Closed-loop control and identification of resistive shell magnetohydrodynamics for the reversed-field pinch

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Abstract

It is demonstrated that control software updates for the magnetic confinement fusion experiment EXTRAP T2R can enable novel studies of plasma physics. Specifically, it is shown that the boundary radial magnetic field in T2R can be maintained at finite levels by feedback. System identification methods to measure in situ magnetohydrodynamic stability are developed and applied with encouraging results. Subsequently, results from closed-loop identification are used for retooling the T2R regulator. The track of research here pursued could possibly be relevant for future thermonuclear fusion reactors.
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1 Preamble

1.1 Sustainable energy

Energy demand on earth is high and keeps soaring. Many densely populated societies are currently underdeveloped and require energy infrastructure, among other things. The contemporary trend seems to be a gradual realization that global politics must successfully negotiate and incorporate climate-change thinking, and action, in the agenda to curb the economy to battle this challenge in a thought-through fashion. This is a result of the emerging scientific consensus that human footprint is disturbing the ecosystem to a dangerous degree. One speaks of sustainability, a concept that signifies long-term perspective. A key component in any sustainable infrastructure is the source of energy, electrical or otherwise. Most plausibly, energy for an “environmentally friendly”, or ecologically integrated (i.e. within some sort of “allowed operating space”), human way of life will be produced from a mixed portfolio: solar power (photovoltaic and thermal), hydroelectric, wind, pressure-retarded osmosis, farmed algae, fast breeder nuclear fission, and so on, and perhaps nuclear fusion. Developments in energy distribution, such as “smart grids”, and progression in techniques for canning and storing of energy, such as fuel cells (using low carbon footprint hydrogen, of course) are also needed.

Technically, fusion power is not renewable (and neither is the sun itself, and so forth). In practice this is only a problem for an unfathomable far-off future. A fusion reactor feeds on the hydrogen isotope deuterium, which can be found in common seawater. There is plenty of seawater.

1.2 Fuel & logistics of a fusion power plant

Suppose the fusion power plant can be realized. Three central questions arise: (i) is the power output sufficient? (ii) what is fuelling the plant? and (iii) what toxic stuff does it output? In answering these questions we will see that the quest for fusion power is a justified venture.

Questions (i) and (ii) are jointly answered in: the most optimistic scenario for fusion power is that the deuterium-deuterium reaction can be controlled in a reactor. Considering that there are, on average, 33 mg of deuterium in each metric tonne of seawater, and that fusing 1 mg deuterium releases $100 \times 10^3$ kWh of energy, it is estimated that the holy-grail deuterium reactor could provide $30 \times 10^3$ kWh per day per person for $1 \times 10^6$ years for $60 \times 10^9$ people. The current average US level of energy consumption is less than $3 \times 10^2$ kWh per day per person.

Deuterium can be extracted from sea water by distillation, electrolysis and harnessing differences in chemical reaction rates. Methods for production of deuterium and heavy water, D$_2$O (a fission moderator), are already developed by the fission reactor industry, among others. Deuterium is routinely used in various scientific measurement techniques (isotope tracer, neutron scattering, nuclear magnetic resonance calibration).

Switching to (iii). Fusion power is very clean. The nuclear reaction itself outputs an energetic neutron and helium (deuterium-tritium fusion). Helium is harmless. Neutrons can however activate materials. Parts of the reactor will...
need to be treated as radioactive waste. In comparison to current generation fission reactor waste, the situation is trivial. Less toxic waste mass, far shorter half-life. In fact, the conditions for fusion are that after 100 years, the radioactive toxicity of a decommissioned fusion power plant would be almost negligible [74].

1.3 Forecasts on and history of fusion power

Uttered by David J.C. MacKay:

_Fusion power is speculative and experimental. I think it is reckless to assume that the fusion problem will be cracked, but I'm happy to estimate how much power fusion could deliver, if the problem is cracked._

in his recent book on sustainable energy [87]. The speculative numbers, which were presented in subsection 1.2, are “good news” to paraphrase [87].

Not surprisingly, scepticism had to counterweight the initial enthusiasm occasionally emitted at the dawn of nuclear fusion energy research. The prognostication

...I venture to predict that a method will be found for liberating fusion energy in a controlled manner within the next two decades.

... by Homi Bhabha, the president of the United Nations Conference on the Peaceful Uses of Atomic Energy 1955 [71], turned out, in honest hindsight, to be wrong. The official declassification of nuclear fusion research occurred at the IAEA Geneva Conference 1958. Since then tremendous progress has been logged [71, 74, 89], but, admittedly, a commercial grade fusion power plant is yet to be engineered. Sceptics say, only half-jokingly, that fusion power will always linger 20 years ahead, a receding and unattainable goal. Fusion power has been achieved in short pulses, but no experiment to this date has breached the break-even condition \( Q = P_{\text{out}}/P_{\text{in}} > 1 \). In 1994 the TFTR experiment in Princeton (New Jersey, US) achieved 11 MW fusion power [3, 117] and in 1997 JET (Culham, UK) set the current world record of 16 MW at peak \( Q = 0.64 \), for about one single second [2, 67]. Research suggests that the tokamak [127], a Russian invention from the 1950s [110], plausibly can be scaled up to a capable reactor [109].

The classic 1993 computer game _Sim City 2000_ (SC2k), an urban infrastructure development simulator, allows the player to build nine different kinds of power plants to electrify his/her carefully nurtured society [5]. Two of these are “futuristic” in character: fusion power and satellite microwave power, the latter an advanced form of solar power. The technology-projection of SC2k is that fusion power can be deployed in year 2050, approximately (exact availability of this “R&D-invention” is controlled by random numbers during gameplay). In addition, SC2k suggests that satellite microwave power plant is available \( \sim 2020 \).

The official forecast of the international thermonuclear experimental reactor (ITER) [109] is to achieve net power gain, at \( Q \approx 10 \), by 2030 [1, 72], in pulses of

\[ Q = \frac{P_{\text{out}}}{P_{\text{in}}} \]
several minutes. Basically, the crews of ITER and SC2k agree on the timeline of commercial fusion power plants. The step beyond ITER, appropriately named DEMO [1], will be connected to the grid around 2040, according to the “fast-track” scenario [1].

Interestingly, the prediction market (PM) [13, 131] Foresight Exchange [10] trades contracts on the claim that a nuclear fusion power plant will sell energy by the end of 2045. At the time of writing the price is 74 (fake-money PM) for the expiry-value of 100 or 0 depending on the outcome. This is essentially the current market consensus on the probability of the claim (~0.74), supposedly reflecting the aggregate crowd intelligence on the matter. This value should react according to the progress and/or failure of on-going fusion research, but possibly also to some extent by PM-trader speculation. A second fusion-related contract trade, on the lightly cynical claim that a major disaster occurs in a fusion research lab before commercial viability, currently sells at 48.

In conclusion, after quoting various prophecies of renowned scientists, computer geeks and the betting market, fusion power (still) elusively lingers a few decades ahead.
2 Thermonuclear fusion physics and reactor engineering

2.1 Fundamentals and feasibility

At very high thermal energies nuclei will occasionally overcome the electrostatic force in the sense that the internucleus distance can become so small (a few femtometers, $10^{-15}$ m) so that electromagnetism is dwarfed by the charge-independent strong interaction, the nuclear force (about $10^2$ times stronger) [122]. If the nuclei are of certain sorts, they will form a stable compound nucleus; they fuse.

Light atomic nuclei can fuse when bumping into each other with a proper velocity [113, 122, 127]. If too slow, the nuclei never come close due to coulomb repulsion. If too fast, the probability of fusion diminishes. The maximum reaction-rate, i.e. number of fusing particles at a given density per unit of time, occurs at a very high temperature [33].

The reactions involving deuterium $D = ^2\text{H}$ and tritium $T = ^3\text{H}$ involve relatively low temperature. Fusion engineering research usually focuses on the nuclear reactions involving $D$ and $T$ [92]

\begin{align}
D + T & \rightarrow ^4\text{He} + n + 17.6 \text{ MeV} \quad (1a) \\
D + D & \rightarrow ^3\text{He} + n + 3.3 \text{ MeV} \quad (1b) \\
D + D & \rightarrow T + H + 4.0 \text{ MeV} \quad (1c) \\
D + ^3\text{He} & \rightarrow ^4\text{He} + H + 18.3 \text{ MeV} \quad (1d)
\end{align}

The first of these, reaction (1a), is plausibly the easiest to achieve and most efficient to maintain [92], yields high energies, and has already been obtained in a few tokamak experiments [67, 117].

A drawback with tritium is that there are no large quantities of it on earth. Tritium is radioactive with a short half-life of $\sim 12.3$ years, hence continuous replenishing is required. For a D-T-reactor, the remedy is to use neutrons from (1a) to drive a breeding reaction

\[ ^6\text{Li} + n \rightarrow ^4\text{He} + T + 4.8 \text{ MeV} \quad (2) \]

for conversion of lithium (injected into the wall of the reactor) to tritium (that can be collected and puffed into the reaction chamber). Lithium repositories exist all over the earth, but ultimately limits the available fuel for (1a). Other industries might compete for the finite lithium resources, predominantly battery producers.

Thinking far ahead then, more desirable are the branching deuterium-only reactions (1b)-(1c). Since $D$ is the abundant fuel source and that the extra engineering required to nurse (2) can be skipped, they could very well be worth the effort, although they require even higher temperature than does (1a).

It is in theory also possible to avoid neutrons. Thermonuclear reactions that are low on neutron production are sometimes called aneutronic [7, 111] and have a great advantage of not activating material surrounding the reaction chamber.
Typical focus is on a reaction involving a proton and a boron nuclei\(^2\) [92]

\[
^{11}\text{B} + \text{p} \rightarrow 3 \times ^4\text{He} + 8.7 \text{MeV} \tag{3}
\]

but unfortunately the peak reaction-rate of (3) occurs at a very high temperature. Also, (3) needs a confinement capability a few orders higher than does (1a). Purportedly, a technique for (3) could be a dense plasma focus (DPF) [81]. Conceptual fusion systems based on \(p - ^{11}\text{B}-\text{DPF}\) [83] claim the potential for direct conversion of the reaction products (charged particles) to electricity (i.e. sidestepping the thermodynamic and mechanical conversion losses in a heat-exchanger/turbine/generator chain). The prospects of (3) have recently been remarke\(\text{red}\) [8].

Finally, cold fusion (not thermonuclear), the notion that nuclei somehow fuse more or less at room temperature, remains highly controversial. No claim to this date has been reproducible.

### 2.2 Enter the plasma

The very high temperature required to obtain fusion of hydrogen isotope nuclei is not practically compatible with any solid, liquid or (normal) gas state of matter. At these energies, all potential fusion atoms are stripped of their electrons and the result is the plasma state: a gas-like state of electrically charged particles. An important requirement here is that the overall charge should be approximately neutral. If \(n_i\) is the density of some type of ion with charge \(q_i\), \(n_e\) the density of electrons and \(q_e = -e\) the electron charge, then it should hold

\[
0 \approx \int_V (n_i q_i + q_e n_e) \, d^3x \quad \text{over the plasma volume } V.
\]

Hence, a proton beam in the LHC particle physics facility CERN does not qualify as a plasma.

The basic difference between a normal gas state and the plasma state is the type of dominant particle interaction [40, 47]. Atoms are seldom ionized in normal gases. Consequently the multibody interactions are very short range. Gas atoms essentially have to collide with each other to change trajectories. Interplay in a plasma, in contrast, is due to the particle charges. The electrostatic force (either attractive or repellent) decays as \(\sim |\Delta \text{x}_{12}|^{-2}\) if \(\Delta \text{x}_{12}\) denotes the vector distance between charged particles 1 and 2. Compare this with e.g. the attractive part of the gradient of the Lennard-Jones potential for neutral atoms \(\sim |\Delta \text{x}_{12}|^{-7}\) [16]. Furthermore, all moving charged particles are in addition deflected by magnetic fields, and they in turn produce magnetic fields.

Since charges are roaming freely, a plasma can sustain electrical currents. Charge motion also screens the electrostatic potential; it is almost zero beyond a distance \(\lambda_D\); the Debye length. Let the macroscopic length dimension of the charged-particle swarm be \(L\) and the number density \(n = n_i \approx n_e\) then if

1. \(\lambda_D \ll L\) (the typical interaction range covers a tiny part of the domain, otherwise the bulk can be dominated by particular boundary effects)

2. \(n \lambda_D^3 \gg 1\) (there must be plenty of particles within the Debye volume for the Debye length to make sense)

\(^2\)Yes, it looks like a weird form of fission, but the hydrogen nuclei (proton) first merge with the boron nuclei and then this newly-fused complex promptly splits up in 3 helium nuclei and release energy.
3. The dominant interaction should be of coulomb type (i.e. the gas must be almost fully ionized).

we basically have a plasma \[40\]. A magnetized plasma is a plasma subject to relatively strong magnetic fields. The particular plasma regime relevant for thermonuclear fusion is studied in the subfield of fusion plasma physics \[113\]. Plasmas need not be extremely hot (as in thermonuclear), they can also be cold, and even complex \[91\]. A newcomer is the misty plasma \[43\].

Research in astrophysics is largely concerned with the plasma state, in various forms, the most proximal being the magnetosphere which is created by the solar wind interacting with the earth’s magnetic field. Actually, we are often told that the plasma state is the most common state of matter in the universe (e.g. stellar interior, nebulae).

2.3 The ballpark according to Lawson

A simple macroscopical condition for thermonuclear fusion can be sketched as follows \[20, 127\]. Consider a volume of fusible matter. Denote the thermal energy density by \(W\). The relation to temperature \(T\) is

\[W = \frac{3}{2} k_B (n_D + n_T + n_e) T = 3n k_B T\]  (4)

in isotropic conditions, where \(2n_D = 2n_T = n_e = n\) and \(k_B\) being the Boltzmann constant. Equation (4) assumes macroscopic charge neutrality and an equal mix of deuterium and tritium and the electron number density \(n\). The rate of change of (4) can be partitioned

\[\dot{W} = \kappa P_{fus} + P_{ext} - P_{loss}\]  (5)

with \(P_{fus}\) the fusion power density, \(\kappa\) the fraction of fusion power that re-heats the volume (the \(\alpha\)-particle), \(P_{ext}\) the power density from external heating sources, and \(P_{loss}\) the power density that escapes the volume (mainly bremsstrahlung \[31\]). Define

\[\tau_E = \frac{W}{P_{loss}}\]  (6)

the energy confinement time, and

\[Q = \frac{P_{fus}}{P_{ext}}\]  (7)

the power amplification ratio. Equations (5), (6) and (7) in a power balance \(\dot{W} = 0\) becomes

\[\frac{W}{\tau_E} = (\kappa + \frac{1}{Q}) P_{fus}\]  (8)

It holds

\[P_{fus} = \frac{1}{4} n^2 \langle \sigma v \rangle_{DT} E_{DT}\]  (9)

where \(\langle \sigma v \rangle_{DT}\) is the temperature-dependent reaction rate \[33\] for fusion and \(E_{DT}\) the energy per fusion reaction, for deuterium-tritium. Plugging in (9) in (8) and refurbishing results in

\[n\tau_E = \frac{12 k_B}{(\kappa + Q^{-1}) E_{DT}} \frac{T}{\langle \sigma v \rangle_{DT}}\]  (10)
which is one form of the Lawson criterion. To see what (10) implies we should plot $T \langle \sigma v \rangle_{DT}^{-1}$ as a function of $T$ [33, 70, 127], which is displayed in figure 1. The limit $Q \to +\infty$ of (10) is called the ignition condition, which means according to (7) that the fusion process is self-maintained (thrives without external heating, $P_{ext} \to 0$). This only makes sense for $\kappa > 0$. For the deuterium-tritium reaction: $E_{DT} = 17.6$ MeV and $\kappa = 3.5/17.6 \approx 0.2$. Boltzmann [94]: $k_B = 8.617385$ eV/K. Hence

$$n\tau E \geq \frac{12k_B}{3.5\text{MeV}} \min_T \left( T \langle \sigma v \rangle_{DT}^{-1} \right) \approx 1.5 \times 10^{20} \text{s/m}^3$$

(11)

for ignition ($Q = \infty$). Note that the “optimal” temperature $T^* \approx 25.7$ keV, illustrated by the vertical dashed black line.

The essence of (11) is that thermonuclear D − T-fusion burn requires a few hundred million degrees kelvin in temperature, and a number density times confinement time product $n\tau E \gtrsim 1.5 \times 10^{20} \text{s/m}^3$. Evidently (11) can be fulfilled in two fundamentally different ways: either low density $n$ and long confinement $\tau E$ or high density and short confinement. Both ways are tricky. See section 2.4.

### 2.4 Reactor engineering approaches

It is not necessary to have an infinite $Q$, see equation (7) for a useful reactor. A power amplification in the range $10 - 100$ could be good enough for commercial energy production. As a consequence, inequality (11) is pushed down by approximately a factor of two [127], say: $n\tau E \gtrsim 0.75 \times 10^{20} \text{s/m}^3$. This is still a difficult task to achieve. Figure 2 displays an assortment of approaches to devise systems to make this happen.
Figure 2: Approaches to achieve thermonuclear fusion. Some major (and fringe) experimental facilities are included.
2.4.1 Gravitational confinement fusion

Loosely speaking, a star is a fusion reactor. It is relying on its enormous mass to confine the reaction; hence *gravitational* confinement fusion (GCF). The thermonuclear chain of reactions proliferating in the environment of the star is different than what is considered relevant for any engineered fusion reactor of smaller dimensions. Of course, solar energy concepts are in a quite direct sense attempts to utilize the radiated power originating from solar fusion. Just outside the atmosphere of planet earth, the incident power density from the sun is a handsome $1.4 \text{ kW/m}^2$.

