Möbius and Loewner energy on curves with corners

Alice Brolin

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Abstract

The Möbius energy and the Loewner energy are two Möbius invariant quantities defined for Jordan curves. We start by introducing some of the basic properties of these two energies. Both are finite if and only if the curves belong to a class called Weil-Petersson. The Weil-Petersson class does not contain curves with corners. In part motivated by recent work of Johansson and Viklund we introduce regularized versions of both the Möbius and Loewner energy which allow for certain curves with isolated corners. We also look at the derivative of the Loewner energy.

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1 Introduction

In this text we look at two energies on simple closed curves on the Riemann sphere, the Loewner energy and the Möbius energy. These two energies are both invariant under Möbius transformations and are both finite at the same time [1]. In particular they are both infinite for curves that have corners. In [2] Johansson and Viklund studied the behaviour of Coulomb gas in a domain bounded by a Jordan curve. They showed that the partition function was related to the Loewner energy of the Jordan curve. They also looked at how it behaved for curves with corners. We compare this with the behaviour of the Möbius and Loewner energy. We also define regularized versions of the Möbius and Loewner energy, which are finite for some curves with corners. For the regularized Möbius energy we look at curves that are C^2 between the corners whereas we only look at polygons for the regularized Loewner energy. Finally we study the derivative of the Loewner energy with respect to smooth perturbation of a curve.

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2 Preliminaries

We start by defining curves, so that we then can define the two energies. A curve in a topological space X is defined by a continuous function $\gamma : [a,b] \to X$. A loop or closed curve is given by a continuous function $\gamma : \mathbb{R}/k\mathbb{Z} \to X$ (with $k \in \mathbb{R}^+$). We will identify two curves γ_1, γ_2 in X if they agree up to parameterization i.e. there is a homeomorphism ϕ between the domains of γ_1 and γ_2 such that $\gamma_2 \circ \phi = \gamma_1$.

Definition 1. *A curve is simple if it is injective. If it is simple but not closed it is an arc. If γ is a simple closed curve in $\mathbb{C}$ it is called a Jordan curve.*

Definition 2. *If a curve has a parameterization $\gamma : D \to \mathbb{R}^n$ which is Lipschitz then it is rectifiable and the length of the curve is $l(\gamma) = \int_D ||\gamma'(t)||dt$*

Note that the derivative is defined a.e.. A useful feature of rectifiable curves is that they have an arc-length parameterization.

Proposition 1. *A simple rectifiable curve has a parameterization, γ such that $||\gamma'|| = 1$ a.e., called the arc-length parameterization.*

Proof. Given a Lipschitz parameterization $\gamma_0 : [a,b] \to \mathbb{R}^n$ define $\phi : [a,b] \to [0, l(\gamma_0)]$ by $\phi(t) = \int_a^t ||\gamma_0'(s)||ds$. Since γ_0 is injective $||\gamma_0'(s)|| > 0$ a.e. so ϕ is a bijection. Set $\gamma = \gamma_0 \circ \phi^{-1}$. Then $||\gamma'(t)|| = ||\gamma_0' \circ \phi^{-1}(t)||\ ||(\phi^{-1})'(t)|| = ||\gamma_0'(\phi^{-1}(t))|| \frac{1}{||\gamma_0'(\phi^{-1}(t))||} = 1$. □

The space we are looking at is primarily the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 3. *A Möbius transformation is a function* $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *of the form* $z \mapsto \frac{az+b}{cz+d}$ *where* $a, b, c, d \in \mathbb{C}$ *and* $ad - bc \neq 0$.

The Möbius functions are the group of holomorphic automorphisms of $\hat{\mathbb{C}}$. Every Möbius transformation can be written as a composition of linear transformation, translation and inversion $z \mapsto z^{-1}$.

2.1 Möbius energy

The Möbius energy is an energy on rectifiable curves originally defined by O'hara in [3]. He defined it with the goal of studying knots.

2.1.1 Knots

A knot is just the image of a simple closed curve in $\mathbb{R}^3$, but we say that two knots K_1 and K_2 are equivalent if there is an ambient isotopy between them. This means that we have a function $H : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that for any fixed $t \in [0, 1]$ $H_t : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism, $H_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity and $H_1(K_1) = K_2$. An equivalence class of knots is called a knot type. The knot type of a circle is called the unknot. Any knot that is not an unknot is called non-trivial.

A knot is a model of a knot on a string, in the sense that if we have a string with a knot (in the common sense) and join the ends together we get a knot (in the mathematical sense). For example if we imagine that we have tied an overhand knot on our string before joining it together we get a trefoil knot. This is a different knot type than the unknot. [4]

Given oriented knots one can define their knot sum. This corresponds to the idea of tying a string twice. We say that a knot type is prime if it can not be written as the sum of two nontrivial knots. Knot types have an (up to ordering) unique decomposition as a sum of nontrivial prime knots. [4]

One important interest in knot theory is establishing when two knots are of the same type.

There are many results relating the Möbius energy and knot types. O'Hara showed that for any $\alpha > 0$ there are only finitely many knot types which can be represented by a simple C^2 loop such that the energy is less than α. It has also been shown that all prime knot types have a knot that minimizes the Möbius energy [5].

2.1.2 Definition of the Möbius energy

The idea behind the Möbius energy is that we want to add a charge to the knot γ and use that to define the energy of a particular knot. The simplest way to do this might be to imagine that we have a uniform charge on the knot which would lead to a potential of the form $z \mapsto \int_\gamma \frac{d|w|}{||z-w||^k}$. The Newtonian potential in $\mathbb{R}^3$ has $k = 1$, but we have nicer properties for $k = 2$ (which is the Newtonian potential in $\mathbb{R}^4$). However this potential does not work as it is infinite for any point z on γ.

Definition 4. *The Möbius energy of a simple rectifiable curve* γ *in* $\mathbb{R}^3 \cup \{\infty\}$ *is given by*

$$E(\gamma) = \iint_{\gamma\times\gamma} \left(\frac{1}{||z-w||^2} - \frac{1}{D(z,w)^2} \right) d|w|d|z|$$

where $D(z,w)$ *is the length of the shortest arc in* γ *between* z *and* w*.*

In our case we are only interested in curves on the Riemann sphere (still with the the Euclidean metric). We can see the Möbius energy as the energy of γ given the regularized potential $E(\gamma, z) := \int_\gamma \left(\frac{1}{||z-w||^2} - \frac{1}{D(z,w)^2} \right) d|w|$. The substitution rule for integrals gives that both $E(\gamma, z)$ and $E(\gamma)$ is independent of parameterization.

The Möbius energy belongs to a family of curve energies of the form

$$E^{k,p}(\gamma) := \iint_{\gamma\times\gamma} \left(\frac{1}{||z-w||^k} - \frac{1}{D(z,w)^k} \right)^p d|z|d|w|$$

where $k, p > 0$. One can note that we have $D(z,w) \geq |z-w|$ with equality if and only if the arc between z and w is a straight line. Therefore we have that all of these energies are greater than or equal to 0 with equality precisely when γ is a line. We only define these energies for simple curves as it would clearly be infinite in the case of an intersection.

Simon Blatt gave necessary and sufficient conditions for $E^{k,p}(\gamma)$ to be finite for certain k, p in [6].

Definition 5. $f \in L^\beta(\mathbb{R}/\mathbb{Z})$ *is in* $W^{\alpha,\beta}(\mathbb{R}/\mathbb{Z})$ *where* $\alpha \in (0,1)$ *if*

$$\int_{\mathbb{R}/\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{||f(t+s) - f(t)||^\beta}{|s|^{1+\alpha\beta}} dsdt < \infty$$

and a function is in $W^{1+\alpha,\beta}(\mathbb{R}/\mathbb{Z})$ *if its weak derivative is in* $W^{\alpha,\beta}(\mathbb{R}/\mathbb{Z})$*.*

Theorem 1. *If* $p \geq 1, kp \geq 2$ *and* $s := \frac{kp-1}{2p}$ *then a unit-length simple rectifiable loop* γ *parameterized by arc length has* $E^{k,p}(\gamma) < \infty$ *if and only if* $\gamma \in W^{1+s,2p}(\mathbb{R}\backslash\mathbb{Z})$*.*

It is easy to see that these energies are invariant under isometries and that when scaling with a factor $r > 0$ we have $E^{k,p}(r\gamma) = r^{2-kp}E^{k,p}(\gamma)$. Thus the energy is invariant under similarities if and only if $kp = 2$. For the Möbius energy we have an even stronger property:

Theorem 2. *For any rectifiable Jordan curve* γ *in* $\mathbb{C}$ *and any Möbius transformation* T *such that* $T^{-1}(\infty)$ *does not lie on* γ $E(T \circ \gamma) = E(\gamma)$*. If* $T^{-1}(\infty)$ *lies on* γ $E(T \circ \gamma) = E(\gamma) - 4$*.*

The Möbius invariance was originally proved in [5] where it was shown for curves and Möbius transformations in $\mathbb{R}^3$ and it is because of this property that it is called Möbius energy.

To show this theorem we start with a lemma.

