Regularity properties of two-phase free boundary problems

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Abstract

This thesis consists of four papers which are all related to the regularity properties of free boundary problems. The problems considered have in common that they have some sort of two-phase behaviour.

In papers I-III we study the interior regularity of different two-phase free boundary problems. Paper I is mainly concerned with the regularity properties of the free boundary, while in papers II and III we devote our study to the regularity of the function, but as a by-product we obtain some partial regularity of the free boundary.

The problem considered in paper IV has a somewhat different nature. Here we are interested in certain approximations of the obstacle problem. Two major differences are that we study regularity properties close to the fixed boundary and that the problem converges to a one-phase free boundary problem.

Sammanfattning

Denna avhandling består av fyra artiklar i vilka vi studerar problem relaterade till frirandsproblem med två faser. Gemensamt för alla artiklar är att den nivåyta som studeras utgör varken ett minimum eller ett maximum för funktionen i fråga. Därför kan vi säga att de alla har ett slags två-fas-beteende.

I de tre första artiklarna studerar vi inre regularitetsegenskaper hos olika frirandsproblem med två faser. Artikel I handlar om regularitetsegenskaperna hos den fria randen, medan artiklarna II och III mestdels fokuserar på funktionens regularitet.

Den fjärde artikeln skiljer sig avsevärt från de tre första. Vi studerar här en approximation av det så kallade hinderproblemet. Skillnaden mot de tre första artiklarna består dels i att vi intresserar oss för regulariteten nära den fixa randen, dels i att problemet konvergerar mot ett en-fas problem.

Acknowledgements

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While being a PhD student I have encountered many different people and personalities. Some of them I am more grateful to than others. In particular, there are certain persons that need to be mentioned.

Obviously, I am deeply indebted to my supervisor Henrik Shahgholian, a born optimist. He has always encouraged me, even at times when I refused to listen to what he said, something that happened every now and then. Many of the ideas I have used originate from him, and without his support, and sometimes also his pressure, this thesis would not exist.

In addition, I would like to thank Henrik's former student Arshak Petrosyan whom I visited in the autumn 2005. Arshak was the best host and tutor you could imagine. He spent an enormous amount of time explaining and discussing things with me. I really learned a lot from him. Thanks to Arshak the visit resulted in a paper contained in this thesis.

Without my fellow students Anders Edquist and Farid Bozorgnia I would not be where I am today. Thank you guys for not making me work. I think both me and Anders can agree on that if we hadn't had the same office for the last three years we probably would have finished a long time ago. Regarding Farid, I believe that if he did not have tea 35 times a day we might actually have gotten some work done.

Furthermore, I wish to thank those who have read my thesis and given me useful suggestions. This has been important for me during the preparation of my thesis.

Evidently, I wouldn't be where I am now without my family. You have always encouraged me to do (almost) whatever I want. For this and for all the other things you have done for me I am most thankful.

Last but not least, thank you Erica. I am very grateful for everything, in particular that you can stand me even when I am as annoying as I sometimes can be.

Erik Lindgren
Stockholm, February 2009

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Chapter 1

Introduction to free boundary problems

In this chapter, the reader will be given a brief introduction to free boundary problems (FBPs). To begin with, some more or less famous examples of FBPs will be given. This is followed by a more mathematical introduction to a large class of FBPs, where the main issues and the usual problems that arise in this area are described.

Regarding the references, I do not in any way claim to present a complete list, but rather a selection of what can be found in the vast literature.

1.1 The obstacle problem

Imagine an elastic string attached to two points and assume that we can neglect the influence of the gravity. Clearly, it will assume the form of a straight line. Suppose now that we push up the string with some kind of object that we call the obstacle. Then, at some points the string and the obstacle will touch. Moreover, if the obstacle is not flat, somewhere the string will stay above the obstacle. Thus, there it will still be "free", which means that it will still be a straight line there. See Figures 1.1 and 1.2. This is a typical free boundary problem. Here the free boundary is the set of points where the string leaves the obstacle. Mathematically, we can formulate this as follows. Suppose that the string is given by the graph of the function f on the interval $[0, 1]$ and that the obstacle is given by the graph of ϕ on $[0, 1]$. Then we fix the end points by saying $f(0) = a$ and $f(1) = b$. The string will

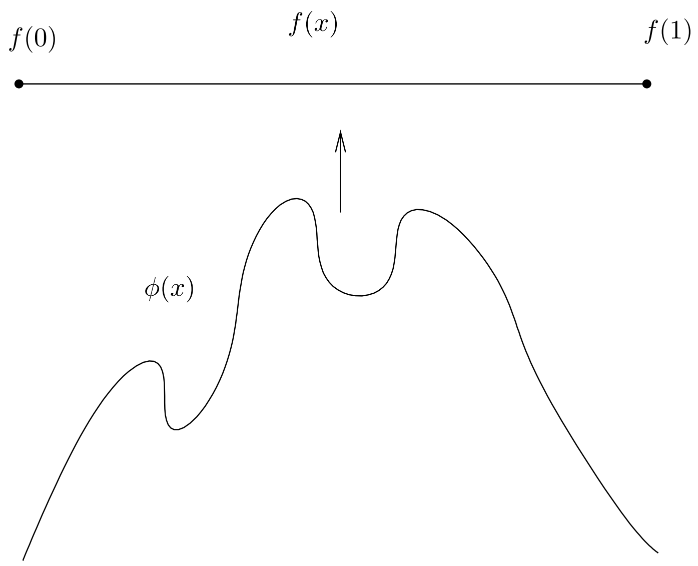

Figure 1.1: When the string does not touch the obstacle.

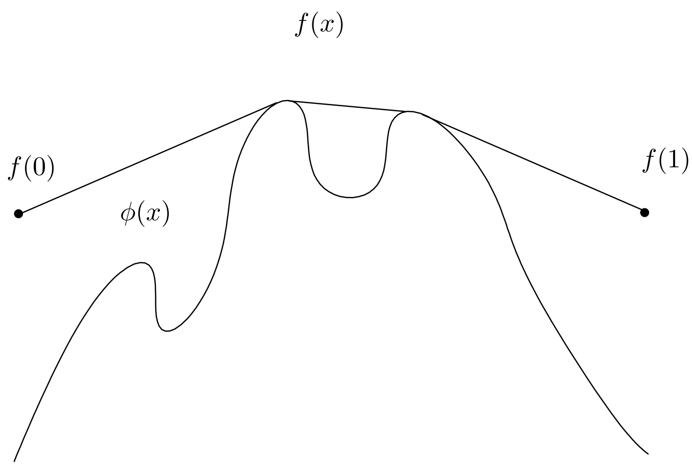

Figure 1.2: When the string is pushed up by the obstacle.

try to minimize the tension energy, which will be proportional to the length of the string, which in turn is given by

$$L = \int_a^b \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x.$$

The minimization will be under the constraints that $f(0) = a$, $f(1) = b$ and $f \geq \phi$. If we assume that we can linearize this, we obtain the approximation

$$L \approx 1 + \frac{1}{2} \int_a^b |f'(x)|^2\, \mathrm{d}x.$$

Then f will instead minimize

$$\int_a^b |f'(x)|^2\, \mathrm{d}x$$

under the constraints that $f(0) = a$, $f(1) = b$ and $f \geq \phi$. In fact, in one dimension this gives the exact same solution as we would get without the approximation.

