Modeling and Model Reduction
by Analytic Interpolation and Optimization

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Invention, it must be humbly admitted, does not consist of creating out of void, but out of chaos.

Mary Wollstonecraft Shelley, 1797-1851.
A nonna Nina e nonno Vitangelo
Abstract

This thesis consists of six papers. The main topic of all these papers is modeling a class of linear time-invariant systems. The system class is parameterized in the context of interpolation theory with a degree constraint. In the papers included in the thesis, this parameterization is the key tool for the design of dynamical system models in fields such as spectral estimation and model reduction.

A problem in spectral estimation amounts to estimating a spectral density function that captures characteristics of the stochastic process, such as covariance, cepstrum, Markov parameters and the frequency response of the process. A model reduction problem consists in finding a small order system which replaces the original one so that the behavior of both systems is similar in an appropriately defined sense.

In Paper A a new spectral estimation technique based on the rational covariance extension theory is proposed. The novelty of this approach is in the design of a spectral density that optimally matches covariances and approximates the frequency response of a given process simultaneously.

In Paper B a model reduction problem is considered. In the literature there are several methods to perform model reduction. Our attention is focused on methods which preserve, in the model reduction phase, the stability and the positive real properties of the original system. A reduced-order model is computed employing the analytic interpolation theory with a degree constraint. We observe that in this theory there is a freedom in the placement of the spectral zeros and interpolation points. This freedom can be utilized for the computation of a rational positive real function of low degree which approximates the best a given system.

A problem left open in Paper B is how to select spectral zeros and interpolation points in a systematic way in order to obtain the best approximation of a given system. This problem is the main topic in Paper C. Here, the problem is investigated in the analytic interpolation context and spectral zeros and interpolation points are obtained as solution of an optimization problem.

In Paper D, the problem of modeling a floating body by a positive real function is investigated. The main focus is on modeling the radiation forces and moment. The radiation forces are described as the forces that make a floating body oscillate in calm water. These forces are passive and usually they are modeled with system of high degree. Thus, for efficient computer simulation it is necessary to obtain a low order system which approximates the original one. In this paper, the procedure developed in Paper C is employed. Thus, this paper demonstrates the usefulness of the methodology described in Paper C for a real world application.

In Paper E, an algorithm to compute the steady-state solution of a discrete-type Riccati equation, the Covariance Extension Equation, is considered. The algorithm is based on a homotopy continuation method with predictor-corrector steps. Although this approach does not seem to offer particular advantage to previous solvers, it provides insights into issues such as positive degree and model reduction, since the rank of the solution of the covariance extension problem coincides with the degree of the shaping filter.
In Paper F a new algorithm for the computation of the analytic interpolant of a bounded degree is proposed. It applies to the class of non-strictly positive real interpolants and it is capable of treating the case with boundary spectral zeros. Thus, in Paper F, we deal with a class of interpolation problems which could not be treated by the optimization-based algorithm proposed by Byrnes, Georgiou and Lindquist. The new procedure computes interpolants by solving a system of nonlinear equations. The solution of the system of nonlinear equations is obtained by a homotopy continuation method.

**Key Words.** Analytic Interpolation theory with a degree constraint, rational covariance extension problem, spectral estimation, model reduction, optimization, passive system.
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Introduction

The topic of this thesis is modeling a certain class of linear time-invariant systems. The system class is parameterized in the context of interpolation theory with a degree constraint. This parameterization is our key tool to design models of dynamical systems in several fields, such as spectral estimation and model reduction. Our major tool for parameter estimation of the models is optimization theory.

The purpose of this introduction is to motivate the problems considered in the papers included in the thesis. In the first part of the introduction, we formulate two related interpolation problems with a degree constraint which play a key role in this thesis: the analytic interpolation problem with a degree constraint and the covariance extension problem with a degree constraint. The second part is devoted to the description of application areas where the analytic interpolation theory and the covariance extension theory play a major role. In this thesis, two areas of applications are considered: spectral estimation and model reduction. First, we describe problems which arise in spectral estimation. Then, we review the basic motivation which lies behind the model reduction theory and present standard techniques to approximate a dynamical system with a simplified model which captures the main features of the original complex system.

Finally, we present short summaries of the six papers that follow and list some related open questions for future research directions.

1 Interpolation Problems with a Degree Constraint

The importance of the interpolation problems comes not only from the richness of mathematics in it, but also from its applicability in engineering areas. For example, many problems in circuit theory, signal processing, identification, robust control and optimal control can be formulated as interpolation problems. These applications normally require models with a low complexity which in general amounts to the interpolants being of low degree. However, the classical interpolation theory [45, 47, 50] can not deal with this problem. Thus, the classical interpolation problem has been modified in order to include constraints on the degree of the interpolant. This new approach has been a topic of studies of many researchers. Fundamental results have been developed by Byrnes, Georgiou, Lindquist and coauthors in the last decades [6, 13–16, 23, 24, 26, 27]. Next, we revisit these results which represent the mathematical background of this thesis.
1.1 Analytic Interpolation with a Degree Constraint

We begin by formulating the analytic interpolation problem with a degree constraint.

Problem 1.1. Given a set of self-conjugate pairs of complex numbers

\[ \{ (z_j, w_j) : z_j \in \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \}^n_{j=0}, \]

\[ z_j \neq z_k, \text{ if } j \neq k, z_0 = 0, \]

\[ w_j = w_k, \text{ if } z_j = \bar{z}_k, \]

(1.1)

determine any function \( f \) which satisfies the following conditions:

**C1** \( f \) is positive real, i.e. \( f \) is analytic and

\[ \text{Re} f(z) \geq 0, \forall z \in \mathbb{D}; \]

**C2** \( f \) satisfies the interpolation conditions

\[ f(z_j) = w_j, \ j = 0, 1, \ldots, n; \]

**C3** \( f \) is real rational of degree at most \( n \).

Note that the classical Nevanlinna-Pick problem requires only conditions **C1** and **C2** and it has a solution if and only if the **Pick matrix**

\[ P := \begin{bmatrix} w_i + \bar{w}_j \n 1 - z_i \bar{z}_j \end{bmatrix}_{i,j=0} \]

(1.2)

is positive semidefinite, see e.g [57]. Moreover the solution set is a singleton if the Pick matrix is singular.

