x_0 \in U_\varepsilon$, so that (35) holds, then

$$\begin{aligned} |f_\varepsilon(x_0) - f(x_0)| &= \Big|\int_{B_\varepsilon(x_0)} \eta_\varepsilon(x_0-y)(f(y)-f(x_0))dy\Big| && (36)\\ &\le \frac{1}{\varepsilon^n}\int_{B_\varepsilon(x_0)} \eta\Big(\frac{x_0-y}{\varepsilon}\Big)|f(y)-f(x_0)|dy && (37)\\ &\le \frac{C}{|B_\varepsilon(x_0)|}\int_{B_\varepsilon(x_0)} |f(y)-f(x_0)|dy, && (38)\end{aligned}$$

which implies that $|f_\varepsilon(x_0) - f(x_0)| \to 0$ as $\varepsilon \to 0$, by Lebesgue's Differential Theorem. Thus, the claim that f_ε converges to f a.e. as $\varepsilon \to 0$ is verified.

(iii) Given $f \in C(U)$, consider f_ε and suppose that V is an open compactly contained subset of U. Then $\overline{V}$ is compact in U, which implies that f is uniformly continuous in V. Thus, for $x \in W$ and for any open, compactly contained $W \subset U$ and $V \subset W$, the limit is

$$\lim_{r\to 0} \int_{B_r(x)} |f(y) - f(x)|dy = 0, \tag{39}$$

holds uniformly. Thus,

$$|f_\varepsilon(x_0) - f(x_0)| \leq \frac{C}{|B_\varepsilon(x_0)|} \int_{B_\varepsilon(x_0)} |f(y) - f(x_0)|dy, \tag{40}$$

and we conclude that f_ε converges uniformly to f on W. Therefore, by W being arbitrary, the claim is verified.

(iv) Assume that $1 \leq p < \infty$, and suppose that $f \in L^p_{loc}(U)$. We claim that f_ε converges to f in $L^p_{loc}(U)$.

We need to show that given any $\delta > 0$, then $||f_\varepsilon - f||_{L^p(V)} < \delta$, where V is an open compactly contained subset of U.

Given open and compactly contained $V, W \subset U$, assume that V is compactly contained in W. Now, for any $\delta > 0$, there exists $g \in C(W)$, such that

$$||f - g||_{L^p(W)} < \delta$$

by $C(W)$ is dense in $L^p(W)$.

Observe that, for $\varepsilon > 0$ sufficiently small, then

$$||f_\varepsilon||_{L^p(V)} \leq ||f||_{L^p(W)}. \tag{41}$$

Indeed, given $x \in V$, then

$$|f_\varepsilon(x)| = \Big| \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y) f(y)dy \Big| \leq \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)|f(y)|dy, \tag{42}$$

which, by Hölder's inequality, implies that

$$|f_\varepsilon(x)| \leq \Big(\int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)dy \Big)^{1-\frac{1}{p}} \Big(\int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)|f(y)|^p dy \Big)^{\frac{1}{p}}. \tag{43}$$

Thus, by $\int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)dy = 1$, we have

$$|f_\varepsilon(x)| \leq \Big(\int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)|f(y)|^p dy \Big)^{\frac{1}{p}}. \tag{44}$$

Hence, by taking the power of p and integrating about V on both sides of the inequality in (44), respectively, we obtain

$$\int_V |f_\varepsilon(x)|^p dx \leq \int_V \Big(\int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)|f(y)|^p dy\Big) dx. \tag{45}$$

Therefore, by Fubini's Theorem and the fact that V is compactly contained in W, we may conclude that

$$\int_V |f_\varepsilon(x)|^p dx \leq \int_W |f(y)|^p \Big(\int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)|f(y)|dy\Big) dx = \int_W |f(y)|^p, \tag{46}$$

whenever $\varepsilon > 0$ is sufficiently small. So, in particular $||f_\varepsilon||_{L^p(V)} \leq ||f||_{L^p(w)}$.

Recall that $g_\varepsilon \to g$ converges uniformly on compact subsets of W, by $g \in C(W)$ and (iii) in Proposition 6.2.5.

Now, given the above observation, consider $||f_\varepsilon - f||_{L^p(V)}$, then

$$||f_\varepsilon - f||_{L^p(V)} = ||f_\varepsilon - g_\varepsilon + g_\varepsilon - g + g - f||_{L^p(V)}, \tag{47}$$

which implies that

$$||f_\varepsilon - f||_{L^p(V)} \leq ||f_\varepsilon - g_\varepsilon||_{L^p(V)} + ||g_\varepsilon - g||_{L^p(V)} + ||f - g||_{L^p(V)}. \tag{48}$$

Thus, by (41), we get

$$||f_\varepsilon - f||_{L^p(V)} \leq 2||f - g||_{L^p(W)} + ||g_\varepsilon - g||_{L^p(V)} \leq 2\delta + ||g_\varepsilon - g||_{L^p(V)} \tag{49}$$

Therefore, we conclude that

$$||f_\varepsilon - f||_{L^p(V)} \leq 2\delta, \tag{50}$$

since g_ε converges uniformly to g in V. Hence, the claim that f_ε converges to f in $L^p_{loc}(U)$ is verified.

□

6.3 Sobolev Space

In the first part of this section, we introduce the notion of a weak derivative, after which we introduce the norm function. From there we show that, under additional assumptions, Sobolev spaces are reflexive separable Banach spaces. The two preceding sections contain results on inequalities, smooth approximation, and weak convergence methods of the Sobolev space. The proofs are formulated using the results communicated in, as mentioned in the introductory section of the appendix. [7], [12], [8] and [1].

6.3.1 The Sobolev space norm and weak derivatives

Definition 6.3.1. Given an open set $U \subset \mathbb{R}^n$, let $C_0^\infty(U)$ denote the space of infinitely differentiable real-functions with compact support in U. Then, for any $\varphi \in C_0^\infty(U)$, we call φ a test function.

Definition 6.3.2. Given $U \subset \mathbb{R}^n$ open, suppose that $u, v \in L^1_{loc}(U)$, where $L^1_{loc}(U)$ denote the space of locally integral function at $U \subset \mathbb{R}^n$, and a multiindex α. If $D^\alpha u = v$, that is,

$$\int_U u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_U v \varphi dx, \tag{51}$$

for any $\varphi \in C_0^\infty(U)$, then v is the $\alpha^{th}-$weak partial derivative of u.

Lemma 6.3.3. Given $u, v, w \in L^1_{loc}(U)$, if $D^\alpha u$ exists, with $D^\alpha u = v$ and $D^\alpha u = w$, then $v - w = 0$ a.e. In particular, if $D^\alpha u$ exists, then $D^\alpha u$ is unique.

Proof. Given $u, v, w \in L^1_{loc}(U)$, if $D^\alpha u = v$ and $D^\alpha u = w$, then, for any $\varphi \in C_0^\infty(U)$, we have

$$\int_U u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_U v \varphi dx,$$

and

$$\int_U u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_U w \varphi dx,$$

which implies that

$$(-1)^{|\alpha|} \int_U w \varphi dx = (-1)^{|\alpha|} \int_U v \varphi dx,$$

thus

$$\int_U (w - v) \varphi dx = 0, \tag{52}$$

for any $\varphi \in C_0^\infty(U)$. Therefore, the claim that $w - v = 0$ a.e. is verified. □

Definition 6.3.4. Given $U \subset \mathbb{R}^n$ open and $1 \leq p \leq \infty$, then the Sobolev space at U, denoted by $W^{m,p}(U)$, is the space consisting of all functions $u \in L^p(U)$, such that, for every α with $|\alpha| \leq m$, the $\alpha^{th}-$weak partial derivative of u exists and $D^\alpha u \in L^p(U)$, where $m \in \mathbb{Z}_+$ and α is a multiindex. So, we get

$$W^{m,p}(U) := \{u \in L^p(U) : D^\alpha u \in L^p(U), \forall\, |\alpha| \leq m, \text{ with multi-index } \alpha \text{ and} m \in \mathbb{Z}_+\}.$$

Remark. In fact, as we shall see later, under an additional assumption, a Sobolev space is a Banach space. However, we need to define a norm function for the corresponding Sobolev space to show this. We do this by first stating some properties of weak derivatives. In particular, we will see that the weak derivative enjoys many of the basic rules of calculus.