Lemma 1. *For any rectifiable Jordan curve γ in $\mathbb{C}$ parameterized by arc length we have a.e. in t*

$$E(\gamma,\gamma(t)) - \frac{4}{l(\gamma)} = \left(E\left(\frac{1}{\gamma},\frac{1}{\gamma(t)}\right) - \frac{4}{l(\frac{1}{\gamma})}\right)\left|\left(\frac{1}{\gamma}\right)'(t)\right|$$

Proof. Assume t is such that $\gamma(t)\neq 0$ and $\left|\left(\frac{1}{\gamma}\right)'(t)\right| = \frac{1}{|\gamma(t)|^2}$. Note that this holds a.e. in t. Let

$$\begin{aligned} A_\epsilon(t) &:= \int_{\epsilon<|t-s|} \frac{|(\frac{1}{\gamma})'(s)|ds}{|\frac{1}{\gamma(t)} - \frac{1}{\gamma(s)}|^2}\left|\left(\frac{1}{\gamma}\right)'(t)\right| \\ &= \int_{\epsilon<|t-s|} \frac{1}{|\frac{1}{\gamma(t)} - \frac{1}{\gamma(s)}|^2}\frac{ds}{|\gamma(s)|^2|\gamma(t)|^2} \\ &= \int_{\epsilon<|t-s|} \frac{ds}{|\gamma(t)-\gamma(s)|^2} \end{aligned}$$

Let Γ be $\frac{1}{\gamma}$ parameterized by arc length. Note that if $u(s) = \int_t^s \frac{dx}{|\gamma(x)|^2}$ is the arc length then $u'(s) = \frac{1}{|\gamma(s)|^2}$ for all s and $u'(t) = |(\frac{1}{\gamma})'(t)|$. If ϵ is sufficiently small then $u(t+\epsilon), -u(t-\epsilon) < \frac{l(\Gamma)}{2}$ for all t so by substituting $v = u(s)$ we get

$$\begin{aligned} B_\epsilon(t) &:= -\left(\int_{\epsilon<|t-s|} \frac{|(\frac{1}{\gamma})'(s)|ds}{D(\frac{1}{\gamma(t)}),\frac{1}{\gamma(s)})^2} - \frac{4}{l(\frac{1}{\gamma})}\right)\left|\left(\frac{1}{\gamma}\right)'(t)\right| \\ &= -u'(t)\int_{\epsilon<|t-s|} \frac{u'(s)ds}{D(\Gamma(u(t)),D(\Gamma(u(s))))^2} - \frac{4u'(t)}{l(\Gamma)} \\ &= -u'(t)\left(\int_{-l(\Gamma)/2}^{u(t-\epsilon)} + \int_{u(t+\epsilon)}^{l(\Gamma)/2}\right)\frac{dv}{v^2} - \frac{4u'(t)}{l(\Gamma)} \\ &= \frac{-u'(t)}{u(t+\epsilon)} + \frac{-u'(t)}{-u(t-\epsilon)}. \end{aligned}$$

This needs to be compared with

$$-\int_{\epsilon<|t-s|} \frac{ds}{D(\gamma(t),\gamma(s))^2} - \frac{4}{l(\gamma)} = \\ -\left(\int_{t-l(\gamma)}^{t-\epsilon} + \int_{t+\epsilon}^{t+l(\gamma)}\right)\frac{ds}{(t-s)^2} - \frac{4}{l(\gamma)} = -\frac{2}{\epsilon}.$$

Next since u' Lipschitz in a neighbourhood around t we have

$$u(t+\epsilon) = u'(t)\epsilon + \int_t^{t+\epsilon} u''(s)(t+\epsilon-s)ds = u'(t)\epsilon + \epsilon^2\int_0^1 u''(t+\epsilon v)(1-v)dv.$$

Then since $\int_t^{t+\epsilon} u''(s)(t-s)ds = O(\epsilon^2)$ one gets for small ϵ

$$\begin{aligned}\frac{u'(t)}{u(t+\epsilon)} &= \frac{1}{\epsilon}\left(1 - \frac{\epsilon}{u'(t)}\int_0^1 u''(t+\epsilon v)(1-v)dv + O(\epsilon^2)\right)\\ &= \frac{1}{\epsilon} - \frac{1}{u'(t)}\int_0^1 u''(t+\epsilon v)(1-v)dv + O(\epsilon).\end{aligned}$$

Similarly

$$\frac{u'(t)}{-u(t-\epsilon)} = \frac{1}{\epsilon} + \frac{1}{u'(t)}\int_0^1 u''(t-\epsilon v)(1-v)dv + O(\epsilon).$$

By the dominated convergence theorem we have that as $\epsilon \to 0$

$$\begin{aligned}&\int_0^1 (u''(t+\epsilon v) - u''(t-\epsilon v))(1-v)dv\\ &\quad= \frac{-2u'(t)}{\epsilon} + \frac{1}{\epsilon}\int_0^1 (u'(t+\epsilon v) + u'(t-\epsilon v))dv\\ &\quad= \int_0^1 v\left(\frac{u'(t+\epsilon v) - u'(t)}{\epsilon v} - \frac{u'(t-\epsilon v) - u'(t)}{-\epsilon v}\right)dv\\ &\quad\to \int_0^1 v(u''(t) - u''(t))dv = 0.\end{aligned}$$

Therefore the monotone convergence theorem gives that

$$\begin{aligned}&\left(E\left(\frac{1}{\gamma}, \frac{1}{\gamma(t)}\right) - \frac{4}{l(\frac{1}{\gamma})}\right)\left|\left(\frac{1}{\gamma}\right)'(t)\right| = \lim_{\epsilon\to 0} A_\epsilon(t) + B_\epsilon(t)\\ &\quad= \lim_{\epsilon\to 0}\int_{\epsilon<|t-s|}\left(\frac{ds}{|\gamma(t)-\gamma(s)|^2} - \frac{ds}{D(\gamma(s), D(\gamma(t))^2}\right) - \frac{4}{l(\gamma)}\\ &\quad\quad+ \frac{1}{u'(t)}\int_0^1 (u''(t+\epsilon v) - u''(t-\epsilon v))(1-v)dv + O(\epsilon)\\ &\quad= E(\gamma, t).\end{aligned}$$

□

With this lemma we can easily prove theorem 2.

Proof. We need to prove invariance of the Möbius energy under linear maps and under inversion. If T is the map $z \mapsto az + b$ then

$$E(T \circ \gamma) =$$

$$\int_0^{l(\gamma)} \int_{t-\frac{l(\gamma)}{2}}^{t+\frac{l(\gamma)}{2}} \left(\frac{|(T\circ\gamma)'(t)||(T\circ\gamma)'(s)|}{|T\circ\gamma(t) - T\circ\gamma(s))|^2} - \frac{|(T\circ\gamma)'(t)||(T\circ\gamma)'(s)|}{D(T\circ\gamma(t), T\circ\gamma(s))^2} \right) dsdt$$

$$= \int_0^{l(\gamma)} \int_{t-\frac{l(\gamma)}{2}}^{t+\frac{l(\gamma)}{2}} \left(\frac{|a\gamma'(t)||a\gamma'(s)|}{|a\gamma(t) - a\gamma(s))|^2} - \frac{|a\gamma'(t)||a\gamma'(s)|}{(at-as)^2} \right) dsdt$$

$$= E(\gamma)$$

Next, by lemma 1 if γ does not pass 0 then

$$E\left(\frac{1}{\gamma}\right) = \int_0^{l(\gamma)} E\left(\frac{1}{\gamma}, \frac{1}{\gamma(t)}\right) \left| \left(\frac{1}{\gamma}\right)'(t) \right| dt$$

$$= \int_0^{l(\gamma)} \left(E(\gamma, \gamma(t)) - \frac{4}{l(\gamma)} + \frac{4}{l(\frac{1}{\gamma})} \left| \left(\frac{1}{\gamma}\right)'(t) \right| \right) dt$$

$$= E(\gamma) - \frac{4l(\gamma)}{l(\gamma)} + \frac{4l(\frac{1}{\gamma})}{l(\frac{1}{\gamma})} = E(\gamma)$$

whereas if γ does pass 0 then $\frac{4}{l(\frac{1}{\gamma})} = 0$ so

$$E\left(\frac{1}{\gamma}\right) = \int_0^{l(\gamma)} E\left(\frac{1}{\gamma}, \frac{1}{\gamma(t)}\right) \left| \left(\frac{1}{\gamma}\right)'(t) \right| dt$$

$$= \int_0^{l(\gamma)} \left(E(\gamma, \gamma(t)) - \frac{4}{l(\gamma)} \right) dt = E(\gamma) - 4.$$

□

2.1.3 Discrete variants

One way of defining the Möbius energy for curves with corners is by looking at discrete variants of the Möbius energy. There exists a number of discrete variants of the Möbius energy which have been used to study the energy numerically. These are energies on a polygon p which converge to the Möbius energy assuming the polygons converge to a sufficiently smooth curve. According to [7] the first such energy was the minimal distance energy defined by Jonathan Simon. It is defined as

$$\sum_{X,Y:\text{ non consecutive segments of } p} \frac{l(X)l(Y)}{(\min\{x-y, x \in X, y \in Y\})^2}.$$

Denise Kim and Rob Kusner used an energy of the form

$$\sum_{i \neq j} \frac{1}{||x_i - x_j||^2} - \frac{1}{D(x_i, x_j)^2}$$

where x_i are the vertices of p. In [8] Sebastian Scholtz proved that it converges to the Möbius energy.

An other discretization is given in [7]. The discretization is based on a formula for the Möbius energy found by Doyle and Schramm. Their formula uses a Möbius invariant angle between two points $\gamma(t), \gamma(s)$ on γ called the conformal angle. To define this angle one looks at the two circles that intersect $\gamma(t)$ and $\gamma(s)$ and are tangent to $\gamma(t)$ and $\gamma(s)$ respectively. Then the conformal angle $\theta_\gamma(\gamma(t), \gamma(s))$ is given by the angle between these two circles at their intersections.

Using this they write the formula for the Möbius energy of a simple closed curve as

$$\iint_{\gamma\times\gamma} \frac{1-\cos(\theta_\gamma(x,y))}{||x-y||^2} dxdy + 4.$$

To create a discrete version of the conformal angle for a polygon p we look at the circles $C_{i,j}$ which intersect x_i, x_{i+1} and x_j. Then we define θ_{ij} as the angle between $C_{i,j}$ and $C_{j,i}$ as well as $\tilde{\theta}_{ij}$ as the angle between $C_{i,j+1}$, $C_{j,i+1}$. This leads to the Möbius invariant energy

$$\sum_{|i-j|>1} \left(1 - \frac{\cos(\theta_{ij}) + \cos(\tilde{\theta}_{ij})}{2}\right) \frac{||x_{i+1}-x_i||\ ||x_{j+1}-x_j||}{||x_i-x_j||\ ||x_{i+1}-x_{j+1}||}$$

where $\frac{||x_{i+1}-x_i||\ ||x_{j+1}-x_j||}{||x_i-x_j||\ ||x_{i+1}-x_{j+1}||}$ is the cross ratio (which is Möbius invariant).

