Valuing the Housing Cooperative Conversion Option

Fredrik Armerin* and Han-Suck Song†

Abstract
Since the 1990s, both private and municipal owners of multifamily rental properties in Sweden have sold a large number of their properties to housing cooperatives established by the property’s tenants. One motivation for the large increase in so called housing cooperative conversions is that the practice of rent regulation causes actual rents to be lower than market-clearing rent levels, especially in attractive areas in larger cities. By selling the property to a housing cooperative, the property owner can take advantage of the positive price difference between the price of housing cooperative dwellings, which are determined by demand and supply, and the value of the property based on the assumption that the rents will continue to be lower than market rents.

In this paper we use a real options approach to derive a closed-form valuation formula for the option an owner of an income producing multifamily property has to sell it to a housing cooperative. In traditional option valuation models, the date when the option matures is known in advance. However, it is common that the property owner does not know in advance when the tenants (through the housing cooperative) will buy the property. In this paper we let the expected time to maturity, which is the day when the tenants purchases the property from their landlord, to be a random variable. The numerical examples suggest that the value of the conversion option increases as expected time to conversion increases, as well as when the volatility of the price of housing cooperative properties increase. The real options approach suggested in this paper may be especially useful to explicitly conceptualize the problem of valuing a rental property with embedded options to switch it to another type of property.

Keywords: Real options, property valuation, rent regulation, housing cooperative conversion

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1 Introduction

1.1 Background

In this paper we develop a valuation model for the option an owner of an income producing multifamily property has to sell it to a housing cooperative established by the property’s tenants. In connection to the sale, the housing cooperative grants cooperative apartments to the members of the association, i.e. the former tenants. The owners of the housing cooperative apartments have a right to use them in perpetuity, to use them as collateral, and to sell them on the free market. This type of sales, called ”housing cooperative conversions”, has increased in Sweden since the 1990s in larger cities, especially in central Stockholm. This large increase in conversions is principally caused by a combination of the rent control system\(^1\), which keeps the actual rent levels below market levels in attractive areas.

Besides causing conversion of rental apartment buildings into housing cooperatives, the rent control system also causes negative effects like long queues for rental apartments in attractive areas, illegal key money, discouragement of new construction of rental apartments, a flourishing market for second-hand contracts, tenants locked into sub-optimal housing arrangements etc (see e.g. Ellingsen and Englund [3]; Lind [10]). These negative effects are most likely strengthening the demand for cooperative apartments, boosting the already high prices, and giving both owners of multifamily rental houses as well as tenants strong incentives to capitalise the high prices through housing cooperative conversions.

Tables 1 and 2 below show that the trend is for fewer rented dwellings and more cooperative owned properties, both through conversion and new construction, as is observed by Magnusson & Turner [12] p. 281. The data clearly shows the popular trend of converting to housing cooperatives in both inner-city Stockholm (Table 1) as well as in whole Stockholm municipality (Table 2). In inner-city Stockholm, the proportion of cooperative apartments in relation to the total number of apartments was about 29 percent in 1990, but had increased to about 45 percent in 2001 and to 56 percent in 2008. The share of cooperative housing has also increased significantly in the whole of Stockholm municipality; from slightly more than 24 percent in 1990 to almost 43 percent in 2008. Both private and municipal owners of rented housing have sold properties to cooperative associations; however the decline in rented dwellings can above all be attributed to the huge number of conversions from private rented housing in inner-city Stockholm.

\(^1\)The rent control system is based on the ”principle of user value” (bruksvärdesprincipen). Rents in private sector apartments are supposed to match, or only be marginally higher, than the rents for comparable apartments in municipal houses. In practice, rents are determined by costs, e.g. production costs, costs for operation, maintenance and interest. This system aims at protecting a sitting tenant against rents higher than the market rent, but also aims at keeping rents in new contracts below the market level.
### Table 1. Inner-city Stockholm
Number of dwellings and percentage share across tenure forms in multi-family properties

<table>
<thead>
<tr>
<th>Year</th>
<th>Municipal (public) rented</th>
<th>Private rented</th>
<th>Cooperative</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>27,858</td>
<td>90,332</td>
<td>47,511</td>
<td>165,701</td>
</tr>
<tr>
<td>Percentage</td>
<td>16.8%</td>
<td>54.5%</td>
<td>28.7%</td>
<td></td>
</tr>
<tr>
<td>1998</td>
<td>29,964</td>
<td>76,190</td>
<td>65,387</td>
<td>171,541</td>
</tr>
<tr>
<td>Percentage</td>
<td>17.5%</td>
<td>44.4%</td>
<td>38.1%</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>27,020</td>
<td>68,256</td>
<td>77,738</td>
<td>173,014</td>
</tr>
<tr>
<td>Percentage</td>
<td>15.6%</td>
<td>39.5%</td>
<td>44.9%</td>
<td></td>
</tr>
<tr>
<td>2004</td>
<td>21,575</td>
<td>57,846</td>
<td>99,555</td>
<td>178,976</td>
</tr>
<tr>
<td>Percentage</td>
<td>12.1%</td>
<td>32.3%</td>
<td>55.6%</td>
<td></td>
</tr>
<tr>
<td>2008</td>
<td>19,594</td>
<td>61,056</td>
<td>98,515</td>
<td>179,165</td>
</tr>
<tr>
<td>Percentage</td>
<td>10.9%</td>
<td>34.1%</td>
<td>55.0%</td>
<td></td>
</tr>
</tbody>
</table>

Source: SCB/USK.

