T violation in neutrino oscillations in matter

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Abstract

We consider the interplay of fundamental and matter-induced T violation effects in neutrino oscillations in matter. After discussing the general features of these effects we derive a simple approximate analytic expression for the T-violating probability asymmetry ΔP_{ab}^T for three-flavour neutrino oscillations in a matter with an arbitrary density profile in terms of the two-flavour neutrino amplitudes. Explicit examples are given for the cases of a two-layer medium and for the adiabatic limit in the general case. We then discuss implications of the obtained results for long baseline experiments. We show, in particular, that asymmetric matter effects cannot hinder the determination of the fundamental CP and T-violating phase δ_{CP} in the long baseline experiments as far as the error in this determination is larger than 1% at 99% C.L. Since there are no T-violating effects in the two-flavour case, and in the limits of vanishing θ_{13} or Δm_{21}^2 the three-flavour neutrino oscillations effectively reduce to the two-flavour ones, studying the T-violating asymmetries ΔP_{ab}^T can in principle provide us with a complementary means of measuring θ_{13} and Δm_{21}^2.

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1 Introduction

The experimental evidence for neutrino oscillations implies that neutrinos are massive and that mixing angles and CP phases exist in the lepton sector. The measurement of these neutrino parameters is very important, not only because it will provide us with an information about neutrino properties, but also because it may have interesting implications for the structure of theories at very high energies, which might "explain" the low energy parameters. Since neutrinos could in general have both Dirac and Majorana masses, they might, in principle, also allow more insight into the flavour problem than quarks do. Extraction strategies for neutrino masses and mixings in current and future experiments are therefore an important subject under study. Future experiments also offer to study CP, T, and CPT properties in the neutrino sector. Local quantum field theories are, however, for general reasons invariant under CPT, and any CP violation implies then a correlated T violation. An independent study of T violation might therefore not appear very interesting, unless it is sensitive to tiny CPT violating effects induced at very high energy scales by physics going beyond local quantum field theory. In this paper, we discuss that this is different for neutrinos and that an independent study of T violation for neutrino oscillations in matter offers interesting insights. The point is that the presence of matter in the experimental setup violates by itself CP as well as CPT and gives thus rise to an extra CP violation in addition to the intrinsic CP violation. Furthermore, there are extra T-violating effects if the matter density profile seen by neutrinos is asymmetric. This extra T violation does not follow directly from CPT because CPT itself is violated by the presence of matter.

One can understand these effects by comparing CP, T, and CPT properties of neutrino oscillations in vacuum and in matter. Using CPT one finds in vacuum $P(\nu_a \rightarrow \nu_b) = P(\bar{\nu}_b \rightarrow \bar{\nu}_a)$. This is no longer true when one studies neutrino oscillations in matter, $i.e.$, a CP- and CPT-asymmetric environment, where the oscillation probabilities of neutrinos and antineutrinos change in a different way due to the differences in coherent forward scattering in a given medium (MSW effect). As a consequence, CPT is violated by matter effects, $i.e.$, $P(\nu_a \rightarrow \nu_b) \neq P(\bar{\nu}_b \rightarrow \bar{\nu}_a)$. Moreover, the total CP violation is now a combined effect, where intrinsic CP violation must be separated in analyses from effects induced by the CP-violating environment. A quantity which measures the total CP violation is $\Delta P_{ab}^{CP} = P(\nu_a \rightarrow \nu_b) - P(\bar{\nu}_a \rightarrow \bar{\nu}_b)$. Such "CP differences" or corresponding asymmetries have been studied for appearance channels like $\nu_e \rightarrow \nu_\mu$ at future long baseline experiments (see, $e.g.$, Refs. [1–3]), and one can nicely see how the CP difference is in general a combination of intrinsic CP effects and matter effects, while in vacuum ΔP_{ab}^{CP} depends only on intrinsic CP violation. Note that one can easily see from CPT and the definition of ΔP_{ab}^{CP} that in vacuum CP violation can only occur in appearance channels, $i.e.$, for $a \neq b$, while in matter one has in general $\Delta P_{aa}^{CP} \neq 0$.

In analogy to the CP difference ΔP_{ab}^{CP} one can also define a "T difference" $\Delta P_{ab}^{T} = P(\nu_a \rightarrow \nu_b) - P(\nu_b \rightarrow \nu_a)$ and the "CPT difference" $\Delta P_{ab}^{CPT} = P(\nu_a \rightarrow \nu_b) - P(\bar{\nu}_b \rightarrow \bar{\nu}_a)$. In vacuum, where CPT holds, one has $\Delta P_{ab}^{CP} + \Delta P_{\bar{a}\bar{b}}^{T} = \Delta P_{ab}^{CPT} \equiv 0$ and $\Delta P_{\bar{a}\bar{b}}^{T}$ is thus given directly by ΔP_{ab}^{CP}. Since in matter CPT is no longer valid, one now has $\Delta P_{ab}^{CPT} \neq 0$. For

this reason, T violation is no longer correlated with CP violation. In particular, matter does not necessarily break T invariance; as was mentioned above, this only happens if its density profile seen by neutrinos is asymmetric.

T violation in neutrino oscillations in matter has been studied in a number of papers (see, *e.g.*, Refs. [4–20]). However, in most of these studies matter of constant density was considered, where only intrinsic T violation is possible. T violation in neutrino oscillations in non-uniform matter was considered in Refs. [16, 20]. In the first of these papers T-violating effects of solar matter on the oscillations of $\mathcal{O}(\mathrm{GeV})$ neutrinos, which could be produced in the annihilation of WIMPs inside the sun, are considered, whereas in the second paper small perturbations of constant matter density profiles were discussed. In the present paper we study T violation in neutrino oscillations in matter of an arbitrary density profile and discuss the interplay of the fundamental and matter-induced T violation. In particular, we discuss where the matter-induced T violation effects may play a role and where they can be safely ignored.

The paper is organized as follows. In Sec. 2 we discuss the general features of T violation in neutrino oscillations in matter. Subsequently, we give approximate analytic results for arbitrary matter density profiles in Sec. 3 and discuss the implications for long baseline experiments, solar, atmospheric, supernova, and cosmological neutrinos in Sec. 4. We discuss the obtained results and conclude in Sec. 5. In Appendix A we give details of our general analytic approach, whereas in Appendix B its application to two particular cases is considered.

2 General features of T violation in neutrino oscillations in matter

Oscillations of neutrino flavour in vacuum or in matter are described by the Schrödinger equation

$$i\frac{d}{dt}|\nu\rangle = H(t)|\nu\rangle\,, \tag{1}$$

where $|\nu\rangle$ is the neutrino vector of state and $H(t)$ is the effective Hamiltonian which in general depends on time t through the t-dependence [1] of the matter density $N(t)$: $H(t) \equiv H[N(t)]$ [2].

We will assume that neutrinos are stable and are not absorbed in matter; in this case the Hamiltonian $H(t)$ is Hermitian. It is convenient to define the evolution matrix $S(t, t_0)$:

$$|\nu(t)\rangle = S(t, t_0)|\nu(t_0)\rangle\,. \tag{2}$$

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[1]We consider the evolution of the neutrino system in time rather than in space (since for relativistic neutrinos $r \simeq t$, both descriptions are equivalent). The coordinate dependence of the matter density profile $N(r)$ therefore translates into its time dependence $N(t)$.

[2]Neutrino flavour evolution in matter depends on the total matter density as well as on the electron number fraction Y_e. Throughout this paper we will for brevity call them collectively "matter density profile".