2.1.4 Gradient flow

The idea with a gradient flow of an energy functional is that you try to transform a function in the direction that fastest decreases the energy. This will then hopefully lead to an energy minimizer. In the context of Möbius energy this means that one would use the gradient flow to hopefully transform a knot into the minimizer for that knot type. In a finite dimensional vector space finding the gradient is easy. However the Möbius energy works on the space of rectifiable curves in $\mathbb{R}^3$, which is an infinite dimensional vector space. This makes studying the gradient and gradient flow significantly harder. In [5] they proved a differentiability result for the Möbius energy

Theorem 3. *If $\gamma, h : \mathbb{R}/l\mathbb{Z} \to \mathbb{R}^3$ have Lipschitz derivatives then there is a derivative*

$$\nabla_h E(\gamma) = \lim_{\epsilon\to 0} \frac{E(\gamma+\epsilon h) - E(\gamma)}{\epsilon} =$$
$$2\iint_{(\mathbb{R}/l\mathbb{Z})^2} \left(\frac{\langle\gamma'(s), h'(s)\rangle}{||\gamma'(t)||^2} - \frac{\langle\gamma(t)-\gamma(s), h(t)-h(s)\rangle}{||\gamma(t)-\gamma(s)||^2}\right) \frac{||\gamma'(s)||\ ||\gamma'(t)||dsdt}{||\gamma(t)-\gamma(s)||^2}$$

and moreover $\nabla_h\gamma$ is linear and bounded in h.

Here the integral is taken as the Cauchy principal value. Let $P_{\gamma'(t)^\perp}$ be the orthogonal projection onto the plane $\gamma'(t)^\perp$. They then defined a curve

Definition 6. *$G_\gamma : \mathbb{R}/l\mathbb{Z} \to \mathbb{R}^3$ is the curve given by*

$$G_\gamma(t) = 2\int_{\mathbb{R}/l\mathbb{Z}} \left(-\frac{\frac{d}{dt}\left(\frac{\gamma'(t)}{||\gamma'(t)||}\right)}{||\gamma'(t)||} + \frac{P_{\gamma'(t)^\perp}(\gamma(s)-\gamma(t))}{||\gamma(t)-\gamma(s)||^2} \right) \frac{||\gamma'(s)||ds}{||\gamma(t)-\gamma(s)||^2}$$

and they proved that G_γ was the L^2 gradient of the Möbius energy. I.e. they showed that

$$\nabla_h E(\gamma) = \langle G_\gamma, h\rangle = \int_{\mathbb{R}/l\mathbb{Z}} \langle G_\gamma(t), h(t)\rangle ||\gamma'(t)||dt.$$

This led to the question of the existence of a gradient flow for the Möbius energy. That is if one given a simple closed curve γ_0 can find a family of curves $(\gamma_t)_{t\in[0,\infty)}$ such that

$$\frac{\partial}{\partial t}\gamma_t = -G_{\gamma_t}.$$

Zheng-Xu He gave in [9] a result related to the short term existence of the gradient flow.

Theorem 4. *If $\gamma_0 : \mathbb{R}/l\mathbb{Z} \to \mathbb{R}^3$ is a smooth simple closed curve with $\gamma_0'(s) \neq 0$, then there is $T > 0$ and a unique smooth function $\gamma_t : [0,T) \times \mathbb{R}/l\mathbb{Z} \to \mathbb{R}^3$ such that $\frac{\partial}{\partial t}\gamma_t = -G_{\gamma_t}$.*

He also proved that the local minimizer of the gradient flow is smooth.

Simon Blatt showed in [10] that for smooth planar curves γ_0 the gradient flow $(\gamma_t)_{t\in[0,\infty)}$ exists and that γ_t converges to a circle as $t \to \infty$.

2.2 Loewner energy

The Loewner energy of a chord was defined by Yilin Wang [11] (see also [12]). The energy was used to study a certain class of stochastic curves in $\mathbb{C}$ called SLE(κ) curves.

2.2.1 SLE(κ)

The SLE(κ) processes have been of interest as they appear as scaling limits of certain conformally invariant stochastic curves in the plane such as SLE(2) being the limit of a loop erased radom walk (LERW) and SLE(3) as the limit of the interfaces in the critical Ising model[13].

For an example we can look at the loop erased random walk. The LERW on a graph G from a vertex a to a vertex b is given by running a simple random walk on the graph starting at a, but whenever this walk forms a loop by returning to a previous vertex we erase that loop. We continue this process until we reach b. This way we define a law on the simple paths in G going from a to b. To turn a domain D in the plane into a graph G_δ we make a set of vertices consisting of D intersected with the vertices of the lattice $\delta\mathbb{Z}^2$ and ∂D intersected with the edges of $\delta\mathbb{Z}^2$. Then we look at the vertices closest to points $a, b \in \partial D$. This defines a law μ_δ given by the LERW for paths in D from near a to near b.

The spherical metric on $\hat{\mathbb{C}}$ gives rise to a Hausdorff distance. Using this we can define a topology of compact subsets of D and view μ_δ as a Borel measure on this topology. Then the scaling limit is $\lim_{\delta\to 0}\mu_\delta$. [14]

The chordal SLE(κ) was originally defined by Oded Schramm who conjectured that the LERW was conformally invariant and showed that this implied that SLE(2) was the scaling limit of the LERW and the uniform spanning tree [14]. It is defined as a chordal Loewner chain.

Definition 7. *A chord in a simply connected domain $\Omega \subset \hat{\mathbb{C}}$ is an arc $\gamma : [0, l] \to \hat{\mathbb{C}}$ such that $\gamma(0), \gamma(l) \in \partial\Omega$ and $\gamma(t) \in \Omega$ for all $t \in (0, l)$.*

Let γ be a chord in $\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$ with $\gamma(0) = 0$ and $\gamma(l) = \infty$. By the Riemann mapping theorem we have for each $t < l$ a bi-holomorphic map $g_t : \mathbb{H}\backslash\gamma([0, t)) \to \mathbb{H}$ with expansion at infinity $g_t(z) = z + \frac{c_t}{z} + o\left(\frac{1}{z}\right)$ where $c_t > 0$. We can parameterize γ such that $c_t = 2t$ [15]. Then the function $W(t) := g_t(\gamma(t))$ is called the driving function of γ and is continuous. [15]

Definition 8. *The chordal Loewner chain with driving function W_t (where W_t is a continuous real valued function) is the family of conformal maps (g_t) solving the differential equation $\frac{\partial}{\partial t}g_t(z) = \frac{2}{g_t(z)-W_t}$ and satisfying $g_0(z) = 0$.*

For a given z this will be defined up to the time

$$\tau(z) = \sup\{t | \inf_{s\in[0,t]} |g_s(z) - W_s| > 0\}.$$

We define $K_t = \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}$. If γ is a chord in $\mathbb{H}$ with driving function W then

$$\gamma([0, t]) = K_t$$

and the Loewner Chain (g_t) at t is the Riemann map $\mathbb{H}\backslash\gamma([0, t)) \to \mathbb{H}$ with expansion at infinity $g_t(z) = z + \frac{c_t}{z} + o\left(\frac{1}{z}\right)$. [13]

Definition 9. *The chordal SLE(κ) is given by the family (K_t) generated by the driving function $\sqrt{\kappa}B_t$, where B_t is Brownian motion.*

The SLE(κ) is almost surely traced out by a non-self-crossing curve γ in the sense that for all $t \geq 0$ $\mathbb{H}\backslash K_t$ is the unbounded connected component of $\mathbb{H}\backslash\gamma([0, t])$. We have a.s. that γ is a chord if $\kappa \leq 4$, γ is not simple if $\kappa > 4$ and γ is space filling if $\kappa \geq 8$. [13]

2.2.2 Definition of the Loewner energy

Definition 10. *The Loewner energy of a chord in D from a to b is given by*

$$I_{D,a,b}(\gamma) = \frac{1}{2}\int_0^l |W_t'|^2 dt$$

where W_t is the driving function of γ under a Riemann map that maps D to $\mathbb{H}$, a to 0 and b to ∞.

This definition does not depend on the chosen Riemann map. The set of chords on $\mathbb{D}$ can be equipped with a topology by giving it the Hausdorff distance

$$(X,Y) \mapsto \; max(\sup_{x\in X} d(x,Y), \sup_{y \in Y} d(y,X))$$

as a metric. Then the set of chords of a domain D get a topology by using a Riemann map $D \to \mathbb{D}$. [15]

The Loewner energy is the large deviation rate functional of the family of probability measures $(\mu_\kappa)_\kappa$ given by SLE(κ) [15]. Essentially this means that the probability that the curve traced by the SLE(κ) is close to γ is approximately $e^{-I(\gamma)/\kappa}$ as $\kappa \to 0$. More precisely it means that for any open set O and closed set F of chords

$$\liminf_{\kappa\to 0} \kappa \ln \mu_\kappa(O) \geq -\inf_{\gamma \in O} I_{D,a,b}(\gamma),$$

$$\limsup_{\kappa\to 0} \kappa \ln \mu_\kappa(F) \leq -\inf_{\gamma \in F} I_{D,a,b}(\gamma).$$

Yilin Wang and Steffen Rohde also defined the Loewner energy of a simple loop $\gamma : \mathbb{R}/\mathbb{Z} \to \hat{\mathbb{C}}$. If $\epsilon \in (0,l)$ then $\hat{\mathbb{C}}\backslash\gamma([0,\epsilon])$ is simply connected and $\gamma|_{[\epsilon,1]}$ is a chord. This leads to the definition

$$I^L(\gamma) = \lim_{\epsilon\to 0} I_{\hat{\mathbb{C}}\backslash\gamma([0,\epsilon]),\gamma(\epsilon),\gamma(1)}(\gamma|_{[\epsilon,1]}).$$

It is clear from the definition that this is Möbius invariant.

Wang proved an other representation of the Loewner energy of loops in [15]. According to the Jordan curve theorem a Jordan curve γ splits the plane into a simply connected interior and exterior. By the Riemann mapping theorem there are Riemann maps f that maps $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto the interior of γ and h that maps $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}$ onto the exterior of γ.

Theorem 5.

$$I^L(\gamma) = \frac{1}{\pi}\iint_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 dA + \frac{1}{\pi}\iint_{\mathbb{D}^*} \left|\frac{h''}{h'}\right|^2 dA + 4\ln\left|\frac{f'(0)}{h'(\infty)}\right|.$$

One can note that $\iint_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 dA$ and $\iint_{\mathbb{D}^*} \left|\frac{h''}{h'}\right|^2 dA$ is the Dirichlet energy of $\ln f'$ and $\ln h'$ in $\mathbb{D}$ and $\mathbb{D}^*$ respectively. It had already been shown in [16] that this expression is finite if and only if $\iint_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 dA < \infty$.