### Table 2. Stockholm municipality
Number of dwellings and percentage share across tenure forms in multi-family properties

<table>
<thead>
<tr>
<th>Year</th>
<th>Municipal (public) rented</th>
<th>Private rented</th>
<th>Cooperative</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>117,914</td>
<td>142,230</td>
<td>84,432</td>
<td>344,576</td>
</tr>
<tr>
<td>Percentage</td>
<td>34.2%</td>
<td>41.3%</td>
<td>24.5%</td>
<td></td>
</tr>
<tr>
<td>1998</td>
<td>113,695</td>
<td>135,216</td>
<td>109,804</td>
<td>351,715</td>
</tr>
<tr>
<td>Percentage</td>
<td>31.7%</td>
<td>37.7%</td>
<td>30.6%</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>110,189</td>
<td>126,260</td>
<td>125,473</td>
<td>361,922</td>
</tr>
<tr>
<td>Percentage</td>
<td>30.4%</td>
<td>34.9%</td>
<td>34.7%</td>
<td></td>
</tr>
<tr>
<td>2004</td>
<td>102,584</td>
<td>110,892</td>
<td>159,246</td>
<td>372,722</td>
</tr>
<tr>
<td>Percentage</td>
<td>27.5%</td>
<td>29.8%</td>
<td>42.7%</td>
<td></td>
</tr>
<tr>
<td>2008</td>
<td>97,389</td>
<td>116,453</td>
<td>159,955</td>
<td>373,797</td>
</tr>
<tr>
<td>Percentage</td>
<td>26.1%</td>
<td>31.2%</td>
<td>42.8%</td>
<td></td>
</tr>
</tbody>
</table>

Source: SCB/USK.
Notably is the conversion of a large number of municipally owned apartments during 1999-2004 initiated by the liberal-conservative majority of Stockholm city: municipal housing companies converted about 12,200 apartments in the whole of Stockholm city during this period. The trend of converting tenancies to cooperative apartments is indeed a highly political issue that completely divides the political arena. In order to prevent municipal properties to be sold to housing cooperatives, the Social Democrats introduced in 2002 a legislation at the national level, the so called "Stopplagen"\(^2\), which aimed at making conversions more difficult by requiring authorization from the local county administrative boards. But the liberal-conservative political alliance, which won the latest Swedish national election in September 2006, terminated the stop legislation in July 2007, which again made it easier for municipal housing companies to initiate conversions. Although the majority of conversions have occurred in attractive inner-city areas such as inner-city Stockholm, it is the intention of the current liberal-conservative government to push forward the conversion of municipal properties in outer town and suburban areas.\(^3\)

1.2 The conversion process

A first important formal step in the conversion process is that the tenants establish a housing cooperative. The initiative for tenants to establish a housing cooperative can either come from the tenants themselves, or the landlord who offers the tenants to buy the property. The property owner and the housing cooperative might – as soon as the housing cooperative has been established - begin the formal process of negotiation, focusing on the terms and conditions of the sale of the property. The conversion process is completed when the housing cooperative has acquired the property and granted the former tenants against an initial payment (which is called the transfer fee, and is equivalent to the acquisition price of the cooperative apartment) the right to use the dwellings.\(^4\) Thereafter, each cooperative apartment owner pays an annual fee (typically paid monthly) to the housing cooperative.

This annual fee will cover the payments the housing cooperative has to do in order to pay debt service (typically the housing cooperative borrows money to finance the purchase), capital improvement as well as technical and financial operating expenses. Thus the major payments a tenant must consider when he or she decides to take part in a conversion process constitute of the initial transfer fee (acquisition price), the annual fees, and annual after-tax debt service payments to the lender. Therefore it is likely that tenants weigh both the acquisition price against their expectations about possible future resale prices, as well as expected size and riskiness of the annual payments against the expected

\(^2\)A direct translation to English would be "Stop law".  
\(^3\)Indeed, the Stockholm municipality has constructed a home page (www.bildabostad.se) that aims at convincing tenants to form cooperative associations in order to buy their housing. 
\(^4\)The owners of the cooperative apartments might later transfer, typically on the open market, the right to use the apartment to another household. In everyday language, we say that a condominium owner sells his/her apartment at market price.
costs and fluctuations of rents. Furthermore, tenants should also consider how their household incomes co-moves with local rents and prices (Ortalo-Magne and Rady [14]). Besides these liquidity and price risks, purchasing a cooperative apartment (or other mode of home ownership) also entails a portfolio choice that affects the wealth composition (Englund et al. [4]).

In addition to these economic considerations, a number of other factors, such as demographic factors like family formation (see e.g. Lauster and Fransson [9]); Áberg [18]), household mobility (see e.g. Özyildirim et al., [19]), choice of location (see e.g. Ortalo-Magn and Rady [15]), as well as sociological and psychological factors might affect households tenure choice (see Englund et al. [5] for an overview of city-specific housing tenure choice in about 50 countries around the world).

Thus there exist a large number of tenure choice determinants that affect whether the conversion process will go quickly, or end up in a lengthy process that even might be terminated. Although time to conversion might vary a lot in more attractive areas, the average time to conversion in such areas are likely to be shorter than in other less attractive areas. For instance, it can take considerably longer time for a majority of the tenants in less attractive areas to form a housing cooperative but also to raise necessary financing to buy the property. Above all, it is enough that more than one-third of the tenants are hesitant or unwilling to support a conversion in order to stop the conversion process, since the law requires that two-thirds of the tenants must be willing to support a conversion.\(^5\)

Furthermore, a property owner that is otherwise willing to sell the property to housing cooperative might be willing to delay the conversion process for different reasons. The property owner might wait for more favorable market conditions concerning the market prices for cooperative dwellings, which affects the price tenants are willing to pay for the property. Property owners that own properties in different areas might have a strategy to only sell those properties that are located in outer town and suburban areas, while keeping the properties located in inner-city areas. By selling the outer-town properties to housing cooperatives, property owners can self-finance acquisitions of inner-city properties and to renovate properties that the landlord choose to keep, thus becoming less dependent on traditional bank lending. Finally, many current property owners’ strategy is to continue to own and manage the property as an income-producing asset, or to sell it to another real estate investor with similar strategy, thus making conversions an unrealistic alternative for the tenants.