Theorem 6.3.5. Given $u, v \in W^{m,p}(U)$, and $|\alpha| \leq m$, then

(i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$,

(ii) $D^{\beta}(D^{\alpha}u) = D^{\alpha}(D^{\beta}u) = D^{\alpha+\beta}u$ for all multiindices α, β with $|\alpha| + |\beta| \leq k$.

(iii) For any $a, b \in \mathbb{R}$, $au + bu \in W^{m,p}(U)$, and $D^{\alpha}(au + bu) = aD^{\alpha}u + D^{\alpha}b$.

(iv) If $V \subset U$ is open, then $u \in W^{m,p}(V)$.

(v) Given $\eta \in C_0^{\infty}(U)$, then $\eta u \in W^{m,p}(U)$, and

$$D^{\alpha}(\eta u) = \Sigma_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\beta}\eta D^{\alpha-\beta}u \tag{53}$$

Remark. Equation (53) is the Leibniz's formula for weak partial derivatives.

Proof. Omitted □

Remark. For a proof of Theorem 6.3.5, we refer the reader to the proof of Theorem 1, Section 5.2.3 in [7]. Alternatively, the proof of Lemma 1.12, Section 1.3 in [12].

We define a function on an arbitrary Sobolev space and show that the function satisfies the condition of a norm.

Lemma 6.3.6. Give any $u \in W^{m,p}(U)$, let $||u||_{W^{m,p}(U)}$ be the function defined by

$$||u||_{W^{m,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leq m} \int_U |D^{\alpha}u|^p dx\right)^{1/p} & (1 \leq p < \infty) \\ \sum_{|\alpha| \leq m} \operatorname{ess\,sup}_U |D^{\alpha}u| & (p = \infty), \end{cases} \tag{54}$$

then $||u||_{W^{m,p}(U)}$ is a norm. In particular, $(W^{m,p}(U), ||\cdot||_{W^{m,p}(U)})$ is a normed space.

Proof. Suppose that $u, v \in W^{m,p}(U)$, with $\lambda \in \mathbb{R}$, and let $||u||_{W^{m,p}(U)}$ be defined as in (54). We claim that $||\cdot||_{W^{m,p}(U)}$ is a norm. We need to show that for any $1 \leq p \leq \infty$, the conditions described in the definition of a norm are satisfied. We observe that $||\lambda u||_{W^{m,p}(U)} = |\lambda| ||u||_{W^{m,p}(U)}$, by properties of weak derivatives and L^p spaces. Furthermore, $||u||_{W^{m,p}(U)} = 0$ if and only if $u = 0$ a.e. Indeed, suppose that $||u||_{W^{m,p}(U)} = 0$, then $||D^{\alpha}u||_{L^p(U)} = 0$, thus $u = 0$ a.e. Now, if $u = 0$ a.e, then

$$\int_U uD^{\alpha}\varphi dx = (-1)^{|\alpha|} \int_U v\varphi dx = 0,$$

for any $\varphi \in C_0^{\infty}(U)$, thus $v = 0$, by Lemma 6.3.3. Hence $D^{\alpha}u = v = 0$ a.e, therefore it suffices to show that $||\cdot||_{W^{m,p}(U)}$ satisfies the triangle inequality.

Fix $1 \leq p \leq \infty$, then

$$||u + v||_{W^{m,p}(U)} \leq ||u||_{W^{m,p}(U)} + ||v||_{W^{m,p}(U)}.$$

Indeed, since

$$||u + v||_{W^{m,p}(U)} = \Big(\sum_{|\alpha| \leq m} \int_U |D^{\alpha}(u + v)|^p dx\Big)^{1/p} = \Big(\sum_{|\alpha| \leq m} ||D^{\alpha}u + D^{\alpha}v||^p_{L^p(U)}\Big)^{1/p},$$

it follows that

$$\Big(\sum_{|\alpha|\le m}||D^\alpha u+D^\alpha v||^p_{L^p(U)}\Big)^{1/p}\le\Big(\sum_{|\alpha|\le m}(||D^\alpha u||_{L^p(U)}+||D^\alpha v||_{L^p(U)})^p\Big)^{1/p},$$

and

$$\Big(\sum_{|\alpha|\le m}(||D^\alpha u||_{L^p(U)}+||D^\alpha v||_{L^p(U)})^p\Big)^{1/p}\le\Big(\sum_{|\alpha|\le m}||D^\alpha u||^p_{L^p(U)}\Big)^{1/p}+\Big(\sum_{|\alpha|\le m}||D^\alpha u||^p_{L^p(U)}\Big)^{1/p},$$

thus

$$\begin{aligned}||u+v||_{W^{m,p}(U)}&\le\Big(\sum_{|\alpha|\le m}||D^\alpha u||^p_{L^p(U)}\Big)^{1/p}+\Big(\sum_{|\alpha|\le m}||D^\alpha u||^p_{L^p(U)}\Big)^{1/p}\\&=||u||_{W^{m,p}(U)}+||v||_{W^{m,p}(U)}.\end{aligned}$$

Hence, we get

$$||u+v||_{W^{m,p}(U)}\le||u||_{W^{m,p}(U)}+||v||_{W^{m,p}(U)}.$$

Therefore, the function $||\cdot||_{W^{m,p}(U)}$ satisfies the conditions of a norm with respect to $W^{m,p}(U)$. □

Definition 6.3.7. Given $u\in W^{m,p}(U)$, then $||u||_{W^{m,p}(U)}$, defined as in (54), is the norm of u in $W^{m,p}(U)$.

Definition 6.3.8. Given a sequence $\{u^k\}_{k=1}^\infty\in W^{m,p}(U)$, then u^k converges to u in $W^{m,p}(U)$, for some $u\in W^{m,p}(U)$, whenever

$$\lim_{k\to\infty}||u^k-u||_{W^{m,p}(U)}=0,$$

denoted by

$$u^k\to u\quad\text{in } W^{m,p}(U).$$

Definition 6.3.9. If $U\subset\mathbb{R}^n$ is open, then the closure of $C_0^\infty(U)$ in $W^{m,p}(U)$ is denoted by $W_0^{m,p}(U)$.

Remark. Let $u\in W^{m,p}(U)$, suppose that $u\in W_0^{m,p}(U)$, then there exists a sequence $\{u^k\}_{k=1}^\infty\in C_0^\infty(U)$, such that $u^k\to u\in W^{m,p}(U)$, as $k\to\infty$, by definition. Now, if there exist $\{u^k\}_{k=1}^\infty\in C_0^\infty(U)$ and $u\in W^{m,p}(U)$, such that $u^k\to u\in W^{m,p}(U)$ as $k\to\infty$, then $u\in W^{m,p}(U)$. So, in particular, we have $\lim_{k\to\infty}||u^k-u||_{W^{m,p}(U)}=0$, thus $u\in W_0^{m,p}(U)$ if and only if $D^\alpha u=0$ on ∂U, with $u\in W^{m,p}(U)$. Therefore, $u\in W_0^{m,p}(U)$, if and only if $\operatorname{supp}(u)$ is compactly contained in U.