2.3 Comparison of the energies

An easy similarity between the two curve energies is that they are both Möbius invariant. In [1] Christopher Bishop shows that for the n:th iterate of a von Koch snowflake with rounded corners $I^L \simeq 4^n \simeq E$, but he also provides a sequence of curves (γ_n) where $I^L(\gamma_n) \simeq n^2$ but $E(\gamma_n) \simeq n^2 \ln(n)$. This shows that there is no simple relation between the magnitude of these energies. However we do have that they are both finite exactly for the same class of curves: the Weil-Petersson curves.

2.3.1 Weil-Petersson curves

The Weil-Petersson curves $T_0(1)$ are a class of quasicircles that was defined by Leon Takhtajan and Lee-Peng Teo in [17] for studying the universal Techimüller space $T(1)$. That the curves are quasicircles means that they are the image of the unit circle under a quasiconformal map. A function $f : D \to \mathbb{C}$ on the domain D is quasiconformal if there exists $\mu \in L^\infty$ such that $\frac{\partial f}{\partial z}\mu = \frac{\partial f}{\partial \bar{z}}$ and $||\mu||_\infty = 1$. μ is called the complex dilation of f. [1]

Given $\mu \in L^\infty(\mathbb{D}^*)$ we can extend it by reflection to $z \in \mathbb{D}$ by taking $\mu(z) = \overline{\mu(\frac{1}{\bar{z}})\frac{z^2}{\bar{z}^2}}$. Then there is a unique quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ such that $\frac{\partial f}{\partial z}\mu = \frac{\partial f}{\partial \bar{z}}$ and such that f fixes $-1, -i, 1$. Then $f|_{\partial\mathbb{D}}$ defines a quasicircle. $T(1)$ can be seen as the space of such quasi circles. Takhtajan and Teo defined a Weil-Petersson metric on this space. Then the Weil-Petersson Techimüller space $T_0(1)$ was defined as the connected component of the identity in $T(1)$ under the Weil-Petersson metric. [17]

They also showed that being Weil-Petersson was equivalent to having

$$\iint_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 dA < \infty$$

where f is a Riemann map from $\mathbb{D}$ to the interior of the curve [16]. This means that the Loewner energy of a curve is finite if and only if the curve is Weil-Petersson. The Möbius energy of a loop γ in $\hat{\mathbb{C}}$ is, by theorem 1, finite if and only if $\gamma \in W^{3/2,2}$. This holds if and only if the curve is a Weil-Petersson curve [1]. Therefore we have that the Loewner and the Möbius energy is finite for the exact same class of curves.

3 Curves with corners

In a recent article [2] Kurt Johansson and Fredrik Viklund studied the behaviour of Coulomb gas in a domain D bounded by a Jordan curve. Coulomb gas is a model of a cloud of charged particles. An associated partition function is

$$Z_n(D) = \frac{1}{n!}\int_{D^n} \prod_{1\leq k\leq l\leq n} |z_k - z_l|^2 dA^n.$$

They normalized the partition function by setting $\overline{Z}_n(D) = \frac{Z_n(D)}{g'(\infty)^{n(n+1)}}$ where g is the Riemann map from $\mathbb{D}^*$ to the outside of the Jordan curve such that $g(\infty) = \infty$ and $g'(\infty) > 0$.

Then $\limsup_{n\to\infty} -\ln \overline{Z}_n$ is finite if and only if the Jordan curve is Weil-Petersson and

$$\limsup_{n\to\infty} -12 \ln \frac{\overline{Z}_n(D)}{\overline{Z}_n(\mathbb{D})} = I^L(\partial D).$$

They also considered domains D bounded by piece-wise analytic Jordan curves with corners $z_1, ..., z_n$ and corresponding exterior angles $\pi\alpha_1, ..., \pi\alpha_n$.

I.e. the angle between the left and right derivative of the Jordan curve at z_k is $\pi\alpha_k$. Then

$$\lim_{n\to\infty} -\frac{1}{\ln n}\ln\frac{\overline{Z}_n(D)}{\overline{Z}_n(\mathbb{D})} = \frac{1}{6}\sum_{k=1}^{n}\frac{\alpha_k^2}{1-\alpha_k}.$$

They note that a similar expression appears in the context of the heat trace. If (λ_j) are the positive eigenvalues of the Laplace operator on D with Dirichlet boundary conditions then the heat trace is $t \mapsto \sum_j e^{-\lambda_j t}$. In the piece-wise analytic case the heat trace is

$$\frac{\text{area}(D)}{4\pi t} - \frac{\text{length}(\partial D)}{8\sqrt{\pi t}} + \frac{1}{6} + \frac{1}{24}\sum_{k=1}^{n}\frac{\alpha_k^2}{1-\alpha_k} + o(1).$$

They also argue that the terms appear if one looks at the Dirichlet energy of $\ln f'$ where f is a Riemann map $\mathbb{D}\to D$. If one takes the Dirichlet energy when we remove disks of radius ϵ around the corners, then we get a divergent term $\pi\sum_{k=1}^{n}\frac{\alpha_k^2}{1-\alpha_k}\ln(\epsilon^{-1})$ as $\epsilon\to 0$.

We will study the behaviour of the Möbius and Loewner energy at corners.

3.1 Möbius energy for curves with corners

For a curve $\gamma : [0,l] \to \mathbb{C}$ with corners the Möbius energy will be infinite.

Example 1. *A rhombus of length 1 with angle $\pi\alpha$ can be parameterized by:*

$$\gamma(t) := \begin{cases} t, & 0 \le t < \frac{1}{4} \\ (t-\frac{1}{4})e^{i\pi\alpha} + \frac{1}{4}, & \frac{1}{4} \le t < \frac{1}{2} \\ (\frac{3}{4}-t) + \frac{1}{4}e^{i\pi\alpha}, & \frac{1}{2} \le t < \frac{3}{4} \\ (1-t)e^{i\pi\alpha}, & \frac{3}{4} \le t < 1 \end{cases}$$

Note that $|\gamma'(t)| = 1$ outside the corners. By symmetry

$$E(\gamma) = 4\int_0^{\frac{1}{4}} E(\gamma,\gamma(t))dt.$$

If $t \in (0,\frac{1}{4})$ then

$$\int_0^{\frac{1}{4}}\left(\frac{1}{|\gamma(t)-\gamma(s)|^2} - \frac{1}{D(\gamma(t),\gamma(s))^2}\right)ds = \int_0^{1}\left(\frac{1}{|t-s|^2} - \frac{1}{|t-s|^2}\right)ds = 0.$$

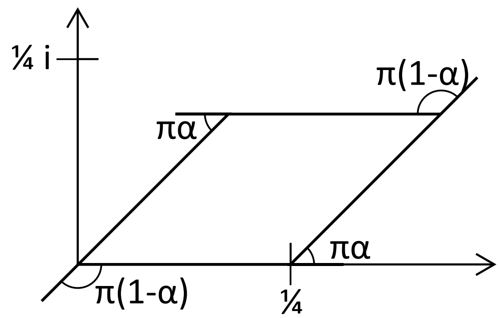

Figure 1: The rhombus given by γ.

Next

$$
\begin{aligned}
&\int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{|\gamma(t)-\gamma(s)|^2} - \frac{1}{D(\gamma(t),\gamma(s))^2} \right) ds \\
&\quad = \int_0^{\frac{1}{4}} \left(\frac{1}{(\frac{1}{4}-t)^2 \sin(\pi\alpha)^2 + (s+(\frac{1}{4}-t)\cos(\pi\alpha))^2} - \frac{1}{(s+\frac{1}{4}-t)^2} \right) ds \\
&\qquad = \frac{\arctan(\frac{\frac{1}{4}+(\frac{1}{4}-t)cos(\pi\alpha)}{(\frac{1}{4}-t)\sin(\pi\alpha)}) - \frac{\pi}{2} + \pi\alpha}{(\frac{1}{4}-t)\sin(\pi\alpha)} + \frac{1}{\frac{1}{2}-t} - \frac{1}{\frac{1}{4}-t} \\
&\qquad = \left(\frac{\pi\alpha}{\sin(\pi\alpha)} - 1 \right) \frac{1}{\frac{1}{4}-t} - \frac{\arctan(\frac{(\frac{1}{4}-t)\sin(\pi\alpha)}{\frac{1}{4}+(\frac{1}{4}-t)\cos(\pi\alpha)})}{(\frac{1}{4}-t)\sin(\pi\alpha)} + \frac{1}{\frac{1}{2}-t}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\int_{\frac{3}{4}}^{1} \left(\frac{1}{|\gamma(t)-\gamma(s)|^2} - \frac{1}{D(\gamma(t),\gamma(s))^2} \right) ds \\
&\qquad = \left(\frac{\pi(1-\alpha)}{\sin(\pi(1-\alpha))} - 1 \right) \frac{1}{t} - \frac{\arctan(\frac{t\sin(\pi\alpha)}{\frac{1}{4}-t\cos(\pi\alpha)})}{t\sin(\pi\alpha)} + \frac{1}{\frac{1}{4}+t}.
\end{aligned}
$$

Lastly

$$\int_{\frac{1}{2}}^{\frac{1}{4}} \left(\frac{1}{|\gamma(t)-\gamma(s)|^2} - \frac{1}{D(\gamma(t),\gamma(s))^2} \right) ds$$

$$= \int_0^{\frac{1}{4}} \frac{ds}{|t-(s+\frac{1}{4}e^{i\pi\alpha})|^2} - \int_{\frac{1}{2}}^{\frac{1}{2}+t} \frac{ds}{(s-t)^2} ds - \int_{\frac{1}{2}+t}^{\frac{3}{4}} \frac{ds}{(1-s+t)^2} =$$

$$= \int_0^{\frac{1}{4}} \frac{ds}{(\frac{1}{4}\sin(\pi\alpha))^2 + (s-t+\frac{1}{4}\cos(\pi\alpha))^2} + 2 - \frac{1}{\frac{1}{2}-t} + 2 - \frac{1}{\frac{1}{4}+t}$$

$$= \frac{\arctan\left(\frac{1-4t+\cos(\pi\alpha)}{\sin(\pi\alpha)}\right) + \arctan\left(\frac{4t-\cos(\pi\alpha)}{\sin(\pi\alpha)}\right)}{\sin(\pi\alpha)} + 4 - \frac{1}{\frac{1}{2}-t} - \frac{1}{\frac{1}{4}+t}.$$

Therefore

$$E(\gamma,\gamma(t)) = \frac{\arctan\left(\frac{1-4t+\cos(\pi\alpha)}{\sin(\pi\alpha)}\right) + \arctan\left(\frac{4t-\cos(\pi\alpha)}{\sin(\pi\alpha)}\right)}{\sin(\pi\alpha)} + 4$$

$$-\frac{\arctan(\frac{(\frac{1}{4}-t)\sin(\pi\alpha)}{\frac{1}{4}+(\frac{1}{4}-t)\cos(\pi\alpha)})}{(\frac{1}{4}-t)\sin(\pi\alpha)} - \frac{\arctan(\frac{t\sin(\pi\alpha)}{\frac{1}{4}-t\cos(\pi\alpha)})}{t\sin(\pi\alpha)} + \frac{\frac{\pi\alpha}{\sin(\pi\alpha)}-1}{\frac{1}{2}-t} + \frac{\frac{\pi(1-\alpha)}{\sin(\pi(1-\alpha))}-1}{t}.$$

This is clearly infinite when integrated on $(0,\frac{1}{4})$.