In essence, if the property owner can sell the property to the housing cooperative formed by the tenants for a price that exceeds the market value of property as income-producing asset, then there exists a major economic incentive for a property owner to complete the conversion process. Similarly, if the housing cooperative can purchase the property for a price that entail the housing cooperative to grant the tenants a right to use the dwellings for a price

\(^5\)Those tenants that have declined to buy a cooperative dwelling have a right to reside as tenants, but this time with the housing cooperative as landlord.
that is attractive in comparison to similar dwellings, then tenants are likewise prepared to buy the property through the housing cooperative they establish. The actual price is a matter of bargaining power.  

We can conclude that, in general, a property owner does not know in advance the time at which his property will be sold to a housing cooperation. In this paper, time to conversion constitute a major variable in the real options analysis and valuation model we develop below.

1.3 The embedded real option to convert

The strong interest among many tenants to acquire their landlords’ properties through conversion of rental dwellings to cooperative dwellings has contributed to the relatively positive increase in the value of income producing multifamily properties, especially in the more attractive areas in Stockholm.

The relatively low valuation yields for residential multifamily properties located in the central areas of Stockholm is an indication that the (long-run) average growth rate in the cash flow that properties in central Stockholm area can generate is significantly higher, than the average growth expectations of similar residential multifamily properties that are located in areas outside inner-city Stockholm (see table 3 below).

Table 3. Valuation yields
Residential multifamily properties

<table>
<thead>
<tr>
<th></th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stockholm Central Area</td>
<td>4.2</td>
<td>3.9</td>
<td>3.5</td>
<td>3.0</td>
</tr>
<tr>
<td>Rest of Greater Stockholm</td>
<td>6.1</td>
<td>5.5</td>
<td>5.0</td>
<td>4.0</td>
</tr>
<tr>
<td>Rest of Sweden</td>
<td>6.4</td>
<td>6.0</td>
<td>5.6</td>
<td>4.8</td>
</tr>
</tbody>
</table>


Put in another way, it is likely that the market takes into consideration the possibility for property owners to either sell their properties to housing cooperatives, or that there will be a change in the legislation that will make it possible for rents to gradually be adjusted to market rent levels. In both cases, the future cash flows from residential multifamily properties, especially those located in the more attractive areas in Stockholm will be higher than the current net operating incomes, due to rent regulation (Lind [11]).

Indeed, the low valuation yields implicitly reflects the embedded real or strategic option an owner of residential properties has to sell their properties.

6A natural extension of the present model is to include this bargaining as a bilateral duopoly.
to housing cooperatives. By adjusting the yield component downwards in their traditional valuation models (reflecting high growth rates in future cash flows), valuers valuation reports better correspond with the transaction prices of comparable properties sold to housing cooperatives.

However, the traditional discounted cash-flow (DCF) approaches to the appraisal of investments in income-producing properties cannot explicitly capture the value of the real estate owners flexibility to switch the property type from multi-family rental property (in which rents are regulated) to housing cooperative property (in which prices reflect market values) by selling it to a housing cooperative. Instead, a real options approach has the potential to conceptualize and quantify the value of a property owners embedded option to convert the property in order to benefit from favorable housing market conditions (cf. Trigeorgis [17]). Indeed, as the yields in Table 3 indicates, the option to convert can constitute a significant part of the overall market values and transaction prices of multi-family rental properties.

The intention of this article is to present a real options model that explicitly conceptualizes and quantifies the value of the housing cooperative conversion option. Below we will develop a contingent-claims model for valuing the housing cooperative conversion option. The next section (2.1) outlines some basic technical and probabilistic preliminaries and assumptions. In section 2.2 we present a general description of the option valuation problem. This description is independent of any specific assumptions about the underlying models. However, in order to determine an explicit formula that might solve the option valuation problem, we need to make some assumptions about the models governing the price processes as well as the distribution of the random time to convert. Therefore, in section 2.3 we present our basic assumptions about the underlying models. Thereafter, we derive in section 2.4, an explicit formula of the option value to convert. In section 3 we present numerical examples of the option values that are determined by the formula we develop. Section 4 concludes this paper.

2 Valuing the option

2.1 Probabilistic preliminaries

Let \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))\) be a filtered probability space. We assume that the probability space is complete, and that the filtration fulfills the usual conditions: \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets of \(\mathcal{F}\) and the filtration \((\mathcal{F}_t)\) is right continuous. We assume that there exists a risk-less bank account \(B\) with constant rate \(r \geq 0\). Hence, \(B\) has dynamics given by

\[
dB(t) = rB(t)dt.
\]

We also assume the existence of an equivalent martingale measure \(Q\), under which the bank account discounted price process of every non-dividend paying traded asset is a martingale. The random time \(\tau\) at which the conversion may
take place is assumed to be independent of every stochastic process occurring in
the modelling.

2.2 A general description of the optionality

Let \( P(t) \) denote the price that the housing cooperative formed by the tenants
is willing to pay for the property at time \( t \), and let \( V(t) \) denote the value of
the property under the assumption that future incomes generated by the property
only consists of the regulated rents. We denote by \( V_0 \) and \( P_0 \) the values of \( V \)
and \( P \) at time \( t = 0 \) respectively. If the tenants want to buy the property from
the owner at time \( t \), the owner will sell if the price \( P(t) \) the tenants are willing to
pay exceeds the value \( V(t) \) of the property with regulated rent. It should be noted
that the time \( \tau \) at which the tenants are ready to buy the property is generally
not known by the owner in advance. As discussed above, there are several
reasons that could explain why tenants don’t buy the property immediately, or
why it can be difficult to accurately predict when a conversion might take place.
For these reasons we will model the time \( \tau \) as a random time, in general not
known at time 0. With \( X \) denoting the net operating income (NOI) per time
unit, the total value of the property (i.e. including the optionality) for the owner
at time 0 is

\[
PV = \mathbb{E}^Q \left[ \int_0^\tau e^{-r s} X(s) ds + e^{-r \tau} \max(P(\tau), V(\tau)) \right] \\
= \mathbb{E}^Q \left[ \int_0^\tau e^{-r s} X(s) ds + e^{-r \tau} V(\tau) + e^{-r \tau} \max(P(\tau) - V(\tau), 0) \right] \\
= V_0 + \mathbb{E}^Q \left[ e^{-r \tau} \max(P(\tau) - V(\tau), 0) \right],
\]
or

\[
PV = \text{Static PV + Option value}.
\]

