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Minimal Cantor Sets

JACOB NORDIN GRÖNING
Abstract

A Cantor set is a topological space which admits a hierarchy of clopen covers. A minimal Cantor set is a Cantor set together with a map such that every orbit is dense in the Cantor set. In this thesis we use inverse limits to study minimal Cantor sets and their properties. In particular, under certain hypothesis we find an upper bound for the number of ergodic measures for minimal Cantor set.

Sammanfattning

En Cantor mängd är ett topologiskt rum med en hierarki av slöppna täcken. En minimal Cantor mängd är en Cantor mängd tillsammans med en avbildning så att varje omlöpsbana är tät. I den här uppsatsen använder vi omvända gränser för att studera minimala Cantor mängder och deras egenskaper. Särskilt under vissa omständigheter hittar vi en övre begränsning på antalet ergodiska mätter på minimala Cantor mängder.


**Introduction**

A minimal Cantor set is a dynamical system \((C, f)\) where \(C\) is a Cantor set and \(f\) a map such that all the orbits of \(f\) are dense. In order to understand minimal Cantor sets, we start by studying Cantor sets. A classical definition of a Cantor set is a compact, totally disconnected, perfect and metrizable space. This definition does not give an idea of the structure of a Cantor set. Thus in Section 2.1 we redefine a Cantor set to be a space which admits a hierarchy of covers consisting of clopen disjoint sets. This definition gives a clearer idea of the structure of a Cantor set. Using methods developed in [3] we prove that these two definitions are in fact equivalent.

In order to study Cantor sets we introduce the powerful tool of inverse limits. It turns out that every Cantor set is homeomorphic to some inverse limit and these inverse limits are easier to work with. This is discussed in Section 2. In this section we also show that Cantor sets and thus inverse limits appear as attractors of iterated function systems.

In Section 3 we discuss how to construct maps on Cantor sets such that all orbits are dense. In order to define a map on a given Cantor set we instead define a map on an appropriate inverse limit. Some inverse limits are easier to define maps on than others. We then prove that all inverse limits are homeomorphic. This allows us to define a map on a easier inverse limit and then using a composition of homeomorphisms we get the desired map on the given Cantor set.

In Section 4 we give a structure theorem stating that all minimal Cantor sets are conjugated to inverse limits of directed graphs, following the methods from [2]. The directed graphs we use are called combinatorial covers. We can think of the combinatorial covers as approximations of the minimal Cantor set. The vertices of the combinatorial covers are clopen sets in the Cantor set and the directed edges show where these sets are mapped to. By defining loops on these graphs we also study the inverse of the structure theorem. That is, if one can get a minimal Cantor set from any inverse limit of combinatorial covers.

Finally in the last section we prove the main result. If the number of loops of the combinatorial covers are uniformly bounded by \(d\), then there are at most \(d\) ergodic measures on the minimal Cantor set. Again we follow the methods from [2]. The strategy of the proof is to show that one can project the space of measures on the inverse limit to the space of measures on a combinatorial cover. It turns out that the space of probability measures on the inverse limit is projected to a convex hull of at most \(d\) points. These points correspond to the ergodic measures on the inverse limit and thus there can be at most \(d\) ergodic measures on the minimal Cantor set.


1 Preliminaries

In this section we start by giving some basic definitions regarding topology and some lemmas so that we can define a Cantor set and prove properties of Cantor sets. For basics facts in topology we do not go into dept of every detail and we suggest the reader to look in for example [4] and [6]. Finally we give some definitions about dynamical systems so that we can define a minimal Cantor set.

1.1 Topological spaces

We want to study the dynamics on Cantor sets which are topological spaces. So we need to define what a topological space is.

Definition 1.1. Given a set $X$ a topology $\mathcal{T}$ on $X$ is a collection of subsets of $X$ such that the following holds:

i $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$,

ii If $U_i \in \mathcal{T}$ for $i \in I$ with $I$ any indexing set then $\bigcup_{i \in I} U_i \in \mathcal{T}$,

iii If $U_i \in \mathcal{T}$ for $i = 1, \ldots, m$ then $\bigcap_{i=1}^{m} U_i \in \mathcal{T}$.

Then the pair $(X, \mathcal{T})$ is a called topological space.

When the specific topology on $X$ is not relevant for our study we call $X$ a topological space omitting what the specific topology is. In order to make it easier to refer to the sets in $\mathcal{T}$ we give those sets a specific name.

Definition 1.2. Given a topological space $(X, \mathcal{T})$ a set $U$ is called open if $U \in \mathcal{T}$.

With the notion of an open set we define a closed set.

Definition 1.3. Given a topological space $X$ a set $U$ is called closed if $X \setminus U$ is open.

From the definition of a topological space and because

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} X \setminus A_i$$

for any indexing set $I$, it follows that any intersection of closed sets are closed.

We now give two examples of topological spaces which will be of interest.

Lemma 1.4. Let $X$ be a set and $\mathcal{T}$ the power set of $X$. That is all the subsets of $X$. Then $\mathcal{T}$ defines a topology on $X$. 
Proof. We check the three conditions from the definition of a topology. Both the empty set and \( X \) will always be subsets of \( X \) and thus the first condition is fulfilled. Any arbitrary union of subsets of \( X \) will be a subset of \( X \) so the second condition also holds. The same reasoning gives us that the third condition holds.

\[ \square \]

**Definition 1.5.** The above topology is called the **discret topology**.

**Lemma 1.6.** Let \((X, \mathcal{T})\) be a topological space and \( Y \subset X \). Then the set

\[ \mathcal{T}_Y = \{ Y \cap U : U \in \mathcal{T} \} \]

defines a topology on \( Y \).

**Proof.** We check the three conditions from the definition of a topological space. Firstly we have that \( Y \in \mathcal{T}_Y \) since \( X \cap Y \in \mathcal{T}_Y \) and \( \emptyset \in \mathcal{T}_Y \) since \( \emptyset \cap Y \in \mathcal{T}_Y \).

Secondly let \( I \) be an indexing set such that \( V_i \in \mathcal{T}_Y \) for \( i \in I \). Then there are sets \( U_i \in \mathcal{T} \) such that

\[ V_i = Y \cap U_i. \]

This gives us that

\[ \bigcup_{i \in I} V_i = \bigcup_{i \in I} Y \cap U_i = Y \cap \bigcup_{i \in I} U_i \in \mathcal{T}_Y \]

because \( \cup U_i \in \mathcal{T} \).

Lastly we let \( I \) be a finite index set such that \( V_i \in \mathcal{T}_Y \). Then there is \( U_i \in \mathcal{T} \) such that

\[ V_i = Y \cap U_i. \]

It follows that

\[ \bigcap_{i \in I} V_i = \bigcap_{i \in I} Y \cap U_i = Y \cap \bigcap_{i \in I} U_i \in \mathcal{T}_Y \]

because \( \cap U_i \in \mathcal{T} \).

\[ \square \]

**Definition 1.7.** The above defined topology \( \mathcal{T}_Y \) is called the **subspace topology**.

If we are given any set \( A \subset X \) then it could be the case that \( A \) is neither open nor closed. Therefore we make the following definition.

**Definition 1.8.** For a set \( A \) in a topological space \( X \) the **interior** of \( A \) denoted by \( A^o \) is defined as

\[ A^o = \bigcup \{ U \subset X : U \text{ is open and } U \subset A \} \]

and the **closure** of \( A \) denoted by \( \overline{A} \) is

\[ \overline{A} = \bigcap \{ U \subset X : U \text{ is closed and } A \subset U \}. \]

The **boundary** is the set

\[ \partial A = \overline{A} \setminus A^o. \]
Observe that if $A$ is a set then the interior $A^\circ$ is always an open set and the closure $\overline{A}$ is always a closed. Moreover $A^\circ$ is always a subset of $A$ and $A$ is always a subset of $\overline{A}$. In some cases the closure of $A$ can be the whole topological space.

**Definition 1.9.** A subset $A$ of a topological space $X$ is said to be **dense** if $\overline{A} = X$.

In a topological space there are some points which are important for open and closed sets

**Definition 1.10.** A point $x$ in $X$ is called **isolated** if there exists an open set $U \subset X$ such that $x \in U$ and no other point of $X$ is in $U$.

**Definition 1.11.** A space $X$ is **perfect** if it is closed and there are no isolated points.

The next definition gives us another defining property of a set begin closed.

**Definition 1.12.** Given a set $A \subset X$, a point $x \in X$ is called an **accumulation point** of $A$ if every open set that contains $x$ also contains a point in $A$ different from $x$. The set of all accumulation points of $A$ is denoted $A'$.

This is related to the closure of $A$ in the following way.

**Lemma 1.13.** Let $A \subset X$, then

$$\overline{A} = A \cup A'$$

**Proof.** We prove this by showing that $\overline{A} \subset A \cup A'$ and $A \cup A' \subset \overline{A}$.

We start by letting $x \in \overline{A}$. Then if $x \in A$ we have that $x \in A \cup A'$ so assume that $x \notin A$. If there is an open set $U$ such that $x \in U$ and $U \cap A = \emptyset$ then the complement $X \setminus U$ is closed, does not contain $x$ and $A \subset X \setminus U$. Then since $X \setminus U$ is closed and contains $A$ we have that

$$\overline{A} \subset X \setminus U$$

but since $x \in U$ we also have that

$$x \notin X \setminus U$$

so then $x \notin \overline{A}$ which is a contradiction. Thus every open set that contains $x$ also contains a point from $A$ so $x \in A'$. Since $x$ was a arbitrary point in $\overline{A}$ we get that $\overline{A} \subset A \cup A'$.

Next we prove that $A \cup A' \subset \overline{A}$. Let $x \in A \cup A'$, if $x \in A$ then we are done so we can assume that $x \notin A$, thus $x \in A'$. Then if $x \notin \overline{A}$ we have that $X \setminus \overline{A}$ is an open set which contains $x$. But $X \setminus \overline{A}$ does not intersect $A$ so $x \notin A'$ which is a contradiction and thus $x \in \overline{A}$. Then we have that $A \cup A' \subset \overline{A}$.

Now we have proven that $\overline{A} = A \cup A'$.
Somewhat unintuitively there are sets which are both open and closed. For example in any topological space $X$ we know that both $X$ and $\emptyset$ are open and closed. Cantor sets are topological spaces with many more sets that are both open and closed. Since such sets will be important we given them their own name.

**Definition 1.14.** Given a topological space $X$ a set $U$ is called **clopen** if it is both open and closed.

Next we give some definitions regarding the connectivity of a space $X$.

**Definition 1.15.** A set $U \subset X$ is called **disconnected** if there are disjoint, nonempty open sets $U_1$ and $U_2$ such that

$$U = U_1 \cup U_2.$$ 

A set $U$ is **connected** if it is not disconnected.

**Definition 1.16.** A space $X$ is **totally disconnected** if all the connected subsets are singletons.

### 1.2 Basis and covers

Given a topological space $(X, T)$ it can be difficult to prove something for all open sets in the topology $T$. Thus we work with topological spaces which have a basis.

**Definition 1.17.** Given a topological space $X$ a basis $B$ for a topology on $X$ is a collection of sets such that

i. Every $V \in B$ is open;

ii. For every open set $U$ there is some collection $\{V_i\}_{i \in I}$ with $V_i \in B$ for $i \in I$ such that

$$U = \bigcup_{i \in I} V_i$$

A basis for a topology is a very powerful tool to prove topological statements. For example one could us it to prove that a function is continuous. First we give the definition of a continuous function.

**Definition 1.18.** A function $f : (X, T_1) \to (Y, T_2)$ is **continuous** if for every open set $V$ in $Y$ we have that $f^{-1}(V)$ is open in $X$.

If $f : (X, T_1) \to (Y, T_2)$ and we have a basis $B$ for $T_2$ then in order to show that $f$ is continuous it is enough to show that the preimage of every set in $B$ is open. Say $f^{-1}(V)$ is open for all $V \in B$ then for any open set $U \subset Y$ we have that there exists $V_i \in B$ such that $U = \bigcup_i V_i$ which gives us that

$$f^{-1}(U) = f^{-1}\left(\bigcup_i V_i\right) = \bigcup_i (f^{-1}(V_i))$$
and since any union of open sets is open we get that $f^{-1}(U)$ is open and $f$ is continuous.

Sometimes we have a space $X$ and some collection $B$ of subsets of $X$ and we would like the sets in $B$ to be open. The following lemma allows us to define a topology on $X$ given some condition on $B$.

**Proposition 1.19.** Let $X$ be a space and let $B$ be a collection of subsets of $X$ such that

1. For each $x \in X$ there is a $V \in B$ such that $x \in V$;
2. If $x \in V_1 \cap V_2$ with $V_1, V_2 \in B$ then there is a $V_3 \in B$ such that $x \in V_3$ and $V_3 \subset V_1 \cap V_2$.

Then the collection of subset $T$ such that $U \in T$ if and only if for all $x \in U$ there is a $V \in B$ such that $x \in V \subset U$, defines a topology on $X$.

**Proof.** Firstly we have that $\emptyset$ is vacuously open and $X$ is open from the first condition in the proposition. Next we let $\{U_i\}_{i \in I}$ be an arbitrary collection of open sets and

$$U = \bigcup_{i \in I} U_i.$$  

Then $U$ is open because for any $x \in U$ there is some index $j$ such that $x \in U_j$ and then there is an element $V \in B$ such that $x \in V \subset U_j \subset U$.

Lastly we prove that any intersection of $n$ open sets is open. It is sufficient to this this in the case with 2 open sets. In the case for $n$ open sets one repeats the argument finitely many times. Let $U_1$ and $U_2$ be open and let $x$ be such that $x \in U_1 \cap U_2$.

Since $U_1$ and $U_2$ are open there is $V_1 \in B$ and $V_2 \in B$ such that $x \in V_1 \subset U_1$ and $x \in V_2 \subset U_2$.

Then from the second condition there is a set $V_3 \in B$ such that $x \in V_3 \subset V_1 \cap V_2 \subset U_1 \cap U_2$ and thus $U_1 \cap U_2$ is open.

We have now proven that $T$ fulfills all the conditions to be a topology.  

Observer that in this case the collection $B$ is a basis for the topology $T$.

We saw earlier that a basis was a helpful tool for proving topological properties of a topological space. Another useful tool is that of a cover.
Definition 1.20. A cover $\mathcal{C}$ of a space $X$ is a collection of sets $\mathcal{C} = \{U_n\}_{n \geq 1}$ such that

$$X \subset \bigcup_{n \geq 1} U_n.$$  

An open (closed) cover is a cover such that all the sets in the collection $\mathcal{C}$ are open (closed).

In some cases we want a special cover, in which case we choose to use a refinement of our cover.

Definition 1.21. Given a topological space $X$ and an open cover $\mathcal{C}$ then we say that $\mathcal{V}$ is a refinement of $\mathcal{C}$ if $\mathcal{V}$ is a cover of $X$ and for every $V_i \in \mathcal{V}$ there exists a $U_n \in \mathcal{C}$ such that $V_i \subset U_n$.

For some sets it is especially useful to use a cover to study them.

Definition 1.22. A space $X$ is compact if every cover $\mathcal{C}$ of $X$ has a finite subcover.

1.3 Metrics

Now we will add more structure to our spaces by defining a metric.

Definition 1.23. A function

$$d : X \times X \to \mathbb{R}$$

is called a metric if for all points $x, y$ and $z$ in $X$ the following holds

i \hspace{1em} $d(x, y) \geq 0$ and equality holds if and only if $x = y$;

ii \hspace{1em} $d(x, y) = d(y, x)$;

iii \hspace{1em} $d(x, y) + d(y, z) \geq d(x, z)$.

We also define the distance from a point to a set.

Definition 1.24. Let $X$ be a metric space and $x \in X$ and $A \subset X$. Then the distance between $x$ and $A$ is

$$d(x, A) = \inf_{y \in A} d(x, y).$$

Using a metric we can introduce the notion of a diameter.

Definition 1.25. Given a set $U$ we define the diameter, denoted as $|U|$, as

$$|U| = \sup_{x,y \in U} d(x, y).$$
The function \( d \) should be thought of as the distance between two points. We will also be interested in the diameter of all sets in a collection.

**Definition 1.26.** Let \( X \) be a collection of sets then the **mesh** of \( X \) is defined as
\[
\text{mesh}(X) = \sup\{|U| : U \in X\}.
\]

**Definition 1.27.** A set \( X \) together with a metric \( d \) is called a **metric space** and is denoted by \((X, d)\).

Given a metric space \((X, d)\) we can induce a topology from the metric \( d \) by defining open balls.

**Definition 1.28.** Let \( X \) be a metric space and let \( x \in X \) and \( r \in \mathbb{R}_{>0} \) then we define the **open ball** with center in \( x \) and radius \( r \) as
\[
B(x, r) = \{y \in X : d(x, y) < r\}
\]

**Proposition 1.29.** Let \((X, d)\) be a metric space and \( B \) the set of all balls centered at point in \( X \). Then \( B \) induces a topology.

**Proof.** We use Proposition 1.19. First we observe that for every \( x \in X \) we have that \( x \in B(x, 1) \). Secondly let \( B_1, B_2 \in B \) and \( x \in B_1 \cap B_2 \). Then we let
\[
r = \min(d(x, B_1), d(x, B_2))
\]
and
\[
B_3 = B(x, r).
\]
Then we must have that \( x \in B_3 \subset B_1 \cap B_2 \). Thus we have fulfills all the conditions for Proposition 1.19 so \( B \) induces a topology on \( X \).

The converse of this statement is more complicated since given a topological space \((X, \mathcal{T})\) we can not always find a metric which induces \( \mathcal{T} \). Topological space for which we can find a metric which induces the topology have a name.

**Definition 1.30.** A topological space \( X \) for which there is a metric \( d \) which induced the topology is called **metrizable**.

Given a metrizable space it could be the case that two different metrics induces the same topology.

**Definition 1.31.** Let \((X, \mathcal{T})\) be a topological space and let \( d_1 \) and \( d_2 \) be metrics on \( X \). We say that \( d_1 \) and \( d_2 \) are **equivalent** if they both induce the topology \( \mathcal{T} \).
1.4 Dynamical systems

The final concept we need is that of dynamical systems.

Definition 1.32. A dynamical system is a pair \((X, f)\) where \(X\) is a set and \(f : X \to X\) is a map.

Given a system \((X, f)\) we will be interested in the iterates of some point. We use the notation
\[ f^{n+1}(x) = f(f^n(x)) \]
and
\[ f^0(x) = x. \]

The collection of all the iterates of a point is the orbit of that point.

Definition 1.33. The orbit of a point \(x\) denoted \(O(x)\) is
\[ O(x) = \{ f^n(x) : n \geq 0 \} \]

With the notion of an orbit we can define a minimal system.

Definition 1.34. A system \((X, f)\) is minimal if
\[ \overline{O(x)} = X \]
for all \(x \in X\).

If \(C\), is a Cantor set and the system \((C, f)\) is minimal then we can use any orbit to study the structure of the Cantor set. This we do in Section 4.
2 Definition of Cantor Sets and Inverse Limits

In this section we define Cantor sets, study an example and prove some of the main properties. With the definition of Cantor set we define minimal Cantor sets. We also define inverse limit and show the relation to Cantor sets. Inverse limits is the main tool we use to study and construct Cantor sets. Lastly we look at iterated function systems since Cantor sets and thus inverse limits appear as attractors of IFS.