From standard methods in the calculus of variations it follows that $f'' = 0$ in the set $\{f > \phi\}$, i.e., in the set where the string stays above the obstacle. This means that the graph of f is a line on every interval in this set. Moreover, it is not hard to prove that $f'' \leq 0$ everywhere.

Introducing the new function $u = f - \phi$, it follows, after some calculations, that

$$u'' = (-\phi'')\chi_{\{u>0\}}.$$

Since $f \geq \phi$, we also have $u \geq 0$.

We can generalize this formulation to higher dimensions. Then for some smooth domain D, we want to minimize

$$\int_D |\nabla f|^2\, \mathrm{d}x$$

over all functions above ϕ with some prescribed boundary values on ∂D. With the same translation as before, $u = f - \phi$, we will have

$$\Delta u = (-\Delta\phi)\chi_{\{u>0\}}.$$

See Figure 1.3 for an example in two dimensions. Both the problem formulated for u and the minimization problem for f are usually called *the obstacle problem*. Sometimes, when one wishes to emphasize the difference from the problem described in the following section, one refers to it as the *one-phase obstacle problem*.

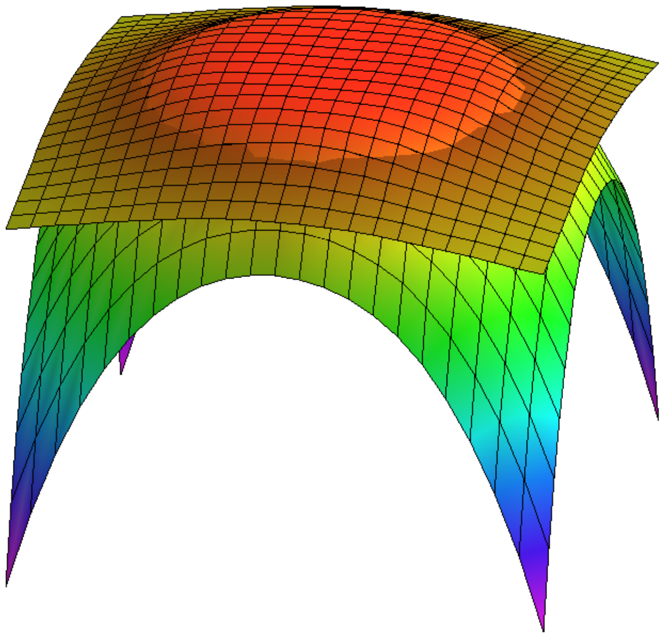

Figure 1.3: A membrane being pushed up by $\phi(x) = 1/4 - (x^2 + y^2)^2/4$.

1.2 Temperature control: the two-phase obstacle problem

Suppose that we have something in a container for which we wish to regulate the temperature. In addition, assume that we can only produce two different heat injections, one positive of size λ_1 and one negative of size λ_2. If we want to keep the temperature u close to zero with this equipment, we could set up the following rules for the regulations:

Inject

$$\begin{cases} \lambda_1 \text{ if } u > 0, \\ -\lambda_2 \text{ if } u < 0. \end{cases}$$

This would lead to the following equation for the temperature:

$$u_t - \Delta u = -\lambda_1 \chi_{\{u>0\}} + \lambda_2 \chi_{\{u<0\}}$$

Under appropriate assumptions we would expect that as $t \to \infty$, the solution will reach some state of equilibrium for which u_t will vanish. Thus, we will have

$$\Delta u = \lambda_1 \chi_{\{u>0\}} - \lambda_2 \chi_{\{u<0\}}.$$

This equation is called the *two-phase obstacle problem* or the *two-phase membrane problem*. The free boundary consists of two parts, $\partial\{u > 0\}$ and $\partial\{u < 0\}$. Whether they have some points in common or not is not a priori known, it depends on the geometry and on the boundary data.

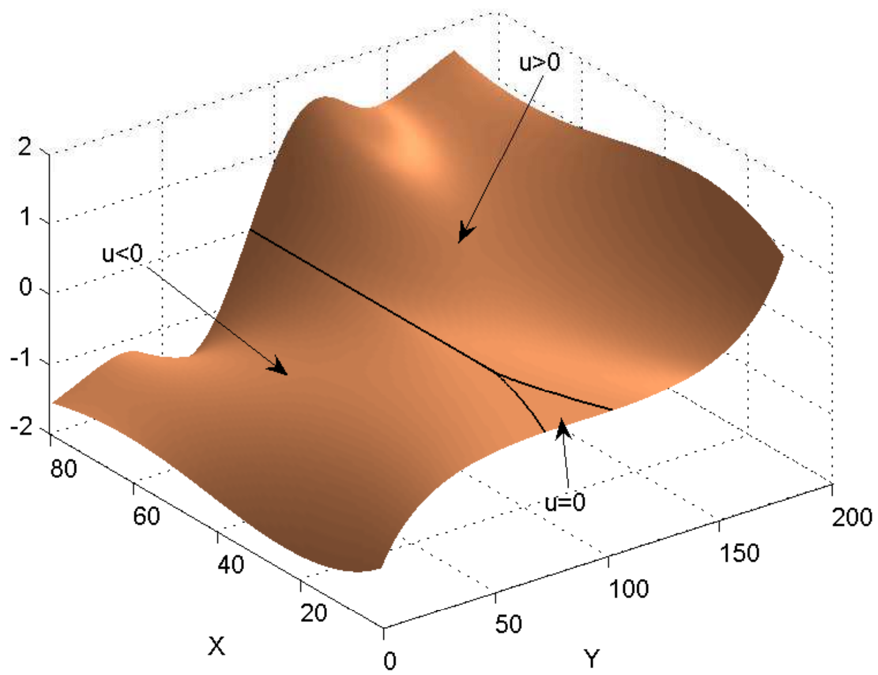

Figure 1.4: An example of the two-phase obstacle problem with $\lambda_1 = \lambda_2 = 1$.

1.3 Ice melting: the Stefan problem

Another example of a FBP is a physical model of a disc of ice that starts to melt due to a heat source placed inside.

For reasons of simplicity, we give a one-dimensional example. Suppose that we are working in the domain $A = [0, 1]$. Moreover, assume that at time t the region $[0, s(t)]$ consists of water and that the region $[s(t), 1]$ consists of ice, where $s(t)$ is an invertible differentiable function. Throughout this section we also assume that the temperature in the ice is zero, and that the heat source keeps the temperature profile to be $g(x)$ for $x \in [0, a]$.

In the region $[0, s(t)]$, we assume that only heat diffusion takes place, i.e., the heat equation governs the behaviour there.