We make the simplifying assumption that if the landlord doesn’t sell the property
to the tenants at time \( \tau \) (i.e. if \( P(\tau) < V(\tau) \)), then there is no possibility
for the conversion to occur at any later date. This assumption can be justified
by the fact that a non-completed conversion is likely to discourage the tenants
and/or the landlord to face the process once more. Since the option value is
non-negative we have

\[
PV \geq V_0.
\]

The price process \( P \) is the market price of a non-dividend paying asset, and
hence \( e^{-r \tau} P(t) \) is a \( Q \)-martingale. Since \( \tau \) is independent of the price
process \( P \) we have

\[
\mathbb{E}^Q \left[ e^{-r \tau} P(\tau) \right] = P_0.
\]

Under the assumption that the NOI is positive we have

\[
PV \geq \mathbb{E}^Q \left[ e^{-r \tau} P(\tau) \right] = P_0,
\]

\[
7
\]
and this implies
\[ PV \geq \max(V_0, P_0) = V_0 + \max(P_0 - V_0, 0). \]
The non-negative difference between PV and \( \max(V_0, P_0) \) is the Time value of
the option, and we can write
\[ PV = V_0 + \max(P_0 - V_0, 0) + \text{Time value}. \]
The second term in this decomposition of PV is the intrinsic value of the option. Thus
\[ PV = \text{Static PV} + \text{Intrinsic value} + \text{Time value}. \]

2.3 The specific model
For the models and theory used in this section see e.g. Björk [1] or Hull [7]. The price \( P(t) \) the tenants have to pay the owner of the property in order to acquire it at time \( t \) is assumed to follow a geometric Brownian motion:
\[ dP(t) = \mu P(t)dt + \sigma P(t)dW^P(t); \quad P(0) = P_0. \]
Here \( W^P \) is a standard \( P \)-Brownian motion. The regulated net operating income (NOI) per time unit \( X \) is assumed to grow with expected rate equal to \( g \). To account for uncertainty, we assume that also the yearly regulated NOI follows a geometric Brownian motion:
\[ dX(t) = gX(t)dt + \nu X(t) \left( \rho dW^P(t) + \sqrt{1-\rho^2}dW^X(t) \right); \quad X(0) = X_0. \]
\( W^X \) is a standard \( P \)-Brownian motion independent of \( W^P \) and \( \rho \) is a constant satisfying \( \rho \in [-1, 1] \). The interpretation of \( \rho \) is that
\[ \text{Corr} \left( \frac{dP(t)}{P(t)}, \frac{dX(t)}{X(t)} \right) = \rho dt. \]
Since \( P \) is a traded asset without dividends, the drift of \( P \) under \( Q \) is equal to \( rP(t) \). The process \( X \) is not a traded asset, so its dynamics under \( Q \) are not given by no-arbitrage arguments. In order to derive \( V \) and the option value we now perform a two-dimensional change of measure. Under \( Q \) the processes
\[ Z^P_t = W^P_t + \frac{\mu - r}{\sigma} t \]
\[ Z^X_t = W^X_t + \lambda_X t \]
are two independent Brownian motions. Here \( \lambda_X \) can be thought of as the market price of NOI risk. The dynamics of \( P \) and \( X \) under \( Q \) are given by
\[ dP(t) = rP(t)dt + \sigma P(t)dZ^P(t) \]
and
\[ dX(t) = gQX(t)dt + vX(t) \left( \rho dZP(t) + \sqrt{1 - \rho^2} dZX(t) \right) \]
respectively, where
\[ gQ = g - v \left( \frac{\mu - r}{\sigma} + \sqrt{1 - \rho^2} \lambda_X \right) \]
is the risk-neutral expected growth rate. To get a well defined present value of the NOI we assume
\[ gQ < r. \]
The value \( V(t) \) at time \( t \) for a property with NOI given by \( X \) is
\[
V(t) = E^Q \left[ \int_t^\infty e^{-r(s-t)} X(s) ds \right]_{\mathcal{F}_t} = \int_t^\infty e^{-r(s-t)} E^Q[X(s) | \mathcal{F}_t] ds = \int_t^\infty e^{-r(s-t)} X(t) e^{gQ(s-t)} ds = \frac{X(t)}{r - gQ}.
\]
This is a version of the Gordon formula. The denominator in the expression of \( V(t) \) is the nominal rate minus the growth rate \( g \) plus the risk premium
\[ RP = v\rho \frac{\mu - r}{\sigma} + v \sqrt{1 - \rho^2} \lambda_X. \]
The product \( v\rho (\mu - r)/\sigma \) is the risk \( v\rho \) times the market price of property price risk, and \( v \sqrt{1 - \rho^2} \lambda_X \) is the risk \( v \sqrt{1 - \rho^2} \) times the market price of NOI risk.

**Example 2.1** Let us look at some special cases of the value \( V(t) \) of the regulated property.

- NOI is uncorrelated with the price \( P \). In this case \( \rho = 0 \) and we get
  \[
  V(t) = \frac{X(t)}{r + v\lambda_X - g}.
  \]
- NOI has correlation 1 with the price process \( P \). We now have \( \rho = 1 \), and the value is given by
  \[
  V(t) = \frac{X(t)}{r + v\frac{\mu - r}{\sigma} - g}.
  \]
- There is no randomness in NOI. This means that \( v = 0 \) and the value is given by.
  \[
  V(t) = \frac{X(t)}{r - g}.
  \]
The dynamics of $V$ under the martingale measure $Q$ is given by
\[
\begin{cases}
dV(t) = g_Q V(t) dt + v V(t) \left( \rho dZ^P(t) + \sqrt{1 - \rho^2} dZ^X(t) \right) \\
V(0) = V_0 = \frac{X_0}{r - \mu Q}
\end{cases}
\]
We must also choose both the distribution of $\tau$ as well as how it is correlated with the other randomness in the model, i.e. two processes $W^P$ and $W^X$ under $Q$ (we have already assumed that it is independent of all involved stochastic processes under $P$). In this paper we make the following two assumptions.