The functions satisfying **C1** are known as **Carathéodory function** in the mathematical literature and as **positive real function** in the circuits and systems theory literature. They are important in the description of the impedance of the RLC circuits, in the study of the stability of linear and nonlinear system and in the characterization of the positivity of probability measures in stochastic systems theory. Thus, positive real interpolants play a major role in applications in circuitry theory, robust stabilization and control, speech synthesis, signal processing and stochastic systems theory. In all these applications it is required interpolants to have low complexity, so that, to be rational and to have a bound on its degree, since the complexity of interpolants directly affects that of the dynamical systems such as filters and controllers. For this reason, degree constraints present new challenges and they need to be included in a systematic way in the classical interpolation theory. The Nevanlinna-Schur recursion algorithm and the linear fractional parameterization of all solutions [57] can be employed for the computation of rational interpolants. However, this does not give any insight on how to parameterize all the rational interpolants with a predefined degree bound. Thus, in order to overcome this drawback, Byrnes, Georgiou, Lindquist and coauthors have proved in various places a theorem which parameterized smoothly the set of rational interpolants with a degree constraint.
**Theorem 1.1.** Assume that $P$ is positive definite. For any element $\sigma$ of the Schur stability region

$$
S := \left\{ \sigma(z) = \sum_{j=0}^{n} \sigma_j z^j, \sigma(0) > 0, \sigma(z) \neq 0, \forall |z| \geq 1 \right\}
$$

(1.3)

there exists a unique (module sign) pair of polynomials $(\alpha, \beta) \in S \times S$ of degree at most $n$ such that $f(z) = \beta(z)/\alpha(z)$ is positive real, $f$ satisfies the interpolation conditions $C2$ and, moreover

$$
f(z) + f(z^{-1}) = \frac{\sigma(z)\sigma(z^{-1})}{\alpha(z)\alpha(z^{-1})}.
$$

In addition, the map sending $\sigma$ to $(\alpha, \beta)$ is a diffeomorphism.

A constructive proof of this theorem was given in [9, 10, 13] where the analytic interpolation problem is studied from differential geometry and optimization viewpoints. The main result in [10, 13] is that for a given Schur polynomial $\sigma$, a rational positive real interpolant of degree at most $n$ can be found as a solution of a convex optimization problem. The objective function of the optimization problem takes the form:

$$
\|\Phi(\theta)\rangle = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{i\theta}) \log[\Phi(e^{i\theta})] d\theta
$$

(1.4)

with $\Phi(e^{i\theta}) = f(e^{i\theta}) + f(e^{-i\theta})$ and $i$ is the unit on the imaginary axis, and

$$
\Psi(e^{i\theta}) := \left| \frac{\sigma(e^{i\theta})}{\tau(e^{i\theta})} \right|^2, \text{ where } \tau(z) := \prod_{j=1}^{n} (1 - z_j z).
$$

(1.5)

Theorem 1.1 completely characterizes all the strictly positive real interpolants of a degree at most $n$ by means of a strictly positive pseudo-polynomial $|\sigma(z)|^2$ on the circle. In [26], Georgiou extended this result to the case of nonnegative pseudo-polynomial on the circle and proved the bijectivity between the class of nonnegative pseudo-polynomials and the class of nonnegative real interpolants. In addition, in Paper F we prove that such map is a homeomorphism. This property together with the result of Theorem 1.1 turns out to be vital in justifying the numerical algorithm proposed in Paper F for computing the interpolant. This algorithm finds the interpolant by solving a system of nonlinear equations which is a new approach and differs from the previous one based on convex optimization [43].

Problem 1.1 can be formulated in a more general way regarding the situation where the interpolation points can have multiplicity higher than 1. In this case, the interpolation condition $C2$ is modified and includes derivative interpolation conditions:

$$
f^{k-1}(z_j) = w_j \text{ with } z_j = z_1, j = 1, \ldots, k.
$$
In particular, if all the interpolation points coincide and they are all placed at 0, then the problem is known as the covariance extension problem with degree constraint, first formulated by Kalman in [32]. In the next section, the formulation of this problem and the main result regarding its solution is presented.

1.2 Rational Covariance Extension with a Degree Constraint

The Rational Covariance Extension problem is the mathematical basis for many engineering problems. For example, the design of the Linear Predictive Coding (LPC) filter used for speech synthesis by most existing cellular telephones involves the rational covariance extension problem [12]. Namely, the filter is modeled as a positive real function which matches a given window of covariances.

In signal and speech processing a signal \( \{y(t) : t \in \mathbb{Z}\} \) is often modeled as the output of a filter obtained by passing white noise\(^1\) \( \{u(t) : t \in \mathbb{Z}\} \) through it, as represented in Fig. 1.

\[
\begin{array}{cc}
\text{u(t)} & \omega(z) & \text{y(t)} \\
\end{array}
\]

Figure 1: Black Box Model

Here \( y \) is a stationary stochastic process with spectral density

\[
\Phi(z) = \omega(z)\omega(z^{-1}),
\]

where \( \omega(z) \) is the transfer function of the filter. The Fourier coefficients

\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\theta, \quad k = 0, 1, 2, \cdots
\]  

(1.6)

are the covariance lags \( c_k = E(y(t+k)y(t)) \). Hereafter, \( E(\cdot) \) denotes the expectation operator which averages over the whole realizations.

In this setting, the rational covariance extension problem can be formulated as follows:

**Problem 1.2.** Given a covariance sequence \( (c_0, c_1, \cdots, c_n) \) such that

\[
R = \begin{bmatrix}
c_0 & c_1 & \cdots & c_n \\
c_1 & c_0 & \cdots & c_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_n & c_{n-1} & \cdots & c_0
\end{bmatrix}
\]

\(^1\)A signal \( u \) is a white noise if \( E(u(t)u(s)) = \delta_{ts} \), where \( \delta_{ts} = 1 \) if \( t = s \) and zero otherwise.
is positive definite, seek a spectral density

\[ \Phi(z) = \tilde{c}_0 + \sum_{k=1}^{\infty} \tilde{c}_k (z^k + z^{-k}) \]

which satisfies the following conditions

- **R1** \( \Phi \) is positive
  \[ \Phi(z) > 0, \ \forall z \in \mathbb{D}; \]

- **R2** \( \Phi \) satisfies the interpolation conditions
  \[ \tilde{c}_k = c_k, \forall k = 0, 1, \ldots, n; \]

- **R3** \( \Phi \) is rational and can be factorized as
  \[ \Phi(z) = \omega(z)\omega(z^{-1}) \]
  with \( \omega \) rational of degree at most \( n \).

The solution set of this problem is given in the following complete parameterization theorem.

**Theorem 1.2.** Given a covariance sequence \((c_0, c_1, \ldots, c_n)\) so that \( R > 0 \). For any \( \sigma \) in the Schur stability region

\[ S := \left\{ \sigma(z) = \sum_{j=0}^{n} \sigma_j z^j, \sigma(0) > 0, \sigma(z) \neq 0, \ \forall |z| \geq 1 \right\} \]

there exists a unique polynomial \( \alpha \in S \) such that the rational spectral density \( \Phi \) of order at most \( n \) satisfying the conditions **R1** and **R2** can be written as

\[ \Phi = \frac{\sigma(z)\sigma(z^{-1})}{\alpha(z)\alpha(z^{-1})}. \]

In addition, the map sending \( \sigma \) to \( \alpha \) is a diffeomorphism.

