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Data Filtering and Control Design for Mobile Robots

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For if every instrument could accomplish its own work, obeying or anticipating the will of others... if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves.

Aristotle

Till min familj

Abstract

In this thesis, we consider problems connected to navigation and tracking for autonomous robots under the assumption of constraints on sensors and kinematics. We study formation control as well as techniques for filtering and smoothing of noise contaminated input. The scientific contributions of the thesis comprise five papers.

In Paper A, we propose three cascaded, stabilizing formation controls for multi-agent systems. We consider platforms with non-holonomic kinematic constraints and directional range sensors. The resulting formation is a leader-follower system, where each follower agent tracks its leader agent at a specified angle and distance. No inter-agent communication is required to execute the controls. A switching Kalman filter is introduced for active sensing, and robustness is demonstrated in experiments and simulations with Khepera II robots.

In Paper B, an optimization-based adaptive Kalman filtering method is proposed. The method produces an estimate of the process noise covariance matrix Q by solving an optimization problem over a short window of data. The algorithm recovers the observations $h(x)$ from a system $\dot{x} = f(x),\ y = h(x) + v$ without a priori knowledge of system dynamics. The algorithm is evaluated in simulations and a tracking example is included, for a target with coupled and nonlinear kinematics.

In Paper C, we consider the problem of estimating a closed curve in $\mathbb{R}^2$ based on noise contaminated samples. A recursive control theoretic smoothing spline approach is proposed, that yields an initial estimate of the curve and subsequently computes refinements of the estimate iteratively. Periodic splines are generated by minimizing a cost function subject to constraints imposed by a linear control system. The optimal control problem is shown to be proper, and sufficient optimality conditions are derived for a special case of the problem using Hamilton-Jacobi-Bellman theory.

Paper D continues the study of recursive control theoretic smoothing splines. A discretization of the problem is derived, yielding an unconstrained quadratic programming problem. A proof of convexity for the discretized problem is provided, and the recursive algorithm is evaluated in simulations and experiments using a SICK laser scanner mounted on a PowerBot from ActivMedia Robotics.

Finally, in Paper E we explore the issue of optimal smoothing for control theoretic smoothing splines. The output of the control theoretic smoothing spline problem is essentially a tradeoff between faithfulness to measurement data and smoothness. This tradeoff is regulated by the so-called *smoothing parameter*. In Paper E, a method is developed for estimating the optimal value of this smoothing parameter. The procedure is based on general cross validation and requires no a priori information about the underlying curve or level of noise in the measurements.

Keywords: formation control, tracking, nonlinear control, optimal smoothing, adaptive filtering

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Stockholm, September 2009

Maja Karasalo

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Chapter 1

Introduction

This thesis explores control design and filtering for autonomous systems under the assumption of noise contaminated feedback from onboard sensors. The focus is on applications such as formation keeping and target tracking for groups of mobile agents, and tracking or estimation of curves from noisy samples. When noisy data is used for feedback, some filtering or smoothing is generally required before applying the control law. In addition, under such circumstances, the control law itself should be robust with respect to measurement errors of some reasonable magnitude.

In this thesis we study both aspects. The particular topics of the thesis are outlined next.

1.1 Thesis Outline

The thesis consists of two introductory chapters and five appended papers. In the remainder of this chapter, a motivation for the work is provided and the appended papers are summarized, while Chapter 2 reviews the relevant results and terminology that constitute the background of the papers. The contents of Chapter 2 should be well known to the initiated reader and the chapter may be skipped at a first reading.

The appended papers may be divided into two categories. The first category is control design and Kalman filtering, under assumptions of sensor constraints. Paper A treats nonlinear control design and switching Kalman filters for feedback from directional sensors, for a team of mobile robots that move in a specified formation. Paper B focuses entirely on adaptive Kalman filtering with no a priori information about the true system dynamics.

The second category is smoothing of noisy data by control theoretic smoothing splines. Theoretical results as well as outcomes of simulations and experiments are reported. This part of the thesis encompasses Papers C, D and E, where Paper C focuses on theoretical aspects, Paper D treats implementation and experiments, and Paper E discusses optimal smoothing.

It should be noted that even though the research presented in this thesis is often motivated by applications, and demonstrated in simulations or experiments, system and sensor models are generally simplifications of the true models. For instance, factors such as slip and traction are overlooked and the pure kinematic equations are used to model robot dynamics.

1.2 Motivation

Early commercial robots were generally designed for industrial applications, such as assembling cars in a controlled environment mostly inhabited by other industrial robots. Such robots are mainly designed for performing pre-determined, repetitive tasks at high speed and with good accuracy.

As the market for domestic robots, autonomous surveillance vehicles, and other automatic agents is expanding, the focus is shifting toward robots with flexible and intelligent behavior that can safely interact with humans and respond appropriately to unexpected events. For mobile robots, the ability to interpret and handle events and objects in their surroundings is essential. One aspect of this concerns the development of better sensors. This is however not the topic of this thesis. Instead, we explore ways of refining information received in form of noise contaminated data, and how to design control signals that are robust to errors in the input.

For many applications, robustness and efficiency can be greatly increased by engaging teams of cooperative mobile robots to carry out tasks together. Examples include mine sweeping, surveillance, lawn mowing and vacuum cleaning. This motivates the current interest in multi-agent systems or networks. Often inter-agent communication or access to global information is a necessity in such operations. One of the topics of this thesis is that of achieving cooperative behavior for a team of agents without global information or communication and with constraints on onboard sensors.

The next section renders a more detailed overview of the appended papers.

1.3 Reader's Guide to the Appended Papers

In this section, the appended papers are presented. An abstract of each paper is provided together with a discussion on contributions, work division, limitations, and suggested extensions. In the five independent papers, notation is introduced separately in each paper. Unless otherwise specified, $\dot{x}$ denotes the time derivative of x and $||x||$ is the euclidian norm of x. The reader is urged to mind notational collision.

Paper A: Robust Formation Control using Switching Range Sensors

Authors: M. Karasalo, T. Gustavi, and X. Hu.

Publication: Submitted to Robotics and Autonomous Systems, April 2009.

Abstract: In this paper, control algorithms are presented for formation keeping and path following for non-holonomic platforms. The controls are based on feedback from onboard directional range sensors, and a switching Kalman filter is introduced for active sensing. Stability is analyzed theoretically and robustness is demonstrated in experiments and simulations.

Contributions: The main result is the globally stable, cascaded formation control. The extensions of this control to adaptable parameters and to monotonic convergence of certain control parameters are novel contributions of this paper. The experimental evaluation of the globally stable control and the switching Kalman filter testifies to the robustness of the approach. Some of the results in this paper have appeared in

[1] T. Gustavi, X. Hu and M. Karasalo,
Multi-Robot Formation Control And Terrain Servoing with Limited Sensor Information,
Proc. of the 16th Congress of the International Federation of Automatic Control (IFAC), 2005.

[2] T. Gustavi, X. Hu and M. Karasalo,
Formation Adaptation with Limited Sensor Information,
invited paper, Proc. of Chinese Control Conference (CCC), 2005.

[3] J. Samuelsson, T. Gustavi, M. Karasalo and X. Hu,
Robust Formation Adaptation for Mobile Platforms with Noisy Sensor Information,
Proc. of the IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), 2006.

[4] T. Gustavi and X.Hu,
Observer Based Leader-Following Formation Control using On-Board Sensor Information,
IEEE Transactions on Robotics, vol. 24, no. 6, pp. 1457-1462, 2008.

Work Division: The basic cascaded tracking algorithm was developed by Hu, as well as the extension that results in monotonic convergence of the parameters d and β. The extension with adaptable parameters was developed by Karasalo. The experimental evaluation has been previously published in [3] and is joint work between Samuelsson, Gustavi, and Karasalo. The simulations were done by Karasalo.

Limitations and Suggested Extensions: In order for the controls to be applicable in practice, some issues need to be addressed. For instance, the cascaded control is sensitive to the scenario that one agent breaks down. The implemented Kalman filter yields a rather rough estimate of the target state. An adaptive Kalman filter that produces better estimates is discussed in Paper B. Simulations and experiments showed that the controls are somewhat sensitive to the method of discretization used on the dynamic system. To work in practice, the controls need feedback with sufficiently high sampling rate.

Paper B: An Optimization Approach to Adaptive Kalman Filtering

Authors: M. Karasalo and X. Hu.

Publication: 48th IEEE Conference on Decision and Control (CDC), 2009.

Abstract: In this paper, an optimization-based adaptive Kalman filtering method is proposed. The method produces an estimate of the process noise covariance matrix Q by solving an optimization problem over a short window of data. The algorithm recovers the observations $h(x)$ from a system $\dot{x} = f(x),\ y = h(x) + v$ without a priori knowledge of system dynamics. Potential applications include target tracking using a network of nonlinear sensors, servoing, mapping, and localization. The algorithm is demonstrated in simulations on a tracking example for a target with coupled and nonlinear kinematics. Simulations indicate superiority over a standard MMAE algorithm for a large class of systems.

Contributions: This particular optimization approach is novel. The method is scalable and applicable to systems where little or no information on the actual dynamics is available. Some of the results in this paper have appeared in

[1] T. Gustavi, M. Karasalo, X. Hu, and C.F. Martin,
Recursive Identification of a Hybrid System,
Proc. of the The European Control Conference (ECC), 2009.