2.1 Definition of Cantor sets

Definition 2.1. A metric space $C$ is a Cantor set if

$C = \bigcap_n \bigcup_{B \in \mathcal{B}_n} B$

such that

1. $\mathcal{B}_n$ is a finite cover of pairwise disjoint clopen sets;

2. $\mathcal{B}_{n+1}$ is a refinement of $\mathcal{B}_n$;

3. for every $B_0 \in \mathcal{B}_n$ we have that $\# \{ B \in \mathcal{B}_{n+1} : B \subset B_0 \} \geq 2$;

4. the function

$$d_c(x, y) = \sup_{n \in \mathbb{Z}_{>0}} \left\{ \frac{1}{n} : \exists B_1, B_2 \in \mathcal{B}_n \text{ such that } x \in B_1 \text{ and } y \in B_2 \right\}.$$  

is a metric equivalent to the one on $C$.

**Definition 2.2.** The metric $d_c$ from the previous definition is called the **Cantor metric**

With the notion of a Cantor set we can define a minimal Cantor set.

**Definition 2.3.** The dynamical system $(C, f)$ is a **minimal Cantor set** if the system is minimal and $f$ is continuous.

Before we study minimal Cantor sets and construct maps on Cantor set we need study Cantor sets and their properties.

Since a Cantor set is a metric space we have a topology generated by the open balls. The next lemma will prove that the open balls are in fact the elements of the covers from the definition of a Cantor set.

**Lemma 2.4.** Let $C$ be a Cantor set. Then the set

$$\mathcal{B} = \{ B : B \in \mathcal{B}_n \}$$

is the collection of all open balls.
Proof. Consider the ball \( B(x, r) \) with \( x \in C \) and \( r > 0 \). Let 
\[
 n_0 = \inf_{n \in \mathbb{Z}^+} \{ n : \frac{1}{n} \leq r \}
\]
and \( B_0 \in \mathcal{B}_{n_0} \) the set such that \( x \in B_0 \). We prove that the ball \( B(x, r) \) is \( B_0 \).

Let \( y \in B_0 \), then 
\[
d_c(x, y) < \frac{1}{n_0} \leq r
\]
so \( y \in B(x, r) \). Next let \( y \in B(x, r) \) and assume that \( y \notin B_0 \). Then there is some \( B_1 \in \mathcal{B}_{n_0} \) such that \( y \in B_1 \). Then we have that 
\[
d_c(x, y) \geq \frac{1}{n_0}
\]
which contradicts \( d_c(x, y) < r \) so \( y \in B_0 \). Thus \( B(x, r) \) is the set \( B_0 \) and since \( B(x, r) \) was an arbitrary ball we get that all balls are contained in \( B \).

The Cantor metric gives us insight about the structure of a Cantor set and its topology. But proving that two metrics are equivalent is somewhat awkward at times. Thus we prove an equivalent definition of a Cantor set.

**Lemma 2.5.** Let \( X \) be a metric space with metric \( d \). Then \( X \) is a Cantor set if and only if
\[
 X = \bigcap_n \bigcup_{B \in \mathcal{B}_n} B
\]
where
1. \( \mathcal{B}_n \) is a finite cover of pairwise disjoint clopen sets;
2. \( \mathcal{B}_{n+1} \) is a refinement of \( \mathcal{B}_n \);
3. for every \( B_0 \in \mathcal{B}_n \) we have that \( \#\{ B \in \mathcal{B}_{n+1} : B \subset B_0 \} \geq 2 \);
4. \( \text{mesh}(\mathcal{B}_n) \to 0 \) as \( n \to \infty \).

**Proof.** Observe that it is only the fourth condition that has changed from the definition of a Cantor set and thus we only need to prove that \( \text{mesh}(\mathcal{B}_n) \to 0 \) as \( n \to \infty \) if and only if the metric \( d \) is equivalent to \( d_c \).

First we prove that if \( X \) is a Cantor set then \( \text{mesh}(\mathcal{B}_n) \to 0 \). Since \( X \) is a Cantor set then we can use the Cantor metric. Thus given \( B_0 \in \mathcal{B}_n \) we have that 
\[
 |B_0| = \frac{1}{n + 1}
\]
and this goes to 0 as \( n \to \infty \).
Next we assume that $X$ and $d$ is as above and let $T_d$ be the topology induced by $d$ and $T_{d_c}$ the topology induced by $d_c$. From Lemma 2.4 we know that the collection
$$
B = \{ B : B \in B_n \}
$$
is a basis for the topology $T_{d_c}$. From the construction of $X$ we know that all sets in $B$ are in $T_d$ so we have that $T_{d_c} \subset T_d$. For the other inclusion let $U \in T_d$ be nonempty. For each $x \in U$ let $n_x$ be an integer such that the set $B_x \in B_{n_x}$ which contains $x$ and $B_x \subset U$. We know such an integer $n_x$ exists because $\text{mesh}(B_n) \to 0$ as $n \to \infty$ and $|U| > 0$. Now we have that
$$
U = \bigcup_{x \in U} B_x
$$
so $U$ is a union of sets which are open in $T_{d_c}$ thus $U$ is open in $T_d$. Since $U$ was an arbitrary nonempty open set in $T_{d_c}$ we have that $T_{d_c} \subset T_d$. Finally we have that $T_d = T_{d_c}$ and $X$ is a Cantor set.

We now give an example of a Cantor set. The middle third Cantor set is the most common example of a Cantor set since it is easily constructed using a recursive definitions. Let $E_0 = [0, 1] \subset \mathbb{R}$. Then we make $E_1$ by removing the middle third. Then $E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ and $E_2$ is constructed by removing the middle third from the two intervals of $E_1$. For $n \geq 1$ we get that $E_n$ is defined as
$$
E_n = E_{n-1} \setminus \left(\bigcup_{k=1}^{3^{n-2}} \left(\frac{k}{3^n}, \frac{k+1}{3^n}\right)\right).
$$
Then the middle third cantor set is defined as
$$
C = \bigcap_{n=0}^{\infty} E_n.
$$
The topology on $C$ will be the subspace topology and $C$ will have the same metric as $\mathbb{R}$. We now prove that the middle third Cantor set is a Cantor set.

Figure 1: The first four iteration for the middle third Cantor set
Lemma 2.6. The middle third Cantor set is a Cantor set.

Proof. We start by observing that each \( E_n \) will be a union of intervals and we denote each of these intervals by \( I_j \). Then we observe that in the construction of \( E_n \) we remove a third of each interval in \( E_{n-1} \) so we have doubled the number of intervals and the length of the \( I_j \) has become a third of the intervals in \( E_{n-1} \). Next since \( E_0 \) is just one interval of length 1 so we get that \( E_n \) is made of \( 2^n \) intervals of length \( 3^{-n} \) and then the distance between two intervals is at most \( 3^{-n} \). We also observe that all intervals in \( E_n \) is closed in \( \mathbb{R} \) because we have removed open sets which is the same as taking the intersection with closed sets.

Next we have that \( C = \cap E_n \) so we get that \( C \cap I_j \) is a cover of \( C \) where the sets are on the form \( C \cap I_j \). Next we observe that \( C \cap I_j \) is disjoint from \( C \cap I_k \) because \( I_j \) disjoint with \( I_k \) when \( j \neq k \). The sets \( C \cap I_j \) will also be closed in the topology \( C \) because \( C \cap I_j \subset I_j \) and \( I_j \) is closed in \( \mathbb{R} \). We can also prove that these sets are open in \( C \). Let \( I_j = [a_j, b_j] \) and \( I_k = [a_k, b_k] \) be two different intervals in \( E_n \). Then the distance between them is at least \( 3^{-n} \) so the set

\[
U = \left( a_j - \frac{3^{-n}}{2}, b_j + \frac{3^{-n}}{2} \right)
\]

will contain \( I_j \) but not \( I_k \). Then since \( U \) only contain \( I_j \) we get that \( C \cap I_j \) is an open set in \( C \). Now we let \( B_n = \{ C \cap I_j \} \), this is a finite cover of pair wise disjoint clopen sets.

Next we check that \( B_{n+1} = \{ C \cap I_j \} \) is a refinement of \( B_n = \{ C \cap J_k \} \). Since \( I_j \subset J_{k_0} \) for some \( k_0 \) we get that \( C \cap I_j \subset C \cap J_{k_0} \) which means that \( B_{n+1} \) is a refinement of \( B_n \).

The third condition in the definition of a Cantor set also holds since each interval \( J_k \) in \( E_n \) is split into two intervals \( I_{j_1} \) and \( I_{j_2} \) in \( E_{n+1} \). So in \( B_{n+1} \) there is two elements \( C \cap I_{j_1} \) and \( C \cap I_{j_2} \) such that they both are subsets of \( C \cap J_k \).

Lastly we check that mesh(\( E_n \)) \( \rightarrow 0 \). Since each interval in \( E_n \) has length \( 3^{-n} \) we get that

\[
\text{mesh}(E_n) = 3^{-n}
\]

and this goes to 0 as \( n \rightarrow \infty \).

Thus by lemma 2.5 we have that the middle third Cantor set is a Cantor set.

Now that we know what a Cantor is we study some properties of Cantor sets.

Proposition 2.7. A set \( X \) is a Cantor set if and only if it is compact, totally disconnected, perfect and metrizable.

The proof of Proposition 2.7 is rather long and requires some lemmas. Thus we split it into two propositions.

Proposition 2.8. Let \( C \) be a Cantor set. Then it is compact, totally disconnected, perfect and metrizable.
Proof. First we prove that $C$ is compact by contradiction. Let $C = \{U_n\}_{n \in I}$ be a cover of $C$ such that there is no finite subcover. Then since $B_1$ is a finite cover of $C$ there must be some element in $B_1$ which can not be cover with finitely many sets from $C$, without loss of generality call it $B_1$. We know such a set exists because otherwise there would exists a finite subcover. With the same reasoning there must be an element in $B_2$ that is a subset of $B_1$ and with no finite cover. We call it $B_2$. We keep doing this and find a $B_n \in B_n$ such that $B_n \subset B_{n-1}$ and there is no finite subcover of $B_n$. Since we have a nested sequence of sets we get that

$$\bigcap_{k=1}^{m} B_k = B_m$$

and since each $B_n$ is nonempty and $\text{mesh}(B_n) \to 0$ we get that there is an $x \in B_n$ for all $n$ such that

$$\bigcap_{i=1}^{\infty} B_k = x.$$  

Since $C$ is a cover of $C$ there exists $U \in C$ such that $x \in U$. Next we choose $N$ such that

$$|B_N| < d(x, C \setminus U)$$

and then $B_N \subset U$. But this contradicts that we can not cover $B_N$ with finitely many sets from $C$ and thus the assumption that $C$ is not compact must be wrong and $C$ is compact.

Secondly we prove that $C$ is totally disconnected again by contradiction. Let $K$ be a connected component with two different points $x$ and $y$. Thus $d(x, y) > 0$ so there is a $B_n$ with $B_1 \in B_n$ such that $x \in B_1$ and $y \notin B_1$. Then since $B_n$ is a cover we get that

$$K = (K \cap B_1) \cup (K \cap (\cup_{B \in B_n \setminus B_1} B))$$

which contradict $K$ being a connected component. Thus the only connected sets are singletons and $X$ is totally disconnected.

Thirdly we prove that $C$ has no isolated points. Let $x \in C$ and $U$ and open set such that $x \in U$. Since we have a basis for $C$ there must be some $n$ and $B$ such that $x \in B \in B_n$ and $B \subset U$. From the definition of Cantor sets we have that there are two sets $B_1, B_2 \subset B$ such that $B_1, B_2 \in B_{n+1}$ and both are nonempty. Thus there must have been at least two elements in $B$ and thus at least two elements in $U$. Since $U$ was an arbitrary open set containing $x$ we get that $x$ is not an isolated point and $C$ is perfect.  

In order to prove that a set is a Cantor set we need to construct a sequence of clopen disjoint covers. The sets in these covers is in some sense separated from each other. The following definition makes this notion precise.

**Definition 2.9.** Given a topological space $X$ then two sets $A, B \subset X$ are separated or can be separated if there exists clopen sets $U$ and $V$ such that

$$A \subset U \quad B \subset V$$
Before we prove the converse of Proposition 2.8 we need some lemmas.

**Lemma 2.10.** Let $X$ be a compact space and $K$ a closed subset of $X$, then $K$ is compact.

**Proof.** Let $\{U_i\}_{i \in I}$ be a cover of $K$. Then since $K$ is closed we get that $U = X \setminus K$ is open so $\{U_i\} \cup U$ is a cover of $X$. Then since $X$ is compact we can find a finite sub cover which will be on the form $\{U_{i_1}, \ldots U_{i_n}, U\}$ or $\{U_{i_1}, \ldots U_{i_n}\}$. Since this is a cover of $X$ it is also a cover of $K$ and since $K$ is disjoint from $U$ we get a finite sub cover of $K$.

The next three lemmas are based on propositions from [3].

**Lemma 2.11.** Let $X$ be a compact space and let $K$ be a closed subset of $X$ and let $x \in X$. Then if for all $y \in K$ the sets $\{y\}$ and $\{x\}$ are separated then $x$ and $K$ are separate.

**Proof.** For each $y \in K$ we can choose clopen sets $U_y$ and $V_y$ such that $x \in U_y$ and $y \in V_y$ and $U_y \cap V_y = \emptyset$. Then we have that $\{V_y\}_{y \in K}$ is a cover of $K$ but since $K$ is closed and therefore compact by Lemma 2.10 we can choose a finite sub cover $\{V_{y_k}\}_{k=1}^n$. Next we consider the two sets

$$V = \bigcup_{k=1}^n V_{y_k}, \quad U = \bigcap_{k=1}^n U_{y_k}. $$

The two sets $V$ and $U$ are clopen since they are union and intersection of finitely many clopen sets respectively. We also have that $U$ and $V$ are disjoint since each $U_{y_k}$ is disjoint from $V_{y_k}$. Thus we have that $x$ and $K$ are separated.

**Lemma 2.12.** Let $X$ be a compact space and let $x \in X$. Then the set

$$M_x = \{y \in X : y \text{ and } x \text{ can not be separated}\}$$

is closed.

**Proof.** We prove this by showing that $X \setminus M_x$ is open. Let $y \in X \setminus M_x$ then we can separate $x$ and $y$. Let $U$ and $V_y$ be disjoint clopen sets such that $x \in U$ and $y \in V_y$. Then $V_y \subset X \setminus M_x$ because if there is a $z \in M_x$ such that $z \in V_y$ we have that $V_y$ and $U$ separate $z$ and $x$. Now we have that

$$\bigcup_{y \in X \setminus M_x} V_y = X \setminus M_x$$

so $X \setminus M_x$ is open and $M_x$ is closed.

**Lemma 2.13.** Let $X$ be a compact metric space and let $x \in X$, then the set $M_x$ is connected.
Proof. We do a proof by contradiction so we assume $M_x$ is disconnected. Then there exists two sets $A$ and $B$ which are nonempty, disjoint, open in $M_x$ and

\[ M_x = A \cup B \]

with $x \in A$. Since $M_x$ is closed in $X$ we get that also $A$ and $B$ are closed in $X$. Next we observe that for each $y \in A$ we have that

\[ d_y = d(y, B) > 0 \]

because if $d(y, B) = 0$ then $y \in B = B$ which contradict $A$ and $B$ being disjoint. So we can create balls

\[ U_y = B \left( y, \frac{1}{2}d_y \right) \]

and these cover $A$. Thus

\[ A \subset \bigcup_{y \in A} U_y \]

but since $A$ is closed it is also compact so we get that

\[ A \subset \bigcup_{k=1}^{n} U_{y_k} = U. \]

Now we also have that $\overline{U} \cap B = \emptyset$ because

\[ \overline{U} = \bigcup_{k=1}^{n} U_{y_k} = \bigcup_{k=1}^{n} U_{y_k} \]

which is disjoint from $B$. Next we have that

\[ \partial U \cap M_x = \partial U \cap (A \cup B) = \emptyset \]

which means that there are no points in $\partial U$ which can not be separated from $x$. Now we can use lemma 2.11 to get a clopen set $V$ such that $\partial U \subset V$ and $x \notin V$. Since $x \in U$ and $x \notin V$ we get that $x \in U \setminus V$. Now we using that $V$ is closed we get that

\[ U \setminus V \]

must be open and since $\partial U \subset V$ we get that

\[ U \setminus V = \overline{U} \setminus V \]

so $U \setminus V$ is a closed set without an open set so it is closed. Thus we have $U \setminus V$ is clopen. But since but we also have that

\[ (U \setminus V) \cap B \subset \overline{U} \cap B = \emptyset \]

so we have separated $x$ with all the points in $B$ which is a contradiction to the definition of $M_x$ and thus $M_x$ is connected. \qed
Using this we can prove a very useful lemma.

**Lemma 2.14.** Let $X$ be a compact and totally disconnected space. Then for every $x \in X$ and open set $U$ such that $x \in U$ there exists a clopen set $V$ such that $x \in V$ and $V \subset U$.

**Proof.** Let $U$ be an open set such that $x \in U$. Then since $U$ is open $X \setminus U$ is closed and therefore compact. Since $X$ is totally disconnected every set $M_y$ with $y \in X \setminus U$ only contains the point $y$ so there are clopen sets $K_y$ and $V_y$ such that $y \in K_y$ and $x \in V_y$. Since $X \setminus U$ is compact we can choose finitely many points $y_k$ such that

$$X \setminus U \subset \bigcup_{k=1}^{n} K_{y_k}.$$ 

Then since $K_{y_k} \cap V_{y_k} = \emptyset$ we get that

$$V = \bigcap_{k=1}^{n} V_{y_k}$$

is clopen and $V \subset U$ which is what we wanted to prove. \qed

We now have everything we need to prove the converse of Proposition 2.8.

**Proposition 2.15.** Let $X$ be compact, totally disconnected, perfect and metrizable. Then $X$ is a Cantor set.

**Proof.** We start with a general observation about clopen sets which we use to construct the covers $B_n$. Let $K \subset X$ be a clopen set with $|K| = \delta$. Then we know that $K$ is compact since it is closed and $K$ contains infinitely many points since it is open. For every $x \in K$ we choose an open set $W_x \subset K$ with $x \in W_x$ and $|W_x| \leq \frac{\delta}{2}$. By using Lemma 2.14 we get that there is a clopen set $V_x \subset W_x$ such that $x \in V_x$. Then the collection $\{V_x\}_{x \in K}$ is a cover of $K$. Using the compactness of $K$ we get a finite subcover $\mathcal{V} = \{V_{x_n}\}_{n=1}^{k}$. Now we can use $\mathcal{V}$ to get a cover of disjoint clopen sets. Let

$$U_1 = V_{x_1}$$

and

$$U_n = V_{x_n} \setminus \bigcup_{i=1}^{n-1} U_i$$

then $\mathcal{C} = \{U_n\}_{n=1}^{k}$ is a cover of $K$ with clopen disjoint. Observer that $k \geq 2$. This is because if $k = 1$ then $K = V_{x_1}$ which gives

$$|K| = |V_{x_1}| = \frac{\delta}{2}$$

and this contradicts $|K| = \delta$ and thus $k \geq 2$. 