The problem of parameterizing the class of all rational spectral densities of degree at most \( n \) by means of Schur polynomials was first formulated by Georgiou in [23, 24]. Georgiou provided a nonconstructive proof based on topological degree theory. He also conjectured that the map sending \( \sigma \) to \( \alpha \) is injective. This conjecture was proved to be true by Byrnes, Lindquist, Gusev and Matveev in [6] using basic results from complex analysis, geometry, system theory and nonlinear dynamics. A constructive proof was given by Byrnes, Gusev and Lindquist in [8] where for a given Schur polynomial \( \sigma \), a rational spectral density which matches a given covariance sequence can be found as the solution of an optimization problem, which is the dual
of the problem corresponding to (1.4). The objective function of this optimization problem takes the form as
\[ J_c(\alpha) = \alpha^T R\alpha - \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma(e^{i\theta})|^2 \log |\alpha(e^{i\theta})|^2 d\theta. \]

The algorithm proposed in [8] for solving the rational covariance extension problem often does not give a solution with acceptable numerical accuracy. This is caused by ill-conditioning in the spectral factorization and the system of linear equations that are involved in the algorithm. In [20] a robust algorithm for solving this optimization problem is proposed.

An alternative approach for the computation of the solution of the rational covariance extension problem is described in [7] where such a solution is obtained by solving a nonstandard Riccati equation, the covariance extension equation. This equation is formulated in terms of the partial covariance data and the choice of zeros of the shaping filter \( \omega \).

In Paper E, we provide an algorithm based on a homotopy continuation method for solving numerically the covariance extension equation so as to find the solution of the rational covariance extension problem. Although this last approach does not offer any computational advantages, in Paper E we point out that it gives some additional insights into other issues such as positive degree and model reduction, which have not been treated in the optimization approach [20].

1.3 Change of domain

In the previous sections both the analytic interpolation problem and the rational covariance extension problem are assumed to find a map from the unit disc into the right-half plane. Problems originating from applications may sometimes have different domain and/or range. However, these assumptions are without loss of generality, since the linear fractional transformation \( s = (az+b)/(cz+d) \), \( ad-bc \neq 0 \) can be applied to both variable and the function, and both domain and range can be transformed to the desired form. Thus, problem formulations and results can be translated easily into different settings.

Example 1.1. The analytic interpolation problem with a degree constraint reviewed in Section 1.1 can be formulated in the case the interpolant \( f \) has the right-half of the complex plane as domain of analyticity. In this new setting the interpolation data are written as:
\[ \{(s_j, w_j) : s_j \in \mathbb{C}_+, \ s_i \neq s_j \text{ if } i \neq j, \ s_0 \text{ real}, \ w_i = \bar{w}_j \text{ if } s_i = s_j, \} \]

Since the formulation of the analytic interpolation problem with degree constraint in discrete time is normalized so that \( z_0 = 0 \), the results obtained in this setting can be transferred to the continuous time one via the bilinear transformation:
\[ s \in \mathbb{C}_+ \mapsto z = \frac{s_0 - s}{s_0 + s} \in \mathbb{D} \]
and its inverse
\[ z \in \mathbb{D} \mapsto s = s_0 \frac{1 - z}{1 + z} \in \mathbb{C}^+. \tag{1.10} \]

In particular, the unit circle \( \{ z = e^{i\theta} \mid \theta \in [-\pi, \pi] \} \) is mapped to the imaginary axis \( \{ s = i\omega \mid \omega \in (-\infty, \infty) \} \).

In Paper B, by employing the above transformation, we rephrase the analytic interpolation problem with degree constraint and Theorem 1.1 in the continuous time setting. The new results are applied to applications in model reduction in Paper B, Paper C and Paper D.

2 Applications

In this thesis, the analytic interpolation theory and the rational covariance extension theory is illustrated by examples in two categories of applications: spectral estimation and model reduction.

2.1 Spectral Estimation Problem

The spectral estimation problem is to estimate a spectral density function that captures characteristics of the stochastic process, such as covariance, cepstrums, Markov parameters, and the frequency response of the process. The main limitation on the quality of such an estimate is due to the small number of available sampled data.

There are two main approaches to estimate a spectral density function: the non-parametric and the parametric. The former estimates the spectral density function as the discrete time Fourier transform of the covariance sequence (1.6):

\[ \Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}. \]

In the latter approach, first an assumption of a given process which leads to a parametric form of the spectral density is made. Then, an estimation of the parameters in the model is done.

In this thesis, the parametric approach is considered. The spectral density function is modeled as a rational function:

\[ \Phi(z) = \left| \frac{\sigma(z)}{\alpha(z)} \right|^2, \tag{2.11} \]

with \( \sigma \) and \( \alpha \) polynomials with degree chosen by the designer. The spectral density (2.11) is associated with a model of a signal called autoregressive moving-average (ARMA) which has been obtained by filtering white noise through a filter with a rational transfer function (see Fig. 1):

\[ \omega(z) = \frac{\sigma(z)}{\alpha(z)}. \]
Thus, our main problem is to estimate the coefficients of the polynomials $\sigma$ and $\alpha$.

An obvious question on how to choose the best method for estimation arises. The meaning of best has to be clarified in connection with applications. Criteria for the choice of the best method can be statistical, predictive ability, computation time, reliability and smoothness of the estimated model with respect to given data.

A large number of estimation methods can be found in the literature. In statistics, a very common method is the Maximum Likelihood (ML) (see [5]) which maximizes the likelihood that the model explains the data. Instead, in system identification, the method mostly adopted is the Predictor Error Method (PEM) [40] which minimizes the prediction error. These methods are equivalent when the driving white noise has a Gaussian distribution.

In the next example, we apply PEM to design a model of a rational spectral density which identifies given data.

**Example 2.2.** (ARMA estimation) We consider real experimental speech data $\{y_k\}_{k=0}^{256}$ taken from [11, Example 3.4]. These data are acquired during the formation of the voiced nasal [ng]. The phonemes have been sampled at a rate of 8000 samples per second and retained 250 sample points for each frame. Thus, each frame represents a time history of speech over a time horizon of 30ms.

