

1. Introduction

1.1 Historical background

In 2001 (in the summer I think) I was contacted by Dr Siv Scheele at the consulting firm Inregia (presently WSP) in Stockholm about some problems they had with computing *Marginal Social Cost* (MSC) equilibria for congestion charging. I and a student of mine (Maria Daneva) went to them for a meeting.

Inregia looked at the computation of equilibria as an asymmetric equilibrium problem, and used the *Method of Successive Averages* (MSA), which resulted in large flow oscillations and slow convergence.

Contemplating on the problem I realized that it could be made symmetric and hence solved as an optimization problem by standard methods, such as the Frank-Wolfe method.

This was the start of the piece of research presented here, done in collaboration with Dr Leonid Engelson at Inregia and the student, resulting in fairly detailed results on MSC-tolls and their implementations; results which to our knowledge were not known before.

1.2 Congestion Charging

Congestion charging is a now accepted means of influencing traffic to behave in a more socio-economic optimal way, like e.g. in the Stockholm project. See e.g. Eliasson et al (2009).

Already early work, e.g. Pigou (1920), showed that road use can be inefficient due externalities, i.e. that users don't experience their own (negative) effect on other users: an extra car on a traffic link causes delays for other cars, but the driver himself does not experience this cost.

Further, already Beckman et al (1956) suggested that, for a congested road network with homogeneous users, each user should be charged a toll equal to the total value of time loss incurred on other users of the network. If we have fixed travel demand, this will induce an equilibrium that is system optimal in the sense that the total cost of network usage is minimal (assuming that all users have fixed and identical time values). To calculate the corresponding toll pattern, one modifies the link cost functions by adding the external cost term and solves for a classical user-optimal solution.

However, tolls typically need to be levied in monetary equivalents. And moreover users mostly have different time values, and thus react differently to monetary tolls. Thus to model effects of congestion charging, we need to introduce multiple user classes differing in their time values.

Thus, in the multi-class case, one would like to introduce, in each link, a *Marginal Social Cost* (MSC) toll, that is equal to the total *value* of time loss incurred, on other users in that link, by an additional driver.

As will be seen in this thesis, the toll results in the single class case do not carry over directly to the multi-class case. Through a not altogether trivial analysis, one can show that the results carry over to certain extent, but with important exceptions.

In the single class case there are unique tolls, whose implementation, as said, gives rise to traffic minimizing the total cost of network usage. This unicity has to do with the convexity of total cost as function of the flows. In the multi-class case the total cost (or rather the total value of travel time) is, as we show, typically non-convex in the class specific flows. Therefore there may be multiple local minima. These will be equilibria under the MSC tolls. There may also be other equilibria, such as saddle points or even maxima.

It may well be that, before the papers in this thesis were published, it was a folklore result in the traffic community that MSC tolls in the multi-class case give results corresponding to the single class case. Comments by Dr J. Eliasson acting as opponent to Daneva's licentiate thesis, and Dial's papers (1999a, b) point to that. Dial himself mentions that he in an earlier version of Dial (1999a) had, an obviously incorrect, proof of the convexity of the total cost.

In this thesis we will restrain ourselves to the stationary, fixed demand case, corresponding to the classical Traffic Assignment Problem in the single class case.

2. Theoretical Models

2.1 Traffic Assignment

All papers build on the model that is common for Traffic Assignment Problems (TAP), see e.g. Patriksson (1994).

In this model we have a *network* consisting of *nodes* $n \in N$ and directed *links* (arcs) $a \in A$, connecting the nodes.

Further let $W \subseteq N \times N$ be the set of *OD* (*origin-destination*) *pairs* (i.e. where traffic originates and ends).

For each OD-pair $w = (o_w, d_w)$ there is a *traffic demand* q_w (e.g. in veh./h) from o_w to d_w .

Further, let R_w be the set of (simple) *routes* from o_w to d_w , and $R = \bigcup_w R_w$ the set of all such routes.