2.4.2 Magnetic confinement fusion

Charged particles in motion are deflected by magnetic field lines \[31, 112\]. The force exerted on the particle is perpendicular to the velocity and the magnetic field. This force results in the particle gyrating along magnetic field lines. So the basic idea is painfully trivial: (i) form closed magnetic field lines (ii) inject charged particles into this field (iii) heat it to thermonuclear temperatures (iv) absorb those conveniently non-charged precious energetic fusion neutrons escaping the magnetic fields. A good idea is to use some sort of blanket as absorber and heat-exchanger, turbines and generators. Subsection 2.5 looks further into this approach.

2.4.3 Inertial confinement fusion

The ballpark of Lawson house a corner with small confinement time, provided the pressure is abnormally high. One can imagine a bunch of ions being compressed so much that the number of fusion neutrons produced in the very brief time this state will persist are significant for a net gain of energy. It costs energy to compress the particles. The particles will stay compressed for a finite time since they have mass, i.e. inertial confinement fusion (ICF). This approach is pursued on large-scale with government funding$^3$ and fresh experimental facilities, in Livermore USA (National Ignition Facility, NIF) and France (Laser Mégajoule, LMJ \[11\]). NIF is, at the time of writing, making fast progress \[63\] and aims for ignition later this year (2010). NIF and LMJ both utilize extremely powerful lasers focused onto a tiny hydrogen pellet in a vacuum chamber. The laser pulse is very short and delivers $\sim 1 \text{ MJ in } \sim 1\text{ ns}$. The state of matter exposed to this kind of infernal battery deserves a dedicated subfield of physics: High energy-density physics (HEDP) \[48\].

The step from successful ICF ignition to a working ICF power plant is very non-trivial.

2.4.4 Magnetized target fusion

Loosely speaking, in a midfield between MCF and ICF there is an area where one can attempt to combine the best parts of the two. Pundits of magnetized

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$^3$Traditionally, at least, ICF research is related to military thermonuclear weapons programmes. This is because the transient conditions with exceptionally high energy-density are also found in hydrogen and conventional atomic payloads, and testing of thermonuclear weapons is forbidden by international agreements (if your government respect this).
target fusion (MTF) venture to compress the fuel as in ICF but ease the pressure requirements by prolonging the fuel confinement by concerting inertial and magnetic confinement [73]. The idea is mainly to focus some compressive action onto a magnetized fuel target. In this way, the argument goes, both “ICF-part” and “MCF-part” of the reactor will be technologically simpler and will cost much less than either of the pure ICF- or MCF-plants. Recent publicity has been aimed at a novel and peculiar acoustic-compression idea [9].

2.5 Main branches in MCF

MCF is unarguably the most developed fusion concept to this date. MCF sports a variety of toroidal creations. Four main types of experiments are presented here. Cartoons of their respective outer magnetic surfaces are seen in figure 3. The illustrations have been generated, where applicable, from the axisymmetric shaped cross-section parameterization [128]

\[
R = R_0 - b + (a + b \cos \theta) \cos (\theta + \delta \sin \theta) \quad (12a)
\]

\[
Z = \kappa a \sin \theta \quad (12b)
\]

with \(R_0, a\) respectively the major and minor radii; \(\kappa\) the ellipticity, \(b\) the indentation and \(\delta\) the triangularity. The stellarator cannot be represented by an axisymmetric shape [12]. In this case some dependence on the toroidal angle \(\phi\) must be incorporated.

2.5.1 Tokamak

The tokamak [15, 127] has almost achieved proper break-even fusion \((Q > 1)\), and the multi-billion-dollar international venture ITER, a tokamak, is designed for \(Q \sim 10\) and 500 MW output power. A fairly typical tokamak is illustrated in figure 3(a). The tokamak has a fundamental problem: its transient operation. The toroidal current that is an integral part of the equilibrium state is inductively driven by ramping the flux through a central solenoid. There are techniques for injecting momentum necessary for driving currents noninductively using electromagnetic wave-power at different frequencies, which all cost lots of energy. Also a fraction of the current can be maintained by the bootstrap effect [127]. Nevertheless, a total solution has not been proven.

2.5.2 Spherical tokamak

By some considered as a more economical tokamak [124]. Two major experiments pursue this high-\(\beta\) variation on the standard tokamak design: NSTX (US) and MAST (UK/Europe, depicted in figure 3(b)), a japanese plant is under development. A spherical tokamak (ST) appears to have some benefits; higher-\(\beta\) aside, it supports a larger fraction of bootstrap current.

2.5.3 Stellarator

Very complex to design and assemble, the stellarator offers an intrinsically MHD-stable and current-free plasma equilibrium. This is due to a cleverly engineered magnetic field produced by non-trivially shaped external coils. The german engineering masterpiece (although highly delayed, and yet to prove
(a) Tokamak; proportions correspond to a typical German ASDEX-U shape; $R_0 = 1.65$ m, $a = 0.5$ m, $\delta = 0.4$, and $\kappa = 1.8$.

(b) Spherical tokamak. A typical cored-apple-shape of the British MAST device; $R_0 = 0.7$ m, $a = 0.5$ m, $\delta = 0.4$, and $\kappa = 2.5$.

(c) Reversed field pinch. Swedish EXTRAP-T2R; always $R_0 = 1.24$ m, $a = 0.18$ m, $\delta = 0$, and $\kappa = 1$.

(d) Stellarator. This is an approximation of the Japanese LHD; essentially a grossly $m/n = 2/10$-perturbed plasma column. Parameterization (12) not appropriate. Dimensions are $R_0 = 3.5$ m, $a = 0.6$ m.

Figure 3: The dominant MCF line-up. The greenish colour, being easy on the eye, has nothing to do with the colour of the plasma, which is transparent in the visible part of the spectrum.
functionable) Wendelstein W7-X, housed in Greifswald, will be the first-ever stellarator based on modular superconducting magnets. It is hard to overstate the complexity of this device [4]. Doubtlessly, the stellarator concept is very appealing for steady-state fusion reactors, but it has yet to show experimental capability to approach the required fusion conditions. The Japanese Large Helical Device (LHD) is cartoonized in figure 3(d).

2.5.4 Reversed-field pinch

Despite recent attention in the journal *Nature* [86], the RFP is by many considered a fringe apparatus when it comes to actual fusion prospects. But contrary opinion exists. Its contributions to progress plasma physics and MHD research are, however, undisputed. Indeed the RFP has breached new ground in e.g. feedback control [37] of RWMs. A specific feature of the RFP is the ohmic heating potential (heating using plasma current only). The chaotic magnetic structure in the RFP results in short energy and particle confinement times. Techniques exist to improve confinement [39] but the RFP is still far from tokamak performance. The RFP has a timeless unpretentious design, see figure 3(c). It can maintain high-β and requires only weak external magnetic fields since it relies on a phenomenon called plasma relaxation [96]. Its design is therefore quite economical.

This licentiate thesis mainly considers magnetic feedback control for RFPs.
3 Magnetic control of tokamaks and RFPs

As seen from Lawson’s criterion, confinement is absolutely central to achieve thermonuclear fusion. Improvement in confinement is dependent on the ability to suppress and mitigate various instabilities that develop as temperature and pressure increase. For tokamaks both equilibrium shaping and auxiliary heating can substantially affect the confinement time. The high-confinement mode, or H-mode, an important operational regime discovered in the early 1990s, is maintained in tokamaks by heating the boundary of the plasma by properly tuned microwaves. However, the emerging picture of a potential tokamak reactor is that it will need further sophistication in terms of active control [126]. During H-mode, as an example, a dangerous type of edge instability tends to burst-out hot plasma somewhat randomly. Unless handled (controlled) the upscaled consequence of this behaviour would, for a reactor-sized machine, be a prohibitive heat- and particle-load on the first-wall material.

Other types of unstable modes that set operational performance limits for tokamak reactors include the resistive-wall mode (RWM) and the neoclassical tearing-mode (NTM). These are of global character and could possibly terminate the entire plasma. RWMs and NTMs are modeled using magnetohydrodynamic (MHD) theory, which is introduced in section 4. MHD equilibrium and stability, in fact, constitute the fundamental toolbox for assessment of MCF designs. Calculations of MHD stability tend to show a discrete spectrum of global plasma eigenmodes, some of which can be controlled by magnetic feedback systems. In particular, the RWM evolves on a time-scale which is easily managed by digital controllers. An RWM-type MHD stability calculation will be written out in section 4 for the RFP, resulting in a simple dynamical model that is manipulated further in a control theory setting, as provided by section 5. This section outlines a less specialized case for control in MCF.

3.1 Fusion plasma control overview

Control of magnetic confinement fusion reactors are typically divided into two distinct topics [20]

**electromagnetic control**: mainly macroscopic stability. Global IMHD stability is a prerequisite for any MCF scheme. However there is no fundamental reason that dictates a magnetic configuration to be *passively* stable [88]. What is crucial is the composite system stability of plasma, external magnetic fields, passive structures and feedback control influence. The central issue of *equilibrium* control belongs to this class. Vertical stabilization of elongated tokamak plasma equilibria is a prime example.

**kinetic control**: mostly regarding transport of heat and particles. For tweaking of kinetic aspects of the MCF plant, global equilibrium MHD stability is obviously a prerequisite, since it provides the environment for the existence of meaningful kinetic properties, such as the global energy confinement time.

We will briefly discuss magnetic control. Subsections 3.2 and 3.3 respectively outline axisymmetric tokamak equilibrium control and RFP RWM control. A short control-oriented introduction to the RFP device EXTRAP T2R, housed
in Stockholm, is provided in subsection 3.4. This section ends with some details on magnetic diagnostics in MCF experiments; 3.5. Descriptions of microwave heating systems, neutral particle injectors and other “actuator” instruments essential for MCF performance [113, 127], are here omitted.

3.2 Tokamak axisymmetric shaping and vertical stabilization

Tokamaks have evolved into vertically elongated, but still bagel-like, shapes [116]. The elongation makes the plasma volume, on average, closer to the high-field side of the toroidal magnetic field. This is beneficial for stability; higher plasma pressure and current can be achieved with these non-circular cross-sections. High elongation also leads to an unstable plasma positioning; and so feedback becomes essential [20, 125, 126]. This unstable mode would make the plasma ring go either up or down (vertical instability) into the vessel roof or floor if not stabilized.

Poloidal field coils can be used for stabilizing the vertical position. The overall objective of being able to shape the plasma surface is typically handled by separation of time-scales. First, the vertical instability is contained by an inner, and obviously mission critical, closed-loop. Second, other poloidal field coils can be considered as tools for morphing the last closed flux surface, on a much longer time-scale [20]. A good example of flexible tokamak plasma shapeshifting is the TCV device in Lausanne, Switzerland.

The problem of plasma shaping inevitably involves real-time reconstruction of tokamak equilibria; a practical nonlinear PDE inverse problem [52].

A tokamak discharge is normally divided into four intervals: (i) breakdown (ii) ramp-up (iii) flat-top and (iv) ramp-down. The flat-top phase is the most important since this is the interval where fusion is supposed to occur. Breakdown deals with the formation/ionization of the plasma. Ramp-up is a steady increase of the plasma current and toroidal magnetic fields. Ramp-down is a gradual shut down of the same quantities. Soft landing and start up of the experiment are very important since vast amount of energy are stored in the current and magnetic fields, if things end abruptly, equipment could shatter. Repair expenses scale with tokamak size; ITER may not afford disruptive termination. Basically then, all the phases (i)-(iv) have dedicated control algorithms.

3.3 RFP success story: multiple RWMs

The “intelligent-shell” (IS), introduced by C.M. Bishop [28], is devised to mimic the boundary of an ideal-conductor. A passive resistively conducting shell is augmented with current carrying filaments controlled by feedback circuits. In its basic mode of operation, the intelligence of IS is very questionable. But recent developments, such as the “clean-mode-control” (CMC) [132], attempt to incorporate physics knowledge into actions.

For RFPs, the simplest basic equation for RWM control inspiration is (128) which was derived above. This model suggests that a proportional feedback should stabilize the system [53], however in reality there are field errors, delays and lags in current amplifiers and nonaxisymmetric shell effects, and so forth. Nevertheless, a classic proportional-integral-derivative (PID) controller does a great job for RFPs. T2R was the first machine to show successful feedback
sustainment of an RFP discharge with multiple unstable RWMs \cite{37}. RFX was not far behind. Prior to installment of feedback control of T2R, the plasma could be maintained for $\tau_{pl} \approx 15\,\text{ms}$, approximately a shell diffusion time, as predicted by \cite{128}. After tuning of the feedback \cite{36}, T2R plasmas can have a life-time up to $\tau_{pl} \approx 90\,\text{ms}$, a limit that does not seem to be related to feedback, but by power supply to the experiment and equilibrium plasma position drifts.

3.4 EXTRAP T2R

This thesis is largely based on experiments with the RFP EXTRAP T2R \cite{35}. Its nonaxisymmetric MHD control capabilities are hinted by the cutaway drawing of figure 4. In this figure, the blue innermost coils are the sensor coils arranged in an equidistributed $4 \times 32$ tile. Any time-varying magnetic flux linking a coil in the array induces a voltage over the coil windings. Time-integration of this signal provides a measure of the surface-averaged radial magnetic field. The red coils, in the same $4 \times 32$ tessellation, are the actuators that are being driven by power audio amplifiers. A current-loop generates a radial linking magnetic flux (and some other stray fields). The yellowish shell is the resistive-shell. It is made of copper and is assembled of two isolated layers, each of width 0.5 mm. The grey toroidal shell is the vacuum vessel.

The coils are connected in a pairwise top/bottom- and outboard/inboard-way that excludes actuation on even-$m$ poloidal mode-numbers. This cuts the number of measured and manipulated signals in half; the result is 64 plant outputs and 64 plant inputs.

3.5 A very short excursion on MCF diagnostics

So far we have not commented on the means of measuring neither MHD activity and topology, nor kinetic properties. The field of plasma diagnostics \cite{127} is indeed a science of its own. Some important methods routinely employed in MCF research will here be outlined.

Classical magnetic measurements are all based on the principle of induction; time-varying magnetic flux induces a voltage along the contour of the linking
surface. The “loop-voltage” $V_l$ is a term reserved for the particular induced voltage around the major circumference of the toroidal plasma. Together with the total plasma current, as inferred from the Rogowski coil voltage encircling the minor period of the plasma, an average plasma resistivity can be estimated. Saddle coils on the inside, or outside, of a resistive shell covering the vacuum vessel, provide measurements of the radial component of the nonaxisymmetric (and axisymmetric, but there are also specialized coils for axisymmetric motion) perturbations of the magnetic fields. In addition a large set of smaller “pick-up” coils measure poloidal and toroidal magnetic field components of the perturbations. Newer sensors can even provide tri-axial magnetic field measurements from the exact same location [118].

Magnetic diagnostics are mainly, currently, representing the “sensor” side of automatic control in MCF. Raw real-time processing of these signals is quite simple. A time-integration gives the magnetic flux. The plasma distortion and/or position can in principle be related to the magnetic flux using the equilibrium equations. As was hinted in subsection 3.2 this equilibrium reconstruction is not entirely trivial nevertheless. A basic limitation is that magnetic measurements alone do not provide enough information to pin down the interior current- and density distributions in the MCF device [100].

A array of diagnostic instruments typically surrounds an MCF experiment. An inexhaustive list follows.

**Interferometry**: Laser interferometric measurements gives line-integrals of plasma density. A set of these can be used for tomographic inversion to density profiles.

**Polarimetry**: Studying the polarization of the laser gives additional information.

**MSE**: (motional Stark effect) is a quite recent technique for measuring the magnetic field internal pitch angle.

**SXR**: (soft X-ray) information on the electron temperature profile.

**ECE**: (electron cyclotron emission) information on electron temperature.

**Visible spectroscopy**: used mostly for studying the plasma edge region where incomplete ionization shows atomic line-emission spectra. Impurity information.

**Thomson scattering**: Laser scattering that provides information on the electron temperature.

**Neutral particle analysis**: yields information on ion temperatures.

**Neutron emission**: information on ion temperatures (and fusion rate).

Possibly, not all of the above instruments would ultimately be required for a reactor. But all of them are needed for scientific development of understanding of plasma behaviour that in the end might lead to a reliable, and ideally, less monitored, reactor design.
4 Magnetohydrodynamics and stability of RFPs

4.1 Kinetic theory of plasmas

Statistical mechanics of multibody charged particle motion dictates the evolution of the number density \( f_s = f_s(t, x, v) \) according to the Boltzmann equation [112]

\[
\frac{\partial f_s}{\partial t} + \sum_i v_i \frac{\partial f_s}{\partial x_i} + \sum_i \frac{F_i}{m_s} \frac{\partial f_s}{\partial v_i} = \left( \frac{\partial f_s}{\partial t} \right)_{\text{coll}}
\]  

(13)

where \( i = 1, 2, 3 \) and \( f_s(t, x, v) \prod_i dx_i \prod_i dv_i \) should be interpreted as the number of particles, of species (e.g., some type of ion, or electrons) \( s \), in a volume \( \prod_i dx_i \) centered around \( x \) and having a velocity within range \( dv_i \) centered at \( v_i \), for all \( i = 1, 2, 3 \), at the time \( t \). In the absence of the collisional operator on the right-hand side, \( f_s \) will be constant along a particle trajectory. Note that equation (13) should be coupled with Maxwell’s electromagnetic field equations. For a plasma, the force term \( F_i \) is the Lorentz force

\[
F(t, x, v) = q_s (E(t, x) + v \times B(t, x))
\]  

(14)

It is computationally demanding to handle a PDE, as (13), with 6 spatial dimensions. There are various ways to reduce (13) by averaging in particular ways. For example one can obtain the gyrokinetic equation by averaging over a gyro-orbit.

A fluid model PDE system has 3 spatial dimensions and can be defined by the first distribution moments

\[
n_s(t, x) = \int f_s(t, x, v) d^3v
\]  

(15a)

\[
v_s(t, x) = \left( \frac{1}{n_s(x)} \right) \int v f_s(t, x, v) d^3v
\]  

(15b)

\[
P_s(t, x) = \int m_s v v f_s(t, x, v) d^3v
\]  

(15c)

Integrating away \( v \) in (13) in the spirit of (15) then gives the highly useful magnetofluid theory of subsection 4.2; which provide the basis for any (global) MCF stability calculation, and also largely dictates the nonlinear macroscopic dynamics of most plasmas. Moment truncation for (13) is quite involved in practice, and require some justifiable means of “closing” the resulting PDE system [64, 68, 113].