At the phase transition between the water and the ice we use Fourier's law which says that the heat flux equals $-k\theta_x$ for some constant k. In the time interval $[t_0, t_1]$, a quantity of $s(t_1) - s(t_0)$ length units of ice has melted, which absorbs the quantity of heat $k'(s(t_1) - s(t_0))$ for some constant k'. This leads to the equality

$$k'(s(t_1) - s(t_0)) = \int_{t_0}^{t_1} -k\theta_x(x, t)\,\mathrm{d}t.$$

Dividing both sides with $t_1 - t_0$ and letting $t_1 \to t_0$, this yields

$$s'(t) = -K\theta_x(x, t)$$

for some constant K. Therefore, θ will satisfy

1. $\theta_{xx}(x, t) - \theta_t(x, t) = 0$, (heat equation)
2. $\theta(x, t) = g(x)$ for $x \in [0, a]$, (heat source)
3. $s'(t) = -K\theta_x(x, t)$ for $x = s(t)$, (condition on the phase transition)
4. $\theta(x, 0) = \theta_0(x)$, (initial condition).

In the simplest one-dimensional case when $g(x) = k$ is constant, there are explicit solutions of the form

$$\theta(x, t) = k - k\frac{\operatorname{erf}(\frac{x}{2\sqrt{t}})}{\operatorname{erf}\lambda},$$

where λ and k are appropriately chosen.

We can write this problem in a more standard way by introducing the new function

$$u(x,t) = \begin{cases} \displaystyle\int_{s^{-1}(x)}^{t} \theta(x,\tau)\,\mathrm{d}\tau \text{ when } t > s^{-1}(x), \text{ i.e., in the water region,} \\ 0 \text{ elsewhere.} \end{cases}$$

Straightforward computations lead to the equation

$$u_{xx} - u_t = \frac{1}{K}\chi_{\{u>0\}} \text{ for } t > 0 \text{ and } x \in [0,1].$$

The problem is often referred to as the *Stefan problem* or the *one-phase Stefan problem* in the literature. The free boundary for this problem is $\partial\{u > 0\}$.

1.4 A more mathematical description of FBPs

A general form of free boundary problems, which includes most of the problems that I have been studying, is the following: Given a smooth domain $D \subset \mathbb{R}^n$ and a function ϕ in some appropriate class, we seek a function u and a set $\Omega(u)$ such that

$$\begin{cases} F(D^2u) = g(u,\nabla u) \text{ in } \Omega(u)\cap D, \\ H(u,\nabla u) = 0 \text{ on } \partial\Omega(u)\cap D, \\ u = \phi \text{ on } \partial D. \end{cases}$$

Here F is some function on symmetric matrices, g is some function which usually is smooth inside $\Omega(u)$ and has discontinuities or even singularities across $\partial\Omega(u)$. The set $\partial\Omega(u)\cap D$ is referred to as the *free boundary*, and is a priori unknown. We note that if we knew that $\partial\Omega(u)$ was enough regular, we could, under certain assumption on F and g, apply boundary estimates and obtain estimates for u inside D. Thus, the behaviour of $\partial\Omega(u)$ is of great interest. The condition that the function H imposes on u is usually referred to as the *free boundary condition.* Moreover, the boundary ∂D is often called the *fixed boundary.*

In general, one calls a FBP *one-phase* if for some B we have $u \geq B$ (or $u \leq B$) and $\Omega(u) = \{u \geq B\}$ (or $\Omega(u) = \{u \leq B\}$). The obstacle problem described in Section 1.1 is a one-phase problem. Indeed, we can identify F to be the trace operator, $g(u) = (-\Delta\phi)\chi_{\{u>0\}}$ and $H(u,\nabla) = |\nabla u|$.

A FBP is called *two-phase* when we can split $\Omega(u)$ into two parts $\Omega_1(u)$ and $\Omega_2(u)$ where $\Omega_1(u) = \{u > A\}$ and $\Omega_2(u) = \{u < A\}$. In this case the free boundary condition is often that u is continuously differentiable, that is u is C^1, over the free boundary. The two-phase obstacle problem (see Section 1.2) can be identified in this setting with $F =$ the trace operator, $g(u) = \lambda_1 \chi_{\{u>0\}} - \lambda_2 \chi_{\{u<0\}}$, and the free boundary condition that u is C^1 over the free boundary.

1.5 The focus of the studies

In general, questions of existence and uniqueness are important and interesting. However, in the problems considered here, the existence usually follows from standard methods. The uniqueness is also, in many cases, easy to prove or disprove by standard methods. More intricate matters are the regularity of the free boundary, $\partial\Omega(u) \cap D$, and the regularity of the function u. Often, the behaviour of u and the free boundary close to the fixed boundary, ∂D, are also intriguing. These are the main issues that I will try to describe briefly in what follows.

1.6 Two important monotonicity formulae

The study of free boundary problems is often very involved and requires many technical tools. Two frequently used tools are the monotonicity formulae due to Weiss (introduced in one particular form in [Wei99]) and Alt, Caffarelli and Friedman (see [ACF84]) respectively.

Theorem 1.1. *(Weiss' monotonicity formula) Suppose that $F(D^2u) = \Delta u$ and that $g = g(u)$ is smooth except at some isolated points. Define*

$$G(t) = \int_0^t g(s)\,\mathrm{d}s,$$

and

$$u_r(x) = \frac{u(rx)}{r^\gamma}$$

where γ is a constant related to the structure of g. Then the function

$$W(r, u, x) = \int_{B_1} |\nabla u_r|^2/2 + G(u)\,\mathrm{d}x - \gamma \int_{\partial B_1} u_r^2\,\mathrm{d}\sigma$$

is, under suitable assumptions on g, a monotonically increasing function in r. Moreover, W is constant if and only if u is a homogeneous function of degree γ.

For the simplest cases it turns out that

$$\frac{\mathrm{d}W(u,r,x)}{\mathrm{d}r} = c_\gamma \int_{\partial B_1} \left(\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)^2 \mathrm{d}\sigma.$$

From this the result follows.

Theorem 1.2. *(ACF monotonicity formula) Let v and w be two non-negative subharmonic functions with disjoint support such that $w(0) = v(0) = 0$. Then the function*

$$\Phi(r) = \frac{1}{r^4}\int_{B_r} \frac{|\nabla v|^2}{|x|^{n-2}}\,\mathrm{d}x \int_{B_r} \frac{|\nabla u|^2}{|x|^{n-2}}\,\mathrm{d}x$$

is a monotonically increasing function in r.

1.7 Interior regularity of the function

The usual procedure in these types of problems is to first attack the regularity of u. An important issue here is the *optimal regularity* which usually is related to the *scaling* of the problem. By the scaling of the problem we mean that there is some α, such that if u solves the problem in B_1, then

$$u_r(x) = \frac{u(rx)}{r^\alpha}$$

solves the problem in $B_{1/r}$. Sometimes u_r does not solve the exact same equation but something very similar, and α is still referred to as the scaling.