The traffic between the OD-pairs gives rise to *route flows* h_r .

Let $H = \{h \in \mathfrak{R}_+^R : \sum_{r \in R_w} h_r = q_w\}$ be the *set of feasible route flows*, i.e. where the route flows between each OD-pair sum up to the appropriate demand. H is a polytope, and hence a convex set.

2.2 Link Flows

The route flows give rise to *link flows* f_a .

Introducing the (link route) *indicator* $\delta_{ar} = \begin{cases} 1 & \text{if route } r \text{ uses link } a \\ 0 & \text{otherwise} \end{cases}$, we see that $f_a = \sum_{r \in R} \delta_{ar} h_r$.

We can also introduce the *set*

$F = \{f = (f_a) : \exists h \in H, f_a = \sum_r \delta_{ar} h_r\}$ of *feasible link flows*, i.e. the set of link flows that arises from feasible route flows. F is convex since H is.

Further, introducing the link-route *incidence matrix* $\Delta = (\delta_{ar})$, with columns indexed by r and using the indexing convention, we see that, $f = \Delta h$ and $F = \Delta H$. (Thus Δ converts route flows to link flows.)

2.3 Costs.

Let $c_a(f)$ be the *arc cost* of traversing link a for a user, as a function of all link flows in the network. In the classical TAP case, $c_a(f) = t_a(f_a)$, the *link travel time*. In more general models $c_a(f)$ can contain also other costs, such as driving costs and tolls. Let $c = (c_a)$ be the vector of arc costs.

Similarly, let $C = (C_r)$ be the vector of total travel costs for users on routes r . We assume that C_r is additive over the links, i.e. that $C_r(h) = \sum_a \delta_{ar} c_a(f) = \sum_a \delta_{ar} c_a(\Delta h)$.

2.4 Equilibria

Definition (Wardrop equilibrium) The route flow vector $\hat{h} \in H$ is an *equilibrium (route flow)* if, for any OD pair w , the class specific costs for routes actually used (i.e. having $h_r > 0$) are equal and not greater than those of any unused route.

Let $\langle *, * \rangle$ denote the inner product between vectors of appropriate dimensions.

One can then see that the route flow $\hat{h} \in H$ is an equilibrium if (and only if)

$$\langle C(\hat{h}), h \rangle = \sum_r C_r(\hat{h}) h_r \geq \sum_r C_r(\hat{h}) \hat{h}_r = \langle C(\hat{h}), \hat{h} \rangle \quad \forall h \in H .$$

This can alternatively be phrased that $\hat{h} \in H$ is an equilibrium if (and only if) it fulfils the *variational inequality* (or VI)

$$\langle C(\hat{h}), h - \hat{h} \rangle \geq 0 \quad \forall h \in H .$$

Using that $f = \Delta h$ we may then see that $\hat{h} \in H$ is an equilibrium if (and only if) $\hat{f} = \Delta \hat{h}$ fulfills the VI

$$\langle c(\hat{f}), f - \hat{f} \rangle \geq 0 \quad \forall f \in F .$$

In this case one also says that $\hat{f} = \Delta \hat{h}$ is an equilibrium link flow.

Note that this equilibrium concept can be viewed as a Nash equilibrium with a continuum of players.

2.5 Multi-class aspects

To this general setting, we add the existence of several user classes, and study the effect that has on equilibria in tolled situations.

3. Results

3.1 The papers

The thesis consists of four papers. In chronological order:

1. **Multi-class user equilibria under social marginal cost pricing** (with L Engelson and M Daneva), *Operations Research Proceedings 2002*, 174-179, Springer 2003
2. **Convexification of the traffic equilibrium problem with social marginal cost tolls**, (with L Engelson) *Operations Research Proceedings 2003*, 141-148, Springer, 2004
3. **Congestion pricing of road networks with users having different time values** (with L. Engelson), *Mathematical and computational models for congestion charging*, 81--104, *Applied Optimization* **101**, Springer, New York, 2006.
4. **A Note on Two Papers by Dial**, Working Paper, Dept. of Transport and Economics, KTH, 2010.