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Next we create the sets $B_n$ using an inductive reasoning. We start by setting $B_0 = \{X\}$. Next we assume that we have the collection $B_n$. Since every $B_k \in B_n$ is clopen we can use the above method to construct a cover $C_k$ of disjoint clopen cover with diameter at most $\frac{|B|}{2}$ and $\#C_k \geq 2$. If we take two elements $B_1, B_2 \in B_n$ and their corresponding covers $C_1$ and $C_2$ we get that these two covers are disjoint since $B_1$ and $B_2$ are disjoint. Thus we get $B_{n+1}$ is the collection

$$B_{n+1} = \{B \in C_k : k = 1, \ldots \}$$

and now we have created all the covers $B_n$. Lastly we observe that

$$\text{mesh}(B_{n+1}) \leq \frac{\text{mesh}(B_n)}{2}$$

and thus we must have that $\text{mesh}(B_n) \to 0$ as $n \to \infty$. Then by Lemma 2.5 we have that $X$ is a Cantor set.

\[\square\]

### 2.2 Inverse limits

We will consider a sequence of topological space $(X_n, T_n)$ and projection maps $\pi_n : X_{n+1} \to X_n$ such that $\pi_n$ is continuous and onto. We also define the projection $\pi_{k,n} : X_k \to X_n$ as

$$\pi_{k,n} = (\pi_n \circ \pi_{n+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k).$$

We are now ready to define the inverse limit.

**Definition 2.16.** Given topological spaces $X_n$ and projections maps $\pi_n$ as described above we define the inverse limit $X$ as

$$X = \lim_{\leftarrow} X_n = \{\{x_n\}_{n \geq 1} | x_n \in X_n \text{ and } \pi_n(x_k) = x_n \text{ where } k > n\}$$

we also define $\pi_n$ on $X$ as $\pi_n(\{x_k\}_{k \geq 1}) = x_n$.

We also define a topology on the inverse limit.

**Lemma 2.17.** Let $X = \lim X_n$ be an inverse limit. Then the set

$$B = \{\pi_{n}^{-1}(x_k) : x_k \in X_n\}.$$

defines a topology on $X$.

**Proof.** We use Proposition 1.19. For the first condition let $\{x_n\}_{n \geq 1} \in X$. Then

$$\pi_k(\{x_n\}_{n \geq 1}) = x_k$$

and thus

$$\{x_n\}_{n \geq 1} \in \pi_{k}^{-1}(x_k)$$

which gives us that $B$ fulfills the first condition of Proposition 1.19.
Next we check the second condition. Let \( \pi_{n_1}^{-1}(x_{k_1}) \) and \( \pi_{n_2}^{-1}(x_{k_2}) \) be two elements in \( B \) with \( x_{k_1} \in X_{n_1} \) and \( x_{k_2} \in X_{n_2} \). Without loss of generality we can assume that \( n_1 \leq n_2 \).

First we do the case \( n_1 = n_2 \). Let \( \{x_n\}_{n \geq 1} \in X \) be such that
\[
\{x_n\}_{n \geq 1} \in \pi_{n_1}^{-1}(x_{k_1}) \cap \pi_{n_2}^{-1}(x_{k_2})
\]
then we have that
\[
k_1 = \pi_n(\{x_n\}_{n \geq 1}) = k_2
\]
which would mean
\[
\pi_{n_1}^{-1}(x_{k_1}) = \pi_{n_2}^{-1}(x_{k_2})
\]
and the second condition is fulfilled.

For the second case we have \( n_1 < n_2 \). Again let \( \{x_n\}_{n \geq 1} \in X \) be such that
\[
\{x_n\}_{n \geq 1} \in \pi_{n_1}^{-1}(x_{k_1}) \cap \pi_{n_2}^{-1}(x_{k_2}).
\]
Next let \( \{y_n\}_{n \geq 1} \in \pi_{n_2}^{-1}(x_{k_2}) \), then
\[
\pi_{n_1}(\{y_n\}_{n \geq 1}) = (\pi_{n_1} \circ \pi_{n_1}^{-1} \circ \cdots \circ \pi_{n_2}^{-1})(\{y_n\}_{n \geq 1}) = (\pi_{n_1} \circ \pi_{n_1}^{-1} \circ \cdots \circ \pi_{n_2}^{-1})(x_{k_2}) = x_{k_1}
\]
and thus
\[
\{y_n\}_{n \geq 1} \in \pi_{n_1}^{-1}(x_{k_1}).
\]
This gives us that
\[
\pi_{n_2}^{-1}(x_{k_2}) \subset \pi_{n_1}^{-1}(x_{k_1})
\]
and then
\[
\pi_{n_1}^{-1}(x_{k_1}) \cap \pi_{n_2}^{-1}(x_{k_2}) = \pi_{n_2}^{-1}(x_{k_2}).
\]
Thus we fulfill the second condition by choosing the third set as \( \pi_{n_2}^{-1}(x_{k_2}) \). \( \square \)

There are different inverse limits depending on the sequence of topological spaces we use.

**Definition 2.18.** An inverse limit \( X = \lim_{n} X_n \) is **non trivial** if for every \( n \) there exists a \( k \) such that
\[
|\pi_{k,n}^{-1}(x_n)| \geq 2
\]
for all \( x_n \in X_n \).

**Definition 2.19.** An inverse limit \( X = \lim_{n} X_n \) is **discrete** if each \( X_n \) contains finitely many elements and the topology \( T_n \) is the discrete topology.

Using an inverse limits we can prove that two Cantor sets are topologically the same space. Two spaces are topologically the same if they are homeomorphic.

**Definition 2.20.** A function \( f : X \to Y \) where \( X \) and \( Y \) are topological space is called a **homeomorphism** if \( f \) is a continuous bijection and \( f^{-1} \) is also continuous. If such an \( f \) exists we say that \( X \) and \( Y \) are **homeomorphic**.
The precise statement of the lemma is as follows.

**Lemma 2.21.** Every Cantor set is homeomorphic.

The proof is postponed until section 3 when we develop maps between inverse limits. Now we show the relation between Cantor sets and inverse limits.

**Proposition 2.22.** Let $X = \lim X_n$ be a discrete non trivial inverse limit. Then $X$ is a Cantor set.

**Proof.** We construct the covers using an inductive reasoning. We start by setting

$$B_1 = \{ \pi_1^{-1}(x) : x \in X_1 \}$$

then given a cover

$$B_m = \{ \pi_n^{-1}(x) : x \in X_n \}$$

we choose $n_{m+1}$ such that

$$|\pi_{n_{m+1},n_m}^{-1}(x)| \geq 2$$

for all $x \in X_{n_{m+1}}$ and let

$$B_m = \{ \pi_{n_{m+1}}^{-1}(x) : x \in X_{n_{m+1}} \}$$

Now we check that these $B_m$ fulfil the definitions for a Cantor set. For the simplicity of notation we write $m$ instead of $n_m$.

For the first condition we start by checking that each $B_m$ is a cover. Let $x \in X$ then we have that $\pi_m(x) \in X_m$ so then there must be a $x_m \in X_m$ such that

$$x \in \pi_m^{-1}(x_m)$$

and since $x$ was arbitrary we have that $B_m$ is a cover. Next we let $y_1, y_2 \in X_m$ such that $y_1 \neq y_2$ and assume by contradiction there is a $x \in X$ such that

$$x \in \pi_m^{-1}(y_1) \cap \pi_m^{-1}(y_2).$$

Then we must have that $\pi_m(x) = y_1$ but also that $\pi_m(x) = y_2$ which would imply that $y_1 = y_2$ which is a contradiction to $y_1 \neq y_2$ which means that no such $x$ can exists and

$$\pi_m^{-1}(y_1) \cap \pi_m^{-1}(y_2) = \emptyset.$$ 

Now we have that

$$X = \bigcup_{x_m \in X_m} \pi_m^{-1}(x_m)$$

which means that

$$X \setminus \pi_m^{-1}(x_m) = \bigcup_{x_m \in X_m} \pi_m^{-1}(x_m) \setminus \pi_m^{-1}(x_m) = \bigcup_{x_m \in X_m \setminus x_m_1} \pi_m^{-1}(x_m)$$

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and since this is a union of open sets we get that $\pi_m^{-1}(x_m)$ is closed and thus it is clopen. Thus $B_m$ fulfils the first condition.

Next we prove that $B_{m+1}$ is a refinement of $B_m$. Let $x_{m+1} \in X_{m+1}$ such that $\pi_m(x_{m+1}) = x_m$ and let

$$x \in \pi_{m+1}^{-1}(x_{m+1}).$$

Then $\pi_{m+1}(x) = x_{m+1}$ which gives us that

$$\pi_m(x) = \pi_m(\pi_{m+1}(x)) = \pi_m(\pi_{m+1}(x_{m+1})) = x_m$$

and thus

$$x \in \pi_{m+1}^{-1}(x_m).$$

This gives us that $B_{m+1}$ is a refinement of $B_m$.

For the third condition we start by letting $x_m \in X_m$ then since $X$ is a non trivial inverse limit we know that there exists $y_1 \neq y_2$ both in $X_{m+1}$ such that

$$\pi_m(y_1) = \pi_m(y_2) = x_m.$$  

But then by the above we get that

$$\pi_{m+1}^{-1}(y_i) \subset \pi_{m+1}^{-1}(x_m)$$

for $i = 1, 2$. Which means that the third condition in the definition of a Cantor set is fulfilled.

Lastly we check that the Cantor metric $d_c$ is a metric on $X$. For the first condition we have that if $x, y \in X$ are different then there is an index $m$ such that

$$\pi_m(x) = x_m \neq y_m = \pi_m(y)$$

Then we have that

$$x \in \pi_m^{-1}(x_m) \quad \text{and} \quad y \in \pi_m^{-1}(y_m)$$

but $\pi_m^{-1}(x_m) \cap \pi_m^{-1}(y_m) = \emptyset$ so $x$ and $y$ must be separated in $B_m$ so

$$d_c(x, y) > 0.$$  

In the case when $x = y$ then they will never be separated in any $B_m$ so

$$d_c(x, y) = 0.$$  

The second condition also holds immediately from the definition of $d_c$. Since if $x$ and $y$ are separated in $B_m$ then also $y$ and $x$ must be separated in $B_m$ so

$$d_c(x, y) = d_c(y, x).$$

For the last condition suppose we have three points $x$, $y$ and $z$ such that

$$d_c(x, y) = \frac{1}{m_1} \quad \text{and} \quad d_c(y, z) = \frac{1}{m_2}$$
and \( d_c(x, z) = \frac{1}{m_3} \)

which means that \( x \) and \( y \) are separated in \( B_{m_1} \), \( y \) and \( z \) are separated in \( B_{m_2} \) and \( x \) and \( z \) are separated in \( B_{m_3} \). Next we argue by contradiction. Suppose that

\[
\frac{1}{m_1} + \frac{1}{m_2} < \frac{1}{m_3}.
\]

Then we get

\[
\frac{1}{m_1} < \frac{1}{m_3} \implies m_3 < m_1
\]

and

\[
\frac{1}{m_2} < \frac{1}{m_3} \implies m_3 < m_2.
\]

Consider now the cover \( B_{m_3} \). Since \( x \) and \( z \) is separated in \( B_{m_3} \) we get that there is two disjoint sets \( B_1, B_2 \in B_{m_3} \) such that \( x \in B_1 \) and \( z \in B_2 \). Since \( m_3 < m_1 \) we have that \( x \) and \( y \) are not separated in \( B_{m_3} \) and thus \( y \in B_1 \). But then \( y \) and \( z \) are separate in \( B_{m_3} \) and thus \( m_2 \leq m_3 \). This is a contradiction to \( m_3 < m_2 \) and thus the assumption

\[
\frac{1}{m_1} + \frac{1}{m_2} < \frac{1}{m_3}
\]

is false and we get that

\[
\frac{1}{m_1} + \frac{1}{m_2} \geq \frac{1}{m_3}.
\]

Now we have that \( d_c \) is a metric on \( X \) and \( X \) is a Cantor set.

The converse is not necessarily true but we have a weaker statement

**Proposition 2.23.** Let \( C \) be a Cantor set. Then there is a discrete non trivial inverse limit \( X = \lim X_n \) which is homeomorphic to \( C \)

**Proof.** We create a sequence of bijections \( h_n : X_n \to B_n \). Without loss of generality we can assume that \( B_1 = \{ C \} \). We start by letting \( X_1 = \{ x_1 \} \) and then \( h_1(x_1) = \{ C \} \). Next we proceed with an inductive reasoning. Assume we have created \( X_n \) and a bijection \( h_n \). Then since \( B_{n+1} \) contains finitely many elements we can define a set \( X_{n+1} \) such that there is a bijection between \( X_{n+1} \) and \( B_n \) and for every \( x_{n+1} \in X_{n+1} \) we have that

\[
\pi_n(x_{n+1}) = x_n \Leftrightarrow h_n(x_n) = B_n \supset B_{n+1} = h_{n+1}(x_{n+1}).
\]

Now we can define the function

\[
h : X \to C
\]

\[
h(\{ x_n \}_{n \geq 1}) = \bigcap_n h_n(x_n).
\]
Now we prove that $h$ is a bijection. First we prove that $h$ is a surjection. Let $x \in C$ then for each $B_n$ there exist a unique set $B_n \in B_n$ with $x \in B_n$. This is because $B_n$ is a cover of disjoint sets. Observe that $B_{n+1}$ can only be a subset of one set in $B_n$ since $B_{n+1}$ is a refinement of $B_n$. Thus since $x \in B_n$ and $x \in B_{n+1}$ we must have that $B_n \subset B_{n+1}$. Since each $h_n$ is a bijection there is a sequence $\{x_n\}_{n \geq 1}$ with $x_n \in X_n$ such that

$$h_n(x_n) = B_n.$$  

Because $B_{n+1} \subset B_n$ we get that $\pi_n(x_{n+1}) = x_n$ and thus $\{x_n\}_{n \geq 1} \in X$. Now we have that

$$h(\{x_n\}_{n \geq 1}) = \bigcap_n B_n = x$$

and thus $h$ is a surjection. To prove that $h$ is an injection let $\{y_n\}_{n \geq 1}, \{x_n\}_{n \geq 1} \in X$ and

$$h(\{y_n\}_{n \geq 1}) = h(\{x_n\}_{n \geq 1}) = x.$$  

By the above there is a unique sequence $\{B_n\}_{n \geq 1}$ such that

$$x = \bigcap_n B_n.$$  

This gives us that

$$h_n(x_n) = h(y_n) = B_n$$

and since each $h_n$ is a bijection we must have that $x_n = y_n$ for all $n$ so

$$\{x_n\}_{n \geq 1} = \{y_n\}_{n \geq 1}.$$  

Thus $h$ is an injection which gives us that $h$ is a bijection.

Next we prove that $h$ and $h^{-1}$ is continuous. We prove that

$$h(\pi_{n_0}^{-1}(x_{n_0})) = B_{n_0}$$

where $x_{n_0} \in X_{n_0}$ and $h_{n_0}(x_{n_0}) = B_{n_0}$. Let $y$ be such that

$$y \in h(\pi_{n_0}^{-1}(x_{n_0})).$$

Then there exist a sequence $\{y_n\}_{n \geq 1} \in \pi_{n_0}^{-1}(x_{n_0})$ such that

$$y = h(\{y_n\}_{n \geq 1}).$$

Since $\{y_n\}_{n \geq 1} \in \pi_{n_0}^{-1}(x_{n_0})$ we have that $y_{n_0} = x_{n_0}$. Thus

$$h_{n_0}(y_{n_0}) = h_{n_0}(x_{n_0}) = B_{n_0}$$

which gives us that $y \in B_{n_0}$. Thus we have that $h(\pi_{n_0}^{-1}(x_{n_0})) \subset B_{n_0}$. Next let $y \in B_{n_0}$. Then there exists a sequence $\{V_n\}_{n \geq 1}$ such that

$$y = \bigcap_n V_n.$$
Since this sequence \( \{V_n\}_{n \geq 1} \) is unique we get that \( V_{n_0} = B_{n_0} \). Now there is a sequence \( \{y_n\}_{n \geq 1} \in X \) such that \( h_n(y_n) = V_n \). Because \( h_n \) is a bijection and

\[
h_{n_0}(y_{n_0}) = B_{n_0}
\]

we get that \( y_{n_0} = x_{n_0} \). Thus \( \{y_n\}_{n \geq 1} \in \pi_{n_0}^{-1}(x_{n_0}) \) and

\[
y = h(\{y_n\}_{n \geq 1}) \in h(\pi_{n_0}^{-1}(x_{n_0}))
\]

giving us that \( B_{n_0} \subset \pi_{n_0}^{-1}(x_{n_0}) \).

Now we have that

\[
h(\pi_{n_0}^{-1}(x_{n_0})) = B_{n_0}
\]

and since \( \pi_{n_0}^{-1}(x_{n_0}) \) is in the basis for the topology on \( X \) we get that \( h^{-1} \) is continuous. But we also have that \( B_{n_0} \) is an arbitrary element in the basis for \( C \). Since \( h \) is a bijection we get that

\[
h^{-1}(B_{n_0}) = \pi_{n_0}^{-1}(x_{n_0})
\]

giving us that \( h \) is continuous.

Finally we have that \( h \) is a bijection and both \( h \) and \( h^{-1} \) are continuous so \( h \) is a homeomorphism.

\[\square\]

### 2.3 Iterated function systems

Next we look at iterated function system as an example of where Cantor sets can appear and thus inverse limits. First we have some definitions and a theorem from [1]. In this subsection we assume that every set is a subsets of \( \mathbb{R}^n \) for some \( n \).

**Definition 2.24.** A mapping \( S \) is called a **contraction** if there exists a \( r \) such that \( 0 < r < 1 \) and

\[
|S(x) - S(y)| \leq r|x - y|.
\]

**Definition 2.25.** A collection of contractions \( \{S_1, S_2, ..., S_m\} \) where \( m \geq 2 \) is called an **iterated function system**, abbreviated IFS.

Given an IFS we will often be interested in the union of the image of all the contractions. We use the following notation

\[
S(E) = \bigcup_{i=1}^{m} S_i(E)
\]

and when we use this function multiple times we use the notation

\[
S^k(E) = S(S^{k-1}(E)).
\]

**Definition 2.26.** An **attractor** \( F \) of an IFS is a compact set such that

\[
F = \bigcup_{i=1}^{m} S_i(F)
\]

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Theorem 2.27. Let \( \{S_1, ..., S_m\} \) be an IFS where
\[
|S_i(x) - S_i(y)| < r_i|x - y|
\]
where \( 0 < r_i < 1 \) for each \( i \). Then the system has a unique attractor \( F \), that is a unique non-empty compact set such that
\[
F = \bigcup_{i=1}^{m} S_i(F).
\]
Moreover,
\[
F = \bigcap_{k=0}^{\infty} S^k(E)
\]
for every non-empty compact set \( E \) such that \( S_i(E) \subset E \) for all \( i \).

The proof of this theorem can be found in Chapter 9 of [1].

Definition 2.28. An IFS satisfy the open set condition if there exists a open set \( V \) such that
\[
V \supset \bigcup_i S_i(V)
\]
and
\[
S_i(V) \cap S_j(V) = \emptyset
\]
for \( i \neq j \).

The open set condition is used to guarantee the one can compute the Hausdorff dimension of an attractor to an IFS. See [1] for details. However the open set condition is not enough to guarantee that the attractor is a Cantor set. We now study when an attractor is a Cantor set and when it is an Inverse limit. We start with a lemma.

Lemma 2.29. Let \( \{S_1, ..., S_m\} \) be an IFS such that each \( S_i \) is injective and there exists a closed and bounded set \( E \subset \mathbb{R} \) such that \( S_i(E) \subset E \) for all \( i \) and
\[
S_i(E) \cap S_j(E) = \emptyset
\]
for \( i \neq j \). Then
\[
(S_{i_1} \circ \cdots \circ S_{i_n}(E)) \cap (S_{j_1} \circ \cdots \circ S_{j_n}(E)) = \emptyset
\]
for two different sequences of contractions i.e \( i_\ell \neq j_\ell \) for some \( \ell \in \{1, \ldots, n\} \).