Theorem 6.3.10. Given $U\subset\mathbb{R}^n$ open and $p\in[1,infty]$, then $W_0^{m,p}(U)$ is a closed subspace of $W^{m,p}(U)$

Proof. Suppose that $U\subset\mathbb{R}^n$ is open and $1\le p\le\infty$, then $W_0^{m,p}(U)$ is the closure of $C_0^\infty(U)\subset W^{1,p}(U)$, by Definition 6.3.9, which implies that $W_0^{m,p}(U)\subset W^{1,p}(U)$, by definition of closure, thus the claim is verified. □

Theorem 6.3.11. For any $m \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$. Let $W^{m,p}(U)$ be the Sobolev space at $U \subset \mathbb{R}^n$, then $W^{m,p}(U)$ is a Banach space.

Proof. For any $m \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$, consider the Sobolev space $W^{m,p}(U)$. We claim that $W^{m,p}(U)$ is a Banach space with respect to the norm $||\cdot||_{W^{m,p}(U)}$. Since $(W^{m,p}(U), ||\cdot||_{W^{m,p}(U)})$ is a normed space, we need to show that $W^{m,p}(U)$ is complete, with respect to $||\cdot||_{W^{m,p}(U)}$, it suffices to show that any Cauchy sequence in $W^{m,p}(U)$ converges to an element in $W^{m,p}(U)$.

Suppose that $\{u^k\}_{k=1}^{\infty} \in W^{m,p}(U)$ is a Cauchy sequence. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $||u^i - u^j||_{W^{m,p}(U)} < \varepsilon$, whenever $i, j \geq N$, thus

$$||u^i - u^j||_{W^{m,p}(U)} = \Big(\sum_{|\alpha| \leq m} ||D^\alpha u^i - D^\alpha u^j||^p_{L^p(U)} \Big)^{1/p} < \varepsilon,$$

whenever $i, j \geq N$. Hence, for every $|\alpha| \leq m$, the sequence $\{D^\alpha u^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^p(U)$. Therefore, for every $|\alpha| \leq m$, there exists $u_\alpha \in L^p(U)$, such that $D^\alpha u^k$ converges to u_α in $L^p(U)$, for every $|\alpha| \leq m$, by $L^p(U)$ is a Banach space. We need to show that $D^\alpha u = u_\alpha$, for some $u \in W^{m,p}(U)$. Fix, $\varphi \in C_0^\infty(U)$, and consider $\lim_{k\to\infty} D^\alpha u^k = u_\alpha$, then

$$\int_U u D^\alpha \varphi dx = \lim_{k\to\infty} \int_U u^k D^\alpha \varphi dx = (-1)^{|\alpha|} \lim_{k\to\infty} \int_U D^\alpha u^k \varphi dx,$$

which implies that

$$\int_U u D^\alpha \varphi dx = (-1)^{|\alpha|} \lim_{k\to\infty} \int_U D^\alpha u^k \varphi dx = (-1)^{|\alpha|} \int_U D^\alpha u_\alpha \varphi dx,$$

thus $u_\alpha = D^\alpha u$. Hence $u \in W^{m,p}(U)$, and therefore $u^k \to u$ in $W^{m,p}(U)$, as k tends to infinity. Indeed, since $D^\alpha u^k \to D^\alpha u$ as $n \to \infty$ in $L^p(U)$, for all $|\alpha| \leq m$, it follows that $\lim_{k\to\infty} ||u^k - u||_{W^{m,p}(U)} = 0$, thus $u^k \to u$ in $W^{m,p}(U)$, as $k \to \infty$. Therefore, $W^{m,p}(U)$ is complete with respect to the norm $||\cdot||_{W^{m,p}(U)}$. □

Corollary 6.3.1. If $1 < p < \infty$, then $W^{1,p}(U)$ is a reflexive separable Banach space.

Proof. Given $U \subset \mathbb{R}^n$ open, suppose that $1 < p < \infty$, and consider $W^{1,p}(U)$. We claim that $W^{1,p}(U)$ is a separable reflexive Banach space, it suffices to show that $W^{1,p}(U)$ is separable and reflexive, by Theorem 6.3.11.

Recall that $L^P(U)$ is a separable reflexive Banach space. So, in particular, any finite product space of $L^P(U)$ is a reflexive separable Banach space, furthermore, any subset of a separable metric space is separable. Moreover, any closed subspace of a reflexive Banach space is a reflexive separable Banach space.

Now, given the above recollection, consider the linear operator

$$T : W^{1,p}(U) \to L^p(U) \times \cdots \times L^p(U) \tag{55}$$

$$u \mapsto (u, (Du)), \tag{56}$$

we claim that T is an isometry.

If T is an isometry, then T is a continuous injective linear operator, which implies that $W^{1,p}(U)$ is isometrically isometric to the image $T(W^{1,p}(U))$. Thus, by the image of a continuous operator of Banach space is a closed subset, the image $T(W^{1,p}(U))$ in $L^p(U) \times \cdots \times L^p(U)$ is closed. Hence, by $W^{1,p}(U)$ is isometrically isomorphic to a closed subspace of a reflexive separable Banach space, we conclude that $W^{1,p}(U)$ is reflexive, by definition, and separable by the properties of separable metric spaces. Therefore, it suffices to convince ourselves that T is, in fact, an isometry.

Indeed, given T, then $||T(u)||_{L^p(U)} = ||(u,(Du))||_{L^p(U)}$, for any $u \in W^{1,p}(U)$, thus

$$||(u,(Du))||_{L^p(U)} = ||u||_{W^{1,p}(U)}, \tag{57}$$

for any $u \in W^{1,p}(U)$. Hence, $||T(u)||_{L^p(U)} = ||u||_{W^{1,p}(U)}$, for any $u \in W^{1,p}(U)$, therefore T is an isometry by definition. So the claim is verified. □

Corollary 6.3.2. If $W^{1,p}(U)$ is a reflexive Banach space, then so is $W_0^{1,p}(U)$.

Proof. Suppose that $W^{1,p}(U)$ is a reflexive Banach space. We claim that $W_0^{1,p}(U)$ is reflexive. In fact, given $W_0^{1,p}(U)$, then $W_0^{1,p}(U)$ is a closed convex linear subspace of $W^{1,p}(U)$, by Definition 6.3.10 and linear subspaces are convex. From this, it follows that $W_0^{1,p}(U)$ is closed under weak convergence. Indeed, since any closed convex subspace of a reflexive space is reflexive, see Theorem 3.7 in [4], it follows that $W_0^{1,p}(U)$ is reflexive. Therefore, the claim is verified. □

Corollary 6.3.3. If $p < \infty$, then $W^{m,p}(U)$ is a separable Banach space.

Proof. The proof is analogous to the proof of Corollary 6.3.1. □

6.3.2 Approximation by smooth functions

Theorem 6.3.12. Given $U \subset \mathbb{R}^n$ open, with $1 \leq p < \infty$, and suppose that $u \in W^{m,p}(U)$. Now, if $u_\varepsilon = \eta_\varepsilon \star u$ in U_ε, then the following hold

(i) $u_\varepsilon \in C^\infty(U_\varepsilon)$, for every $\varepsilon > 0$,

(ii) $u_\varepsilon \to u \in W^{m,p}_{loc}(U)$ as $\varepsilon \to 0$.

