Enabling Variable Phase-Pole Drives with the Harmonic Plane Decomposition

GUSTAF FALK OLSON\textsuperscript{1}, (Student Member, IEEE), YIXUAN WU\textsuperscript{2}, (Student Member, IEEE), OMER IKRAM UL HAQ\textsuperscript{3}, (Member, IEEE), and LUCA PERETTI\textsuperscript{4}, (Senior Member, IEEE)

\textsuperscript{1}\textsuperscript{2}\textsuperscript{4}Division of Electric Power and Energy Systems, Royal Institute of Technology, Stockholm, Sweden (e-mail: \{gufo, yixuanw, lucap\}@kth.se)

\textsuperscript{3}ABB Corporate Research, Västerås, Sweden (e-mail: omer.ikramulhaq@se.abb.com)

Corresponding author: Gustaf Falk Olson (e-mail: gufo@kth.se).

**ABSTRACT** Magnet-free variable phase-pole machines are competitive alternatives in electric vehicles where torque-speed operating region, reliability, cost, and energy efficiency are key metrics. However, their modeling and control have so far relied on existing fixed-phase and pole-symmetrical models, limiting their drive capabilities, especially when switching the number of poles on the fly. This paper establishes the harmonic plane decomposition theory as a space-discrete Fourier transformation interpretation of the Clarke transformation, decomposing all pole-pair fields into a fixed number of orthogonal subspaces with invariant parameters. The model remains unaltered for all phase-pole configurations, guaranteeing continuity even under phase-pole transitions. Relations of the state and input space vectors, and model parameters to those of the vector space decomposition theory used for multiphase machines are established via the use of the complex winding factor. Experiments confirm the modeling theory and demonstrate its practical usefulness by performing a field-oriented-controlled phase-pole transition. Non-trivial configurations with more than one slot/pole/phase and a fractional phase number are also demonstrated.

**INDEX TERMS** discrete Fourier transform, harmonic plane decomposition, multiphase electric machines, variable phase-pole machine, vector space decomposition

**Nomenclature**

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<td>Pitch angle between winding groups or slots</td>
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<td>$\ell$</td>
<td>Indicator for magnetic axes spacing</td>
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<td>Maximum resolvable space-harmonic order</td>
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I. INTRODUCTION

THE so-called variable phase-pole machines (VPPMs) are a subgroup of multiphase machines (MPMs) that are capable of varying the number of magnetic poles and phases by controlling the current in independent stator-winding groups, without the need for hardware reconfiguration [1], [2]. Their main advantage is the ability to extend the torque-speed region or improve energy efficiency by reducing the number of magnetic poles for higher speeds, analogously to an electric gearbox [3]–[5]. Thus, the phase-pole configuration (PPC) can be chosen to fulfill an optimality criterion, e.g. maximum torque per ampere (MTPA) [2].

The benefits of VPPMs have been studied in various papers. Already in 1997, Osama and Lipo identified VPPMs as an interesting candidate for electric vehicles as they envisioned them to extend the torque-speed operating range [3]. Since then, a case study in [4] demonstrated that a VPPM accomplishes better energy efficiency in the high-speed, low-torque operating range (corresponding to cruising speed in automotive applications) and has comparable torque density to an interior permanent-magnet synchronous machine (IPM). Reference [5] compares key performance metrics of a phase-pole-configurable driveline to that of a fixed-pole, three-phase counterpart. The most remarkable results from the case study are that the combined machine-inverter losses in the phase-pole-configurable drive were reduced by 45% at partial load conditions, while the DC-bus capacitance shrunk by more than 60%. Although synchronous VPPM conceptions are possible [6], a common denominator of the aforementioned VPPMs is that they are of induction type with a squirrel cage rotor and thus free of magnets. In traction applications, a common demur regarding induction machines is their assumed lower energy efficiency compared to IPMs. However, over the entire driving cycle of an electric vehicle, even a fixed phase-pole induction machine can be on par with an IPM [7]. The mentioned virtues are all desirable in traction applications, making VPPMs a contender in this area.

The vector space decomposition (VSD) is the state-of-the-art method for modeling space-vectors in MPMs [8]. In the VSD, time-domain space-vector quantities map into orthogonal vector spaces. Hence, time-harmonic components are separated into well-defined subspaces. The virtue of the VSD is that one harmonic component per subspace can be controlled independently. Therefore, conventional field-oriented control algorithms of three-phase machines can be applied in each subspace [9].

A shortcoming is that the number of vector spaces, \([m_s/2]\), increases with the number of phases, \(m_s\). As a result, the number of controllers in an MPM with phase-changing capabilities ought to change with the number of phases [10]. That the number of subspaces depends on the number of phases stems from the fact that the VSD assumes \(m_s\) fixed magnetic axes. This is viable when the currents in a phase belt, comprising multiple stator slots, are equal.

On the contrary, VPPMs do not necessarily generate equal and opposite currents in the slots shifted by \(\pi\) rad electrically. For instance, this is the case for toroidally-wound VPPMs by their construction [2], [10]. Such VPPMs provide the highest torque density among different winding types, according to [11]. Moreover, the number of magnetic axes depends on the excitation in a VPPM.

An attempt to fixate the dimension of the Clarke matrix is suggested in [12], where the model relies on the conventional VSD and, therefore, uses a weighted sum of the different Clarke and parameter matrices originating from different PPCs. This could be used for low numbers of viable PPCs but becomes more complicated as the number grows. A key concern with this approach is its underlying modeling lapse: it artificially averages transitions between different PPC models using an arbitrary weighting factor. This cannot represent the actual physical behavior of the electric machine.

Another method using so-called phase-pole modulation (PPM) is proposed in [13], [14] to perform VSD-based control of VPPMs. However, PPM as presented in [5] is inflicted by the limitation that the ”pole count influences the model because the Clarke transformation matrix must be adapted and because model parameters […] need to be adjusted” [5, Sec. II.B]. Although PPM as presented in [14] intrinsically allows for simultaneous modeling and control of multiple pole-pair fields, two questions linger. Firstly, a description...
and working control of PPCs with \( q_s > 1 \) slots per pole per phase is yet to be demonstrated. Secondly, the VSD assumes half-wave symmetry of the stator current distribution, making the strategy impractical for machines with individually controlled stator slots. Moreover, high-order harmonics can easily be excited in VPPMs just as in MPMs due to the low impedance of these subspaces [15], and they should not be ignored as in [14].

In summary, the VSD does not offer a straightforward way to actively control the currents in adjacent slots belonging to the same phase belt, especially during PPC transitions. A generalized VSD, termed the harmonic plane decomposition (HPD), was introduced in [10]. It establishes a model with a fixed number of subspaces which in practice results in a fixed number of current controllers. This model is consistent independently of the PPC, de facto removing the inherent discontinuity problem of VSD-based modeling for different PPCs. The HPD accomplishes this by considering all possible magnetic pole numbers in isolation. The number of subspaces is \( \lfloor Q_s/2 \rfloor \) for a \( Q_s \) stator-slot VPPM with independent coil excitation. It is thus only dependent on \( Q_s \). Similarly to the VSD, each harmonic plane in the HPD can be described independently with a lumped parameter model, such as the inverse-\( \Gamma \) or T-equivalent model. In fact, the VSD and HPD parameters are related, as detailed in [16].