4.2 Magnetohydrodynamic theory

As hinted in subsection 4.1, the kinetic description can be reduced by taking moments of the particle species distribution functions. A fluid model then pops out [59, 64, 113]. The magnetohydrodynamic (MHD) model (also known as hydromagnetic) comes in an assortment of flavours [29, 45]. For example, the following representation of resistive single-fluid MHD can be derived [64] from
In (16) \( \rho \) denotes density, \( v \) velocity, \( p \) pressure, \( b \) magnetic induction, \( e \) electric field, \( j \) current density, \( \eta \) resistivity, and \( \gamma \) is the gas constant. Faraday’s equation (16d) dictates that \( \nabla \cdot (\nabla \times b) / (\partial t) = \nabla \cdot \nabla b / (\partial t) = -\nabla \cdot (\nabla \times e) = 0 \). Thus it is sufficient to enforce (16g) as an initial condition.

Ideal MHD (IMHD) is the most fundamental fluid model \([59]\). IMHD is obtained from (16) by setting \( \eta = 0 \) in (16c) and (16f). A stationary IMHD solution \( 0 = \partial (\cdot) / (\partial t) \) with zero flow \( v = 0 \) to (16) must therefore fulfil the equilibrium conditions

\[
\begin{align*}
\nabla \cdot b & = 0 \quad \text{(17a)} \\
\nabla p - j \times b & = 0 \quad \text{(17b)} \\
\mu_0 j - \nabla \times b & = 0 \quad \text{(17c)}
\end{align*}
\]

A slight refurbishment of (16) yields

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) & = 0 \quad \text{(18a)} \\
\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v + \nabla p - \frac{1}{\mu_0} (\nabla \times b) \times b & = 0 \quad \text{(18b)} \\
\frac{\partial p}{\partial t} + v \cdot \nabla p + \gamma p \nabla \cdot v & = (\gamma - 1) \frac{\eta}{\mu_0} |\nabla \times b|^2 \quad \text{(18c)} \\
\frac{\partial b}{\partial t} - \nabla \times (v \times b) & = -\frac{\eta}{\mu_0} \nabla \times (\nabla \times b) \quad \text{(18d)}
\end{align*}
\]

Equation (18d) can be written

\[
\frac{\partial b}{\partial t} = \nabla \times (v \times b) + \frac{\eta}{\mu_0} \nabla^2 b
\]

since \( \nabla \cdot b = 0 \). (19) is known as the advection-diffusion equation. The two right-hand side terms in (19) express how magnetic field lines are dragged around by the background flow and smeared out by a finite conductivity (reciprocal resistivity). An important quantity is the \textit{Lundquist} number (a.k.a. the magnetic Reynolds number)

\[
S = \frac{\mu_0 V L}{\eta} \sim \frac{|\nabla \times (v \times b)|}{|\eta/\mu_0| \nabla^2 b}| \quad \text{(20)}
\]

which governs the relative strength of the above mentioned effects. In (20) \( V \) and \( L \) respectively are the characteristic velocity and the characteristic length.
of the system. We now ignore resistivity (η = 0) in (18) and denote a flowless equilibrium solution to (17) with the fields: \( v^0, b^0, \rho^0 \) and \( p^0 \). We then perturb the equilibrium \((\cdot)^0\) by quantities \((\cdot)^1\) such that

\[
\begin{align*}
v &= v^0 + v^1 = v^1 \\
b &= b^0 + b^1 \\
p &= p^0 + p^1 \\
\rho &= \rho^0 + \rho^1
\end{align*}
\]  

(21a) (21b) (21c) (21d)

Stuffing (21) into (18), noting the relation \( \nabla p^0 - (1/\mu_0)(\nabla \times b^0) \times b^0 = 0 \) which follows from (17b) and (17c), and discarding all terms nonlinear in \((\cdot)^1\) gives

\[
\begin{align*}
\frac{\partial \rho^1}{\partial t} &= -\nabla \cdot (\rho^0 v^1) \quad (22a) \\
\rho^0 \frac{\partial v^1}{\partial t} &= -\nabla p^1 + \frac{1}{\mu_0} ((\nabla \times b^0) \times b^1 + (\nabla \times b^1) \times b^0) \quad (22b) \\
\frac{\partial p^1}{\partial t} &= -v^1 \cdot \nabla \rho^0 - \gamma p^0 \nabla \cdot v^1 \quad (22c) \\
\frac{\partial b^1}{\partial t} &= \nabla \times (v^1 \times b^0) \quad (22d)
\end{align*}
\]

often referred to as linearized IMHD, for reasons quite obvious. (22a) appears decoupled from the remaining equations in (22). Introduce the displacement \( \xi = \int t^1 v^1 d\tau \)

\[
\frac{\partial \xi}{\partial t} = v^1 \Rightarrow \frac{\partial v^1}{\partial t} = \frac{\partial^2 \xi}{\partial t^2}
\]  

(23)

and plug (23) into (22a), (22c) and (22d). After commuting spatial and time differentiations and integrating timewise it is found

\[
\begin{align*}
\rho^1 &= -\nabla \cdot (\rho^0 \xi) + C_p \\
p^1 &= -\xi \cdot \nabla \rho^0 - \gamma p^0 \nabla \cdot \xi + C_p \\
b^1 &= \nabla \times (\xi \times b^0) + C_b
\end{align*}
\]  

(24a) (24b) (24c)

where the constants of integration are \( C_p = \rho^1(t = 0), C_p = p^1(t = 0) \) and \( C_b = b^1(t = 0) \), provided \( \xi(t = 0) = 0 \). The particular choice \( C_p = 0, C_b = 0 \) is more or less standardized in the literature [23, 59]. We adopt this standard and merge (22b) with (24b) and (24c). Lo and behold

\[
\rho^0 \frac{\partial^2 \xi}{\partial t^2} = F(\xi) \quad (25)
\]

where

\[
F(\xi) = \nabla (\xi \cdot \nabla p^0 + \gamma p^0 \nabla \cdot \xi) \\
+ \frac{1}{\mu_0} (\nabla \times b^0) \times \{\nabla \times (\xi \times b^0)\} \\
+ \frac{1}{\mu_0} (\nabla \times \{\nabla \times (\xi \times b^0)\}) \times b^0
\]  

(26)
A digestible representation of (25) is obtained by explicitly writing the equilibrium and perturbed current densities
\[
\mu_0 j^0 = \nabla \times b^0 \\
\mu_0 j^1 = \nabla \times b^1 \\
= \nabla \times \{\nabla \times (\xi \times b^0)\}
\] (27b)
so that (22b) apparently is
\[
\rho^0 \frac{\partial \xi}{\partial t} = -\nabla p^1 + j^0 \times b^1 + j^1 \times b^0
\] (28)
Notably equations (18) and (22) are eight-order nonlinear and linear PDEs respectively. As noted (22a) becomes decoupled, so we can leave \(C_p\) in (24a) unspecified. Thus (22) essentially is a seventh-order linear PDE. Curbing the initial-condition by (here) selecting \(C_b = 0, C_p = 0\) and \(\xi(t = 0) = 0\) results in (25) dropping to sixth-order. The initial velocity field
\[
\frac{\partial \xi}{\partial t} \bigg|_{t=0}
\] (29)
is arbitrary for (25).

Suppose now that time- and space-dependencies of \(\xi = \xi(t, x)\) in (25) can be separated such that \(\xi = e^{-i\omega t} \xi(x)\). An eigenvalue equation
\[
-\rho^0 \omega^2 \xi(x) = F(\xi(x))
\] (30)
is obtained since (26) is linear in \(\xi\) and the \(e^{-i\omega t}\) can be factored out. From (30) it is evident how one could program general IMHD linear stability number-crunching. Given an equilibrium, first express (26) in some sort of truncated pseudospectral basis, finite-differences, finite-elements or a mixed concoction, and secondly assemble matrix elements for the finite-dimension linear algebra instance of (30). Of course, all relevant boundary conditions must be respected when doing this.

It is readily verified that any displacement vector field \(\xi(x) = c b^0(x)\), where \(c\) is an arbitrary scalar, is a particular solution to the zero eigenvalue \(\omega^2 = 0\) of (30). For this “parallel” displacement, the terms in (26) all vanish independently: \(c b^0 \cdot \nabla \rho^0 = 0\) follows from (17b), \(\nabla \cdot c b^0 = 0\) according to (17a) and finally \(c b^0 \times b^0 = 0\).

It turns out that the so-called force operator formalism for IMHD, (25), (26) and (29) launch a quite handsome (albeit limited, not to be used with e.g. nonzero flow equilibria, and resistivity is often very important) theoretical framework. In MHD it is basically mandatory to namedrop the energy principle [23, 29, 59, 64, 113, 127]. Basically it works like this [128]. Write the kinetic energy \(K\) and potential energy \(V\)
\[
K = \frac{1}{2} \int \rho^0 \dot{\xi} \cdot \dot{\xi} \, dV \quad (31)
\]
\[
V = -\frac{1}{2} \int \xi \cdot F(\xi) \, dV \quad (32)
\]
compute
\[
\frac{\partial}{\partial t} K = \frac{1}{2} \int \rho^0 \frac{\partial}{\partial t} \dot{\xi} \cdot \dot{\xi} dV \\
= \frac{1}{2} \int \rho^0 (\dot{\xi} \cdot \dot{\xi} + \dot{\xi} \cdot \dot{\xi}) dV \\
= \int \dot{\xi} \cdot \rho^0 \dot{\xi} dV = \int \dot{\xi} \cdot F(\xi) dV
\]
(33)
and
\[
\frac{\partial}{\partial t} V = -\frac{1}{2} \int \frac{\partial}{\partial t} \xi \cdot F(\xi) dV \\
= -\frac{1}{2} \int \{\xi \cdot F(\xi) + \xi \cdot F(\xi)\} dV \\
= [\text{self-adjoint}] = -\int \dot{\xi} \cdot F(\xi) dV \\
= -\frac{\partial}{\partial t} K
\]
(34)
seen from (33), hence (34) claims \(\partial/\partial t) (V + K) = 0\) or
\[V + K = \text{constant}\]
(35)
The key property of self-adjointness\(^4\), used in (34), of force-operator \(F\) is as follows.
\[
\int \eta \cdot F(\xi) dV = \int \xi \cdot F(\eta) dV
\]
(36)
for all \(\eta, \xi\) fulfilling appropriate boundary conditions (consistent with IMHD).
Suppose again \(\xi = e^{-i\omega t} \xi(x)\) and let the constant in (35) be zero, then eigenvalues can be found by extremizing
\[
\omega^2 = \frac{V}{(1/2) \int \rho^0 \xi \cdot \xi dV}
\]
(37)
which follows from (35). In the literature the notation \(\delta W = V\) has been adopted.

The energy principle states that an equilibrium is stable if and only if \(V\), as defined in (32), is non-negative for all (including “unphysical”) displacements \(\xi\) respecting the boundary conditions.

This can be understood from (37) and the Rayleigh-Ritz method for eigenvalues as extrema [108]. Set \(u = \sqrt{\rho^0} \xi\), the linear operator \(T : = -(1/\rho^0)F(\cdot)\) and the scalar product \((u, v) = \int u \cdot v dV\) then (37) takes the form
\[
\omega^2 = \frac{(Tu, u)}{(u, u)} \equiv r(u)
\]
(38)
which is a Rayleigh quotient \(r(u)\). The argument (loosely) goes like this; for a (bounded) self-adjoint operator \(T\) it holds [80] for any eigenvalue \(\lambda\) to \(T\) \((Tv = \lambda v, v\) the eigenvector), that
\[
\lambda \in \left[ \min_{u \neq 0} r(u), \max_{u \neq 0} r(u) \right]
\]
(39)
\(^4\)This is not trivial to verify. See e.g. [59] for some brave vector calculus.
i.e. $\lambda \in \mathbb{R}$ and $\lambda$ is bracketed by the minimum and maximum of $r(u)$ over all "test-vectors" $u$. This means that if $\min_{u \neq 0} r(u) > 0$, $\omega = \sqrt{\lambda}$ must be real-valued and represent an oscillating solution. But if $\min_{u \neq 0} r(u) < 0$, $\omega$ attains complex-conjugate values, and will represent an exponentially growing solution. The marginal case $\omega = 0$ is not disastrous (and should be expected, as shown earlier).

The detailed MHD spectrum is more complicated than the story (39) seems to tell. Continua, algebraic growth, eigenvalue accumulation and non-normalizable eigenfunctions require quite serious analysis [64].

### 4.3 Non-plasma MHD

MHD is also the prevailing theory for general liquid metal [45] dynamics, levitation melting, aluminium casting, and other related industrial processes. In these situations the parameter regimes are obviously quite different. Liquid MHD flow is believed to be important also for fusion reactors [76], or more specifically, for absorption and transport of heat in the blanket.

### 4.4 Addressing finite-$\beta$ equilibria and resistive-shell mode stability

#### 4.4.1 Outline

We will here derive the most simple dynamical finite-$\beta$ model possible for MHD dynamics on the resistive-shell eddy time-scale in RFPs. The equilibrium calculations in subsection 4.4.2 basically follow [90], but is more articulate and fill in the gaps. Symmetric one-dimensional profiles in cylinder coordinates are assumed. For equilibrium we impose: $\mathbf{b} = \mathbf{b}(r) = b_r(r)\hat{e}_r + b_\theta(r)\hat{e}_\theta + b_z(r)\hat{e}_z$ where $r$ is the minor radius and $\hat{e}_i$ are unit basis $\mathbb{R}^3$-vectors. We will denote by $(\cdot)'$ the radial coordinate derivative $\partial(\cdot)/\partial r$. The spatial dimension along $\hat{e}_z$ is periodic with the period $L = 2\pi R$ where $R$ is the major toroidal radius. Stability calculations are substantially more involved than equilibria. Chapters focusing on cylindrical plasmas can be found in e.g. [23, 29, 64]. Subsection 4.4.3 mobilize an admixture of the material within these references.

For the purpose of this thesis, the main motivation of presenting the calculations of this subchapter 4.4 is to justify, in a self-contained way, the sketchy physics model (128), finally presented in subsection 4.4.5, which has been invoked more or less von oben in a majority of the author’s papers.

#### 4.4.2 Equilibrium

Recollect equilibrium conditions (17). First, vanishing divergence (17a) implies $b_r = 0$. In general, taking the cross-product $\mathbf{b} \times \text{eq. (17b)}$, reshuffling and using (17c) one obtains

$$\nabla \times \mathbf{b} = \mu_0 \left( \hat{b} \cdot \mathbf{j} \right) \hat{b} + \mu_0 \frac{\mathbf{b} \times \nabla p}{\mathbf{b} \cdot \mathbf{b}} \tag{40}$$
where \( \hat{b} = b/|b| = b/\sqrt{b \cdot b} \). Now, the RFP is typically parameterized \(^{[90]}\) with a prescribed profile as follows

\[
\mu(r) = \mu_0 \frac{b \cdot j}{b \cdot b} = \mu(0) \left( 1 - \frac{r}{a} \right)^{\alpha} = \frac{2\Theta_0}{a} \left( 1 - \frac{r}{a} \right)^{\alpha}
\]

(41)

where the dimensionless quantity \( \Theta_0 \equiv \mu(0)a/2 = (\mu_0 a/2) j_{||}(0)/|b(0)| \) essentially is a normalized on-axis current-density to toroidal magnetic field ratio. Eq. (41) is known as the \( \Theta_0-\alpha \)-model and implies that the first term on the right hand side of (40) reads as \( \mu(r)b \). A heuristic argument now follows for parameterization of the remaining pressure-gradient term in (40). Introduce

\[
q(r) = \frac{r b_z}{R b_\theta}
\]

(42)

and note \( q'/q = 1/r + b_z'/b_z - b_\theta'/b_\theta \). A well-known necessary condition for stability of interchange modes is Suydam's criterion \(^{[23, 29, 64]}\) (this inequality will be derived later)

\[
\left( \frac{q'}{q} \right)^2 > -\frac{8\mu_0 p'}{r b_\theta^2}
\]

(43)

that imposes a local constraint on the plasma pressure gradient. Using (40) and (42) in (43) we get

\[
\mu_0 p' > -\frac{1}{2} r \left( \frac{b_z}{r} - \frac{\mu_0 p'}{b_\theta^2} \right)^2
\]

(44)

The idea of \(^{[90]}\) is to change the inequality of (44) to an equality and multiply the right hand side by a parameter \( 0 < \chi(r) < 1 \). This parameter \( \chi \) can be used to control the average \( \beta \) of the equilibrium while enforcing (43). In summary, (40), (41) and the modified (44) determine a finite-\( \beta \) RFP equilibrium as follows.

\[
b_z' = 0
\]

(45a)

\[
b_\theta' = -\frac{b_\theta}{r} + \mu(r)b_z - \frac{\mu_0 p'}{b_z^2} b_\theta
\]

(45b)

\[
b_z' = -\mu(r)b_\theta + \frac{\mu_0 p'}{b_z^2} b_z
\]

(45c)

\[
\mu_0 p' = -\chi(r) \frac{1}{2} r \left( \frac{b_z}{r} - \mu(r) \frac{b_z^2}{2b_\theta} \right)^2
\]

(45d)

where \( b_z^2 = b_\theta^2 + b_z^2 \) and \( \mu(r) \) is given by (41). It is convenient to introduce a normalization \( x_1 = b_\theta/b_z(0) \), \( x_2 = b_z/b_z(0) \) and \( x_3 = 2p\mu_0/b_z^2(0) \) and integrate the ODE

\[
x_1' = -\frac{1}{r} x_1 + \mu(r)x_2 - \frac{x_3}{2} \frac{x_1}{x_1^2 + x_2^2}
\]

(46a)

\[
x_2' = -\mu(r)x_1 + \frac{x_3}{2} \frac{x_2}{x_1^2 + x_2^2}
\]

(46b)

\[
x_3' = -\chi(r) r \left( \frac{x_2}{r} - \mu(r) \frac{x_3 x_2^2}{2x_1^2} \right)^2
\]

(46c)
from $r = 0$ to $r = a$. The initial condition should be $x_1(0) = 0, x_2(0) = 1$. Apparently $x_3(0)$ is arbitrary but should be adjusted after integration so that $x_3(a) = 0$, as a boundary condition on $p(a)$. In starting-off the ODE solver it is useful to notice $x'_1(0) = \mu(0)/2$ and $x'_2(0) = x'_3(0) = 0$. From the solution (46) we can then calculate

$$\beta = \frac{2\mu_0 p}{b^2} = \frac{x_3}{x_1^2 + x_2^2}$$

i.e. the local $\beta$ and its volume-average

$$\langle \beta \rangle = \frac{1}{\pi a^2} \int_0^a 2\pi r \beta(r) dr = \frac{2}{a^2} \int_0^a r \frac{x_3}{x_1^2 + x_2^2} dr$$

Fig. 5 shows an example solution of (45) with model parameters and resulting volume averaged $\beta$ as displayed. The example can be compared to fig. 1 in [90]. The characteristic field reversal in fig. 5, first explained by J.B. Taylor [119], is a result of magnetic helicity-constrained [26] plasma relaxation [96]. This curious equilibrium is a sort of minimum energy state.

$$\alpha = 3.58, \ \Theta_0 = 1.62, \ \chi = 0.75 \implies \langle \beta \rangle = 0.0501$$

![Figure 5: Example RFP equilibrium profiles. Here $a/R = 0.183/1.24$. The vertical dash-dotted line annotated $r_z$ shows the surface of field reversal.](image)

Experimentally, an RFP equilibrium is mainly characterized by two normalized parameters defined as follows.

$$F = b_z(a)/\langle b_z \rangle$$

$$\Theta = b_\theta(a)/\langle b_z \rangle$$
The volume-average is \( \langle b_z \rangle = (2/a^2) \int_0^a rb_z \, dr \). \( F \) is the reversal-parameter. \( \Theta \) is the pinch-parameter. Both (49) and (50) are computable from measurements of the magnetic field outside the plasma boundary.