Proof. Given $U \subset \mathbb{R}^n$ open and $1 \leq p < \infty$. Suppose that $u \in W^{m,p}(U)$, then u is locally integrable, which implies that u_ε is in $C^\infty(U_\varepsilon)$, by property (i) of Theorem 6.2.5. So it suffices to verify the claim that u_ε converges to u in $W^{m,p}_{loc}(U)$ as $\varepsilon \to 0$.

Given u and u_ε as above, then, for any $|\alpha| \leq m$, we have $D^\alpha u_\varepsilon = \eta_\varepsilon \star D^\alpha u$ in U_ε, which implies that $D^\alpha u_\varepsilon \to D^\alpha u$ in $L^p_{loc}(U)$ as $\varepsilon \to 0$, see (iii) Theorem 6.2.5. Thus, for any open compactly contained $V \subset U$, we have

$$||u_\varepsilon - u||^p_{W^{m,p}(V)} = \sum_{|\alpha| \leq m} ||D^\alpha u_\varepsilon - D^\alpha u||^p_{L^p(U)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{58}$$

Hence, we have $u_\varepsilon \to u \in W^{m,p}_{loc}(U)$ as $\varepsilon \to 0$, therefore, it suffices to convince ourselves that $D^\alpha u_\varepsilon = \eta_\varepsilon \star D^\alpha u$ is in U_ε, for any $|\alpha| \le m$.

Give $x \in U_\varepsilon$, consider $D^\alpha u_\varepsilon(x)$, then

$$\begin{aligned}
D^\alpha u_\varepsilon(x) &= D^\alpha \int_U \eta_\varepsilon(x-y)u(y)dy \\
&= \int_U D^\alpha_y \eta_\varepsilon(x-y)u(y)dy \\
&= (-1)^{|\alpha|} \int_U D^\alpha_y \eta_\varepsilon(x-y)u(y)dy \\
&= (-1)^{|\alpha|+|\alpha|} \int_U \eta_\varepsilon(x-y)D^\alpha u(y)dy \\
&= (\eta_\varepsilon \star D^\alpha u)(x).
\end{aligned}$$

Thus, we have $D^\alpha u_\varepsilon = \eta_\varepsilon \star D^\alpha u$ in U_ε, for any $|\alpha| \le m$. Therefore, the original claim is verified. □

Theorem 6.3.13. Given $U \subset \mathbb{R}^n$ open and $1 \le p < \infty$. Suppose that $u \in W^{m,p}(U)$, then there exists $\{u^k\}_{k=1}^\infty \in C^\infty(U) \cap W^{m,p}(U)$, such that

$$u^k \to u \quad \text{in } W^{m,p}(U) \quad \text{as } k \to \infty. \tag{59}$$

Proof. Assume that $U \subset \mathbb{R}^n$, with $1 < p < \infty$. For any $u \in W^{m,p}(U)$, we claim that there exists a $\{u^k\}_{k=1}$ in $C^\infty(U) \cap W^{m,p}(U)$, such that u^k converges to u in the norm of $W^{m,p}(U)$.

Recall that if U is an open subset of $\mathbb{R}^n$, then U is a smooth submanifold of the smooth manifold $\mathbb{R}^n$, which implies that U admits a smooth partition of unity $\{\theta^k\}_{k=1}^\infty$ subordinate to any open cover $\{U_i\}_{i=1}^\infty$, such that, for each $k \ge 1$, we have $1 \le \theta^k \le 1$, and the support of θ^k is compact, with $\operatorname{supp}(\theta^k) \subset U_i$, for some $i \ge 1$ and for any $k \ge 1$, by Theorem 2.23 in [14].

So, given $U \subset \mathbb{R}^n$ open, suppose that $\{U_k\}_{k=1}^\infty$ is a refinement of an open cover of U, then there exists a smooth partition of unity $\{\theta^k\}_{k=1}^\infty$ subordinate to open $\{U_k\}_{k=1}^\infty$, such that $\operatorname{supp}(\theta^k)$ is locally finite for each $k \ge 1$, with $\sum_{k=1}^\infty \theta^k = 1$. Now, if $\theta^k \in C_0^\infty(U_k)$, for all $k \ge 1$, then $\theta^k u \in W^{m,p}(U)$, for any $u \in W^{m,p}(U)$, where $U = \cup_{k=1}^\infty U_k$, with $\operatorname{supp}(\theta^k u) \subset U_k$, by (v) in Theorem 6.3.5.

Fix some $\delta > 0$ and choose $\varepsilon_k > 0$ sufficiently small, then we find that the mollification of $\theta^k u$ in $U_{\varepsilon_k} := \{x \in U_k : \operatorname{dist}(x, \partial U_k) > \varepsilon_k\}$, defined by $u_{\varepsilon_k} := \eta_\varepsilon \star \theta^k u$ satisfies

$$\begin{cases} ||u_{\varepsilon_k} - \theta^k u||_{W^{k,p}(U)} \le \frac{\delta}{2^{k+1}} \\ ||\operatorname{supp}(u_{\varepsilon_k}) \subset U_k \end{cases}. \tag{60}$$

In fact, since $U_{\varepsilon_k} \subset U_k$, it follows that $\operatorname{supp}(\eta_{\varepsilon_k}) \subset U_k$, therefore, by (ii) Theorem 6.3.12, we have u_{ε_k} satisfies (60).

Now, let $v = \sum_{k=1}^{\infty} u_{\varepsilon_k}$, then $v \in C^{\infty}(U)$. Indeed, since the support of θ^k is locally finite, with $\operatorname{supp}(u_{\varepsilon_k})$ and $\operatorname{supp}(\theta^k u)$ contained in U_k, it follows that, for any open and compactly contained V in U, no more than a finite number of u_{ε_k} in the sum are non-zero about V, thus v is well defined. Hence, we have $v \in C^{\infty}(U)$. Recall that $\sum_{k=1}^{\infty} \theta^k = 1$, by properties of smooth partition of unity. So, in particular $\sum_{k=1}^{\infty} \theta^k u = u$, which implies that

$$||v-u||_{W^{m,p}(V)} \leq \sum_{k=1} \infty ||u_{\varepsilon_k} - \theta^k u||_{W^{m,p}(U)} \tag{61}$$

$$\leq \sum_{k=1}^{\infty} \frac{\delta}{2^{k+1}} \tag{62}$$

$$= \delta. \tag{63}$$

Thus, by taking the supremum of sets $V \subset U$, we get $||v-u||_{W^{m,p}(U)}$. Therefore, the claim is verified. □

Theorem 6.3.14. Given $u \in W^{1,p}(U)$, and let $f \in C^1(\mathbb{R})$, such that $f' \in L^{\infty}(\mathbb{R})$, and $f(0) = 0$, then $f \circ u \in W^{1,p}(U)$ and

$$D_j(f \circ u) = f'(u) \circ D_j u \ \text{ a.e in } U \quad j = 1,2,\ldots,n. \tag{64}$$

Proof. Given $u \in W^{1,p}(U)$, and f as above, we claim that $f(u) \in W^{1,p}(U)$ and that $D_j f(u) = f'(u) D_j u$, for $j = 1,2,\ldots,n$. Indeed, since

$$|f(u)| = |f(u) - 0| = |f(u) - f(0)| = \Big| \int_0^u f'(t)dt \Big| \leq \int_0^u |f'(t)|dt \leq ||f'||_{L^{\infty}(\mathbb{R})} |u|,$$

it follows that

$$\Big(\int_U |f(u)|^p \Big)^{\frac{1}{p}} \leq ||f'||_{L^{\infty}(\mathbb{R})} \Big(\int_U |u|^p \Big)^{\frac{1}{p}}. \tag{65}$$

Thus, $f(u) \in L^p(U)$. Therefore, it suffices to show that $D_j f(u) = f'(u) D_j u \in L^p(U)$, for $j = 1,2,\ldots,n$.