By the optimal regularity, we mean the best regularity that could possibly hold. In a lot of problems, the scaling determines the optimal regularity, so that the optimal regularity is $C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}$ when the scaling is α. The main reason is that one can usually find explicit solutions that are homogeneous of degree α. Nevertheless, this property is not true for all free boundary problems. In [MW07] and [Jia07], there are examples of FBPs, for which this does not hold.

When trying to prove the optimal regularity, there are in many cases standard steps to go through. The first ones, proving for example C^β-regularity or $C^{1,\beta}$-regularity for some $0 < \beta < 1$ (depending of course on the specific problem) are normally quite straightforward. For instance, in the case of the one-phase (see Section 1.1) or the two-phase (see Section 1.2) obstacle problem, we have

$$\Delta u \in L^\infty(D).$$

This implies by standard elliptic theory (see the Calderon-Zygmund inequality in for instance [GT83]) that $u \in C^{1,\beta}(K)$ for any $K \subset\subset D$ and $0 < \beta < 1$. Other frequently used tools are Morrey's lemma (see [GT83]) and also some Nash-Moser iterative techniques (see [MZ97]).

The main result needed in order to prove the optimal regularity is usually some growth result of the type

$$\sup_{B_r(y)} |u| \leq Cr^\alpha$$

whenever r is small enough and $y \in \partial\Omega(u) \cap D$. In general, this has to be adjusted so that if α is bigger than some integer k, we would then also have to remove the k first terms in the Taylor expansion of u.

Probably there is no method for proving the optimal regularity that works as an all-round solution to all kinds of problems. However, in many cases, similar contradictory scaling arguments can give a partial solution of the problem. We will illustrate these arguments by giving the proof of the optimal growth in the case of the obstacle problem.

Proving the optimal interior regularity in the case of the obstacle problem

In this section, we illustrate a technique which applies to many FBPs, by giving a proof of the optimal regularity for solutions to the obstacle problem. The method was introduced in [KS00].

Let u be a non-negative solution to

$$\Delta u = (-\Delta\phi)\chi_{\{u>0\}} \text{ in } B_1$$

where

$$||\phi||_{C^{2,\alpha}(B_1)} \leq C.$$

Suppose moreover that $0 \in \partial\{u > 0\} \cap B_{1/2}$ and $\sup_{B_1} u \leq 1$. Then we claim that there is a constant $C > 0$ such that for any $r > 0$ either

$$\sup_{B_r} u \leq Cr^2$$

or there is a $k > 0$ such that

$$\sup_{B_r} u \leq 4^{-k} \sup_{B_{2^k r}} u.$$

This would imply

$$\sup_{B_r} u \leq C' r^2,$$

for some C' and all $r > 0$, which in turn would be enough to prove the interior $C^{1,1}$-regularity.

To prove this we argue by contradiction, so assume that there are sequences $r_j \to 0$ and $C_j \to \infty$ such that

$$\sup_{B_{r_j}} u \geq C_j r_j^2$$

and

$$\sup_{B_{r_j}} u \leq 4^{-k} \sup_{B_{2^k r_j}} u$$

for all $k \in \mathbb{N}$. Define

$$v_j(x) = \frac{u(r_j x)}{\sup_{B_{r_j}} u}.$$

Then we observe that

1. $|\Delta v_j| = C C_j^{-1}$ in B_{1/r_j},
2. $\sup_{B_{2^k}} v_j \leq 4^k$ for any $k > 0$,
3. $v_j(0) = |\nabla v_j(0)| = 0$,
4. $v_j \geq 0$,
5. $\sup_{B_1} v_j = 1$.

By standard elliptic regularity theory (see for instance [GT83]), properties (1) and (2) imply that the sequence v_j is uniformly bounded in $C^{1,\beta}(K)$ for any $K \subset B_{1/r_j}$ and $0 < \beta < 1$. Therefore we can extract a subsequence again labelled v_j that converges in $C^{1,\beta}(K)$ for any compact $K \subset \mathbb{R}^n$. Let $v_0 = \lim v_j$. Then v_0 satisfies

1. $\Delta v_0 = 0$ in $\mathbb{R}^n$,

2. $v_0(0) = |\nabla v_0(0)| = 0$,

3. $v_0 \geq 0$,

4. $\sup_{B_1} v_0 = 1$.

Since v_0 attains a minimum at the origin, the strong minimum principle implies that v_0 vanishes identically. This contradicts (4).

For other examples of how to prove the optimal regularity in different free boundary problems, see [BK74], [Caf98], [KS80], [Sha03] and [Ura01].

1.8 Interior regularity of the free boundary

Once the optimal interior regularity is settled, the attention is usually turned to the properties of the free boundary. The aim depends of course on the specific problem, even though one, under certain assumptions, often expects that the free boundary is the graph of a C^1-function.

A first step can be to prove that the Lebesgue measure of the free boundary is zero, or that it has finite $(n-1)$-dimensional Hausdorff measure. These kinds of results usually follow from a combination of the optimal regularity, non-degeneracy (see Eq. (1.8.1)), and that g has some special structure.

If we want to prove that something is the graph of a C^1-function, and in particular that it has a unique tangent plane, we actually need to show that if we zoom in close enough, this will be very close to a plane. See Figure 1.5 to get an intuitive picture of what happens when we zoom in for different cases.

Recalling the definition of u_r in the previous section, we can intepret u_r as the $1/r$-times zoomed in version of u. Hence, an important question is to study what happens when we zoom in infinitely many times, i.e., to study what happens when $r \to 0$. The observant reader might have noticed that if all derivatives of u up to order $\lfloor \alpha \rfloor$ vanishes at some point x_0, u_r will remain locally bounded for small r if the optimal regularity is settled. In most cases we also have some kind of compactness, which will allow us to pass to subsequential limits. Therefore, we introduce the following notion:

We say that u_0 is the *blow-up* of u at x_0 if for some sequence $r_j \to 0$

$$u_0(x) = \lim_j u_{r_j}(x) = \lim_j \frac{u(r_j x + x_0)}{r_j^\alpha}.$$

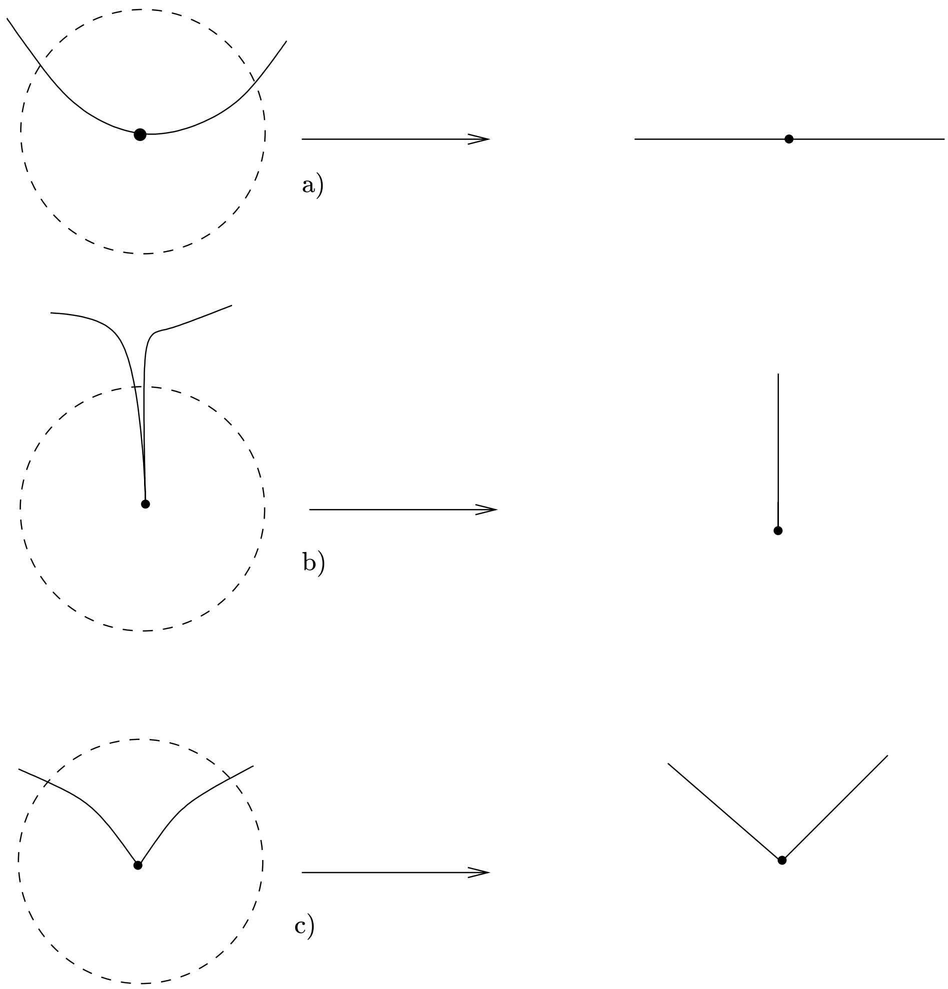

Figure 1.5: Examples of what typically happens when zooming in: a) a differentiable point, b) a cusp and c) a corner point.

As mentioned before, each u_r will be a solution in $B_{1/r}$. Thus, in the limit one would expect each blow-up to be a solution in the whole $\mathbb{R}^n$. So to study what happens when we zoom in will lead us to the study of solutions in $\mathbb{R}^n$. We call such solutions *global solutions*.

We need to rule out the possibility that the blow-up vanishes identically.

If this were the case, we would have lost all information in the limiting process. This is equivalent to saying that the function does not grow too slow around free boundary points. It turns out that in most cases, the function grows exactly at the rate it is permitted to do by the optimal regularity. If α denotes the optimal regularity, one usually has an inequality of the type

$$\sup_{B_r(y)\cap\Omega(u)} (u - u(y))^{\pm} \geq Cr^{\alpha} \tag{1.8.1}$$

for some constant $C > 0$, whenever r is small enough. This property is usually referred to as *non-degeneracy*. The exact form of this depends of course on the specific problem. However, there are some problems that lack this property, see for example [MW07].

Another important property is that, using Weiss' monotonicity formula (Theorem 1.1), one can often conclude that any blow-up must be a homogeneous function of degree α. Indeed, we have

$$W(s, u_0, x) = \lim_j W(s, u_{r_j}, x) = \lim_j (sr_j, u, x) = \lim_j (r_j, u, x) = W(1, u_0, x),$$

where we used the fact that W is monotonically increasing, and therefore has a limit from above. Hence, $W(u_0)$ is constant and u_0 must be homogeneous.

Some further interesting material concerning the interior regularity of free boundary problems can for instance be found in [AC85], [Caf77], [Caf98], [KS80] and [SUW07].

1.9 Regularity properties near the fixed boundary

In addition to the interior regularity properties, the behaviour close to the fixed boundary, ∂D, is of great interest. The boundary conditions and the properties of the fixed boundary can have a regularizing effect, in the sense that the free boundary will always stay somewhat regular close to the fixed boundary.

Briefly speaking, the tools and methods used to study the behaviour near the fixed boundary are very akin to those in the interior case. Similar scaling techniques can usually prove the optimal regularity up to the fixed boundary, under suitable boundary conditions.

Concerning the behaviour of the free boundary near the fixed boundary, we want to study what happens at points in ∂D. This requires the study of blow-ups around those points instead of around points in the interior.

To get the general picture, think of the fixed boundary as a plane. The property of interest is then how the free boundary approaches the fixed one. In the regular case, when $\partial\Omega(u)$ touches the plane in a tangential manner, we would expect that after the blow-up, we will have a solution on one side of the plane, for which $\Omega(u_0)$ is the whole half-space above the plane. Papers related to the behaviour of the free boundary near the fixed boundary are for instance [Mat05], [AMM06], [And07], [KKS07] and Paper IV (see Chapter 5).

Chapter 2

Overview of Paper I

This paper is concerned with the study of minimizers of the functional

$$\int_D \frac{|\nabla u|^2}{2} + \lambda_1 (u^+)^p + \lambda_2 (u^-)^p \, \mathrm{d}x \tag{2.0.1}$$

for $p \in (0,1)$, $\lambda_1, \lambda_2 > 0$, where $D \subset \mathbb{R}^2$ is a smooth and bounded domain. The main result is that the free boundary, $\partial\{u \neq 0\} \cap D$, can be decomposed into two pieces, which both are locally the graph of a C^1-function.

2.1 Introduction and basic properties

It is straightforward to prove that in the set $\{u \neq 0\} \cap D$, any minimizer of (2.0.1) satisfies the equation

$$\Delta u = p[(u^+)^{p-1} - (u^-)^{p-1}]. \tag{2.1.1}$$

Furthermore, it is proved in [GG84] that the minimizers will be in $C^{1,\beta-1}_{\mathrm{loc}}(D)$ where $\beta = \beta(p) = \frac{2}{2-p}$. Observing that there are one dimensional solutions of the form

$$c_p (x_1^+)^\beta - c_p (x_1^-)^\beta$$

we realize that this must be the optimal regularity. Having that in mind we can identify the free boundary condition, i.e., the function H in Section 1.4, to be that ∇u is continuous over $\partial\Omega(u) \cap D$ which in this case is $\partial\{u \neq 0\} \cap D$. In what follows we will use the notation $\Gamma(u) = \partial\{u \neq 0\} \cap D$, and $\Gamma^\pm(u) = \partial\{\pm u > 0\} \cap D$.

The properties around so called one-phase points have already been studied in [AP86], where the authors in particular prove that in dimension two, the free boundary is the graph of a $C^{1,\alpha}$-function for some $0 < \alpha < 1$, and in fact that it is analytic. This means that if $x_0 \in \Gamma^{\pm}(u)$ and $\pm u > 0$ in some ball $B_\delta(x_0)$ then there is a constant $c_0 > 0$ such that $\Gamma^{\pm}(u) \cap B_{c_0\delta}$ is the graph of an analytic function.