### 4.4.3 Stability of equilibrium

Simply put, we wish to perturb the equilibrium defined through (45) in various ways, and study what happens. In particular we are interested in the resistive-shell configuration. We start by writing out the linearized eigenvalue equation (30) in a general cylindrically symmetric situation. This requires some patience.

Recall from subsection 4.4.2 that \( b^0 = b^0_\theta(r) \hat{e}_\theta + b^0_z(r) \hat{e}_z \) and \( p^0 = p^0(r) \). The fact that the equilibrium profiles are independent of \( \theta \) and \( z \) is important. It then suffices to study Fourier-decomposed displacements \textit{individually}. The total perturbation is thus \( \xi = \sum_{m,k} \xi^{mk} \) with

\[
\xi^{mk}(x) = e^{im\theta + ikz} (\xi^{mk}_r(r) \hat{e}_r + \xi^{mk}_\theta(r) \hat{e}_\theta + \xi^{mk}_z(r) \hat{e}_z)
\]

and \( \xi^{mk}(x,t) = e^{-i\omega t} \xi^{mk}(x) \). Using (51) we evaluate the perturbed quantities (24b) and (24c).

\[
(p^1)^{mk} = -\xi^{mk} \cdot \nabla p^0 - \gamma p^0 \nabla \cdot \xi^{mk} = -\xi^{mk} \cdot \frac{1}{\mu_0} \left( \nabla \times b^0 \right) \times b^0 - \gamma p^0 \nabla \cdot \xi^{mk}
\]

(52)

where \( \nabla p^0 = \frac{1}{\mu_0} \left( \nabla \times b^0 \right) \times b^0 \) is the equilibrium condition (17). Therefore \( \nabla p^0 = (1/\mu_0) \hat{e}_r \{ -(b^0_\theta)'b^0_z - (1/r)b^0_0 (rb^0_\theta)' \} \) and it follows

\[
\xi^{mk} \cdot \nabla p^0 = -\frac{1}{\mu_0} e^{-i\omega t} e^{im\theta + ikz} \xi^{mk}_r \left( (b^0_\theta)'b^0_z + (1/r)b^0_0 (rb^0_\theta)' \right)
\]

(53)

The compression term becomes

\[
\nabla \cdot \xi^{mk} = e^{-i\omega t} e^{im\theta + ikz} \left\{ \frac{1}{r} (r \xi^{mk}_r)' + \frac{im}{r} \xi^{mk}_\theta + ik \xi^{mk}_z \right\}
\]

(54)

and the perturbed magnetic field is

\[
(b^1)^{mk} = \nabla \times \left( \xi^{mk} \times b^0 \right) = e^{-i\omega t} \nabla \times \left\{ e^{im\theta + ikz} \left\{ (-\xi^{mk}_r b^0_\theta + \xi^{mk}_\theta b^0_z) \hat{e}_r \\
+ (-\xi^{mk}_r b^0_\theta) \hat{e}_\theta + (\xi^{mk}_r b^0_\theta) \hat{e}_z \right\} \right\} = (b^1)^{mk}_r \hat{e}_r + (b^1)^{mk}_\theta \hat{e}_\theta + (b^1)^{mk}_z \hat{e}_z
\]

(55)

where

\[
(b^1)^{mk}_r = e^{-i\omega t} e^{im\theta + ikz} \left\{ \frac{im}{r} \xi^{mk}_r b^0_\theta + ik \xi^{mk}_r b^0_z \right\}
\]

(56a)

\[
(b^1)^{mk}_\theta = e^{-i\omega t} e^{im\theta + ikz} \left\{ ik (-\xi^{mk}_r b^0_\theta + \xi^{mk}_\theta b^0_z) - (\xi^{mk}_r b^0_\theta)' \right\}
\]

(56b)

\[
(b^1)^{mk}_z = e^{-i\omega t} e^{im\theta + ikz} \left\{ -\frac{1}{r} (r \xi^{mk}_r b^0_\theta)' - \frac{im}{r} (\xi^{mk}_r b^0_\theta + \xi^{mk}_\theta b^0_z) \right\}
\]

(56c)
The initially pedagogical reason for writing out superscript \( n \) and the spatial- and time-dependence factor \( e^{-i\omega t} e^{im\theta + ikz} \) has now played out its role. First introduce

\[
f = \frac{m}{r} b_0^0 + k b_z^0
\]  

(57)

Thus, in lightweight notation we rewrite (56a) as

\[
ib_r^1 = -f \xi_r
\]  

(58)

Now, notice that equations (56b), (56c) and (54) all have a \( \xi' \)-term. Since

\[
\frac{1}{r}(r\xi_r b_z^0)' = \frac{b_0^0}{r} (r\xi_r)' + \frac{(b_0^0)'}{r} (r\xi_r)
\]

(59)

\[
(b_1\xi_r^0)' = \frac{b_0^0}{r} (r\xi_r)' + \left( \frac{b_1^0}{r} \right)' (r\xi_r)
\]

slimmer versions of (56b) and (56c) are obtained by plugging in \( (r\xi_r)' = r \nabla \cdot \xi - im\xi_\theta - ikr\xi_z \), doing the algebra and employing (57)

\[
b_1^0 = fi\xi_\theta - b_0^0 \nabla \cdot \xi - (r\xi_r) \left( \frac{b_0^0}{r} \right)'
\]

(59)

\[
b_z^1 = fi\xi_z - b_0^0 \nabla \cdot \xi - \xi_r \left( b_0^0 \right)' 
\]

(60)

Continuing with the terms in (30) we need

\[
(\nabla \times b^0) \times b^1 = \left( -(b_z^0)' b_z^1 - \frac{1}{r}(r b_0^0)' b_r^1 \right) \hat{e}_r
\]

\[
+ \left( \frac{1}{r}(r b_0^0)' b_r^1 \right) \hat{e}_\theta
\]

\[
+ \left( (b_z^0)' b_z^1 \right) \hat{e}_z
\]

(61)

and

\[
(\nabla \times b^1) \times b^0 = \left( b_z^0 (i k b_k^1) - (b_z^1)' - \frac{1}{r} b_0^0 ((r b_0^0)' - im b_0^1) \right) \hat{e}_r
\]

\[
+ \left( -b_0^0 \left( \frac{1}{r} im b_z^1 - i k b_0^1 \right) \right) \hat{e}_\theta
\]

\[
+ \left( b_0^0 \left( \frac{1}{r} im b_z^1 - i k b_0^1 \right) \right) \hat{e}_z
\]

(62)

Stuffing (30) with (61) and (62) and formally cancelling the (notationally omitted) exponential factors we get three algebraic equations, for \( r, \theta \) and \( z \) components respectively. The details of \( p^1 \), provided by (52), are hidden at this stage by using

\[
\nabla p^1 = (p^1)' \hat{e}_r + \frac{im}{r} p^1 \hat{e}_\theta + ikp^1 \hat{e}_z
\]

(63)
Hence

\[-\rho^0 \omega^2 \xi_r = -(p^1)' + \frac{1}{\mu_0} \left\{ -(b_z^0)'b_r^1 - \frac{1}{r}(rb_0^0)'b_r^1 + b_0^0(ikb_r^1 - (b_z^1)') \right\} \]

\[-\rho^0 \omega^2 \xi_\theta = -\frac{im}{r} p^1 + \frac{1}{\mu_0} \left\{ \frac{1}{r}(rb_0^0)'b_\theta^1 - b_0^0(\frac{1}{r}imb_\theta^1 - ikb_\theta^1) \right\} \]  \hspace{1cm} (64a)

\[-\rho^0 \omega^2 \xi_z = -ikp^1 + \frac{1}{\mu_0} \left\{ (b_z^0)'b_r^1 + b_0^0(\frac{1}{r}imb_\theta^1 - ikb_\theta^1) \right\} \]  \hspace{1cm} (64b)

The \(r\)-component equation (64a) can be refurbished

\[-\rho^0 \omega^2 \xi_r = -(p^1)' + \frac{1}{\mu_0} \left\{ -(b_z^0)'b_r^1 - \frac{1}{r}(rb_0^0)' - \frac{1}{r}b_0^0(rb_0^0)' - f^2 \xi_r \right\} \]

\[= -(p^1)' + \frac{1}{\mu_0} \left\{ -(b_z^0)'b_r^1 - (b_0^1)'b_r^1 - 2\frac{b_z^1}{r}b_\theta^0 - f^2 \xi_r \right\} \]

\[= - (p^*)' - \frac{1}{\mu_0} \left\{ \frac{2}{r} b_\theta^0 b_\theta^0 + f^2 \xi_r \right\} \]  \hspace{1cm} (65)

using (57), (58), product differentiation and the definition

\[p^* = p^1 + \frac{1}{\mu_0} (b_\theta^0 b_\theta^1 + b_z^0 b_z^1) = p^1 + \frac{1}{\mu_0} b^0 \cdot b^1 \]  \hspace{1cm} (66)

i.e. the sum of perturbed thermodynamic and perturbed magnetic pressure [23]. Substituting for \(p^1\) in (64b) and (64c) with (66) and again identifying f:s (57) popping out, it appears

\[-\rho^0 \omega^2 \xi_\theta = -\frac{im}{r} p^* + if \frac{1}{\mu_0} b_\theta^1 + \frac{1}{\mu_0} \frac{1}{r}(rb_0^0)'b_r^1 \]  \hspace{1cm} (67a)

\[-\rho^0 \omega^2 \xi_z = -ikp^* + if \frac{1}{\mu_0} b_z^1 + \frac{1}{\mu_0} \frac{1}{r}(b_0^0)'b_r^1 \]  \hspace{1cm} (67b)

Combining (67a) with (58) and (59), and similarly (67b) with (58) and (60), then yields

\[-\rho^0 \omega^2 + f^2/\mu_0 \right i \xi_\theta = \frac{m}{r} p^* + \frac{f}{\mu_0} b_\theta^1 - \frac{2f b_\theta^0}{r} \xi_r \]  \hspace{1cm} (68a)

\[-\rho^0 \omega^2 + f^2/\mu_0 \right i \xi_z = kp^* + \frac{f}{\mu_0} b_\theta^1 \nabla \cdot \xi \]  \hspace{1cm} (68b)

respectively. It is now apparent that a coupled system of first-order ODEs in \(p^*, r \xi_r\) can be distilled from (54) and (65); here rewritten as

\[(r \xi_r)' = r \nabla \cdot \xi - m [i \xi_\theta] - kr [i \xi_z] \]  \hspace{1cm} (69)

\[(p^*)' = \frac{1}{r} \left\{ \frac{f^2}{\mu_0} + \rho^0 \omega^2 \right\} r \xi_r - \frac{2\rho b_\theta^0}{\mu_0 r} b_z^1 \]  \hspace{1cm} (70)

for clarity. It remains the task of expressing \(\nabla \cdot \xi\) in terms of \(p^*\) and \(r \xi_r\). Having done so, the right hand sides of (69) and (70) will also unfold to terms in \(p^*\) and \(r \xi_r\) alone. Equation (66) can be retooled to

\[p^* - \frac{1}{\mu_0} b^1 \cdot b^0 = p^1 = -\xi \cdot \nabla p^0 - \gamma p^0 \nabla \cdot \xi \]  \hspace{1cm} (71)
Invoking (53) converts (71) to

\[ p^* + \xi_r \left( (b_z^0/b_x^0) + (1/r)b_z^0(rb_0^0)' \right) - \frac{1}{\mu_0} \left( b_{0b}^0 b_0^0 + b_{0b}^0 b_x^0 \right) = -\gamma p^0 \nabla \cdot \xi \tag{72} \]

which can be rearranged by employing (59), (60), (68) and some algebra. Finally

\[ (\nabla \cdot \xi) = \frac{-\rho^0 \omega^2 \left( p^* - \frac{2}{r\mu_0} \xi b_z^0 \right)}{\gamma \rho \left( \rho^0 \omega^2 - \frac{\mu_0}{\rho} \left( b_{0b}^0 + b_{0b}^0 \right) \right)} \tag{73} \]

and so putting (73), (58) and (68) into (69) and (70) yields a useful form of the cylinder-1D eigenvalue problem. This final step is omitted in this presentation. The reader is encouraged to do this him- or herself. The cute equation (30) turns out to be quite squalid when looking under its hood, even in one spatial dimension. Tidying up the algebra we obtain \[ 19, 23, 42, 65 \]

\[ Dr (\xi_r)' = c_{11} r \xi_r + c_{12} p^* \quad \tag{74a} \]
\[ Dr (p^*)' = c_{21} r \xi_r - c_{11} p^* \quad \tag{74b} \]

where

\[ D = -\rho^0 \omega^2 + \frac{f^2}{\mu_0} \quad \tag{75a} \]
\[ \Lambda = \frac{\rho^0 \omega^4}{-D \gamma p^0 + \rho^0 \omega^2 \left( b_{0b}^0 + b_z^0 \right) / \mu_0} \quad \tag{75b} \]
\[ c_{11} = \frac{2}{\mu_0} \left( b_{0b}^0 \frac{m}{r} - \Lambda b_{0b}^0 \right) \quad \tag{75c} \]
\[ c_{12} = r^2 \Lambda - m^2 - r^2 k^2 \quad \tag{75d} \]
\[ c_{21} = -D^2 + 2D \frac{b_{0b}^0}{\mu_0} \left( \frac{b_{0b}^0}{r} \right) \quad \tag{75e} \]
\[ + 4 \left( \frac{b_{0b}^0}{r \mu_0} \right)^2 \left( f^2 - \Lambda b_{0b}^0 \right) \]

Figure 6 summarize the problem geometry. The limited plasma extends from \( r = 0 \) to \( r = a \). A description of linearized perturbed cylindrically symmetric IMHD is given by (74). In \( a < r < r_w \) there is an approximate vacuum. At \( r = r_w \) there is a resistive shell. In \( r_w < r \) there is air, which is also approximately vacuum, from the magnetostatic point of view anyway. At \( r = r_c \) we have saddle coils that can produce magnetic field perturbations actively. Therefore we will need to consider three interfaces with boundary conditions: plasma-vacuum, vacuum-shell, and vacuum-current-sheet. Also, for (74), \( r = 0 \) needs special attention. And the region \( r_c < r < +\infty \) must be “physical”.

However, first consider the vacuum domain magnetic field. From (16) in a region where \( \mathbf{j} = 0 \) it must hold \( \nabla \cdot \mathbf{b} = 0 \) and \( \nabla \times \mathbf{b} = 0 \) (pre-maxwell, no displacement current [94]). For \( \mathbf{b} = b_z \) fourier modes this is equivalent to four scalar equations: \( (rb_z)''/r + imb_{z0}/r + ikb_{z1} = 0 \), \( im b_{z1}'/r - ikb_z = 0 \), \( i k b_{z1}'/(b_x^0) = 0 \) and \((rb_z)''/r - imb_{z0}/r = 0 \). It is possible to bundle these to a second-order ODE

\[ (b_z^0)'' + (1/r)(b_z^0)' - (b_z^0)(m^2/r^2 + k^2) = 0 \quad \tag{76} \]
Figure 6: Cross-section of the cylinder geometry considered. Four domains (i)-(iv): plasma-vacuum-vacuum-vacuum.
The solution of this ODE is written in terms of modified Bessel functions \( I_m(\cdot) \), \( K_m(\cdot) \) [101]. In vacuum we therefore have, for \( k \neq 0 \)

\[
\begin{align*}
ib_r^1 &= |k| \left\{ C_1 I'_m(|k|r) + C_2 K'_m(|k|r) \right\} \\
b_\theta^1 &= \frac{m}{r} \left\{ C_1 I_m(|k|r) + C_2 K_m(|k|r) \right\} \\
b_z^1 &= k \left\{ C_1 I_m(|k|r) + C_2 K_m(|k|r) \right\}
\end{align*}
\]

(77a) - (77c)

where \((\cdot)'\) in (77a) denotes differentiation with respect to the argument of the modified Bessel function. For \( k = 0 \), we similarly find \( b_z^1 = 0 \) and \((rb_r^1)' + imb_\theta^1 = 0, (rb_r^1)' - imb_\theta^1 = 0\). These can be converted to the ODE \( r^2 b_\theta'' + 3rb_\theta' + (1 - m^2)\left| b_\theta \right| = 0 \), with the general solution \( b_\theta = C_1 r^{-1+m} + C_2 r^{-1-m} \). So for \( k = 0 \)

\[
\begin{align*}
ib_r^1 &= C_1 r^{m-1} - C_2 r^{-m-1} \\
b_\theta^1 &= C_1 r^{-1+m} + C_2 r^{-1-m} \\
b_z^1 &= 0
\end{align*}
\]

(78a) - (78c)

The constants \( C(\cdot) \) in (77) and (78) must be solved for such that they patch together a composite domain solution according to appropriate interface boundary conditions. If \( m = k = 0 \) we instead get \( b_1^1 = C_1/r, b_1^0 = C_2/r \) and \( b_1^1 = C_3 \) (do not follow from (78) with \( m = 0 \)).