It has the obvious properties

$$S(t, t_0) = S(t, t_1)S(t_1, t_0) \,, \qquad S(t_0, t_0) = \mathbf{1} \,, \qquad S(t, t_0)S(t, t_0)^\dagger = \mathbf{1} \,, \tag{3}$$

where the last property (unitarity) follows from the hermiticity of $H(t)$. From Eqs. (1) and (2) it follows that $S(t, t_0)$ satisfies the equation

$$i\frac{d}{dt}S(t, t_0) = H(t)S(t, t_0) \,. \tag{4}$$

It is also sometimes useful to consider the evolution equation of $S(t, t_0)$ with respect to its second argument, t_0. Differentiating the equality $S(t, t_0)S(t_0, t) = \mathbf{1}$ with respect to t_0 and using Eq. (4), one finds

$$i\frac{d}{dt_0}S(t, t_0) = -S(t, t_0)H(t_0) \,. \tag{5}$$

The amplitudes of neutrino flavour transitions are just the elements of the evolution matrix S:

$$A[\nu_a(t_0) \to \nu_b(t)] = [S(t, t_0)]_{ba} \,. \tag{6}$$

We are interested in the properties of the solutions of the evolution equations (1) or (4) with respect to the time reversal transformation T. Unlike in the case of CP transformation, these properties cannot be directly experimentally tested as one cannot change the direction of time. However, time reversal can be studied by simply interchanging the initial and final neutrino flavours. This can be readily seen in the case of neutrino oscillations in vacuum, for which

$$[S(t, t_0)]_{ba} = \sum_j U_{aj}\, e^{-iE_j(t-t_0)}\, U_{bj}^* \,, \tag{7}$$

where U and E_i are the lepton mixing matrix and energy eigenvalues in vacuum, respectively. Indeed, time reversal interchanges the initial and final times, t_0 and t, respectively, which leads to the complex conjugation of the exponential factors in Eq. (7). On the other hand, the interchange of the initial and final neutrino flavours $a \leftrightarrow b$ (*i.e.*, $S \to S^T$, where the superscript T denotes transposition) is equivalent to the complex conjugation of the matrix elements of the leptonic mixing matrix U in Eq. (7). The action of these two operations on the elements of the evolution matrix S in Eq. (7) differ from each other only by complex conjugation. Since the transition probabilities $P_{ab} \equiv P(\nu_a \to \nu_b)$ are just the squares of the moduli of the matrix elements S_{ba}, these two procedures are physically equivalent. This means, as is well known, that in vacuum, instead of studying neutrino oscillations "backward in time", one can study the oscillations forward in time, but with the initial and final flavours interchanged.

The situation is, however, less obvious in the case of neutrino oscillations in matter of non-constant (and, in general, asymmetric) matter density. Time reversal means, in particular, that the matter density profile has to be traversed in the opposite direction. That is to say, one has to consider the evolution in the "reverse" profile $\tilde{N}(t)$ rather than in the

"direct" profile $N(t)$. Here $\tilde{N}(t)$ should be understood as the profile seen by neutrinos when the positions of the neutrino source and detector are interchanged. Therefore, the following question arises: When one reduces the probability of the time-reversed neutrino oscillations in matter to that for the interchanged neutrino flavours, does one have to simultaneously replace the direct profile by the reverse one? There is some confusion in the literature regarding this issue, and therefore we believe that it is worth clarifying it here. In general, in the case of neutrino oscillations in matter of non-constant density, no closed-form expressions for the transition probabilities exist, and the arguments similar to those applied to Eq. (7) cannot be used. There is, however, a very simple and general argument which does not depend on whether neutrinos oscillate in vacuum or in matter and what the matter density profile is.

Indeed, under time reversal the arguments of the evolution matrix $S(t, t_0)$ are interchanged:

$$T: \qquad S(t, t_0) \to S(t_0, t).$$

From Eq. (3) one then finds

$$S(t_0, t) = S(t, t_0)^{-1} = S(t, t_0)^{\dagger} = [S(t, t_0)^T]^* . \tag{8}$$

Therefore, the operations of time reversal and interchange of initial and final neutrino flavours [i.e., transposition of $S(t, t_0)$] are related by complex conjugation and so lead to the same transition probabilities in matter with an arbitrary density profile as well as in vacuum. This, in particular, means that instead of considering neutrino oscillations in matter "backward in time", one can consider the oscillations between interchanged initial and final neutrino flavours forward in time and in the same (i.e., direct) matter density profile.

In vacuum, due to CPT invariance, T violation is equivalent to CP violation. In particular, the CP-odd and T-odd differences of neutrino oscillation probabilities

$$\Delta P_{ab}^{CP} \equiv P_{ab} - P_{\bar{a}\bar{b}} \qquad \text{and} \qquad \Delta P_{ab}^{T} \equiv P_{ab} - P_{ba}$$

are equal to each other. In the case of $n \geq 3$ neutrino species they are expressed through $(n-1)(n-2)/2$ Dirac-type CP-violating phases $\{\delta_{CP}\}$ [3].

The situation with neutrino oscillations in matter is drastically different. Ordinary matter is both CP and CPT-asymmetric as it consists of particles (electrons and nucleons) and not of their antiparticles or, in general, of unequal numbers of particles and antiparticles. The violation of CPT by matter implies that CP and T violation effects are in general different. CP is violated by the very existence of matter. This violation manifests itself even in the two-flavour neutrino systems (or, in general, even in the absence of fundamental CP violation, i.e., when $\delta_{CP} = 0$) – matter may enhance the oscillations between neutrinos and suppress those between antineutrinos, or vice versa [21, 22]. Moreover, the survival pro-

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[3]If neutrinos are Majorana particles, there are $n - 1$ additional (so-called Majorana) phases, but they have no effect on neutrino oscillations.

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babilities P_{aa}, which are CP-symmetric in vacuum due to CPT invariance, are no longer CP-symmetric in matter.

In contrast to this, the presence of matter does not necessarily break T invariance. In particular, matter of constant density (or, in general, matter with a density profile that is symmetric with respect to the interchange of the positions of the neutrino source and detector) does not induce any T violation. In addition, the survival probabilities P_{aa} are always T-symmetric, since the initial and final neutrino flavours coincide.

The point that symmetric matter density profiles do not induce any T violation seems intuitively rather obvious; here we will give an explicit proof of this statement and also will derive several useful properties of the evolution matrix $S(t, t_0)$. To study these properties, it is convenient to consider the evolution over the symmetric time interval $[-t,\, t]$. This does not lead to any loss of generality as any time interval can be reduced to a symmetric one by the proper choice of the point $t = 0$. Since both arguments of the evolution matrix $S(t, -t)$ depend on t, it does not satisfy Eq. (4). It is, however, not difficult to derive the evolution equation for $S(t, -t)$. Using Eq. (3) one can write $S(t, -t) = S(t, 0)S(0, -t)$. The matrix $S(t, 0)$ satisfies the usual evolution equation

$$i\frac{d}{dt}S(t, 0) = H(t)S(t, 0)\,. \tag{9}$$

Taking its Hermitian conjugate and substituting $-t$ for t, one finds

$$i\frac{d}{dt}S(0, -t) = S(0, -t)H(-t)\,. \tag{10}$$

Using Eqs. (9) and (10), one finally obtains

$$i\frac{d}{dt}S(t, -t) = H(t)S(t, -t) + S(t, -t)H(-t)\,. \tag{11}$$

The action of time reversal on this equation is given by the substitution $t \to -t$.

Let us now assume that the fundamental CP and T violation is absent, $i.e.$, all $\{\delta_{CP}\} = 0$. In this case the Hamiltonian of the neutrino system is real (or can be made real by a rephasing of the neutrino states). Since it is real and Hermitian, it is also symmetric: $H^T = H$. Transposition of Eq. (11) yields

$$\begin{aligned} i\frac{d}{dt}S(t, -t)^T &= S(t, -t)^T H(t)^T + H(-t)^T S(t, -t)^T \\ &= S(t, -t)^T H(t) + H(-t)S(t, -t)^T\,. \end{aligned} \tag{12}$$

The evolution matrix for neutrinos passing through the reverse profile $\tilde{N}(t) = N(-t)$ can be obtained from Eq. (11) by replacing $H(t)$ with $H(-t)$. Comparing the resulting equation with Eq. (12), one obtains

$$S_{\mathrm{dir}}(t, t_0)^T = S_{\mathrm{rev}}(t, t_0) \qquad (\{\delta_{CP}\} = 0)\,. \tag{13}$$

Here the subscripts dir and rev denote propagation in the direct and reverse profiles, $N(t)$ and $\tilde{N}(t)$, respectively, and we have reinstated the time interval $[t_0, t]$ in the arguments of the evolution matrices. From Eq. (13) it immediately follows that in the case of a symmetric matter density profile the evolution matrix $S(t, t_0)$ is symmetric and there is no matter-induced T violation. For the particular case of matter consisting of a number of constant density layers, Eq. (13) was derived by Fishbane and Kaus [23].