Proof. Let \( U_1 = S_{i_1} \circ \cdots \circ S_{i_n}(E) \) and \( U_2 = S_{j_1} \circ \cdots \circ S_{j_n}(E) \) then since the sequence of contractions is different there is a smallest integer \( k \) such that \( i_k \neq j_k \). Let
\[
T = S_{i_1} \circ \cdots \circ S_{i_{k-1}} = S_{j_1} \circ \cdots \circ S_{j_{k-1}}
\]
where the equality holds since $S_i = S_j$ for $t < k$. This give us that

$$U_1 = T \circ S_{i_k} \circ \cdots \circ S_{i_n}(E)$$

and $U_2 = T \circ S_{j_k} \circ \cdots \circ S_{j_n}(E)$. Since each $S_i$ is injective we get that also $T$ must be injective. Next we use that $S_{i_n}(E) \subset E$ and get

$$S_{i_{n-1}} \circ S_{i_n}(E) \subset S_{i_{n-1}}(E)$$

and then repeatedly using that $S_i(E) \subset E$ for all $i$ we get that

$$U_1' = S_{i_k} \circ \cdots \circ S_{i_n}(E) \subset S_{i_k}(E).$$

Analogously we also get

$$U_2' = S_{j_k} \circ \cdots \circ S_{j_n}(E) \subset S_{j_k}(E).$$

Since $S_{i_k}(E) \cap S_{j_k}(E) = \emptyset$ we get that $U_1' \cap U_2' = \emptyset$. Now we have that

$$U_1 = T(U_1') \quad \text{and} \quad U_2 = T(U_2')$$

and since $U_1 \cap U_2 = \emptyset$ and $T$ is injective we get that also $U_1$ and $U_2$ must be disjoint.

This lemma will be used both to prove that attractors are Cantor sets and inverse limits.

**Proposition 2.30.** Let $\{S_1, \ldots, S_m\}$ be an IFS with attractor $F$. If each $S_i$ is injective and there exists a closed and bounded set $E \subset \mathbb{R}$ such that $S_i(E) \subset E$ for all $i$ and $S_i(E) \cap S_j(E) = \emptyset$ for $i \neq j$ then $F$ is a Cantor set.

**Proof.** We start by letting

$$\mathcal{B}_n = \{S_{i_1} \circ \cdots \circ S_{i_n}(E) : 1 \leq i_k \leq m\}.$$  

and prove that this is a cover of $F$ of finitely many disjoint clopen sets. Since we have finitely many contractions each $\mathcal{B}_n$ contains finitely many elements. From Theorem 2.27 we get that

$$F = \bigcap_{k=1}^{\infty} S^k(E) \subset S^n(E) = \bigcup S_{i_1} \circ \cdots \circ S_{i_n}(E)$$

so each $\mathcal{B}_n$ is a cover of $F$. Moreover Lemma 2.29 gives us that the sets in $\mathcal{B}_n$ are disjoint.

Next we prove that $\mathcal{B}_n$ only contains clopen sets. Since $F$ is a subset of $\mathbb{R}^n$ for every $B_0 \in \mathcal{B}_n$ there is an open set $U$ such that $B_0 \subset U$ and for every $B \in \mathcal{B}_n$ such that $B \neq B_0$ we have that

$$U \cap B = \emptyset$$
so each $B_0$ is open in $F$. Moreover since $B_n$ is an open cover of $F$ we get

$$F \setminus B_0 = \left( \bigcup_{B \in B_n} B \cap F \right) \setminus B_0 = \bigcup_{B \in B \setminus B_0} B \cap F$$

so $B_0$ is also closed in $F$. Now we have that $B_n$ is a cover of pairwise disjoint clopen sets.

Next we check that $B_{n+1}$ is a refinement of $B_n$. Let $B_0 \in B_{n+1}$. Then we have that

$$B_0 = S_{i_1} \circ \cdots \circ S_{i_{n+1}}(E) = S_{i_1}(B_1) \subset B_1$$

for some element $B_1 \in B_n$. Thus we have that $B_{n+1}$ is a refinement of $B_n$.

For the third condition we need to prove that for each $B_0 \in B_n$ there is at least two sets $B_1, B_2 \in B_{n+1}$ which are subsets of $B_0$. Let

$$B_0 = S_{i_1} \circ \cdots \circ S_{i_n}(E)$$

then every element in $B_{n+1}$ of the form

$$S_{i_1} \circ \cdots \circ S_{i_t}(E)$$

with $1 \leq t \leq m$ will be a subset of $B_0$. This follows from

$$S_{i_1}(E) \subset E \Rightarrow S_{i_1} \circ \cdots \circ S_{i_n}(E) \subset S_{i_1} \circ \cdots \circ S_{i_n}(E).$$

Since $m \geq 2$ we get that there is at least two such sets.

Lastly we need to prove that $\text{mesh}(B_n) \to 0$ as $n \to \infty$. Let $B \in B_n$ such that

$$B = S_{i_1} \circ \cdots \circ S_{i_n}(E).$$

Then using that each $S_i$ is a contraction we get that

$$|B| = \sup_{x,y \in B} |x - y| = \sup_{x,y \in E} |S_{i_1} \circ \cdots \circ S_{i_n}(x) - S_{i_1} \circ \cdots \circ S_{i_n}(y)| < \ldots$$

$$\cdots < \sup_{x,y \in E} r^n|x - y| = r^n|E|$$

for some $0 < r < 1$. Since $E$ is bounded we get that $|E| < \infty$ we get that

$$|B| < r^n|E| \to 0 \text{ as } n \to \infty$$

and thus also $\text{mesh}(B_n) \to 0$ as $n \to \infty$.

Now we have by Lemma 2.5 that $F$ is a Cantor set. \qed

We have already proven that every Cantor set is homeomorphic to an inverse limit and thus if an attractor of an IFS is a Cantor set it is homeomorphic to an inverse limit. We give an immediate proof for when an attractor is homeomorphic to an inverse limit for completion.
Proposition 2.31. Let \( \{S_1, \ldots, S_m\} \) be an IFS with attractor \( F \). If each \( S_i \) is injective for all \( i \) and \( E \) a closed bounded set such that
\[
S_i(E) \cap S_j(E) = \emptyset
\]
for \( i \neq j \). Then there is a discrete non trivial inverse limit \( X \) which is homeomorphic to \( F \).

Proof. We construct the spaces \( X_n \) using induction. Let \( X_1 \) be a space with \( m \) elements and the discrete topology. To get \( X_{n+1} \) from \( X_n \) let \( \pi^{-1}_n(x_n) \) be a set with \( m \) elements for every \( x_n \in X_n \). Then let
\[
X_{n+1} = \bigcup_{x_n \in X_n} \pi^{-1}_n(x_n).
\]
Observe that \( \#X_n = m^n \).

Next we construct a bijection from \( X \) to integer sequences \( \{i_n\}_{n \geq 1} \) where \( 1 \leq i_n \leq m \). Let \( x = \{x_n\}_{n \geq 1} \in X \) then since \( X_1 \) contains \( m \) elements we can find a bijection from \( X_n \) to \( \{1, \ldots, m\} \) so we can send \( x_1 \) to an integer \( i_1 \in \{1, \ldots, m\} \). The element \( x_2 \) is in \( \pi^{-1}_1(x_1) \) which contains \( m \) element so there is a bijection from \( \pi^{-1}_1(x_1) \) into \( \{1, \ldots, m\} \) so we can send \( x_2 \) to the integer \( i_2 \). We continue this process to get a bijection
\[
h_1 : X \to \{\{i_n\}_{n \geq 1} : 1 \leq i_n \leq m\}.
\]

Next let \( x \in F \) then from theorem 2.27 we have that
\[
x \in F = \bigcap_{k=0}^{\infty} S^k(E)
\]
we also have that
\[
S^k(E) = \bigcup_{I_k} \cap S_{i_1} \circ \cdots \circ S_{i_k}(E)
\]
where \( I_k \) is the set of all \( k \) terms sequences. Thus there must be a sequence of integers \( \{i_n\}_{n \geq 1} \) such that
\[
x \in S_{i_1} \circ \cdots \circ S_{i_k}
\]
for all \( k \) and this sequence is unique by Lemma 2.29. Thus we have a bijection from
\[
h_2 : F \to \{\{i_n\}_{n \geq 1} : 1 \leq i_n \leq m\}.
\]
Now if we compose \( h_1 \) with \( h_2^{-1} \) we get a bijection \( h = h_1 \circ h_2^{-1} \) from \( X \) to \( F \).

Next we prove that \( h \) and \( h^{-1} \) are continuous. First we let \( U = \pi_k^{-1}(x_k) \) then
\[
h_1(U) = \{(i_n) : 1 \leq i_n \leq m \text{ for } n \geq k + 1\}
\]
and \( \{i_n\}_{n \geq 1} \) is a fixed sequence. If we then apply \( h_2^{-1} \) we get
\[
h(U) = S_{i_1} \circ \cdots S_{i_k}(E)
\]
which is open so $h^{-1}$ is continuous. Next for an open set in $F$ we only need to consider sets on the form

$$U = S_{i_1} \circ \cdots \circ S_{i_k}(E)$$

where $\{i_n\}_{n \geq 1}$ is some sequence of integers between 1 and $m$. This is because these sets are open and we can make the diameter of such sets arbitrarily small. If we then apply $h_2$ we get

$$h_2(U) = \{(i_n) : 1 \leq i_n \leq m \text{ for } n \geq k + 1\}$$

but the preimage of this set under $h_1$ is precisely $\pi^{-1}_k(x_k)$ for some $x_k \in X_k$. So we get

$$h^{-1}(U) = \pi^{-1}_k(x_k)$$

and thus $h$ is continuous. Now we have a homeomorphism between $F$ and $X$ so they are homeomorphic.

We end this section studying some examples of IFS where the attractor is not a Cantor set. Consider the IFS of the two contractions

$$S_1(x) = \frac{x}{2} \quad \text{and} \quad S_2(x) = \frac{1}{2} + \frac{x}{2}.$$ 

This IFS satisfies the open set with open interval $(0, 1)$. Doing the computation we get

$$S_1((0, 1)) = \left[0, \frac{1}{2}\right) \quad \text{and} \quad S_2((0, 1)) = \left(\frac{1}{2}, 1\right)$$

which gives us that

$$S_1((0, 1)) \cup S_2((0, 1)) = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \subset (0, 1).$$

Thus the IFS satisfies the open set condition. But if we compute the attractor we get that

$$S_1([1, 0]) \cup S_2([1, 0]) = \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right] = [0, 1]$$

and thus the attractor is $[0, 1]$ but this is not a Cantor set.

Next we study an example where the contractions are not injective. Let

$$S_1(x) = 0 \quad \text{and} \quad S_2(x) = 1$$

then we have that

$$S_1([0, 1]) = 0 \quad \text{and} \quad S_2([0, 1]) = 1$$

which gives

$$S_1([0, 1]) \cup S_2([0, 1]) = \{0\} \cup \{1\} = \{0, 1\} \subset [0, 1].$$
Thus there is a closed set $E$ such that

$$S_i(E) \subset E$$

for $i = 1, 2$ and

$$S_1(E) \cap S_2(E) = \emptyset.$$

But the attractor is the set $\{0, 1\}$ which is not a Cantor set.
3 Maps on Inverse Limits

In this section we construct maps on Cantor sets. Rather than constructing a map immediately on a Cantor set we will use that every Cantor set is homeomorphic to some Inverse limit and then we define a map on that Inverse limit instead. The general idea for defining a map on an inverse limit \( X = \lim X_n \) is to define maps \( f_n \) on each \( X_n \) and then define \( f \) to be

\[
 f(\{x_n\}_{n \geq 1}) = \{f_n(x_n)\}_{n \geq 1}.
\]

In the first subsection we prove that every discrete non trivial inverse limit is homeomorphic. Using this we prove in the second subsection that for every Cantor set \( C \) there is a map \( f \) such that \((C, f)\) is a minimal Cantor set.

3.1 Maps from \( X \) to \( Y \)

Throughout this section let \( X = \lim X_n \) and \( Y = \lim Y_n \) be two discrete non trivial inverse limits with projective maps \( \pi_n \) and \( \sigma_n \) respectively and most functions \( f : X \rightarrow Y \) are defined using some sequence of function \( f_n \) as

\[
 f(\{x_n\}_{n \geq 1}) = \{f_n(x_n)\}_{n \geq 1}.
\]

Defining \( f \) in this way does not immediately guarantee that it is well defined in the sense that \( f(\{x_n\}_{n \geq 1}) \in Y \) for every \( \{x_n\}_{n \geq 1} \in X \) because given a sequence \( \{y_n\}_{n \geq 1} \) with \( y_n \in Y_n \) then \( \{y_n\}_{n \geq 1} \in Y \) if and only if \( \sigma_n(y_{n+1}) = y_n \) for all \( n \). The next lemma gives us sufficient conditions for \( f \) to be well defined and be continuous.

![Figure 2: Commutative diagram for Lemma 3.1](image)

Lemma 3.1. If the diagram in Figure 2 commutes for all \( n \) then \( f : X \rightarrow Y \) is well defined and continuous.

Proof. Firstly we prove that \( f \) is well defined. We have that \( f_{n+1}(x_{n+1}) \in Y_{n+1} \) for all \( n \) and since the diagram in Figure 2 commutes we get that

\[
 \sigma_n(f_{n+1}(x_{n+1})) = f_n(\pi_n(x_{n+1})) \in Y_n
\]

for all \( n \) and thus \( \{f_n(x_n)\}_{n \geq 1} \in Y \).
To prove that $f$ is continuous consider an arbitrary open set in the basis of the topology on $Y$. By Lemma 2.17 we get that this set is of the form $\sigma_k^{-1}(y)$ for some $y \in Y_k$ with $k \geq 1$. Next let $\{x_n\}_{n \geq 1} \in f^{-1}(\sigma_k^{-1}(y))$ then we have

$$f(\{x_n\}_{n \geq 1}) = \{f_n(x_n)\}_{n \geq 1} \in \sigma_k^{-1}(y)$$

by definition we know that $\{z_n\}_{n \geq 1} \in \sigma_k^{-1}(y)$ if and only if $z_k = y$, which gives us that

$$f_k(x_k) = y.$$ 

With the same reasoning as above we have that $\{x_n\}_{n \geq 1} \in \pi_k^{-1}(x_k)$ giving us that

$$\{x_n\}_{n \geq 1} \in \bigcup_{x_k \in X_k \atop f_k(x_k)=y} \pi_k^{-1}(x_k).$$

And thus

$$f^{-1}(\sigma_k^{-1}(y)) \subset \bigcup_{x_k \in X_k \atop f_k(x_k)=y} \pi_n^{-1}(x_n).$$

Next we let

$$\{x_n\}_{n \geq 1} \in \bigcup_{x_k \in X_k \atop f_k(x_k)=y} \pi_k^{-1}(x_k)$$

then we have that $f_k(x_k) = y$ which means that

$$\sigma_k(f(\{x_n\}_{n \geq 1})) = y.$$ 

Then by taking $\sigma_k^{-1}$ and $f^{-1}$ we get that

$$\{x_n\}_{n \geq 1} \in f^{-1}(\sigma_k^{-1}(y_k))$$

and thus

$$f^{-1}(\sigma_k^{-1}(y)) \supset \bigcup_{x_k \in X_k \atop f_k(x_k)=y} \pi_n^{-1}(x_n).$$

With the above we get that

$$f^{-1}(\sigma_k^{-1}(y)) = \bigcup_{x_k \in X_k \atop f_k(x_k)=y} \pi_n^{-1}(x_n)$$

and thus we get that the preimage of $\sigma_k^{-1}(y)$ under $f$ is a union of open sets in $X$ so also $f^{-1}(\sigma_k^{-1}(y))$ must be open in $X$. Since $\sigma_k^{-1}(y)$ was an arbitrary open in the basis of the topology on $Y$ we get that $f$ is continuous. \(\square\)

The main result we prove in this section is that all inverse limits are homeomorphic. Our main tool for proving this will be the following lemma.
Lemma 3.2. Let $X$ and $Y$ be such that
\[ \# Y_2 \geq \# X_1 \geq \# Y_1 \]
and for all $x_n \in X_n, y_n \in Y_n$ and $x_{n-1} \in X_{n-1}$ we have that
\[ \# \pi_n^{-1}(x_n) \geq \# \sigma_n^{-1}(y_n) \geq \# \pi_{n-1}^{-1}(x_{n-1}) \]
for $n \geq 2$. Then $X$ and $Y$ are homeomorphic.

The conditions in the above Lemma might seem ad hoc. One could loosen the conditions in order to include more inverse limits. We have chosen these conditions for the lemma to make the proof less technical while still giving us what we need.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be defined using the sequence of functions $g_n : Y_{n+1} \to X_n$.

We construct two maps $f : X \to Y$ and $g : Y \to X$ and then show that they are each other's inverses. The construction of $f$ and $g$ is done by defining two sequences of functions, $f_n : X_n \to Y_n$ and $g_n : Y_{n+1} \to X_n$ and then let
\[ f(\{x_n\}_{n \geq 1}) = \{f_n(x_n)\}_{n \geq 1} \]
and
\[ g(\{y_n\}_{n \geq 1}) = \{g_n(y_{n+1})\}_{n \geq 1}. \]

In order for $f$ and $g$ to be well defined we need the diagram in Figure 3 to commute otherwise the image of $f$ and $g$ will not be in $X$ or $Y$ respectively.

![Figure 3: Commutative diagram for between two inverse limits](image)

We use an inductive reasoning to construct $f_n$ and $g_n$. First we observe that $\# X_1 \geq \# Y_1$ and they contain finitely many elements we can choose $f_1$ to be a surjective function. Secondly we observe that
\[ \# X_1 \leq \# Y_2 = \sum_{y_1 \in Y_1} \sigma_1^{-1}(y_1) \]
and thus we can choose $f_1$ such that for all $y_1 \in Y_1$, $f_1$ does not send more than $\# \sigma_1^{-1}(y_1)$ to $y_1$. For $g_1$ we start by observing that the construction of $f_1$ gives us that
\[ \# f_1^{-1}(y_1) \leq \# \sigma_1^{-1}(y_1) \]
for all \( y_1 \in Y_1 \) and thus we can choose \( g_1 \) to be a function such that 
\[
g_1 (\sigma_1^{-1}(y_1)) = f_1^{-1}(y_1)
\]
for all \( y_1 \in Y_1 \).