The boundary conditions for the plasma-vacuum interface [23, 29, 64] can be cast

\[
\begin{align*}
p^* \big|_{r=a^-} &= \frac{1}{\mu_0} \left( b_\theta^0 b_\theta^1 + b_\theta^0 b_z^1 \right) \big|_{r=a^+} \\
b_r^1 \big|_{r=a^-} &= b_r^1 \big|_{r=a^+}
\end{align*}
\]

(79a) - (79b)

where \( f|_{r_1} = f(r_1) \). The boundary conditions for the vacuum-shell interface follows from modeling the shell as a thin layer of current with a surface current density \( k_w \)

\[
\hat{n} \cdot \left[ b_1^1 \right]_{r_w} = 0 \\
\hat{n} \times \left[ b_1^1 \right]_{r_w} = \mu_0 k_w
\]

(80a) - (80b)

where \( [f]_{r_1} = f(r_1) - f(r_2) \). In (80) \( \hat{n} \) is the surface normal vector, in our case \( \hat{n} = \hat{e}_r \), and \( r_w \) denotes the radial position of the interface. The current-sheet interface (80) is valid also for our actuator (set \( r_w = r_e \)), with the difference that the active current layer (set \( k_w = k_e \)) is a source of energy, not a passive eddy harmonic. Finally at \( r = +\infty \) the vacuum field must vanish. Hence \( C_1 = 0 \) in (77) for the domain \( r_e < r < +\infty \).

Boundary conditions can be written

\[
\left[ b_r^1 \right] = 0, \left[ b_\theta^1 \right] = -\mu_0 k_w, b_\theta^0 = \mu_0 k_w, \quad \left[ b_z^1 \right] = \mu_0 k_w, z
\]

(81)

Setting \( k_w = d j_w \) where \( d \) is the resistive-shell thickness and \( j_w \) the supposedly homogeneous volume current density in the shell, and using \( j_w = \sigma \hat{e} \) (\( \sigma \) the shell electrical conductivity) and \( \nabla \times \hat{e} = -\partial b / \partial t \), combine to

\[
\nabla \times k_w = -\sigma d \frac{\partial b_r^1}{\partial t}
\]

(82)
Introduce the stream function $J_w$ such that

$$\mu_0 k_w = \nabla J_w \times \hat{e}_r = ik J_w \hat{e}_\theta - \frac{im J_w}{r} \hat{e}_z$$

(83) in (82) gives the $e_r$-component (the others are not applicable) relation

$$\left( \frac{m^2}{r^2} + k^2 \right) J_w = -\mu_0 \sigma \frac{\partial b^1_r}{\partial t}$$

(83) in (81) gives two relations: $[b^1_1] = -ik J_w$ and $[b^1_2] = -(im/r) J_w$. A linear combination of these is $ik [b^1_1] + (im/r) [b^1_2] = (m^2/r^2 + k^2) J_w$. However $(rb^1_1)/r + imb^1_2/r + kkb^1_2 = 0$ and so using (84) it holds

$$\mu_0 \sigma d r_w \frac{\partial b^1_r}{\partial t} = \left[ (rb^1_1)' \right] r_w^{+} - \left[ rb^1_1 \right] r_w^{-} = - \left[ imb^1_2 + ikrb^1_2 \right] r_w^{+} - r_w^{-}$$

where the second equality follows from (80a). From Ampere’s “low-frequency” law $\nabla \times \mathbf{b} = \mu_0 \mathbf{j} = (1/d) \mu_0 k_w$ it follows from (83)

$$b_r = J_w/d$$

for the radial magnetic field.

Relation (85) is the time-dependent boundary condition that must be solved together with (74). It remains to express (85) in terms of the state-variables in (74). A common algebraic form of (85) is

$$-i \omega \tau_w = \left[ \begin{array}{c} b^1_1' \\ r_w \end{array} \right] r_w^{+} \left[ \begin{array}{c} b^1_1 \\ r_w \end{array} \right] r_w^{-}$$

with

$$\tau_w \equiv \mu_0 \sigma d r_w$$

being the characteristic resistive-shell time.

Denote the vacuum region between the plasma and the shell by (ii) (the plasma-region is region (i)). Then (79) and (77) can be combined to a linear equation for the constants of (77) for $k \neq 0$

$$\left( \begin{array}{cc} kI'_m(ka) & kK'_m(ka) \\ f(a)I_m(ka) & f(a)K_m(ka) \end{array} \right) \left( \begin{array}{c} C^{(ii)}_1 \\ C^{(ii)}_2 \end{array} \right) = \left( \begin{array}{c} -f(a) \xi_r(a) \\ \mu_0 p^*(a) \end{array} \right)$$

(89)

For $k = 0$, (78) must be used analogously. In region (iii), for the passive case, we have vacuum extending from $r_w$ out to $r = +\infty$. This means that $C_{1}^{(iii)} = 0$ in (77). Then from (80a) it follows

$$C_{2}^{(iii)} = C_{1}^{(iii)} \frac{I'_m(kr_w)}{K'_m(kr_w)} + C_{2}^{(ii)}$$

(90)

(85) can then be written ($k \neq 0$)

$$-i \omega \tau_w = \frac{C_{1}^{(iii)} I'_m(kr_w)}{C_{1}^{(iii)} I'_m(kr_w) + C_{2}^{(iii)} K'_m(kr_w)} \left\{ \frac{I'_m(kr_w)}{K'_m(kr_w)} K_m(kr_w) - I_m(kr_w) \right\}$$

$$= \frac{C_{1}^{(iii)} I'_m(kr_w) + C_{2}^{(iii)} K'_m(kr_w) \frac{1}{K'_m(|kr_w|)}}{C_{1}^{(iii)} I'_m(kr_w) + C_{2}^{(iii)} K'_m(kr_w)}$$

(91)
The handy Wronskian identity \([99]\)

\[
I_m(x)K'_m(x) - I'_m(x)K_m(x) = -1/x
\]  

(92)

for solutions to (76) was used in the last algebraic step of (91). In this case, the solution is unstable (stable) if the sign of the right-hand side of (91) is positive (negative). Define the growth-rate \(\dot{\gamma}\)

\[
\dot{\gamma} = -i\omega
\]

(93)

Strictly, we need to run some sort of shooting code to iteratively match the solution of (74) with the condition (91) for \(\gamma \in \mathbb{R}\) (note that in (74) \(\omega^2 = -\gamma^2\)). The lazy and much-used approach is however to neglect plasma inertia altogether. Putting a massless plasma, \(\rho^0 = 0\), in (74) renders (74) independent of \(\omega\) so \(\gamma\) (93) can be computed directly, without iteration, from (91) using the edge-values of (74) entering through (89). Introduce the variables \(y_1 = r\xi_r\) and \(y_2 = \mu_0 p^*\). Massless (74)-(75) can then be rehashed to

\[
rf^2y_1' = \tilde{c}_{11}y_1 + \tilde{c}_{12}y_2
\]

(94a)

\[
rf^2y_2' = \tilde{c}_{21}y_1 - \tilde{c}_{11}y_2
\]

(94b)

with \(f = (m/r)b_0^0 + kb_2^0\) (repeating (57)) and

\[
\tilde{c}_{11} = \frac{2m}{r}f b_0^0
\]

(95a)

\[
\tilde{c}_{12} = -m^2 - r^2k^2
\]

(95b)

\[
\tilde{c}_{21} = -f^4 + 2f^2b_0^0 \left(\frac{b_0^0}{r}\right)' + 4f^2 \left(\frac{b_0^0}{r}\right)^2
\]

(95c)

Not only does (94) lack \(\omega\), but also \(\gamma\) (thermodynamic) disappeared altogether with the profile \(\rho^0(r)\) when \(\rho^0 = 0\). Indeed the compression term, obtained through (73), vanishes: \(\nabla \cdot \chi = 0\). Evidently, a plethora of interesting physics were neglected (\(\mu_0\) also vanishes from the equations). On the other hand, finite-beta effects are still somewhat included since \(b_0^0(r)\) and \(b_2^0(r)\) depend on \(\chi\) in the equilibrium ODE (45).

To determine the appropriate initial conditions (i.e. the boundary condition at \(r = 0\)) for (94) we insert the series expansions \(y_1 = a_0 + a_1 r + a_2 r^2 + \ldots + a_j r^j + \ldots\) and \(y_2 = b_0 + b_1 r + b_2 r^2 + \ldots + b_j r^j + \ldots\) in (94) and identify the indicial equations \(j = 0, 1, 2, \ldots\)

\[
 j f^2a_j = \tilde{c}_{11}a_j + \tilde{c}_{12}b_j, \quad j f^2b_j = \tilde{c}_{21}a_j - \tilde{c}_{11}b_j
\]

(96)

For (96) to have a nonzero solution it is required that \((\tilde{c}_{11} - jf^2)(-\tilde{c}_{11} - jf^2) - \tilde{c}_{12}\tilde{c}_{21} = 0\). It can be shown that at \(r = 0\) it holds: \(\tilde{c}_{11} = \mu(0) f(0)/2 + k\) and \(\tilde{c}_{12} = -m^2\) and \(\tilde{c}_{21} = -f^4(0) + f^2(0)\mu^2(0)\) where \(f(0) = m\mu(0)/2 + k\) and \(\mu(0) = 2\Theta_0/a\) (see (41)). The vanishing-determinant condition for (96) can be written \(f^4(0)(j^2 - m^2) = 0\) which implies

\[
y_1 \sim r^m \quad \text{and} \quad y_2 \sim \frac{f(0)}{m} (\mu(0) - f(0)) y_1
\]

(97)

as \(r \to 0^+\). Since (94) is linear in \(y(.)\) the amplitudes of \(y(.)\) are arbitrary.
Eyeballing (94) we soon spot a potential problem. A radial coordinate \( r = r_s \) that solves \( f(r) = 0 \) turns out to exist for some combinations of \( m:s \) and \( k:s \). Such \( r_s:s \) are called resonant surfaces. We will need to compute the external \( r_s < r < a \) side of such resonant modes in order to determine its growth-rate. To solve (94) we therefore have to sketch the initial (boundary) condition at \( r = r_s \). While at it, we will also derive Suydam’s criterion (43) as a side-result, and relate system (94) to a seminal paper [93].

The classic Newcomb equation [93], given by (100) below, can be obtained by rewriting (94) to a single second-order ODE. Use \( y_1' = \xi_r + r\xi_r' \) in (94a) and refurbish the result to

\[
f_0\xi_r' = A\xi_r - ry_2
\]

where

\[
f_0 = \frac{r^3f^2}{c_{12}} = \frac{r^3f^2}{m^2 + r^2k^2}
\]

and \( A = (r^2f^2 - r^2c_{11})/c_{12} \). Differentiating (98) gives \( (f_0\xi_r)' = A'\xi_r + A\xi_r' - y_2 - ry_2' \). (94b) is equivalent to \( rf^2y_2' = r\bar{c}_{21}\xi_r - \bar{c}_{11}y_2 \) so \( (f_0\xi_r)' = (A' - \bar{c}_{21}r/f^2)\xi_r + A\xi_r' + (\bar{c}_{11}/f^2 - 1)y_2 \). Using (98) to eliminate \( y_2 \) then yields [64]

\[
(f_0\xi_r)' - g_0\xi_r = 0
\]

where

\[
\begin{align*}
g_0 &= \left( \frac{r^2f^2 - r^2c_{11}}{c_{12}} \right)' - \frac{\bar{c}_{21}r}{f^2} + \frac{\bar{c}_{11} - f^2}{c_{12}} \left( \frac{r^2f^2 - r^2c_{11}}{f^2} \right) \\
&= \frac{1}{c_{12}} \left[ 2r(f^2 - \bar{c}_{11}) + r^2(f^2 - \bar{c}_{11})' - \frac{r\bar{c}_{12}\bar{c}_{21}}{f^2} - \frac{r}{f^2}(f^2 - \bar{c}_{11})^2 \right] \\
&- \frac{1}{c_{12}^2} \left[ r^2c_{12}'(f^2 - \bar{c}_{11}) \right] \\
&= \frac{2k^2r^2\mu_0p'}{m^2 + r^2k^2} + \frac{rf^2(m^2 + r^2k^2 - 1)}{m^2 + r^2k^2} - \frac{2fr^3k^2(mb_0^2/r - kb_0^2)}{(m^2 + r^2k^2)^2}
\end{align*}
\]

(101)
after a session of algebra. The particular form (100) with (99) and (101) can be found in [64] but other flavours also exist in the literature [23, 29]. To finalize (101) we used the equilibrium relation \( \mu_0p' + b_0^0b_0'' + b_0^0b_0'' + (b_0^0)^2/r = 0 \), that follows from \( \nabla p = \nabla (b^0 \cdot b^0)/(2\mu_0) \), and the identity \( r_0^0(b_0^0/r)' = b_0^0b_0'' - (b_0^0)^2/r \).

It is also handy to note \( f^2 - \bar{c}_{11} = (f - 2mb_0^2/r)f = -(mb_0^2/r)^2 + (kb_0^2)^2 \) while doing the calculations.

We are now in a position to derive condition (43), that we invoked out-of-the-blue in subsection 4.4.2, from (100) at the singular position \( r = r_s \) where \( f(r) = 0 \). Introduce \( s = r - r_s \) and calculate the lowest order coefficients in the expansions of (99) and (101) at \( s = 0 \). Then (100) transforms to \((s^2\xi_r')' - w\xi_r = 0\) where \( w = 2k^2\mu_0p'/(rf^2) \). Taking the ansatz \( \xi_r \sim s^v \) implies that \( v(v+1) - w = 0 \) with the solution \( v = -1/2 \pm (1/2)\sqrt{1+4w} \). Non-oscillatory solutions require \( 1 + 4w > 0 \). Recall \( q \) from (42). It holds, where \( f = 0 \), that \( f'/kb_0^2 = q'/q \), so \( 1 + 4w > 0 \) is equivalent to

\[
1 + 8\mu_0p' \frac{1}{b_0^2} \left( \frac{q}{q'} \right)^2 > 0
\]

(102)
which is just (43).

Since our finite-\(\beta\) equilibrium is designed by enforcing (44) we will not have oscillatory \(\xi\)'s near \(r_s\), on the other hand, for both branches: \(v_1, v_2 < 0\). This follows from 1 + 4\(w = 1 - \chi\), 1 > \(\chi\) ≥ 0 as (103) below shows. Both branches diverge as \(s \to 0\) (we are interested in the external part \(s > 0\)) and are manifestly singular. We can evaluate

\[
w = \frac{x'_3}{4r} \left( \frac{x_2}{r} - \mu(r) \frac{x_1^2 + x_2^2}{2x_1} \right)^{-2} = \frac{1}{4} \chi
\]

at \(r = r_s\), as is seen from the normalized equilibrium quantities \(x(\cdot)\) of subsection 4.4.2. Indeed, by construction \(\chi = 1 \Rightarrow 1 + 4w = 0\). It can be argued that the so-called ‘small’ branch \(\xi_r \sim s^{v_1}\) where \(-1/2 < v_1 = -1/2 + (1/2)\sqrt{1 - \chi} < 0\) since \(0 < \chi < 1\) is applicable for our problem (the ‘large’ branch does not represent finite energies [64]). Even tough \(\xi_r\) diverge near \(r = r_s\) it follows from (58) that \(ib_r^1 = -f\xi_r \sim f'(r_s)s s^{v_1} \sim s^{v_1 + 1}\) and \(1/2 < v_1 + 1 < 1\). Therefore \(b_r^1 \to 0\) as \(r \to r_s\).