An example of the issues related to VSD-based modeling of VPPMs is shown in Fig. 1 (reproduced from [17]). The dotted lines represent the simulated VSD currents for an open-loop transition between \( [m_s = 3, p = 1] \) and \( [m_s = 3, p = 2] \), where the parameterized model corresponding to \( p = 1 \) is used until \( t = 1 \) when it is abruptly replaced by the \( p = 2 \) model. The upper subplot displays the currents in six adjacent slots of a \( Q_s = 36 \)-slot machine.

In an electrical machine where the inductances enforce continuous currents, the profiles of the VSD-modeled currents in the two lower subplots are non-physical due to the inherent discontinuity of the approach. The addition of a weighted average between the \( p = 1 \) and the \( p = 2 \) models, like in [12], will only add arbitrariness. For comparison, a different response is overlapped in the plots by the continuous HPD-model which is examined in this paper. Here, the slot-current transition (especially visible in the second sub-plot) is physically achievable.

This paper aims to give an in-depth and machine-independent derivation of the HPD presented in [10]. It includes theoretical interpretations missing in previous articles, and highlights differences to the VSD; especially when applying the two decompositions to VPPMs. Thereby, it serves to enable VPPM drives independent of machine and winding topology. Examples are provided to show how the HPD is implemented in an indirect rotor field oriented control (IRFOC) to produce desired PPCs and perform phase-pole transitions. For these purposes, the paper is organized as follows.

Section II derives the HPD from the discrete Fourier transformation (DFT) theory and juxtaposes it to the VSD. Section III presents the links between the VSD and the HPD. Particularly, we describe how time harmonics in the subspaces of the VSD map into the subspaces, representing space harmonics, of the HPD. Examples are provided to demonstrate how the HPD acts as a time-to-space transformation, its relation to complex winding factors, and how to perform parameter transformations between reference frames. Section IV provides experimental support and highlights the practical usefulness of the HPD by performing a controlled PPC transition, and by producing a PPC with \( q_s > 1 \), which requires the generation of non-trivial (unbalanced) voltages previously not shown in the literature for VPPMs. Section V summarizes the main findings.

II. THEORY

This section starts by reviewing the fundamental reference frame introduced in [18] and extends it to comprise machines without pole symmetry. Next, the VSD is reviewed and interpreted as a DFT while its shortcomings are highlighted. The HPD is then developed and scrutinized. Important terminology is introduced throughout this section.

A. THE 123 FUNDAMENTAL REFERENCE FRAME

Reference [18] introduces the fundamental 123 reference frame as a means of generalizing the modeling of arbitrary \( m_s \)-phase machines. It modifies the classical Fortescue symmetrical-component modeling by introducing a simple configuration matrix that maps the machine’s magnetic axes into a half-plane, inferring pole symmetry. In contrast to the practical \( abc \) reference frame, which sorts the magnetic axes by their electrical phase shift, the 123 fundamental reference frame sorts the magnetic axes by their physical location in the stator or rotor. The difference can be expressed mathematically by introducing the complex rotational operator \( q \) and a winding-configuration function \( p \) (explained later), as in (1). They determine the position of the magnetic axis for
the space-vector quantity $x$.

$$x_{(k+1)} = p(k+1)x_{(k+1)}(t) \cdot \alpha^k$$

$$k \in \{0, 1, \ldots, m_s - 1\}, \quad \alpha = e^{j \frac{2\pi}{m_s} t} \quad (1)$$

In the practical abc reference-frame, the magnetic axes are separated by $2\pi/m_s$ ($\ell = 1$), whereas they are separated by $\pi/m_s$ ($\ell = 2$) when the fundamental 123 reference-frame is adopted. $x_{(k+1)}$ represents the instantaneous value of the phase quantity producing the flux whose direction in electrical radians is determined by $\alpha^k$ when $p(k+1) \equiv 1$. Thus, $k = 0$ corresponds to phase $a$, $k = 1$ phase $b$, etc. Reference [18] points out that $\ell = 1$ poses a problem when developing a generic electromagnetic machine model because an even phase-order machine will have linearly dependent magnetic axes. For example $\alpha^k = -\alpha^{(k+m_s/2)}$ whenever $m_s$ is even, so the space vectors $x_{(k+1)}$ and $x_{(k+1+m_s/2)}$ oppose each other.

If the machine coils are arranged such that slots $k = q_s (n - 1 + 2m_s(p - 1) + q)$ carry the current $i_k$ and slots $k = q_s (n - 1 + m_s(2p - 1) + q)$ for $\text{cyl} \in \{1, \ldots, p\}$, $n \in \{1, \ldots, m_s\}$, $q \in \{1, \ldots, q_s\}$ carry the return currents, pole symmetry is guaranteed and the direction of the magnetic field space-vector in the $\pi$-symmetrical model can be incorporated by a scalar $p_n \in \{-1, 1\}$. This function is referred to as the winding configuration function in [18] and simply reflects phase quantities in the lower half-plane into the upper half-plane:

$$p_n = \begin{cases} -1 & \forall n \text{ such that } \frac{2\pi n}{m_s} > \pi \\ 1 & \text{else} \end{cases}$$

This machine model is capable of accommodating any arbitrary phase number and winding configuration [18].

If the currents in the slots separated by one pole pitch are not equal and opposite, on the other hand, even the model proposed in [18] fails, due to the asymmetry. However, the 123 reference frame can be used to our benefit with a different interpretation. When the symmetry assumption is disposed of, we may choose to define sampling points around the entire machine circumference (rotation around its symmetry axis) separated in space by the angle $\delta = 2\pi/n_{mw}$. Here, $n_{mw}$ is introduced to denote the number of independent minimum windings, which is interpreted as the number of independent current measurements. Hence, if each slot of the machine is independently excited, $n_{mw} = Q_s$, as an example. Simply put, the fundamental 123 reference frame is expanded to the full stator circumference. Now, the index $k \in \{1, \ldots, n_{mw}\}$ represents a sampling point in space of the space-continuous variable $x_\delta$:  

$$x[k+1] = x(\delta k) \quad (2)$$

For now, we only acknowledge that the extension of the 123 reference-frame to the full circumference lends itself very well to VPPMs as the physical locations of the sampling points remain independent of the PPCs. Furthermore, no assumption on symmetry is necessary. Section II-B continues by showing that this sampling interpretation applies also to the ordinary Clarke transformation.

The difference between the abc and the $2\pi$-symmetrical 123 reference-frames for the $[m_s = 6, p = 1]$ configuration with $n_{mw} = 12$ is depicted in Fig. 2.