To get something that can be finite for piece-wise C^1 functions we define a regularized version of Möbius energy.

Definition 11. *Assume* γ *is a simple closed curve in* $\mathbb{R}^3 \cup \{\infty\}$ *and assume further that there are* $z_1, ..., z_n \in \mathbb{R}^3$, $\pi\alpha_1, ..., \pi\alpha_n \in (-\pi,\pi)$ *such that the* z_k *lie on* γ,γ *is* C^1 *outside the* z_k *and such that the exterior angle of* γ *at* z_k *is* $\pi\alpha_k$. *Then we can define the regularized Möbius energy as*

$$\hat{E}(\gamma) := \lim_{\epsilon\to 0} \int_{z\in\gamma:\forall j:\epsilon<||z-z_j||} E(\gamma,z)d|z| + 2\sum_{k=1}^{n} \mu_k ln(\epsilon)$$

where

$$\mu_k := \frac{\pi\alpha_k}{\sin(\pi\alpha_k)} - 1.$$

It will be practical to take the labeling of the z_k such that $z_{k+nj} = z_k$ for any integer j and also take them such that γ is C^1 on the arc between z_k and z_{k+1} for all k.

Lemma 2. *If* γ *is bounded and* $E(\gamma,z) - \sum_{k=1}^{n} \frac{\mu_k}{D(z,z_k)}$ *is integrable along* γ

then

$$\hat{E}(\gamma) = \int_\gamma \left(E(\gamma, z) - \sum_{k=1}^{n} \frac{\mu_k}{D(z, z_k)} \right) d|z| + 2 \sum_{k=1}^{n} \mu_k \ln\left(\frac{l(\gamma)}{2}\right) =$$

$$\iint_{\gamma\times\gamma} \left(\frac{1}{||z-w||^2} - \frac{1}{D(z,w)^2} - \frac{1}{2}\sum_{k=1}^{n} \mu_k \frac{1}{(D(z,z_k)+D(w,z_k))^2} \right) d|w|d|z| + 2\sum_{k=1}^{n} \mu_k \ln\left(\frac{l(\gamma)}{4}\right).$$

Proof. Let γ be parameterized by arc length and let t_k be such that $z_k = \gamma(t_k)$. Then $||\gamma(t_k) - \gamma(t)|| = |t_k - t| + O(t_k - t)^2$. Since the integrand is $O(t_k - t)$ we get

$$\int_{\epsilon<||\gamma(t)-\gamma(t_k)||} \frac{dt}{D(\gamma(t),\gamma(t_k))} = \int_{\epsilon<|t-t_k|} \frac{dt}{D(\gamma(t),\gamma(t_k))} + O(\epsilon).$$

Then

$$\begin{aligned}
&\int_{\forall j:\epsilon<||\gamma(t)-\gamma(t_j)||} \frac{dt}{D(\gamma(t),\gamma(t_k))} \\
&\quad= \left(\int_{\epsilon<||\gamma(t)-\gamma(t_k)||} - \int_{\forall j\neq k:\epsilon<||\gamma(t)-\gamma(t_j)||} \right) \frac{dt}{D(\gamma(t),\gamma(t_k))} \\
&\quad= \int_{\epsilon<|t-t_k|} \frac{dt}{D(\gamma(t),\gamma(t_k))} + O(\epsilon) \\
&\quad= 2\int_{\epsilon}^{l(\gamma)/2} \frac{dt}{t} + O(\epsilon) = 2\ln\left(\frac{l(\gamma)}{2}\right) - 2\ln(\epsilon) + O(\epsilon),
\end{aligned}$$

so by the dominated convergence theorem

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int_{\forall j:\epsilon<||z-z_j||} E(\gamma,z)d|z| + 2\sum_{k=1}^{n} \mu_k \ln(\epsilon) \\
&= \lim_{\epsilon\to 0} \int_{\forall j:\epsilon<||z-z_j||} \left(E(\gamma,\gamma(t)) - \sum_{k=1}^{n} \frac{\mu_k}{D(\gamma(t),\gamma(t_k))} \right) dt + 2\sum_{k=1}^{n} \mu_k \ln\left(\frac{l(\gamma)}{2}\right) \\
&\quad= \int_\gamma \left(E(\gamma,z) - \sum_{k=1}^{n} \frac{\mu_k}{D(z,z_k)} \right) d|z| + 2\sum_{k=1}^{n} \mu_k \ln\left(\frac{l(\gamma)}{2}\right).
\end{aligned}$$

□

Theorem 6. *Assume that* γ *is a bounded simple closed curve in* $\mathbb{R}^3$ *with corners* z_k *of exterior angle* $\pi\alpha_k$ *and that it is* C^2 *on the segments between the corners. Then* $\hat{E}(\gamma) < \infty$

Proof. γ has an arc length parameterization such that $0 < t_1 < ... < t_n = l - t_1$ with $\gamma(t_k) = z_k$ and $\gamma'(t_k+) = R_k\gamma'(t_k-)$ where R_k is a rotation matrix of angle $\pi\alpha_k$. Let $l_1 = 2t_1, l_k = t_k - t_{k-1}$. We may assume $l_k < \frac{l(\gamma)}{4}$ by adding angle 0 corners. Define

$$E_k(\gamma, z) = \int_{\gamma|_{(t_k - \frac{l_k}{2}, t_k + \frac{l_{k+1}}{2})}} \left(\frac{1}{||z-w||^2} - \frac{1}{D(z,w)^2} \right) d|w|.$$

Then by the above lemma it is sufficient to show that for real $0 < |t| < \frac{l}{2}$

$$E_k(\gamma, \gamma(t_k + t)) - \frac{\mu_k}{D(\gamma(t_k+t), \gamma(t_k))} = E_k(\gamma, \gamma(t_k+t)) + \frac{1}{|t|} - \frac{\pi\alpha_k}{\sin(\pi\alpha_k)|t|}$$

is integrable. It will be shown for $t > 0$, the case $t < 0$ is similar. If t is not near 0 then $E_k(\gamma, \gamma(t_k + t))$, $\frac{1}{D(\gamma(t_k+t),\gamma(t_k))^2}$ are bounded so it is sufficient to look at $t \leq t_k + \frac{l_{k+1}}{2}$.

First since $\gamma|_{[t_k, t_k + \frac{l_{k+1}}{2}]}$ is C^2 properties of the Möbius energy give

$$\int_0^{\frac{l_{k+1}}{2}} \left(\frac{1}{||\gamma(t_k+t) - \gamma(t_k+s)||^2} - \frac{1}{D(\gamma(t_k+t) - \gamma(t_k+s))^2} \right) ds \in L^1(\gamma).$$

Next

$$\frac{1}{t} - \int_0^{\frac{l_k}{2}} \frac{ds}{D(\gamma(t_k+t), \gamma(t_k-s))^2} = \frac{1}{t} - \int_0^{\frac{l_k}{2}} \frac{ds}{(s+t)^2} = \frac{1}{t + \frac{l_k}{2}} \in L^1(\gamma).$$

Now what remains is to show that

$$\int_0^{\frac{l_k}{2}} \frac{ds}{||\gamma(t_k+t) - \gamma(t_k-s)||^2} - \frac{\pi\alpha_k}{\sin(\pi\alpha_k)t} \in L^1(\gamma).$$

Note that

$$\begin{aligned} \gamma(t_k+t) - \gamma(t_k-s) &= \gamma(t_k+t) - \gamma(t_k) - (\gamma(t_k-s) - \gamma(t_k)) \\ &= t\gamma'(t_k+) + O(t^2) + s\gamma'(t_{k+1}-) + O(s^2) \\ &= \gamma'(t_k-)(R_k t + s) + O((s+t)^2). \end{aligned}$$

By rotating we see that $||\gamma'(t_k-)(s + R_k t)|| = |s + te^{i\pi\alpha_k}|$. One has

$$s + t \geq |s + te^{i\pi\alpha_k}| \geq |Re(se^{-i\pi\frac{\alpha_k}{2}} + te^{i\pi\frac{\alpha_k}{2}})| \geq cos(\pi\frac{\alpha_k}{2})(s+t)$$

where $\cos(\pi\frac{\alpha_k}{2}) > 0$. Then $||\gamma(t_k+t) - \gamma(t_k-s)||^2 = |s + te^{i\pi\alpha}|^2 + O((s+t)^3)$ and

$$\begin{aligned} \frac{1}{||\gamma(t_k+t) - \gamma(t_k-s)||^2} &= \frac{1}{|s+te^{i\pi\alpha}|^2}\frac{1}{1+O(s+t)} \\ &= \frac{1}{|s+te^{i\pi\alpha}|^2}(1 + O(s+t)) = \frac{1}{|s+te^{i\pi\alpha}|^2} + O\left(\frac{1}{s+t}\right). \end{aligned}$$