### 4.4.4 An RWM shooting procedure

To implement a shooting code for the RWM problem is is useful to properly normalize the eigenvalue equation. It is straightforward, but boring, to show that (74) and (75) can be rebranded using

\[
\begin{align*}
v_0 &= b_0/\sqrt{\mu_0 \rho_0} \\
\tau_0 &= a/v_0 \\
\hat{\omega} &= \omega \tau_0 \\
\hat{r} &= r/a \\
\hat{\rho} &= \rho^0/\rho_0 \\
\hat{p} &= p^0/(\rho_0 v_0^2) = \mu_0 p^0/b_0^2 = (1/2)\hat{\beta} \\
\hat{k} &= k a \\
\hat{b}_\theta &= b_\theta^0/b_0 \\
\hat{b}_z &= b_z^0/b_0 \\
\hat{\xi} &= \xi_r/a
\end{align*}
\]

as

\[
\begin{align*}
\hat{D}_r \frac{\partial}{\partial \hat{r}} \hat{y}_1 &= \hat{c}_{11} \hat{y}_1 + \hat{c}_{12} \hat{y}_2 \\
\hat{D}_r \frac{\partial}{\partial \hat{r}} \hat{y}_2 &= \hat{c}_{21} \hat{y}_1 - \hat{c}_{11} \hat{y}_2
\end{align*}
\]
where

\[ \dot{y}_1 = \dot{r}\dot{\xi} = (r/a)(\xi_r/a) \]  
\[ \dot{y}_2 = p_v/(\rho_v^2) = \mu op^2/b_0^2 \]  
\[ \dot{c}_{11} = 2\left\{b_0 \frac{m_0}{r} \dot{f} - \dot{b}_0^2 \hat{\lambda}\right\} \]  
\[ \dot{c}_{12} = \dot{r}^2 \hat{\lambda} - m^2 - \hat{k}^2 \dot{r}^2 \]  
\[ \dot{c}_{21} = -\dot{D}^2 + 2\dot{D}b_0 \frac{\partial}{\partial r} \left(\frac{b_0}{r}\right) \]  
\[ + 4\left(\frac{\dot{b}_0}{r}\right)^2 \left\{\dot{f}^2 - \hat{\Lambda}\dot{b}_0^2\right\} \]  
\[ \dot{D} = -\hat{\rho}\hat{\omega}^2 + \dot{f}^2 \]  
\[ \dot{f} = \frac{m_0}{r} b_0 + \dot{k}b_z \]  
\[ \hat{\lambda} = \frac{\hat{\rho}^2\hat{\omega}^4}{\hat{\rho}\hat{\omega}^2 \left(b^2 + \gamma \rho\right) - \gamma \hat{\rho}f^2} \]  

with \( \hat{b}^2 = \dot{b}_0^2 + \dot{b}_z^2 \). In words, we have normalized \( \hat{\omega} \) to the Alfvén transit time \( \tau_0 \). Let’s rewrite the similarly normalized equilibrium quantities (it is practical to integrate these in parallel to make good use of the adaptive-step ODE library solvers).

\[ \frac{\partial}{\partial \hat{r}} \dot{b}_0 = -\frac{\dot{b}_0}{\hat{r}} + \hat{\mu} \hat{b}_z - (1/2) \frac{\dot{b}_0}{\hat{b}_0^2} \frac{\partial}{\partial \hat{r}} \hat{\beta} \]  
\[ \frac{\partial}{\partial \hat{r}} \dot{b}_z = -\hat{\mu} \hat{b}_0 + (1/2) \frac{\dot{b}_z}{\hat{b}_0^2} \frac{\partial}{\partial \hat{r}} \hat{\beta} \]  
\[ \frac{\partial}{\partial \hat{r}} \hat{\beta} = -\hat{\chi} \hat{f} \left(\frac{\dot{b}_z}{\hat{r}} - \hat{\mu} \dot{b}_0^2/(2\hat{b}_0)\right)^2 \]  

In (107) we have introduced \( \hat{\mu}(\hat{r}) = a\mu(r) = 2\Theta_0 (1 - \hat{r}^\alpha) \), and redefined

\[ \hat{\beta} = 2\mu \hat{\rho}^2/b_0^2 = \hat{\rho} \]  

and used \( \dot{b}^2 = \dot{b}_0^2 + \dot{b}_z^2 \). Boundary condition for (107) is

\[ \dot{b}_0/\hat{r} \rightarrow \hat{\mu}(0)/2 \text{ as } \hat{r} \rightarrow 0^+ \]  

and \( \dot{b}_z(0) = 1 \) as \( b_0 = \dot{b}_z(0) \) (if \( b_0 \) is the on-axis toroidal field). Series expansion for (105) near \( \hat{r} = 0 \) shows that the initial (boundary) condition is

\[ \dot{y}_1 \sim \hat{r}^m, \quad \dot{y}_2 = -\frac{1}{m} \left(-\hat{\rho}\hat{\omega}^2 + \dot{f}(0) \left\{\dot{f}(0) - \hat{\mu}(0)\right\}\right) \dot{y}_1 \]  

as \( \hat{r} \rightarrow 0^+ \). Recall \( \dot{f}(0) = m\hat{\mu}(0)/2 + \dot{k} \), \( \hat{\mu}(0) = 2\Theta_0 \) using (109).

The continuous spectra of system (105) is given by [19]

\[ 0 = \left(-\hat{\rho}\hat{\omega}^2 + \dot{f}^2\right) \left\{\hat{\rho}\hat{\omega}^2 \left(b^2 + \gamma \hat{\rho}\right) - \gamma \hat{\rho}\dot{f}^2\right\} \]  

43
which give the only true singularities of (105). The the speed of sound
\[ v_1 = \sqrt{\gamma p_0 / \rho_0} = v_0 \sqrt{\gamma} \]  
(112)
is useful when elaborating on the compressible MHD-wave solutions (in (112) use \( p_0 = \rho_0 c_0^2 \)). Anyway, we will however be most concerned with the low-frequency discrete spectra of the RWM. As suggested in the previous section (105) should be integrated in \( 0 < \hat{r} < 1 \), for some trial \( \hat{\omega}^2 \), and the quantities \( \hat{y}_{1,2} \) at the plasma edge \( \hat{r} = 1 \) must fulfil an algebraic relation that includes a resistive term; \( \hat{\omega} \) to a unity power. In general this problem can have complex-valued \( \hat{\omega} \)s as solutions.

Modifying a bit yields (for \( \hat{k} = k a \neq 0 \))
\[
\begin{pmatrix} |\hat{k}| l'_m(\hat{k}) & |\hat{k}| K_m'(\hat{k}) \\ f l_m(\hat{k}) & f K_m(\hat{k}) \end{pmatrix} \begin{pmatrix} \hat{c}_2'^{(ii)} \\ \hat{C}_2' \end{pmatrix} = \begin{pmatrix} -\hat{f} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} \]  
\( \hat{r} = 1 \)  
(113)
and the RWM dispersion-relation can be cast
\[
- i \hat{\omega} \frac{\tau_w}{\tau_0} \frac{|\hat{k}| \hat{r}}{m^2 + k^2 \hat{r}^2} = \frac{\hat{c}_2'^{(ii)} K_m(\hat{k} \hat{r}) - \hat{c}_1'^{(ii)} l_m(\hat{k} \hat{r}) - \hat{c}_2^{(ii)} K_m(\hat{k} \hat{r})}{\hat{c}_1'^{(ii)} l_m(\hat{k} \hat{r}) + \hat{c}_2^{(ii)} K_m'(\hat{k} \hat{r})} 
\]  
(114)
where \( \hat{k} \hat{r} = k r_w \), the \( \hat{C}_{(i)} \)'s being the solution to (114) and
\[
\hat{c}_2'^{(ii)} = \hat{c}_1'^{(ii)} \frac{l_m'(\hat{k} \hat{r})}{K_m'(\hat{k} \hat{r})} + \hat{c}_2'^{(ii)} \text{, at } \hat{r} = r_w / a 
\]  
(115)
in parallel with subsection 4.4.3.

The resistive-shell boundary condition can therefore explicitly be written in the promised algebraic form
\[ 0 = g(\hat{\omega}) \]  
(116)
such that (114), (115), (92) combine to
\[
g(\hat{\omega}) = i \hat{\omega} \frac{\tau_w}{\tau_0} \frac{|\hat{k}| \hat{r}}{m^2 + k^2 \hat{r}^2} + \frac{\hat{c}_1'^{(ii)} (K_m(\hat{k} \hat{r}) l_m'(\hat{k} \hat{r}) - l_m(\hat{k} \hat{r}) \hat{r} \hat{C}_m(\hat{k} \hat{r}) + \hat{C}_2^{(ii)} K_m'(\hat{k} \hat{r}) \hat{C}_m'(\hat{k} \hat{r})}{\hat{c}_1'^{(ii)} l_m(\hat{k} \hat{r}) + \hat{c}_2^{(ii)} K_m'(\hat{k} \hat{r})} 
\]  
\[
= i \hat{\omega} \frac{\tau_w}{\tau_0} \frac{|\hat{k}_w|}{m^2 + k_w^2} + \left( l_m(\hat{k}_w) - \hat{C}_m(\hat{k}_w) \frac{l_m'(\hat{k}_w)}{K_m'(\hat{k}_w)} \right) f_\hat{\omega} (\hat{\omega}^2) 
\]  
\[
= i \hat{\omega} \frac{\tau_w}{\tau_0} \frac{|\hat{k}_w|}{m^2 + k_w^2} - \frac{1}{|\hat{k}_w| K_m'(\hat{k}_w)} f_\hat{\omega} (\hat{\omega}^2) 
\]  
(117)
with
\[
f_\hat{\omega} (\hat{\omega}^2) = \frac{K_m a}{(- l_m' + K_m' + K_m'' \hat{y})} f \hat{\omega} + \frac{K_m a}{(- l_m' + K_m' + K_m'' \hat{y})} f \hat{\omega} \hat{k} \hat{y}_2 
\]  
(118)
where (·)_w, (·)_w imply evaluation at \( \hat{k}_w, \hat{k} \) respectively, and \( \hat{k}_w = k r_w, \hat{k} = k a \). Of course, \( \hat{f} \) and \( \hat{y}_{1,2} \) are evaluated at \( \hat{r} = 1 \). That’s it. Let’s implement an old-school shooting code. By the way, the recurrence formulas [6, 101]
\[
2 l_m' = l_{m+1} + l_{m-1}, \quad -2 K_m' = K_{m+1} + K_{m-1} 
\]  
(119)
for the modified Bessel functions are handy when implementing.

From (117) one can anticipate that the order of the ratio $\tau_w/\tau_0$ plays a crucial role. For $\omega^2 < 0$ (105) should be free from singularities. Typical plasma parameters for T2R are electron number density $n_e \sim 1 - 2 \times 10^{19}$ m$^{-3}$ and on-axis toroidal field strength $b_0 \sim 0.1$ T. Recalling the proton rest mass [94] $m_p \approx 1.6726 \times 10^{-27}$ kg we can approximate $\tau_0 = (a/b_0)\sqrt{\mu_0 n_e m_p}$ as somewhere ($a = 0.183$ m) between .27 and .38 microseconds (µs). Since, for T2R $\tau_w \sim 10 \times 10^{-3}$ seconds we find that it is defendable to assign $\tau_w/\tau_0 = 3 \times 10^4$ in (114). This was done to obtain figure 8 below. The solutions to (118) we are interested in are particular to the resistive-shell setup. The RWMs emerge as a new discrete spectrum from $\omega^2 = 0$ whereas the ideal marginally stable oscillations are slightly damped (shifted spectra) by the presence of the resistive shell [59].

For more general computations of cylindrical plasma eigenmode solutions, manifestly resistive spectra, complex growth-rates and so forth, see e.g. [78].

4.4.5 Linearized mode dynamics with active external coils

We open this finalizing subsection by considering a plasmaless cylindrical resistive shell. It is an illustrating calculation and warm-up. It is useful to compare the resulting field diffusion equation to the plasma-filled cylinder equation subsequently derived below.

Denote the three vacuum regions $(i)$, $(iii)$ and $(iv)$. The intervals are $0 < r < r_w$, $r_w < r < r_c$ and $r_c < r < +\infty$, respectively. A passive resistive shell resides at $r_w$. At $r_c$ we have a source current-sheet. Consult figure 6. The scenario is readily condensed to four scalar boundary conditions.

$$
\tau_w b_r^w = \left[imb_0 + ikrb_z^c r_w^+ \right]_{r_w} (120a)
$$

$$
0 = \left[ib_r^+ r_w\right]_{r_w} (120b)
$$

$$
\left( m^2/r^2 + k^2 \right) J_c = \left[i(m/r)b_0 + ikb_z^+ r_c^+ \right]_{r_c} (120c)
$$

$$
0 = \left[ib_r^+ r_c\right]_{r_c} (120d)
$$

From equations (120) and (77) it follows

$$
\tau_m k b_r^w + \theta_r^w = \left(-k^2 r_c I_m^w K_m^c \right) J_c (121)
$$

after some algebra. The modal wall time is

$$
\tau_{mk} = -\tau_w \frac{I_m^w K_m^c}{1 + \frac{m_m}{k^2 \tau_w}} > 0 (122)
$$

depicted in figure 7. For T2R: $r_w = 0.198$ m so the modified Bessel functions in (122) should be evaluated in $kr_w = \tilde{k}(r_w/a) \approx 1.08\tilde{k}$.

The exact same boundary conditions at $r = r_w$ and $r = r_c$, i.e. equations (120) also apply to the plasma-filled cylinder. In addition, the previously treated plasma-vacuum boundary condition at $r = a$ must also be used. Denote the four domains of this problem $(i)$, $(ii)$, $(iii)$ and $(iv)$. The respective intervals, sketched in figure 6, are this time $0 < r < a$, $a < r < r_w$, $r_w < r < r_c$ and
$r_c < r < +\infty$, where (i) is plasma and the remaining vacuum. Recycling a substantial portion of the algebra invoked to obtain (121), we arrive at e.g.

$$\tau_{mk} \dot{b}_r + b_r = |k| K_m \dot{w} i C_2^{(ii)} = \left(-k^2 r_c \dot{I}_m^w K_m^{w'}\right) J_c$$ (123)

which appears to be the vacuum expression fortified with a bonus term on the left-hand side. Indeed, $C_2^{(ii)}$ is one of the constants for the vacuum magnetic field (77) in the region $a < r < r_w$. The “free” growth-rate of (123) is obtained by zeroing $J_c$ and solving for $\omega$ using the shooting-code as detailed in subsection 4.4.4. Below we derive various remixes of (117).

The RWM dispersion relation

$$1 - i\omega \tau_{mk} - \frac{C_2^{(ii)} K_m^{w'}}{C_1^{(ii)} I_m^{w'} + C_2^{(ii)} K_m^{w'}} = 0$$ (124)

follows from $J_c = 0$ in (123). Recall that $C_1^{(i)}, i$ are functions of $\omega^2$. A relation equivalent to (125) using the shooting-code normalization (104) is

$$0 = 1 - i\omega \frac{\tau_{mk}}{\tau_0} = \left\{1 - \left(I_m^{w'}/K_m^{w'}\right) \frac{\hat{y}_1 \hat{f}^2 K_m^a + \hat{y}_2 |k| K_m^{a'}}{\hat{y}_1 \hat{f}^2 I_m^a + \hat{y}_2 |k| I_m^a}\right\}^{-1}$$ (125)

yet another version of (117). Indeed equation (116) is equivalent to

$$0 = i\omega \frac{\tau_{mk}}{\tau_0} + I_m^{w'} f_\delta(\hat{\omega}^2)$$ (126)

which follows by multiplication of (117) by $-|k_w| I_m^{w'} K_m^{w'}$. It is straightforward to verify that (126) is the exact same equation as (125).

A solution $\tilde{\gamma}_{mk} = -i\omega$ to (125) dictates the exponential growth described by

$$\dot{b}_r = \tilde{\gamma}_w b_r \quad \Leftrightarrow \quad \tau_{mk} \dot{b}_r - \tau_{mk} \tilde{\gamma}_{mk} b_r = 0$$ (127)

In conclusion, the “prototype” RWM model is (128).

$$\tau_{mk} \dot{b}_r - \tilde{\gamma}_{mk} b_r = a_{mk} b_r$$ (128)

where

$$\tilde{\gamma}_{mk} = \tau_{mk} \tilde{\gamma}_{mk}$$ (129)

is a normalized modal growth-rate and

$$a_{mk} = -dk^2 r_c I_m^{w'} K_m^{c'} > 0$$ (130)

a dimensionless factor of proportionality. Some basic properties of (128) are

$$\tilde{\gamma}_{mk} > 0 \quad \text{instability}$$

$$0 > \tilde{\gamma}_{mk} > -1 \quad \text{amplification}$$

$$-1 > \tilde{\gamma}_{mk} \quad \text{attenuation}$$

and the particular case $\tilde{\gamma}_{mk} = -1$ renders (123) equivalent to (121). Hence for all $(m, k)$ such that $\tilde{\gamma}_{mk} = -1$ the temporal mode dynamics is not affected by the plasma (although the eigenmodes’ profiles are different).
A plot of $\hat{\gamma}_{mk}$ is provided in figure 8. The dispersion relation in this plot corresponds to the equilibrium depicted in figure 5. The annotations in figure 8 employ RFP nomenclature: *internal* modes have $k < 0$, *external* $k > 0$. *Resonant* modes have $f = 0$ somewhere in $0 < \hat{r} < 1$ which means $-(a/R)m/k = q$.

The wavenumbers for internal and external resonance are respectively given by $\hat{k} < \hat{k}_{\text{int}}$ and $\hat{k} > \hat{k}_{\text{ext}}$, where $\hat{k}_{\text{int}} = -(a/R)m/q(0) < 0$ and $\hat{k}_{\text{ext}} = -(a/R)m/q(a) > 0$. *Nonresonant* modes reside in the interval $\hat{k}_{\text{int}} < \hat{k} < \hat{k}_{\text{ext}}$.

Figure 7: Modal wall-time for $m = 1, 2, 3$, for $r_w/a = 1.08$ (corresponding to T2R). Higher wavenumbers $(m, k)$ imply shorter wall-time.

### 4.5 Magnetic islands: friend or foe?

It can be argued that IMHD breaks down at the surface $r = r_s$ in the cylindrical plasma where $f(r_s) = 0$. The basic problem is that the resistive term in the diffusion equation should not be ignored when the spatial gradients are large.

It turns out that resistive resonant perturbations can tear up the nested flux surfaces of the equilibrium [60]. We will show illustrations of this in subsection 4.5.2. A monochromatic resonant perturbation typically creates a *magnetic island*\(^5\). Multiple resonant modes can completely shatter the orderly excursions of equilibrium field lines. If this happens, confinement is more or less destroyed.

\(^{5}\text{Incidentally, there exists a real Magnetic Island, a 52 km}^2\text{ tourist magnet at best, barely off the Australian north-east coast [12]. Allegedly, James Cook had some issues with his compass while passing nearby 1770. The island, having approximately 2000 permanent residents, is of course also blessed with the Magnetic Times daily online newspaper.}
Figure 8: RFP stability example summarized. As $|k\alpha| \to \infty$ the (internal or external) resonant surface approaches the reversal surface $r_s \to r_z$. 
in fusion-grade plasmas [105]. Cartoon MHD modes, as developed in subsection 4.5.1, will be used for the illustrations.