**B. THE VECTOR-SPACE DECOMPOSITION AS A DISCRETE FOURIER TRANSFORMATION**

The Clarke transformation matrix of the VSD transforms quantities from the fundamental 123 to the stationary $\alpha\beta0$ reference frame. Equation (3) defines it and its inverse $T_{123\rightarrow\alpha\beta0}$ and $T_{\alpha\beta0\rightarrow123}$, respectively [18]. The pitch angle $\delta$ is the spatial phase shift between the magnetic axes in the $\pi$-symmetrical fundamental 123 reference-frame. For an amplitude-invariant transformation, $K = 1$, and for a power-invariant transformation, $K = 1/2$.

$$\bar{x}_{\alpha\beta0} = \left(\frac{2}{m_s}\right)^K C \cdot \bar{x}_{123} \quad \bar{x}_{123} = \left(\frac{2}{m_s}\right)^{1-K} C^T \cdot \bar{x}_{\alpha\beta0}$$

$$C = \begin{bmatrix} \cos(0\delta) & \cos(1\delta) & \cos(2\delta) & \ldots & \cos((m_s - 1)\delta) \\ \sin(0\delta) & \sin(1\delta) & \sin(2\delta) & \ldots & \sin((m_s - 1)\delta) \\ \cos(0\delta) & \cos(3\delta) & \cos(6\delta) & \ldots & \cos((m_s - 1)3\delta) \\ \sin(0\delta) & \sin(3\delta) & \sin(6\delta) & \ldots & \sin((m_s - 1)3\delta) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos(0\delta) & \cos(\xi\delta) & \cos(2\xi\delta) & \ldots & \cos((m_s - 1)\xi\delta) \\ \sin(0\delta) & \sin(\xi\delta) & \sin(2\xi\delta) & \ldots & \sin((m_s - 1)\xi\delta) \end{bmatrix}$$

$$\delta = \pi/m_s \quad (3)$$

In the VSD, each phase is represented by one magnetic axis, leading to $[m_s/2]$ odd-order vector spaces in a $m_s$-phase machine [8]. These vector spaces are denoted $\nu \in \{1, 3, \ldots, m_s\}$. In case $m_s$ is an odd number, the ultimate vector space is a one-dimensional axis. The vector spaces, $\nu$, are electromagnetically decoupled as they represent different harmonic orders (pole-pair numbers).
It is possible to reformulate the $\alpha-$ and $\beta-$ components for one vector space, $\nu$, as in (4) using an amplitude-invariant transformation, $K = 1$.

$$x_{\alpha,t}(t) = \frac{2}{m_s} \sum_{k=0}^{m_s-1} x_{k+1}(t) \cos(k\nu \delta)$$

$$x_{\beta,t}(t) = \frac{2}{m_s} \sum_{k=0}^{m_s-1} x_{k+1}(t) \sin(k\nu \delta)$$

In (4), the space-vector quantity $\bar{x}_{123}$ is considered at one time instant, $t$. The $(k + 1)^{th}$ element of $\bar{x}_{123}(t) = [x_1(t) x_2(t) \ldots x_{m_s}(t)]^T$ is the instantaneous value of $x_{k+1}(t)$ at the electrical angle $\theta_s = k\delta$, which is the location of the $(k + 1)^{th}$ magnetic axis. For example, $x_1(0)$ on the right side of (4) is the value of the space vector at location 1, corresponding to $\theta_s = 0$, at time $t = 0$ (see the left plot of Fig. 2). To simplify (4), (5) introduces the complex notation of the space vector $\bar{X}_{\alpha\beta0,\nu}$. It may also be represented in polar form with a modulus $\bar{X}_{\alpha\beta0,\nu}$, and phase $\gamma_{\nu}$. Using (4) and (5), it is shown in (6) that the VSD transformation is equivalent to a single-sided DFT of $\bar{x}_{123}$, $\tilde{\mathcal{F}}(\bar{x}_{123})$.

$$\bar{X}_{\alpha\beta0,\nu}(t) = x_{\alpha,t}(t) + jx_{\beta,t}(t) = \bar{X}_{\alpha\beta0,\nu} e^{j(\nu \pi f_s t + \gamma_{\nu})}$$

$$\tilde{\mathcal{F}}(\bar{x}_{123}(t)) = \bar{X}_{\alpha\beta0,\nu}(t) = \frac{2}{m_s} \sum_{k=0}^{m_s-1} x_{k+1}(t) e^{j(\pi k \nu / m_s)}$$

$$\nu \in \{1, 3, \ldots, m_s\}$$

The VSD samples the $m_s$-phase quantities, i.e. magnetic axes, along the machine circumference with an evenly distributed sampling interval of $\delta$. In fact, the VSD samples the compounded space vectors once at the magnetic symmetry axis of each phase. In this way, it uses electric angles and assumes half-wave symmetric windings, with equal magnitude but opposite polarity of the quantity in two slots shifted by $\pi$ electrical radians. The spectrum of a half-wave symmetric signal consists only of odd harmonics, corresponding to $[m_s/2]$ odd vector spaces up to the number of phases, $m_s$.

### C. EXTENSION OF THE VECTOR-SPACE DECOMPOSITION

Relating to the discussion about the $2\pi$-symmetric fundamental 123 reference-frame in Section II-A, the DFT interpretation of the VSD allows a reformulation of the Clarke transformation by using mechanical angles without any assumptions about the windings. The resulting Clarke transformation matrix is formed by taking all independent windings into account. In the case where each slot winding is individually wound, there are $Q_s$ independent windings. Substituting $m_s \mapsto Q_s$, (6) becomes (7). We refer to (7) as the harmonic plane decomposition.

$$\bar{X}_{\alpha\beta0,h}(t) = X_{\alpha\beta0,h} e^{j(\nu \pi f_s t + \gamma_{h})}$$

$$\bar{X}_{\alpha\beta0,h}(t) = \left\{ \begin{array}{ll}
\frac{2}{Q_s} \sum_{k=0}^{Q_s-1} x_{k+1}(t) e^{j(\nu \pi k \delta / Q_s)}, & 0 < h < \frac{Q_s-2}{2} \\
\frac{1}{Q_s} \sum_{k=0}^{Q_s-1} x_{k+1}(t)(-1)^{2h/k}, & h \in \{0, \frac{Q_s-2}{2}\}
\end{array} \right.$$ 

(7)

Rather than consisting of $[m_s/2]$ vector spaces, $\nu$, the HPD generates $Q_s/2 + 1$ harmonic planes, $h \in \{0, 1, 2, \ldots, [Q_s/2]\}$. They exist for all harmonic orders up to $[Q_s/2]$, i.e. both odd and even, and a DC-component, $h = 0$. It is noted that the DC component $h = 0$ and the homopolar component $h = [Q_s/2]$ are axes and in case the number of independent windings is odd, only the DC component is retained. This is a consequence of the two-sided DFT that (7) constitutes. Frequencies $h_1 = \{1, \ldots, Q_s/2 - 1\}$ and $h_2 = \{Q_s/2 + 1, \ldots, Q_s - 1\}$ are complex conjugate pairs with $\bar{X}_{\alpha\beta,h_1} = \bar{X}_{\alpha\beta,h_2}$ for a real-valued signal, whereas the components at $h = 0$ and $h = Q_s/2$ only exist once [19, p.12]. Hence, the different scalar multipliers in (7). The HPD Clarke transformation is formulated in (8) for the more general case of $n_{m_{wu}}$ independent windings. It corresponds to a real-valued Fourier matrix [19, (1.17)] whose harmonic order has been reduced to the Nyquist frequency, $[n_{m_{wu}}/2]$.

Each harmonic plane describes one space harmonic along the air gap. The extended sampling from $\pi$ to $2\pi$ corresponds to a longer sampling length and results in narrower frequency bins in the spectrum compared to the half-wave symmetric VSD. This is equivalent to an increase in frequency resolution of the DFT. Furthermore, the sampling frequency is increased if $n_{m_{wu}} > 2m_s$ leading to a higher Nyquist frequency. Consequently, higher-order harmonics are resolved.