It is easy to generalize Eq. (13) to the case when $\{\delta_{CP}\} \neq 0$. Following the lines that led to Eq. (13), one obtains, in this case,

$$S_{\mathrm{dir}}(t, t_0)^T = S_{\mathrm{rev}}(t, t_0)|_{\{\delta_{CP}\} \to -\{\delta_{CP}\}} . \tag{14}$$

This expression generalizes Eq. (13) of Ref. [23], which was obtained for matter consisting of constant-density layers, to the arbitrary matter density profile. Eq. (14) has a rather obvious physical meaning. It just reflects the fact that there are two kinds of effects that contribute to T violation (*i.e.*, to the difference between S and S^T) in matter – intrinsic T violation, due to the non-vanishing CP and T violating phases $\{\delta_{CP}\}$, and extrinsic T violation, due to the asymmetry of the density profile with respect to the interchange of the positions of the neutrino source and detector.

Eq. (13) means that in the case of $\{\delta_{CP}\} = 0$ the difference

$$\Delta P_{ab} \equiv P_{\mathrm{dir}}(\nu_a \to \nu_b) - P_{\mathrm{rev}}(\nu_b \to \nu_a) \tag{15}$$

vanishes. Any deviation of this difference from zero is therefore a measure of non-vanishing fundamental CP and T violation and can, in principle, be used for their experimental searches. Fishbane and Kaus [23] have stressed that one can, in principle, probe the effects of $\{\delta_{CP}\} \neq 0$ even by studying the survival probabilities P_{aa} ($a \neq e$) if one compares these probabilities for direct and reverse profiles. This is a very interesting observation, even though the experiments with interchanged positions of neutrino source and detector would certainly be difficult to perform.

The point that the survival probabilities can be used for looking for fundamental CP and T violation is easy to understand. The probabilities $P(\nu_a \to \nu_a)$ are T-symmetric as the initial and final neutrino flavours coincide. Therefore, the contributions to their T asymmetry coming from the fundamental T violation and from the asymmetry of the matter density profile must cancel each other exactly. This means that by measuring the asymmetry $P_{\mathrm{dir}}(\nu_a \to \nu_a) - P_{\mathrm{rev}}(\nu_a \to \nu_a)$ (with $a \neq e$) one directly measures, up to the sign, the asymmetry due to the fundamental CP and T-violating phases $\{\delta_{CP}\}$.

The asymmetry of the ν_e survival probability P_{ee} with respect to density profile reversal vanishes (up to the tiny terms due to radiative corrections to matter-induced potentials of ν_μ and ν_τ [24]) due to the specific way the neutrino evolution equation depends on matter density [23]. Indeed, it was shown in Refs. [4, 25] that P_{ee} is independent of the phase δ_{CP}; from Eq. (14) it then immediately follows that $P(\nu_e \to \nu_e)_{\mathrm{dir}} = P(\nu_e \to \nu_e)_{\mathrm{rev}}$. It is interesting to note that this result depends crucially on the assumption that the matter-induced potentials of all neutrino species except ν_e are the same. This is no longer true if there are sterile neutrinos, and so in that case in general $P(\nu_e \to \nu_e)_{\mathrm{dir}} \neq P(\nu_e \to \nu_e)_{\mathrm{rev}}$.

An important point to notice is that there is no T violation (either fundamental or matter-induced) in two-flavour neutrino systems. Mathematically, this follows from the fact that the off-diagonal elements of any 2×2 unitary matrix have the same absolute values – for this reason the probabilities of the transitions $\nu_a \to \nu_b$ and $\nu_b \to \nu_a$, which are given by the squares of the moduli of the elements S_{21} and S_{12} of the evolution matrix S, coincide. Physically, this is related to the fact that conservation of probability (*i.e.*, unitarity) puts rather stringent constraints in the two-flavour neutrino case. For example, in the (ν_e, ν_μ) system the conditions that the probabilities of the transitions from ν_e to all states (including ν_e itself) and from all states to ν_e must both be equal to unity are

$$P_{ee} + P_{e\mu} = 1, \qquad P_{ee} + P_{\mu e} = 1. \tag{16}$$

From this one immediately obtains that

$$P_{e\mu} = P_{\mu e}, \tag{17}$$

i.e., neutrino oscillations are T-invariant irrespective of whether they take place in vacuum or in matter and whether the matter density profile is symmetric or not.

Another consequence of unitarity in the two-flavour neutrino systems is that in the case of a symmetric matter density profile the off-diagonal elements of the evolution matrix S are pure imaginary. This is well known in the case of constant matter density, and it is easy to see that this in fact also holds for an arbitrary symmetric matter density profile. Indeed, in the two-flavour case the most general form of the unitary evolution matrix S (up to phase factors which can be absorbed into redefinitions of the fields) is

$$S = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \qquad \text{where} \quad |\alpha|^2 + |\beta|^2 = 1. \tag{18}$$

From Eq. (13) it follows that when $\tilde{N}(t) = N(t)$, the evolution matrix is symmetric, *i.e.*, β is pure imaginary.

Unitarity is much less constraining in the case of more than two neutrino flavours. For example, for three neutrino flavours one obtains from equalities similar to Eq. (16) only the condition

$$\Delta P_{e\mu}^T + \Delta P_{e\tau}^T = 0, \tag{19}$$

and the T-odd asymmetries $\Delta P_{e\mu}^T$ and $\Delta P_{e\tau}^T$ need not vanish. Considering different initial neutrino states, one can also find $\Delta P_{\mu\tau}^T + \Delta P_{\mu e}^T = 0$ and $\Delta P_{\tau e}^T + \Delta P_{\tau\mu}^T = 0$, which together with Eq. (19) give [5]

$$\Delta P_{e\mu}^T = \Delta P_{\mu\tau}^T = \Delta P_{\tau e}^T. \tag{20}$$

This relation coincides with the corresponding relation in the case of neutrino oscillations in vacuum; it implies that in a three-flavour neutrino system there is only one independent T-odd probability difference for neutrinos (and similarly one for antineutrinos). In a few recent publications [15,20], this relation was obtained (in some approximations) for neutrino

oscillations in matter of constant density. However, as should be clear from its derivation, it is exact and does not depend on the matter density profile.

It is well known that in the limits of vanishing mixing angle θ_{13} (*i.e.*, vanishing element U_{e3} of the lepton mixing matrix U) or vanishing mass squared difference Δm_{21}^2, three-flavour neutrino oscillations effectively reduce to the two-flavour ones [26]. Since, as was discussed above, there are no T-violating effects in the two-flavour neutrino case, any such effect can be considered as a measure of the genuine three-flavourness of the neutrino system. Studying T-violating effects in neutrino oscillations in matter can thus, in principle, provide us with important complementary means of measuring the parameters θ_{13} and Δm_{21}^2, even in the absence of fundamental CP and T violation.

3   Approximate analytic description for an arbitrary matter density profile

We will give here a simple approximate expression for T-violating effects in neutrino oscillations in a matter with an arbitrary matter density profile. This will enable us to study the interplay of the fundamental and matter-induced T-violating effects in neutrino oscillations.

We will consider a three-flavour neutrino system $(\nu_e, \nu_\mu, \nu_\tau)$ and use the parameterization of the leptonic mixing matrix U which coincides with the standard parameterization of the quark mixing matrix [27]. The leptonic mixing angle θ_{13} is constrained by the CHOOZ reactor neutrino experiment and is known to be small [28]:

$$\sin^2 2\theta_{13} \lesssim 0.10 \quad \Rightarrow \quad \sin\theta_{13} \lesssim 0.16\,.$$

Analyses of solar and atmospheric neutrino data [29] also show that the ratio $\Delta m_\odot^2/\Delta m_{\rm atm}^2 \equiv \Delta m_{21}^2/\Delta m_{31}^2 \lesssim 0.1$. As was discussed above, T-violating effects (both fundamental and matter-induced) disappear in the limit of vanishing θ_{13} or Δm_{21}^2 and therefore they must be suppressed by both these small factors. This means that they can be calculated to a very good accuracy within perturbation theory. One can either consider $\Delta m_{21}^2/\Delta m_{31}^2$ as a small parameter and treat θ_{13} exactly, or vice versa; the T-odd asymmetry will automatically be suppressed by both these parameters. We choose to treat $\sin\theta_{13}$ as a small parameter, while making no further approximations. The details of our calculations are described in Appendix A; here we just describe the main idea and the results.