1. $\tau$ is exponentially distributed with parameter $\gamma$ both under $P$ and $Q$:
   \[ P(\tau > t) = Q(\tau > t) = e^{-\gamma t}, \quad t \geq 0. \]

2. $\tau$ is independent under both probability measure $P$ and $Q$ of $W^P$ and $W^X$.

In the language of incomplete markets, the assumption that $\tau$ is independent of $W^P$ and $W^X$ both under $P$ and $Q$ means that the option value is calculated under what is known as the minimal martingale measure. This measure was introduced in Föllmer & Schweizer [6] (see also Schweizer [16]). Møller [13] uses the minimal martingale measure in an insurance application that resembles the situation considered here.

2.4 The closed-form conversion option valuation formula

Under the assumptions given in Section 2.3 we have
\[
\text{Option value} = E^Q \left[ e^{-rt} \max(P(\tau) - V(\tau), 0) \right] = \int_0^\infty E^Q \left[ e^{-rx} \max(P(x) - V(x), 0) \right] \gamma e^{-\gamma x} dx
\]
Now introduce
\[ q = r - g_Q \]
(we have assumed this to be a strictly positive parameter). Using standard methods for valuing options on the maximum of two assets (see e.g. Hull [7] Section 27.7) we get
\[
e^{-rx} E^Q \left[ V(x) \max \left( P(x) - 1, 0 \right) \right] = V_0 e^{-(r - g_Q)x} E^{Q'} \left[ \max \left( \frac{P(x)}{V(x)} - 1, 0 \right) \right],
\]
where $Q^V$ is a probability measure equivalent to $Q$, and under the measure $Q^V$ the process $P(t)e^{-qt}/V(t)$ is a martingale. Since

$$d(P(t)e^{-qt}/V(t)) = -qP(t)e^{-qt} \frac{1}{V(t)} dt + e^{-qt} \left( dP(t)/V(t) + P(t)d(1/V(t)) + d(P,1/V(t)) \right),$$

we have

$$d(P(t)e^{-qt}/V(t)) = \left( -qP(t)e^{-qt} \frac{1}{V(t)} dt + \sigma dZ^P(t) - \left( \nu P(t) + \sqrt{1 - \rho^2} dZ^X(t) \right) \right),$$

in order for $P(t)e^{-qt}/V(t)$ to be a martingale under $Q^V$ it must have drift equal to zero, and we can write

$$d(P(t)e^{-qt}/V(t)) = (P(t)e^{-qt}/V(t)) \sqrt{\sigma^2 - 2\rho\sigma v + v^2} dZ^Y(t),$$

where $Z^Y$ is a standard one-dimensional $Q^V$-Brownian motion. It follows that

$$d \left( \frac{P(t)}{V(t)} \right) = q \frac{P(t)}{V(t)} dt + \sqrt{\sigma^2 - 2\rho\sigma v + v^2} \frac{P(t)}{V(t)} dZ^Y(t).$$

The Black-Scholes formula for the value of a European call option is given by

$$c(S,K,t,T,r,\delta,\sigma) = S \Phi \left( \ln(S/K) + (r - \delta + \sigma^2/2)(T - t) \right) \sqrt{T - t} \Phi \left( \ln(S/K) + (r - \delta - \sigma^2/2)(T - t) \right),$$

and using this formula we can write

$$e^{-rx} E^Q \left[ V(x) \max \left( \frac{P(x)}{V(x)} - 1, 0 \right) \right] = V_0 e^{-qx} E^{Q^V} \left[ \max \left( \frac{P(x)}{V(x)} - 1, 0 \right) \right]$$

$$= c(P_0, \delta, q, x, V_0, \sqrt{\sigma^2 - 2\rho\sigma v + v^2}).$$

With

$$\bar{\sigma} = \sqrt{\sigma^2 - 2\rho\sigma v + v^2}$$

we have

$$e^{-rx} E^Q \left[ V(x) \max \left( \frac{P(x)}{V(x)} - 1, 0 \right) \right] = P_0 \Phi \left( \frac{1}{\sqrt{x}} - \frac{1}{\sigma} \ln \left( \frac{P_0}{V_0} \right) + \left( \frac{q}{\sigma} + \frac{\sigma}{2} \right) \sqrt{x} \right)$$

$$- e^{-qx} V_0 \Phi \left( \frac{1}{\sqrt{x}} - \frac{1}{\sigma} \ln \left( \frac{P_0}{V_0} \right) + \left( \frac{q}{\sigma} - \frac{\sigma}{2} \right) \sqrt{x} \right)$$

Let

$$I(k,L,M) = \int_0^\infty \Phi \left( M \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} dx$$

and define the parameters

$$\alpha = \frac{1}{\sigma} \ln \left( \frac{P_0}{V_0} \right) \quad \text{and} \quad \beta = \left( \frac{q}{\sigma} + \frac{\bar{\sigma}}{2} \right).$$

We can now state the following, which is the main Theorem in this paper.
Theorem 2.2 With parameter values $P_0$, $V_0$, $\alpha$, $\beta_+$, $\beta_-$ and $\gamma$ as above, the value of the conversion option is

$$\text{Option value} = \gamma [P_0 \cdot I(\gamma, \alpha, \beta_+) - V_0 \cdot I(\gamma + r, \alpha, \beta_-)],$$

where $I(k, L, M)$ is given by

$$I(k, L, M) = \begin{cases} e^{-L(M-\sqrt{M^2+2k})} \left(\frac{M}{\sqrt{M^2+2k}} + 1\right) & \text{if } L < 0 \\ \frac{1}{\sqrt{2\pi k}} e^{-\frac{1}{2} - L(M+\sqrt{M^2+2k})} \left(\frac{M}{\sqrt{M^2+2k}} - 1\right) & \text{if } L \geq 0 \end{cases}$$

The following lemma, which is crucial in the proof of the previous Theorem, is proved in Appendix A.