Although the sum of currents into the neutral point is zero, the sampled stator currents may not necessarily total zero in the defined current direction. This is particularly relevant in toroidally wound VPPMs, where slots may conduct time-alternating currents, generating a space-domain DC component. This results in a circulating flux in the stator back, as shown in Fig. 3, which does not cross the air gap and cannot contribute to useful torque. However, it impacts steel saturation and induces Joule losses from the current inflicting the flux. Despite this, like the VSD approach, we assume magnetic linearity and overlook the potential saturation of the stator back iron.
D. ROTOR MODELLING

To obtain a full parametric model of the MPM, e.g. the inverse-Γ or the T-model in Fig. 4, rotor quantities must be referred to the stator side and transformed to the stationary αβ0 and rotating dθ0 reference-frames.

Whereas [20] adopts a transformation using complex base functions to transform rotor quantities to three stator phases similar to the Fortescue transformation, (9) develops the idea further to the rotor side Clarke transformation, C_r. On the rotor side, a 2π-symmetric fundamental 123r reference frame encompasses all rotor slots. In a squirrel-cage rotor, the rotor slot pitch determines the sampling interval, δ_r = 2π/Q_r, whereas δ_ψ may need to be adapted for a wound rotor, depending on the winding configuration.

To comply with the stator model, C_r has the dimension Q_s × Q_r as it computes Q_r/2 + 1 Fourier coefficients based on Q_r sampling points. The stator DC component is trapped in the yoke and does not interact with the rotor. Furthermore, any rotor that obeys ∑_k=1^Q_r i_r,k = 0 along the rotor periphery cannot generate a circulating flux itself. This is true for a squirrel-cage rotor. Therefore, the row corresponding to the DC component contains all zeros. In case Q_s > Q_r, the entries for harmonic planes h > [Q_r/(2p_mw)] need to be filled with zeros. Otherwise, excited stator space-harmonics of order Q_r/2 + q, q ∈ {1, 2, ..., (Q_r−Q_r/2)/2}, would map into h = Q_r/2−q of the rotor and vice versa, corresponding to spatial aliasing. The Nyquist-Shannon theorem supports the claim because a sampling frequency of Q_r samples per revolution has a Nyquist frequency of Q_r/2 cycles per revolution corresponding to the same pole-pair number. Physically, it is also sensible to concatenate the zero rows. A Q_r-slot winding can only generate magnetic fields with maximally [Q_r/2] pole-pairs (neglecting slot harmonics), so rotor harmonics of order h > Q_r/2 cannot be excited. If these rows were not filled with zeros, aliasing would be inevitable, according to the Nyquist-Shannon criterion, and would correspond to a coupling of different pole-pair fields. However, such a coupling cannot take place, which can be concluded from the winding-function approach used to calculate the mutual stator-rotor inductance [21]:

\[ L_{sr}(h_s,h_r) = \mu_0 l_{eff} \int_0^{2\pi} 2 \pi N_{h_s,h_r} \cos(h_s \theta) N_{h_r} \cos(h_r \theta) d\theta, \]

(10)

The trigonometric functions \( \cos(h_s \theta) \) and \( \cos(h_r \theta) \) are orthogonal on the interval [0, 2π] whenever \( h_s \neq h_r \). Thus, for a constant air-gap length, \( g(\theta) = g, L_{sr} = 0, \forall h_s \neq h_r \).

Consequently, the equivalent circuit of the harmonic planes \( h > [Q_r/(2p_mw)] \) simplifies to Fig. 4(b) comprising stator parameters only.

Finally, the Clarke transformation for rotor quantities is completed in (11) by taking the transformer ratio into account.

\[ \bar{x}_{\alpha r} = n_t \cdot \left( \frac{2}{Q_r} \right)^{\frac{1}{2}} C_r \cdot \bar{x}_{123r} \]

\[ \bar{x}_{123r} = \left( \frac{2}{Q_r} \right)^{\frac{1}{2}} C_r \cdot \bar{x}_{\alpha r} \]

(11)

Notice that the DFT interpretation of the HPD lends itself to any general rotor geometry. For example, the permanent magnet flux-linkage can be decomposed into space harmonic orders whose relative position to a reference angle changes over time. In this case, permanent magnet harmonics can be decomposed into \( [n_m/2] \) subspaces.

III. APPLICATION OF THE HARMONIC PLANE DECOMPOSITION

This section explains in detail the links between the VSD and the HPD. We outline how the components of the VSD vector spaces distribute into the harmonic planes and vice versa, suggest how current references shall be selected in a controller to accomplish a certain PPC, and highlight some peculiarities of VPPMs that the HPD is able to model.
A. VECTOR SPACE DISTRIBUTION

The VSD’s dimension is defined by the number of phases in which the machine is excited. Slots carrying the current $\pm i_k(t)$ at and in the neighborhood of time $t$ are said to belong to phase $k \in \{1, 2, \ldots, m_s\}$. A phase must consist of at least two slots. Obviously, this concept is pertinent in conventionally wound machines. However, in a VPPM the number of phases may change and there may be more independent currents than phases. This becomes particularly apparent in a phase-pole transition. Therefore, the concept of phases becomes less useful in the HPD theory and should rather be replaced by a slot or winding index representing each of the independent windings, i.e. $n_{mw}$ independent currents. For interpretability and to allow for a better description of the excitation, the term phase is kept here, but we define it as

$$m_s \triangleq \frac{Q_s}{2pq_s}.$$  

(12)

With this definition, an odd or fractional number of slots may constitute a pole-pair leading to non-integer phase numbers. For example, there are two such PPCs in a $n_{mw} = Q_s = 36$ VPPM obtained for $q_s = 1: \{m_s = 4.5, p = 4\}$ and $[m_s = 3.6, p = 5]$. Their vector space distributions are shown in Fig. 5. The current of slot $k$ for these PPCs at steady state is

$$i_k(t) = \hat{i} \cos \left[ \omega_m t - \frac{2\pi(k-1)}{n_{mw}} \right].$$  

(13)

For the cases when the VSD is applicable for a fixed PPC of a VPPM, the VSD and HPD must be equivalent. It was shown in [17] by simulations that this is indeed the case when the model parameters and the applied voltage per solenoid are appropriately transformed from VSD to HPD. The question is which current references that should be commanded to the current controller to emulate a certain $[m_s, p]$ PPC as if the machine was wound in that particular way.

To answer this question, we begin by examining how vector spaces and harmonic planes are linked quantitatively. In the VSD, the rank of the Clarke matrix is related to $m_s$, i.e. it is related to the degrees of freedom in the drive. On the contrary, the rank of the Clarke matrix in the HPD is always higher than $m_s$, meaning that vector spaces $\nu$ distribute over the harmonic planes $h$. The relation between $h$ and $\nu$ in (14) from [10] describes this vector-space distribution for a balanced $[m_s, p]$ PPC. To make notations compact, we introduce $h_\nu$ as the set of harmonic planes $h$, which contains the same vector space, $\nu$. Conversely, $\nu_h$ is the vector space $\nu$, that hosts the harmonic plane, $h$. Note that one vector space may be distributed over several harmonic planes. The mapping in (14a), effectively describes the space-harmonic orders produced by a $j^{th}$ time-harmonic order current, which has also been reported in [22, (12)]. The sign of the addition in (14a) indicates the sequence. Conversely, the vector space $\nu$ occupied by a harmonic plane $h$ is given in (14b).

$$h \in h_\nu \quad \text{if} \quad h = (\nu \pm m_s q) \frac{p}{p_{mw}} \quad \text{with} \quad \nu \in \{1, 3, \ldots, m_s\}, q \in \{0, 2, 4, \ldots\}.$$  