In the standard parameterization, the leptonic mixing matrix U can be written in the form $U = O_{23}V_{13}O_{12}$, where O_{ij} are orthogonal matrices describing rotations by the angles θ_{ij} in the corresponding (i,j)-planes, and V_{13} is the unitary matrix which describes the rotation in the (1,3)-plane and in addition includes the Dirac-type CP and T-violating phase δ_{CP} [see Eq. (A3)]. In the following, we will use the notation $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$,

$$\delta \equiv \frac{\Delta m_{21}^2}{2E}\,, \qquad \Delta \equiv \frac{\Delta m_{31}^2}{2E}\,, \qquad \tilde{\Delta}(t) \equiv \Delta - \frac{1}{2}[\delta + V(t)]\,. \tag{21}$$

It is convenient to perform the rotation according to $\nu' = O_{23}^T \nu$, where ν is the neutrino vector of state in flavour basis. To zeroth order in perturbation theory in the small parameter $s_{13} \equiv \sin \theta_{13}$, the evolution matrix in the rotated basis can be written as

$$S_0'(t, t_0) = \begin{pmatrix} \alpha(t, t_0) & \beta(t, t_0) & 0 \\ -\beta^*(t, t_0) & \alpha^*(t, t_0) & 0 \\ 0 & 0 & f(t, t_0) \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1, \tag{22}$$

where

$$f(t, t_0) = \exp\left\{ -i \int_{t_0}^t \tilde{\Delta}(t') \, dt' \right\} \tag{23}$$

and the parameters $\alpha(t, t_0)$ and $\beta(t, t_0)$ are to be determined from the solutions of the two-flavour neutrino problem in the $(1,2)$-subsector. It is now easy to obtain the correction to the evolution matrix S_0' to order s_{13}, from which, upon rotation back to the unprimed basis, one finds the following expression for the T-odd probability difference $\Delta P_{e\mu}^T = \Delta P_{\mu\tau}^T = \Delta P_{\tau e}^T$ [4]:

$$\Delta P_{e\mu}^T \simeq -2 s_{13} s_{23} c_{23} \left(\Delta - s_{12}^2 \delta \right) \mathrm{Im} \left[e^{-i\delta_{CP}} \beta^* (A_a - C_a^*) \right], \tag{24}$$

where

$$A_a \equiv \alpha \int_{t_0}^t \alpha^* f \, dt' + \beta \int_{t_0}^t \beta^* f \, dt', \qquad C_a \equiv f \int_{t_0}^t \alpha f^* \, dt'. \tag{25}$$

Eqs. (24) and (25) give the T-odd probability difference in the three-flavour neutrino system in terms of f and the two-flavour neutrino amplitudes α and β for an arbitrary matter density profile. Note that this simplifies the problem considerably as the two-flavour neutrino problems are generally much easier to solve than the three-flavour ones. For $\delta_{CP} = 0$ it can be shown that the right-hand side of Eq. (24) vanishes for any symmetric matter density profile, as it must (see Appendix A).

We will now consider two special cases: first, matter consisting of two layers of constant densities, and second, the adiabatic approximation for an arbitrary matter density profile. For the first case with two layers of constant densities we will use the following notation: layer widths L_1 and L_2, electron number densities N_1 and N_2, the corresponding matter-induced potentials V_1 and V_2, the values of the mixing angle in the $(1,2)$-subsector in matter θ_1 and θ_2 [5], respectively,

$$\omega_i \equiv \frac{1}{2} \sqrt{(\cos 2\theta_{12} \, \delta - V_i)^2 + \sin^2 2\theta_{12} \, \delta^2}, \tag{26}$$

and

$$s_i \equiv \sin(\omega_i L_i), \qquad c_i \equiv \cos(\omega_i L_i), \qquad (i = 1, 2). \tag{27}$$

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[4]Eqs. (24) and (25) were obtained neglecting the (very small) corrections of the order of $(\Delta m_{21}^2 / \Delta m_{31}^2)^2$. Expressions which do not use this approximation are given in Appendix A.

[5]In the limit of vanishing densities N_1 and N_2, one has $\theta_1 = \theta_2 = \theta_{12}$.

For the T-odd difference of the neutrino oscillation probabilities one then finds

$$\Delta P_{e\mu}^T \simeq -2s_{13}s_{23}c_{23}\left(\Delta - s_{12}^2\delta\right)\left[\cos\delta_{CP}(X_1 Z_R - X_2 Z_I) + \sin\delta_{CP}(X_1 Z_I + X_2 Z_R)\right]. \quad (28)$$

Here X_1, X_2, X_3, and Y are the parameters which define the two-flavour evolution matrix for neutrinos passing through the two-layer medium, $s = Y - i\boldsymbol{\sigma}\cdot\mathbf{X}$, with [30]

$$Y = c_1 c_2 - s_1 s_2 \cos(2\theta_1 - 2\theta_2), \qquad X_1 = s_1 c_2 \sin 2\theta_1 + s_2 c_1 \sin 2\theta_2,$$
$$X_2 = -s_1 s_2 \sin(2\theta_1 - 2\theta_2), \qquad X_3 = -(s_1 c_2 \cos 2\theta_1 + s_2 c_1 \cos 2\theta_2),$$

and

$$\begin{aligned}
Z_R &\equiv \operatorname{Re}(A_a - C_a^*) = -[D_- - (\Omega_1 \cos 2\theta_1 - \Omega_2 \cos 2\theta_2)] \\
&\times \ [\sin(\Delta_1 L_1 + \Delta_2 L_2) - c_2 \sin(\Delta_1 L_1) - c_1 \sin(\Delta_2 L_2)] \\
&- \ [D_- \cos 2\theta_1 + \Omega_2 \cos(2\theta_1 - 2\theta_2) - \Omega_1]s_1 \cos(\Delta_2 L_2) \\
&- \ [D_- \cos 2\theta_2 - \Omega_1 \cos(2\theta_1 - 2\theta_2) + \Omega_2]s_2 \cos(\Delta_1 L_1) \\
&- \ D_- X_3 - \Omega_1[s_1 c_2 + s_2 c_1 \cos(2\theta_1 - 2\theta_2)] + \Omega_2[c_1 s_2 + s_1 c_2 \cos(2\theta_1 - 2\theta_2)], \quad (29)
\end{aligned}$$

$$\begin{aligned}
Z_I &\equiv \operatorname{Im}(A_a - C_a^*) = [D_+ - (\Omega_1 \cos 2\theta_1 + \Omega_2 \cos 2\theta_2)]\cos(\Delta_1 L_1 + \Delta_2 L_2) \\
&- \ [D_- - (\Omega_1 \cos 2\theta_1 - \Omega_2 \cos 2\theta_2)][c_1 \cos(\Delta_2 L_2) - c_2 \cos(\Delta_1 L_1)] \\
&- \ [D_- \cos 2\theta_1 + \Omega_2 \cos(2\theta_1 - 2\theta_2) - \Omega_1]s_1 \sin(\Delta_2 L_2) \\
&+ \ [D_- \cos 2\theta_2 - \Omega_1 \cos(2\theta_1 - 2\theta_2) + \Omega_2]s_2 \sin(\Delta_1 L_1) \\
&- \ D_+ Y + \Omega_1(c_1 c_2 \cos 2\theta_1 - s_1 s_2 \cos 2\theta_2) + \Omega_2(c_1 c_2 \cos 2\theta_2 - s_1 s_2 \cos 2\theta_1). \quad (30)
\end{aligned}$$

Here we have used the notation

$$\Delta_i = \Delta - \frac{1}{2}(\delta + V_i), \quad D_\pm = \frac{\Delta_1}{\Delta_1^2 - \omega_1^2} \pm \frac{\Delta_2}{\Delta_2^2 - \omega_2^2}, \quad \Omega_i = \frac{\omega_i}{\Delta_i^2 - \omega_i^2} \quad (i = 1, 2). \quad (31)$$

The term proportional to $\cos\delta_{CP}$ in Eq. (28) describes the matter-induced T violation, whereas the $\sin\delta_{CP}$ term is due to the fundamental T violation. It is easy to see that the parameters X_2 and Z_R are antisymmetric with respect to the interchange of the two layers, while X_1, X_3, Y, and Z_I are symmetric, and therefore the condition (14) is satisfied.