We would like to shape the spectral density of this process by a rational spectral density function $\Phi$ written as in (2.11) with $\sigma$ and $\alpha$ polynomials of degree 6. This means that we assume that the process is an ARMA(6,6) signal, namely in time domain

$$\alpha(q)y(t) = \sigma(q)e(t)$$

with $e(t)$ white noise. To estimate $\sigma$ and $\alpha$, ARMAX in MATLAB has been used. The command ARMAX implements PEM, see [41] and outputs the following estimates for the coefficients of $\sigma$ and $\alpha$:

$$\begin{align*}
\sigma &= [1.0000, -0.1167, 0.8931, 0.1345, -0.4632, 0.2329, -0.6466] \\
\alpha &= [1.0000, -1.8142, 1.8321, -1.1497, 0.2104, 0.0939, -0.0609]
\end{align*}$$

In Fig. 2, the spectral density of the ARMA(6,6) model which shapes the data is plotted together with the spectrum of the data.

This example shows that for this particular application PEM implemented in MATLAB can be considered a fairly good estimator. However, the estimation of the spectral density of a process can be made through alternative methods that catch other properties than the frequency response as PEM does.

In [52], the authors propose a method to estimate a spectral density which simultaneously matches covariance and Markov parameter. In [11], the Cepstral Covariance Matching estimator was proposed; it computes a spectral density which matches covariance and cepstrum. In [10], the estimation of the frequency response of the spectral density is done using a filter bank and spectral zeros.

In all of these works, the question of designing a spectral density which optimally approximates covariances and the frequency response of a process simultaneously
has not been investigated. In Paper A an attempt to answer this question is presented, based on the rational covariance extension theory with a degree constraint. The basic idea of the methodology shown in Paper A is the same as the one developed for sensitivity shaping in [44]. However, in [44] the situation of uncertainty on the interpolation conditions was not dealt with. In Paper A the uncertainty on the interpolation conditions means uncertainty on the computation of the covariance.

In spectral estimation, the covariances are estimated from a finite record of observed data which leads to statistical errors. The mathematical interpretation of it is that the covariance lie in an uncertainty region. In [51], the uncertainty region is modeled as a polyhedral set \([c_k^- , c_k^+]\), \(k = 0, 1, \cdots , n\) in which the covariance are constrained to be.

Figure 2: Speech data and Arma(6,6)
In Paper A, the set \([c^-, c^+]\) has two interpretations: the set of an "admissible" error on \(c_k\) to improve the approximation of data and the set of uncertainty on \(c_k\). The latter interpretation is typical when experimental data are used to estimate covariances. In view of this, the new approach of Paper A can be employed for the estimation of the spectral density of \(\{y_k\}_{k=0}^{250}\) in Example 2.2. This example is revisited in Paper A and the spectral density of the speech data is approximated by a spectral density function robustly, i.e. for any covariance sequence in the uncertainty region.

### 2.2 Model Reduction

In recent years, many physical models have become more complex due to either increased system size or an increased desire for details. Thus, there has grown a need to construct simplified models which approximate the original complex one. The design of a simplified model requires two steps: the *modeling phase* and the *model reduction phase*. In the modeling phase, we try to derive a set of first order differential equations which describes the physical system under investigation. The model reduction step consists in replacing the original set of equations with a much smaller one so that the behavior of both systems is similar in an appropriately defined sense. Such situations often arise when a physical system needs to be simulated or controlled.

In this thesis, the *modeling phase* consists of determining a single input-single output (SISO) linear dynamical system represented as

\[
\Sigma := \begin{cases} 
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) 
\end{cases}
\]  

(2.12)

where \(x(t) \in \mathbb{R}^n\), \(u(t) \in \mathbb{R}\) and \(y(t) \in \mathbb{R}\), and the matrices \(A\), \(B\), \(C\) and \(D\) have compatible dimensions. The system \(\Sigma\) has the properties:

- **P1** the matrix \(A\) has all eigenvalues which lie in the open left half of the complex plane,
- **P2** \((A, B)\) is reachable,
- **P3** \((C, A)\) is observable,
- **P4** \(D + DT > 0\).

In the model reduction step, we seek a reduced order system \(\hat{\Sigma}\)

\[
\hat{\Sigma} := \begin{cases} 
\hat{x}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t) \\
\hat{y}(t) &= \hat{C}\hat{x}(t) + Du(t) 
\end{cases}
\]

(2.13)

where \(\hat{x}(t) \in \mathbb{R}^k\), \(k < n\) and \(\hat{y}(t) \in \mathbb{R}\) so that the following properties are satisfied:

1. the approximation error, in some suitable norms, \(||y - \hat{y}||\) is small.
(2) system properties (stability and passivity) are preserved,
(3) the procedure is computational efficient and numerically stable.

Several methods to reduce the order of a system have been studied in the literature. A method is more preferable than another if it performs not only a good approximation of the original system but also maintains crucial properties of the original physical system in the reduction phase.

In this thesis, we consider systems which are stable and passive. A classical example of passive and stable system is represented by the RLC circuit, i.e. a circuit consisting only of resistors, inductors and capacitors.

A system is stable if the eigenvalues \( \{ \lambda_j \}_{j=1}^n \) of the matrix \( A \) in (2.12) are in \( \mathbb{C}_- \), i.e. \( \text{Re}(\lambda_j) < 0, \ j = 0, 1, \cdots, n \). 

A system is passive if it does not generate energy internally. Mathematically, a system \( \Sigma \) is passive if
\[
\int_0^T u(t)^T y(t) dt \geq 0
\]
for all \( T > 0 \) and all square-integrable inputs \( u \). The passivity property of a system can be expressed as a property of its transfer function. The system \( \Sigma \) is passive if and only if its transfer function
\[
G(s) = C(sI - A)^{-1} B + D
\]
is positive real, i.e.
\[
G(i\omega) + G(-i\omega) \geq 0, \ \omega \in \mathbb{R}.
\]

A well-known model reduction procedure which preserves the stability and passivity properties is the stochastically balanced truncation. It was originally proposed by Desai and Pal in [18,19] and in the context of stochastic realization theory by Lindquist and Picci in [38]. This method transforms the system into a basis where the states are equally difficult to reach and observe. The reduced model is obtained simply by truncating the states which are the most difficult to reach and observe.

The stochastically balanced truncation belongs to the class of approximation methods which are related to the singular value decomposition. This family of methods preserves important properties of the original system but it can only be applied to relatively low dimensional systems (a few hundred states).

Up to now, many researchers have been investigating other methods which can perform model reduction for high order systems. In this respect, there are interesting works that try to establish a connection between iterative Krylov methods in numerical analysis [29,49] and model reduction. Some examples are: the implicitly restarting algorithm (IRA) [53] which has been applied to obtain a stable reduced order model [30]; the approximate balancing method introduced in [1] which iteratively computes a \( k \)-th order balanced system without computing the full order balanced model. However, none of these methods can maintain the passivity property in the reduction phase.
A few years ago, Antoulas and Sorensen came out with a new procedure based on Krylov methods which keeps the stability and passivity properties in the model reduction step [2, 54]: the passivity preserving model reduction by Krylov method. This method belongs to the family of rational Krylov methods which achieve model reduction by moment matching (see Section 2.4 for a detailed overview).