Let u_{ε} denote the mollification of u in U_{ε}, and consider $Df(u)$, then, by Theorem 6.2.5, we have

$$\int_U f(u) D_j \varphi = \int_V f(u) D_j \varphi \tag{66}$$

$$= \lim_{\varepsilon \to 0} \int_V f(u_{\varepsilon}) D_j \varphi \tag{67}$$

$$= (-1) \lim_{\varepsilon \to 0} \int_V f'(u_{\varepsilon}) D u_{\varepsilon,j} \varphi \tag{68}$$

$$= -\int_V f'(u) D u_j \varphi \tag{69}$$

$$= -\int_U f'(u) D u_j \varphi, \tag{70}$$

$$\tag{71}$$

for any open compactly V in U, thus

$$\Big(\int_U |f'(u)D_j u|^p\Big)^{\frac{1}{p}} \le ||f'||_{L^\infty(\mathbb{R})}\Big(\int_U |Du|^p\Big)^{\frac{1}{p}}. \tag{72}$$

Therefore, $D_j f(u) \in L^p(U)$, for any $j = 1,\dots,n$, and hence the claim that $f(u) \in W^{1,p}(U)$ is now verified. □

Theorem 6.3.15. Let $U \subset \mathbb{R}^n$ be any open subset and suppose that $u \in W^{1,p}(U)$. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is bi-Lipschitz, then $u(T) = u \circ T \in W^{1,p}(V)$, where $V = T^{-1}(U)$ and

$$Du[(T(x))] \cdot dT(x,\xi) = Du(T(x)) \cdot \xi \tag{73}$$

a.e in U, for any $\xi \in \mathbb{R}^n$.

Proof. We recall Rademacher's Theorem; see Theorem 3.2 of Section 3 in [8], which tells us that any Lipschitz function defined on an open set is differentiable a.e, along with the generalized definition of the Jacobian for Lipschitz function. This, along with the observation that the chain rule holds for any $u \in W^{1,p}(U)$, see Theorem 6.3.13, together with the inverse property of T, implies that

$$\int_U |Du(T)|^p \le \mathrm{Lip(T)} \int_U |Du[(T(x))]|^p |dT(x,\xi)| = \int_V |Du|^p, \tag{74}$$

where Lip(T) > 0 is the Lipschitz constant of T. The argument is analogous for the converse direction. Therefore, the claim is verified. □

Theorem 6.3.16. Given $u \in W^{1,p}(U)$, then $|u| \in W^{1,p}(U)$.

Proof. Suppose that $u \in W^{1,p}(U)$, we claim that $|u| \in W^{1,p}(U)$, it suffices to find a function f, such that $f(u) = |u|$, with $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$ and $f(0) = 0$, by Theorem 6.3.14. Given $\varepsilon > 0$, consider $f_\varepsilon : \mathbb{R} \to \mathbb{R}$, defined by $f_\varepsilon(t) = \sqrt{t^2+\varepsilon^2} - \varepsilon$, then $f(0) = 0$ and $\lim_{\varepsilon\to 0} f_\varepsilon(t) = |t|$, for any $t \in \mathbb{R}$.

Observe that $f_\varepsilon \in C^1(\mathbb{R})$, furthermore $||f'_\varepsilon|| \le 1$, for any $\varepsilon > 0$, by $f_{\varepsilon'}(t) = \frac{t}{\sqrt{t^2+\varepsilon^2}}$ for any $t \in \mathbb{R}$. Moreover, we have

$$\lim_{\varepsilon\to 0} f'_\varepsilon(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}. \tag{75}$$

This, along with Theorem 6.3.14, implies that $\lim_{\varepsilon\to 0} f_\varepsilon(u) = |u|$ and $f_\varepsilon(u) \in W^{1,p}(U)$, where the sign of $D|u|$ is given by the value of the function in (75), that is,

$$D|u| = \begin{cases} Du \text{ a.e} & \text{in } \{u > 0\} \\ 0 \text{ a.e} & \text{in } \{u = 0\} \\ -Du \text{ a.e} & \text{in } \{u < 0\} \end{cases}. \tag{76}$$

□

Corollary 6.3.4. Given $u \in W^{1,p}(U)$, then $\max(u,0), \min(u,0) \in W^{1,p}(U)$.

Proof. Given $u \in W^{1,p}(U)$, we claim $\max(u,0), \min(u,0) \in W^{1,p}(U)$. Indeed, given any $u \in W^{1,p}(U)$, then $\max(u,0) = \frac{u+|u|}{2}$ and $\min(u,0) = \frac{|u|-u}{2}$, thus, by Theorem 6.3.16, we have $\max(u,0), \min(u,0) \in W^{1,p}(U)$. Therefore, the claim is verified. □

Corollary 6.3.5. Given $u, v \in W^{1,p}(U)$, then $\max(u,v), \min(u,v) \in W^{1,p}(U)$. Furthermore, if $u, v \in W_0^{1,p}(U)$, then $\max(u,v), \min(u,v) \in W_0^{1,p}(U)$.

Proof. Given $u, v \in W^{1,p}(U)$, we claim that $\max(u,v), \min(u,v) \in W^{1,p}(U)$, it suffices to note that $\max(u,v) = \frac{(u+v+|u-v|)}{2}$ and $\min(u,v) = \frac{(u+v-|u-v|)}{2}$, therefore, the claim is verified by $W^{1,p}(U)$ being closed under linear operations and absolute values, Theorem 6.3.16.

Thus, the first part of the claim is verified. Given any $u, v \in W_0^{1,p}(U)$, assume without loss of generality that $u \geq v$ in U, then $\max(u,v) = u$ and $\min(u,v) = v$, which implies that $D(\max(u,v)) = Du$ and $D(\min(u,v)) = Dv$, respectively, in U, thus $D(\max(u,v))$ and $D(\min(u,v))$ exist, with the support of $\max(u,v)$ and $\min(u,v)$ compactly contained in U, respectively, by $u, v \in W_0^{1,p}(U)$. Hence, the claim that $\max(u,v)$ and $\min(u,v)$ is in $W_0^{1,p}(U)$, respectively, for any $u, v \in W_0^{1,p}(U)$, is verified.

□

Lemma 6.3.17. Given $u \in W^{1,p}(U)$, then $Du = 0$ a.e on $\{u = 0\}$. In particular, $Du = 0$ a.e on any constant set $\{u = c\}$, for any constant c.

Proof. The claim follows from $Du = Du^+ + Du^-$, along with the definition (76). □

Lemma 6.3.18. Given $u \in W_0^{m,p}(U)$, assume that $\tilde{u}$ is the zero extension of u, then $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$ and $D^\alpha \tilde{u} = \widetilde{D^\alpha u}$.

Proof. Given $u \in W_0^{m,p}(U)$, consider the zero extension of u to $\mathbb{R}^n \setminus U$, then $\tilde{u}$ is defined by

$$\tilde{u} := \begin{cases} u & \text{in } U \\ 0 & \text{in } \mathbb{R}^n \setminus U \end{cases} . \tag{77}$$

We claim that $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$ and $D^\alpha \tilde{u} = \widetilde{D^\alpha u}$.