In the low energy regime $\delta = \Delta m_{21}^2/2E \gtrsim V_{1,2}$ (which for the LMA-MSW solution of the solar neutrino problem and matter densities typical for the upper mantle of the earth corresponds to $E \lesssim 1$ GeV) the main contributions to Eq. (28) come from the D_+ terms in Z_I, and the expression for the T-odd probability difference simplifies significantly:

$$\begin{aligned}
\Delta P_{e\mu}^T &\simeq \cos\delta_{CP} \cdot 8s_{12}c_{12}s_{13}s_{23}c_{23} \frac{\sin(2\theta_1 - 2\theta_2)}{\sin 2\theta_{12}}\{s_1 s_2[Y - \cos(\Delta_1 L_1 + \Delta_2 L_2)]\} \\
&+ \ \sin\delta_{CP} \cdot 4s_{13}s_{23}c_{23} X_1[Y - \cos(\Delta_1 L_1 + \Delta_2 L_2)]. \quad (32)
\end{aligned}$$

Here the $\cos \delta_{CP}$ term has a remarkably simple structure: it is given by an oscillating term multiplied by an effective Jarlskog invariant

$$J_{\text{eff}} \equiv s_{12} c_{12} s_{13} s_{23} c_{23} \frac{\sin(2\theta_1 - 2\theta_2)}{\sin 2\theta_{12}} . \tag{33}$$

This has to be compared with the usual Jarlskog invariant [31]

$$J \equiv s_{12} c_{12} s_{13} c_{13}^2 s_{23} c_{23} \sin \delta_{CP} . \tag{34}$$

Note that the factor $\sin(2\theta_1 - 2\theta_2)/\sin 2\theta_{12}$ in J_{eff} plays the same role as $\sin \delta_{CP}$ in J (which is zero in the absence of fundamental CP and T violation). The factor $\sin(2\theta_1 - 2\theta_2)/\sin 2\theta_{12}$ is a measure of the asymmetry of the matter density profile ($\theta_1 \neq \theta_2$). Note also the absence of the c_{13}^2 factor which is equal to unity in our approximation. The $\sin \delta_{CP}$ term in Eq. (32) can also be expressed in terms of the (usual) Jarlskog invariant if one writes X_1 as

$$X_1 = \sin 2\theta_{12} \left(s_1 c_2 \frac{\sin 2\theta_1}{\sin 2\theta_{12}} + s_2 c_1 \frac{\sin 2\theta_2}{\sin 2\theta_{12}} \right) . \tag{35}$$

Note that the ratios $\sin 2\theta_1/\sin 2\theta_{12}$ and $\sin 2\theta_2/\sin 2\theta_{12}$ are finite in the limit $\sin 2\theta_{12} \to 0$.

Let us now consider an arbitrary matter density profile in the adiabatic approximation (see Appendix B). In this case the parameters α and β describing the two-flavour neutrino evolution in the (1,2)-subsector are

$$\alpha(t, t_0) = \cos \Phi \cos(\theta - \theta_0) + i \sin \Phi \cos(\theta + \theta_0) , \tag{36}$$
$$\beta(t, t_0) = \cos \Phi \sin(\theta - \theta_0) - i \sin \Phi \sin(\theta + \theta_0) , \tag{37}$$

whereas $f(t, t_0)$ is given by Eq. (23) as before. Here θ_0 and θ are the values of the mixing angle in the (1,2)-subsector in matter at the initial and final points of the neutrino evolution, t_0 and t, respectively,

$$\Phi \equiv \int_{t_0}^{t} \omega(t') \, dt' , \tag{38}$$

and $\omega(t)$ is given by Eq. (26) with the substitution $V_{1,2} \to V(t)$. We will also assume that $V(t) \lesssim \delta \ll \Delta$. The integrals in Eq. (25) can then be done approximately (see Appendix B) and one obtains

$$\Delta P_{e\mu}^{T} \simeq 2 s_{13} s_{23} c_{23} \{ \cos \delta_{CP} [\sin(2\theta - 2\theta_0) \cos^2 \Phi - 2 \sin(\theta - \theta_0) \cos \Phi \cos(\Delta(t - t_0))] +$$
$$\sin \delta_{CP} [\sin(\theta + \theta_0) \cos(\theta - \theta_0) \sin 2\Phi - 2 \sin(\theta + \theta_0) \sin \Phi \cos(\Delta(t - t_0))] \} . \tag{39}$$

In the regime in which the oscillations governed by large $\Delta = \Delta m_{31}^2/2E$ are fast and therefore can be averaged over, the above expression simplifies to

$$\Delta P_{e\mu}^{T} \simeq 2 s_{13} s_{23} c_{23} \{ \cos \delta_{CP} \sin(2\theta - 2\theta_0) \cos^2 \Phi + \sin \delta_{CP} \sin(\theta + \theta_0) \cos(\theta - \theta_0) \sin 2\Phi \} . \tag{40}$$

Note that the $\cos\delta_{CP}$ contribution can again be written in terms of the effective Jarlskog invariant J_{eff}:

$$(\Delta P_{e\mu}^T)_{\cos\delta_{CP}} \simeq 4\cos\delta_{CP} \cdot s_{12}c_{12}s_{13}s_{23}c_{23}\frac{\sin(2\theta - 2\theta_0)}{\sin 2\theta_{12}}\cos^2\Phi = 4\cos\delta_{CP}\, J_{\text{eff}}\cos^2\Phi\,. \quad (41)$$

Interestingly, the effective Jarlskog invariant appears both in the adiabatic regime and in the case of the two-layer matter density profile, which is an example of an extreme non-adiabatic case. If the oscillations due to a smaller energy difference ω are also in the averaging regime, the $\sin\delta_{CP}$ contribution in Eq. (40) vanishes and only the matter-induced T-violating term proportional to $\cos\delta_{CP}$ survives. Thus, we make the interesting observation that matter-induced T violation, unlike the fundamental one, does not disappear when the neutrino oscillations are in the regime of complete averaging.

4   Implications

In this section we compare our approximate analytic formulas with numerical calculations of $\Delta P_{e\mu}^T$ and discuss the relevance of matter-induced T violation in neutrino oscillations for long baseline experiments, and also for solar, atmospheric and supernova neutrinos and for neutrinos in the early universe.

4.1   Accuracy of the analytic approximation

In order to estimate the accuracy and domain of validity of our results we have to distinguish two cases: $L/E \lesssim 10,000$ km/GeV and $L/E \gtrsim 10,000$ km/GeV. In the first case the oscillating structure of ΔP_{ab}^T can, in general, be resolved, as it is shown in Fig. 1. As can be seen from the figure, the size of $\Delta P_{e\mu}^T$ is reproduced rather accurately, but there is an error in the oscillation phase which accumulates with distance and increases with increasing θ_{13} and Δm_{21}^2.

In the second case, the oscillations governed by the large $\Delta m_{31}^2 = \Delta m_{\text{atm}}^2$ are very fast. We illustrate this case in Fig. 2, where $\Delta P_{e\mu}^T$ is plotted as a function of the total distance L traveled by neutrinos for matter consisting of two layers with densities $\rho_1 = 0$, $\rho_2 = 6.4$ g/cm^3 and widths $L_1 = L_2 = L/2$. We have chosen for the left plot the same parameters as those in Fig. 3a of Ref. [23] [6]. The baseline values are rather unrealistic and serve illustrative purposes only. In the right plot of Fig. 2 we plot $\Delta P_{e\mu}^T$ for larger values of θ_{13} and Δm_{21}^2. The result obtained using our analytic formula (28) (grey curve in Fig. 2) reproduces the results of numerical calculations of Ref. [23] and of our own numerical calculations very well: the difference can be barely seen. In fact, the fast oscillations cannot be resolved by any

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[6]In Ref. [23] the value $\rho_2 = 8$ g/cm^3 instead of 6.4 g/cm^3 was erroneously quoted for Fig. 3a, but the quoted value of the potential V_2 was correct. Note that in Fig. 3a of Ref. [23] the asymmetry of the transition probability $P_{\mu e}$ with respect to density profile reversal rather than $\Delta P_{e\mu}^T$ was plotted. However, as follows from Eq. (13), in the case of $\delta_{CP} = 0$ this asymmetry coincides with $\Delta P_{\mu e}^T = -\Delta P_{e\mu}^T$.

realistic detector due to its limited energy resolution; for this reason in Fig. 2 we compare the averaged over these fast oscillations values of $\Delta P_{e\mu}^T$ obtained using approximate analytic formula (28) with those calculated numerically (black solid and dashed curves, respectively). Since the error in the phase is irrelevant due to the averaging, the accuracy of the analytic approach is very good. For $\theta_{13} = 0.1$ and $\Delta m_{21}^2 = 5 \cdot 10^{-5}$ eV^2 the maximal relative error is about 5%; it increases with increasing θ_{13} and Δm_{21}^2, and for $\theta_{13} = 0.16$ and $\Delta m_{21}^2 = 2 \cdot 10^{-4}$ eV^2 the maximal error is about 10%.