Despite the different approach and performance of the stochastically balanced truncation and the passivity preserving model reduction by Krylov method, both perform model reduction by projection. This means that the reduction of the order of a system is achieved by constructing two matrices \( V, U \in \mathbb{R}^{n \times k} \) such that \( U^T V = I_k, I_k \) the identity matrix in \( \mathbb{R}^{k \times k} \) and defines the reduced order system \( \hat{\Sigma} \) by:

\[
\hat{A} = U^T A V, \quad \hat{B} = U^T B, \quad \hat{C} = CV. \quad (2.16)
\]

In the next two sections, an overview of the two methods mentioned above is given.

### 2.3 Stochastically Balanced Truncation

Let us consider the system (2.12) with transfer function

\[
G(s) = C(sI - A)^{-1}B + D. \quad (2.17)
\]

\( G(s) \) is positive-real if and only if it satisfies the positive real lemma. Thus, there exist matrices \( P = P^T > 0 \) and \( Q = Q^T > 0 \) which satisfy two Riccati equations:

\[
A^T P + PA + (C^T - PB)(D + D^T)^{-1}(C - B^T P) = 0
\]

\[
AQ + QA^T + (B - QC^T)(D + D^T)^{-1}(B^T - CQ) = 0. \quad (2.19)
\]

The essential idea of stochastic balancing is to find a state transformation so that the solutions of the two Riccati equations are equal and diagonal. Thus, we seek a transformation \( T \) of \( G(s) \) so that

\[
P = Q = S = \text{diag}(s_1, \cdots, s_n), \quad s_i \geq s_{i+1}
\]

The algorithm to compute the transformation matrix \( T \) has been presented in various places in the literature [28, 38]:

1. Compute a Cholesky factorization of \( Q = RR^T \);
2. Do the Singular Value Decomposition of \( RT PR \), i.e. compute the factorization \( RT PR = US^2U^T \), with \( U^T U = I \);
3. Define \( T := S^{-1/2}U^T R^T \).

The transformation \( T \) will balance the system \( \Sigma \), i.e.

\[
TPT^T = S = T^{-T}Q^{-1}T^{-1},
\]
and $S$ is a diagonal matrix with the singular values of $PQ^{-1}$ on its diagonal. The transfer function of the balanced system is

$$G(s) = D + \tilde{C}(sI - \tilde{A})^{-1}\tilde{B},$$

(2.20)

where

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = CT^{-1}.$$  

Since $S$ is diagonal it can be partitioned as follows

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \quad S_1 = \text{diag}(s_1, \ldots, s_k), \quad S_2 = \text{diag}(s_{k+1}, s_{k+2}, \ldots, s_n).$$  

(2.21)

Then, we can partition $A, B, C, D$ conformally with $S$:

$$G(s) = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}$$

and the truncated system

$$G_T(s) = D + C_1(sI_k - A_{11})^{-1}B_1$$

(2.22)

is of degree $k << n$ and keeps the stability and positive realness property of the full-order system.

The advantage of the stochastically balanced truncation is that not only does it preserve important properties of the system but also provides an explicit bound for the approximation error [28].

Although the stochastically balanced truncation keeps the stability and the passivity of the full-order system in the model reduction phase and it comes with a bound, for large-scale setting it is expensive to implement because it requires dense matrix factorizations. It results in a computational complexity of $O(n^3)$ and storage requirement of $O(n^2)$.

Therefore, since the order of many physical models has increased, there has been a growing research field on the investigation of new numerical efficient methodologies for model reduction. In the next section, we briefly describe one of these methods, namely the rational Krylov projection method on which is based the passivity preserving model reduction by Krylov method in [2,54].

### 2.4 Krylov projection method

In the recent years, there has been a vast number of studies to connect two areas of research: model reduction and numerical linear algebra. Its motivation comes from the need to solve numerical problems which arise using the standard methods in model reduction, namely methods based on the singular value decomposition.
of which the stochastically balanced truncation is one. On the other side, in numerical linear algebra there has been a proliferation of iterative algorithms, and in particular, Krylov iterative algorithms \cite{21,48,49}. These iterative methods lead to approximate solutions with low computational effort.

The key part of Krylov algorithms is the computation of certain subspaces which are known as the Krylov subspace in the linear algebra community and as the reachability or controllability subspaces in the control system community. In linear algebra these algorithms have been used in different applications:

- To find the iterative solution of $Ax = b$.
- To compute iteratively the eigenvalues of a matrix $A$.
- To approximate a linear system by another system which satisfies certain interpolation conditions.

In this section, we focus our attention on the use of the last issue in the list, since it is closely related to the main problem in model reduction.

The first attempt to explore the extension and development of the iterative Krylov algorithms for the analysis and approximation of a dynamical system is done in \cite{31}. Grimm showed in his thesis \cite{31} that the Krylov projection method achieved model reduction via rational interpolation. In the Krylov projection method, the reduced order system is obtained by constructing in an appropriate way the projection matrices $U$ and $V$ in (2.16). Depending on how these matrices are obtained, the reduced order system satisfies certain interpolation conditions.

**Proposition 2.1.** Suppose that the projection matrices $U$ and $V$ are constructed as follows:

$$V = [B, AB, \ldots, A^{k-1}B],$$

and $U$ is any left inverse matrix\(^2\) of $V$. Then, the reduced order system $\hat{\Sigma} = (U^TAV, U^TB, CV, D)$ matches the first $k$ Markov parameters of the full order system $\Sigma = (A, B, C, D)$.

**Proposition 2.2.** Suppose that a point $s_0 \in \mathbb{C}$ is given and that the projection matrices $U$ and $V$ are constructed as follows:

$$V = [(s_0I_n - A)^{-1}B, \ldots, (s_0I_n - A)^{-1}I]$$

and $U$ is any left inverse matrix of $V$. Then, the reduced order system $\hat{\Sigma} = (U^TAV, U^TB, CV, D)$ interpolates the full order system $\Sigma = (A, B, C, D)$ at $s_0$ together with $k - 1$ derivatives at the same point

**Proposition 2.3.** Suppose that $k$ distinct points $\{s_j\}_{j=1}^k$ are given in the complex plane and that the projection matrices $U$ and $V$ are constructed as follows:

$$V = [(s_1I_n - A)^{-1}B, \ldots, (s_kI_n - A)^{-1}B]$$

\(^2\)A matrix $U$ is a left inverse of a matrix $V$ if $U^TV = I$, with $I$ the identity matrix.
and $U$ is any left inverse matrix of $V$. Then, the reduced order system $\tilde{\Sigma} = (UT AV, U^T B, CV, D)$ interpolates the full order system $\Sigma = (A, B, C, D)$ at points \( \{s_j\}_{j=1}^k \).