Work Division: The idea and method of optimization based adaptive Kalman filtering was developed by Karasalo. Hu designed the tracking example to connect the theory with applications, and provided valuable comments on the presentation of the paper.

Limitations and Suggested Extensions: The cost function in the optimization problem was chosen because of its appealing simplicity and performance rather than its theoretical properties. Theoretical results in general are still lacking for the method and alternative cost functions should be investigated. The method is developed to work for systems with unknown dynamics. An interesting extension would be to enable incorporation of known traits of the system for increased performance.

Remark 1.3.1 *Due to the page limitation for publication in conference proceedings, some of the figures appearing in Paper B have been removed in the published paper.*

Paper C: Periodic and Recursive Control Theoretic Smoothing Splines

Authors: M. Karasalo, X. Hu, and C.F. Martin.

Publication: Submitted to Communications in Information and Systems, August 2009.

Abstract: In this paper, a recursive control theoretic smoothing spline approach is proposed for reconstructing a closed contour. Periodic splines are generated by minimizing a cost function subject to constraints imposed by a linear control system. The optimal control problem is shown to be proper, and sufficient optimality conditions are derived for a special case of the problem using Hamilton-Jacobi-Bellman theory.

The filtering effect of the smoothing splines allows for usage of noisy sensor data. An important feature of the method is that several data sets for the same closed contour can be processed recursively so that the accuracy can be improved stepwise as new data becomes available.

Contributions: The main contribution is the formulation of the recursive spline problem, which is appealing since it can be transformed so that it is identical to the closed form smoothing spline problem. A connection is made between regular periodic smoothing splines and optimal control, opening up for the formulation of more advanced smoothing problems. Some of the results in this paper have appeared in

[1] M. Karasalo, X. Hu, and C.F. Martin,
Closed Contour Reconstruction using Iterated Smoothing Splines,
Proc. of the third Swedish Workshop on Autonomous Robotics (SWAR), 2005

[2] M. Karasalo, X. Hu, and C.F. Martin,
Contour Reconstruction and Matching using Recursive Smoothing Splines,
Modeling, Estimation and Control, Springer, pp. 193–206, 2007.

Work Division: The idea of this particular form of recursive smoothing splines was the result of a collaboration between Hu and Martin. The specific recursion formula was developed by Hu. The connection with Hamilton-Jacobi-Bellman theory and the convergence results were derived by Karasalo.

Limitations and Suggested Extensions: Sufficient optimality conditions are only derived for the case $N \to \infty$. Conditions for a finite N are still lacking. Although simulation results indicate fast convergence of the recursive problem, theoretical conditions for convergence, such as level and nature of the added noise, and features of the underlying curve, have yet to be investigated. Some of the results are applicable to optimal control problems with other dynamic constraints than those examined in this paper. Closer investigation of such problems is of interest.

Remark 1.3.2 *In [2], the definitions of boundary conditions to some differential equations contain errors. In Paper C they should however be correct.*

Paper D: Contour Reconstruction using Recursive Smoothing Splines - Algorithms and Experimental Validation

Authors: M. Karasalo, G. Piccolo, D. Kragic and X. Hu.

Publication: Robotics and Autonomous Systems, no. 57, pp. 617–628, 2009.

Abstract: In this paper, a recursive smoothing spline approach for contour reconstruction is studied and evaluated. Periodic smoothing splines are used by a robot to approximate the contour of encountered obstacles in the environment. The splines are generated through minimizing a cost function subject to constraints imposed by a linear control system and accuracy is improved iteratively using a recursive spline algorithm. The filtering effect of the smoothing splines allows for usage of noisy sensor data and the method is robust with respect to odometry drift. The algorithm is extensively evaluated in simulations for various contours and in experiments using a SICK laser scanner mounted on a PowerBot from ActivMedia Robotics.

Contributions: The recursive spline problem formulated in Paper C is thoroughly evaluated in simulations and experiments with real sensor data. A discretization is derived, which transforms the optimal control problem to a simple, unconstrained quadratic programming problem. Some of the results in this paper have appeared in

[1] G. Piccolo, M. Karasalo, D. Kragic, and X. Hu,
Contour Reconstruction using Recursive Smoothing Splines - Experimental Validation,
Proc. of the IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), 2007.

[2] M. Karasalo, X. Hu, and C.F. Martin,
Localization and Mapping using Recursive Smoothing Splines,
Proc. of the European Control Conference (ECC), 2007.

[3] M. Karasalo, X. Hu, and C.F. Martin,
Contour Reconstruction and Matching using Recursive Smoothing Splines,
Modeling, Estimation and Control, Springer, pp. 193–206, 2007.

Work Division: The implementation and experiments were joint work between Piccolo, Kragic and Karasalo. The simulation results and analysis are due to Karasalo. The recursive formulation is due to Hu, who also offered invaluable support during troubleshooting of the simulation code. Kragic's feedback contributed greatly to the presentation of the material.

Limitations and Suggested Extensions: A weakness of the approach is the need to find a suitable level of smoothing by manual tuning. A systematic way of determining the smoothing is presented in Paper E. Most of the evaluation is performed on simulated data. Experiments with more challenging contours would be of interest, as well as an evaluation of the lower limit on the number of added data points at each iteration to achieve convergence.

Remark 1.3.3 *After publication of this paper, it has been brought to the authors' attention that there was an error in the proof of Proposition D.4.1. The result is however correct. A remedied proof is provided in this thesis. Some typos have also been corrected.*

Paper E: An Estimated General Cross Validation Function for Periodic Control Theoretic Smoothing Splines

Authors: M. Karasalo, X. Hu, and C.F. Martin.

Publication: Perspectives in Mathematical System Theory, Control, and Signal Processing, Lecture Notes in Control and Information Sciences, Springer, to appear 2010.

Abstract: In this paper, a method is developed for estimating the optimal smoothing parameter ε for periodic control theoretic smoothing splines. The procedure is based on general cross validation (GCV) and requires no a priori information about the underlying curve or level of noise in the measurements. The optimal ε is the minimizer of a GCV cost function, which is derived based on a discretization of the L_2 smoothing problem for periodic control theoretic smoothing splines.

Contributions: The main contribution is the derivation of the estimated GCV cost function for the particular periodic control theoretic smoothing spline problem. Simulation results suggest that with this estimate, the error convergence in the limit $N \to \infty$ corresponds to the convergence for the analytic GCV function.

Work Division: The method presented in this paper is an adaptation of a general method for smoothing splines in a statistical setting. Martin and Hu contributed with their knowledge in statistics and control, suggested relevant references and provided constructive feedback on the text. The estimate of the influence matrix was derived by Karasalo, based on a discretization of the problem presented in Paper D.

Limitations and Suggested Extensions: This method should be extended to the recursive problem discussed in Papers C and D. It would be desirable to find an estimate of the GCV cost function such that error convergence can be obtained using the estimated ε as the number of recursions k increases. At present, convergence is only apparent when increasing the number of data points N.

Remark 1.3.4 *Proposition E.3.1 is essentially equivalent with Proposition D.4.1. The proposition and the proof are included in Paper E since the proof provided in the published version of Paper D was incomplete.*

1.4 Formulations of the Smoothing Spline Problem in Papers C, D and E

The smoothing spline problems in Papers C, D and E, although closely related, are somewhat differently formulated. In this section, the distinctions are pointed out and explained. First, the three problems are stated.

Problem 1.4.1 ***Control Theoretic Splines in Paper C***

$$\underset{u\in L_2[0,T]}{\text{minimize}}\ J(u,x) = \frac{1}{2}\int_0^T u(t)^T Q^{-1} u(t)dt + \frac{1}{2}\sum_{i=1}^{N}(t_i - t_{i-1})(z_i - Cx(t_i))^T R^{-1}(z_i - Cx(t_i)) \tag{1.1}$$

$$\text{subject to} \quad \dot{x} = Ax + Bu \tag{1.2}$$

$$x(0) = x(T), \tag{1.3}$$

with data input defined by $z_i = z(t_i)$, $t_i \in [0,T]$, $z(T) = z(0)$ *and* $z_i = Cx(t_i) + \xi_i$, *where* ξ_i *is a symmetric, zero-mean iid noise with bounded variance.* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$ *and* $C^T \in \mathbb{R}^n$, *where* *the pair (A,B) is controllable and (A,C) is observable. Q and R are positive definite matrices of suitable dimension.*

Problem 1.4.2 ***Control Theoretic Splines in Paper D***

$$\underset{u\in L_2[0,T]}{\text{minimize}}\ J(u,x) = x(0)^T P_0^{-1} x(0) + \int_0^T u(t)^T Q^{-1} u(t)dt + \sum_{i=1}^{N}(t_i - t_{i-1})(z_i - Cx(t_i))^T R^{-1}(z_i - Cx(t_i)) \quad (1.4)$$

$$\begin{aligned} \text{subject to} \quad \dot{x} &= Ax + Bu && (1.5)\\ x(0) &= x(T), && (1.6) \end{aligned}$$

with data input defined by (t_i,z_i), such that $t_i \in [0,T]$ is the polar coordinate angle, $T = 2\pi$ and z_i is the radius in polar coordinates. Further, $z_i = Cx(t_i) + \xi_i$ where ξ_i is a symmetric, zero-mean iid noise with bounded variance. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$ and $C^T \in \mathbb{R}^n$, where the pair (A,B) is controllable and (A,C) is observable. P_0, Q and R are positive definite matrices of suitable dimension.