Moreover, since the implicit function theorem implies that $\Gamma(u)$ is the graph of a C^1 function around points where the gradient does not vanish, we turn our focus to points in the set $\Gamma^+(u) \cap \Gamma^-(u) \cap \{|\nabla u| = 0\}$, which are usually referred to as branching points.

The main result of the paper is:

Theorem 2.1 (C^1 regularity of $\Gamma^{\pm}$ in $n = 2$)**.** *Let u be a minimizer of (2.0.1) with $0 < p < 1$ in dimension $n = 2$. Then the free boundaries $\Gamma^{\pm}(u)$ are C^1 regular.*

2.2 Blow-ups

Throughout the paper, the main issue is to classify the blow-ups around branching points. In this case this will be the subsequential limits of u_{r_j} where

$$u_r(x) = \frac{u(rx + y)}{r^\beta}$$

for $y \in \Gamma^+(u) \cap \Gamma^-(u) \cap \{|\nabla u| = 0\}$.

As mentioned in Section 1.6, Weiss' monotonicity theorem (Theorem 1.1) implies that any blow-up will be homogeneous of degree β. In this case the monotone function Φ takes the form

$$\Phi(u, r, x_0) = \int_{B_1} \frac{|\nabla u_r|^2}{2} + \lambda_1 (u_r^+)^p + \lambda_2 (u_r^-)^p \, dx - \frac{\beta}{2} \int_{\partial B_1} u_r^2 \, d\sigma.$$

By comparing $(u^{\pm})^{\frac{2}{\beta}}$ and $c_p|x|^2$ for some suitable constant c_p, we are able to prove non-degeneracy, which in this case is

$$\sup_{B_r(y)\cap\{\pm u>0\}} u^{\pm} \geq c_p r^\beta$$

for any $y \in \Gamma^{\pm}(u)$ such that $B_r(y) \subset D$.

Observing that u is subharmonic and using that $|u|^{1/\beta}$ is superharmonic whenever u is a two dimensional homogeneous solution, we can then show that up to rotations and translations, there is only one possibility of a blow-up:

$$u_0(x) = C_p[(x_1^+)^\beta - (x_1^-)^\beta]$$

2.3 Differentiability and C^1 regularity

The first step we take towards any regularity of the free boundary is to prove that the blow-up does not depend on the subsequence r_j, which means that the blow-up is unique. The idea is to use the Alexandrov reflection-comparison methods, which are used and introduced in [SW06].

Briefly speaking, we study the difference between reflections. Consider

$$\xi_{r_j}^\omega(\theta) = \phi_{r_j}(\omega + \theta) - \phi_{r_j}(\omega - \theta)$$

where $\phi_{r_j} = u_{r_j}(\cos\theta, \sin\theta)$. A key result here is the following reflection-comparison lemma:

Lemma 2.2. *If $x = (x_1, x_2)$ define $x^* = (x_1, -x_2)$. Assume that $u(x) < u(x^*)$ for $x \in (\partial B_r)^+$. Then $u(x) \leq u(x^*)$ for $x \in (B_r)^+$.*

If one blow-up is u_0, then by the C^1-convergence

$$\phi_{r_j} \to \phi_0 = u_0(\cos\theta, \sin\theta),$$

$\xi_{r_j}^\omega$ is non-negative for all θ and for some ω. The reflection-comparison lemma applied to u restricted to $B_{r_j}^+$ implies that this is also true for ξ_r^ω whenever $r \leq r_j$.

Assuming there is another blow-up, equal to u_0 rotated by an angle ω_1, and repeating these arguments for $\xi_r^{\omega-\omega_1}$, we arrive at a contradiction unless $\omega_1 = 0$.

Exploring these methods even further, we are able to conclude that more generalized blow-ups of the type

$$\lim_{r\to 0, z\to y} \frac{u(rx+z)}{r^\beta}$$

where y is a branching point, indeed exist and that they are unique. This implies that around branching points the normal is actually continuous.

Chapter 3

Overview of Paper II

In this paper we study the p-Laplace version of the two-phase obstacle problem. We prove that for $p \simeq 2$ the minimizers have the optimal growth of order r^{p^*} around branching points, where $p^* = \frac{p}{p-1}$. Using this result, we obtain, as a by-product, estimates of the $(n-1)$-dimensional Hausdorff measure of the free boundary for $p > 2$ and p close to 2.

3.1 Introduction and general properties

Given a smooth and bounded domain D, positive constants λ_1 and λ_2, we are concerned with minimizers of

$$\int_D \frac{|\nabla u|^p}{p} + \lambda_1 u^+ + \lambda_2 u^- \,\mathrm{d}x \tag{3.1.1}$$

and solutions of the corresponding Euler-Lagrange equation

$$\Delta_p u = \lambda_1 \chi_{\{u>0\}} - \lambda_2 \chi_{\{u<0\}}. \tag{3.1.2}$$

For $p \geq 2$ it turns out that that minimizers of (3.1.1) and solutions of (3.1.2) are in one-to-one correspondence.

When $p = 2$ there are many known results, see [Ura01], [Wei01], [Sha03], [SUW04], [SW06] and [SUW07]. For instance, it is known that solutions and minimizers are in $C^{1,1}_{\mathrm{loc}}(D)$. Moreover, Shahgholian, Weiss and Uraltseva prove in [SUW04] that the only global solutions with quadratic growth at infinity that have a branching point, i.e., solutions for which the set

$\partial\{u>0\}\cap\partial\{u<0\}\cap\{|\nabla u|=0\}\cap D$ is not empty, are (up to rotations)

$$u(x)=\frac{\lambda_1}{2}(x_1^+)^2-\frac{\lambda_2}{2}(x_1^-)^2.$$

In particular these global solutions are uniformly bounded in L^∞ on any compact set. This property is central for the methods used in the paper.

Since we are concerned with the interior properties it suffices to study what happens in any ball contained in D. The main result of the paper is the theorem below.

Theorem 3.1. *Let u be a minimizer of (3.1.1) for $D=B_1$ such that $y\in\partial\{u>0\}\cap\partial\{u<0\}\cap\{|\nabla u|=0\}\cap B_{1/2}$ and*

$$\sup_{B_1}|u|\le M.$$

Then there are constants C, δ and r_0, depending on λ_i, M and the dimension such that $|p-2|<\delta$ implies

$$\sup_{B_r(y)}|u|\le Cr^{p^*}$$

for $r<r_0$.

3.2 Known results

Using by now almost standard methods we can prove non-degeneracy, i.e., that

$$\sup_{B_r(y)\cap\{\pm u>0\}} u^\pm\ge c_p r^{p^*}$$

for any free boundary point y such that $B_r(y)\subset D$.

Moreover, standard estimates for the p-Laplace operator imply, since $\Delta_p u\in L^\infty$, that u has local $C^{1,\beta}$-estimates for some $0<\beta<1$. See Theorem 1 in [Tol84].