**Remark 2.1.** In the numerical linear algebra community, the projection matrices $U$ and $V$ coincide with the Krylov subspaces and can be computed by iterative algorithms which are numerically very efficient. Moreover, the above propositions highlight an important propriety of the Krylov method for model reduction: the interpolation conditions are satisfied without computation of the interpolation values which can be numerically very expensive and not accurate for very large order systems.

In the rational Krylov methods, the location of the interpolation points and the interpolation conditions satisfied are the central factor in determining the accuracy and the dimension of the reduced order model. This well-known result is shown through the following example.

**Example 2.3.** We consider the minimal realization of a system $\Sigma = (A, B, C, D)$ of degree 5:

\[
A = \begin{bmatrix}
-2 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & -5
\end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = 1.
\]

The system is approximated by a system of degree 3 obtained by the projection method described in Proposition 2.3. Two sets of three interpolation points have been chosen:

\[
s_1^1 = 0.5, \quad s_2^1 = 4, \quad s_3^1 = 20
\]

\[
s_1^2 = 0.5, \quad s_2^2 = 0.04 + 0.8i, \quad s_3^2 = 0.04 - 0.8i.
\]

In Fig. 3, the original model together with the two reduced order systems is plotted. The reduced order system corresponding to the second set of interpolation points approximates better the original one around the frequency 1 rad/sec since we have imposed that the reduced order system has to interpolate the original one at the points $(s_2^2, s_3^2)$.

Example 2.3 shows how important is the appropriate placement of the interpolation points for approximation purposes in the rational Krylov method for model reduction. In [2, 54] the authors proved that placing the interpolation points in the mirror images of stable zeros of $G + G^*$ with $G$ transfer function of a given system, it is possible to achieve not only a good approximation of the full order system

\[\text{For a complex-valued matrix function } G, \quad G^* (s) := G(-\bar{s})^T, \quad \forall s \in \mathbb{C}.\]
Figure 3: Full order model and reduced order model by two different set of interpolation points

but also to maintain critical properties of the original model: the stability and the passivity.

**Proposition 2.4.** Suppose that a system $\Sigma = (A, B, C, D)$ of degree $n$ is given. Let us consider $\{s_j\}_{j=1}^k$ points in the complex plane which are $k$ generalized eigenvalues of the matrices

$$A := \begin{bmatrix} A & -A^T & B \\ -C & B^T & C \\ C & B^T & D + D^T \end{bmatrix}, \mathcal{E} := \begin{bmatrix} I_n & I_n \\ \end{bmatrix}.$$
Then a reduced order system $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, D)$ which interpolates $\Sigma$ at $\infty$, at $\{s_j\}_{j=1}^{k}$ and at $\{\bar{s}_j\}_{j=1}^{k}$ is passive, stable and of degree $k$.

**Remark 2.2.** In the control system theory community, the generalized eigenvalues in Proposition 2.4 are known as the spectral zeros of the system $\Sigma$, i.e. the zeros of $G + G^*$ with $G$ transfer function of $\Sigma$.

In [54], Sorensen developed an algorithm to obtain the system $\hat{\Sigma}$ by a projection method without computation of all generalized eigenvalues of $\Sigma$. The projection matrices are constructed by solving a $k$-th order partial real Schur decomposition:

$$AQ = EQR.$$ 

For very large order system the partial real Schur decomposition is efficiently computed by using the command `eigs` in MATLAB which implements the implicitly restarted Arnoldi (IRA) method.

Despite the numerically efficiency of the algorithm in [54], in Paper B we observe that the system $\hat{\Sigma}$ output from such algorithm can be regarded as a special solution in the solutions set of the analytic interpolation problem with a degree constraint. As a matter of fact, the methodology in [2,54] for the computation of a low order passive system only works if the interpolation points are chosen at the mirror points of some spectral zeros of the original system. In addition, the spectral zeros of the reduced order system has to be a subset of the spectral zeros of the original one. However, these are constraints that are not considered in the analytic interpolation theory for the computation of a positive real function.

The freedom in the analytic interpolation theory for the placement of spectral zeros and interpolation points can be regarded as an important tool to design a reduced order system. This idea has been further developed in [33] where spectral zeros and interpolation points are interpreted as tuning parameters to design a bounded analytic function of low degree which shapes a given one. In Paper C, the same approach as the one in [33] has been discussed in the setting of positive real function in order to model passive systems which can be important in many engineering applications.

Next, we describe in detail two *real world* examples, a CD player and a floating body, which have been used to show the efficiency of the methodology suggested in Paper B and Paper C.

### 2.5 "Real world" examples for model reduction

In this section, we briefly describe two *real world* examples that we use in this thesis to show the performance of different model reduction methods: the passivity preserving model reduction by Krylov method, the approximation method based on analytic interpolation discussed in Paper C and the stochastically balanced truncation.
2.5.1 CD player

This example has been taken from [56]. In Fig 4, a schematic view of the important part of the Compact Disc mechanism is shown.

![Diagram of a Compact Disc mechanism](image)

Figure 4: Schematic view of a rotating arm Compact Disc mechanism [56].

The most important part of this system is the optical unit (lenses, laser diode and photo detectors) and its actuators. The main task in this system is to control the arm holding the optical unit to read the required track on the disk and to adjust the position of the focusing lens adapting the depth of the laser beam penetrating the disc. The system has been modeled as a 2 input-2 output system of order 120. Due to the fact that the disc is not perfectly flat and due to irregularities in the tracks, the challenge is to find a low-cost controller that can make this system faster and less sensitive to external shocks. In Fig. 5 the frequency response of the full ordered system is plotted. In Paper B and Paper C, the transfer function from the 2nd input to the 1st output has been approximated by a 12th order system.

2.5.2 Floating body

In this example, we consider the mathematical model of a floating body. There are several different application areas where the model of a floating body at zero speed is of interest: dynamical positioned (DP) vessels, real time simulators of marine systems, wave power plants.

The equations of motion for a seagoing vessel is in general based on the Newton-Euler equations of motion for a rigid body and kinematic transformation [22]. By this approach, the forces and moments working on the floating body can be incorporated into the model by means of force and moment superposition:

$$M_{RB} \ddot{\xi}(t) = \tau_R + \tau_H + \tau_{ext} + \tau_{visc} + \tau_A.$$  (2.23)

Here, $M_{RB}$ is the rigid body system inertia matrix, $\tau_R$ represents the radiation forces and moments, $\tau_H$ represents the hydrostatic forces and moments, $\tau_{ext}$ rep-
Figure 5: Frequency response of arm position controller.
resents the external forces and moments, \( \tau_A \) represents the actuator forces and moments and \( \tau_{\text{visc}} \) the viscous forces and moments. A more precise description of the forces and moments is given below.