Problem 1.4.3 ***Control Theoretic Splines in Paper E***

$$\underset{u(t)\in L_2[0,T]}{\text{minimize}} \quad J(u,r) = \int_0^{2\pi} u(t)^2 dt + \frac{\varepsilon^2}{N}\sum_{i=1}^{N}(r(t_i) - z_i)^2 \quad (1.7)$$

$$\begin{aligned} \text{subject to} \quad r''(t) &= u(t) && (1.8)\\ r(0) &= r(2\pi) && (1.9)\\ r'(0) &= r'(2\pi), && (1.10) \end{aligned}$$

with data input defined by (t_i,z_i), such that $t_i \in [0,2\pi]$ is the polar coordinate angle and z_i is the radius in polar coordinates. Further, $z_i = r(t_i) + \xi_i$, $\xi_i \in \mathbf{N}(0,\sigma^2)$, with σ unknown.

The main distinction of Problem 1.4.1 is the inclusion of the factor $1/2$ in the cost function. The motivation is simply that with this factor, the differential equations resulting from the Hamilton-Jacobi-Bellman equation become neater.

Problem 1.4.2 includes the term $x(0)^T P_0^{-1} x(0)$. This is a remnant from early formulations of control theoretic smoothing splines and is motivated by the fact that it may facilitate solution of the problem. In Paper D, this term is mainly included because it guarantees that the problem has a well defined, unique solution even for empty data sets. This is stated and proved in the paper. Due to the application focus of the paper, the formulation of the problem is slightly less generic than in Paper C, assuming polar coordinates.

In Paper E, the focus is not on the spline problem itself but on finding a suitable level of smoothing. Therefore the presentation of the spline problem in its general form is skipped, and Problem 1.4.3 is expressed directly in polar coordinates. The assumption on normally distributed data, as well as the inclusion of the factor $1/N$ instead of $t_i - t_{i-1}$, is a prerequisite for some results for general cross validation for regular smoothing splines, and the method presented in Paper E is an adaptation of this.

Chapter 2

Preliminaries

This chapter offers an overview of fundamental concepts in control theory, mobile robotics, smoothing and filtering. The purpose of the chapter is to give a brief introduction for readers who are unfamiliar with some of the theory. The definitions and results of this chapter are established and well known, thus proofs of theorems are omitted. The focus of the following sections will be on results and techniques that are relevant for the appended papers. For a comprehensive treatment of the subjects in this chapter, the reader is invited to study the references given at the end of each section.

The outline of this chapter is as follows. In Section 2.1 we briefly introduce terminology and results in mathematical systems theory. Section 2.3 gives an overview of the field of mobile robotics. Section 2.5 concerns filtering and smoothing of data. These sections are independent and the reader may well skip one if familiar with its topics. The notation used should be unambiguous within each section but readers are urged to mind notational collision between sections.

2.1 Mathematical Systems Theory

Control systems have been invented and applied since ancient times, such as wind mills and water supply networks. The scientific field of systems and control theory emerged in the mid 1800s, with the development of complex machines and engines. Frequency domain techniques dominated the field of control theory during the first half of the 20^{th} century. State space approaches emerged mainly during the second half of the century within the field of mathematical systems theory.

The state space description of a control system is a set of differential equations for a *state vector* x, including a *control signal* u that is to be designed such that the evolvement of the state meets some specified control objective.

The *output* y of the control system contains the measurements on the system, which is a function of the state x and possibly also of the control u. In this section, basic definitions and results in the field of mathematical systems theory are summarized. Fundamental concepts such as controllability and observability are introduced for the special case of linear systems and then extended to nonlinear systems. First, basic definitions of different types of systems, in the mathematical sense, are stated.

Definition 2.1.1 *In the context of mathematical systems theory, a* ***control system*** *is the set of equations*

$$\dot{x}(t) = f(x(t), u(t), t) \tag{2.1}$$

$$y(t) = h(x(t), u(t), t), \tag{2.2}$$

or, in discrete time

$$
\begin{aligned}
x(t_{k+1}) &= \hat{f}(x(t_k), u(t_k), t_k) && (2.3)\\
y(t_{k+1}) &= \hat{h}(x(t_k), u(t_k), t_k), && (2.4)
\end{aligned}
$$

where $x \in X \subset \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output and t is the time. If $m = p = 1$ the system is called a ***SISO*** *(single input, single output) system. If $m > 1, p > 1$ it is called a* ***MIMO*** *(multiple input, multiple output) system. Correspondingly, the types $m = 1, p > 1$ and $m > 1, p = 1$ are called* ***SIMO*** *and* ***MISO****.*

Definition 2.1.2 *An* ***autonomous system****, in the mathematical sense, is a system that does not depend explicitly on the time t:*

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) && (2.5)\\
y(t) &= h(x(t), u(t)). && (2.6)
\end{aligned}
$$

The control systems appearing in this thesis are usually autonomous and affine or even linear. Definitions of such systems follow.

Definition 2.1.3 *An* ***affine control system*** *is a system of the form*

$$
\begin{aligned}
\dot{x}(t) &= f(x(t)) + g(x(t))u(t) && (2.7)\\
y(t) &= h(x(t)). && (2.8)
\end{aligned}
$$

Definition 2.1.4 *A* ***linear control system*** *is a system of the form*

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) && (2.9)\\
y(t) &= C(t)x(t) + D(t)u(t), && (2.10)
\end{aligned}
$$

where $A(t), B(t), C(t)$ and $D(t)$ are matrices of suitable dimensions.

Definition 2.1.5 *A* ***time-invariant linear system*** *is a system of the form*

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) && (2.11)\\
y(t) &= Cx(t) + Du(t), && (2.12)
\end{aligned}
$$

where A, B, C and D are constant matrices of suitable dimensions.

Now we move on to definitions concerning certain properties that are relevant for control and observer design for control systems.

Definition 2.1.6 *A system* (2.1) - (2.2) *is called* **controllable** *if, for any two points x_0 and x_1 in $\mathbb{R}^n$, there exists an admissible control u, such that u drives x from x_0 to x_1 in some finite time T.*

Remark 2.1.1 *In the literature, if $x_0 = 0$,* (2.1) - (2.2) *is sometimes called* ***reachable****, while if $x_1 = 0$ it is called null controllable. In the present context, we use these terms without distinction.*

Definition 2.1.7 *Consider the system* (2.1) - (2.2)*. Two states x_0 and x_1 are called* ***distinguishable*** *if it holds that*

$$y(\cdot, x_0) \neq y(\cdot, x_1) \tag{2.13}$$

where $y(\cdot, x)$ is the output trajectory with initial condition x. Furthermore, the system is called ***locally observable at*** *x_0 if there is a neighborhood $N(x_0)$ such that* (2.13) *holds for all $x_1 \in N(x_0)$. The system is called* ***locally observable*** *if it is locally observable at every $x \in X$.*

Finally, we introduce some terminology regarding stability of control systems.

Definition 2.1.8 *Consider the autonomous system*

$$\dot{x} = f(x), \tag{2.14}$$

and let $x(t, x_0)$ denote the state x at time t with initial condition $x(t_0) = x_0$. We say that

1) *$x = x_0$ is an* ***equilibrium*** *of* (2.14) *if $f(x_0) = 0$. In the following, without loss of generality, we assume $x_0 = 0$.*

2) *$x = 0$ is* ***stable*** *if $\forall\ \varepsilon > 0\ \exists\ \delta(\varepsilon) > 0$ such that*

$$\|x_0\| < \delta(\varepsilon) \Rightarrow \|x(t, x_0)\| < \varepsilon \quad \forall t \geq 0.$$

3) *$x = 0$ is* ***unstable*** *if it is not stable.*

4) *$x = 0$ is* ***attractive*** *if $\exists\ \eta > 0$ such that $\|x_0\| < \eta \Rightarrow \lim_{t\to\infty} x(t, x_0) = 0$.*

5) *$x = 0$ is* ***asymptotically stable*** *if it is stable and attractive.*

6) *$x = 0$ is* ***exponentially stable*** *if $\exists\ k > 0, r > 0$ and a neighborhood $N(0)$ of the origin such that*

$$\|x(t, x_0)\| < k\|x_0\|e^{-rt} \quad \forall\ t \geq 0, \quad x_0 \in N(0).$$

2.1.1 Linear Systems

In this section, the theory of linear control systems is reviewed. Properties and control design of linear systems is by now well understood, and a study of this particular type of systems may facilitate understanding of control systems in general. Here, results on observability, reachability and stability are presented. First, we introduce the concept of transition matrices.

The Transition Matrix

The transition matrix provides a nice means of expressing the solution of a differential equation. A definition is provided next.

Definition 2.1.9 *Let $\mathbf{e}_j \in \mathbb{R}^n$ denote the $j : th$ unit vector in $\mathbb{R}^n$. Consider the linear, uncontrolled system*

$$\dot{x}(t) = A(t)x(t) \tag{2.15}$$

$$x(t_0) = \mathbf{e}_j, \tag{2.16}$$

and let $\Phi_j(t, t_0) \in \mathbb{R}^n$ denote the unique solution of (2.15) - (2.16)*. Then the* ***transition matrix*** *$\Phi(t, t_0) \in \mathbb{R}^{n\times n}$ is defined by*

$$\Phi(t, t_0) = [\Phi_1(t, t_0), \ldots, \Phi_n(t, t_0)]. \tag{2.17}$$

The following lemma lists properties of $\Phi(t,t_0)$.