3.3 General idea in the proof of the optimal growth

The idea is to use the information for $p=2$ and some sort of continuity argument to pass this on to a neighbourhood of 2. We observe that for any branching point y,

$$\sup_{B_r(y)} |u_p| < Cr_j^{p^*} + \tau(p),$$

where u_p denotes a solution to the problem for the p-Laplace operator. In addition, $\tau \to 0$ as $p \to 2$. Define

$$r_j = \sup\{r : \sup_{B_r(y)} |u_{p_j}| > 2Cr^{p_j^*}\}$$

where p_j is any sequence such that $p_j \to 2$ and C is some constant to be chosen later. The aim is to prove that $r_j = 0$ for j large enough. We do a blow-up with respect to the sequence r_j, i.e., we let

$$v_j(x) = \frac{u_{p_j}(r_j x + y)}{r_j^{p^*}}.$$

It turns out that a subsequence of v_j converges to a global solution of the two-phase obstacle problem having a branching point at the origin. Moreover, by the definition of the r_js, we have

$$\lim_j \sup_{B_1} |v_j| = 2C.$$

If we choose C big enough this will contradict the fact that these global solutions are uniformly bounded on B_1.

3.4 $(n-1)$-dimensional Hausdorff measure estimates

Using the optimal growth together with the non-degeneracy, we are able to prove certain growth and non-degeneracy properties of for instance the quantities $|\nabla u|$ and

$$\int_{B_r} |\nabla u|^{p-2} (D^2 u)^2 \, \mathrm{d}x$$

for $p > 2$. Using a synthesis of the methods in [LS03] and [SW06], we can estimate

$$\mathcal{H}^{n-1}(\partial\{u \neq 0\} \cap \{|\nabla u| = 0\} \cap K),$$

for any $K \subset\subset D$.

Chapter 4

Overview of Paper III

In this paper we are concerned with solutions of the two-phase obstacle problem introduced in Section 1.2, where the coefficients λ_i are only assumed to be Hölder continuous. We obtain local $C^{1,1}$-estimates and estimates of the Reifenberg flatness module close to branching points.

4.1 Introduction and some basic properties

Given a smooth and bounded domain D we study solutions of

$$\Delta u = \lambda_1 \chi_{\{u>0\}} - \lambda_2 \chi_{\{u<0\}} \text{ in } D, \tag{4.1.1}$$

where λ_1 and λ_2 are strictly positive and Hölder continuous functions. When the λ_is are assumed to be Lipschitz continuous, the solutions are known to have local $C^{1,1}$-estimates, see [Ura01] or [Sha03]. Moreover, when the λ_is are constants, global solutions having branching points are classified. Both these properties are of great importance for the methods used here. Non-degeneracy is in general known for the two-phase obstacle problem without any continuity assumptions on the coefficients λ_i. We denote the free boundaries $\partial\{\pm u > 0\} \cap B_1$ by $\Gamma^{\pm} = \Gamma^{\pm}(u)$ and we will as before refer to points in the set $\Gamma^+ \cap \Gamma^- \cap \{|\nabla u| = 0\}$ as branching points.

The main result is the following theorem:

Theorem 4.1. *Assume that u is a solution of (4.1.1) in B_1 such that*

$$\sup_{B_1} |u| \leq M_0.$$

Suppose moreover that the coefficients λ_i *satisfy* $M > \lambda_i > 1/M$*, and has a modulus of continuity* ω *such that* $\omega(r) \leq Mr^\alpha$ *for some* $0 < \alpha < 1$*. Then there are* r_0 *and* C *depending on* M*,* M_0 *and the dimension such that*

$$||u||_{C^{1,1}(B_{r_0})} \leq C.$$

Theorem 4.2. *Assume that the hypotheses in Theorem 4.1 hold. Then there are* r_0 *and* γ *depending on* M*,* M_0 *and* ω *such that* $|\nabla u(y)| \leq \gamma$ *and* $\operatorname{dist}(y, \Gamma^\pm(u)) \leq \gamma$ *imply that* $\Gamma(u)^+ \cap B_{r_0}(y)$ *and* $\Gamma(u)^- \cap B_{r_0}(y)$ *are both Reifenberg vanishing sets. In particular they both admit Hölder parameterizations.*

4.2 The scaling argument

The idea of how to prove the optimal $C^{1,1}$-regularity is very similar to the one in Paper II. The difference here is that instead of varying the parameter p in the p-Laplace operator, we vary the oscillations of the coefficients λ_i. Then we can use the properties known for the case of constant coefficients.

As before, we will for y being a branching point consider rescalings of the form

$$\frac{u_j(r_j x + y)}{r_j^2}.$$

Here r_j is chosen in a way similar to Section 3.3, and u_j corresponds to the coefficients λ_i^j, which have smaller and smaller oscillations. It turns out that these rescalings converge, up to a subsequence, to a global solution of the two-phase obstacle problem with quadratic growth at infinity having a branching point at the origin. Therefore, we obtain a similar contradiction as in Paper II.

In this case, we want to obtain $C^{1,1}$-estimates. The method described above only gives growth estimates around branching points. Hence, we also need to treat the points where the gradient does not vanish.

When the gradient does not vanish but has a small modulus, it is possible to modify the method above to work. The difference will be that the limit function will solve

$$\Delta u = \lambda_1 \chi_{\{u > \nu \cdot x\}} - \lambda_2 \chi_{\{u < \nu \cdot x\}} \text{ in } \mathbb{R}^n$$

instead of the two-phase obstacle problem. It turns out that we can classify these global solutions and that they are uniformly bounded. See the lemma below.

Lemma 4.3. *Let u be a solution of the following problem*

$$\begin{cases} \Delta u = \lambda_1 \chi_{\{u>-ax_1\}} - \lambda_2 \chi_{\{u<-ax_1\}} & \text{in } \mathbb{R}^n, \\ u(0) = |\nabla u(0)| = 0, \\ 0 \in \partial\{u \neq 0\}, \end{cases} \tag{4.2.1}$$

where $a > 0$, and λ_1 and λ_2 are constants. Assume also that for some $C > 0$

$$\sup_{B_r} |u| \leq Cr^2$$

whenever $r > 1$. Then one of the following holds

1. $u(x) = \frac{\lambda_1}{2}(x_1^+)^2 - \frac{\lambda_2}{2}(x_1^-)^2$,
2. $u(x) = \frac{\lambda_1}{2}(x_1^+)^2$,
3. $u(x) = -\frac{\lambda_2}{2}(x_1^-)^2$.

The proof of this lemma is based on the ACF monotonicity formula (Theorem 1.2).

For the points where the gradient has a big modulus, the implicit function theorem implies that the free boundary is locally a $C^{1,\alpha}$-graph. Using the Hodograph transform as in [KS80], we obtain an equation for which there are $C^{2,\alpha}$-estimates up to the boundary (see [KNS78]).