In both cases the accuracy of the predicted size of matter-induced T violation is quite good, the error being $\lesssim 10\%$; however, the phase is not reproduced to the same accuracy.
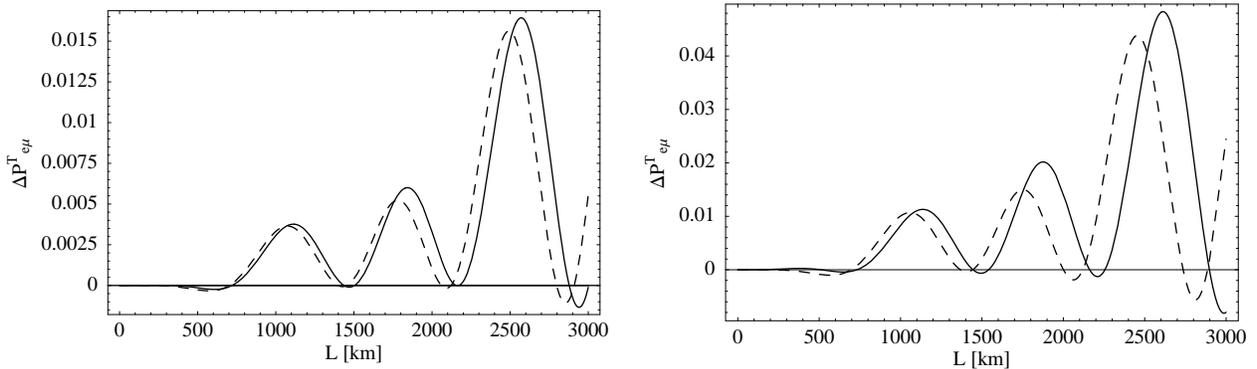

Figure 1: Comparison of the results of the analytic (solid curve) and numerical (dashed curve) calculations of $\Delta P_{e\mu}^T$ for two layers of widths $L_1 = L_2 = L/2$, densities 1 g/cm^3 and 3 g/cm^3 and electron number fractions $(Y_e)_1 = (Y_e)_2 = 0.5$. In both plots $E = 1$ GeV, $\Delta m_{31}^2 = 3.5 \cdot 10^{-3}$ eV^2, $\delta_{CP} = 0$, $\theta_{12} = 0.56$, and $\theta_{23} = \pi/4$ are used. The remaining parameters are $\theta_{13} = 0.1$ and $\Delta m_{21}^2 = 5 \cdot 10^{-5}$ eV^2 for the left plot and $\theta_{13} = 0.16$ and $\Delta m_{21}^2 = 2 \cdot 10^{-4}$ eV^2 for the right plot.

4.2   Long baseline experiments

One possible implication of the discussed effects may be in future long baseline neutrino oscillation experiments. Neutrino factory experiments with thousands of kilometers baselines appear feasible, leading to beams which traverse the matter of the earth. The matter density profile of the earth is approximately spherically symmetric; however, the deviations from exact spherical symmetry may be considerable, which in turn can give rise to T-violating matter density profiles seen by neutrino beams. These fine details of the earth structure are, unfortunately, not very well known, but from seismological data one can infer some upper bounds on the possible asymmetry. A conservative estimate of these deviations is to assume a 10% variation in density on the length scales of several thousand kilometers [32] somewhere along the neutrino path. Another possibility to have an asymmetric matter density profile exists for shorter baselines $L \lesssim 1,000$ km where the neutrino path goes only some
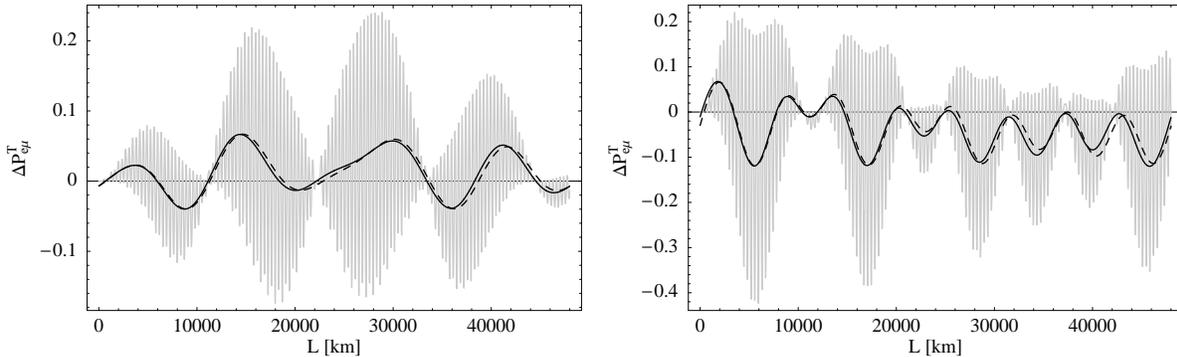
Figure 2: Comparison of the analytic and numerical calculations of $\Delta P_{e\mu}^T$ for two layers of widths $L_1 = L_2 = L/2$, densities 0 and 6.4 g/cm^3 and electron number fractions $(Y_e)_1 = (Y_e)_2 = 0.5$. The grey curve is the result of our analytic calculation, the black solid and dashed curves show the results averaged over the fast oscillations of the analytic and numerical calculation, respectively. In both plots $E = 0.5$ GeV, $\Delta m_{31}^2 = 3.5 \cdot 10^{-3}$ eV^2, $\delta_{CP} = 0$, $\theta_{12} = 0.56$, and $\theta_{23} = \pi/4$ are used. The remaining parameters are $\theta_{13} = 0.1$ and $\Delta m_{21}^2 = 5 \cdot 10^{-5}$ eV^2 for the left plot and $\theta_{13} = 0.16$ and $\Delta m_{21}^2 = 2 \cdot 10^{-4}$ eV^2 for the right plot.

ten kilometers below the surface of the earth. In this case one half of the neutrino path could be in the sea ($\rho \simeq 1$ g/cm^3) and the other half in the continental earth crust ($\rho \simeq 3$ g/cm^3). The experimental setups under discussion will have a remarkable precision, and it is thus important to know when the discussed matter-induced T-violating effects become relevant. In other words, we will discuss when it is necessary to include the discussed extra T-violating effects and when the usual analysis is justified.

We consider the usual setup for neutrino factory experiments with a beam energy of 50 GeV. The energy threshold of the detector is 4 GeV and its energy resolution is 10%. The detector is capable of charge identification, thus the wrong sign muon signal is available. We assume $2 \cdot 10^{21}$ useful muon decays for both polarities and a detector mass of 40 kton. This luminosity coincides with the one used in Ref. [1] and is a factor of 40 larger than the one used in Ref. [2]. We include only statistical errors. All fits include both polarities and both appearance and disappearance rates. Further details can be found in Ref. [2].

We assume two different asymmetric matter density profiles. The first one consists of two layers of equal widths, with densities 1 g/cm^3 and 3 g/cm^3 (Figs. 3 and 4). This corresponds to the sea-earth scenario, which of course can be realized on the earth only up to baselines of $\sim 1,000$ km. The second one also consists of two layers of equal widths, but with densities 3 g/cm^3 and 3.3 g/cm^3 as an example of density perturbations which could arise in real very long baseline experiments (Fig. 5).