Next we construct \( f_{n+1} \) and \( g_{n+1} \) assuming that we have \( f_n \) and \( g_n \), we start with \( f_{n+1} \). First we observe that we can partition \( X_{n+1} \) into sets on the form \( \pi_n^{-1}(x_n) \) with \( x_n \in X_n \). All the elements in \( \pi_n^{-1}(x_n) \) are projected to \( x_n \) by \( \pi_n \) so \( f_{n+1} \) need to send the elements in \( \pi_n^{-1}(x_n) \) to the elements in \( Y_{n+1} \) that are mapped to \( x_n \) by \( g_n \). Since the diagram commutes we get that the elements in \( Y_{n+1} \) which are mapped to \( x_n \) by \( g_n \) are those which are projected to \( f_n(x_n) = y_n \) under \( \sigma_n \). Thus we need that 
\[
f_{n+1} (\pi_n^{-1}(x_n)) \subset \sigma_n^{-1}(y_n).
\]
But we also have that 
\[
\#\pi_n^{-1}(x) \geq \#\sigma_n^{-1}(y_n)
\]
so we can choose \( f_{n+1} \) such that 
\[
f_{n+1} (\pi_n^{-1}(x_n)) = \sigma_n^{-1}(y_n)
\]
for all \( x_n \in X_n \). The construct of \( g_{n+1} \) is very similar to that of \( f_{n+1} \). First we partition \( Y_{n+2} \) into sets of the form \( \sigma_{n+1}^{-1}(y_{n+1}) \) with \( y_{n+1} \in Y_{n+1} \). Let \( g_n(y_{n+1}) = x_n \), then \( f_{n+1} \) send all the elements in \( \pi_n^{-1}(x_n) \) to \( y_{n+1} \). Thus we have that \( g_{n+1} \) must be such that 
\[
g_{n+1} (\sigma_{n+1}^{-1}(y_{n+1})) \subset \pi_n^{-1}(x_n)
\]
but since we have that 
\[
\#\sigma_{n+1}^{-1}(y_{n+1}) \geq \#\pi_n^{-1}(x_n)
\]
we can choose \( g_{n+1} \) such that 
\[
g_{n+1} (\sigma_{n+1}^{-1}(y_{n+1})) = \pi_n^{-1}(x_n)
\]
for all \( y_{n+1} \in Y_{n+1} \). Now we have that \( f \) and \( g \) are well defined functions.

Next we show that \( g = f^{-1} \). First let \( \{x_n\}_{n \geq 1} \in X \) then we get 
\[
g(f(\{x_n\}_{n \geq 1})) = \{g_n(f_{n+1}(x_{n+1}))\}_{n \geq 1} = \{\pi_n(x_n)\}_{n \geq 1} = \{x_n\}_{n \geq 1}.
\]
Next let \( \{y_n\}_{n \geq 1} \in Y \) then 
\[
f(g(\{y_n\}_{n \geq 1})) = \{f_n(g_n(y_{n+1}))\}_{n \geq 1} = \{\sigma_n(y_{n+1})\}_{n \geq 1} = \{y_n\}_{n \geq 1}.
\]
Thus we have that \( g = f^{-1} \) is a bijection.

Next we check that \( f \) and \( g \) are continuous. We get that \( f \) is continuous immediately from Lemma 3.1. Proving that \( g \) is continuous requires a more work. Lemma 2.17 gives us that the elements in the basis of the topology of
Y is of the form $\sigma_n^{-1}(y_n)$ with $y_n \in Y_n$ for $n \geq 1$. For $n \geq 2$ an analogous argument as the proof for Lemma 3.1 gives us that the preimage of $\sigma_n^{-1}(y_n)$ is open. For $n = 1$ we observe that

$$\sigma_1^{-1}(y_1) = \bigcup_{\substack{y_2 \in Y_2 \\ \sigma_1(y_2) = y_1}} \sigma_2^{-1}(y_2)$$

which gives

$$g^{-1}(\sigma_1^{-1}(y_1)) = \bigcup_{\substack{y_2 \in Y_2 \\ \sigma_1(y_2) = y_1}} g^{-1}(\sigma_2^{-1}(y_2)).$$

The preimage of $\sigma_2^{-1}(y_2)$ is open for each $y_2 \in Y_2$ so the preimage of $\sigma_1^{-1}(y_1)$ is a union of open set and thus open. Now we have that the preimage of $\sigma_n^{-1}(y_n)$ is with $y_n \in Y_n$ is open for all $n$ so $g$ is continuous. Thus we have that both $f$ and $g$ are continuous so $f$ is a homeomorphism and $X$ is homeomorphic to $Y$.

We use these conditions since given two inverse limits $X$ and $Y$ we can find two sub limits which satisfy these conditions.

**Definition 3.3.** Let $X = \lim \leftarrow X_n$ be an inverse limit. Then a sub limit $X' = \lim \leftarrow X_{n_i}$ of $X$ is an inverse limit where the sequence of topological spaces we consider $(X_{n_i}, T_{n_i})$ is a subsequence of $(X_n, T_n)$ and the projective maps $\sigma_i$ is defined by

$$\sigma_i(x_{n_i}) = \pi_{n_i}(x_{n_i}).$$

Observe that the projections $\sigma_i$ are not the same projections as $\pi_{n_i}$ because they have different preimages. The sub limit $X'$ can be written as the set

$$X' = \{\{x_{n_i}, \{i \geq 1 : x_{n_i} \in X_{n_i} \text{ and } \sigma_i(x_{n_{i+1}}) = x_{n_i}\}\}.\]

The reason we can use a sub limit $X'$ instead of the limit $X$ is because we can reconstruct $X$ from $X'$ by using the projections $\pi_{n_i}$. This idea lets us prove that every inverse limit is homeomorphic to all its sub limits.

**Lemma 3.4.** If $X$ is an inverse limit then it is homeomorphic to every sub limit $X'$.

**Proof.** Let

$$X = \lim \leftarrow X_n = \lim X_n,$$

with projection $\pi_n$ and $\sigma_i$ respectively. Next we let $h : X \to X'$ defined as

$$h(\{x_n\}_{n \geq 1}) = \{x_n\}_{i \geq 1}$$

so $h$ removes the elements from the sequence $\{x_n\}_{n \geq 1}$ such that we get a sequence $\{x_{n_i}\}_{i \geq 1}$ which is in $X'$. Then we can define an inverse of $h$. Let $g : X' \to X$ be the function

$$g(\{x_{n_i}\}_{i \geq 1}) = \{\pi_{n_i,i}(x_{n_i})\}_{i \geq 1} = \{x_n\}_{n \geq 1}.$$
Then we get that

\[ g(h \{ x_n \}_{n \geq 1}) = g(\{ y_n \}_{i \geq 1}) = \{ \pi_{n,i}(x_n) \}_{i \geq 1} = \{ x_n \}_{n \geq 1} \]

and

\[ h(g(\{ x_n \}_{i \geq 1})) = h(\{ \pi_{n,i}(x_n) \}_{i \geq 1}) = h(\{ x_n \}_{n \geq 1}) = \{ x_n \}_{i \geq 1} \]

which proves that \( g = h^{-1} \). Thus \( h \) has an inverse so \( h \) is a bijection. Next we check that \( h \) is continuous. From Lemma 2.17 we get that all the elements in the basis of the topology on \( X' \) are on the from \( \sigma_{n,j}^{-1}(x_{n_j}) \) for some \( x_{n_j} \in X_{n_j} \). Thus we only need to check that the image of such sets under \( g \) is open. Let \( \{ y_{n_i} \}_{i \geq 1} \in \sigma_{n,j}^{-1}(x_{n_j}) \) then we must have that \( y_{n_j} = x_{n_j} \). Hence we get that

\[ g(\{ y_{n_i} \}_{i \geq 1}) = \{ y_n \}_{n \geq 1} \in \pi_{n,j}^{-1}(x_{n_j}) \]

and thus

\[ g(\sigma_{n,j}^{-1}(x_{n_j})) \subset \pi_{n,j}^{-1}(x_{n_j}). \]

Next let \( \{ y_n \}_{n \geq 1} \in \pi_{n,j}^{-1}(x_{n_j}) \) which gives \( y_{n_j} = x_{n_j} \). Since \( h \) is a bijection we get that

\[ g(h(\{ y_{n_j} \}_{n \geq 1})) = g(\{ y_{n_j} \}_{i \geq 1}) = \{ y_{n_j} \}_{n \geq 1} \]

but since \( y_{n_j} = x_{n_j} \) we get that \( \{ y_{n_j} \}_{i \geq 1} \in \sigma_{j}^{-1}(x_j) \). Hence we get

\[ g(\sigma_{n,j}^{-1}(x_{n_j})) \supset \pi_{n,j}^{-1}(x_{n_j}) \]

and finally we have that

\[ g(\sigma_{n,j}^{-1}(x_{n_j})) = \pi_{n,j}^{-1}(x_{n_j}). \]

Thus the image of \( \sigma_{j}^{-1}(x_{n_j}) \) is open under \( g \) so \( h \) is continuous.

Next we check that \( h^{-1} = g \) is continuous. Again using Lemma 2.17 we only need to show that the image of sets of the from \( \pi_{m}^{-1}(x_m) \) for some \( x_m \in X_m \). Let \( \{ y_n \}_{n \geq 1} \in \pi_{m}^{-1}(x_m) \) then we have that \( y_m = x_m \). Consider now

\[ h(\{ y_{n_j} \}_{n \geq 1}) = \{ y_{n_j} \}_{i \geq 1} \]

then

\[ \pi_{n,m}(y_{n_m}) = y_m = x_m. \]

Thus we get

\[ h(\{ y_n \}_{n \geq 1}) = \{ y_{n_j} \}_{i \geq 1} \in \bigcup_{\pi_{n,m}(y_{n_m}) = x_m} \sigma_{n,m}^{-1}(y_{n_m}). \]

Thus we have that

\[ h(\pi_{m}^{-1}(x_m)) \subset \bigcup_{\pi_{n,m}(y_{n_m}) = x_m} \sigma_{n,m}^{-1}(y_{n_m}). \]
Next let \( \{y_n\}_{i \geq 1} \in \bigcup_{\pi_{n,m}(y_{n,m}) = x_m} \sigma_m^{-1}(y_{n,m}) \) and consider
\[
g(\{y_n\}_{i \geq 1}) = \{y_n\}_{n \geq 1}.
\]
Then we have that
\[
x_m = \pi_{n,m}(y_{n,m}) = y_m
\]
so \( \{y_n\}_{n \geq 1} \in \pi_m^{-1}(x_m) \). Hence we get
\[
\{y_n\}_{i \geq 1} = h(\{y_n\}_{n \geq 1}) \in h(\pi_m^{-1}(x_m))
\]
which means
\[
h(\pi_m^{-1}(x_m)) \supset \bigcup_{\pi_{n,m}(y_{n,m}) = x_m} \sigma_m^{-1}(y_{n,m}).
\]
Finally we have that
\[
h(\pi_m^{-1}(x_m)) = \bigcup_{\pi_{n,m}(y_{n,m}) = x_m} \sigma_m^{-1}(y_{n,m})
\]
so the image of \( \sigma_m^{-1}(x_m) \) is open under \( h \) so \( h^{-1} = g \) is continuous and \( h \) is a homeomorphism. Thus \( X \) and \( X' \) are homeomorphic.

With all of the above we can now prove that all inverse limits are homeomorphic.

**Proposition 3.5.** Any two discrete non trivial inverse limits are homeomorphic.

**Proof.** The main idea is to use that
\[
\pi_{n+2,n}(x_n) = \bigcup_{\pi_{n+1,n}(x_{n+1}) = x_n} \pi_{n+2,n+1}(x_{n+1})
\]
and since this is a disjoint union we get that
\[
\#\pi_{n+2,n}^{-1}(x_n) = \sum_{\pi_{n+1,n}(x_{n+1}) = x_n} \#\pi_{n+2,n+1}^{-1}(x_{n+1}) \geq \sum_{\pi_{n+1,n}(x_{n+1}) = x_n} 2 \geq 2 \cdot 2 = 4
\]
where we have used that the preimage contains at least two elements. With an inductive reasoning this gives us that
\[
\#\pi_{n+k,n}^{-1}(x_n) \geq 2^k.
\]
This means that we can find a sub limit \( X' \) of \( X \) such that the preimage of each projection is arbitrarily large. With this we can create sub limits \( X' \) and \( Y' \) such that we can use Lemma 3.2. First we let
\[
Y_{m_1} = Y_1
\]
and then we chose \( n_1 \) as the smallest integer such that

\[
\#X_{n_1} \geq \#Y_{m_1}
\]

we know such and \( n_1 \) exists due to the above observation. Next we can proceed with an inductive reasoning. We choose \( Y_{m_1} \) such that

\[
\#\sigma_{m_1,m_1-1}^{-1}(y_{m_1-1}) \geq \#\pi_{n_1-1,n_1-2}^{-1}(x_{n_1-2})
\]

and then we can choose \( n_i \) such that

\[
\#\pi_{n_i-1,n_i}^{-1}(x_{n_i}) \geq \#\pi_{n_i-1,n_i}^{-1}(x_{n_i})
\]

Now we have that \( X' \) and \( Y' \) satisfies the conditions for Lemma 3.5 so they are homeomorphic. But from Lemma 3.4 we get that \( X \) is homeomorphic to \( X' \) and \( Y \) is homeomorphic to \( Y' \) and thus we get that \( X \) is homeomorphic to \( Y \).

From this we can easily prove Lemma 2.21 which stated that every two Cantor sets are homeomorphic.

Proof of Lemma 2.21. Let \( C_1 \) and \( C_2 \) be two Cantor sets. From Proposition 2.23 we get that there exists two inverse limits \( X \) and \( Y \) such that \( C_1 \) is homeomorphic to \( X \) and \( C_2 \) is homeomorphic to \( Y \). Then from Proposition 3.5 we get that \( X \) and \( Y \) are homeomorphic and thus also \( C_1 \) and \( C_2 \) must be homeomorphic.

3.2 Construction of minimal Cantor set

Recall that we defined minimal Cantor sets in Definition 2.3. We restate the definition here. A minimal dynamical system \((C,f)\) is a minimal Cantor set if \( C \) is a Cantor set and \( f \) is continuous.

When we prove the existence of minimal Cantor set we use that every Cantor set is homeomorphic to an inverse limit and then define a function on the inverse limit. Given an inverse limit \( X = \lim X_n \) we again define a function \( f \) by defining a sequence of functions \( f_n : X_n \to X_n \). In order to assure that every orbit of \( f \) is dense we can make every orbit of \( f_n \) dense. This can be difficult to achieve with some inverse limits as we still need to make sure that the image is in \( X \).

For example, let \( X \) be an inverse limit such that \( X_1 \) contains two elements \( a \) and \( b \) and the \( X_2 \) contains five elements with the following structure

\[
\pi_1^{-1}(a) = \{a_1,a_2\}
\]

and

\[
\pi_1^{-1}(b) = \{b_1,b_2,b_3\}.
\]

If \( f \) is to be well defined we need that for a given point in \( \{x_n\}_{n \geq 1} \) in \( X \) that

\[
\pi_1(f_2(x_2)) = f_1(x_1)
\]
and if we want every orbit of every \( f_n \) to be dense we must have that
\[
 f_1(a) = b \quad \text{and} \quad f_1(b) = a.
\]

This means that \( f_2 \) must take elements in \( \pi_1^{-1}(a) \) to elements in \( \pi_1^{-1}(b) \) and vice versa but \( \pi_1^{-1}(a) \) contains 2 elements and \( \pi_1^{-1}(a) \) contains 3 elements and thus all orbits of \( f_2 \) can not be dense.

In order to circumvent this problem we define a map on the 2-inverse limit and then use Lemma 3.5 to get that every Cantor set is homeomorphic to the 2-inverse limit.

**Definition 3.6.** The **2-inverse limit** is an inverse limit \( P = \lim P_n \) with projective maps \( \rho_n \) such that
\[
\# P_n = 2^n
\]
and
\[
\rho_n^{-1}(p_n) = 2
\]
for all \( p_n \in P_n \) and for all \( n \geq 0 \).

The 2-inverse limit has a very nice structure which we can easily define a function on such that all orbits are dense on it. We define a function \( s \) in the same fashion as we did before by defining a sequence of functions \( s_n \) such that the orbit of each \( p_n \in P_n \) is dense. Since each \( P_n \) contain finitely many elements if the orbit of \( p_n \) is to be dense we must have that the orbit of \( p_n \) is all of \( P_n \). We again do an inductive construction. First we observe that \( P_0 \) contains only one element so \( s_0 \) is the identity function and all the orbits must be all of \( P_0 \).

![Figure 4: Commutative diagram for the 2-inverse limit](image)
we know that we can enumerate the elements in $P_{n-1}$ in this way since all the orbits of every element must be all of $P_{n-1}$. Then we can denote the elements in $P_n$ like

\[ p_{k,i} \text{ with } k = 1, 2, \ldots, 2^{n-1} \text{ and } i = 1, 2 \]

such that

\[ \rho_{n-1}(p_{k,i}) = q_k. \]

The index $k$ of $p_{k,i}$ should be thought of which element $p_{k,i}$ is projected to and the index $i$ as which if the two preimages of $q_k$ it is.

Now we define $s_n$ as

\[ s_n(p_{k,i}) = \begin{cases} p_{k+1,i} & \text{if } k \neq 2^{n-1} \\ p_{1,2} & \text{if } k = 2^{n-1} \text{ and } i = 1 \\ p_{1,1} & \text{if } k = 2^{n-1} \text{ and } i = 2. \end{cases} \]

Computing the orbit of $p_{k,i}$ we get

\[ \{s_n(p_{k,i})^m : m \geq 0\} = \{p_{k,i}, p_{k+1,i}, \ldots, p_{2^{n-1},i}, p_{1,j}, \ldots, p_{2^{n-1},j}, p_{1,i}, \ldots, p_{k-1,i}\} \]

where $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$ and thus the orbit of $p_{k,i}$ is dense.

We also check that the diagram commute. Let $k \neq 2^{n-1}$, then we get

\[ s_{n-1}(\rho_{n-1}(p_{k,i})) = s_{n-1}(q_k) = q_{k+1} = \rho_{n-1}(p_{k+1,i}) = \rho_{n-1}(s_n(p_{k,i})) \]

the case when $k = 2^{n-1}$ is proven analogously. Thus we have that

\[ s(\{p_n\}_{n\geq 1}) = \{s_n(p_n)\}_{n\geq 1} \]

is a well defined function.

**Definition 3.7.** The pair $(P, s)$ is called the **minimal 2-inverse limit**

It follows immediately from Lemma 3.1 that $s$ is a continuous function. Another property of $s$ which we prove in a lemma is that every orbit is dense.

**Lemma 3.8.** Let $\{p_n\}_{n\geq 1} \in P$ then the orbit of $\{p_n\}_{n\geq 1}$ under $s$ is dense in $P$

**Proof.** Let $\{q_n\}_{n\geq 1} \in P$ and $U$ any open set such that $\{q_n\}_{n\geq 1} \in U$. Then we know that there is some $q_k$ such that

\[ \{q_n\}_{n\geq 1} \in \pi^{-1}_k(q_k) \subset U. \]

Since the orbit of $p_d$ under $s_n$ is dense we get that there is some $n_k$ such that

\[ s^{n_k}_k(p_k) = q_k \]

and thus

\[ s^{n_k}(\{p_n\}_{n\geq 1}) \in \pi^{-1}_k(q_k) \subset U. \]
But since $U$ was any open set containing $\{q_n\}_{n \geq 1}$ we get that $\{q_n\}_{n \geq 1}$ is an accumulation point of $\{p_n\}_{n \geq 1}$ but $\{q_n\}_{n \geq 1}$ was an arbitrary point in $P$ so all points in $P$ are accumulation points of the orbit of $\{p_n\}_{n \geq 1}$. Thus we have that

$$\overline{O}(\{p_n\}_{n \geq 1}) = \overline{O}(\{p_n\}_{n \geq 1}) \cup O(\{p_n\}_{n \geq 1})' = P$$

which is what we wanted to prove. \hfill \Box

Now we have all the tools we need to prove the existence of minimal Cantor sets.

**Proposition 3.9.** Let $C$ be a Cantor set. Then there exists a continuous function $f$ such that $(C, f)$ is a minimal Cantor set.