Recall that if $u \in W_0^{m,p}(U)$, then, by Definition 6.3.9, there exists $\{u^k\}_{k=1}^\infty \in C_0^\infty(U)$, such that u^k converges to u in $W_0^{m,p}(U)$ as $k \to \infty$.

Given any $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \tilde{u} D^\alpha \varphi = \int_U \tilde{u} D^\alpha \varphi + \int_{\mathbb{R}^n \setminus U} \tilde{u} D^\alpha \varphi = \int_U u D^\alpha \varphi + 0 = \int_U \tilde{u} D^\alpha \varphi, \tag{78}$$

which, by the above recollection, implies that

$$\int_{\mathbb{R}^n} \tilde{u} D^\alpha \varphi = \int_U u D^\alpha \varphi \tag{79}$$

$$= \lim_{k\to\infty} \int_U u^k D^\alpha \varphi \tag{80}$$

$$= (-1)^{|\alpha|} \lim_{k\to\infty} \int_U D^\alpha u^k \varphi \tag{81}$$

$$= (-1)^{|\alpha|} \int_U D^\alpha u \varphi \tag{82}$$

$$= (-1)^{|\alpha|} \Big(\int_U \widetilde{D^\alpha u} \varphi + \int_{\mathbb{R}^n \setminus U} \widetilde{D^\alpha u} \varphi \Big) \tag{83}$$

$$= \int_{\mathbb{R}^n} u \widetilde{D^\alpha} \varphi, \tag{84}$$

$$\tag{85}$$

Thus, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have $D^\alpha \tilde{u} = \widetilde{D}^\alpha u$. Hence, by Lemma 6.3.3, we obtain $D^\alpha \tilde{u} = \widetilde{D}^\alpha u$ a.e. in $\mathbb{R}^n$, so we conclude that $||\tilde{u}||_{W^{m,p}(\mathbb{R}^n)} = ||u||_{W^{m,p}(U)}$. Therefore, the claim that $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$, is now verified. □

6.3.3 Poincare Inequality

In this section, we state and prove the classical Poincaré inequality, see Theorem 6.3.20. The proof uses, among other methods, results from the Gagliardo–Nirenberg–Sobolev inequality, see Theorem 6.3.19, and arguments in the proof of Theorem 6.3.19.

Theorem 6.3.19. Suppose that $1 \le p < n$, then there exists a constant C, depending only on p,n, such that

$$||u||_{L^{p^*}(\mathbb{R}^n)} \le C||Du||_{L^p(\mathbb{R}^n)}, \tag{86}$$

for all $u \in C_0^1(\mathbb{R}^n)$, with $p^* = \frac{np}{n-p} > p$.

Proof. Given any $u \in C_0^1(\mathbb{R}^n)$, suppose that $p \in [1,n)$, with $p^* = \frac{np}{n-p}$. We claim that there exists a constant $C := C(n,p) > 0$, such that

$$||u||_{L^{p^*}(\mathbb{R}^n)} \le C||Du||_{L^p(\mathbb{R}^n)}. \tag{87}$$

To verify the claim, we need to show that the inequality in (86) holds, it suffices to show that the inequality holds for $p = 1$, and $1 < p < n$. If $p = 1$, then $p^* = \frac{n}{n-1}$, thus we need to show that

$$||u||_{L^{p^*}(\mathbb{R}^n)} \le C||Du||_{L^1(\mathbb{R}^n)}. \tag{88}$$

Given $u \in C_0^1(\mathbb{R}^n)$, then, for each $i = 1,\dots,n$ and $x \in \mathbb{R}^n$,

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1,\dots,x_{i-1},y_i,x_{i+1}\dots,x_n)dy_i,$$

by the fundamental theorem of calculus and u has compact support in $\mathbb{R}^n$. This implies that

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1,\dots,x_{i-1},y_i,x_{i+1}\dots,x_n)|dy_i,$$

thus

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} |Du(x_1,\dots,x_{i-1},y_i,x_{i+1}\dots,x_n)|dy_i \Big)^{\frac{1}{n-1}}. \tag{89}$$

Therefore, if we integrate with respect to x_1 in (89), then

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} |Du(x_1,\dots,x_{i-1},y_i,x_{i+1}\dots,x_n)|dy_i \Big)^{\frac{1}{n-1}} dx_1. \tag{90}$$

Observe that if $i=1$, then $|Du(x_1,\dots,x_{i-1},y_i,x_{i+1}\dots,x_n)| = |Du(y_1,x_2\dots,x_n)|$, thus $|Du(y_1,x_2\dots,x_n)|$ is independent of x_1. Therefore, the inequality in (90) is equivalent to

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \Big(\int_{-\infty}^{\infty} |Du|dy_i \Big)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \Big(\int_{-\infty}^{\infty} |Du|dy_i \Big)^{\frac{1}{n-1}} dx_1, \tag{91}$$

with Du the shorthand notion for $Du(x_1,\dots,x_i)$, for $i=1,\dots n$. Recall the general Hölder inequality, see Appendix B in [7], and consider

$$\int_{-\infty}^{\infty} \prod_{i=2}^{n} \Big(\int_{-\infty}^{\infty} |Du|dy_i \Big)^{\frac{1}{n-1}} dx_1.$$

Then, by the general Hölder inequality, we have that

$$\int_{-\infty}^{\infty} \prod_{i=2}^{n} \Big(\int_{-\infty}^{\infty} |Du|dy_i \Big)^{\frac{1}{n-1}} dx_1 \leq \Big(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|dx_1 dy_i \Big)^{\frac{1}{n-1}}, \tag{92}$$

thus

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \Big(\int_{-\infty}^{\infty} |D(u)|dy_i \Big)^{\frac{1}{n-1}} \Big(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|dx_1 dy_i \Big)^{\frac{1}{n-1}}. \tag{93}$$

Recall that if $p=1$, then $p^* = \frac{n}{n-1}$, which implies that the inequality in (86) is equivalent to

$$\Big(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \Big)^{\frac{n-1}{n}} \leq C \int_{\mathbb{R}^n} |Du|dx, \tag{94}$$

for all $u \in C_0^1(\mathbb{R}^n)$, where $C>0$ is a constant that depends only on p and n. To obtain the inequality in (94). We consider, for each $i=2,\dots,n$, the analogous process described above. If for each $i=,2\dots n$, we first integrate with respect to x_i, secondly we extract the term independent of x_i, thirdly we apply the general Hölder inequality, then eventually we obtain

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx \leq \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du|dx_1\dots dy_i \dots dx_n \Big)^{\frac{1}{n-1}} = \Big(\int_{-\infty}^{\infty} |Du|dx \Big)^{\frac{n}{n-1}}, \tag{95}$$

thus,

$$\Big(\int_{\mathbb{R}^n}|u|^{\frac{n}{n-1}}dx\Big)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n}|Du|dx. \tag{96}$$

Therefore, for $p=1$, the inequality in (86) is verified.