\textit{Radiation forces and moments (}\( \tau_R \)). These forces and moments appear as a consequence of the change in the momentum of the fluid and the waves generated due to the motion of the hull. These forces and moments are linearly related to the acceleration and the velocities of the vessel.

\textit{Hydrostatic force and moments (}\( \tau_H \)). The hydrostatic forces and moments are restoring forces and moments due to the gravity and the buoyancy. By assuming that the restoring forces and moments are linear, it is a good approximation to express them as proportional to \( \xi \) [46]:

\[
\tau_H(t) = -C_H \xi(t),
\]

where \( C_H \) is the matrix of the restoring forces and moments.

\textit{External forces and moments (}\( \tau_{\text{ext}} \)). The external forces and moments can be composed of

- Time varying forces and moments due to waves;
- Time varying forces and moments due to wind;
- Restoring forces and moments from mooring systems.

\textit{Actuator forces and moments (}\( \tau_A \)). For vessels, the actuator forces and moments can be due to propellers or thrusters. In other applications, such as wave energy plant, the actuator forces and moments can be generated by other devices.

\textit{Viscous forces and moments (}\( \tau_{\text{visc}} \)). The viscous forces and moments are nonlinear damping forces and moments appearing due to the nonlinear non-conservative phenomena, and kinetic energy of the hull is transferred to heat due to the viscous effects (skin friction, flow separation and eddy making). These forces and moments depend on the relative velocities between the hull and the fluid [46].

Numerical software can be used to obtain the data set used in the equations of motion for a vessel. The coordinate frame used in the software for computing the parameters in the equations of the motion are not in agreement with the coordinate frames used in control, for observers and simulators for a vessel. The equation of motion in numerical hydrodynamics are usually expressed in the \( h \)-frame (see Fig. 6). In this formulation the generalized position vector \( \xi = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]^T \) is defined in the \( h \)-frame and \( \{\xi_i\}_{i=1}^6 \) represents the positions and the angles with respect to the moving body (\( b \)-frame).

The \( b \)-frame and the \( h \)-frame are illustrated in Fig. 6 and are described as follows:
Introduction

Figure 6: Ship motions in b-frame and h-frame.

- **Body-Fixed frame (b-frame: \( o_b, x_b, y_b, z_b \))**
  The frame is assumed fixed to the ship hull. The origin \( o_b \) is determined by letting the axes of this frame coincide with the principal axes of inertia, with \( x_b, y_b, z_b \) pointing respectively towards the bow, starboard and down.

- **Hydrodynamic frame (h-frame: \( o_h, x_h, y_h, z_h \))**
  The frame moves with the average speed of the vessel following its path. The positive axes \( x_h, y_h \) and \( z_h \) are pointing respectively towards forward, starboard and downwards, whereas the horizontal plane spanned by \( x_h \) and \( y_h \) correspond to the mean free surface of the ocean. The origin \( o_h \) is placed such that the \( z_h \) axis is placed on the same axis as the center of gravity (CG).

In Paper D this application has been studied extensively. In this paper the main focus has been on the modeling of the radiation force \( \tau_R \) in (2.23). The state-space model of the radiation force is of high order. This makes it necessary to obtain a low order model for efficient computer simulation and control synthesis. Traditionally the reduced order model has been computed using standard model reduction techniques. In Paper D, the original model has been approximated by a low order system computed using the methodology developed in Paper C.

3 Summary and contribution of the papers

This thesis contains the six papers listed below. In Paper A a spectral estimation procedure is proposed based on the rational covariance extension theory. Paper B and Paper C discuss model reduction problems in the analytic interpolation setting. Moreover, in Paper C an algorithm for the computation of a low order passive system is presented. This algorithm is employed to model the radiation forces of a floating body in Paper D.

Paper E and Paper F discuss numerical aspects of the computation of the solution of the covariance extension problem and the analytic interpolation problem, respectively. In Paper E the solution of the rational covariance extension problem is
computed numerically by solving a discrete-time matrix Riccati-type equation. Paper F proposes a numerical algorithm for the computation of a scalar real rational Nevanlinna-Pick interpolants of a bounded degree by solving a system of nonlinear equations.

Paper A: "Spectral Estimation by least-squares optimization based on rational covariance extension"


This paper proposes a new spectral estimation technique based on rational covariance extension with degree constraint. In spectral estimation, the main problem is to estimate the spectral density function that captures characteristics of a stochastic process. Traditionally, the maximum entropy solution for the covariance extension problem is one of the most popular spectral estimates. However, there is no guarantee that the maximum entropy solution possesses more favorable characteristics than other covariance matching. In recent years, a lot of studies have been done in order to find alternative methodologies for spectral estimation [10, 11, 52]. The novelty of the new approach is in designing a spectral density that optimally approximates covariance and the frequency response of a process simultaneously.

In the rational covariance extension theory, all the rational spectral densities of bounded degree which match a given covariance sequence, are parameterized in terms of Schur polynomials. Regarding the Schur polynomial coefficients as design parameters, we minimize a least-square distance between given data and a spectral density function under some constraints on the covariances.

We consider two different constraints on the covariances: we can trust the covariance accuracy and we assume certain uncertainty on a given covariance sequence. By taking into account these two different situations, the spectral density approximation can be formulated as a "min-min" problem and "min-max" problem respectively. The basic idea in this paper is the same as the one in [44]. However, neither "min-min" nor "min-max" types of optimization problems mentioned above, have been dealt with in [44]. In that respect, this paper deals with a much wider and more interesting class of problems.

Paper B: "Passivity-Preserving Model Reduction by Analytic Interpolation"


Many engineering applications deal with problems of approximating a positive real function with one of a lower degree, i.e. model reduction problems. In this paper, these problems are discussed in the analytic interpolation setting.

In the literature, there are several methods to perform model reduction. We consider the class of model reduction procedures based on rational interpolation.
In these procedure the rational transfer function of the reduced order model takes the same value as the original one in a set of suitable points. In addition, our attention is focused on model reduction methods that preserve certain properties of the original system. In this paper, systems which have stable and positive real transfer functions are taken into consideration.

Recently, Antoulas and Sorensen in [2, 54] have proposed a procedure to approximate a positive real transfer function with one of a lower degree. The method is based on an observation by Antoulas according to which if the approximant is preserving a subset of spectral zeros and takes the same value as the original transfer function in the mirror points of such spectral zeros, then the approximant is also positive real. However, in this paper, we have proved that this procedure outputs an approximant which can be interpreted as a special solution in the analytic interpolation theory, namely the \textit{central solution}.