**Proof.** From Lemma 2.23 we know that there is a homeomorphism from $C$ to some inverse limit $X$ and from Proposition 3.5 there is a homeomorphism from $X$ to $P$. Thus there is a homeomorphism $h$ from $C$ to $P$. Next we let

$$f = h^{-1} \circ s \circ h.$$ 

Then $f$ is a continuous function because $h$, $h^{-1}$ and $s$ are continuous.

Next to prove that every orbit under $f$ is dense we first observe that for any $x \in C$

$$f^n(x) = (h^{-1} \circ s \circ h)^n(x) = (h^{-1} \circ s^n \circ h)(x).$$

Now let $y \in C$ and $U$ any open set containing $y$. Then from Lemma 3.8 we get that there is a $n$ such that

$$s^n(h(x)) \in h(U) \Rightarrow h^{-1}(s^n(h(x))) \in U$$

but since $U$ was arbitrary we must have that $y$ is an accumulation point of

$$(h^{-1} \circ s^n \circ h) = (h^{-1} \circ s \circ h)^n.$$ 

And since $y$ was an arbitrary point we get

$$C = \{(h^{-1} \circ s \circ h)^n : n \geq 0\} = \{f^n(x) : n \geq 0\}$$

which means that $(C, f)$ is a minimal Cantor set. \hfill \Box
4 Combinatorial Covers

In this section we give a structure theorem for all minimal Cantor sets using combinatorial covers. Then we study when combinatorial covers gives rise to minimal Cantor set. We follow the methods outlined in [2].

4.1 Combinatorial covers from Minimal Cantor sets

In order to define combinatorial covers we need to know what a graph is.

**Definition 4.1.** A set $G = \{V, E\}$ is called a **directed topological graph** if $V$ is a topological space of points called vertices and $E$ is a set of pairs of elements from $V$ called edges.

**Definition 4.2.** Let $G$ be a graph then a sequence of edges $\{E_n\}_{n \geq 1}$ is called a **path** if $E_n = (x_n, x_{n+1})$ for some sequence of vertices $\{x_n\}_{n \geq 1}$.

**Definition 4.3.** Let $G$ be a graph then we say that two vertices $x$ and $y$ are **connected** if there exists a path starting in $x$ and ending in $y$.

Let $(C, f)$ be a minimal Cantor set and $B_k$ a cover from the definition of the Cantor set. We construct a cover $\mathcal{X}$ of $C$ which take the function $f$ into consideration. In some sense $\mathcal{X}$ will be a approximation of the minimal Cantor set. The construction is made inductively. First choose a clopen $U_0 \subset B_0 \in \mathcal{B}$ and then letting

$$\mathcal{X}(0) = \{U_0\}.$$ 

Then to get $\mathcal{X}(n+1)$ we look at the preimage of every set in $\mathcal{X}(n)$ under $f$ and intersect with the elements from $B_k$ we must also remove all elements that might be in $U_0$. Formally we get

$$\mathcal{X}(n+1) = \{f^{-1}(V) \cap B \setminus U_0 : V \in \mathcal{X}(n) \text{ and } B_0 \in \mathcal{B}_k\}.$$ 

Then we let the cover $\mathcal{X}$ be the collection

$$\mathcal{X} = \bigcup_{n \geq 0} \mathcal{X}(n).$$

Some immediate consequences of this are given in this lemma.

**Lemma 4.4.** Let $(C, f)$ be a minimal Cantor set and $\mathcal{X}$ a cover constructed as above. Then the following holds

1. $\mathcal{X}$ is a pairwise disjoint clopen cover of $C$,

2. there is a number $N$ such that for all $n \geq N$

   $$\mathcal{X}(n) = \emptyset,$$
3. the diameter of every $V \in \mathcal{X}$ is less than mesh($\mathcal{B}_k$).

Proof. 1. To observe that $\mathcal{X}$ is a cover choose any

$$x \in C \setminus U_0.$$  We can assume $x \notin U_0$ since if $x \in U_0$ for all $x \in C$ then $\mathcal{X}$ is a cover. Then we know that

$$\overline{O(x)} = C$$

so every point of $C$ is in the orbit of $x$ or is an accumulation point of the orbit of $x$. In either case there must be a smallest integer $n$ such that

$$f^n(x) \in U_0 \iff x \in f^{-n}(U_0).$$

Next we make the following observations

$$f^{n-1}(x) \in f^{-1}(U_0)$$

and since $\mathcal{B}_k$ is a cover there is some $B_{n-1} \in \mathcal{B}_k$ such that

$$f^{n-1}(x) \in B_{n-1}$$

and finally since $n$ is the smallest integer such that $f^n(x) \in U_0$ we have that

$$f^{n-1}(x) \notin U_0.$$  

Then we have that

$$f^{n-1}(x) \in \{ U_0 \cap B_{n-1} \} \setminus U_0$$

which is precisely an element of $\mathcal{X}(1)$. With an analogous reasoning we get that $f^{n-2}(x)$ is in some element of $\mathcal{X}(2)$. Continuing this reasoning we get that $x$ must be in some element of $\mathcal{X}(n)$. Thus we have that $\mathcal{X}$ is a cover. To see that $\mathcal{X}$ is cover of disjoint clopen sets we start by showing that all the sets in $\mathcal{X}(n)$ are pairwise disjoint using induction. Let $V_1, V_2 \in \mathcal{X}(1)$ then we have that

$$V_1 = (f^{-1}(U_0) \cap B_1) \setminus U_0$$

and

$$V_2 = (f^{-1}(U_0) \cap B_2) \setminus U_0$$

for some $B_1, B_2 \in \mathcal{B}_k$. Since $B_1$ and $B_2$ must be disjoint we get that also $V_1$ and $V_2$ are disjoint. Now we assume that the statement is true for $\mathcal{X}(n)$ and prove that it is true for $\mathcal{X}(n+1)$. Let $W_1, W_2 \in \mathcal{X}(n+1)$, then there are elements $W_1', W_2' \in \mathcal{X}(n)$ such that

$$W_1 = (f^{-1}(W_1') \cap B_1') \setminus U_0$$

and

$$W_2 = (f^{-1}(W_2') \cap B_2') \setminus U_0$$

where $B_1'$ and $B_2'$ are some elements in $\mathcal{B}_k$ not necessarily distinct. But since $W_1'$ and $W_2'$ are disjoint we must also have that $f^{-1}(W_1')$ and $f^{-1}(W_2')$ are disjoint.
and thus $W_1$ and $W_2$ must be disjoint. Thus by induction we have that all the sets of each $\mathcal{X}(n)$ must be disjoint.

Next assume that we have to sets $V_1 \in \mathcal{X}(n)$ and $V_2 \in \mathcal{X}(m)$ with $n \neq m$ and

$$x \in V_1 \cap V_2.$$ 

With out loss of generality we can assume that $n < m$. Since $x \in V_1 \in \mathcal{X}(n)$ we must have that $f^n(x) \in U_0$ and since $x \in V_2 \in \mathcal{X}(m)$ we must have that

$$f^n(x) \in \mathcal{X}(m - n).$$

Thus $m = n$ which is a contradiction. Thus $\mathcal{X}$ is a cover of disjoint sets.

To observe that $\mathcal{X}$ is made of clopen sets we again use induction. For $\mathcal{X}(0)$ we only have the set $U_0$ which is clopen. Now we assume that all the sets in $\mathcal{X}(n)$ are clopen and show that all the sets in $\mathcal{X}(n + 1)$ are clopen. Let $V \in \mathcal{X}(n + 1)$ then there is a $V' \in \mathcal{X}(n)$ such that

$$V = (f^{-1}(V') \cap B) \setminus U_0$$

for some $B \in B_k$. This is a finite intersection of clopen sets so $V$ must also be clopen. Thus by induction we have that every $\mathcal{X}(n)$ is made of clopen sets.

Now we have that $\mathcal{X}$ is a cover of disjoint clopen sets.

2. Since $\mathcal{X}$ is a cover of $C$ and $C$ is compact we know that there is a $N$ such that

$$C = \bigcup_{n \geq 0} \mathcal{X}(n).$$

Now let $V \in \mathcal{X}(k)$ for some $k > N$ then from 1 we have that $V$ is disjoint from all the sets in $\mathcal{X}(n)$ for $n \leq N$ which means that

$$C = V \cup \bigcup_{n \geq 0} \mathcal{X}(n) \Leftrightarrow V = C \setminus \bigcup_{n \geq 0} \mathcal{X}(n) = \emptyset.$$ 

3. Given $V \in \mathcal{X}(n + 1)$ there is some $V' \in \mathcal{X}(n)$ such that

$$V = (f^{-1}(V') \cap B) \setminus U_0 \subset B$$

for some $B \in B_k$. Thus the diameter of $V$ must be less than the diameter of $B$ so the diameter of all the sets in $\mathcal{X}$ must be less than $\text{mesh}(B_k)$. □

Now we realise the cover $\mathcal{X}$ as a graph. Let all the nonempty sets of $\mathcal{X}$ be the vertices and draw a edge from $V$ to $V'$ if

$$f(V) \cap V' \neq \emptyset.$$ 

**Definition 4.5.** A directed topological graph $G = \{V, E\}$ is called a **combinatorial cover** if

1. $V$ contains finitely many vertices and has the discrete topology ;
2. every two vertices can be connected by a path;
3. except for one vertex $0_G \in V$, every vertex only has one outgoing edge.

The vertex $0_G$ will be called the **splitting vertex**.

Figure 5: A combinatorial cover. The splitting vertex is denoted by $U$.

**Lemma 4.6.** When $\mathcal{X}$ is realised as a graph it is a combinatorial cover.

**Proof.**
1. Since $B_k$ contains finitely many elements we get that each $\mathcal{X}(n)$ must contain finitely many elements and thus $\mathcal{X}$ is finite.
2. Let $V, V' \in \mathcal{X}$ and $x \in V$ then since $f$ is minimal there is a $n$ such that
   \[ f^n(x) \cap V' \neq \emptyset. \]

Then we can choose $V_{n-1}$ such that
\[ f^{n-1}(x) \cap V_{n-1} \neq \emptyset \]
which means that $f(V_{n-1}) \cap V' \neq \emptyset$ so there is an edge between $V_{n-1}$ and $V'$. In the same way we can choose $V_{n-2}$ which has an edge from to $V_{n-1}$. Continuing this we get a path from $V$ to $V'$.
3. Let $V \in \mathcal{X}$ and $V \neq 0$. Then there must be some $V \in \mathcal{X}$ and $B_0 \in B_k$ such that
   \[ V = (f^{-1}(V') \cap B_0 = \emptyset). \]

From this we get that
\[ V \subset f^{-1}(V') \]
which means that all the elements in $V$ must be sent to $V'$ by $f$ and thus there is only one outgoing edge from $V$.

Given two vertices $x, y \in V$ such that there is an edge $(x, y)$ we also say that $y$ is an **image** of $x$. This is because we should think of the edge as applying $f$ to $x$.

Since the cover $\mathcal{X}$ can naturally be thought of as a graph we call $\mathcal{X}$ a combinatorial cover and depending on the context refer to the cover itself or the graph. If we see $\mathcal{X}$ as graph we already know what a path is but we formalise it for $\mathcal{X}$ as a cover aswell.
Definition 4.7. Let \( V_n, U_n \in \mathcal{X}(n) \) then the path of \( V_n \) to \( U_n \) is
\[
\lambda(V_n, U_n) = \{V_n, V_{n-1}, \ldots, V_1, U_n\}
\]
where \( V_k \in \mathcal{X}(k) \) and \( f(V_{k+1}) \subset V_k \).

One way of thinking about a path of \( V_n \) on \( \mathcal{X} \) is to consider the path from the vertices associated with \( V_n \) in \( G \) to the vertex associated with \( U_0 \).

Lemma 4.8. Let \((C, f)\) be a minimal Cantor set and \( \mathcal{X} \) a combinatorial cover. Then let \( U_1, U_2, \ldots, U_d \in \mathcal{X} \) be the sets such that \( f(U_0) \cap U_j \neq \emptyset \) for \( j = 1, \ldots, d \), then
\[
C = \bigcup_{j=1}^d \lambda(U_j, U_0)
\]

Proof. Let \( x \in C \) and \( x \in V_n \in \mathcal{X}(n) \). Then either \( V_n = U_j \) for some \( j \) or there is a \( B \in B_k \) such that
\[
V_{n+1} = (f^{-1}(V_n) \cap B) \setminus U_0 \in \mathcal{X}_{n+1}.
\]
But the same reasoning is true for \( V_{n+1} \), either there is some \( V_{n+2} \) or \( V_{n+1} = U_j \) for some \( j \). But there must be some number \( m \) such that \( V_m = U_j \) for some \( j \) because by Lemma 4.4 \( \mathcal{X}(n) \) is empty for \( n \geq N \) for some \( N \). Thus we get that \( x \in \lambda(U_j, U_0) \) and the result follows. \( \square \)

The graph \( \mathcal{X} \) should be thought of as an approximation of some minimal Cantor set \((C, f)\) where the vertices are clopen sets of \( C \) and the edges an approximation of \( f \). All but one of the vertices will only have one outgoing edge and thus the image of those sets under \( f \) is another vertices. But for the splitting edge \( 0_X \) there can be multiple outgoing edges and thus the image may not be one vertices. We solve this problem by creating a refinement of \( \mathcal{X} \).

Let \((C, f)\) be a minimal Cantor set and \( \mathcal{X}_n \) a combinatorial cover with the partition \( B_k \) and \( U_0 \in \mathcal{X}_n \) as splitting vertex. Then choose \( B_j \) such that
\[
\text{mesh}(B_j) < \min\{|V| : V \in \mathcal{X}_n\}
\]
and \( U_1 \in \mathcal{X}_n \) such that
\[
(f^{-1}(U_1) \cap B) \setminus U_0 = \emptyset
\]
for all \( B \in B_j \) which means that
\[
f^{-1}(U_1) \subset U_0.
\]
Now we can choose \( B_0 \in B_j \) such that
\[
U' = f^{-1}(U_1) \cap B_0 \subset U_0.
\]
Lastly we choose some clopen set \( U \subset U' \) and construct \( \mathcal{X}_{n+1} \) using \( U \) and \( B_j \). Observe that \( f(U) \subset U_1 \). We can also define a projection \( \pi_n : \mathcal{X}_{n+1} \to \mathcal{X}_n \) induced by inclusion. That is
\[
\pi_n(V) = U \iff V \subset U.
\]
The construction of $\pi_n$ was done such that it would be a combinatorial refinement.

**Definition 4.9.** Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a map between two combinatorial covers. Then we say that $\pi$ is a **combinatorial refinement** if

1. $\pi$ preserves the graph structure. That is if there is an edge from $V$ to $W$ in $\mathcal{X}$ then there is an edge from $\pi(V)$ to $\pi(W)$ in $\mathcal{Y}$;

2. $\pi(0_\mathcal{X}) = 0_\mathcal{Y}$;

3. There is an image $1_\mathcal{Y} \in \mathcal{Y}$ of $0_\mathcal{Y}$ such that

$$1_\mathcal{Y} = \pi(\{V : V \text{ is an image of } 0_\mathcal{X} \in \mathcal{X}\})$$

**Lemma 4.10.** The above constructed projection $\pi_n : \mathcal{X}_{n+1} \to \mathcal{X}_n$ is a combinatorial refinement.

**Proof.** 1. If there is an edge from $V$ to $W$ then there exist an $x$ such that

$$x \in f(V) \cap W.$$

Now we have that $V \subset \pi(V)$ and $W \subset \pi(W)$ thus we must have that

$$x \in f(\pi(V)) \cap \pi(W)$$

which means that there is an edge from $\pi(V)$ to $\pi(W)$. Thus $\pi$ preserves the graph structure.

2. The construction of $\mathcal{X}_{n+1}$ was such that $U$, the splitting vertex of $\mathcal{X}_{n+1}$ is a subset of the splitting vertex $U_0$ of $\mathcal{X}_n$. Thus we have that

$$\pi_n(U) = U_0.$$

3. From construction we know that $U_1$ is an image of $U_0$ such that $U \subset f^{-1}(U_1)$. Thus $f(U) \subset U_1$ so all images of $U$ are subsets of $U_1$ and

$$U_1 = \pi_n(\{V : V \text{ is an image of } U\}).$$

\[ \square \]

Using these projective maps we can construct a new Cantor set by

**Lemma 4.11.** The sequence of combinatorial covers $\mathcal{X}_n$ with projective maps $\pi_n$ constructed from a minimal Cantor sets makes a non-trivial discrete inverse limit.

**Proof.** Firstly we prove that $\pi_n$ is continuous and onto using that $\mathcal{X}_{n+1}$ is a refinement of $\mathcal{X}_n$. Let $V \in \mathcal{X}_n$ then since $\mathcal{X}_{n+1}$ is a refinement we know that there is a $U \in \mathcal{X}_{n+1}$ and $U \subset V$ and thus $\pi_n(U) = V$. The preimage of $V$ is a
union of sets in $X_{n+1}$ and all the sets in $X_{n+1}$ are open so the preimage must be open and $\pi_n$ is continuous.

Secondly the preimage of $\pi_n$ must contain at least two elements since each element in $X_n$ intersects at least two elements in the partition used to construct $X_{n+1}$. Thus for each element in $V \in X_n$ there is at least two elements in $X_{n+1}$ which are subsets of $V$.

Thirdly let $k > n$ and $U \in X_k$ and $V \in X_n$ such that
\[
\pi_{k,n}(U) = V.
\]
Next let $V_{n+1} \in X_{n+1}$ such that
\[
\pi_{k,n+1}(U) = V_{n+1}
\]
then we have that $U \subset V_{n+1}$ but we also know that $U \subset V$ so $V_{n+1} \cap V \neq \emptyset$ and $V_{n+1}$ can only be a subsets of one set in $X_n$ and thus $V_{n+1} \subset V$. Applying this reason inductively we get that
\[
\pi_{k,n}(U) = (\pi_{k,k-1} \circ \pi_{k-1,k-2} \circ \cdots \circ \pi_{n+1,n})(U).
\]
Thus we have that
\[
\mathcal{X} = \lim_{\leftarrow} X_n
\]
is a non trivial discrete inverse limit. \hfill \square

Now by applying Lemma 2.22 we get that $\mathcal{X}$ is a Cantor set.

Next we define a minimal function $g$ on $\mathcal{X}$. First we observe that we can not define a function from $X_n \to X_n$ by following the edges because the splitting vertex has multiple outgoing edges. But all the images of the splitting vertex are projected to the same element thus the function
\[
g_n(U) = \pi_n(\{V : V \text{ is an image of } U\})
\]
is a well defined function from $X_{n+1} \to X_n$. Then we defined the function $g : \mathcal{X} \to \mathcal{X}$ as
\[
g(U_n) = \{g_n(U_{n+1})\}_{n \geq 1}
\]

Proposition 4.12. Let $(C, f)$ be a minimal Cantor set and use it to construct $(\mathcal{X}, g)$. Then $(\mathcal{X}, g)$ is a minimal Cantor set.

In order to prove this proposition we use a lemma.

Lemma 4.13. Let $\mathcal{X} = \lim_{\leftarrow} X_n$ be an inverse limit of combinatorial covers and $g$ defined by the vertices. Then
\[
\overline{\mathcal{O}(\{U_n\}_{n \geq 1})} = \mathcal{X}
\]
for all $\{U_n\}_{n \geq 1} \in \mathcal{X}$ if and only if
\[
\overline{\mathcal{O}(\{0x_n\}_{n \geq 1})} = \mathcal{X}.
\]
Proof. For the first implication we have that
\[ \overline{O(\{U_n\}_{n \geq 1})} = \mathcal{X} \]
for all \( \{U_n\}_{n \geq 1} \) hence it is also true for \( \{0_{\mathcal{X}_n}\}_{n \geq 1} \).