Lemma 2.1.1 *The transition matrix* (2.17) *satisfies the following properties:*

$$\begin{aligned}
\frac{\partial\Phi(t,s)}{\partial t} &= A(t)\Phi(t,s) && (2.18)\\
\Phi(s,s) &= I && (2.19)\\
\frac{\partial\Phi(t,s)}{\partial s} &= -\Phi(t,s)A(s) && (2.20)\\
x(t_0) &= a \quad \Rightarrow \quad x(t) = \Phi(t,t_0)a && (2.21)\\
\Phi(t,s) &= \Phi(t,\tau)\Phi(\tau,s) \quad \forall \ (t,s,\tau). && (2.22)
\end{aligned}$$

Now consider the linear, controlled system

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t)+B(t)u(t) && (2.23)\\
x(t_0) &= x_0 && (2.24)\\
y(t) &= C(t)x(t). && (2.25)
\end{aligned}$$

It is easy to show that, with $\Phi(t,t_0)$ defined by (2.17), the solution can be written

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,s)B(s)u(s)ds. \tag{2.26}$$

Lemma 2.1.2 *For a time-invariant system* (2.11) - (2.12) *it holds that*

$$\Phi(t,s) = e^{A(t-s)} = \sum_{k=0}^{\infty}\frac{1}{k!}A^k(t-s)^k. \tag{2.27}$$

We will now move on to discuss criteria for the essential properties reachability, observability and stability of linear systems.

Reachability

Investigating reachability (controllability) of a control system is essential before designing a control. States that are reachable can be manipulated by a feedback control to meet some performance criteria, which are usually expressed for a particular state component as a reference value. Reachability is defined below.

Definition 2.1.10 *The* ***reachability gramian*** *of* (2.23) - (2.25) *is the $n \times n$-matrix*

$$W(t_0,t_1) = \int_{t_0}^{t_1}\Phi(t_1,s)B(s)B(s)^T\Phi(t_1,s)^T ds. \tag{2.28}$$

Definition 2.1.11 *For a time-invariant system, the* ***reachability matrix*** *is the matrix*

$$\Gamma = [B\ AB\ A^2B\ \dots\ A^{n-1}B]. \tag{2.29}$$

Theorem 2.1.1 ***Reachability.*** *For the system* (2.23) - (2.25), *the state transfer from* $x_0 = x(t_0)$ *to* $x_1 = x(t_1)$ *is possible if and only if*

$$x_1 - \Phi(t_1,t_0)x_0 \in \text{Im } W(t_0,t_1). \tag{2.30}$$

Furthermore, if (2.23) - (2.25) *is time-invariant, it is completely reachable if* rank $\Gamma = n$.

The results about observability of linear systems are in many ways analogous to the reachability results. They are discussed next.

Observability

Usually all states of a control system are not measurable directly. They may however be present in the output implicitly. The observability properties of a system tell us which states can be reconstructed from the output. For linear systems, observable states can be reconstructed or estimated using an observer or a filter. This will be discussed further in Section 2.5. Such estimates often play a vital role when designing feedback controls. Definitions of observability are given below.

Definition 2.1.12 *The* ***observability gramian*** *of* (2.23) - (2.25) *is the matrix*

$$M(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_1,s)^T C(s)^T C(s)\Phi(t_1,s)ds. \tag{2.31}$$

Definition 2.1.13 *For a time-invariant system, the* ***observability matrix*** *is the matrix*

$$\Omega = [C\, CA\, CA^2\, \dots\, CA^{n-1}]^T. \tag{2.32}$$

Theorem 2.1.2 ***Observability.*** *For the system* (2.23) - (2.25), *the initial states* $x_0 = x(t_0)$ *and* $x_1 = x(t_1)$ *produce the same output on* $[t_0,t_1]$ *if and only if*

$$x_0 - x_1 \in \ker\ M(t_0,t_1). \tag{2.33}$$

Furthermore, if (2.23) - (2.25) *is time-invariant, it is completely observable if* rank $\Omega = n$.

Finally, we will cover stability and stabilization of linear systems.

Stability and Stabilization

Stable systems have the nice property that the state converges to, or stays close to, some reference value. If a linear system is controllable, a stabilizing feedback can be designed so that the closed loop system is stable. Stability is a property often desired for the error dynamics of a control system - that is, the deviation from reference values should be bounded or even converge to 0 as $t \longrightarrow \infty$. In the following, some relevant definitions are stated.

Definition 2.1.14 *A linear system* (2.9) - (2.10) *is called* ***input-to-output stable*** *if there is a k such that, for all initial times* t_0,

$$\left.\begin{array}{l} x(t_0) = 0, \\ \|u(t)\| \le 1, \quad t \in [t_0,\infty) \end{array}\right\} \quad \Rightarrow \quad \|y(t)\| \le k, \quad t \in [t_0,\infty). \tag{2.34}$$

Definition 2.1.15 *Consider the linear, time-invariant system*

$$\begin{aligned}\dot{x} &= Ax &(2.35)\\ x(0) &= x_0. &(2.36)\end{aligned}$$

This system is called ***stable*** *if the solution is bounded for $t \in [0, \infty)$ for all initial states x_0 and* ***asymptotically stable*** *if $x(t) \to 0$ as $t \to \infty$ for all x_0.*

Definition 2.1.16 *A matrix A is called a* ***stable matrix*** *if the real parts of all eigenvalues of A are negative.*

Theorem 2.1.3 *The system* (2.35) - (2.36) *is asymptotically stable if and only if A is a stability matrix. If at least one eigenvalue is positive, the system is unstable.*

Theorem 2.1.4 *The linear, time-invariant system* (2.11) - (2.12) *is input-to-output stable if the matrix A is a stability matrix.*

For linear systems, a common control objective is to find a feedback $u = Kx$ such that the closed loop system $\dot{x} = (A - BK)x$ is stable, *i.e.* the matrix $(A - BK)$ is a stability matrix.

For nonlinear systems, analysis is often much more complicated than for linear systems. In the next section, fundamentals of nonlinear control theory are summarized.

2.1.2 Nonlinear Systems

For nonlinear systems, the existing results on controllability, observability and stability are in general weaker than for linear systems. In this section, the focus is on affine nonlinear systems (2.7) - (2.8). Control design for robots with nonlinear kinematics is discussed in Paper A. Although most of the topics in this section are outside the scope of the appended papers they are included for completeness of the systems theory overview. Often, a linearization of the nonlinear system is used for local analysis. A linearization of (2.7) at x_0 has the form

$$\dot{z} = \frac{\partial f}{\partial x}(x_0)z + g(x_0)v. \tag{2.37}$$

In the following, terminology and results for nonlinear systems are presented both for local analysis and in general.

Controllability

Locally, controllability of an affine system can be analyzed by studying a linearization.

Theorem 2.1.5 ***Local Controllability***. *Consider the system* (2.7) - (2.8). *Suppose $f(x_0) = 0$ and $u = 0$. If the linearization* (2.37) *is controllable, then the set of points that can be reached from x_0 in finite time contains a neighborhood of x_0.*

For a more general result, some mathematical tools are needed. They are introduced in the definitions below.

Definition 2.1.17 *Let N be an open set in $\mathbb{R}^n$. Define, for a set of smooth functions λ_i, the set*

$$M = \{x \in N : \lambda_i(x) = 0, i = 1, \ldots, n-m\}. \tag{2.38}$$

If

$$\operatorname{rank}\left[\frac{\partial \lambda_1}{\partial x}, \ldots, \frac{\partial \lambda_{n-m}}{\partial x}\right]^T = n - m \,\forall\, x \in M, \tag{2.39}$$

then M is a ***hypersurface****, which is a* ***smooth manifold****, of dimension m.*

Definition 2.1.18 *Let $x \in M$ and attach at x a copy of $\mathbb{R}^n$ tangential to M. The resulting structure is called the **tangent space** of M at x, and is denoted by T_xM.*

Definition 2.1.19 *A **vector field** f on M is a mapping assigning to each point $p \in M$ a tangent vector $f(p)$ in T_xM.*

Definition 2.1.20 *Let λ be a smooth real-valued function on M. The **Lie derivative** $L_f\lambda$ of λ along f is a function $M \mapsto \mathbb{R} : (L_f\lambda)(p) = f(p)\lambda$. In local coordinates, it is represented by*

$$(L_f\lambda)(p) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i. \tag{2.40}$$

Definition 2.1.21 *For any two vector fields f and g on M, let the new vector field $[f,g]$ on M be defined by*

$$[f,g]\lambda = L_f L_g \lambda - L_g L_f \lambda. \tag{2.41}$$

*This vector field is the **Lie bracket** of f and g. In local coordinates, the expression for $[f,g]$ is given by*

$$\frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g. \tag{2.42}$$

Definition 2.1.22 *An affine system* (2.7) - (2.8) *has **relative degree** r at x_0 if*

$$\begin{aligned} L_g L_f^k h(x) &= 0 \quad \forall x \text{ in a neighborhood of } x_0 \text{ and } k \leq r-2 && (2.43)\\ L_g L_f^{r-1} h(x_0) &\neq 0. && (2.44)\end{aligned}$$

Definition 2.1.23 *A **distribution** D on M is a map which assigns to each $p \in M$ a vector subspace $D(p)$ of T_pM.*

Definition 2.1.24 *A distribution D is **invariant** under the vector field f if $d \in D \Rightarrow [d, f] \in D$.*

Definition 2.1.25 *The **strong accessibility distribution** R_c of an affine control system* (2.7) - (2.8) *is the smallest distribution that contains* $\mathrm{span}\{g_1, \dots, g_m\}$ *and is invariant under the vector fields $f, g_1, \dots, g_m$.*

Finally we arrive at a controllability result for nonlinear systems.