These papers will be commented on in separate subsections below. Paper 3 contains the main theoretical work so we will start with that paper.

3.2 Paper 3: Congestion pricing of road networks with users having different time values

The paper starts by giving (in §2) definitions of general multi-class traffic equilibria. Such definitions existed before, but not in readily available sources. The equilibrium conditions are given in route flow as well as link flow versions. We also introduce a stability concept in analogy with that in Sandholm (2002). There is, however, some impreciseness in the reference to Sandholm, and as a consequence also some inappropriateness in our stability concepts. I will therefore return to this matter later. We also give symmetry conditions guaranteeing that the equilibrium problem can be viewed as an optimization problem (or rather a problem of finding points fulfilling necessary conditions for a local optimum of such a problem).

Realizing that MSC tolls need to be implemented as flow-independent tolls, the paper goes (in §3) on to study multi-class equilibria under fixed tolls. For such, the disutilities of the users can be expressed either in time or in money. It turns out that the cost version gives asymmetric equilibrium problems, whereas the time version gives symmetric ones, which thus may equivalently be stated as optimization problems. The multi-class equilibrium problem under fixed tolls turns out to correspond to a convex optimization problem. It may however have non-unique solutions in terms of class flows. But as we show, in spite of this non-uniqueness, the *total (perceived) value of travel time*, V , is unique, given that link travel time functions are strictly increasing. Under the same conditions we also show that the total link flows depend continuously on the link tolls.

Next (in §4) we study equilibria under MSC tolls. In this case, the version with disutilities expressed in time turns out to be asymmetric, whereas the cost version is symmetric, and hence corresponding to an optimization problem. The objective of the latter turns out to be the *total value of travel time*, V . Local or global minima of the total value of travel time are thus MSC equilibria, but so are also other stationary points, e.g. even local maxima.

Implementing the equilibrium MSC tolls as fixed tolls, the MSC flow equilibria are equilibria also for the fixed toll equilibrium problem. But as noted above, the fixed toll equilibrium problem may have other equilibria; equilibria that need not be MSC equilibria. But as also noted above all these fixed toll equilibria have the same value of the total travel time, which makes the non-uniqueness less severe. (We proved this under strictly increasing link travel times. But going through the papers

of the thesis recently I realized that one only needs non-decreasing travel times. I will return to this matter below.) Finally in this section, we use the continuity of the total link flows as functions of fixed tolls to demonstrate that implementing close to equilibrium MSC tolls as fixed tolls lead to equilibria with values of total travel time close to the equilibrium value.

In the next section (§5) we give a simple two link example, demonstrating the possibility of several equilibria, some of which not being local minima to the total value of travel time. This example is then expanded to a rather general case, showing that the total value of travel time is in general non-convex.

We go on (in §6) to give an algorithm of Frank-Wolfe (FW) type for computing MSC equilibria. It is applied (in §7) to the two link example, Sioux Falls, and Stockholm. For the two link example the algorithm converges to a saddle point of V , when started from free flow conditions. For the other two cases, it seems to converge to one and the same equilibrium, irrespective of the starting point. The convergence rate is of the same order as for the standard Traffic Assignment problem.

3.3 Paper 1: Multi-Class User Equilibria under Social Marginal Cost Pricing

This was the first paper. It contains the symmetrization mentioned above, as well as the two link example and applications to that example and to Sioux Falls.

3.4 Paper 3: Convexification of the traffic equilibrium problem with social marginal cost tolls.

One way to handle to non-convexity of V , the total value of travel time, is to convexify it, i.e. to compute the convex hull $\text{conv } V$ of V . This is done in Paper 3. The derivation of $\text{conv } V$ is rather technical and somewhat messy. We have used $\text{conv } V$ to compute lower bounds on V . We have also used it in the early phases of the FW algorithm, to speed up the computations.