These illustrations unequivocally seems to show that resistive resonant modes can never be a good thing for any MCF reactor. It might however be a most useful thing to provoke stochasticity in the magnetic field, in some situations. Ongoing research attempts to mitigate (and provoke) ELMs by applying resonant magnetic perturbations (RMPs) to the edge of tokamak plasmas [24, 38, 51, 97]. Stellarators are on purpose designed with islands and stochastic fields working as divertors [102]. In addition, achieving a healthy dominant, nearly monochromatic, magnetic island would possibly be a blessing for the RFP [86].

4.5.1 Sketchy resistive MHD modes

Perhaps the easiest way to sketch resistive eigenfunctions is to utilize equations (105) and (106) with \( \rho = 0 \) and avoid the resonant position \( r = r_s \) where \( j(r_s) = 0 \). On an intermediate time-scale \( \tau_s \), significantly slower than the Alfvén transit time \( \tau_0 \) but faster than the resistive-shell time \( \tau_w \) the magnetic field is expected to diffuse through the “barrier” at \( r_s \). Suppose we insert a current-layer at the resonant surface \( r_s \), where a finite plasma resistivity \( \eta_s \) is assumed. Let the layer have width \( \delta_s > 0 \). Introduce a resistive-layer time

\[
\tau_s = \mu_0 r_s \delta_s / \eta_s
\]  

(131)

For our purposes its order of magnitude is important. Pretension of an “exact” value of \( \tau_s \) is most probably meaningless due to the crudeness of the model. Using \( \delta_s = 1 \text{ mm}, \eta_s = 10^{-6} \Omega/\text{m} \) and \( r_s = 0.1 \text{ m} \) yields (131) \( \tau_s = 10^{-2} \mu_0 \approx 10^{-4} \text{ sec} \). So \( \tau_s \) is about two orders of magnitude larger than our estimate of \( \tau_0 \), and two orders of magnitude smaller than \( \tau_w \). The approximate value for \( \eta_s \) is based on a resistive time-scale of T2R [56, 57]: \( \tau_R = a^2 \mu_0 / \eta \approx 0.1 \text{ s} \) which yields a global average \( \eta \approx 10a^2 \mu_0 \) of the order \( \eta_s \).

In order to stay away from \( r \in [r_s - \delta_s / 2, r_s + \delta_s / 2] \) we will do integration of (105) forward \( r = 0 \ldots r_w^- \), and backward \( r = 1 \ldots r_w^+ \) respectively, and scale the two solutions such that \( b_v(r_w^-) = b_v(r_w^+) \) (continuous radial magnetic field component), where \( r_w^\pm = r_s \pm \delta_s / 2 \). In general, the tangential magnetic field will jump over the resistive layer (surface current).

If we prescribe \( b_v(r_w) \) and \( b_\theta(r_w) \) for the solution we find that the boundary condition for (105) at \( r = a \) follows from (77) and (113)

\[
\left( \begin{array}{c}
-\hat{f} \hat{y}_1 \\
\hat{y}_2
\end{array} \right)_{\hat{r} = 1} = \left( \begin{array}{c}
|\hat{k}| I_m^a \hat{f} \hat{I}_m^a \\
|\hat{k}| K_m^a \hat{f} K_m^a
\end{array} \right)
\] 

(132)

\[
\times \left( \begin{array}{c}
|\hat{k}| I_m^{w'} m \hat{I}_m^{w'} m \\
|\hat{k}| K_m^{w'} m \hat{K}_m^{w'} m
\end{array} \right)^{-1} \left( \begin{array}{c}
\hat{b}_r^1 \\
\hat{b}_\theta^1
\end{array} \right)_{\hat{r} = r_w / a}
\]

whereas the condition (110) at \( \hat{r} = 0 \) is unchanged. The matching condition for the sketched solution is

\[
0 = \left[ -\hat{y}_1 / \hat{f} \hat{f} \right]_{\hat{r} = r_w^-}
\]  

(133)

apparently. The linear stability can be estimated in analogy with (87).

More proper treatment of the cylindrical-plasma resistive modes are found in e.g. [78, 90]. A shooting code with resistivity was used for a finite-\( \beta \) RFP
study in [90]. A problem with the RFP is that resonant resistive MHD modes are not linearly stable in experiments; and so their finite amplitude nonlinear evolution comes in.

4.5.2 Field-line trajectories in (un-)perturbed RFP

Nothing will be withheld from them that walk randomly.

We will in this subsection show the character of magnetic field-line trajectories, based on our example RFP equilibrium. First the unperturbed equilibrium will be assessed. Second, this equilibrium will be perturbed by a resonant ideal MHD eigenfunction. Recall that these ideal modes have \( b_r = 0 \) at the resonant position where \( f = 0 \). These perturbations will appear rather benign, from the Poincaré point of view. Third, the iffy resistive modes sketched in subsection 4.5.1 will be overlaid to the RFP, first one then another, then both of them concurrently. For these resistive modes, where \( b_r \neq 0 \) at \( f = 0 \), the magnetic field will start to look quite different [31, 46, 50, 60, 68, 129].

Field-line integration is straightforward. Using the arc-length as the parameter \( s \) we obtain for the cylinder coordinates \((\hat{r}, \theta, \hat{z})\)

\[
\begin{align*}
\frac{d\hat{r}}{ds} &= \alpha B_r(\hat{r}, \theta, \hat{z}) \\
\frac{d\theta}{ds} &= (\alpha/\hat{r}) B_\theta(\hat{r}, \theta, \hat{z}) \\
\frac{d\hat{z}}{ds} &= \alpha B_z(\hat{r}, \theta, \hat{z})
\end{align*}
\]

where \( \alpha = \alpha(\hat{r}, \theta, \hat{z}) = (B_r^2 + B_\theta^2 + B_z^2)^{-1/2} \) and \( B_r = b^0_r(r) + b^1_r(\hat{r}, \theta, \hat{z}) \) (analogously for the other components). The simplified massless case \( \rho = 0 \) implies that the eigenfunction components can be expressed in the nondimensional shooting variables \((104), (106)\), as follows.

\[
\begin{align*}
\hat{b}^1_r,_{mn} &= -\hat{f}\hat{y}_1/\hat{r} \\
\hat{b}^1_\theta,_{mn} &= -\left(2\hat{y}_1/\hat{r}^2 + \frac{\partial}{\partial \hat{r}} \left( \hat{b}^0_\theta/\hat{r} \right) \right) \hat{y}_1 + m\hat{y}_2/(f\hat{r}) \\
\hat{b}^1_z,_{mn} &= -\hat{y}_1 \frac{\partial}{\partial \hat{r}} \left( \hat{b}^0_z \right) /\hat{r} + (\hat{k}/\hat{f})\hat{y}_2
\end{align*}
\]

Combining (135) with perturbation amplitudes \( A_{mn} \) and phases \( \varphi_{mn} \) results in a total vector field

\[
\begin{align*}
B_r &= \hat{b}^0_r(\hat{r}) + \sum_{mn} A_{mn} \cos \left( -\frac{\pi}{2} + \phi_{mn} \right) \left( \hat{b}^1_r,_{mn} \right) \quad (136a) \\
B_\theta &= \hat{b}^0_\theta(\hat{r}) + \sum_{mn} A_{mn} \cos (\phi_{mn}) \hat{b}^1_\theta,_{mn} \quad (136b) \\
B_z &= \hat{b}^0_z(\hat{r}) + \sum_{mn} A_{mn} \cos (\phi_{mn}) \hat{b}^1_z,_{mn} \quad (136c)
\end{align*}
\]

for equation (134), where \( \phi_{mn} = m\theta + \hat{k}\hat{z} + \varphi_{mn} \).

Poincaré-maps defined by the intersection of the field-lines (134) with the plane(s) \( 0 = \mod (\hat{z}, 2\pi R/a) \) are displayed in figure 9. The exact same initial
Figure 9: Intersections of magnetic field lines with the “periodic” plane $z = 0$. 
conditions have been used for all figures; identity coded by cheerful colours. A total of 16 field lines have been followed for each figure. Each field line has been traced the exact same distance, i.e. \( s = 0 \ldots s_{\text{max}} \) for system (134), with the same \( s_{\text{max}} \) for all trajectories. Here, \( s_{\text{max}} = N_0 \times 2\pi(R/a) \), which means that the magnetic-axis field line \( \hat{r} = 0 \) would do \( N_0 \) intersections with \( 0 = \text{mod}(\hat{z}, 2\pi R/a) \) for the unperturbed (equilibrium) field \( \mathbf{b}_0(\hat{r}) \). \( N_0 = 1000 \) for the maps in figure 9.

Compared to the central field, not many intersections occur near the edge. The reason is that \( b_z \) is close to zero near the reversal surface so field lines have to be followed for a long distance before a return to the plane of intersection. Also note that if it happens that a magnetic surface has a \( q \)-value close to a rational number, the surface will not be traced out very densely since the field line map jumps orderly between a discrete set of point on this surface. Notice the complete anarchy, figure 9(e), produced by adding two magnetic islands that occupies a shared volume of space. This volume is approximately the impressionistic potpourris encompassing the core in figure 9(e).

A transformation, and plot, of the complete field-line trajectory to the modeled toroidal shape in cartesian coordinates \((\bar{x}, \bar{y}, \bar{z})\)

\[
\bar{x} = (\kappa + \hat{r} \cos \theta) \cos \phi \\
\bar{y} = (\kappa + \hat{r} \cos \theta) \sin \phi \\
\bar{z} = \hat{r} \sin \phi \\
\kappa = R/a \\
\phi = z/R = \hat{z}/\kappa
\]

hints the three-dimensional geometry of the RFP magnetic field. This is shown in figure 10. Parts of the magnetic island of figure 9(d) corresponds to figure 10(a). The stochastic field of figure 10(b) appears no good for particle confinement; figure 9(e) corresponds to this fully three-dimensional taglialette. The bichromatic resistively perturbed line in 10(b) fills a volume if traced an infinite distance. The monochromatic field line 10(a) covers an island surface if followed an infinite distance.

The structure of the field lines is not, of course, the full story of plasma confinement. Obviously, field-lines are only followed by charged particles approximately. The exact motion is more complicated, and incentivizes the use of guiding-center drift theories. Nevertheless, if the magnetic field is “stochastic” in a substantial volume to begin with, energy confinement will inevitably be doomed serious degradation.

An user-friendly approximate criterion for “stochastic” (the term “deterministic chaos” might be more appropriate) onset has been conjured up [41]: if individual magnetic islands “overlap” (as quantified by the Chirikov parameter) then they provoke ergodic field lines in a nonzero volume. Time-evolution of magnetic islands and tearing-modes are highly nonlinear [104] and must incorporate plasma flow velocity for quantitative relevance [54].

4.5.3 Nonlinear RFP dynamics

The self-organized magnetic field in the RFP is loosely explained as the nonlinear interaction of a bunch of never-settling tearing-modes: the “RFP-dynamo” [30, 96]. It is usually described as a competition between magnetic diffusion and
Figure 10: Snazzy 3D toroidal magnetic field lines in the perturbed RFP. Figures 10(a) and 10(b) each show a portion of the same single field line; starting from the same position.
plasma relaxation. Diffusion drives the system away from a particular state that turns out to be a minimum-energy state, given a constraint on global magnetic-helicity \[119\]. Therefore the plasma state falls back (relaxes) to this configuration repeatedly. This state is characterized by a reversed toroidal field direction close to the plasma edge. Simulations have shown \[107\] that resistive-MHD more-or-less explains the global RFP nonlinear plasma dynamics. Given a driving toroidal electric field that supplies energy to this leaky system, the magnetic configuration seems to be indefinitely self-sustained. Without an ideal shell, a feedback-stabilized boundary could probably provide the same indefinite sustainment.
5 Automatic control primer

A substantial part of this thesis can be considered as applied automatic control; in particular feedback control \cite{22, 62, 66}. To aid readers not trained in control theory this section covers some relevant basics. Others can read it as a short review. As an appropriate example and illustration, in subsection 5.2, we will linger and elaborate a bit on the prototype cylindrical-symmetry toy modal RWM model \eqref{128} that was painstakingly derived in subsection 4.4 earlier.

5.1 Condensed story of control

Some credit J.C. Maxwell for co-hatching control theory, for which he nevertheless is much less celebrated than for classical electromagnetism. A Governor \cite{88}, the story went, “is a part of a machine by means of which the velocity of the machine is kept nearly uniform, notwithstanding variations in the driving-power or the resistance”. Modern day “governors” are more casually referred to as controllers and are often to a very large extent defined in software, instead of being some ingenious 19th-century mechanical contrivance/contraption made of some shining polished alloy. Feedback control has been a key idea throughout the history of technology \cite{27}, gradually being more formalized.

The original topic of control theory was the stability of differential equations. This remains central. Control theory traditionally resides in an interdisciplinary layer between mathematics and engineering sciences. Present day control researchers study the general behaviour of dynamical systems and how to affect their evolution by realistically sized inputs, as well as observing their internals by incompletely sized outputs.

Control theory also tangents the modeling of systems, which often is regarded as physics. Control-oriented modeling endeavours to formulate model structures and models with minimal complexity (that still achieves acceptable accuracy) that are manageable with contemporary controller synthesis and analysis methods. In practice this is very important. Feedback can often handle model inaccuracy (which in some sense also is a Maxwellian “notwithstanding variation”), but feedback must be quickly computable. A first-principles, ab-initio, or otherwise grand approach to modeling can result in systems too large to handle. However, distilling the essentials of the model can be good-enough for a control system when closing the loop. Modern terminology for the task of Maxwell's governor, by the way, is disturbance rejection.

5.2 Some concepts by a worked example

5.2.1 Modeling and representation

Recall \eqref{128}:

$$\tau_{mk} \dot{b}_r^w - \gamma_{mk} b_r^w = a_{mk} b_c^c$$ \quad (138)

Model \eqref{138} represents the time evolution of a linearized global MHD mode on the time-scale set by the non-ideally conducting external shell surrounding the plasma vessel, in a periodic cylinder plasma, assuming perfect cylindrical symmetry not only for the plasma column equilibrium but also for the resistive shell. Be that as it may, from the control point of view we merely need to sort-out what is the input and what is the output.
The externally applied magnetic field (from actuator coils) \( b_c \) is the input, and the sensor coils essentially measures \( b_c \); the output. Aiming for a standard-looking representation we rewrite (138):

\[
\begin{align*}
\dot{x} &= Ax + Bu & \text{(139a)} \\
y &= Cx & \text{(139b)}
\end{align*}
\]

where \( x = b_c \), \( C = 1 \), \( A = \gamma_{mk}/\tau_{mk}, B = a_{mk}/\tau_{mk} \) and \( u = b_c \). Formulation (139) is in typical state-space notation. This example is trivial with state \( x \) a scalar, and therefore the system “matrix” \( A \) is also a scalar. Since \( u \) is a scalar, the system (139) is single-input. Similarly \( y \) is scalar; single-output. Thus (139) is SISO (single-input-single-output). Equations (139a) and (139b) are respectively the state dynamics and the measurement relation.

To hint at the flexibility of state-space representations, let’s accept that it might not be realistic to manipulate \( b_c \) directly as an input. Rather, to change the magnetic field linking the actuator coil one must feed a voltage \( v \) to a power amplifier that drives a current through the windings of the coil. This current, which generates the radially directed magnetic field \( b_c \), cannot be instantaneous. Suppose then that the voltage-to-magnetic-field system can be modeled by the first-order system

\[
\tau_a u + u = \alpha v
\]

where \( \tau_a \) is a time-constant (should be \( 0 < \tau_a < \tau_{mk} \)) and \( \alpha \) an amplification. If we define a vector state \( \mathbf{x} = (x \ u) \) it is a simple task to show that the cascade actuator/RWM system can be cast

\[
\begin{align*}
\dot{\mathbf{x}} &= \bar{A} \mathbf{x} + \bar{B} v & \text{(140a)} \\
y &= \bar{C} \mathbf{x} & \text{(140b)}
\end{align*}
\]

where

\[
\begin{align*}
\bar{A} &= \begin{pmatrix} \gamma_{mk}/\tau_{mk} & a_{mk}/\tau_{mk} \\ 0 & -1/\tau_a \end{pmatrix} & \text{(141a)} \\
\bar{B} &= \begin{pmatrix} 0 \\ \alpha/\tau_a \end{pmatrix} & \text{(141b)} \\
\bar{C} &= \begin{pmatrix} 1 & 0 \end{pmatrix} & \text{(141c)}
\end{align*}
\]

Of course, (140) is still SISO. The powerfulness of state-space approaches becomes more clear when defining control-objectives and for plants with multiple-inputs and/or multiple outputs (SIMO/MISO/MIMO). Alternatively, the SISO transfer function of (140) can be obtained by Laplace transformation [101] (signal transforms denoted by \( \tilde{\cdot} \), and Laplace transform argument by \( s \))

\[
s \tilde{\mathbf{x}} = A \tilde{\mathbf{x}} + B \tilde{v}, \quad \tilde{y} = C \tilde{\mathbf{x}}
\]

which results in

\[
\tilde{y} = G(s) \tilde{v}, \quad G(s) = C (sI - \bar{A})^{-1} \bar{B}
\]

so plugging in (141) yields

\[
G(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s - \gamma_{mk}/\tau_{mk} & -a_{mk}/\tau_{mk} \\ 0 & s + 1/\tau_a \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \alpha/\tau_a \end{pmatrix}
\]

\[
= \frac{(a_{mk}/\tau_{mk})(\alpha/\tau_a)}{(s - \gamma_{mk}/\tau_{mk})(s + 1/\tau_a)}
\]

(144)
which is just the product $G(s) = G_1(s)G_2(s)$ of two first-order transfer functions

$$G_1(s) = \frac{a_{mk}/\tau_{mk}}{s - \hat{\gamma}_{mk}/\tau_{mk}} \quad (145)$$

$$G_2(s) = \frac{\alpha/\tau_a}{s + 1/\tau_a} \quad (146)$$

since their interconnection is of cascade-type. Transfer function (145) represents the RWM model (128) whereas (146) represents the actuator dynamics.