For the second implication let \( \{U_n\}_{n \geq 1} \in \mathcal{X} \). Then for every \( U_{n_0} \in \mathcal{X}_{n_0} \) there is a path of length \( t_0 \) from \( U_{n_0} \) to \( 0_{\mathcal{X}_{n_0}} \). This gives us that
\[ g^{t_0}(\{U_n\}_{n \geq 1}) \in \mathcal{X}_{n_0} \]
and thus \( \{0_{\mathcal{X}_n}\}_{n \geq 1} \) is an accumulation point of
\[ \overline{O(\{U_n\}_{n \geq 1})} \]
and thus
\[ \overline{O(\{0_{\mathcal{X}_n}\}_{n \geq 1})} \subset \overline{O(\{U_n\}_{n \geq 1})} \]
and this gives us the desired result. \( \square \)

Now we can prove Proposition 4.12

Proof of Proposition 4.12. We start by proving that \( g \) is continuous. Let \( U \in \mathcal{X}_{n+1} \) and \( V \in \mathcal{X}_{n+1} \) such that there is an edge from \( U \) to \( V \). Then
\[ g_n(U) = \pi_n(V) \]
and since \( \pi_n \) is a combinatorial refinement there must be an edge from \( \pi_n(U) \) to \( g_n(U) \). This gives us that
\[ g_{n-1}(\pi_n(U)) = \pi_n^{-1}(g_n(U)) \]
and the diagram in Figure 6 commutes.

![Figure 6:](image)

So by Lemma 3.1 we get that \( g \) is continuous. Next let
\[ 0_{\mathcal{X}} = \bigcap_{n=1}^{\infty} 0_{\mathcal{X}_n} \]
we know this point exist because its a nested sequence of closed sets and its unique because the diameter goes to 0. Next we let \( \{U_n\}_{n \geq 1} \in \mathcal{X} \), then an
arbitrary open set from the basis of the topology on $X$ which contain $\{U_n\}_{n \geq 1}$ is on the form $\pi_{n_0}^{-1}(U_{n_0})$. Since $(C, f)$ is a minimal Cantor set there is a $m$ such that $f^m(0_{X}) \in U_{n_0}$. Then from the definition of $g$ we have that

$$f(0_{X}) \in g_{X_{n_0-m}}(0_{X_{n_0-m}}).$$

Thus by an inductive reasoning we get that

$$f^m(0_{X}) \in (g_{X_{n_0-1}} \circ \cdots \circ g_{X_{n_0-m}})(0_{X_{n_0-m}}).$$

But $X_{n_0}$ is a cover of disjoint sets and thus we must have that

$$(g_{X_{n_0-1}} \circ \cdots \circ g_{X_{n_0-m}})(0_{X_{n_0-m}}) = U_{n_0}$$

which gives us that

$$g^m(\{0_{x_n}\}_{n \geq 1}) \in \pi_{n_0}^{-1}(U_{n_0}).$$

Since this was an arbitrary open set containing $\{U_n\}_{n \geq 1}$ we get that $\{U_n\}_{n \geq 1}$ is a accumulation point of

$$O(\{0_{x_n}\}_{n \geq 1})$$

and since $\{U_n\}_{n \geq 1}$ was an arbitrary point in $X$ we get that

$$O(\{0_{x_n}\}_{n \geq 1}) = X.$$
Proof. Let \((X, g)\) be a combinatorially obtained minimal Cantor set obtained using \((C, f)\). Then we define a sequence of maps \(h_n : C \to X_n\) defined by

\[ h_n(x) = \{V\} \iff x \in V. \]

and the define the function \(h\) as

\[ h(x) = \{h_n(x_n)\}_{n \geq 1}. \]

![Diagram](image)

Figure 8: Commutative diagram of a Cantor set \(C\) and combinatorial covers.

Next we check that \(h\) is well defined. Let \(x \in V \in X_n\) and \(x \in W \in X_{n+1}\). Since \(X_{n+1}\) is a refinement of \(X_n\) there must be some set in \(X_n\) which \(W\) is a subset of moreover this set must be unique since \(X_n\) is a cover of disjoint sets and thus \(W\) can only intersect with one set in \(X_n\). We have that \(x \in V \cap W\) and therefor \(W \subset V\) and by the definition of \(\pi_n\) we get that \(\pi_n(W) = V\). Thus the diagram in Figure 8 commutes and \(h\) is a well defined function.

Now we show that \(h\) is a bijection. Each \(X_n\) is a cover of \(C\) and thus \(h_n\) is a surjective function and thus \(h\) is also surjective. To observe that \(h\) is an injection let \(x, y \in C\). Then from Lemma 4.4 we know that \(\text{mesh}(X_n) \to 0\) and thus there is a \(n_0\) with \(V, W \in X_{n_0}\) such that \(x \in V\) and \(y \in W\). This gives \(h_n(x) \neq h_n(y)\) and thus \(h(x) \neq h(y)\). Now we have that \(h\) is a bijection.

Next we observe that \(h\) and \(h^{-1}\) is continuous. Consider the set \(\pi^{-1}_k(V_k)\) which is in the basis of the topology on \(X\) and \(x \in C\), then

\[ h(x) \in \pi^{-1}_k(V_k) \iff h_k(x) = V_k \iff x \in V_k. \]

This gives us that

\[ h^{-1}(\pi^{-1}_k(V_k)) = V_k \]

and since \(V_k \in \mathcal{X}_k\) we get that \(V_k\) is open. Let \(B\) be from the basis of \(C\). Then there is a \(\mathcal{X}_k\) such that \(\text{mesh}(\mathcal{X}_k) < |B|\), then for every \(V \in \mathcal{X}_k\) either \(V \cap B = \emptyset\) or \(V \subset B\). Thus there is a collection \(\{V_j\}_{k \geq 1}\) of set from \(\mathcal{X}_k\) such that

\[ B = \bigcup_{j \geq 1} V_j. \]

Next we observer that

\[ h(V_j) = \pi^{-1}_k(V_j) \]
and thus the image of $B$ is a union of open set and $h^{-1}$ is continuous. Now we have that $h$ is a homeomorphism from $C$ to $X$.

Lastly we need to prove that $h$ conjugates $(C, f)$ to $(X, g)$. Let $x \in C$ and $h(x) = \{U_n\}_{n \geq 1}$, then $x \in U_n$ for all $n$. Then we know that for every $U_n$ there is an image $V_n$ such that $f(x) \in V_n$ but then we also have that $f(x) \in \pi_{n-1}(V_n)$. This gives us that if $x \in U_{n+1}$ then $f(x) \in g_n(U_{n+1})$ since $g_n(U_{n+1})$ is the projection of all the images of $U_{n+1}$ for all $n$. Thus we have that $h(f(x)) = \{g_n(U_n)\}_{n \geq 1}$ which gives us that the diagram in Figure 7 commutes and $h$ conjugates $(C, f)$ to $(X, g)$.

This proposition gives us an understanding of the structure of minimal Cantor by conjugating them to a combinatorially obtained minimal Cantor set and thus allowing us to use the graph structure of $X$. We mainly use this in order to define invariant measure on Cantor sets.

4.2 Minimal Cantor sets from Combinatorial covers

We know that we can use minimal Cantor set to get combinatorial covers. However the converse is not necessarily true. There are two things which can fail, either the combinatorial refinements gives a trivial inverse limit or the function is not minimal. We study two example of when both of the things fail and then develop theory for sufficient and necessary condition for when we get a minimal Cantor set.

Example 4.16. Consider the combinatorial cover $X$ which is just one vertices $V$ with one edge from $V$ to $V$. Then the identity map is a combinatorial refinement. This does not give us a Cantor set since the preimage of the projection only contain one element and thus we get a trivial inverse limit.

Example 4.17. The example with an inverse limit of combinatorial covers such that we do not get a minimal Cantor set is more involved. In order to make sure we get a Cantor set we need that the preimage of each projection contains at least to elements and in order to make sure that the function is not minimal we make a loop between two points. For the other edges the only thing we need is the they make a combinatorial cover. The simplest way to achieve this is to have one long loop going through all of them. Let $X_1$ be a combinatorial cover such that we have three vertices, $0_{X_1}, 1_{X_1}$ and $U$ with edges going from $0_{X_1}$ to $1_{X_1}$ and $1_U$ and two more edges from $1_{X_1}$ and $U_1$ back to $0_{X_1}$. In general $X_n$ contains $2^{n+1} - 1$ vertices with edges going from $0_{X_n}$ to $1_{X_n}$ and back and the other edges make a path from $0_{X_n}$ back to itself going through all the other edges see Figure 9.

In order to construct $X_{n+1}$ from $X_n$ let there be three vertices $0_{X_{n+1}}, V_1$ and $W_1$ which are projected to $0_{X_n}$ by $\pi_n$, two vertices $1_{X_{n+1}}$ and $U$ which are projected to $1_{X_n}$ and for every other vertex in $X_n$ there are two in $X_{n+1}$ which are projected to that vertex. For the edges there is one going from $0_{X_{n+1}}$ to
1,\mathcal{X}_{n+1} and one going back. Then there is an edge from 0,\mathcal{X}_{n+1} to \tilde{U} and one from \tilde{U} to V_1. Next we have a long loop going through all the other vertices. There is edges making a path from V_1 to V following the path of \mathcal{X}_n, an edge from V to W_1, edges making a path from W_1 to W again following the paths of \mathcal{X}_n and finally an edge from W to 0,\mathcal{X}_{n+1}. See Figure 10.

Now we can take the inverse limit and get that \mathcal{X} = \lim_{n \to \infty} \mathcal{X}_n is a Cantor set. For the function g on \mathcal{X} we have that

\[ g^2(\{0,\mathcal{X}_n\}_{n \geq 1}) = g(\{1,\mathcal{X}_1\}_{n \geq 1}) = \{0,\mathcal{X}_1\}_{n \geq 1} \]

which gives us that the orbit of (0,\mathcal{X}_n) is only 0,\mathcal{X}_n and 1,\mathcal{X}_n and thus (\mathcal{X}, g) is not a minimal system.

For the theory we start by formalising what we mean with a loop.

**Definition 4.18.** Let \mathcal{X} be a combinatorial cover then a **loop** is a path starting at any of the images of 0,\mathcal{X}_n and ending at 0,\mathcal{X}_n we use the notation

\[ \lambda(U, 0,\mathcal{X}_n) = \lambda(U). \]

Consider the inverse limit \mathcal{X} obtained from combinatorial covers \mathcal{X}_n and projective maps \pi_n together with the function g defined with the edges. Given \mathcal{X}_n we can label the loops by the images of 0,\mathcal{X}_n. Let L_n be the collection of images of 0,\mathcal{X}_n, that is U_1, \ldots, U_d_n where U_1 is the one containing g(\{0,\mathcal{X}_n\}_{n \geq 1}).

**Definition 4.19.** Let U_i \in L_n and V_j \in L_{n+1}, then the **winding matrix** W_n corresponding to \pi_n is the matrix with entries

\[ w_{ij} = \#\{T \in \lambda(V_j) : T \subset U_i\}. \]

The winding matrix counts how many times a loop in \mathcal{X}_{n+1} passes through a loop in \mathcal{X}_n. Using a product of winding matrices we can find how many times a loop in \mathcal{X}_{m+1} passes through a loop in \mathcal{X}_n.

**Lemma 4.20.** The winding matrix \[ W_{mn} \text{ corresponding to the projection } \pi_{m,n} \]

is given by

\[ W_{mn} = \prod_{j=n}^{m-1} W_j \]
Figure 10: $X_{n+1}$ with projections in Example 4.17
Proof. The proof is with induction. In the case \( m = n + 1 \) we have

\[
W_{n+1,n} = \prod_{j=n}^{n} W_j = W_n
\]

which is true by definition. Now we assume that the statement is true for some \( m \) and prove that it is true for \( m + 1 \). Let

\[
W_{mn} = \prod_{j=m}^{m-1} W_j
\]

with entries \( a_{ij} \) and \( W_m \) have entries \( b_{jk} \). Then for a loop \( W_k \in L_{m+1} \) we have that it passes \( b_{jk} \) times through the loop to \( V_j \in L_m \). Likewise the loop \( V_j \) passes through the loop \( U_i \in L_n \) a total of \( a_{ij} \) times. Thus the loop \( W_k \) must pass through the loop \( U_i \) a total of

\[
\sum_{j=1}^{\#L_m} a_{ij} b_{jk}
\]

times. But this is precisely the entries of the matrix

\[
W_{m+1,n} = \prod_{j=m}^{m} W_j.
\]

Thus by induction the lemma follows.

**Proposition 4.21.** Let \( \mathcal{X} \) be obtained by an inverse limit of combinatorial covers. Then \( (\mathcal{X}, g) \) is a minimal Cantor set if and only if for very \( n \) there is an \( m \) such that the first column of the winding matrix \( W_{mn} \) only contains numbers greater or equal to 2.

Proof. We start by assuming that \( (\mathcal{X}, g) \) is a minimal Cantor set. The general idea to prove that the first column of \( W_{mn} \) contains numbers greater or equal to 2 is not very difficult but the proof itself is very technical with a many indices. So we start by giving the general idea. Since \( (\mathcal{X}, g) \) is minimal the orbit of \( \{0_{\mathcal{X}_n}\}_{n \geq 1} \) is all of \( \mathcal{X} \). So if we fix a cover \( \mathcal{X}_n \) then \( \mathcal{O}(\{0_{\mathcal{X}_n}\}_{n \geq 1}) \) will intersect all of \( \mathcal{X}_n \). Thus we only need to look at the path of \( W_1 \in L_m \) for \( m \) sufficiently large. This will ensure that the loop of \( W_1 \) will pass trough all the sets in \( \mathcal{X}_n \), so the first column of \( W_{mn} \) will contain numbers greater or equal to 1. To show that the entries of the first column of \( W_{mn} \) are at least 2 we use that that \( \mathcal{X} \) is a Cantor set and thus we can find an integer \( m \) such that for every set in \( \mathcal{X}_n \) there are at least two sets in \( \mathcal{X}_m \).

Now we do the formal proof. Firstly we fix a cover \( \mathcal{X}_n \). Then there is a cover \( \mathcal{X}_{m'} \) such that for all \( U_i \in L_n \) there is at least two sets in \( \mathcal{X}_{m'} \) which are subset of \( U_i \). Let \( K \subset \mathcal{X}_{m'} \) contain all the subsets of the sets in \( L_n \). Next let \( m_i \) be the smallest integer such that

\[
g^{m_i}(\{0_{\mathcal{X}_n}\}_{n \geq 1}) = \{(g_n \circ \cdots \circ g_{n+m_i-1})(0_{\mathcal{X}_{n+m_i}})\}_{n \geq 1} \in \pi_{m'}^{-1}(V_i)
\]
we know such an integer exists because \((X, g)\) is minimal. In particular we have that
\[
(g_{m'} \circ \cdots \circ g_{m'+m_i-1})(0_{X_{m'+m_i}}) = V_i.
\]
Next let
\[
A_k^i = (g_k \circ \cdots \circ g_{m'+m_i-1})(0_{m'+m_i}).
\]
Then \(A_k^i \neq 0_{X_k}\) for all \(k = m', \ldots, m' + m_i - 1\). To get a contradiction suppose \(A_k^i = 0_{X_k}\) for some \(k\). Then we have that
\[
V_i = A_{m'}^i = (g_{m'} \circ \cdots \circ g_{k-1})(A_k^i) = (g_{m'} \circ \cdots \circ g_{k-1})(0_{X_k})
\]
which implies that
\[
g^k(\{0_{X_n}\}_{n \geq 1}) \in \pi^{-1}_{m'}(V_i)
\]
which is a contradiction to \(m_i\) being the smallest integer with this property.

Next let
\[
m = m' + \max_i m_i
\]
and consider the set \(W_1 \in L_m\). Since \(W_1\) is an image of \(0_{X_m}\) and the projective maps preserve the graph structure we get that
\[
\pi_{m,m'+m_i-1}(W_1) = g_{m'+m_i-1}(0_{X_{m'+m_i}}) = A_{m'+m_i-1}^i.
\]
Next we let \(B_2\) be the image of \(W_1\). By the same reasoning as above we get that
\[
\pi_{m,m'+m_i-2}(B_2) = A_{m'+m_i-2}^i.
\]
We continue this process and get a path \(\lambda(W_1, B_{m'})\). Observe that \(B_k \neq 0_{X_m}\) for any \(k\). Suppose by contradiction there is some \(k\) such that \(B_k = 0_{X_m}\). Then
\[
\pi_{m,m'+m_i-k}(B_k) = \pi_{m,m'+m_i-k}(0_{X_m}) = 0_{X_{m'+m_i-k}}
\]
but this implies that
\[
A_{m'+m_i-k}^i = 0_{X_{m'+m_i-k}}
\]
which is a contradiction.

This gives us that the path \(\lambda(W_1, B_{m_i})\) must be contained in the loop \(\lambda(W_1)\).

We also have that
\[
\pi_{m,m'}(B_{m_i}) = A_{m'}^i = V_i
\]
which implies that
\[
B_{m_i} \subset V_i.
\]
Thus the path \(\lambda(W_1, B_{m_i})\) intersects \(V_i\) and thus also the loop \(\lambda(W_1)\) intersect \(V_i\).

The above argument can be repeated for every \(V_i \in K\) giving us that the loop of \(W_1\) intersects every element in \(K\). Since each \(U_i \in L_n\) has at least two subsets in \(K\) we get that the loop of \(W_1\) must go through the loop of \(U_i\) at least twice. Then the first column of the winding matrix \(W_{mn}\) only contain numbers greater or equal to 2.
For the other implication we start by proving that $\mathcal{X}$ is a Cantor set. Since $\mathcal{X}$ is a discrete inverse limit we only need to prove that it is non trivial i.e for every $U_n \in \mathcal{X}_n$ there is an $m$ such that $\# \pi_{m,n}^{-1}(U_n) \geq 2$. Let $m$ be such that $W_{mn}$ then $V_1 \in L_m$ passes through $U_n$ at least twice and thus there must at least two sets in $\mathcal{X}_m$ which are subsets of $U_n$. Since $U_n$ was an arbitrary set we get that $\mathcal{X}$ is a Cantor set.

Next to prove that $(\mathcal{X}, g)$ is a minimal Cantor set. Let $\{U_n\}_{n \geq 1} \in \mathcal{X}$, then the open set containing $\{U_n\}_{n \geq 1}$ is on the form $\pi_{n_0}^{-1}(U_{n_0})$. Choose $m$ such that $W_{mn}$ is, thus there is a path of length $t$ going from $V_1 \in L_m$ to $V' \subset U_{n_0}$. This gives us that

$$(g_t \circ \cdots \circ g_{m+1})(0_{\mathcal{X}_m}) \subset U_{n_0}$$

and thus

$$g^t(0_{\mathcal{X}_m}) \in \pi_{n_0}^{-1}(U_{n_0})$$

and since $\{U_n\}_{n \geq 1}$ was an arbitrary point we get from Lemma 4.13 that $(\mathcal{X}, g)$ is a minimal Cantor set. \qed
5 Measures on Cantor sets

We now put the final structure on our Cantor sets by making them into measure spaces. The goal is to give a description of all ergodic measures on Cantor sets. We again follow the methods from [2]. Firstly we give some standard definitions related to measure. Secondly we give a description of all the probability measures which turns out to be a simplex of the ergodic measures. Finally we give a theorem about the number of ergodic measures on Cantor sets.