Lemma 2.3 Define for $k > 0$ and $L, M \in \mathbb{R}$

$$I(k, L, M) = \int_0^{\infty} \Phi \left( M \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} dx,$$

where $\Phi$ is the distribution function of a standard normal random variable. Then

$$I(k, L, M) = \begin{cases} e^{-L(M-\sqrt{M^2+2k})} \left(\frac{M}{\sqrt{M^2+2k}} + 1\right) & \text{if } L < 0 \\ \frac{1}{\sqrt{2\pi k}} e^{-\frac{1}{2} - L(M+\sqrt{M^2+2k})} \left(\frac{M}{\sqrt{M^2+2k}} - 1\right) & \text{if } L \geq 0 \end{cases}$$

The conversion option value is equal to that of a European-Canadian option with interest rate $q$, zero dividend yield, volatility $\sqrt{\sigma^2 - 2\rho\sigma v + v^2}$ and strike price $V_0$. The Canadian options were introduced by Carr [2] as a way of finding approximate values of the vanilla American put option. See also Kimura [8], and in particular his Proposition 2.

3 Numerical results

This section investigates the effects of changing four key parameters on the conversion option value:

- The expected time to conversion ($E[\tau] = 1/\gamma$).
- The size of $P_0$ relative to the size of $V_0$, i.e. if the option at $t = 0$ is in-, at-, or out-of-the-money.
- The volatility of the housing cooperative property value ($\sigma$).

Table 4, 5 and 6 illustrate the effects of changing expected time to conversion $E[\tau]$ and changing volatility $\sigma$ when the static present value for the regulated property is set to the normalized size of $V_0 = 100$. Table 4 illustrates different option values when the value of $P_0 = 150$, i.e. when the option is (deep) in-the-money. This might be the case in very attractive areas in central Stockholm.

7The other model parameters, which are assumed to be the same in the tables 4, 5 and 6, are set to: the risk-free rate $r = 1\%$, the expected growth rate of the NOI $g = 2\%$, the volatility of the NOI $\nu = 5\%$, the expected growth rate of the housing cooperative property value $\mu = 8\%$, the correlation $\rho = 0.2$ and the market price of NOI risk $\lambda_X = 0.2.$
Table 4. Conversion option values when option is (deep) in-the-money.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$E$</th>
<th>1 week</th>
<th>1 month</th>
<th>1 year</th>
<th>2 year</th>
<th>3 year</th>
<th>4 year</th>
<th>5 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>50.03</td>
<td>50.11</td>
<td>51.36</td>
<td>52.69</td>
<td>53.98</td>
<td>55.24</td>
<td>56.46</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>50.01</td>
<td>50.06</td>
<td>50.69</td>
<td>51.42</td>
<td>52.19</td>
<td>52.97</td>
<td>53.76</td>
<td></td>
</tr>
<tr>
<td>15%</td>
<td>50.01</td>
<td>50.04</td>
<td>50.56</td>
<td>51.38</td>
<td>52.31</td>
<td>53.26</td>
<td>54.22</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>50.01</td>
<td>50.03</td>
<td>50.75</td>
<td>52.04</td>
<td>53.43</td>
<td>54.79</td>
<td>56.12</td>
<td></td>
</tr>
<tr>
<td>25%</td>
<td>50.01</td>
<td>50.02</td>
<td>51.23</td>
<td>53.20</td>
<td>55.17</td>
<td>57.04</td>
<td>58.80</td>
<td></td>
</tr>
<tr>
<td>30%</td>
<td>50.00</td>
<td>50.02</td>
<td>51.96</td>
<td>54.72</td>
<td>57.32</td>
<td>59.70</td>
<td>61.90</td>
<td></td>
</tr>
</tbody>
</table>

The initial price of housing cooperative property is $P_0 = 150$, and $V_0 = 100$. Different option values as expected time to conversion and the volatility of the housing cooperative property value vary.

Table 5. Conversion option values when option is at-the-money.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$E$</th>
<th>1 week</th>
<th>1 month</th>
<th>1 year</th>
<th>2 year</th>
<th>3 year</th>
<th>4 year</th>
<th>5 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>0.32</td>
<td>0.70</td>
<td>2.97</td>
<td>4.65</td>
<td>6.13</td>
<td>7.48</td>
<td>8.76</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>0.51</td>
<td>1.07</td>
<td>3.95</td>
<td>5.77</td>
<td>7.23</td>
<td>8.50</td>
<td>9.65</td>
<td></td>
</tr>
<tr>
<td>15%</td>
<td>0.73</td>
<td>1.53</td>
<td>5.45</td>
<td>7.80</td>
<td>9.64</td>
<td>11.20</td>
<td>12.59</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>0.97</td>
<td>2.02</td>
<td>7.07</td>
<td>10.05</td>
<td>12.34</td>
<td>14.27</td>
<td>15.97</td>
<td></td>
</tr>
<tr>
<td>25%</td>
<td>1.20</td>
<td>2.51</td>
<td>8.74</td>
<td>12.37</td>
<td>15.14</td>
<td>17.46</td>
<td>19.48</td>
<td></td>
</tr>
<tr>
<td>30%</td>
<td>1.44</td>
<td>3.01</td>
<td>10.43</td>
<td>14.72</td>
<td>17.96</td>
<td>20.66</td>
<td>23.00</td>
<td></td>
</tr>
</tbody>
</table>

The initial price of housing cooperative property is $P_0 = 100$, and $V_0 = 100$. Different option values as expected time to conversion and the volatility of the housing cooperative property value vary.