Putting all the pieces together yields the desired $C^{1,1}$-estimate independently of the modulus of the gradient.

4.3 Estimates of the flatness module

Definition 4.4. *(Reifenberg-flatness) A compact set S in $\mathbb{R}^n$ is said to be δ-Reifenberg flat if, for any compact set $K \subset \mathbb{R}^n$, there exists an $R_K > 0$ such that, for every $x \in K \cap S$ and every $r \in (0, R_K]$, we have a hyperplane $L(x, r)$ such that*

$$\text{dist}(L(x,r) \cap B_r(x), S \cap B_r(x)) \leq 2r\delta.$$

We define the modulus of flatness as

$$\theta_K(r) = \sup_{0<\rho\leq r} \left(\sup_{x \in S\cap K} \frac{\text{dist}(L(x,\rho) \cap B_\rho(x), S \cap B_\rho(x))}{\rho} \right).$$

By the $C^{1,1}$-estimates and Lemma 4.3 it follows that each blow-up at points near branching points will converge to some rotation of

$$\frac{\lambda_1}{2}(x_1^+)^2 - \frac{\lambda_2}{2}(x_1^-)^2.$$

Using some stability properties it turns out that we can trap u between the two solutions u_1 and u_2 satisfying

$$\Delta u_1 = \inf \lambda_1 \chi_{\{u>0\}} - \sup \lambda_2 \chi_{\{u<0\}}$$

and

$$\Delta u_2 = \sup \lambda_1 \chi_{\{u>0\}} - \inf \lambda_2 \chi_{\{u<0\}}$$

with the same boundary conditions as u. From the results in [SUW07] we know that $\Gamma^\pm(u_1)$ and $\Gamma^\pm(u_2)$ are locally C^1-graphs. Moreover, the distance between the free boundaries of u_1 and u_2 can be estimated as

$$\operatorname{dist}(\Gamma^\pm(u_1) \cap B_r, \Gamma^\pm(u_2) \cap B_r) \leq C\sqrt{\max(\operatorname{osc} \lambda_i)}.$$

Applying this to the rescalings, we obtain the desired estimate. However, the estimate we get is not as good as the estimate one can get using the same method for the one-phase obstacle problem. In that case it yields the C^1 regularity of the free boundary, see [Bla01].

Chapter 5

Overview of Paper IV

This paper is devoted to the study of the penalized obstacle problem in a unit half ball. The penalized obstacle problem is an approximation of the classical obstacle problem. We prove that when the approximation is good enough and when certain level sets are close to the fixed boundary, they are the graphs of uniformly C^1 regular functions.

5.1 Introduction

As mentioned in Section 1.1, solutions of the obstacle problem is given by non-negative solutions to the equation

$$\Delta u = f\chi_{\{u>0\}},$$

for some function f. The simplest case $f = 1$ exhibits typical behaviour, and it has been studied extensively, see [Caf98] and [KS80]. The penalized obstacle problem is a sort of approximation of this equation. For instance, one can take

$$\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon),$$

where β_ε is a function that in some sense converges to the Heaviside function. One reason to study the penalized obstacle problem is that in order to do numerical computations, it is necessary to consider reasonable approximations of the problem. A good penalization might be useful for numerical purposes if it has good regularity and stability properties.

This problem has been studied in for instance [KS80], [BK74], [Red93a] and [Red93b]. Breziz and Kinderlehrer proved that solutions are locally uniformly $C^{1,1}$ regular. Redondo proved, using the result of Brezis and Kinderlehrer, that under a certain flatness assumption, the level sets of u_ε are locally uniform $C^{1,\alpha}$-graphs in some direction. However, all these estimates are local inside the interior, i.e., away from the fixed boundary.

We study this equation in a half unit ball

$$\begin{cases} \Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon) \text{ in } B_1^+ = B_1 \cap \{x_1 > 0\}, \\ u_\varepsilon = g \geq 0 \text{ on } \partial B_1 \cap \{x_1 > 0\}, \\ u_\varepsilon = 0 \text{ on } B_1 \cap \{x_1 = 0\}, \end{cases}$$

for some given non-negative function $g \in L^\infty(B_1^+) \cap H^1(B_1^+)$. We are mainly interested in the behaviour of the level sets of u_ε close to the hyperplane $\{x_1 = 0\}$, which we refer to as the fixed boundary. The main results of this paper are:

Theorem 5.1. *There is a constant $C = C(n, M)$ such that if*

$$\sup_{B_1^+} u_\varepsilon \leq M$$

then

$$||u_\varepsilon||_{C^{1,1}(B_{1/2}^+)} \leq C.$$

Theorem 5.2. *Assume that $y \in \partial\{u_\varepsilon > \varepsilon^\gamma\} \cap B_{1/2}^+$ for some $0 < \gamma < 1$. Then there are constants $\varepsilon_0, \rho > 0$ such that*

$$\max(\varepsilon, y_1) < \varepsilon_0$$

implies that $B_\rho^+(y) \cap \partial\{u_\varepsilon > \varepsilon^\gamma\}$ is a C^1-graph with the C^1-norm bounded independently of ε.

The proof of Theorem 5.2 is very much inspired by [SU03].

5.2 $C^{1,1}$-estimates up to the fixed boundary

In order to prove that u_ε has $C^{1,1}$-estimates up to the fixed boundary we prove that

$$\sup_{B_r^+} |u_\varepsilon(x) - x \cdot \nabla u_\varepsilon(0)| \leq Cr^2,$$

for r small enough. This is done in a way very similar to the one presented in Section 1.7.

5.3 C^1 regularity close to the fixed boundary

The first step towards the regularity of the level sets is to prove that away from the boundary, the ε^γ-level set is the graph of a C^1-function whenever it is close enough to the plane fixed boundary. The idea is to use the results in [Red93a, Red93b]. In order to do that we need to make sure that the set $\{u_\varepsilon \leq \varepsilon^\gamma\}$ is not too small.

We consider the rescalings

$$v_j(x) = \frac{u_{\varepsilon_j}(y_1^j x + y^j)}{(y_1^j)^2}$$

for certain sequences y^j such that $y_1^j \to 0$. We are able to prove that all subsequential limits of v_j are one-dimensional and monotone. With a bit of work it turns out that this implies that the set $\{u_\varepsilon \leq \varepsilon^\gamma\}$ will be big enough in order to use the result from [Red93a, Red93b].

5.4 Uniform C^1 regularity close to the fixed boundary

The main step in order to prove the uniform C^1 regularity is to prove that as the level sets approach the fixed boundary, they will do this in a tangential manner. The idea of proving this is very similar to the one in the previous section. We study different blow-up limits and from there we can conclude that this tangential approach must indeed hold. From here it is quite straightforward to obtain the uniform C^1 regularity. The result in the previous section gives us that there is a normal at each point, and that it is uniformly continuous away from the fixed boundary. Moreover, from the tangential property we know that as the level sets approach the fixed boundary, the normal will converge to e_1. Combining these properties, we obtain that the normal is uniformly continuous.

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