(14a)

$$\nu \in \nu_h \quad \text{if} \quad \nu = \left\{ \begin{array}{l} \frac{\nu_{mw} h}{p} \quad \text{if} \quad \nu_{mw} h \leq m_s \\ \frac{\nu_{mw} h - 2m_s}{p} \quad \text{if} \quad m_s < \frac{\nu_{mw} h}{p} \leq 3m_s \\ \frac{\nu_{mw} h - 4m_s}{p} \quad \text{if} \quad 3m_s < \frac{\nu_{mw} h}{p} \leq 5m_s \\ \vdots \end{array} \right.$$  

(14b)

Next, the magnitude and angular dependencies of the harmonic planes are considered. Since $x_{\alpha0,0,h}$ and $x_{\alpha0,0,\nu}$ represent complex space vectors at any time, their relative positions and lengths can be defined as a ratio:

$$k_{\nu, h} \triangleq \frac{x_{\alpha0,0,h}}{x_{\alpha0,0,\nu}} = |k_{\nu, h}| e^{j\varphi_h}, \quad \text{if} \quad h \in h_\nu.$$  

(15)

A special case occurs when $\nu$ only contains one time-harmonic frequency. Energy conservation demands:

$$\sum_{h \in h_\nu} x_{\alpha0,0,h} \cdot x_{\alpha0,0,h} = |x_{\alpha0,0,\nu}|^2.$$  

(16)

Dividing (16) by $|x_{\alpha0,0,\nu}|^2$, renders

$$k'_{\nu, h} \triangleq \frac{k_{\nu, h}}{|x_{\alpha0,0,\nu}|} = |k'_{\nu, h}| e^{j\varphi'_h}, \quad \text{if} \quad h \in h_\nu.$$  

(17)

$k'_{\nu, h}$ is a complex winding factor, whose modulus corresponds to the winding factor known from classical machine theory. This interpretation is consistent with the complex winding factor introduced in [23, (49)].
In short, $k_{w,h}$ describes the relation between the space vectors $\underline{X}_{30,h}$ in HPD to the space vectors $\underline{X}_{30,\nu}$ in VSD, where the modulus $|k_{w,h}|$ is the relative size and the argument $\varphi_h$ is the spatial angular displacement.

Table 1 contains the modulus, $|k_{w,h}|$, and non-zero argument, $\varphi_h$, of each complex winding factor for all harmonic planes and achievable configurations of a VPPM with $n_{mw} = Q_s = 36$ independent solenoidal stator windings. Because all balanced operations exhibit half-wave symmetric current distributions over one pole-pitch, odd pole-pair configurations only distribute into odd harmonic planes whereas even pole-pair configurations only distribute into even harmonic planes.

The sets $h_{\nu}$ and $\nu_{h}$, defined in (14), are formally not represented by functions. A graphical aid in representing these mappings of vector spaces to harmonic planes is presented in Fig. 5 with examples from four different PPCs of a VPPM with $n_{mw} = Q_s = 36$ individual currents. It resembles [24], which uses graphs for linking time harmonics to space harmonics in MPMs. These graph representations remain valid and can be combined with the following analysis.

To start with, a grid is formed with horizontal lines representing harmonic planes and vertical lines representing vector spaces. Although neglected in most modeling, it is important to include the homopolar components in $h = 0$ and $\nu = 0$. Drawing a line in this grid allows to determine the vector-space distribution graphically. The point $(\nu = 0, h = 0)$ marks the start. Next, a line of steepness $\nu/p_{mw}$ is drawn until the boundary of the grid. Intersections of the line with the underlying grid correspond to the mapping of vector spaces into harmonic planes. When the line reaches the boundary, the direction reverses until the opposing edge is reached. The procedure repeats until the line reaches $h = [n_{mw}/2]$. The direction of the line implies a positive or negative sequence of the space-harmonic. In Fig. 5, positive sequences are indicated by a red area and negative sequences by a green area. It is observed that the number of harmonic planes, $|h_{\nu}|$, excited with a certain time harmonic, $\nu$, corresponds to the width of the phase belt, i.e. the number of slots-per-phase-per-pole $q_s = Q_s/(2m_{sp})$.

Comparing the VSD to the HPD, a major difference is that the VSD creates a minimal model for a given phase number,
whereas the HPD creates a model with seemingly redundant subspaces. However, these subspaces are excitable and can be used for harmonic injection [10]. Possible use cases are peak flux minimization and torque maximization [25], parameter estimation by injection [26], bearingless rotor suspension [27], and fault tolerant control [28].

**B. EXTENDED PARK TRANSFORMATION MATRIX**

Equations (8) and (9) establish the space vectors in the stationary \(\alpha\beta\) reference-frame. A generalized Park transformation brings these into a rotational \(dq0\) reference-frame to enable synchronous reference-frame control. Leveraging the independence of the harmonic planes, a rotation transformation can be applied to each of the space vectors separately. The resulting Park transformation matrix in (18) is a block diagonal matrix with \(2 \times 2\) blocks. The first row and column of the generalized Park transformation correspond to the DC-component \(h = 0\) and thus do not require a rotation. Moreover, in case \(n_{mw}\) is even, the second homopolar component exists. Therefore, an additional column and row are added to the right and bottom respectively, marked in gray in (18).

\[
T_{\alpha\beta0\rightarrow dq0} = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & \mathcal{A}_{h=1} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \mathcal{A}_{h=n_{mw}} & 0 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]  

(18)

\[
\mathcal{A}_h = \begin{bmatrix}
\cos(2\pi n_{sh}f_s t + \gamma_h) & \sin(2\pi n_{sh}f_s t + \gamma_h) \\
-\sin(2\pi n_{sh}f_s t + \gamma_h) & \cos(2\pi n_{sh}f_s t + \gamma_h)
\end{bmatrix}
\]

According to (14), the space vectors host time harmonics \(\nu_h\) of different orders and sequences depending on the PPC. Consequently, the \(dq0\) reference-frame must rotate according to the sign and ordinal of \(\nu_h\). In addition to the rotation, the harmonic planes may be shifted relative to each other. The shift angle should be chosen as \(\gamma_h = \phi_h(t = 0) + \gamma_{\nu} = \arg(k_{w,h}) + \gamma_{\nu}\) to facilitate the IRFOC of PPCs with \(q_s > 1\) when more than one harmonic plane is excited. The rotor field orientation angle of each plane is thus the mechanical angle \(\theta_h = 2\pi n_{sh}f_s t + \gamma_h\).

Fig. 6 illustrates the VSD and HPD space vectors, and their trajectories in the stationary reference frame with time as a parameter for an \([m_s = 6, p = 1]\) PPC. The VSD space vector is described by the normalized \(x_{\alpha\beta,0} = e^{i\theta_{\bar{i}_d}}(0)\) whereas the corresponding HPD space vectors are given by (15). Moreover, the alignments and rotations of the three synchronous reference frames \((h = 1, 11, 13)\) required to produce the PPC using IRFOC are depicted in Fig. 7. The sequences and arguments declared in Table 1 are indicated.