We simulate such experiments numerically for these types of profiles and perform fits to the obtained event rates. We compare this with the fits performed for symmetrized versions of the corresponding profiles, which are modeled by replacing the transition probabilities by

14

the symmetrized ones:

$$P_S = \frac{1}{2} \left(P_{\text{dir}} + P_{\text{rev}} \right) . \tag{42}$$

Thus we are only sensitive to the errors induced by the asymmetry of the matter density profile and not to the errors in the average density, which of course should not be ignored in
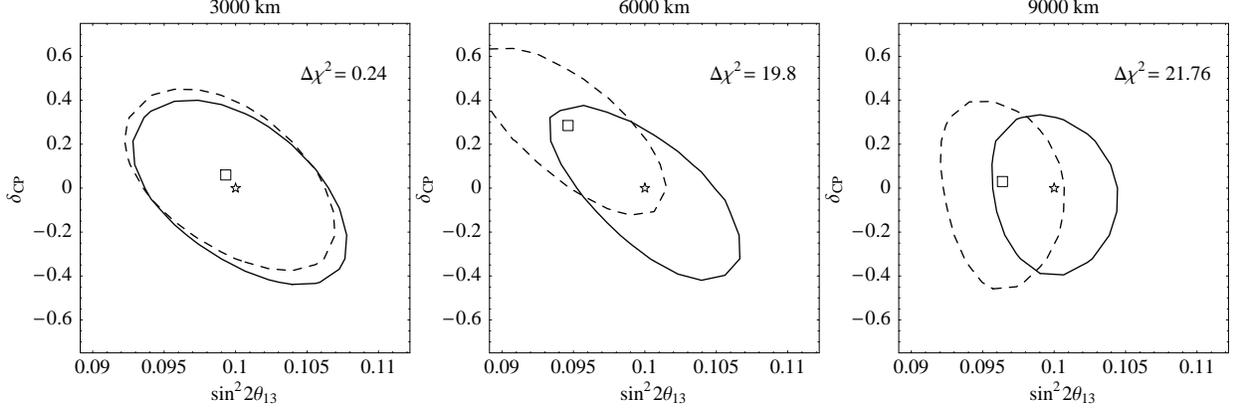

Figure 3: A fit of $\sin^2 2\theta_{13}$ and δ_{CP} for different baselines in a neutrino factory experiment with a beam energy of 50 GeV (see the text for details). The solid curve is the 99% C.L. contour for the asymmetric matter density profile with two layers of widths $L_1 = L_2 = L/2$, densities 1 g/cm^3 and 3 g/cm^3 and electron number fractions $(Y_e)_1 = (Y_e)_2 = 0.5$. The dashed curve is the 99% C.L. contour for the symmetrized matter density profile as defined in Eq. (42). The star and square indicate the best fit points for the asymmetric and symmetrized profiles, respectively. The $\Delta\chi^2$ value given in each plot is the difference of χ^2 (2 d.o.f.) between the two best fit points. The parameters are $\theta_{12} = \pi/4$, $\theta_{23} = \pi/4$, $\Delta m^2_{31} = 3.5 \cdot 10^{-3}$ eV^2, $\Delta m^2_{21} = 10^{-4}$ eV^2, $\sin^2 2\theta_{13} = 0.1$, and $\delta_{CP} = 0$.
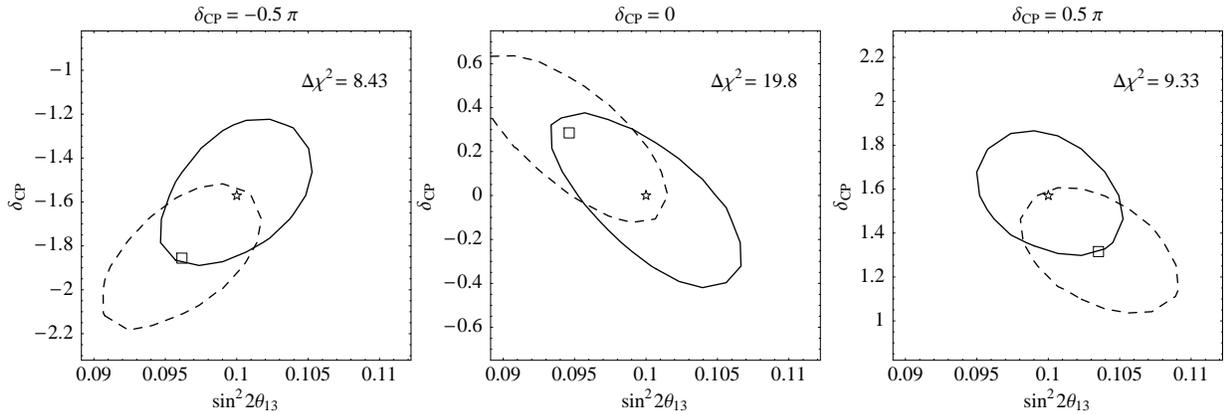

Figure 4: Same as in Fig. 3, but for the fixed baseline $L = 6,000$ km and three different values of δ_{CP}: $-\pi/2$, 0, and $\pi/2$.

15

reality. The difference between the minimal χ^2 values for the asymmetric and symmetrized profiles is a direct measure of the sensitivity to the matter-induced T violation.
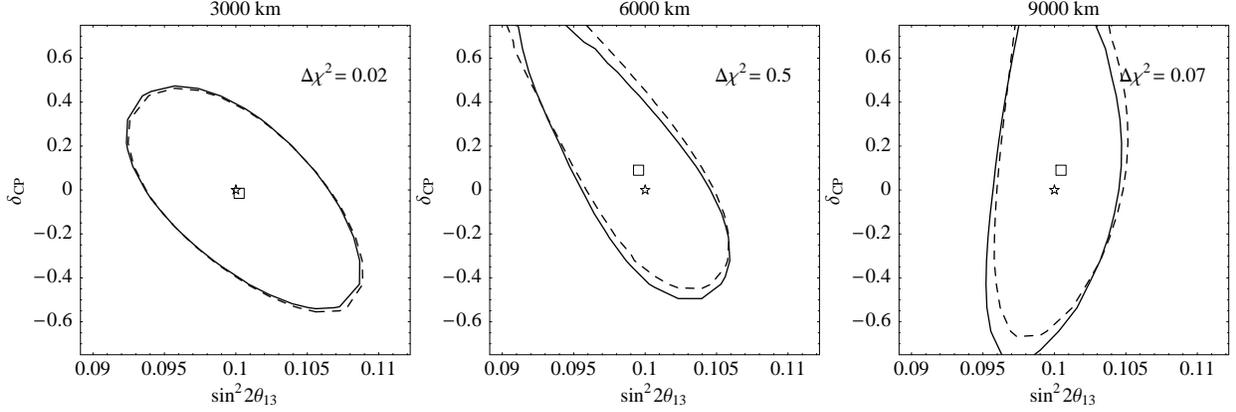

Figure 5: Same as in Fig. 3, but for the layer densities $\rho_1 = 3$ g/cm^3 and $\rho_2 = 3.3$ g/cm^3.

The effect can be quite sizable for matter density profiles of the sea-earth type (see Fig. 3), however, only for baselines above 1,000 km, which cannot be realized on the earth. For large baselines the errors in the determination of the fundamental CP and T-violating phase δ_{CP} induced by asymmetric matter are comparable with the statistical errors in the case of symmetric matter. This behavior is quite similar for all possible values of δ_{CP} as can be seen in Fig. 4.
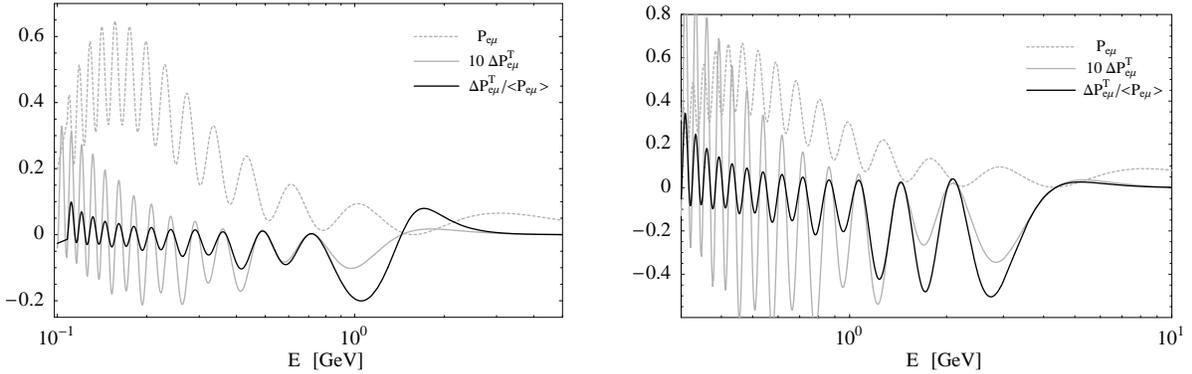

Figure 6: Energy dependence of the transition probability $P_{e\mu}$ (grey dashed curve), T-odd probability difference $\Delta P_{e\mu}^T$ multiplied by 10 (grey solid curve), and the ratio $\Delta P_{e\mu}^T / \langle P_{e\mu} \rangle$, where $\langle P_{e\mu} \rangle$ is the average of $P_{e\mu}$ over fast oscillations (black solid curve) in the case of two-layer density profile with $\rho_1 = 1$ g/cm^3, $\rho_2 = 3$ g/cm^3 and electron number fractions $(Y_e)_1 = (Y_e)_2 = 0.5$. The layer widths are $L_1 = L_2 = 500$ km (left plot) and $L_1 = L_2 = 1,500$ km (right plot). The values of neutrino parameters are $\Delta m_{21}^2 = 2 \cdot 10^{-4}$ eV^2, $\theta_{12} = 0.559$, $\Delta m_{31}^2 = 3.5 \cdot 10^{-3}$ eV^2, $\theta_{23} = \pi/4$, $\theta_{13} = 0.16$, and $\delta_{CP} = 0$.