### 5.2.2 Stability and feedback stabilization

Feedback is formally a flow of information from the output of a system back into the input, according to some computational rule or other mechanism. For a designed feedback, the signal rerouting/filtering should achieve some objective. A prime control objective is stabilization [114]. A more generic is performance functional optimization.

It is a fact that the system (140) is asymptotically stable if, and only if, the eigenvalues of $\bar{A}$ have strictly negative real part [115]. The upper triangular form (69) of (141a) implies that the open-loop eigenvalues are simply: $\lambda_1 = \hat{\gamma}_{mk}/\tau_{mk}$ and $\lambda_2 = -1/\tau_a$. Whenever $\hat{\gamma}_{mk} > 0$ this system is unstable. This will inevitably happen, according to the MHD stability calculation in subsection 4.4. How does feedback help? We will determine what can be achieved by a proportional feedback scheme

$$v = Ky \quad (147)$$

applied to (140). Before assessing the prospects of scheme (147) a slightly more general framework will be introduced.

Suppose we devise a feedback system that uses two signals $r$, $y$ to generate a control action $v$. Let’s choose a very specific form and some SISO transfer function $F(s)$ to produce

$$\tilde{v} = F(s)(\tilde{r} - \tilde{y}) \quad (148)$$

Suppose our plant $G(s)$ generates the output

$$\tilde{y} = G(s)\tilde{v} + H(s)\tilde{w} \quad (149)$$

where $H(s)\tilde{w}$ is a perturbing signal; $w$ is a white-noise-type signal and $H(s)$ a filter that shapes this noise into some process that represents the way the plant is believed to be jogged, jerked and nudged unpredictably. Putting (148) in (149) and solving for $\tilde{y}$ results, for SISO systems, in the form

$$\tilde{y} = \frac{G(s)F(s)}{1 + G(s)F(s)}\tilde{r} + \frac{H(s)}{1 + G(s)F(s)}\tilde{w} \quad (150)$$

The first term in (150) represents the closed-loop dynamics from the reference $r$ to output $y$. The second term dictates how the disturbances will affect the output $y$. For good output-tracking, the first transfer function should be close to unity. For good disturbance rejection the second transfer function should be close to zero. It impossible to achieve this [62] for all frequencies.
A classic choice of $F(s)$ is the versatile proportional-integral-derivative (PID) regulator \cite{21}. Let the $e = r - y$, then the ideal\footnote{In practice the derivative term should be filtered for various reasons.} PID is defined by

$$v = k_p e + k_i \int_{-\infty}^{t} e(p) dp + k_d \dot{e}$$ \hfill (151)

for the given regulator parameter triple $k_p$, $k_i$, $k_d$. The ideal PID transfer function follows from the properties of Laplace transforms

$$F_{\text{pid}}(s) = k_p + k_i \frac{1}{s} + k_d s$$ \hfill (152)

Back to the original task then; (147) is contained in the PID structure (151) by assigning $r = 0$, $k_i = k_d = 0$, and $k_p = -K$. Let also $H(s) = 1$ and $G(s)$ be system (144). The transfer function $T_{yw}(s)$ from $w$ to $y$ then takes the form

$$T_{yw}(s) = \frac{1}{1 + G(s) F(s)} = \frac{p(s)}{p(s) - qK}$$ \hfill (153)

since $F(s) = -K$; where $p(s) = (s - \hat{\gamma}_{mk}/\tau_{mk})(s + 1/\tau_a)$ and $q = \frac{a_{mk} \alpha}{\tau_{mk} \tau_a}$. Stability of the closed loop requires that the roots of $p(s) - qK = 0$ are located in the left-hand half of the complex plane. This is the equation of the poles of $T_{yw}(s)$. This equation is exactly the secular equation of the feedback-modified system matrix of (140). To see this, put (147) in (140) to obtain the autonomous linear system

$$\dot{x} = \bar{A}_K x$$ \hfill (154)

where

$$\bar{A}_K = \bar{A} + \bar{B} K \bar{C} = \begin{pmatrix} \hat{\gamma}_{mk} / \tau_{mk} & a_{mk} / \tau_{mk} \\ K \alpha / \tau_a & -1 / \tau_a \end{pmatrix}$$ \hfill (155)

Indeed, it holds

$$p(s) - qK = |sI - \bar{A}_K|$$ \hfill (156)

where $|\cdot|$ denotes matrix determinant. The feedback gain $K$ thus parameterizes the stability properties of $T_{yw}(s)$ by tracing out a root locus

$$s_{1,2} = \frac{\hat{\gamma}_{mk} \tau_a - \tau_{mk}}{2 \tau_{mk} \tau_a} \pm \sqrt{\left( \frac{\hat{\gamma}_{mk} \tau_a - \tau_{mk}}{2 \tau_{mk} \tau_a} \right)^2 + \frac{4 \tau_{mk} \tau_a (K \alpha \alpha_{mk} - \hat{\gamma}_{mk})}{2 \tau_{mk} \tau_a}}$$ \hfill (157)

i.e. the solution to $p(s) - qK = 0$. Assume that $\hat{\gamma}_{mk} \tau_a - \tau_{mk} < 0$. Then define $\kappa = \hat{K}_a \alpha_{mk}$ and observe:

$$\kappa > \kappa_1 \quad \text{unstable}$$
$$\kappa_2 < \kappa < \kappa_1 \quad \text{stable}$$
$$\kappa < \kappa_2 \quad \text{stable/oscillatory}$$

where

$$\kappa_1 = \frac{\hat{\gamma}_{mk} + (\hat{\gamma}_{mk} \tau_a - \tau_{mk})(\hat{\gamma}_{mk} \tau_a - \tau_{mk} - 1)}{4 \tau_{mk} \tau_a}$$
$$\kappa_2 = \frac{\hat{\gamma}_{mk} - (\hat{\gamma}_{mk} \tau_a - \tau_{mk})^2}{4 \tau_{mk} \tau_a}$$
The stable/oscillatory condition means that, transiently, a system will settle to zero while oscillating (repeatedly crossing zero while vanishing in amplitude).

If there is an additional lag in the actuator (such as a time-delay and/or a more realistic model with both coil and power amplifier finite time-constants), then pushing $\kappa$ too far beyond the oscillatory limit $\kappa_2$ would, almost ironically, turn the system unstable again. Selecting a “good” $K$, is commonly known as tuning the controller. Tuning rules (and algorithms) are popular for PID setup [79, 82, 133] for (mostly SISO) subsystems in industrial process control. In many situations, only very approximate dynamics of the plant need to be measured to be able to satisfactorily tune a PID.

A feedback that bases its actions on the plant output only; e.g. (147), is known as output-feedback. A feedback that somehow have access to the full system state, say

$$v = Kx$$

is dubbed state-feedback. For our example system $\tilde{K}$ would belong to $\mathbb{R}^{1 \times 2}$, since $x \in \mathbb{R}^{2 \times 1}$ and $v \in \mathbb{R}^{1 \times 1}$. State feedback is a very powerful technique, if applicable. A common approach to be able to apply (158) is to reconstruct the state $x$ online using an observer. For linear systems the Kalman filter [17, 77, 115] is a widely used observer.

Suppose, for simplicity, that we can directly measure the full state $x$ of (140). Then output-feedback and state-feedback coincide. Closing (140) with (158) gives

$$\dot{x} = (\tilde{A} + \tilde{B}K)x = \tilde{A}_K x$$

where $\tilde{K} = (k_1 \ k_2)$ gives

$$\tilde{A}_K = \begin{pmatrix} \gamma_{mk}/\tau_{mk} & a_{mk}/\tau_{mk} \\ k_1 \alpha/\tau_a & -1/\tau_a + k_2 \alpha/\tau_a \end{pmatrix}$$

Eigenvalue placement is more free this time with 2 regulator parameters $k_1$ and $k_2$. Indeed the secular equation

$$0 = \left( s - \frac{\gamma_{mk}}{\tau_{mk}} \right) \left( s + \frac{1}{\tau_a} - \frac{k_2 \alpha}{\tau_a} \right) - \frac{k_1 \alpha a_{mk}}{\tau_{mk} \tau_a}$$

can here be mapped to any second order equation $s^2 + Es + F = 0$. Hence, the pair of closed-loop complex-conjugate eigenvalues can be placed completely at will, in this case. There is a caveat though: with no constraints on $\tilde{K}$ it is likely that (158) will give actuator commands $v$ that are out-of-bounds. The actuator might saturate, break-down, clamp, and so forth; it will never be governed by the assumed linear model for all magnitudes of $v$.

A classic method for optimizing state feedback for linear systems with penalty on input size is LQR; linear system (L) quadratic penalty (Q), regulator (R). The basic idea is very simple; pose an optimal control [115] problem for (140):

$$\min_{K \in \mathbb{R}^{1 \times 2}} \int_0^\infty \left( x^T(t)Qx(t) + Rv^2(t) \right) dt$$

subject to

$$\begin{cases} \dot{x} = \tilde{A}x + \tilde{B}v \\ v = \tilde{K}x \\ x(0) = x_0 \end{cases}$$

This is actually true for EXTRAP T2R since the state $x$ is essentially the coil current and the sensor coil flux; both are measured in real-time.
and solve it. $Q \in \mathbb{R}^{2 \times 2}$ should be symmetric and positive definite and scalar $R \geq 0$. Program (162) is of infinite-horizon type. Note that $\bar{K}$ must stabilize system (140) for the functional in (162a) to converge at all. One way to solve (162) is to employ dynamic programming [25, 115]. Introduce a cost-to-go functional $V = \int_0^\infty l(\tau) d\tau$ where $l = x^TQx + v^TRv$ (to obtain a more general solution, pretend that $v$ is a vector and $R$ a symmetric positive semidefinite matrix).

The principle of dynamic programming states $0 = \min_v \{l + \frac{\partial V}{\partial x} \dot{x}\}$. Assuming a form $V = x^TPx$ we then find that it must hold

$$0 = \min_v \{x^TQx + v^TRv + 2x^TP(\bar{A}x + \bar{B}v)\}$$  \hspace{1cm} (163)

Differentiation with respect to $v$ and setting the result to zero implies $0 = 2v^TR + 2x^TP\bar{B}$ with the solution

$$v = -R^{-1}\bar{B}^TPx$$  \hspace{1cm} (164)

and substituting (164) in (163) results in the matrix equation for $P = P^T$

$$0 = Q + P\bar{A} + \bar{A}^TP - P\bar{B}R^{-1}\bar{B}^TP$$  \hspace{1cm} (165)

after a short calculation where the trick rewrite $2x^TP\bar{A}x = x^TP\bar{A}x + x^T\bar{A}^TPx$ is used and positive definiteness is invoked to eliminate $x$. The LQR for (140) is thus defined by solving (165) and equating $\bar{K} = -R^{-1}\bar{B}^TP$ as seen from (158), (164). Equation (165) is, in fact, an algebraic Riccati equation.

Recent developments in control theory include Robust control [66] that attempts to systematically handle structured and unstructured plant uncertainties. It can then be more appropriate to consider worst-case “costs” instead of integrals-of-squares such as (162a).

### 5.2.3 Identification and estimation

A problem, or challenge, highly related to automatic control is system identification [55, 75, 85]. Depending on what is searched for, and the method to do so, alternative labels are: parameter identification, system estimation, parameter estimation, recursive identification, adaptive filtering.

Yet again consider (140). Having full knowledge of the system matrices it is quite easy to simulate state evolution $x(t)$ and output $y(t)$ given some input $v(t)$ and initial condition $x_0$. System identification deals with inverse-type problems; given measurements of $y(t)$ and $v(t)$ (polluted by noise), determine the system matrix elements. One particular topic of system identification, experiment design [75], tries to synthesize input perturbations $v(t)$ that give optimally informative data from the plant, to enable as small parameter estimate variance as possible, for finite records of signal data.

A popular, and general, method for system identification is the prediction error method (PEM) [85]. For some model structures PEM reduces to explicit systems of linear equations. The generic situation is however complicated by the need to solve nonlinear optimization programs. For large systems this can be prohibitive.

System identification is intimately connected to controller tuning [61]. Some approaches consider the plant and the controller to be one single entity. A standard, and pragmatic, perspective is however that system identification provides
models that are beneficially used for controller tuning. Since the basis is experimental data, this makes good pragmatic sense; theoretical aesthetics aside, a model-based tuning has to adjust to the real world to function well.

5.3 Distributed parameter systems

The RFP system, modeled by MHD, resistive current decay and magnetostatic field equations (PDEs) is technically a distributed parameter system, i.e. having a infinite-dimensional state-space, in its original formulation. A particular subfield of control theory deals with the properties and control prospects of such systems.

The gross simplification obtained in going from MHD to modal dynamics (138) could be regarded as an example of control-oriented modeling. The final result suggests a structure that captures the essential global plasma response and stability on a particular time-scale of interest; the resistive-shell time. Intermediate behaviour on shorter time-scales (and longer) are however neglected by this stability model. Distributed parameter system theory also prove handy for various phenomena in MCF devices [126, 130].
6 Main contribution in this thesis

6.1 System identification methods and practice in MCF research

The importance of engineering mechanics and electromagnetics on MHD stability is receiving more attention [98, 123]. Real-world geometry and external structures surrounding the plasma lack the ideal symmetry often imposed in theoretical models, for various reasons, such as (semi-) analytic tractability and/or finite availability of numerical horse-power. Pragmatic ways of dealing with real-world plants have been and are being developed in the automatic control field known as system identification. Half of the bulk of this thesis attempts to adapt and apply such methods to the RFP. Primary work is reported in the following peer-reviewed publications

   Online: [http://dx.doi.org/10.1109/CCA.2009.5281183](http://dx.doi.org/10.1109/CCA.2009.5281183).

   Online: [http://dx.doi.org/10.1109/CDC.2009.5400016](http://dx.doi.org/10.1109/CDC.2009.5400016)

6.2 Experimental control system capabilities in MHD research

Control system software updates have been necessary both in order to implement means to realize system identification experiments, subsection 6.1, and to retool the closed-loop to work in accordance with the results of these experiments. Controller synthesis is thus the subject of the second half of the bulk of this thesis. Main work is reported in the following peer-reviewed publications

   Online: [http://dx.doi.org/10.1016/j.fusengdes.2008.11.052](http://dx.doi.org/10.1016/j.fusengdes.2008.11.052)

2. Erik Olofsson, Per Brunsell, Emmanuel Witrant and James Drake, *Synthesis and operation of an FFT-decoupled fixed-order RFP plasma control system based on identification data*, Plasma Physics and Controlled Fusion, special issue on “MHD mode control in toroidal devices”, accepted March 2010
   Online: t.b.a.
The first of these papers outlines a process control style rejigging of the IS implementation of T2R. It turns out that the thus refurbished IS, or RIS, demonstrate an experimental proof-of-principle: radial magnetic field boundary-tracking. This might be useful for (i) development of ELM mitigation techniques [24] by inducing edge magnetic islands by clamping a resonant magnetic perturbation to some desired amplitude and wavenumber [51, 106] and (ii) providing nonaxisymmetric boundary conditions to study RFP dynamics, such as nonresonant plasma flow breaking and, perhaps, MHD relaxation to allegedly nonaxisymmetric minimum-energy states [44, 120]. For the RFP it is also interesting to provoke near-axis resonant modes to study e.g. heat transport effects, fancy new core-MHD states [86].

RMP effects on tearing-mode dynamics have already been studied using this technique of boundary tracking [58]. Many of these studies could qualify as reactor relevant, or more specifically ITER-relevant [32].

6.3 Contributions not included

The author has also written, co-written, and enabling experimental means for, the following peer-reviewed papers.


2. Erik Olofsson and Per Brunsell, Closed-loop system identification and controller reconfiguration for EXTRAP T2R reversed-field pinch, IEEE Transactions on Plasma Science, 99, pages 1-6, January 2010. Online: http://dx.doi.org/10.1109/TPS.2009.2038380


Three additional proceedings are


- Erik Olofsson and Per Brunsell, Closed-loop system identification and controller reconfiguration for EXTRAP T2R reversed-field pinch, Proceedings of the Symposium on Fusion Engineering (SOFE), San Diego, June 2009 Online: http://dx.doi.org/10.1109/FUSION.2009.5226479
where the first one reports control-simulations; to a large extent a spin-off from
master thesis work [95]. The second proceeding is a short version of the journal
article with the same name.

Oral presentations have been held at the annual November U.S. MHD
Control Workshops 2007 (Columbia University, New York), 2008 (University of
Texas, Austin) and 2009 (invited, Princeton Plasma Physics Laboratory, New
Jersey); at the RFP Workshop in Stockholm 2008, the EFDA feedback control
group kick-off meeting at JET in Culham (UK) 2009. Seminars have been given
at EAST in Hefei (China, December 2009) and in GIPSA-lab Automatique INP
in Grenoble (France, April 2010). All attended IEEE control conferences (CDC
2008/2009, MSC 2009) have also involved oral presentation. The author was
chairman at a regular system identification session in the MSC in St. Petersburg,
July 2009.
7 Conclusion

It has been shown, experimentally, that advancements in control theory and practice can be useful for magnetic confinement fusion research. First, a re-programming of the reversed-field pinch EXTRAP T2R enabled a new type of study for the detailed dynamics of rotating magnetic islands. The basic idea is to sustain, accurately, resonant magnetic perturbations by closed-loop output-tracking of the sensor coil array. Second, T2R was programmed for injected randomized nonaxisymmetric dithering in stabilized operation, i.e. a closed-loop system identification setup (control jargon), which is a novelty in MCF. It was seen that (most plausibly) essential magnetohydrodynamics can be “detected” in this fashion. Such experimental results strongly incentivize further studies with reactor control in mind. Third, system identification data was, exclusively, used for another control system reinstallment. This part constitute a proof-of-principle; measure the response of the MHD system in situ and subsequently compute the control system.
8 Bibliography

References