To prove the last part of the statement, assume that $1<p<n$. Given any $u\in C_0^1(\mathbb{R}^n)$, set $v:=|u|^\gamma$, for $\gamma>1$, then $v\in C_0^1(\mathbb{R}^n)$, which, by applying estimate (96) to v, implies that

$$\Big(\int_{\mathbb{R}^n}|v|^{\frac{n}{n-1}}dx\Big)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n}|Dv|dx, \tag{97}$$

thus, by $v=|u|^\gamma$, we obtain

$$\Big(\int_{\mathbb{R}^n}|u|^{\frac{\gamma}{n-1}}dx\Big)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n}|D|u|^\gamma|dx = \gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}|Du|dx.$$

Hence, by $\frac{p-1}{p}+\frac{1}{p}=1$ and the Hölder inequality, we get

$$\gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}|Du|dx \leq \gamma\Big(\int_{\mathbb{R}^n}|u|^{(\gamma-1)\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^n}|Du|^p dx\Big)^{\frac{1}{p}},$$

therefore

$$\Big(\int_{\mathbb{R}^n}|u|^{\frac{\gamma}{n-1}}dx\Big)^{\frac{n-1}{n}} \leq \gamma\Big(\int_{\mathbb{R}^n}|u|^{(\gamma-1)\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^n}|Du|^p dx\Big)^{\frac{1}{p}}. \tag{98}$$

Now, choose $\gamma:=\frac{p(n-1)}{n-p}$, then

$$(\gamma-1)\frac{p}{p-1}=\frac{p(n-1)}{n-p}\frac{p}{p-1}-\frac{p}{p-1}=\frac{(n-1)p^2-p(n-p)}{(n-p)(p-1)}=\frac{np}{n-p}=p^*,$$

which implies that $\frac{\gamma n}{n-1}=p^*$, thus we get $\frac{n-1}{n}=\frac{\gamma}{p^*}$ and $\frac{p-1}{p}=\frac{\gamma-1}{p^*}$. Hence, (98) is equivalent to

$$\Big(\int_{\mathbb{R}^n}|u|^{p^*}dx\Big)^{\frac{n}{n-1}} \leq \gamma\Big(\int_{\mathbb{R}^n}|u|^{p^*}dx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^n}|Du|^p dx\Big)^{\frac{1}{p}},$$

therefore

$$\Big(\int_{\mathbb{R}^n}|u|^{p^*}dx\Big)^{\frac{n}{n-1}-\frac{p-1}{p}} \leq \gamma\Big(\int_{\mathbb{R}^n}|Du|^p dx\Big)^{\frac{1}{p}},$$

which is equivalent to

$$\Big(\int_{\mathbb{R}^n}|u|^{p^*}dx\Big)^{\frac{1}{p^*}} \leq \gamma\Big(\int_{\mathbb{R}^n}|Du|^p dx\Big)^{\frac{1}{p}}, \tag{99}$$

with $\gamma=C(n,p)>0$. Hence, we conclude that (99) satisfies the inequality in (86). □

Theorem 6.3.20. Assume that $U \subset \mathbb{R}^n$ is open and bounded. Given $u \in W_0^{1,p}(U)$, with $p \in [1,n)$, then

$$||u||_{L^q(U)} \leq C||Du||_{L^p(U)}, \tag{100}$$

for every $q \in [1,p^*]$, where $p^* = \frac{np}{n-p}$, with constant $C := C(p,q,n,U) > 0$, depending only on p,q,n and U. Moreover, we have the following

$$||u||_{L^p(U)} \leq C||Du||_{L^p(U)}, \tag{101}$$

for all $p \in [1,\infty]$, where $C := C(p,U) > 0$ In particular, the statement is valid whenever $U \subset \mathbb{R}^n$ is open and bounded in at least one direction.

Proof. Given $U \subset \mathbb{R}^n$ open and bounded, let $1 \leq p < n$ and assume that $u \in W_0^{1,p}(U)$. We claim that u satisfies the inequality in (100). Consider $\tilde{u} : U \to \mathbb{R}^n$, the zero extension of u, defined by

$$\tilde{u} = \begin{cases} u(x) & \text{x } \in U \\ 0 & \text{x } \in \mathbb{R}^n \setminus U \end{cases} .$$

If $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$, then

$$||u||_{L^{p^*}(U)} \leq C||Du||_{L^p(U)}.$$

Indeed, since $U \subset \mathbb{R}^n$ is open, it follows that $v \in W^{1,p}(U)$, for any $v \in W^{1,p}(\mathbb{R}^n)$, by Theorem 6.3.5, thus

$$||u||_{L^q(U)} \leq C||Du||_{L^p(U)}, \tag{102}$$

for any $q \in [1,p^*]$, with $p^* = \frac{np}{n-p}$, by Theorem 6.3.19. So, in particular,

$$||u||_{L^p(U)} \leq C||Du||_{L^p(U)},$$

by U is a bounded subset of $\mathbb{R}^n$. Hence, it suffices to show that $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$. Given $u \in W^{1,p}(U)$, then the zero extension of u, denoted by $\tilde{u}$, is in $W^{1,p}(\mathbb{R}^n)$, by Lemma 6.3.18. Therefore, by $U \subset \mathbb{R}^n$ being open and bounded, and $\tilde{u} = u$ in U, it follows that

$$||u||_{L^q(\mathbb{R}^n)} \leq C||Du||_{L^p(\mathbb{R}^n)}, \tag{103}$$

and the claim is verified for any $q \in [1,p^*]$. For the case where $U \subset \mathbb{R}^n$ is open and bounded in some x_n direction, it suffices to integrate with respect to x_n in the proof of 6.3.19 to obtain the inequality in (102). □

6.3.4 Weak convergence

Definition 6.3.21. Given $p \in (1,\infty)$, suppose that $U \subset \mathbb{R}^n$ is open. Let $\{f^k\}_{k=1}^{\infty}$ be a sequence in $L^p(U)$, then $\{f^k\}_{k=1}^{\infty}$ converges weakly to a function $f \in L^p(U)$, denoted by

$$f^k \rightharpoonup f \quad \text{ in } L^p(U) \quad \text{as } k \to \infty,$$

provided that

$$\lim_{k\to\infty} \int_U f^k g = \int_U fg,$$

for any $g \in L^{p'}(U)$, where $p' = \frac{p}{p-1}$.

Remark. In the proof of Theorem 4.0.1, we used the connection between bounded sequences in the Sobolev space $W^{1,2}(\mathbb{R}^n_+)$ to obtain a weakly convergent sequence in $L^2(\mathbb{R}^n_+)$.

Definition 6.3.22. Suppose that $\{u^k\}_{k=1}^{\infty} \in W^{1,p}(U)$, with $U \subset \mathbb{R}^n$ open, and $1 < p < \infty$. If there exists a subsequence $\{u^{k_j}\}_{j=1}^{\infty} \subset \{u^k\}_{k=1}^{\infty}$, and $u \in W^{1,p}(U)$, such that

$$\begin{cases} u^{k_j} \rightharpoonup u \quad \text{in } L^p(U) \quad \text{as } j \to \infty \\ Du^{k_j} \rightharpoonup Du \quad \text{in } L^p(U\mathbb{R}^n) \quad \text{as } j \to \infty, \end{cases} \tag{104}$$

then $\{u^{k_j}\}$ converges weakly to u in $W^{1,p}(U)$, denoted by,

$$u^{k_j} \rightharpoonup u \quad \text{in } W^{1,p}(U) \quad \text{as } j \to \infty.$$

Remark. The definition of weak convergence in Sobolev spaces is justified by Definition 6.3.7. In fact, we will see that some of the properties of weak convergence in $L^p(U)$ have desirable implications in the corresponding Sobolev space, which we take advantage of in the proof of Theorem 4.0.1.

Definition 6.3.23. Given a function $F : W^{1,p}(U) \to [-\infty, \infty]$, assume that

$$F[u] \le \liminf_{k\to\infty} F[u^k],$$

whenever

$$u^k \rightharpoonup u \quad \text{in } W^{1,p}(U), \tag{105}$$

then F is (sequentially) weakly lower semicontinuous on $W^{1,p}(U)$.