3.5 Paper 4: A Note on Two Papers by Dial.

This paper discusses some shortcomings of two papers by Dial (1999a, b) on MSC tolls. Dial's papers show much insight, but they are marred with serious errors and a general sloppiness.

Dial starts in the other end compared to us. He sets out to minimize V , the total value of travel time. Then (modulo some errors) he "derives" our conditions for an equilibrium, which he erroneously believes are necessary and sufficient for a minimum of V . Further, instead of attempting to minimize V , Dial finds a solution satisfying the mentioned conditions using methods for complementarity problems.

Moreover his papers lack clear definitions. For instance he talks repeatedly about the "optimal tolls problem", without defining it.

Finally Dial phrases his problem in an infinite-dimensional setting, without having the tools for dealing with that.

The origin of this note is a previous appendix of Paper 3. See under discussion below.

4. Discussions and Conclusions

Here I will give some comments in relation to the papers in the thesis.

4.1 Stability

As said above, Paper 3 is somewhat imprecise in referring to Sandholm's (2001) stability concept.

In the paper we say that Sandholm's adjustment processes converge to an equilibrium. This is not quite true. Sandholm proves that his adjustment processes converge to *a connected set of equilibria*. And that is what is true in our case too. Despite of this our Theorem 1 is still true.

Definition 3 of the paper is however not appropriate in the case when the adjustment process converges to a connected equilibrium set, that is not a singleton. In such a case the following definition is more natural.

Definition 3'. A connected equilibrium flow set F is locally stable if for any f in F and any rational adjustment process started in a sufficiently small neighbourhood of any route flow h corresponding to f , will converge to F .

Observing that all limit points of adjustment process will have the same value of the primitive function one can then derive:

Theorem 1'. Under the assumptions of Theorem 1, all locally stable equilibrium sets are sets of local minima of the primitive function.

4.2 On strictly increasing link travel times.

As said in the summary of Paper 3, we used the assumption of strictly increasing link travel times to derive that, when implementing an MSC equilibrium as a fixed toll equilibrium then all such fixed toll equilibria have the same total value of travel time. This was also used to derive that when implementing an MSC equilibrium as a fixed toll equilibrium, then all other, possibly non-MSC-equilibria have the same total value of travel time.

As further mentioned above, rereading Paper 3 I realized that one only needs non-decreasing travel times for the latter conclusion.

The argument goes along the following lines:

Suppose we have an MSC equilibrium \bar{f} , and implement that as a fixed toll equilibrium. Then \bar{f} is an equilibrium to this fixed toll problem.

In links a where \bar{f}_a is a point of strict increase of the link travel time, then the total link flow is unique, and hence also the collected toll in the link, by the arguments of §3 of Paper 3.

On the other hand, in a link a , where \bar{f}_a is not a point of strict increase of the link travel time, then the equilibrium *travel times* in the links are still unique, although the link flows need not be unique. (This is by default also true in the standard TAP case. I have not managed to find out whether this observation is new.) Further, in this case, the derivative of the link travel time is zero, whence also the MSC toll is zero. Thus the collected toll in this link in the fixed toll problem will be zero too, and hence unique. Thus the total collected toll will be unique. Then by the arguments of Proposition 3 of Paper 3 the total value of travel time will be unique.

4.3 On the note/appendix on Dial's papers.

Paper 3 was first submitted to Operations Research. It then contained an appendix on Dial's two papers (1999a, b), highlighting their shortcomings. The referee report was rather negative. However, 80 percent of it concerned our appendix, and how clever Dial was, having written very insightful papers earlier. There was no substantial criticism of our paper itself.

I was rather upset, and I contacted Don Hearn for consultation. He directly invited our paper to the specialized volume Hearn and Lawphongpanich (2006), which we accepted after some hesitation.

In retrospect this may have been a bad choice, since rather few authors have cited our paper. Some of the few are Clark et al (2009), Guo and Yang (2007, 2009), Clark (2007).

5. References

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