Theorem 2.1.6 ***Local Strong Accessibility**. Consider the system* (2.7) - (2.8)*. If at a point x_0, it holds that* $\dim(R_c(x_0)) = n$*, then the system is locally strongly accessible from x_0. This means that for any neighborhood of x_0, the set of reachable points in some sufficiently small finite time T contains a non-empty open set.*

Further, if $f = 0$ and $\dim(R_c(x)) = n$ $\forall x \in X$*, then the system is controllable.*

Observability

For nonlinear systems, observability is not a trivial issue. In general, observability properties depend on control input as well as initial conditions and observability of a system does not imply the existence of an observer. A result on local observability for affine systems is given in this section.

Definition 2.1.26 *Consider the system* (2.7) - (2.8). *The* ***observation space*** *O is the linear space over $\mathbb{R}$ of functions on X in the form of*

$$L_{v_1}L_{v_2}\dots L_{v_k}h_j, \quad j\in[1,\dots,p], \quad k=1,2,\dots, \tag{2.45}$$

where $v_i\in\{f,g_1,\dots,g_m\}$. Further, the ***observability codistribution*** *is defined by*

$$dO=\operatorname{span}\{dH : H\in O\}, \tag{2.46}$$

where dH is the exterior derivative of H, which in local coordinates has the form

$$\sum_{i=1}^{n}\frac{\partial H}{\partial x_i}dx_i. \tag{2.47}$$

Theorem 2.1.7 ***Local Observability****. Consider the system* (2.7) - (2.8). *If*

$$\dim\ dO(x_0)=n, \tag{2.48}$$

then the system is locally observable at x_0.

Stability is an important issue in nonlinear control and several approaches exist to investigate whether a system is stabilizable. In the next section, we briefly introduce some of the key results.

Stability and Stabilization

We begin by studying stability locally for nonlinear affine systems. First, two theorems regarding local stability are stated.

Theorem 2.1.8 ***Local Stability****. Consider the system*

$$\dot{x}=f(x) \tag{2.49}$$

and its linearization

$$\dot{z}=Az. \tag{2.50}$$

If the equilibrium $z=0$ of (2.50) *is exponentially stable, then $x=0$ of* (2.49) *is locally exponentially stable. If A is a constant matrix with at least one eigenvalue with positive real part, then $x=0$ is unstable.*

Theorem 2.1.9 ***Local Feedback Stabilization****. Consider the linearization* (2.37) *of the affine system* (2.7) - (2.8). *Define $A=\partial f(0)/\partial x$, $b=g(0)$. Then a necessary condition for* (2.7) - (2.8) *to be stabilizable by a differentiable feedback control is that*

1) (A,b) does not have uncontrollable states associated with unstable eigenvalues.

2) The map $(x,u)\mapsto f(x)+g(x)u$ maps onto a neighborhood of x_0.

For more general results, more advanced approaches are needed, such as center manifold theory and Lyapunov functions. We will not go into detail about stabilizability for nonlinear systems here, but give one result regarding Lyapunov stability.

Definition 2.1.27 *A continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$ is said to be a **locally positive definite function (lpdf)** if $V(t,0) = 0 \ \forall \ t \geq 0$ and there exists a strictly increasing function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and $\varepsilon > 0$ such that for x in the ε-ball around the origin it holds that*

$$\alpha(\|x\|) \leq V(t,x) \quad \forall \ t \geq 0. \tag{2.51}$$

Theorem 2.1.10 *Suppose $x = 0$ is an equilibrium of the system*

$$\dot{x} = f(x,t), \quad f \in C^1, \quad x \in \mathbb{R}^n. \tag{2.52}$$

Then $x = 0$ is uniformly stable if there exists a C^1, decrescent lpdf $V(t,x)$ such that for x in a neighborhood of the origin, it holds that

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(x,t) \leq 0 \quad \forall t \geq 0. \tag{2.53}$$

It is hard to find a systematic approach to determining such an $lpdf$. Generally, methods of clever guessing and trial-and-error are applied.

The final topic of this section is control design by means of optimization.

2.1.3 Optimal Control

Optimal control simply means controlling a system in a way that is optimal with respect to a specified criterion, usually expressed as a cost function or performance index. This is a very useful method for control design and widely used in fields such as economics, aeronautics and robotics. The field of optimal control emerged in the 1950s mainly due to the rapid development of the space industry.

This section will give a brief introduction to the area by studying the following common form of optimal control problem:

Problem 2.1.1

$$\begin{aligned} \underset{u \in U}{\text{minimize}} \quad J(u) &= \int_{t_0}^{t_f} f_0(t,x(t),u(t))dt + \phi(x(t_f)) && (2.54)\\ \text{subject to } \ \dot{x}(t) &= f(t,x(t),u(t)) && (2.55)\\ x(t_0) &= x_0 && (2.56)\\ & && (2.57) \end{aligned}$$

In general, f_0, ϕ, and f are assumed to be C^1. While the initial state $x(t_0)$ is fixed, the terminal state $x(t_f)$ is free, but deviations from some desired terminal state are penalized by the term $\phi(x(t_f))$ in the cost function. We will state results for this formulation and for a special case where the cost function is quadratic and the constraints are linear. Variations of this formulation can be solved by similar methods.

First, we state the principle of optimality.

Lemma 2.1.3 ***The principle of optimality***. *Let $u^*(t)$ be the optimal solution of Problem 2.1.1 and let $x^*(t)$ be the corresponding optimal trajectory. Then, for any $t' \in (t_0, t_f]$ the optimal pair (u', x') on the interval $[t', t_f]$ is $(u^*(t), x^*(t))$, where $t \in [t', t_f]$.*

This lemma simply states the rather intuitive result that the optimal control on an interval of time is exactly the optimal control for the entire problem, restricted to that time interval. The principle of optimality leads to the following result.

Lemma 2.1.4 ***The dynamic programming equation***. *Let $J^*(t, x(t))$ denote the optimal cost-to-go, which is the optimal cost from the time t to t_f. Then, from the principle of optimality, it holds that*

$$J^*(t, x(t)) = \int_t^{t+\Delta t} f_0(s, x(s), u(s)) ds + J^*(t + \Delta t, x(t + \Delta t)), \tag{2.58}$$

which means that the optimal control can be computed backwards from the terminal time.

From (2.58), we can derive the Hamilton-Jacobi-Bellman Equation (HJBE), which is essential for the dynamic programming approach to optimal control problems:

$$\begin{aligned} -\frac{\partial J^*(t,x(t))}{\partial t} &= \underset{u \in U}{\text{minimize}} \left\{ f_0(t,x(t),u(t)) + \frac{\partial J^*(t,x(t))}{\partial x}^T f(t,x(t),u(t)) \right\} && (2.59) \\ J^*(t_f, x(t_f)) &= \phi(x(t_f)). && (2.60) \end{aligned}$$

The fact that the HJBE holds for the optimal control input $u^*(t)$ leads to the following theorem, which is useful for solving Problem 2.1.1.

Theorem 2.1.11 ***The Verification Theorem for Dynamic Programming***. *Suppose that the function $V : [t_0, t_f] \times \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable in t and x and solves the HJBE:*

$$\begin{aligned} -\frac{\partial V(t,x(t))}{\partial t} &= \underset{u \in U}{\text{minimize}} \left\{ f_0(t,x(t),u(t)) + \frac{\partial V(t,x(t))}{\partial x}^T f(t,x(t),u(t)) \right\} && (2.61) \\ V(t_f, x(t_f)) &= \phi(x(t_f)). && (2.62) \end{aligned}$$

Further, suppose that

$$\mu(t, x(t)) = \underset{u \in U}{\arg\min} \left\{ f_0(t,x(t),u(t)) + \frac{\partial V(t,x(t))}{\partial x}^T f(t,x(t),u(t)) \right\} \tag{2.63}$$

is an admissible control. Then $V(t, x(t)) = J^*(t, x(t))$ for all $(t, x) \in [t_0, t_f] \times \mathbb{R}^n$, and $\mu(t, x(t)) = u^*(t)$ is the optimal control.

It should be noted that the conditions in Theorem 2.1.11 are sufficient but not necessary. A solution scheme derived from Theorem 2.1.11 follows.

1) Define the *Hamiltonian* $H(t, x, u, \lambda) = f_0(t, x, u) + \lambda^T f(t, x, u)$, where λ is a parameter vector of suitable dimension.

2) Find $\tilde{\mu}(t, x, \lambda) = \underset{u \in U}{\arg\min}\ H(t, x, u, \lambda)$.

3) Solve $-\frac{\partial V(t,x)}{\partial t} = H\left(t, x, \tilde{\mu}\left(t, x, \frac{\partial V(t,x)}{\partial x}\right), \frac{\partial V(t,x)}{\partial x}\right)$ subject to $V(t_f, x) = \phi(x)$.