Table 6. Conversion option values when option is out-of-the-money.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$E$</th>
<th>1 week</th>
<th>1 month</th>
<th>1 year</th>
<th>2 year</th>
<th>3 year</th>
<th>4 year</th>
<th>5 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.10</td>
<td>0.29</td>
<td>0.58</td>
<td>0.93</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>0.00</td>
<td>0.00</td>
<td>0.08</td>
<td>0.36</td>
<td>0.74</td>
<td>1.19</td>
<td>1.65</td>
<td></td>
</tr>
<tr>
<td>15%</td>
<td>0.00</td>
<td>0.00</td>
<td>0.32</td>
<td>1.02</td>
<td>1.79</td>
<td>2.57</td>
<td>3.33</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>0.00</td>
<td>0.00</td>
<td>0.78</td>
<td>2.04</td>
<td>3.27</td>
<td>4.43</td>
<td>5.52</td>
<td></td>
</tr>
<tr>
<td>25%</td>
<td>0.00</td>
<td>0.01</td>
<td>1.45</td>
<td>3.32</td>
<td>5.02</td>
<td>6.57</td>
<td>8.00</td>
<td></td>
</tr>
<tr>
<td>30%</td>
<td>0.00</td>
<td>0.02</td>
<td>2.26</td>
<td>4.77</td>
<td>6.95</td>
<td>8.88</td>
<td>10.61</td>
<td></td>
</tr>
</tbody>
</table>

The initial price of housing cooperative property is $P_0 = 75$, and $V_0 = 100$. Different option values as expected time to conversion and the volatility of the housing cooperative property value vary.

Table 4 shows, that if the expected time to conversion is only one week, the value of the option is close to its intrinsic value ($= P_0 - V_0 = 150 - 100 = 50$). Furthermore, we can see that the option value increases with longer expected time to conversion, implying that the time value of the option increases with expected time to conversion. This characteristic is also similar to how the value of basic plain vanilla options (e.g. European call or put options) behave as time until expiration increases, since more time to expiration allows more chance for the option to be even more in-the-money.
The volatility of the housing cooperative property value ($\sigma$) is an interesting parameter. In contrast to how basic plain vanilla options behave, the conversion option value does not increase monotonically with increasing volatility; for low expected time to conversion, the value of the option actually decreases with higher volatility. However the changes in option values are nearly economic insignificant when expected time to conversion is one year or less. The impact of changing volatilities increases slightly as the expected time to conversion increases. As expected time to conversion increases from 2 to 5 years, we can notice an increasing ‘smile’ effect. For instance, when the expected time to conversion is 5 years, the option value is 56.46 when the volatility is 5%. Then it decreases to 53.76 as the volatility increases to 10%. Then the option value starts to increase with increasing volatility, and reaches the highest value of 61.90 for the highest assumed values of expected time to conversion and volatility.

Although Table 4 indicates that the option value increases with expected time to conversion, the change is not very dramatic: the increase from the lowest value of 50 to 61.90 represents an increase of only slightly more than 20%. Indeed, when the expected time to conversion ranges from one week to five years, and the option is deep in the money as in this case, the increase in option value is modest.

In contrast, the percentage effect of changing the expected time to conversion on the option value is much higher when the option is at-the-money (recall that for basic plain vanilla options, the time value reaches its maximum when the option is at-the-money). Table 5 shows that when the option is at-the-money (here, when $P_0 = V_0 = 100$), the option value is almost negligeble when the expected time to conversion is one month or less. However, as the expected time to conversion increases, the option value (represented by the time value only, since the intrinsic value max($P_0 - V_0$, 0) is zero), increases fastly. The time values in Table 5 are about two to three times higher than those in Table 4.

We can also notice that the option values seem to increase monotonically with increasing volatility, which is a general characteristic with natural economic interpretation of basic plain vanilla options.

Finally, Table 6 presents option values when we assume that the option is (rather deep) out-of-the-money: ($P_0 - V_0 = 75 - 100 = -25$), which probably represents an odd case. However, the economic interpretation is quite intuitive: when the option is (deep) out-of-the-money, the value of the option is given by its time value only, since the intrinsic value is zero. When the option is deep out-of-the-money, it is not worth anything when the expected time to conversion is small – not much positive can happen even if the volatility of the housing cooperative property is high. In other words, although the combined effect of higher volatility and larger expected time to conversion seem to increase the option value monotonically, we must assume very high price volatility in order to obtain somewhat economically significant option values, only then can an unlikely event happen.
4 Conclusion

The relatively low valuation yields of multifamily rental properties located in the attractive areas, especially those located in inner-city Stockholm, indicate that the market takes into consideration the possibility for property owners to either sell their properties to housing cooperatives, or that there will be a change in the legislation that will make it possible for rents to gradually be adjusted to market rent levels.

In this paper we use a real options approach to derive a closed-form valuation formula for the option an owner of an income producing multi-family property has to sell it to a housing cooperative. A housing cooperative conversion process consists of several steps involving important tenure choice, financial, legal, and technical aspects, as well as political dimensions. Such factors make it difficult for both property owners (and tenants) to know in advance when a housing cooperative conversion might occur. Therefore, in contrast to option valuation formulas that assumes that an option has a predetermined expiration date, we assume that the expiration date (i.e. when the tenants buy the property) is not known in advance. In order to consider this, we let the expected time to maturity, which is the day when the tenants purchases the property from their landlord, to be a random variable. Our model allows the expected time to conversion to vary from one day to practically an infinite length of time.

Similar to the Black-Scholes model, our model also makes a number of assumptions that might be restrictive. For instance, the option valuation model assumes that the volatilities are constant for the life of the option. Given such restrictions, it is therefore important to perform sensitivity analysis that examines the effect of changing parameter values. With our numerical examples, we obtain insights about how the conversion option values changes with respect to changes in four key parameters. We find that when the option is deep in-the-money, that is, when the price households are willing to pay to live in the property’s dwellings sharply exceeds the regulated rent levels, the intrinsic value of the option seems to constitute the major share of the total option value. In other words, the time value of the option seems to be relatively insensitive to the choice of price volatility of housing cooperative properties, as well as the expected time to conversion. However, when the option is at-the-money, the time-value of the options seems to be more sensitive to changes in the price volatility and the expected time to conversion. Finally, when we assume that the option is (rather deep) out-of-the-money, the time value again becomes less sensitive to changes in the price volatility and expected time to conversion. Thus, the findings above indicate that our option valuation model has characteristics that are similar to those of basic plain vanilla options.