**C. EXAMPLE: CURRENT REFERENCE GENERATION FOR AN \([m_s = 6, p = 1]\) PPC**

The first step in generating the current references of a \(q_s > 1\) PPC for a synchronous reference frame vector-controller is to set the magnetization level of the fundamental, corresponding to \(i_{d,1}\) in an \([m_s = 6, p = 1]\) PPC. Torque production is generated in \(h = 1\) so \(i_{q,1}\) may be generated from a speed controller or an operator if in torque control. From (14a), \(h_{\nu=1} = \{1, 11, 13\}\) are occupied for the \([m_s = 6, p = 1]\) PPC. To reproduce the desired PPC, the moduli and phases of these harmonic planes need to be fixated. Using (15), (19) assures that the modulus requirement is met. The matrix multiplication mirrors the \(q\)-axis reference of the negative sequence/s. Alternatively, the \(q\)-axis itself can be mirrored so that it always leads the \(d\)-axis in the direction of rotation.

\[
\bar{i}_{dq0,h} = \frac{k_{w,h}}{|k_{w,1}|} \begin{bmatrix}
1 & 0 \\
0 & \text{sgn}(\nu_h)
\end{bmatrix} \bar{i}_{dq0,1}
\]

(19)

The output voltage of the subsequent current controller can then be transformed to the \(123\) reference frame by multiplication with \(T_{\alpha\beta0\rightarrow 123}^{-1} T_{\alpha\beta0\rightarrow dq0}(\theta_h)\). By including the appropriate static displacement of \(\gamma_h\) in the Park transformation, the phase requirement of the harmonic planes is fulfilled. The resulting voltage vector may then be sent to the modulator after appropriate compensation of inverter nonlinearities [29].

**D. EXAMPLE: SHORT-PITCHED MACHINE**

The following example intends to showcase the differences between the VSD and HPD using a well-known case of the current distribution in a \([m_s = 3, p = 1]\), \(Q_s = 36\) slot machine with a \(y_p = 17/18\) short-pitched, double-layer winding. At least three terminals are needed to create this
distribution. However, the current distribution does not necessarily follow from the winding layout but can be created by exciting more independent windings than three, using an appropriate control strategy, e.g., as suggested in [10].

The 2π-symmetric fundamental current vector of the bottom layer is given in (20). The top layer current vector, \( \bar{x}_{123,t} \), is shifted angularly by \( (1 - y_p) \pi \), which corresponds to translating \( \bar{x}_{123,b} \) by one element (slot pitch) for the given \( y_p = \frac{17}{18} \). The fundamental current vector resulting from the contributions of both layers is given in (21).

\[
\bar{x}^T_{123,b} = [\bar{x}_{\text{base}}, -\bar{x}_{\text{base}}]
\]
\[
\bar{x}_{\text{base}} = [x_1, x_2, \ldots, -x_1, -x_2, \ldots].
\]
\[
x_{123} = \bar{x}_{123,b} + \bar{x}_{123,t}
\]

At \( t = 0 \), the normalized phase currents are \( \bar{x}_{abc} = [1, -0.5, -0.5] \) implying \( \gamma_{t=1} = 0 \). Applying the HPD transformation in (8) results in the complex space vector

\[
\bar{x}_{\alpha\beta0} = [k_{w,h}][x_{abc}]e^{jkw_{p,h}}
\]
\[
= \begin{bmatrix} 0.9525e^{j30^\circ}, & 0, & 0, & 0.1787e^{-j150^\circ}, \\
0, & 0.1190e^{j30^\circ} \end{bmatrix}^T,
\]

where we have truncated the vector at \( h = 7 \) and, informally, used degrees rather than radians in the complex exponential. In the full-pitch case, the slots conducting the positive a-phase current are located at positions of \( 0^\circ, 10^\circ, \ldots, 50^\circ \). Thus, the phase belt symmetry axis extends in the direction of \( 25^\circ \) and the peak of the space vector corresponding to \( h = p = 1 \) is turned \( 25^\circ \) counter-clockwise in relation to slot one. The short pitch results in another \( (1-y_p)\pi/2 \) rotation. Furthermore, the magnitude of the winding factor for the \( h \)th space-harmonic is [30, (2.35)]

\[
k_{w,h} = \frac{\sin (h \omega_0 \delta/2)}{q_h \sin (h \delta/2)} \sin \left( y_p \frac{\pi}{2} \right)
\]
\[
= \begin{bmatrix} 0.9525, & 0, & 0, & 0.1787, & 0, & 0.1190, \ldots \end{bmatrix}^T,
\]

corresponding to the scaling of the space vector in (22).

The winding layout fixes the magnetic axes in space of a conventionally wound \( m_s \)-phase machine. The effect of the fixed winding can later be incorporated through the winding factor [30, Ch.2]. In contrast, it is observed that such geometric factors of a non-sinusoidal winding layout are incorporated in the HPD transformation itself.

Fig. 8 displays the difference between a three-phase VSD of the current waveform and the HPD counterpart indicating the higher space-harmonic resolution of the latter.

**E. Example: Parameter Transformations**

A standard procedure in transforming physical machine parameters to a certain reference frame is to apply the transformation matrices directly to the physical parameters. For example, this means that the stator current contribution to the stator flux linkage can be expressed as \( \psi_{ss,abc} = L_{ss,abc}i_{s,abc} \):

\[
\psi_{ss,\alpha\beta0} = \mathbf{T}_{abc\to\alpha\beta0}L_{ss,abc}\mathbf{T}_{\alpha\beta0\to abc}i_{s,\alpha\beta0}.
\]

The \( L_{ss,abc} \) matrix contains the self inductance of a phase-coil and their mutual inductances. If symmetry is assumed in a three-phase machine, \( L_{uv} = L_{sm} \cos \left( \frac{2\pi}{3} \right) \) for \( u \neq v \) and \( L_{uv} = L_{sm} \cos (0) + L_{sr} \) for \( u = v \) (\( u \) and \( v \) denote phases), the resulting inductance matrix using \( K = 2/m_s \) in the stator reference frame decouples the vector spaces:

\[
L_{ss,\alpha\beta0} = \begin{bmatrix}
\frac{2}{3}L_{sm} + L_{sr} & 0 & 0 \\
0 & \frac{2}{3}L_{sm} + L_{sr} & 0 \\
0 & 0 & L_{sr}
\end{bmatrix}.
\]

Here, \( L_{sm} \) can be calculated analytically [30, (4.10)], which demonstrates that \( L_{sm} \propto |k_{w1}|^2 \).

Now, consider the fundamental 123-reference-frame stator inductance matrix, \( L_{ss,123}(x_1, x_2) \), which contains the mutual inductances between any two coils, indexed \( x_1 \) and \( x_2 \). For a toroidally wound VPPM, both \( x_1 \) and \( x_2 \) are in the domain \( \{0, 1, \ldots, Q_s - 1\} \). As pointed out in [31], such a matrix will have a Toeplitz structure provided equally spaced solenoids and isotropic, linear magnetic conditions. This implies that any two solenoids with an equal distance between them, \( |x_1 - x_2| \), will have equal mutual inductance. Therefore, each row (column) in the matrix is merely a one-slot shifted replica of the previous row (column) with the self-inductance along the matrix diagonal. Using (8) and (23), we arrive at the HPD stator-inductance matrix in (25).

\[
L_{ss,\alpha\beta0}^{\text{HPD}} = \frac{2}{Q_s}C_s L_{ss,123}C_s^T
\]

The proof in Appendix shows that \( L_{ss,\alpha\beta0}^{\text{HPD}} \) is diagonal provided \( L_{ss,123} \) is symmetric and Toeplitz. Its diagonal elements consist of magnetizing inductances, independent of any winding factor as noted in [10] and proven by (31), in addition to a leakage inductance comprising slot, tooth-tip, and end-winding components [16]. The winding factor is missing due to an absence of air-gap leakages. On the other hand, \( L_{sm,h>\alpha/2} \) are not coupled to the rotor and purely...
TABLE 2: \( \Gamma^{-1} \) electrical parameters of the VPPM. Resistances in [\( \text{m}\Omega \)] and inductances in [\( \mu\text{H} \)].