5.1 Definitions related to measures

Given some space $X$ we define measures on some subset of the power set of $X$. This subset is a $\sigma$-algebra.

**Definition 5.1.** Let $X$ be a set and $\mathcal{A}$ a collection of subsets of $X$. Then $\mathcal{A}$ is called a $\sigma$-algebra if

1. $X \in \mathcal{A}$;
2. if $E \in \mathcal{A}$ then $X \setminus E \in \mathcal{A}$;
3. if $E_i \in \mathcal{A}$ for $i = 1, 2, \ldots$ then $\bigcup_{i} E_i \in \mathcal{A}$.

Now we can define a measure.

**Definition 5.2.** Let $X$ be a space and $\mathcal{A}$ a $\sigma$-algebra on $X$. Then a measure $\mu$ is a function such that

1. $\mu : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$;
2. $\mu(\emptyset) = 0$;
3. If $\{E_i\}_{i \geq 1}$ is a collection of disjoint set in $\mathcal{A}$ then

$$\mu \left( \bigcup_{i} E_i \right) = \sum_{i} \mu(E_i).$$

**Definition 5.3.** We say that a measure is a probability measure if $\mu(X) = 1$.

With this we define a measure space.

**Definition 5.4.** The triplet $(X, \mathcal{A}, \mu)$ is called a measure space.

A measure of a set is always non negative but we will briefly be interested in signed measures which can give a negative value to some sets.

**Definition 5.5.** Let $\mu_1$ and $\mu_2$ be two measure on a $\sigma$-algebra $\mathcal{A}$ such that at least one of them is finite. Then

$$\mu = \mu_1 - \mu_2$$

is a signed measure.
Given a minimal Cantor set \((C, f)\) we have a function \(f : C \to C\) and the next two definitions take this function into consideration.

**Definition 5.6.** A set \(E\) is **invariant** if
\[
f^{-1}(E) = E.
\]

**Definition 5.7.** Let \((X, A, \mu, f)\) be a measure space with a function \(f : X \to X\). Then \(\mu\) is an **invariant measure** if
\[
\mu(f^{-1}(E)) = \mu(E)
\]
for all \(E \in A\).

**Definition 5.8.** Let \(\mu\) be an invariant measure. Then \(\mu\) is an **ergodic measure** if
\[
\mu(E) \in \{0, 1\}
\]
for all invariant sets \(E\).

One should think of an ergodic measure as measuring only one piece. Assume there are two invariant sets \(E\) and \(F\) with \(E \cap F = \emptyset\). Then if \(\mu\) is an ergodic measure the measure of \(E\) and \(F\) can not both be 1. Supposed \(\mu(E) = \mu(F) = 1\) then
\[
\mu(E \cup F) = \mu(E) + \mu(F) = 2 > 1 = \mu(X)
\]
which is a contradiction. Thus if \(\mu(E) = 1\) only sets which intersects \(E\) have measure greater than 0.

If one would start with a measure space \((X, A, \mu)\) one could be interested in functions \(f : X \to X\) such that \(\mu(f^{-1}(E)) = \mu(E)\). These functions are called invariant functions. In our case we start with a minimal Cantor set \((C, f)\) and thus already have a fixed function which is why we define invariant measure rather than invariant functions. The same remark hold for the definition of an ergodic measure.

### 5.2 Measures on Minimal Cantor sets

Let \((X, f)\) be a combinatorially obtained minimal Cantor set with \(X = \lim X_n\). The \(\sigma\)-algebra \(A\) we consider is the Borel \(\sigma\)-algebra generated by all the open sets. In order to find all ergodic measures on \(X\) and we start by considering the space \(\mathcal{M}(X)\) which contain all signed invariant measures on \(X\). To understand \(\mathcal{M}(X)\) we construct an inverse.

**Lemma 5.9.** Let \(\mu\) be a signed invariant measure on \(X_n\) and \(A \in X_n\). Then \(\mu(A)\) is uniquely determined by the measure on \(U_1, \ldots, U_d \in L_n\).

**Proof.** Since \(\mu\) is invariant we get that \(\mu(A) = \mu(f^{-1}(A))\). Formally for a point \(\{x_n\}_{n \geq 1} \in X\) the function \(f\) is defined as \(f(\{x_n\}_{n \geq 1}) = \{f_n(x_n)\}_{n \geq 1}\). Thus if we consider the image of an open set \(\pi_k^{-1}(\{x_n\}_{n \geq 1})\) under \(f\) we get
\[
f(\pi_k^{-1}(x_k)) = \{\{y_n\}_{n \geq 1} \in X : y_k = f_{k-1}(x_k)\} = \pi_{k-1}^{-1}(f_{k-1}(x_k)).
\]
This gives us that \( f^{-1}(A) \) is a union of some sets in \( X_{n+1} \). Let \( U \in X_{n+1} \) such that \( f(U) = A \) then we have that \( A \) is an image of \( \pi_n(U) \) but \( f^{-1}(A) \) contains all the sets \( U \) with this property. Thus if we let \( B_1, \ldots, B_m \in X_n \) be the sets such that \( A \) is an image of \( \pi_n(U) \) then we have that

\[
\begin{align*}
f^{-1}(A) &= \bigcup_{U \in X_{n+1}} \bigcup_{A \text{ image of } \pi_n(U)} U = \bigcup_{i=1}^{m} B_i. 
\end{align*}
\]

Then using the properties of a measure we get that

\[
\begin{align*}
\mu(A) &= \mu(f^{-1}(A)) = \mu(\bigcup_{i=1}^{m} B_i) = \sum_{i=1}^{m} \mu(B_i).
\end{align*}
\]

Continuing the above reasoning we get that

\[
\mu(A) = \sum_{i=1}^{k} U_i
\]

where \( U_1, \ldots, U_k \) corresponds to the loops which intersect \( A \).

With this in mind we can describe every measure on \( X_n \) using the measure on the sets in \( L_n \). Let

\[
\nu^n_j(A) = \sum_{U \in L_n} \chi_{\lambda(U)}(U)
\]

with \( U_j \in L_n \). Then \( \nu^n_j \) is an invariant measure since the preimage of every set in a loop \( \lambda(U_j) \) only contains one set in that loop. Next we let \( H_1(X_n) \) be the vector space over \( \mathbb{R} \) generated by the measures \( \nu^n_j \). Since Lemma 5.9 gave us that every measure is determined by the measure on the sets \( U_j \) we get that every measure is a linear combination of the measures \( \nu^n_j \) and thus a point in \( H_1(X_n) \).

Given a measure in \( H_1(X_n) \) we can construct a map \( \varphi : H_1(X_n) \to \mathbb{R}^{d_n} \) defined by

\[
\varphi \left( \sum_{j=1}^{d_n} \alpha_j \nu^n_j \right) = (\alpha_1, \ldots, \alpha_{d_n}).
\]

Then \( \varphi \) also has an inverse give by

\[
\varphi^{-1}(\alpha_1, \ldots, \alpha_{d_n}) = \sum_{j=1}^{d_n} \alpha_j \nu^n_j.
\]

So we can think of \( H_1(X_n) \) as \( \mathbb{R}^{d_n} \) where a point in \( \mathbb{R}^{d_n} \) is a measure in \( H_1(X_n) \).

The map \( \pi_n : X \to X_n \) induces a map

\[
(\pi_n)_* : \mathcal{M}(X) \to H_1(X_n)
\]

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defined as
\((\pi_n)_*(\mu) = \mu(\pi_n) = \mu^n\).

We say that \(\mu\) is projected to \(\mu^n\). One can also see this as restricting \(\mu\) to sets in the \(\sigma\)-algebra on \(X_n\). Likewise the map \(\pi_n : X_{n+1} \to X_n\) also induces a map in the same way.

**Lemma 5.10.** The map
\[(\pi_n)_* : H_1(X_{n+1}) \to H_1(X_n)\]
represented using the measure \(\nu^n_j\) is the winding matrix of \(\pi_n\),
\[(\pi_n)_* = W_n.

**Proof.** Let \(W_n\) be the winding matrix with entries \(w_{ij}\) and \(\mu^{n+1} \in H_1(X_{n+1})\), then
\[\mu^{n+1} = \sum_{j=1}^{d_{n+1}} \alpha_j \nu_j^{n+1}\]
and
\[(\pi_n)_*(\mu^{n+1}) = \mu^n = \sum_{i=1}^{d_n} \beta_i \nu_i^n.

We need to compute the \(\beta_i\) which is the same a computing the the measures of \(U_i \in L_n\). Let \(\{V_k\}_k\) be the collection of sets in \(X_{n+1}\) such that \(V_k \subset U_i\). Then we get
\[
\mu^n(U_i) = \mu^{n+1}(U_i) = \mu^{n+1}(\cup_k V_k) = \sum_k \mu^{n+1}(V_k) = \sum_k \sum_j \alpha_j \nu_j^{n+1}(V_k) = \sum_j \alpha_j \sum_k \nu_j^{n+1}(V_k).
\]
The sum \(\sum_k \nu_j^{n+1}(V_k)\) counts how many times the loop \(\lambda(\tilde{U}_j)\) with \(\tilde{U}_j \in L_{n+1}\) intersects \(U_i\) which is exactly the number \(w_{ij}\). This gives us that
\[
\mu^n(U_i) = \sum_j \alpha_j w_{ij}
\]
and thus
\[
\mu^n = \sum_{i=1}^{d_n} (\sum_j \alpha_j w_{ij} \nu^n_i)
\]
which gives us the desired result. \(\square\)

Next we observe that for \(\mu \in M(X)\)
\[
(W_n \circ (\pi_{n+1})_*)(\mu) = (\mu(\pi_n(\pi_{n+1}))) = \mu(\pi_n) = (\pi_n)_*(\mu)
\]
which makes the inverse limit
\[ \lim_{\leftarrow W_n} H_1(X_n) = \{ (\mu^n)_{n \geq 1} : W_n \mu^{n+1} = \mu \} \]
well defined and we can extend the map \((\pi_n)_*\) to a map
\[ \pi_* : \mathcal{M}(X) \to \lim_{\leftarrow W_n} H_1(X_n) \]
defined by
\[ \pi_*(\mu) = \{ \mu^n \}_{n \geq 1}. \]

We now consider the set \(I(X) \subset \mathcal{M}(X)\) consisting of all invariant measure and define
\[ H_1^+(X) = \left\{ \sum_{j=1}^{d_n} \alpha_j \nu_j^n : \alpha_j \geq 0 \right\} \]
which is the measures in \(H_1(X_n)\) or the points in \(\mathbb{R}^{d_n}\) with positive coordinates.

Now we consider the set \(I(X_n) = \{ W_n(W_{j,n} \mu_j^n) : \mu_j \geq 0 \} \)
where we have used that \(W_n\) is linear. Thus we get that every \(W_n(\mu)\) is a linear combination of the measures \(W_n(\nu_j^n)\) with non-negative coefficients. Thus we get that \(W_n(H_1^+(W_{n+1}))\) is a cone in \(H_1^+(X_n)\).

We now consider the set
\[ I(X_n) = \bigcap_{j=n+1}^{\infty} W_{j,n}(H_1^+(X)). \]
Recall the \(W_{mn}\) is the composition \(W_n W_{n+1} \ldots W_{n+1} W_n\). From the above observation we get that \(I(X_n)\) is a cone in \(H_1(X_n)\). We also observe that
\[ W_n(I(X_{n+1})) = W_n \left( \bigcap_{j=n+2}^{\infty} W_{j,n+1}(H(X_j)) \right) = \]
\[ = \bigcap_{j=n+2}^{\infty} W_{n} W_{j,n+1}(H(X_j)) = \bigcap_{j=n+1}^{\infty} W_{j,n}(H(X_j)) = I(X_n) \]
which allows us to define the inverse limit \(\lim_{W_n} I(X_n)\).

Finally we let \(\mathcal{P}(X) \subset \mathcal{I}(X)\) be the set of probability measure on \(X\) and \(\mathcal{P}(X_n) \subset I(X_n)\) be the probability measure in \(I(X_n)\).
Lemma 5.11. The map

$$\pi^*: \mathcal{I}(X) \to \lim_{W_n} I(X_n)$$

is an isomorphism. In particular the map

$$\pi^*: \mathcal{P}(X) \to \lim_{W_n} P(X_n)$$

is an isomorphism.

Proof. See section 3 in [2].

It could be the case that the measure $\nu^n_j$ is not a probability measure since

$$\nu^n_j(X) = \# \{ U \in \lambda(U_j) \} \geq 1$$

and thus we normalise these measure by letting

$$\mu^n_j = \frac{1}{t^n_j} \nu^n_j$$

where $t^n_j$ is the length of the loop $\lambda(U_j)$. Let $P_n \subset H^+_1(X_n)$ be the set of probability measures and if $\mu$ is a probability measure we get that

$$(\pi_n)_* \mu(X) = \sum_{j=1}^{d_n} \alpha_j \mu_j^n(X) = \sum_{j=1}^{d_n} \alpha_j = 1$$

thus $P_n$ is a convex hull of the measures $\{\mu^n_j\}$.

 Proposition 5.12. Let $(X, f)$ be a combinatorially obtained minimal Cantor set such that $d_n \leq d$. Then there is at most $d$ ergodic measure on $(X, f)$. 
The proof this proposition is based on a number of lemmas on of which uses Birkhoff ergodic theorem. We first state a simplified version of Birkhoff ergodic theorem.

**Proposition 5.13.** Let \((X, \mathcal{A}, \mu, f)\) be a measure space such that \(\mu\) is an ergodic measure. Then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(f^k(x)) = \mu(A)
\]
for \(\mu\) a.e.

For a full statement of the theorem and a proof see [5].

Using Birkhoff we can show that there is invariant sets.

**Lemma 5.14.** Given two different ergodic measure \(\mu_1\) and \(\mu_2\) there exists two disjoint invariant sets \(B_1\) and \(B_2\).

**Proof.** Since \(\mu_1\) and \(\mu_2\) are different measure there exists a set \(A\) such that
\[
\mu_1(A) \neq \mu_2(A).
\]

Then we let
\[
B_i = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(f^k(x)) = \mu_i(A) \right\}
\]
for \(i = 1, 2\). Then \(B_i\) is nonempty by Birkhoff ergodic theorem. To see that \(B_i\) is invariant we make the following computations. Let \(y \in f^{-1}(B_i)\) then
\[
\frac{1}{n} \sum_{k=0}^{n-1} 1_A(f^k(x)) = \frac{1}{n} \left( 1_A(y) + \sum_{k=0}^{n-2} 1_A(f^k(x)) \right) \leq \frac{1}{n} 1_A(y) + \frac{1}{n-1} \sum_{k=0}^{n-2} 1_A(f^k(x)) \to \mu_i(A)
\]
as \(n \to \infty\) and
\[
\frac{1}{n} \left( 1_A(y) + \sum_{k=0}^{n-2} 1_A(f^k(x)) \right) \geq \frac{1}{n} 1_A(y) + \frac{1}{n} \left( \sum_{k=0}^{n-1} 1_A(f^k(x)) - 1 \right) \to \mu_i(A)
\]
as \(n \to \infty\). Thus \(B_i\) is invariant. Finally to assume that \(y \in B_1 \cap B_2\). Then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(f^k(y)) = \mu_1(A)
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(f^k(y)) = \mu_2(A)
\]
which implies that
\[
\mu_1(A) = \mu_2(A)
\]
and that is a contradiction and thus \(B_1 \cap B_2 = \emptyset\). \qed
From Lemma 5.11 we already know that all probability measure is projected onto the space \( P(X_n) \). The next three lemmas describe this space \( P(X_n) \) and where in \( P(X_n) \) ergodic measures are projected.

**Lemma 5.15.** The set \( P(X_n) \) is the convex hull of at most \( d \) points.

*Proof.* Let \( m > n \) and \( P_m \) the set of probability measure in \( H_1(X_m) \). We know that \( P_m \) is the convex hull of the measures \( \{ \mu^n \} \) which there is at most \( d \) of. Then the set \( P^n_m = W_{mn}(P_m) \) is the convex hull of \( \mu^m_j = W_{nm}(\mu^n_m) \). Now we can realise the measures \( \mu^m_j \) as points in \( \mathbb{R}^d \) as

\[
\varphi(\mu^m_j) = a_m \in [0, 1]^d
\]

then we can take the limit as \( m \to \infty \) and get that \( a_m \to a \in [0, 1]^d \) by possibly taking a subsequence. Then we must have that the measure \( \mu^m_j \) converges to a measures \( \mu_j \in P_n \). Next we have that \( P(X_n) = \cap P^n_m \) by Lemma 5.11 thus \( P(X_n) \) is the convex hull of the measure of at most \( d \) points. \( \square \)

**Lemma 5.16.** Only ergodic measure can be projected onto extremal points of \( P(X_n) \).

*Proof.* Fix an \( n \) and assume that \( \mu \) is a probability measure which is not ergodic. Then there is an invariant set \( B \) such that

\[
0 < \mu(B)
\]

and

\[
0 < \mu(X \setminus B).
\]

Next we let

\[
\mu_1 = \frac{1}{\mu(B)} \mu|_B
\]

and

\[
\mu_2 = \frac{1}{\mu(X \setminus B)} \mu|_{X \setminus B}
\]

then both \( \mu_1 \) and \( \mu_2 \) are probability measure and \( \mu \) is a linear combination of \( \mu_1 \) and \( \mu_2 \). Using Lemma 5.11 we get that both \( \mu_1 \) and \( \mu_2 \) are projected into \( P(X_n) \). Then using the linearity of \( W_n \) we have that

\[
W_n(\mu) = \mu(B)W_n(\mu_1) + \mu(X \setminus B)W_n(\mu_2)
\]

thus \( \mu \) can not be projected onto a extremal point. \( \square \)

**Lemma 5.17.** Ergodic measure are always projected onto extremal points.

*Proof.* Assume that \( \mu \) is projected onto \( \mu^{n+1} \) then there is \( \alpha_1, \ldots, \alpha_d \) such that

\[
\mu^{n+1} = \alpha_1 \mu_1^{n+1} + \cdots + \alpha_d \mu_d^{n+1}
\]

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with at least two of $\alpha_1 \ldots, \alpha_d$ non zero. Applying the map $W_n$ yields

$$
\mu^n = W_n(\alpha_1^{n+1} \mu_1^{n+1}) + \cdots + W_n(\alpha_d^{n+1} \mu_d^{n+1}) = \alpha_1^{n+1} \mu_1^n + \cdots + \alpha_d^{n+1} \mu_d^n
$$

thus $\alpha_i^n = \alpha_i^{n+1}$ for all $n$. From Lemma 5.16 we have that each $\mu^n$ is the projection of a different ergodic measure with support on different invariant sets. Thus $\mu$ must have support on at least two disjoint invariant sets and therefore $\mu$ can not be ergodic.

Now the proof of Proposition 5.12 follows.

**Proof of Proposition 5.12.** Assume there is more than $d$ ergodic measure. Since all the ergodic measure are projected onto extremal points of $P(X_n)$ which there is a most $d$ of, we get that at least two measure $\mu_1$ and $\mu_2$ must always be projected onto the same element in $P(X_n)$ for all $n$. Then they must be the same element in

$$
l_\lim W_n P(X_n)
$$

and by Lemma 5.11 they must be the same element in $\mathcal{P}(X)$ and thus they must be the same measure which is a contradiction. Thus there is at most $d$ measure.
References