Moreover, we point out that in the analytic interpolation theory the freedom in the placement of spectral zeros and interpolation points can be employed for designing a positive real approximant of a low degree which shapes the given high order model. This observation is supported by two benchmark examples: RLC circuits and CD player.

**Paper C: "Passive system with degree bound designed by Analytic Interpolation"**

Paper C is based on the following publication:


This paper develops a methodology for generating a stable and positive real approximant of degree $n$ of a given system. The mathematical background lies on the analytic interpolation theory with a degree constraint. In the analytic interpolation theory, all the positive real functions which satisfy certain interpolation conditions are parameterized in terms of the polynomial of spectral zeros and they can be computed by minimizing a weighted entropy functional.

In this paper, we study the connection between the minimizer of the entropy functional and the weight. Due to this connection, the approximant of a passive system can be obtained by solving an optimization problem with respect to the polynomial of the spectral zeros and the polynomial of the interpolation points. This means that spectral zeros and interpolation points are employed as tuning parameters for the computation of an approximant of the original system.

The main idea of this paper is the same as the one in [33] where the focus was in computing the approximant of a bounded real function satisfies certain interpolation condition. Thus, this paper deals with a different class of functions and, then, different theoretical results are achieved. The efficiency of the proposed approach is supported by numerical examples.
Paper D: "Low order radiation forces by Analytic Interpolation with degree constraint"

Paper D is coauthored with K. Unneeland and it is based on the following publication:

- "Low order radiation forces by Analytic Interpolation with degree constraint" in Proceeding of the 46th IEEE Conference on Control and Decision, New Orleans, LA, USA, pp. 2405-2410, December 2007.

This paper deals with the problem of positive real modeling of a floating body. The main focus is on the modeling of the radiation forces and moments. The radiation forces are described as the forces that make a floating body oscillate even if the surrounding fluid is calm. This type of mathematical model is of great interest in control and simulation of dynamical positioned vessel, i.e. ships, offshore platforms, and wave plants.

It is well-known that the radiation forces can be modeled with systems that have a positive real transfer function. The state-space models are usually of high degree. Thus, a reduced order model which approximates the original one is needed in order to have computational efficiency for simulation and control synthesis.

Traditionally, this reduced order model is obtained by using standard techniques for model reduction, as the balanced truncation, $H_2$ model reduction, to mention a few of them. In this paper, we have adopted a different approach. The reduced order model is computed by approximating data obtained by hydrodynamical software. The approximation technique used here is the same as the one proposed in Paper C.

The numerical example shows that this approach performs efficiently. This paper is a valuable contribution that supports the usefulness of the methodology described in Paper C in a real world application.

Paper E: "A homotopy continuation solution of the covariance extension equation"

Paper E is coauthored with C.I. Byrnes and A. Lindquist, and it is published in New Directions and Applications in Control Theory, Springer Verlag, 2005, pp. 27–42.

This paper studies the steady-state form of a discrete-type Riccati equation, called the Covariance Extension Equation. In [7], this matrix equation is formulated in terms of the partial covariance data and a choice of spectral zeros of a desired shaping filter. It is a nonstandard Riccati equation since it has a unique positive semidefinite solution and not a positive definite solution as the standard Riccati equations has.

In this paper, we prove that the set of solutions of the covariance extension equation is diffeomorphic to the set of pairs of positive covariance sequence and Schur polynomials. This observation implies that the homotopy continuation method can be applied to solve numerically the covariance extension equation.

The homotopy continuation method consists of two steps: the predictor step which uses Euler’s approximation, and the corrector step which applies Newton’s
iteration. Although the proposed approach does not seem to offer particular advantage to a previous solver [20], it provides insights into issues such as positive degree and model reduction, since the rank of the solution of the covariance extension equation coincides with the degree of the shaping filter. The usefulness of the proposed approach is supported by examples of academic character.

**Paper F: "Computation of bounded degree Nevanlinna-Pick interpolants by solving nonlinear equations"**


This paper develops a new algorithm for computing scalar real rational Nevanlinna - Pick interpolants of a bounded degree. It applies to the class of positive real interpolants and it is capable of treating the case with boundary spectral zeros. In this respect, the results are more reliable than the previous optimization-based algorithm [43]. The new procedure computes interpolants by solving a system of nonlinear equations. Such a system can be constructed immediately using the assertion of bijectivity between the class of positive real numerator/denominator polynomials and a class of positive real interpolants in [26]. In this paper, we show that this bijective map is also a homeomorphism. Moreover, if the domain of such a map is restricted to the set of strictly positive real numerator/denominator pairs of polynomials and if its range is restricted to the set of strictly positive real interpolants, then the map is actually a diffeomorphism [9]. These two properties allow us to solve the system of nonlinear equations by a continuation method on a homotopy from the equation of the *central solution* to the system of our interest. The efficiency of the proposed procedure is illustrated with an example where an interpolant with spectral zeros on the unit circle is computed.

4 Open problems and future research directions

The work presented in this thesis leaves several questions open. Some of them are listed and commented below.

- In Paper B, we point out that the algorithm suggested by Sorensen in [54] can solve tangential interpolation problems. Yet a more general approach which involves matrixial and tangential interpolation would be valuable to explore. As a matter of fact, the model reduction problem for multi-input/ multi-output systems are very important in engineering applications. For example, the modeling of the radiation forces of a vessel can be done considering the overall system which is more challenging for marine applications. In [35], the author studies the generalization of the Nevanlinna-Pick interpolation theory with a degree constraint for multivariable systems. In this respect, it would
be interesting to extend the algorithm in Paper C to the multi-input/multi-output case in view of the results achieved in [35].

- In Paper A we propose a method for spectral estimation based on least-squares. Despite the efficiency of the proposed algorithm, the least-square method is based on a local search technique for the optimum by using a gradient-following technique. Moreover, initial information on the system parameters is needed a priori for the convergence of the algorithm. In view of this, in system identification literature, there are several attempts to apply other algorithms for parameter estimation. In [34], the genetic algorithm is implemented for parameter estimation of ARMAX models. It would be valuable to investigate the efficiency of the genetic algorithm for problems treated in this thesis.

- In all the papers included in this thesis, the interpolant has been computed by using the numerical algorithm in Paper F. Due to the efficiency of this algorithm, it would be important to extend it for multivariable cases, that is, both matricial and tangential analytic interpolation problems. As a matter of fact, in several applications treated in this thesis, the multivariable case has been decomposed in single-input/single output subproblems. In this way, a partial analysis of the problem is done. Thus, with a theory for the multivariable case available the full problem could be addressed directly.

5 References