<table>
<thead>
<tr>
<th>( h )</th>
<th>( R_s )</th>
<th>( L_s )</th>
<th>( L_M )</th>
<th>( R_R )</th>
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<tr>
<td>1</td>
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</table>

constitute air-gap leakage components for those planes. This is not the case in a VSD model whose higher-order air-gap fields map into lower-order vector spaces, where they constitute the air-gap leakage. The degree of coupling is expressed by the winding factors [30, Ch. 4.3.2]. Reference [16] details the segregation of \( L_{ss,h} \) into different leakage components and derives the exact relationships between VSD and HPD parameters.

IV. EXPERIMENTAL VALIDATION

This section presents experiments using a VPPM with \( n_{mw} = Q_s = 36 \) independent stator toroids. Each of them is connected to a half-bridge inverter leg as indicated in Fig. 9 and connected to a common star point. However, we remark that the HPD would be equally applicable even if the condition \( \sum_{k=1}^{n_{mw}} i_k \) did not apply, e.g. in case the toroids were supplied from a full bridge-inverter. Fig. 10 shows the test rig and Fig. 11 a high-level block-diagram of the driveline. The VPPM is a rewound cast iron - ABB M3BP 160MLB 2 induction machine [32]. Its parameters were identified in [16] and are summarized in Table 2.

Firstly, the steady state of a \( [m_s = 6, p = 1] \) PPC showcases the vector space distribution and the implication of the complex winding factor \( k_{w,h} \). This also validates the current-reference generation for the IRFOC in the non-trivial case of \( q_s > 1 \).

Secondly, a phase-pole transition highlights the benefits of the continuous HPD-model in transient phase-pole reconfigurations. The specific PPCs for this experiment were carefully chosen to display the following properties of the HPD:

- Vector space distribution of the fundamental stator current.
- Ability to emulate non-integer phase numbers.
- Unified, constant rank HPD model for different PPCs.

It is pointed out that the VPPM under test does not guarantee equal currents in slots shifted by \( \pi \text{ rad} \) other than through the appropriate control. Since the VSD assumes half-wave symmetry of the slot currents through the winding configuration, it is not possible to directly compare a control strategy based on the VSD to the HPD on this type of machine.

A. STEADY-STATE OF AN \([m_s = 6, p = 1]\) PPC WITH \( q_s > 1 \)

Controlling the currents in a VPPM to emulate a PPC with \( q_s > 1 \) requires synchronized control of several harmonic planes, as described in Section III. Attempts to do so have previously not been demonstrated in the literature, to the authors’ best knowledge.

Using the IRFOC strategy described in [10], an \([m_s = 6, p = 1]\) PPC without time-harmonic injection is controlled and demonstrated in practice in Fig. 12. The current references were generated using (19) and \( \vec{i}_{q,1} = [1.26, 1.8] \). A while the speed was regulated with the load machine to 500 rpm. The magnetization was selected to keep the VPPM within its linear magnetic operating range.

The figure shows that the eighteen first 123-currents form \( m_s \) groups of \( q_s = 3 \) slots per pole per phase. Similarly, the remaining eighteen 123-currents (not shown to avoid clutter) form another \( m_s \) groups with opposite polarity to those shown. These correspond to the return currents of \( i_{1,18} \). Hence, an \([m_s = 6, p = 1]\) PPC is formed.

Consequently, it is confirmed that the HPD control scheme is able to reproduce a \( q_s > 1 \) PPC provided appropriate references and reference frame generation.

Table 3 lists the experimental complex winding factors \( k_{w,h} \) obtained from applying (15) to the result of an FFT.
on \( i_{α,β,h_1} \) in Fig. 12. Comparing them to the theoretical ones in Table 1 indicates a high level of agreement. Thus, the interpretation of the HPD as a time-instantaneous DFT of the current distribution along the inner stator periphery and its relation to the complex winding factor have been demonstrated.

**B. PHASE-POLE TRANSITION**

To continue with, we present experimental results showing a controlled transition from \([m_s = 4.5, p = 4]\) to \([m_s = 3, p = 6]\) using the strategy described in [10]. This experiment demonstrates the HPD theory’s ability to model the pole transition continuously, which is a prerequisite for controlling the transition.

The VPPM was operated in speed control with \( ω_{ref}^{m} = 1000 \text{ rpm} \) and the load machine imposed \( τ_{shaft} = 4.5 \text{ Nm} \). A PI speed-controller identical to the one in [10] was adopted implying that a single harmonic plane at a time generated torque by acting on \( i_{ref}^{q,h} \) issued by the speed controller, while \( i_{ref}^{q,h} = 0 \) for all other harmonic planes. The pole transition was carried out in three stages:

- Predemagnetization: The rotor flux linkage \( ψ_{R,h=4} \) was first reduced. \( i_{q,h=4} \) increased to retain the torque.
- Premagnetization: \( ψ_{R,h=6} \) increased. Magnetic saturation was mitigated by not magnetizing both rotor flux linkages to nominal simultaneously.
- Pole Transition: The \( q \)-currents shifted from \( h = 4 \) to \( h = 6 \).

Fig. 13 shows the mechanical rotor speed and torque throughout the transition. Fig. 14 shows the currents \( \{i_{d,q,h=4}, i_{d,q,h=6}\} \) in the two harmonic planes \( h = \{4, 6\} \) corresponding to the pole-pair numbers before and after the pole transition. Fig. 15 plots the currents \( \{i_1, i_{10}\} \) in the windings \( \{1, 10\} \), which are mechanically shifted by 90°. The mechanical quantities indicate that the presented experiment behaves similarly to the results in [10] with a momentary torque drop of \( ≈ 50\% \). A close look at Fig. 15 reveals that \( i_1 \) and \( i_{10} \) are in phase before and opposite after the reconfiguration, suggesting a successful transition.

This experiment shows two virtues of the HPD. First, the model is continuous throughout the pole transition. Looking at Fig. 14, the currents are modeled and controlled at all times. Second, the PPC \([m_s = 4.5, p = 4]\) operates cus-
tomary, suggesting that the term *phase* in this context is less useful than the term *independent minimum windings*.

V. CONCLUSIONS

Variable phase-pole machines (VPPMs) are afflicted with certain challenges when it comes to modeling their dynamic operation. Particularly, previous attempts have typically relied on the vector space decomposition (VSD), but this model assumes pole symmetry, which is not guaranteed in VPPMs, especially during a pole transition. Moreover, the number of subspaces and their parameters is phase-number dependent, leading to model discontinuities. This paper developed the harmonic plane decomposition (HPD) to create a model of electrical machines that dispose of symmetry assumptions and remain fixed independent of the phase-pole configuration (PPC). It was interpreted as a space-domain DFT and was shown to be related to the VSD through the complex winding factor, which scales and rotates the space-harmonic content in the VSD space vector. The HPD retains many desirable features of the VSD model, e.g., decoupling of the harmonic planes and similar equivalent circuits. The paper was able to verify the elaborated theory through experiments implementing field-oriented control of the established model. It demonstrated the generality and usefulness of the HPD by performing a continuous phase-pole transition, and emulating non-trivial PPCs with more than one slot per pole per phase and fractional phase numbers.