Theorem 6.3.24. Given $U \subset \mathbb{R}^n$ open and $1 < p < \infty$. Suppose that $\{f^k\}_{k=1}^{\infty}$ in $L^p(U)$ converges weakly to f in $L^p(U)$, then $\{f^k\}_{k=1}^{\infty}$ is bounded in $L^p(U)$, and

$$||f||_{L^p(U)} \le \liminf_{k\to\infty} ||f^k||_{L^p(U)} \tag{106}$$

Proof. Given $U \subset \mathbb{R}^n$ and $1 < p < \infty$, assume that $\{f^k\}_{k=1}^{\infty} \in L^p(U)$ is a weakly convergent sequence. We claim that $\{f^k\}_{k=1}^{\infty}$ is bounded in $L^p(U)$, and that the norm $||\cdot||_{L^p(U)}$ is sequentially weakly lower semicontinuous, it suffices to show that the norm L^p is sequentially weakly lower semicontinuous. Indeed, since weakly convergent sequences are norm bounded, by (iii) of Proposition 3.13 [4], it follows that $\{f^k\}_{k=1}^{\infty}$ is bounded in $L^p(U)$, so the first part of the claim is verified. Hence, it suffices to show that the norm is sequentially weakly lower semicontinuous, that is,

$$||f||_{L^p(U)} \le \liminf_{k\to\infty} ||f^k||_{L^p(U)}. \tag{107}$$

If $\{f^k\}_{k=1}^{\infty}$ is weakly convergent in $L^p(U)$, then, by Definition 6.3.21, we can choose $g \in L^{p'}(U)$, so that $||g||_{L^{p'}(U)} = 1$, with $p' = \frac{p}{p-1}$, and

$$||f||_{L^p(U)} = \int_U fg, \tag{108}$$

where the equality in (108) follows from the Riesz representation theorem. This, along with Cauchy–Schwarz's and Hölder's inequality, implies that

$$||f||_{L^p(U)} = \int_U fg = \lim_{k\to\infty}\int_U f^k g \le \liminf_{k\to\infty}\int_U |f^k||g| \le \liminf_{k\to\infty}||f^k||_{L^p(U)}||g||_{L^{p'}(U)}, \tag{109}$$

thus

$$||f||_{L^p(U)} \le \liminf_{k\to\infty}||f^k||_{L^p(U)}||g||_{L^{p'}(U)} = \liminf_{k\to\infty}||f^k||_{L^p(U)}. \tag{110}$$

Hence, the inequality in (107) is satisfied, therefore, the claim that the norm $||\cdot||_{L^p(U)}$ is sequentially weakly lower semicontinuous is verified. □

Theorem 6.3.25. Let $U \subset \mathbb{R}^n$ be open and $1 < p < \infty$. Given a bounded sequence $\{f^k\}_{k=1}^{\infty} \in L^p(U)$, there exists a subsequence $\{f^{k_j}\} \subset \{f^k\}_{k=1}^{\infty}$ and f in $L^p(U)$, such that f^{k_j} converges weakly to f in $L^p(U)$, that is,

$$f^{k_j} \rightharpoonup f \quad \text{in } L^P(U) \quad \text{as} \quad j\to\infty. \tag{111}$$

Proof. Given $U \subset \mathbb{R}^n$ open and $1 < p < \infty$, let $\{f^k\}_{k=1}^{\infty} \in L^p(U)$ be a bounded sequence. We claim that there exists a subsequence $\{f^{k_l}\}_{l=1}^{\infty}$ in $\{f^k\}_{k=1}^{\infty}$ and a function $f \in L^p(U)$, such that f^{k_l} converges weakly to f in $L^p(U)$. We recall that $L^p(U)$ is a reflexive Banach space; see Theorem 4.10 [4]. So, given $L^p(U)$ as above, $L^p(U)$ is a reflexive Banach space, which implies that every bounded sequence contains a weakly convergent subsequence, and thus the claim is verified. □

Theorem 6.3.26. Given $U \subset \mathbb{R}^n$ open and $1 < p < \infty$. Let $\{u^k\}_{k=1}^{\infty}$ be a bounded sequence in $W^{1,p}(U)$, then there exists a subsequence $\{u^{k_l}\}_{l=1}^{\infty}$ in $\{u^k\}_{k=1}^{\infty}$, and $u \in W^{1,p}(U)$, such that $u^{k_l} \rightharpoonup u$ and $Du^{k_l} \rightharpoonup Du$ in $L^p(U)$, respectively, as $l\to\infty$. In particular, if $\{u^k\}_{k=1}^{\infty} \in W_0^{1,p}(U)$, for each $k \ge 1$, then $u \in W_0^{1,p}(U)$.

Proof. Given $U \subset \mathbb{R}^n$ open and $p \in (1,\infty)$, then $W^{1,p}(U)$ is a reflexive separable Banach space, which implies that any bounded sequence contains a convergent subsequence, by the analogous argument in the proof of Theorem 6.3.26, thus the first part of the claim is verified. Therefore, it suffices to prove the second claim.

Given a bounded sequence $\{u^k\}_{k=1}^{\infty}$ in $W^{1,p}(U)$, if $u^k \in W_0^{1,p}(U)$, for each $k \ge 1$, then $u \in W_0^{1,p}(U)$, that is, $W_0^{1,p}(U)$ is closed with respect to weak convergence. Indeed, we recall that $W_0^{1,p}(U)$ is a closed and convex subspace of the reflexive Banach space $W^{1,p}(U)$. Therefore, $W_0^{1,p}(U)$ is reflexive. Moreover, $W_0^{1,p}(U)$ closed under weak convergence, as any reflexive normed space is weakly complete, see Theorem 3.7 [4]. Therefore, by the argument above, there exist $u \in W^{1,p}(U)$ and a weakly convergent subsequence $\{u^{k_l}\}_{l=1}^{\infty}$ in $\{u^k\}_{k=1}^{\infty}$, such that u^k converges weakly to u in $W_0^{1,p}(U)$, by $W_0^{1,p}(U)$ is closed under weak convergence. Therefore, the last claim of the statement is verified. □

Corollary 6.3.6. Given $\{u^k\}_{k=1}^{\infty} \in W^{1,p}(U)$, suppose that $u^k \rightharpoonup u \in W^{1,p}(U)$, for some $u \in W^{1,p}(U)$, then

$$||u||_{W^{1,p}(U)} \le \liminf_{k\to\infty}||u^k||_{W^{1,p}(U)}. \tag{112}$$

Proof. Given a sequence $\{u^k\}_{k=1}^{\infty} \in W^{1,p}(U)$, suppose that $\{u^k\}_{k=1}^{\infty}$ converges weakly to some $u \in W^{1,p}(U)$. We claim that the norm $||\cdot||_{W^{1,p}(U)}$ is sequentially weakly lower semicontinuous. We need to show that the inequality in (112) is satisfied, it suffices to show that $||Du||_{L^p(U)} \leq \liminf_{k\to\infty} ||Du^k||_{L^p(U)}$ holds. Now, recall Theorem 6.3.24, furthermore, by Definition 6.3.26, we see that u^k and Du^k converge weakly to u and Du in the norm of $L^p(U)$, respectively, which implies that

$$||D^\alpha u||_{L^p(U)} \leq \liminf_{k\to\infty} ||D^\alpha u^k||_{L^p(U)},$$

for all $|\alpha| \leq 1$, thus, by the definition of the Sobolev norm, we have

$$||u||_{W^{1,p}(U)} \leq \liminf_{k\to\infty} ||u^k||_{W^{1,p}(U)}. \tag{113}$$

Therefore, the claim is verified. □

7 References

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