Naturally, the other parameters of the option valuation model, besides the parameters we have focused on in this paper, affect the option value. Although not presented above, our model indicates that the option value increases if the net operating income risk (volatility of NOI) increases. Furthermore, the option value decreases for higher values of the expected growth rate of net operating incomes, illustrating that the housing cooperative conversion option
value becomes less attractive when the growth rate of rents is high.

The insights we have gained from this paper is that the real options approach suggested here may be especially useful to explicitly conceptualize the problem of valuing a rental property with embedded options to switch it to another type of property.
References


A Proof of Lemma 2.3

Lemma A.1 Define for $k > 0$ and $L, M \in \mathbb{R}$

\[ I(k, L, M) = \int_0^\infty \Phi \left( M \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} dx, \]

where $\Phi$ is the distribution function of a standard normal random variable. Then

\[ I(k, L, M) = \begin{cases} \frac{e^{-L(M-\sqrt{(M^2+2k)})}}{2k} \left( \frac{M}{\sqrt{M^2+2k}} + 1 \right) & \text{if } L < 0 \\ \frac{1}{k} + \frac{e^{-L(M+\sqrt{(M^2+2k)})}}{2k} \left( \frac{M}{\sqrt{M^2+2k}} - 1 \right) & \text{if } L \geq 0 \end{cases} \]

In the proof of the lemma we will use the following two results. For $a > 0$ and $b \geq 0$ we have

\[ \int_0^\infty x^{-\frac{1}{2}} e^{-\frac{1}{2}(ax+\frac{a}{2})} dx = \sqrt{\frac{2\pi}{a}} e^{-\sqrt{ab}}. \]

For $a > 0$ and $b > 0$ we have

\[ \int_0^\infty x^{-\frac{3}{2}} e^{-\frac{1}{2}(ax+\frac{a}{2})} dx = \sqrt{\frac{2\pi}{b}} e^{-\sqrt{ab}}. \]

Proof. We divide the proof depending on the value of the parameter $L$. First we take $L = 0$. Integration by parts in this case yields

\[ \int_0^\infty \Phi \left( M \sqrt{x} \right) e^{-kx} dx = \left[ \Phi \left( M \sqrt{x} \right) \frac{e^{-kx}}{-k} \right]_0^\infty - \int_0^\infty \frac{M}{2\sqrt{x}} \varphi \left( M \sqrt{x} \right) e^{-kx} dx \]

\[ = \frac{1}{2k} + \frac{M}{2k\sqrt{2\pi}} \int_0^\infty x^{-\frac{1}{2}} e^{-\frac{1}{2}(M^2+2k)x} dx \]

\[ = \frac{1}{2k} \left( 1 + \frac{M}{\sqrt{M^2+2k}} \right). \]
Now take $L > 0$. In this case integration by parts implies
\[ \int_{0}^{\infty} \phi \left( \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} \, dx = \left[ \phi \left( \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} \right]_{0}^{\infty} \]
\[ - \int_{0}^{\infty} \left( \frac{M}{2 \sqrt{x}} - \frac{L}{2 \sqrt{x}^3} \right) \varphi \left( \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} \, dx \]
\[ = \frac{1}{k} + \frac{1}{2k \sqrt{2 \pi}} \int_{0}^{\infty} \left( M x^{-\frac{1}{2}} - L x^{-\frac{3}{2}} \right) e^{-\frac{1}{2} \left( M^2 x + 2 ML + \frac{L^2}{x} \right) - k^2 x} \, dx \]
\[ = \frac{1}{k} + \frac{e^{-ML}}{2k \sqrt{2 \pi}} \int_{0}^{\infty} \left( M x^{-\frac{1}{2}} - L x^{-\frac{3}{2}} \right) e^{-\frac{1}{2} \left( M^2 + 2k \right) x^{-\frac{1}{2}} - \frac{L^2}{2x}} \, dx \]
\[ = \frac{1}{k} + \frac{e^{-L(M + \sqrt{(M^2 + 2k) \pi})}}{2k \sqrt{2 \pi}} \left( \frac{M}{\sqrt{M^2 + 2k}} - 1 \right). \]

Finally we look at $L < 0$. In this case
\[ \int_{0}^{\infty} \phi \left( \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} \, dx = \left[ \phi \left( \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} \right]_{0}^{\infty} \]
\[ - \int_{0}^{\infty} \left( \frac{M}{2 \sqrt{x}} - \frac{L}{2 \sqrt{x}^3} \right) \varphi \left( \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} \, dx \]
\[ = 0 + \frac{1}{2k \sqrt{2 \pi}} \int_{0}^{\infty} \left( M x^{-\frac{1}{2}} - L x^{-\frac{3}{2}} \right) e^{-\frac{1}{2} \left( M^2 x + 2ML + \frac{L^2}{x} \right) - k^2 x} \, dx \]
\[ = \frac{e^{-ML}}{2k \sqrt{2 \pi}} \int_{0}^{\infty} \left( M x^{-\frac{1}{2}} - L x^{-\frac{3}{2}} \right) e^{-\frac{1}{2} \left( M^2 + 2k \right) x^{-\frac{1}{2}} - \frac{L^2}{2x}} \, dx \]
\[ = \frac{e^{-L(M - \sqrt{(M^2 + 2k) \pi})}}{2k \sqrt{2 \pi}} \left( \frac{M}{\sqrt{M^2 + 2k}} + 1 \right). \]

Combining the three parts proves the lemma. \( \square \)