Then $u^*(t) = \mu(t, x) = \tilde{\mu}\left(t, x, \frac{\partial V(t,x)}{\partial x}\right)$.

Now we move on to the special case of linear quadratic (LQ) control.

2.1.4 LQ Optimal Control

First, we formulate the linear-quadratic special case of (2.55) - (2.56):

Problem 2.1.2

$$\begin{aligned} \underset{u \in U}{\text{minimize}} \quad J(u) &= \int_{t_0}^{t_f} [x(t)^T Q x(t) + u(t)^T R u(t)] dt + x(t_f)^T Q_0 x(t_f) \qquad (2.64) \\ \text{subject to } \dot{x}(t) &= Ax + Bu \qquad (2.65) \\ x(t_0) &= x_0, \qquad (2.66) \end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$, and (A, B) is controllable. $Q_0 \geq 0$, $Q \geq 0$, and $R > 0$ are symmetric matrices of suitable dimensions. Problems similar to Problem 2.1.2 appear in Papers C, D and E. We get:

1) $H(t, x, u, \lambda) = x^T Q x + u^T R u + \lambda^T (Ax + Bu)$.

2) $\tilde{\mu}(t, x, \lambda) = -\frac{1}{2} R^{-1} B^T \lambda$.

3) Since $H(t, x, \tilde{\mu}(t, x, \lambda), \lambda) = x^T Q x - \frac{1}{4} \lambda^T B R^{-1} B^T \lambda + \lambda^T A x$, and $V(t_f, x) = x^T Q_0 x$, a suitable guess is $V(t, x) = x^T P(t) x$, for some positive semi-definite matrix $P(t)$. Then the HJBE becomes

$$x^T [\dot{P} + Q - PBR^{-1}B^T P + PA + A^T P] x = 0, \quad x^T P(t_f) x = x^T Q_0 x, \qquad (2.67)$$

or

$$\dot{P} + Q - PBR^{-1}B^T P + PA + A^T P = 0, \quad P(t_f) = Q_0, \qquad (2.68)$$

which is a matrix Riccati equation. The optimal control is $u^*(t, x) = \tilde{\mu}\left(t, x, \frac{\partial V(t, x)}{\partial x}\right) = -R^{-1} B^T P(t) x(t)$ and the optimal cost is $V(t_0, x_0) = x_0^T P(t_0) x_0$.

This concludes the overview of the field of mathematical systems theory. In the subsequent sections, we will discuss mobile robotics from a control perspective as well as smoothing and filtering of data.

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2.3 Mobile Robotics

What is a robot? Depending on the context, several different definitions are applicable. Some examples follow:

- A device that responds to sensory input.
- A program that runs automatically without human intervention.
- A device that automatically performs complicated and often repetitive tasks.
- An electro-mechanical system which conveys a sense that it has intent or agency of its own.

In the context of this thesis, robots are mechanical systems that can gather information about their environment via sensors and respond to that information in an intelligent manner. Further, the robots that appear in this thesis are *mobile*, meaning that they, as opposed to industrial robots which are usually stationary, can move about in their environment in a controlled fashion. An *autonomous* mobile robot is a robot that can act and react to events without interference or guidance from human beings. Control design for autonomous mobile robots is one of the topics of this thesis.

There are several kinds of mobile robots. Some of the most common types are listed below.

- **Humanoids.** These robots are constructed to resemble human beings and usually move about by walking on two legs. A well known example would be the fictional humanoids in the Terminator movies, but real humanoids are represented by for instance ASIMO by Honda.
- **Unmanned Ground Vehicles (UGVs).** These robots generally move on wheels and often resemble cars. Robots of this type are by now commercially available, in the form of automatic vacuum cleaners such as Trilobite by Electrolux, or lawn mowers such as Robomow by Friendly Robotics. The robots that appear in this thesis are all UGVs.
- **Unmanned Air Vehicles (UAVs).** These robots come in the shapes of helicopters or airplanes. Many existing UAV models today are research models designed for laboratory use. A growing area of application for UAVs is military reconnaissance. An example model is the Luna X 2000 UAV of the German Army.

- **Autonomous Underwater Vehicles (AUVs).** These robots are often designed as unmanned submarines and used commercially for instance by oil companies to map the ocean floor. Examples include Sapphires by the Swedish Defense Research Agency and SAAB, and the Norwegian HUGIN 3000 by Kongsberg Maritime and the Norwegian Defense Research Establishment. AUV models inspired by underwater animals are available for research, such as AquaJelly by Festo.

The focus of the remainder of this section will be on UGVs. Control layers and design are discussed next.

2.3.1 Control

The control of autonomous robots can be broken down into several layers. The division and degree of precision should be adapted for the intended applications. In the context of this thesis, a suitable layering is depicted in Figure 1. One can distinguish between high-level and low-level control.

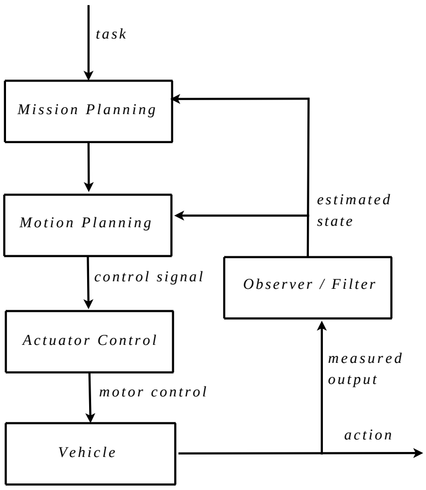

Figure 1: A model of control layers for an autonomous mobile robot.

Usually motor or actuator control are regarded as low level, while mission planning is high level. One may view the intermediate level of motion planning as a black box that translates mission goals to executable commands. In this thesis, the focus is usually on the level of motion planning. The scenarios we study generally concern translative motion of the robot, either planning of the motion such as trajectory estimation, or motion relative to other agents, such as formation control. This text is adapted to the control objectives relevant for the appended papers.

Designing control signals for robotic systems is an important discipline within the field of mathematical systems theory. To apply the theory and tools reviewed in Section 2.1, we must relate the terminology of systems theory to the physical features and control objectives for the mobile robot. We define the *state* of the robot as a vector x containing relevant information such as current position, velocity and heading. The state can be manipulated via a *control signal* u that should be based both on the control objective and the current state of the robot. To this end, the robot uses *sensors* to measure its state. The measurements are collected in the *output vector* y. Most of the time, some components of the state vector x cannot be directly measured. This introduces the need of an *observer*, whose purpose is to compute a *state estimate* $\hat{x}$, which can be used for feedback control. Furthermore, as there are no perfect sensors, measurements are generally noise contaminated, calling for some preprocessing before the feedback is computed. The preprocessing, and sometimes also the estimation of unmeasured states, is done by a *filter* or *smoother*. This is a major topic of this thesis. A brief introduction to relevant filtering and smoothing techniques is given in Section 2.5. An example is provided to illuminate the terminology introduced in this section.

Example 2.3.1 *A common state model for mobile robots is unicycle kinematics. Unicycle robots appear in Papers A, B and D. This is a suitable model for robots that have two parallel wheels that can be controlled, and possibly additional passive wheels. See Figure 2. Let the state vector be*

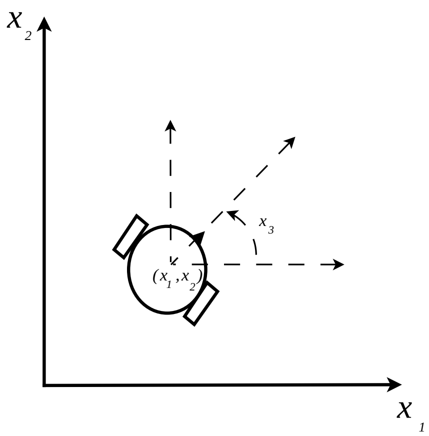

Figure 2: A unicycle robot.

$\mathbf{x} = [x_1\ x_2\ x_3]^T$*, where* (x_1,x_2) *is the location of the center of the robot with respect to some global coordinate system, and* x_3 *is the heading, defined as the angle between the* x_1*-axis and the robot's motion vector. Then the equations of motion for the robot are*

$$\dot{x}_1 = u_1 \cos x_3 \tag{2.69}$$
$$\dot{x}_2 = u_1 \sin x_3 \tag{2.70}$$
$$\dot{x}_3 = u_2, \tag{2.71}$$

where $u = [u_1\ u_2]^T$ *is the control input.* (2.70) *is an example of a* ***non-holonomic*** *system, meaning that the motion is restricted in some directions. For instance, a vehicle with kinematics determined by* (2.70) *can not move sideways.*

Again, sensing is a prerequisite to feedback control. Naturally, without feedback from surroundings and events the robot cannot be controlled in an autonomous manner. Sensing is the topic of the next section.

2.3.2 Sensing

A device that can measure physical quantities such as temperature, distance or density, is called a sensor. Usually, a sensor also converts the measurement to a signal that is suitable for data processing. Sensor information is crucial for feedback control for mobile robots. In order to react properly to unexpected events, the robot must be able to gather information about itself and its environment in real time. In the field of robotics, a typical classification of sensors is as follows.

- **Enteroception.** These sensors measure the inner state, such as pressure and temperature.
- **Proprioception.** These sensors measure quantities such as location and heading, both of the robot itself and possibly of attached manipulators.
- **Exteroception.** These sensors measure the state of the robot's surroundings, such as the distance to objects or shape of obstacles.