If $\pi\alpha_k = 0$ one gets $|s + te^{\pm i\pi\alpha_k}| = s + t$ and $\frac{\pi\alpha_k}{\sin \pi\alpha_k} = 1$ so

$$\int_0^{\frac{l_k}{2}} \left(\frac{1}{|s + te^{\pm i\pi\alpha_k}|^2} + O\left(\frac{1}{s+t}\right)\right) ds - \frac{\pi\alpha_k}{\sin(\pi\alpha_k)t}$$
$$= -\frac{1}{\frac{l_k}{2} + t} + O\left(\ln\left(t + \frac{l_k}{2}\right) - \ln(t)\right),$$

and otherwise

$$\int_0^{\frac{l_k}{2}} \left(\frac{1}{|s + te^{\pm i\pi\alpha}|^2} + O\left(\frac{1}{s+t}\right)\right) ds - \frac{\pi\alpha_k}{\sin(\pi\alpha_k)t}$$
$$= \int_0^{\frac{l_k}{2}} \frac{1}{(s + t\cos(\pi\alpha_k))^2 + (t\sin(\pi\alpha_k))^2} ds + O\left(\ln\left(1 + \frac{l_k}{2t}\right)\right) - \frac{\pi\alpha_k}{\sin(\pi\alpha_k)t}$$
$$= \frac{1}{t\sin\pi\alpha_k}\left[\arctan\left(\frac{s + t\cos(\pi\alpha_k)}{t\sin(\pi\alpha_k)}\right)\right]_{s=0}^{\frac{l_k}{2}} + O\left(\ln\left(1 + \frac{l_k}{2t}\right)\right) - \frac{\pi\alpha_k}{\sin\pi\alpha_k t}$$

$$= -\frac{1}{\sin\pi\alpha_k t}\arctan\left(\frac{t\sin(\pi\alpha_k)}{\frac{l_k}{2} + t\cos(\pi\alpha_k)}\right) + O\left(\ln\left(1 + \frac{l_k}{2t}\right)\right).$$

In both cases it is integrable. □

When looking at curves in the plane we unlike the normal Möbius energy have that this energy is not Möbius invariant. Instead we have

Theorem 7. *Assume γ is a Jordan curve with corners z_k of exterior angle $\pi\alpha_k$ and is C^2 on the segments between the corners. For any Möbius transformation T such that $T^{-1}(\infty)$ does not lie on γ*

$$\hat{E}(T \circ \gamma) = \hat{E}(\gamma) + 2\sum_{k=1}^{n} \mu_k \ln|T'(z_k)|,$$

and for any Möbius transformation T such that $T^{-1}(\infty)$ is on γ but not on any corner

$$\hat{E}(T \circ \gamma) = \hat{E}(\gamma) - 4 + 2\sum_{k=1}^{n} \mu_k \ln|T'(z_k)|.$$

Proof. Assume γ is parameterized by arc length and $\gamma(t_k) = z_k$. Since Möbius transformations are conformal $T \circ \gamma$ has the same angle at t_k that γ has. If

$T(z) = az + b$ then $|T'| = |a|$ and

$$\hat{E}(T \circ \gamma) = \lim_{\epsilon \to 0} \int_{\forall j: \epsilon < |T\circ\gamma(t) - az_j|} E(T \circ \gamma, T \circ \gamma(t))|(T \circ \gamma)'(t)|dt + 2\sum_{k=1}^{n} \mu_k \ln(\epsilon)$$

$$= \lim_{\epsilon \to 0} \int_{\forall j: \frac{\epsilon}{|a|} < |\gamma(t) - z_j|} E(\gamma, \gamma(t))|\gamma'(t)|dt + 2\sum_{k=1}^{n} \mu_k \ln\left(\frac{\epsilon}{|a|}\right) + 2\sum_{k=1}^{n} \mu_k \ln|a|$$

$$= \hat{E}(\gamma) + 2\sum_{k=1}^{n} \mu_k \ln|a|.$$

Next assume $I(z) = \frac{1}{z}$. We have that

$$|I(z) - I(z_k)| = |I'(z_k)(z - z_k) + O(z - z_k)^2| = |I'(z_k)||z - z_k| + O(z - z_k)^2$$

and we know that $E(\gamma, z) = O(z - z_k)$ for z near z_k. Hence for small ϵ

$$\int_{\forall j: \frac{\epsilon}{|I'(z_k)|} < |z - z_k|} E(\gamma, z)d|z| = \int_{\forall j: \epsilon < |I(z) - (z_k)|} E(\gamma, z)d|z| + O(\epsilon).$$

Combining this with lemma 1 we get that if $l(I \circ \gamma) < \infty$ then

$$\hat{E}(I \circ \gamma)$$

$$= \lim_{\epsilon \to 0} \int_{t: \forall j: \epsilon < |I\circ\gamma(t) - I(z_k)|} E(\gamma, I \circ \gamma(t))\,|(I \circ \gamma)'(t)|\,dt + 2\sum_{k=1}^{n} \mu_k \ln(\epsilon)$$

$$= \lim_{\epsilon \to 0} \int_{t: \forall j: \epsilon < |I\circ\gamma(t) - I(z_k)|} \left(E(\gamma, t) - \frac{4}{l(\gamma)} + \frac{4|(I \circ \gamma)'(t)|}{l(I \circ \gamma)}\right) dt + 2\sum_{k=1}^{n} \mu_k \ln(\epsilon)$$

$$= \lim_{\epsilon \to 0} \int_{t: \forall j: \frac{\epsilon}{|I'(z_k)|} < |z - z_k|} E(\gamma, t)dt + 2\sum_{k=1}^{n} \mu_k \ln\left(\frac{\epsilon}{|I'(z_k)|}\right) + 2\sum_{k=1}^{n} \mu_k \ln|I'(z_k)|$$

$$= \hat{E}(\gamma) + 2\sum_{k=1}^{n} \mu_k \ln|I'(z_k)|,$$

whereas if $l(I \circ \gamma) < \infty$ then $\frac{4|(I\circ\gamma)'(t)|}{l(I\circ\gamma)} = 0$ so

$$\hat{E}(I \circ \gamma) = \hat{E}(\gamma) - 4 + 2\sum_{k=1}^{n} \mu_k \ln|I'(z_k)|.$$

Now an arbitrary Möbius transformation T can be written as $T_2 \circ I \circ T_1$. So

$$E(T \circ \gamma) =$$

$$E(\gamma) + 2\sum_{k=1}^{n} \mu_k \ln|T_1'(z_k)| + 2\sum_{k=1}^{n} \mu_k \ln|I'(T_1(z_k))| + 2\sum_{k=1}^{n} \mu_k \ln|T_2'(I(T_1(z_k)))|$$

$$= E(\gamma) + 2\sum_{k=1}^{n} \mu_k \ln|T'(z_k)|$$

or

$$E(T\circ\gamma) =$$
$$E(\gamma)+2\sum_{k=1}^{n}\mu_k \ln|T_1'(z_k)|-4+2\sum_{k=1}^{n}\mu_k \ln|I'(T_1(z_k))|+2\sum_{k=1}^{n}\mu_k \ln|T_2'(I(T_1(z_k)))|$$
$$= E(\gamma) - 4 + 2\sum_{k=1}^{n}\mu_k \ln|T'(z_k)|.$$

□

Example 2. *With γ parameterizing the rhombus as in Example 1 we see that it has corners of exterior angle $\pi(1-\alpha)$ at $0, \frac{1}{2}$ and $\pi\alpha$ at $\frac{1}{4}, \frac{3}{4}$. Then*

$$\hat{E}(\gamma) =$$
$$4\int_0^{\frac{1}{4}}\left(E(\gamma,\gamma(t)) - \frac{\frac{\pi\alpha}{\sin(\pi\alpha)}-1}{\frac{1}{2}-t} - \frac{\frac{\pi(1-\alpha)}{\sin(\pi(1-\alpha))}-1}{t}\right)dt + \left(\frac{2\pi}{\sin(\pi\alpha)}-4\right)\ln(2)$$
$$= 4\int_0^1\left(\frac{\arctan\left(\frac{t+\cos(\pi\alpha)}{\sin(\pi\alpha)}\right)+\arctan\left(\frac{t-\cos(\pi\alpha)}{\sin(\pi\alpha)}\right)}{\sin(\pi\alpha)}+1\right.$$
$$-\frac{\arctan(\frac{t\sin(\pi\alpha)}{1+t\cos(\pi\alpha)})+\arctan(\frac{t\sin(\pi\alpha)}{1-t\cos(\pi\alpha)})}{t\sin(\pi\alpha)}$$
$$\left.-\frac{\frac{\pi}{\sin\pi\alpha}-2}{1+t}\right)dt + \left(\frac{2\pi}{\sin(\pi\alpha)}-4\right)\ln(2)$$
$$= 4\frac{(1+\cos(\pi\alpha))\arctan\left(\frac{1+\cos(\pi\alpha)}{\sin(\pi\alpha)}\right)+(1-\cos(\pi\alpha))\arctan\left(\frac{1-\cos(\pi\alpha)}{\sin(\pi\alpha)}\right)}{\sin(\pi\alpha)}$$
$$-\frac{4\cos(\pi\alpha)(\pi-2\pi\alpha)}{\sin(\pi\alpha)} - 4\ln(2\sin(\pi\alpha)) + 16 - \left(\frac{2\pi}{\sin(\pi\alpha)}-4\right)\ln(2)$$
$$+\frac{2i}{\sin(\pi\alpha)}\sum_{j=1}^{\infty}\frac{e^{ni\pi\alpha}-e^{-ni\pi\alpha}+e^{ni(\pi(1-\alpha))}-e^{ni(\pi(1-\alpha))}}{2i\sin(\pi\alpha)n^2}$$

3.2 Loewner energy for polygons

The Loewner energy will have a slightly different behaviour near corners then the Möbius energy. This we can see by looking at how it behaves for polygons, since for polygons we know what the Riemann maps look like.