APPENDIX. ORTHOGONALITY OF HARMONIC PLANES

Provided a symmetric and Toeplitz \( L_{m,x,123} \), we may decompose it in a mutual inductance, \( L_m(x_1, x_2) \), whose element \((x_1, x_2)\) is dependent on \(|x_1 - x_2|\) (see [33]), and a leakage inductance as follows:

\[
L_{m,x,\alpha\beta}^{\text{HPD}} = \frac{Q_s}{Q_x} C_x L_m(x_1, x_2) C_\alpha^T C_x L_{L_{m,x,123}}(x_1, x_2). \quad (26)
\]

Apparent, \( \frac{2Q_x}{Q_s} C_x I_{Q_s \times Q_s} C_\alpha^T = L_{\text{ss}} I_{Q_s \times Q_s} \) since \( C_x \) is the inverse of \( C_\alpha^T \). We now investigate the first term in (26), Equation (27) declares the elements of the matrix resulting from the multiplication \( C_x L_m(x_1, x_2) C_\alpha^T \):

\[
L_{m,x,\alpha\beta}^{\text{HPD}}(i,j) = \begin{cases} 
\frac{1}{2} \sum_{m=1}^{Q_s} \sum_{k=1}^{Q_s} L_{m,k} \cos(h(k-1)\delta), & \text{if } i = j = h, \\
\frac{1}{2} \sum_{m=1}^{Q_s} \sum_{k=1}^{Q_s} L_{m,k} \sin \left( \frac{i}{2} (k-1)\delta \right), & \text{if } i \neq j
\end{cases}
\]

\[
\lfloor . \rfloor \text{ is the floor operator, } j \text{ the column index and } i \text{ the row index. The latter corresponds to the harmonic plane } h \text{ of the same row. We now show that the off-diagonal elements are identically equal to zero when the } L_m \text{-matrix is symmetric and Toeplitz due to the symmetry of the coils and their locations. Using the addition formulae for sin and expanding the expression for (27) when } i \neq j \text{ renders:}
\]

\[
\frac{1}{2} \sum_{m=1}^{Q_s} \sum_{k=1}^{Q_s} \sin ((k-m)\delta) + \sin ((k+m-2)\delta) = \frac{1}{2} \sum_{m=1}^{Q_s} \sum_{k=1}^{Q_s} \left( L_{i,j} \sin [(i-j)\delta] - L_{j,i} \sin [(i-j)\delta] \right) + \sum_{i=1}^{Q_s} \sum_{j=1}^{Q_s} \left( L_{i,j} \sin [(i+j-2)\delta] + L_{Q_s-i+1,j} \sin [Q_s-i-j+2\delta] \right) + \sum_{i=1}^{Q_s} L_{i,i} \sin [2(i-1)\delta] = 0
\]

In the expansion, the first term equates to zero due to the odd symmetry of the sin function and the symmetry of the inductance matrix. The third term is zero due to the periodicity of the sin function and the equal sampling length, \( \delta = \frac{2\pi}{Q_s} \). Similarly, the second term is zero because \( L_{i,j} = L_{Q_s-i-j} \) (equal distance between coils) and \( \sin [(i+j-2)\delta] + \sin [Q_s-i-j+2\delta] = 0 \).

The diagonal elements are non-zero as can be concluded by expanding the expression for (27) when \( i = j = h \). Again,

![Figure 15: Slot currents \( i_1 \) and \( i_{10} \) and estimated rotor flux linkages \( \psi_{R,h=4} \) and \( \psi_{R,h=6} \) during the pole transition from \([m_s = 4.5, p = 4]\) to \([m_s = 3, p = 0]\).](Image 448x747 to 539x770)
we utilize the summation formulae for $\cos$:

$$
\sum_{m=1}^{Q_s} \sum_{k=1}^{Q_s} L_{m,k} \cos (h(k-1)\delta) \cos (h(m-1)\delta)
$$

$$
= \sum_{m=1}^{Q_s} \sum_{k=1}^{Q_s} Q_s L_{m,k} \cos (m+k-2)\delta)
$$

$$
S_1 + L_{m,k} \cos (k-h)m\delta)
$$

S_2

(29)

Thus, the stator inductance matrix can be written as in (31):

$$
L_{m,k}= L_{1,1}+ (1)^k L_{q_2}+ 2 \sum_{k=1}^{Q_s} L_{1,k-1} \cos (k-h)\delta)
$$

REFERENCES


GUSTAF FALK OLSON was born in Kristinehamn, Sweden. He received his M.Sc. in electrical power engineering from the Royal Institute of Technology KTH, Stockholm, Sweden in 2016. After graduation, he worked as a Field Application Engineer for Texas Instruments. Since 2019, he is working towards a Ph.D. in parameter estimation of multiphase electrical machines at the Division for Electrical Power and Energy Systems (EPE) at KTH Royal Institute of Technology, Stockholm, Sweden. His research encompasses modeling, control, and parameter estimation of multiphase machines.

YIXUAN WU received his M.Sc. in electrical engineering from the RWTH Aachen University, Aachen, Germany and KTH Royal Institute of Technology, Stockholm, Sweden in 2019. Since 2019, he is working towards a Ph.D. in fault tolerance of multiphase electrical machines at the Division for Electrical Power and Energy Systems (EPE) at KTH Royal Institute of Technology, Stockholm, Sweden. His research encompasses modeling and control of variable phase-pole drives, power electronics, fault-tolerance and electromobility.

OMER IKRAM UL HAQ was born in Multan, Pakistan on 8th May 1985. He received his Bachelor’s in Electrical Engineering from Bahauddin Zakariya University (BZU), Multan, Pakistan in 2008 and a Master’s in Electrical Power Engineering from The Royal Institute of Technology (KTH), Stockholm, Sweden in 2010. He is currently pursuing an industrial Ph.D. degree at The Royal Institute of Technology (KTH), Stockholm, Sweden in collaboration with ABB AB, Sweden (2020-2025).

He worked at Orascom Telecom (Mobilink), Pakistan as an Intelligent Network engineer in 2007. Then he moved to Pak-American Fertilizers as a boiler control engineer in 2008. While pursuing his Master’s degree at KTH, he joined ABB Corporate Research center, Sweden in 2010 and is presently working as a Senior Scientist. His major field of interest includes position sensorless control of asynchronous machines, multiphase machine modeling, and phase pole modulation.

LUCA PERETTI received the M.Sc. degree in Electronic Engineering in 2005 from the University of Udine, Italy, and the Ph.D. degree from the University of Padova, Italy, in 2009. Between November 2007 and March 2008, he was a visiting Ph.D. student at ABB Corporate Research, Västerås, Sweden. From August 2010 to August 2018, he was with ABB Corporate Research, Västerås, Sweden in different roles as principal scientist, project leader and strategy coordinator.

He has also been an Affiliated Faculty member at KTH, division of Electric Power and Energy Systems, since July 1, 2016. From September 2018 Luca is an Associate Professor at KTH, division of Electric Power and Energy Systems, in the field of Electric Machines and Drives. His main scientific interests relate to the automatic parameter estimation in electric machines, sensorless control, loss segregation in drive systems, multiphase drives, condition monitoring of machines and drives, in the context of industrial, wind energy, and traction applications.