In this thesis, the focus is on proprioception and exteroception. Below we list some common sensor types for these purposes. The sensors appearing in the appended papers are laser range sensors, infrared (IR) sensors and wheel encoders. The specific types are described further in the papers.

- **Proprioception.**
  - *Gyros.* A gyro measures orientation by utilizing the principle of the preservation of angular momentum. An example is the XV-3500CB by Seiko Epson Corporation.
  - *Accelerometers.* An accelerometer usually consists of a cover layer that is fixed in the robot and an inner core which is not. Due to the inertia of the core, changes in the robot's speed can be detected. An example of an accelerometer is the MTN/1100 Series by Monitran.
  - *Wheel Encoders.* Encoders measure the position of an UGV compared to its initial position by converting the number of times the wheels have turned to physical distance. An example is the WW-01 WheelWatcher Encoder Kit by Nubotics. Wheel encoders are used for localization of robots in Papers A and D.
- **Exteroception.**
  - *Laser range sensors.* A laser range sensor measures the distance to objects by sending out a laser beam and measuring the time until the reflected beam returns. An example is the FG21-LR Long-Range Rangefinder by RIEGL. Laser range sensors are used for range measurements in Paper D.
  - *Vision/cameras.* Due to the increasing availability of quality low-price digital cameras the field of computer vision has grown significantly over the past few years. Computer vision software translates the visual input to digital information on shape, size and location of objects surrounding the robot. A lot of open source code is available for computer vision, for instance Blepo from RoboRealm.
  - *Active IR.* An active IR sensor works in much the same way as a laser sensor, but sends out a beam of infrared light. They are usually more sensitive to ambient light and therefore have lower accuracy. An example is the GP series by Acroname robotics. Active IR is used for range measurements in Paper A.
  - *Active Sonar.* An active sonar sensor sends out a sound impulse and computes distance based on the time it takes for the echo to return. Sonar is a popular technique for underwater applications, but is also used for robot navigation in air. An example of a sonar sensor is the Mini-A from SonaSwitch.

2.4 References

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2.5 Smoothing and Filtering

Smoothing and filtering of measurement data are two closely related concepts. One may view them as two approaches to the same problem, namely removing disturbances from a data set to refine the estimate of an underlying signal. Smoothing is a well known topic in statistics, while filtering is a classical tool in signal processing and systems theory.

Often, smoothing is applied to a complete data set, while filtering is sometimes performed online and pointwise. Smoothing may also be considered as a form of low-pass filtering, since in most applications, a smoother removes high-frequency fluctuations from the input signal.

There is a rich literature on both data smoothing and data filtering. A thorough treatment is beyond the scope of this text. Here, we introduce the techniques relevant for the appended papers, namely smoothing splines and Kalman filtering.

2.5.1 Smoothing Splines

Classical splines were introduced in the 1940s as functions defined piecewise in terms of low-degree polynomials, with the main purpose to interpolate between points of a given data set. The idea itself, of using polynomials for interpolation, is far older, dating back to the mid 1700s. Interpolating between n points using a single polynomial requires a polynomial of degree $n-1$ and has disadvantages such as the well known Runge's phenomenon. Using polynomial splines remedies this problem. A formulation of the spline problem follows.

Problem 2.5.1 ***Interpolating Splines.*** *Let $z_i \in \mathbb{R}$ be data sampled at times $t_i \in [0,T]$, $i \in [1,N]$, and define the set F of twice differentiable functions that interpolate (t_i, z_i), i.e.*

$$F = \{ f \in C^2[0,T] \ : \ f(t_i) = z_i \}, \tag{2.72}$$

which is a Banach space under the supremum norm. Then the interpolating spline is the solution of

$$\underset{f \in F}{\text{minimize}} \ \left\{ \max_{t \in [0,T]} \left| \frac{d^2 f(t)}{dt^2} \right| \right\}. \tag{2.73}$$

The solution of Problem 2.5.1 is the *cubic* spline, which is a piecewise polynomial of degree three.

Interpolating splines are an excellent means of estimating curves from discrete samples, if the samples are exact. However, this is seldom the case for data sets in real applications. Also, even with exact samples, an interpolating estimate is not always desirable. An illustrating example is in aircraft applications, where exact tracking of way points often requires large control gain and increased fuel usage. This motivates the introduction of smoothing splines. Smoothing splines became a major topic in the field of mathematical statistics in the 1970s. The smoothing spline problem is formulated below.

Problem 2.5.2 ***Regular Smoothing Splines***. *Let z_i be data sampled at times $t_i \in [0,T]$, $i \in [1,N]$, and let L_2 be the Hilbert space of square integrable functions. Then the smoothing spline is the solution of*

$$\underset{f:\frac{d^2 f(t)}{dt^2}\in L_2}{\text{minimize}} \int_0^T \left(\frac{d^2 f(t)}{dt^2}\right)^2 dt + \lambda \sum_{i=1}^{N} (f(t_i) - z_i)^2. \tag{2.74}$$

The output of this problem is also a cubic spline, but the spline does not necessarily interpolate directly through the data points. $\lambda > 0$ determines the tradeoff between smoothness and faithfulness to the data. Letting $\lambda \to \infty$ results in an interpolating spline.

Control theoretic splines may be viewed as a generalization of regular smoothing splines. They are discussed next.

Control Theoretic Smoothing Splines

Control theoretic smoothing splines were introduced in the early 2000s and the theory is therefore still emerging. As the name indicates, control theoretic smoothing splines makes a connection between the fields of mathematical statistics and control theory, or more specifically, between regular smoothing splines and optimal control. Here, a derivation of the control theoretic smoothing spline problem is given.

Consider the linear, time invariant, controllable and observable SISO system

$$\dot{x} = Ax + Bu \tag{2.75}$$
$$y = Cx, \tag{2.76}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and B and and C are vectors of compatible dimensions. Since

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds, \tag{2.77}$$

we can write

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-s)}Bu(s)ds. \tag{2.78}$$

The aim of control theoretic smoothing splines is to produce a control law $u(t)$ that drives the output trajectory $y(t)$ close to a fixed set of data points

$$D = \{(t_i, z_i) : t_i < t_{i+1},\ t_i \in [0,T],\ i \in [1,N], z_i \in \mathbb{R}\}. \tag{2.79}$$

A natural approach to achieve this objective is optimal control. Define a cost function

$$J(u) = \int_0^T u(t)^2 dt + \lambda \sum_{i=1}^{N} w_i (y(t_i) - z_i)^2, \tag{2.80}$$

where w_i are non-negative weights. The desired control is the function $u^*(t)$ that minimizes $J(u)$ subject to the affine constraint (2.78). A formal statement of the control theoretic smoothing spline problem follows.

Problem 2.5.3 ***Control Theoretic Smoothing Splines***. *Let z_i be data sampled at times $t_i \in [0,T]$, $i \in [1,N]$, and let L_2 be the Hilbert space of square integrable functions. Then the control theoretic smoothing spline is the output $y(t)$ of the system* (2.77) - (2.78), *whose input $u(t)$ is the solution of*

$$\begin{aligned} \underset{u \in L_2}{\text{minimize}} \quad \int_0^T & \quad u(t)^2 dt + \lambda \sum_{i=1}^{N} (y(t_i) - z_i)^2 && (2.81) \\ \text{subject to } \dot{x} &= Ax + Bu && (2.82) \\ y &= Cx. && (2.83) \end{aligned}$$

Lemma 2.5.1 *The optimal solution of Problem 2.5.3 has the form*

$$\begin{aligned} u(t) &= \sum_{i=1}^{N} \tau_i g_{t_i}(t) && (2.84) \\ \text{where } g_{t_i}(t) &= \begin{cases} Ce^{A(t_i - t)}B, & t \leq t_i \\ 0 & \text{otherwise}, \end{cases} && (2.85) \end{aligned}$$

and τ_i are scalar coefficients.

The computation of the coefficients τ_i is non trivial and will not be discussed further here. As a special case, note that with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (1 \quad 0), \tag{2.86}$$

we obtain

$$g_{t_i}(t) = t_i - t, \tag{2.87}$$

so that the resulting spline $y(t)$ computed from (2.78) is again a cubic smoothing spline.

The following theorem follows from Hilbert's projection theorem (which is stated further down in Theorem 2.5.2).

Theorem 2.5.1 ***Existence of solutions***. *Let $\mathscr{C}$ be a closed, affine subspace of $L_2[0,T]$. Minimizing* (2.80) *subject to* (2.77) - (2.78) *and $u \in \mathscr{C}$ yields a unique solution $u^*(t)$.*

Control theoretic smoothing splines are studied in Papers C, D and E. In the next section, we move on to filtering by means of the Kalman filter. This is a well known and widely used filter. A derivation of the Kalman recursions is presented below.

2.5.2 Kalman Filtering

Although named after Rudolf E. Kalman, the Kalman filter was developed simultaneously by several researchers in the late 1950s and early 1960s. It is probably the most well known filter in the field of mathematical systems theory and has many nice properties. A derivation of the Kalman filter is supplied in this section.