The Schwarz-Christoffel formula states that a Riemann map $f:\mathbb{D}\to D$ where D is a polygon with corners $w_1,...,w_n$ and exterior angles $\alpha_k\pi$ will be of the form $f(z) = A\int_0^z\prod_{k=1}^n(\zeta-z_k)^{-\alpha_k}d\zeta + f(0)$, where A is a constant and $f(z_k) = w_k$. [18]

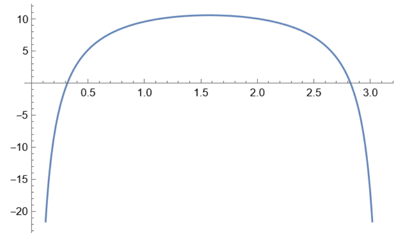

Figure 2: Energy of a unit rhombus depending on its angle plotted in wolfram mathematica

As noted in the beginning of this section they argued in [2] that the Dirichlet energy of $\ln f'$ has a divergence $\pi \sum_{k=1}^n \frac{\alpha_k^2}{1-\alpha_k} \ln(\epsilon^{-1})$. More precisely we get a regularized Dirichlet energy $J(D)$ given by:

Theorem 8. *For a Riemann map f which maps $\mathbb{D}$ to a polygon D with corners $f(z_k) = w_k$ and outer angles $\alpha_k \pi$ we have that*

$$J(D) := \lim_{\epsilon \to 0} \frac{1}{\pi} \iint_{\{z \in \mathbb{D} | \forall k : |f(z) - w_k| \geq \epsilon\}} \left| \frac{f''}{f'} \right|^2 dA - \sum_{k=1}^n \frac{\alpha_k^2}{1-\alpha_k} \ln(\epsilon^{-1}) =$$
$$\sum_{k=1}^n \left(\frac{\alpha_k^2}{1-\alpha_k} \ln \left| \frac{1-\alpha_k}{C_k} \right| + \alpha_k^2 \left(\ln(2) - \frac{2}{\pi} \int_0^1 \frac{\arcsin(r)}{r} dr \right) \right)$$
$$- \sum_{1 \leq j < k \leq n} \alpha_j \alpha_k \ln(2 - 2Re(z_j \overline{z_k}))$$

where $C_k = \lim_{z \to z_k} (z - z_k)^{\alpha_k} f'(z)$.

Proof. We have that $\frac{f''(z)}{f'(z)} = -\sum_{j=1}^n \frac{\alpha_j}{z - z_j}$ so

$$\left| \frac{f''(z)}{f'(z)} \right|^2 = \sum_{k=1}^n \frac{\alpha_k^2}{|z - z_k|^2} + \sum_{j<k} Re \left(\frac{2\alpha_j \alpha_k}{(z - z_j)(\overline{z} - \overline{z_k})} \right).$$

To see how it behaves we cut out a disk of radius ϵ around z_k. This means that we cut out from $\mathbb{D}$ the z with $|f(z) - f(z_k)| < \epsilon$. Since $(z - z_k)^{\alpha_k} f'(z)$ analytic near z_k we have

$$|f(z) - f(z_k)| = \left| \int_{z_k}^z f'(z) dz \right|^2 = \left| \int_{z_k}^z (z - z_k)^{-\alpha_k} (C_k + O(z - z_k)) dz \right|$$
$$= \left| \frac{C_k}{1-\alpha_k} (z - z_k)^{1-\alpha_k} + O(|z - z_k|^{2-\alpha_k}) \right|$$
$$= \left| \frac{C_k}{1-\alpha_k} (z - z_k)^{1-\alpha_k} \right| + O(|z - z_k|^{2-\alpha_k}).$$

If $z - z_k$ is sufficiently small the remainder term will be smaller than $|\frac{C_k}{1-\alpha_k}(z-z_k)^{1-\alpha_k}|$ which gives that $|z-z_k|^{1-\alpha_k} = O(\epsilon)$. Thus there is a function $R_k(z,\epsilon) = O\left(\epsilon^{\frac{2-\alpha_k}{1-\alpha_k}}\right)$ such that $|f(z) - f(z_k)| < \epsilon$ if and only if

$$|z - z_k| \leq \left|\frac{1-\alpha_k}{C_k}\epsilon\right|^{\frac{1}{1-\alpha_k}} + R_k(z,\epsilon).$$

We define $\delta = \left|\frac{1-\alpha_k}{C_k}\epsilon\right|^{\frac{1}{1-\alpha_k}}$. Note that the area of

$$\{z : |f(z) - w_k| < \epsilon \text{ or } |z - z_k| < \delta\} \backslash \{z : |f(z) - w_k| < \epsilon \text{ and } |z - z_k| < \delta\}$$

is $O(\epsilon^{\frac{1}{1-\alpha_k}}\epsilon^{\frac{2-\alpha_k}{1-\alpha_k}}) = O(\epsilon^{\frac{3-\alpha_k}{1-\alpha_k}})$. Also we have $|f''/f'| = O(\epsilon^{\frac{-2}{1-\alpha_k}})$ in that region. Thus the difference between integrating over $\{z \in \mathbb{D} | \forall k : |f(z) - w_k| \geq \epsilon\}$ and $\{z \in \mathbb{D} | \forall k : |z - z_k| \geq \epsilon\}$ is $O(\epsilon)$. Therefore we can instead look at the behaviour of the Dirichlet energy when we cut out disks of radius δ around z_k.

First we have that

$$\begin{aligned}
\iint_{\mathbb{D}} \sum_{j<k} Re\left(\frac{2\alpha_j\alpha_k}{(z-z_j)(\overline{z}-\overline{z_k})}\right) dA &= \sum_{j<k} Re\left(\int_0^1 \oint_{|\zeta|=1} \frac{\alpha_j\alpha_k 2(-i)\frac{r}{\zeta} d\zeta dr}{(r\zeta - z_j)(\frac{r}{\zeta} - \overline{z_k})}\right) \\
&= \sum_{j<k} \alpha_j\alpha_k Re\left(\int_0^1 \frac{4\pi r dr}{z_j\overline{z_k} - r^2}\right) \\
&= -2\pi \sum_{j<k} \alpha_j\alpha_k \ln|z_j\overline{z_k} - 1| \\
&= -\pi \sum_{j<k} \alpha_j\alpha_k \ln(2 - 2Re(z_j\overline{z_k}))
\end{aligned}$$

Next we look at the terms $\frac{\alpha_k^2}{|z-z_k|^2}$. By going to polar coordinates around z_k we have

$$\begin{aligned}
\iint_{\{z\in\mathbb{D}:|z-z_k|\geq\epsilon\}} \frac{\alpha_k^2}{|z-z_k|^2} dA &= \int_\delta^2 \int_{-\arccos(r/2)}^{\arccos(r/2)} \frac{\alpha_k^2}{r^2} r d\theta dr \\
= \alpha_k^2 \int_\delta^\infty \frac{2\arccos(\frac{r}{2})}{r} dr &= \alpha_k^2 \int_\delta^2 \frac{\pi}{r} dr - \alpha_k^2 \int_\delta^2 \frac{2\arcsin(\frac{r}{2})}{r} dr \\
&= \alpha_k^2 \ln(\delta^{-1}) + \alpha_k^2 \ln(2) - \alpha_k^2 \int_\delta^2 \frac{2\arcsin(\frac{r}{2})}{r} dr.
\end{aligned}$$

We have that $\frac{2\arcsin(\frac{r}{2})}{r} \to 1$ as $r \to 0$ so that integral is bounded.

□

The Schwarz-Christoffel formula for a mapping g from $\mathbb{D}$ to D^* the exterior of the polygon will be of the form $g(z) = a\int_0^z \frac{1}{\zeta^2}\prod_{k=1}^n (\zeta - \zeta_k)^{\alpha_k} d\zeta + b$, where

a and b are constants and $g(\zeta_k) = z_k$ [18]. Thus $h(z) = g(\frac{1}{z})$ is a Riemann map $\mathbb{D}^* \to D^*$. We have $\frac{h''(z)}{h'(z)} = -\frac{2}{z} - \frac{1}{z^2}\frac{g''(\frac{1}{z})}{g'(\frac{1}{z})}$ so

$$\iint_{\{z\in\mathbb{D}^*|\forall k:|h(z)-w_k|\geq\epsilon\}} \left|\frac{h''(z)}{h'(z)}\right|^2 dA$$
$$= \iint_{\{z\in\mathbb{D}|\forall k:|g(z)-w_k|\geq\epsilon\}} \left|2z + z^2\frac{g''(z)}{g'(z)}\right|^2 \frac{dA}{|z|^2}$$
$$= \iint_{\{z\in\mathbb{D}|\forall k:|g(z)-w_k|\geq\epsilon\}} \left|\sum_{k=1}^{n} \frac{\alpha_k}{z-\zeta_k}\right|^2 dA.$$

Thus we see that we get essentially the same integral. However instead of removing disks of radius $\left|\frac{1-\alpha_k}{C_k}\epsilon\right|^{\frac{1}{1-\alpha_k}}$ we want to remove disks of radius $\left|\frac{1+\alpha_k}{c_k}\epsilon\right|^{\frac{1}{1+\alpha_k}}$ where $c_k = \lim_{z\to\zeta_k}(z-\zeta_k)^{-\alpha_k}g'(z)$. I.e.

$$J(D^*) := \lim_{\epsilon\to 0}\frac{1}{\pi}\iint_{\{z\in\mathbb{D}^*|\forall k:|h(z)-w_k|>\epsilon\}} \left|\frac{h''}{h'}\right|^2 dA - 2\sum_{k=1}^{n}\frac{\alpha_k^2}{1+\alpha_k}\ln(\epsilon^{-1}) =$$
$$\sum_{k=1}^{n}\left(\frac{\alpha_k^2}{1+\alpha_k}\ln\left|\frac{1+\alpha_k}{c_k}\right| + \alpha_k^2\left(\ln(2) - \frac{2}{\pi}\int_0^1 \frac{\arcsin(r)}{r}dr\right)\right)$$
$$- \sum_{1\leq j<k\leq n}\alpha_j\alpha_k\ln(2-2Re(\zeta_j\overline{\zeta_k})).$$

This leads to a regularized Loewner energy

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\iint_{\{z\in\mathbb{D}:\forall k:|f(z)-w_k|>\epsilon\}}\left|\frac{f''}{f'}\right|^2 dA + \frac{1}{\pi}\iint_{\{z\in\mathbb{D}^*|\forall k:|h(z)-w_k|>\epsilon\}}\left|\frac{h''}{h'}\right|^2 dA$$
$$+ 4\ln\left|\frac{f'(0)}{h'(\infty)}\right| - 2\sum_{k=1}^{n}\frac{\alpha_k^2}{1-\alpha_k^2}\ln(\epsilon^{-1})$$
$$= J(D) + J(D^*) + 4\ln\left|\frac{f'(0)}{h'(\infty)}\right|.$$

4 Derivative of the Loewner energy

When writing this text it was not known if one could define the derivative of the Loewner energy. However a recent article by Bridgeman, Bromberg, Vargas Pallete and Wang proves the existence of the derivative as well as a gradient flow in the Weil-Petersson Teichmüller space [19]. Here we show that we can define the derivative of the Loewner energy in a similar way as for the Möbius Energy. To do this we first need this theorem due to Paileve [20].