Consider a linear discrete-time system

$$x(t_{k+1}) = Ax(t_k) + Bw(t_k), \tag{2.88}$$

where $x(t_k) \in \mathbb{R}^n$ and a measurement $y(t_k) \in \mathbb{R}^m$, governed by

$$y(t_k) = Cx(t_k) + Dv(t_k), \tag{2.89}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times s}$ $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times p}$. The signals $w(t_k)$ and $v(t_k)$ are random process noise and measurement noise, respectively, and are assumed to be independent, zero-mean, gaussian noise with covariance matrices

$$Q(t_k) = E(w(t_k)w(t_k)^T) \tag{2.90}$$
$$R(t_k) = E(v(t_k)v(t_k)^T). \tag{2.91}$$

Here, $E(\cdot)$ is the expectation value of $(\cdot)$ and $Q(t_k) \in \mathbb{R}^{s\times s}$, $R(t_k) \in \mathbb{R}^{p\times p}$. Then the Kalman filter is an observer that gives an optimal estimate $\hat{x}(t_k)$ of the state $x(t_k)$ at time step k, given the estimate $\hat{x}(t_{k-1})$ and the observation $y(t_k)$, in a least squares sense. In other words, if we define the estimation error as

$$e(t_k) = x(t_k) - \hat{x}(t_k), \tag{2.92}$$

then the Kalman filter is a *linear* filter that produces an estimate $\hat{x}(t_k)$ which minimizes $E(e(t_k)^T e(t_k))$ at each time step k. Now let

$$P(t_k) = E(e(t_k)e(t_k)^T) \tag{2.93}$$

denote the covariance matrix of $e(t_k)$ In the following, a derivation of the Kalman recursions is provided. The estimate $\hat{x}(t_k)$ should be a linear function of the previously gathered information, namely the sequence of observations

$$\{y(t_1), \ldots, y(t_{k-1})\}. \tag{2.94}$$

Furthermore, $\hat{x}(t_k)$ should minimize (2.93). Define the finite-dimensional inner product space H that consists of all linear combinations of the stochastic variables generated by (2.88) - (2.89) Define the inner product on H as

$$< \xi, \eta >= E(\xi\eta) \tag{2.95}$$

and the norm as

$$\|\xi\| =< \xi, \xi >^{1/2}. \tag{2.96}$$

Let $H_k(y)$ denote the space of all linear combinations of the sequence (2.94). Then it holds that

$$H_0(y) \subset H_1(y) \subset \ldots H_{k-1}(y) \subset H_k(y) \subset H \tag{2.97}$$

and $\hat{x}(t_k)$ is the element in $H_{k-1}(y)$ that minimizes $\|x(t_k) - \hat{x}(t_k)\|$. The existence of such a minimizer follows from the following theorem.

Theorem 2.5.2 ***Projection***. *For a subspace H_k of the finite-dimensional inner product space H, and $x \in H$, there exists a unique element $\hat{x} \in H_k$ such that $\|x - \hat{x}\|$ is minimized. Furthermore, $\hat{x}$ has the property that $(x - \hat{x}) \perp H_k$, i.e. with the inner product defined by* (2.95), $E((x - \hat{x})h_k) = 0 \quad \forall h_k \in H_k$.

Define the map $E^{H_k} : x \mapsto \hat{x}$, which is the orthogonal projection of x onto H_k. The estimate can now be written $\hat{x} = E^{H_k}x$. Some properties of E^{H_k} are stated in the following lemma.

Lemma 2.5.2 *The map E^{H_k} has the following properties:*

- *E^{H_k} is linear.*
- *For a matrix A such that the product Ax is defined, it holds that $E^{H_k}Ax = AE^{H_k}x$.*
- *For $H_k \perp H_j$ subspaces of H, it holds that $E^{H_k \oplus H_j} = E^{H_k} + E^{H_j}$.*

The following lemma follows from Theorem 2.5.2 and is useful for the derivation of the Kalman recursions.

Lemma 2.5.3 ***Linear Least Squares***. *Let x, y be random vectors and suppose the components of y are linearly independent. The linear least-squares estimate $\hat{x}$ of x given by y is*

$$\hat{x} = E^{H_k}x = E(xy^T)E(yy^T)^{-1}y. \tag{2.98}$$

Using the properties in Lemma 2.5.2, we can now write

$$\hat{x}(t_{k+1}) = E^{H_k(y)}x(t_{k+1}) = E^{H_k(y)}(Ax(t_k) + Bw(t_k)) = \{w(t_k) \perp H_k(y)\} = AE^{H_k(y)}x(t_k). \tag{2.99}$$

Now define the vector

$$\begin{aligned}\tilde{y}(t_k) = y(t_k) - E^{H_{k-1}(y)}y(t_k) = y(t_k) - CE^{H_{k-1}(y)}x(t_k) - E^{H_{k-1}(y)}Dv(t_k) = \\ \{v(t_k) \perp H_{k-1}(y)\} = y(t_k) - C\hat{x}(t_k) = C(x(t_k) - \hat{x}(t_k)) + Dv(t_k) = Ce(t_k) + Dv(t_k),\end{aligned} \tag{2.100}$$

and let $[\tilde{y}(t_k)]$ denote the space spanned by $\tilde{y}(t_k)$, so that

$$E^{H_k(y)}x(t_k) = E^{H_{k-1}(y)}x(t_k) + E^{[\tilde{y}(t_k)]}x(t_k) = \hat{x} + E^{[\tilde{y}(t_k)]}x(t_k). \tag{2.101}$$

From Lemma 2.5.3 it follows that

$$E^{[\tilde{y}(t_k)]}x(t_k) = E(x(t_k)\tilde{y}(t_k))^T(E(\tilde{y}(t_k)\tilde{y}(t_k)^T))^{-1}\tilde{y}(t_k). \tag{2.102}$$

From the definition of $\tilde{y}(t_k)$, $R(t_k)$ and $P(t_k)$, we get

$$E(\tilde{y}(t_k)\tilde{y}(t_k)^T) = CP(t_k)C^T + DR(t_k)D^T. \tag{2.103}$$

Now we can define the Kalman gain as

$$K(t_k) = AP(t_k)C^T(CP(t_k)C^T + DR(t_k)D^T)^{-1}, \tag{2.104}$$

so that (2.99) becomes

$$\hat{x}(t_{k+1}) = A\hat{x}(t_k) + K(t_k)\tilde{y} = A\hat{x}(t_k) + K(t_k)(y(t_k) - C\hat{x}(t_k)). \tag{2.105}$$

Now, since

$$e(t_{k+1}) = x(t_{k+1}) - \hat{x}(t_{k+1}) = (A - K(t_k)C)e(t_k) + K(t_k)Dv(t_k) + Bw(t_k), \tag{2.106}$$

we get

$$\begin{aligned}P(t_{k+1}) = E(e(t_{k+1})e(t_{k+1})^T) = \\ (A - K(t_k)C)P(t_k)(A - K(t_k)C)^T + K(t_k)DR(t_k)D^TK(t_k)^T + BQ(t_k)B^T.\end{aligned} \tag{2.107}$$

Summarizing, given initial estimates $\hat{x}_0$ and P_0, and introducing the intermediate variables $\hat{x}(t_k)^-$ and $P(t_k)^-$ for ease of notation, the discrete Kalman filter is the recursive process

$$\hat{x}(t_k)^- = A\hat{x}(t_{k-1}) \tag{2.108}$$

$$P(t_k)^- = AP(t_{k-1})A^T + BQ(t_{k-1})B^T \tag{2.109}$$

$$K(t_k) = P(t_k)^-C^T\left(CP(t_k)^-C^T + DR(t_k)D^T\right)^{-1} \tag{2.110}$$

$$\hat{x}(t_k) = \hat{x}(t_k)^- + K(t_k)\left(y(t_k) - C\hat{x}(t_k)^-\right) \tag{2.111}$$

$$P(t_k) = (I - K(t_k)C)P(t_k)^-. \tag{2.112}$$

Adaptive Kalman Filtering

$R(t_k)$ and $Q(t_k)$ play important roles in the recursions. Convergence to the optimal estimate $\hat{x}(t_k)$ requires accurate values of both matrices. Since these matrices represent uncertainties in the model and measurements, correct values may be hard to come by. A Kalman filter that tries to estimate these matrices and possibly adjust them online is called an adaptive Kalman filter.

The adaptive Kalman filtering schemes most frequently found in the literature are Innovation-based Adaptive Estimation (IAE) and Multiple Model Adaptive Estimation (MMAE). IAE methods estimate the covariance matrix of the process noise Q and/or the measurement noise R utilizing the fact that for the right values of Q and R the innovation sequence of the Kalman filter is white noise. By tuning Q and/or R and studying the resulting innovation sequence one can get an idea of the appropriate values of the covariance matrices. However convergence to the "right" values of Q and R is not guaranteed with IAE and most algorithms require estimation made over rather large windows of data to achieve reliable covariance measurements, making the method impractical for rapidly changing systems.

MMAE methods handle model uncertainty by implementing a bank of several different models and computing the bayesian probability for each model to be the true system model given the measurement sequence and under the assumption that one of the models in the model bank is the correct one. The state estimate can be either the output of the most probable model or a weighted sum of the outputs of all models. This method is suitable for applications such as fault detection, where you have some a priori information on the system dynamics. For instance, if the dynamics of an engine is well known, each model in the bank can represent the engine dynamics if one or several components fail. With this information, if the probability of one of the failure models gets to high an alarm is raised.

Adaptive Kalman filtering is studied in Paper B.

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