Theorem 9. *If γ is a smooth Jordan curve and f is a Riemann map from $\mathbb{D}$ to the interior of γ then f, f^{-1} extend to smooth functions on the boundary.*

Corollary 1. *If γ is a smooth Jordan curve, f is a Riemann map from $\mathbb{D}$ to the interior of γ and g is a Riemann map from $\mathbb{D}^*$ to the exterior of γ. Then f, g are bi-Lipschitz.*

Next assume that $\gamma : \mathbb{R}/l\mathbb{Z} \to \mathbb{C}$ is a smooth Jordan curve and f, g are Riemann maps from $\mathbb{D}$ and $\mathbb{D}^*$ to the interior and exterior of γ respectively such that $f'(0), g'(0) > 0$. If $h_\epsilon : \mathbb{R}/l\mathbb{Z} \to \mathbb{C}$ is any family of smooth function such that it and all its derivatives tend uniformly to 0 as $\epsilon \to 0$ then for all sufficiently small ϵ $\gamma + h_\epsilon$ will be a smooth Jordan curve and $f(0)$ will be in the interior of $\gamma + h_\epsilon$. Thus we can for those ϵ define Riemann maps f_ϵ, g_ϵ with $f_\epsilon(0) = f(0)$ and $f'_\epsilon(0) > 0, g'_\epsilon(\infty) > 0$. For these maps we have the following result [20]

Theorem 10. *There exists $\epsilon_0 > 0$ such that $f_\epsilon : [0, \epsilon_0] \times \overline{\mathbb{D}}$ is smooth.*

Corollary 2. *If $h : \mathbb{R}/l\mathbb{Z} \to \mathbb{C}$ is smooth and f_ϵ, g_ϵ are Riemann maps from $\mathbb{D}$ and $\mathbb{D}^*$ to the interior and exterior of $\gamma+\epsilon h$ respectively such that $f'_\epsilon(0), g'_\epsilon(0) > 0$ and $f_\epsilon(0) = f(0)$ then there exists $\epsilon_0 > 0$ such that $f_\epsilon : [0, \epsilon_0] \times \overline{\mathbb{D}} \to \mathbb{C}, g_\epsilon : [0, \epsilon_0] \times \overline{\mathbb{D}^*} \to \mathbb{C}$ are smooth.*

Proof. It follows immediately from the theorem that we have $\epsilon_0 > 0$ such that $f_\epsilon : [0, \epsilon_0] \times \overline{\mathbb{D}} \to \mathbb{C}$ is smooth.

$\tilde{g}_\epsilon(w) = \frac{1}{g_\epsilon(\frac{1}{w}) - f(0)}$ is a Riemann map from $\mathbb{D}$ to the interior of $\frac{1}{\gamma+\epsilon h - f(0)}$. Let $\tilde{\gamma} = \frac{1}{\gamma - f(0)}$ and $\tilde{h}_\epsilon = \frac{1}{\gamma+\epsilon h - f(0)} - \frac{1}{\gamma - f(0)} = \epsilon \frac{h}{(\gamma - f(0))(\gamma+\epsilon h - f(0))}$. Clearly $\tilde{\gamma}$ is a smooth Jordan curve and $\tilde{h}_\epsilon$ is a smooth family of functions such that it and and all its derivatives tend uniformly to 0 as $\epsilon \to 0$. Therefore we have ϵ_0 such that $\tilde{g}_\epsilon : [0, \epsilon_0] \times \overline{\mathbb{D}} \to \mathbb{C}$ is smooth. Then $g_\epsilon : [0, \epsilon_0] \times \overline{\mathbb{D}^*}$ is smooth. ☐

Using this we can show that the Loewner energy is differentiable for some curves.

Theorem 11. *Let $\gamma : \mathbb{R}/l\mathbb{Z} \to \mathbb{C}$ be a smooth Jordan curve and $h : \mathbb{R}/l\mathbb{Z} \to \mathbb{C}$ any smooth function. Let f, f_ϵ be Riemann maps from $\mathbb{D}$ to the interior of γ and $\gamma + \epsilon h$ respectively. Let g, g_ϵ be Riemann maps from $\mathbb{D}^*$ to the exterior of γ and $\gamma+\epsilon h$ respectively. Assume $f(0) = f_\epsilon(0)$ and $f'(0), f'_\epsilon(0), g'(\infty), g'_\epsilon(\infty) > 0$. Then*

$$\nabla_h I^L(\gamma) = \lim_{\epsilon\to 0} \frac{I^L(\gamma+\epsilon h) - I^L(\gamma)}{\epsilon} =$$
$$\frac{2}{\pi} \iint_{\mathbb{D}} \left\langle \frac{f''}{f'}, \frac{f' \frac{\partial}{\partial\epsilon} f''_0 - f'' \frac{\partial}{\partial\epsilon} f'_0}{(f')^2} \right\rangle dA + \frac{2}{\pi} \iint_{\mathbb{D}^*} \left\langle \frac{g''}{g'}, \frac{g' \frac{\partial}{\partial\epsilon} g''_0 - g'' \frac{\partial}{\partial\epsilon} g'_0}{(g')^2} \right\rangle dA$$
$$+ 4 \left(\frac{\frac{\partial}{\partial\epsilon} f'_0(0)}{f'(0)} - \frac{\frac{\partial}{\partial\epsilon} g'_0(\infty)}{g'(\infty)} \right).$$

Proof. As $\epsilon \to 0$

$$\begin{aligned}\frac{1}{\epsilon}\left(\frac{|f_\epsilon''|^2}{|f_\epsilon'|^2}-\frac{|f''|^2}{|f'|^2}\right) &= \frac{|f_\epsilon'' f'|^2-|f_\epsilon' f''|^2}{\epsilon|f_\epsilon'|^2|f'|^2}\\ &= \frac{\langle f_\epsilon'' f' + f'' f_\epsilon', \frac{f_\epsilon'' f' - f'' f_\epsilon'}{\epsilon}\rangle}{|f'|^2|f_\epsilon'|^2}\\ &\to 2\frac{\langle f'' f', f'\frac{\partial}{\partial\epsilon}f_0'' - f''\frac{\partial}{\partial\epsilon}f_0'\rangle}{|f'|^4}\end{aligned}$$

Since $f, f_\epsilon$are bi-Lipschitz and $f_\epsilon' \to f'$ we can find m such that $|f_\epsilon'|, |f'| > m$. Then it is easy to see that we can apply dominated convergence to get

$$\lim_{\epsilon\to 0}\iint_{\mathbb{D}}\frac{1}{\epsilon}\left(\frac{|f_\epsilon''|^2}{|f_\epsilon'|^2}-\frac{|f''|^2}{|f'|^2}\right)dA = 2\iint_{\mathbb{D}}\left\langle\frac{f''}{f'}, \frac{f'\frac{\partial}{\partial\epsilon}f_0''-f''\frac{\partial}{\partial\epsilon}f_0'}{(f')^2}\right\rangle dA.$$

For g we use

$$\iint_{\mathbb{D}^*}\frac{1}{\epsilon}\left(\frac{|g_\epsilon''|^2}{|g_\epsilon'|^2}-\frac{|g''|^2}{|g'|^2}\right)dA = \iint_{\mathbb{D}}\frac{1}{\epsilon}\left(\frac{|g_\epsilon(\frac{1}{z})''|^2}{|g_\epsilon(\frac{1}{z})'|^2}-\frac{|g''(\frac{1}{z})|^2}{|g'(\frac{1}{z})|^2}\right)\frac{dA}{|z|^2}.$$

We have

$$\begin{aligned}\left(\frac{|g_\epsilon''(\frac{1}{z})|^2}{|z|^2|g_\epsilon'(\frac{1}{z})|^2}-\frac{|g''(\frac{1}{z})|^2}{|z|^2|g'(\frac{1}{z})|^2}\right) &= \frac{\langle g_\epsilon'' g' + g'' g_\epsilon', g'\frac{g_\epsilon''-g''}{\epsilon} - g''\frac{g_\epsilon'-g'}{\epsilon}\rangle}{|z|^2|g'|^2|g_\epsilon'|^2}\\ &\to 2\frac{\langle g'' g', g'\frac{\partial}{\partial\epsilon}g_0'' - g''\frac{\partial}{\partial\epsilon}g_0'\rangle}{|z|^2|g'|^4}.\end{aligned}$$

We know $g''(\infty) = 0, g_\epsilon''(\infty) = 0$ so $\frac{g''(\frac{1}{z})}{z}, \frac{g_\epsilon''(\frac{1}{z})}{z}, \frac{g_\epsilon''(\frac{1}{z})-g''(\frac{1}{z})}{\epsilon z}$ is bounded in $\overline{\mathbb{D}}$ and for small ϵ. Hence we can again use dominated convergence to get

$$\lim_{\epsilon\to 0}\iint_{\mathbb{D}^*}\frac{1}{\epsilon}\left(\frac{|g_\epsilon''|^2}{|g_\epsilon'|^2}-\frac{|g''|^2}{|g'|^2}\right)dA = 2\iint_{\mathbb{D}^*}\left\langle\frac{g''}{g'}, \frac{g'\frac{\partial}{\partial\epsilon}g_0''-g''\frac{\partial}{\partial\epsilon}g_0'}{(g')^2}\right\rangle dA.$$

Lastly

$$\lim_{\epsilon\to 0}\frac{\ln|f_\epsilon'(0)| - \ln|f'(0)|}{\epsilon} = \frac{\frac{\partial}{\partial\epsilon}f_0'(0)}{f'(0)}$$

and

$$\lim_{\epsilon\to 0}\frac{\ln|g_\epsilon'(\infty)| - \ln|g'(\infty)|}{\epsilon} = \frac{\frac{\partial}{\partial\epsilon}g_0'(\infty)}{g'(\infty)}.$$

□

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