16

The energy dependence of the transition probability $P_{e\mu}$, T-odd probability difference $\Delta P_{e\mu}^T$, and their ratio in the case of two-layer matter density profile corresponding to the sea-earth scenario is shown in Fig. 6. One can see that the relative size of the T-violating effect is largest at energies of about 1 GeV. A dedicated low energy experiment capable of measuring the matter-induced T violation with the sea-earth type matter density profile at a baseline of $\sim 1,000$ km would, however, require enormous luminosities, at least six orders of magnitudes higher than an initial stage of a neutrino factory ($2 \cdot 10^{19}$ muons/year, 10 kton).

For much more realistic matter density profiles with only 10% density variation, the matter-induced T violation effects are small at any baseline (see Fig. 5). The statistical errors in the determination of δ_{CP} and θ_{13} are much larger than the errors induced by replacing the asymmetric matter density profile by the symmetrized one. This again holds for all values of δ_{CP}. Therefore we conclude that the determination of the fundamental CP and T-violating phase δ_{CP} cannot be spoiled by an unknown T asymmetry of the matter density profile until the luminosity (and therewith the accuracy) is increased by at least two orders of magnitude, bringing the error in δ_{CP} down to $\sim 1\%$ at 99% C.L.

4.3 Solar, supernova and atmospheric neutrinos and neutrinos in the early universe

Matter-induced T-violating effects can, in principle, manifest themselves in a number of other situations when oscillating neutrinos propagate through asymmetric matter. In particular, solar neutrinos traveling from the center of the sun towards its surface traverse a matter density profile which is highly asymmetric and characterized by a large contrast of densities (from $\rho \simeq 150$ g/cm^3 to $\rho \simeq 0$). There are no muon or tau neutrinos originally produced in the nuclear reactions in the sun, and so one cannot study the quantities like $\Delta P_{e\mu}^T$ or $\Delta P_{e\tau}^T$ with solar neutrinos [7]. However, genuine three-flavour effects (including those of asymmetric matter) in general contribute terms of the order of s_{13} to the transition probabilities which can be large compared to the usual terms proportional to s_{13}^2 which are present in the effective two-flavour approach. In particular, the survival probability of solar neutrinos averaged over the fast oscillations due to the large $\Delta m_{\mathrm{atm}}^2$ is approximately given by "quasi two-flavour" formula [33]

$$P_{ee}^{3\nu} = c_{13}^4 P_{ee}^{2\nu} + s_{13}^4 \,, \tag{43}$$

where $P_{ee}^{2\nu}$ is the two-flavour ν_e survival probability in the (1,2) sector calculated with the effective matter-induced neutrino potential $V_{\mathrm{eff}} = c_{13}^2 V$. It contains only second and higher order in s_{13} corrections to the standard two-flavour result. One can then ask if this result is modified by asymmetric matter, leading to the order s_{13} corrections.

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[7]Neutrinos of all flavours with energies $\mathcal{O}(\mathrm{GeV})$ can be produced in the annihilation of weakly interacting massive particles (WIMPs) inside the sun. T-violating effects of solar matter on the oscillations of such neutrinos were discussed in Ref. [16].

In the case of the LMA-MSW solution of the solar neutrino problem (which leads to the largest allowed values of Δm^2_{21}) the evolution of neutrinos inside the sun is adiabatic, and Eqs. (39) or (40) can be used. For maximal values of θ_{13} allowed by the CHOOZ experiment one finds that the corrections of the order s_{13} to the transition probabilities $P_{e\mu}$ and $P_{e\tau}$ of solar neutrinos can be as large as $(5-10)\%$. However, these corrections have opposite sign and the same absolute value, and therefore they cancel exactly in the total transition (or survival) probability of electron neutrinos. The fact that the survival probability of ν_e is symmetric with respect to matter density profile reversal has already been discussed above (see Sec. 2); since one cannot distinguish ν_μ from ν_τ in low energy experiments, this makes the asymmetric matter effects in oscillations of solar neutrinos inside the sun unobservable. Eq. (43) therefore does not have to be modified. We have checked this by comparing its predictions with the results of numerical calculations of the full three-flavour evolution equation for solar neutrinos and found that the accuracy of the approximation in Eq. (43) is extremely good (error $\leq 10^{-3}$).

Another potentially interesting implication of matter-induced T-violating effects could be in oscillations of supernova neutrinos propagating from the supernova's core outwards. However, these effects again cancel because of the absence of the matter-induced asymmetry of P_{ee} and the fact that the fluxes and spectra of the supernova ν_μ and ν_τ are practically identical.

The expansion of the universe implies that neutrino oscillations in the early universe take place in a time-dependent, asymmetric environment. One can then ask if T violation effects in neutrino oscillations can result in a lepton flavour asymmetry. Such an asymmetry, *e.g.*, in electron neutrinos could have important consequences for the big bang nucleosynthesis. However, it is easy to make sure that if the original numbers of neutrinos of all species are equal, this does not happen: the summation over the contributions of all flavours makes the asymmetries in the individual lepton flavours vanish.

Thus, we see that even though there are, in principle, interesting extra T-violating effects in asymmetric situations like in the sun, in the early universe and in supernovae explosions, these effects cancel whenever a summation over neutrino flavours is present, and a naive analysis without these effects leads to correct results.

Matter-induced T violation can, in principle, also influence atmospheric neutrino oscillations. This is related to the fact that neutrinos are produced in the atmosphere at an average height of about 15 km and so the neutrinos coming to the detector from the lower hemisphere travel first in air and then in the earth. For neutrinos from just below the horizon this can result in nearly equal pathlengths (of the order of a few hundred kilometers) in air and in the earth. The effect is, however, very small, which can be seen through a simple estimate. The asymmetric matter effects can be looked for through the distortions of the zenith angle distributions of the e-like events (the distributions of the μ-like events are mainly governed by the large θ_{23} and Δm^2_{31}, and the relative T-violating effects for them are smaller than those for the e-like events). The probability difference $\Delta P^T_{e\mu}$ is of the order of a few percent of $P_{e\mu}$, and $P_{e\mu}$ itself is of the order of a few percent of P_{ee}; therefore the matter-induced T-violating effect is of the order $10^{-4}-10^{-3}$ at best. Thus, one would need

$10^6 - 10^8$ events to achieve a statistical accuracy of the same order. Super-Kamiokande at the moment has roughly 10^4 events. Including the flux uncertainties would make the situation even worse. Therefore matter-induced T-violating effects in atmospheric neutrinos can be safely neglected. This result is also confirmed by numerical analysis.

5 Discussion and conclusions

In the present paper we studied T violation in neutrino oscillations in asymmetric matter, with a special emphasis on matter-induced T violation. To this end, we derived a simple approximate analytic expression for T-odd probability differences ΔP_{ab}^T in the case of a general matter density profile. We have shown that our analytic expressions reproduce the results of direct numerical integration of the neutrino evolution equation very well.

Matter-induced T violation has two aspects. First, it is an interesting matter effect, which is present only in asymmetric matter (and therefore is absent in the most studied case of constant-density matter). It can manifest itself not only in the T-odd differences of oscillation probabilities but also in specific modification of the probabilities themselves. Since this effect can only exist in systems of three or more neutrino flavours, it is sensitive to the parameters that discriminate between genuine three-flavour and effectively two-flavour oscillations, such as θ_{13} and Δm_{21}^2, and therefore can, in principle, be used for their determination. Second, matter-induced T violation can fake the fundamental one and so impede the determination of the fundamental CP and T-violating phase δ_{CP} in the long baseline experiments.

We have studied both these aspects and have shown that in the case of three neutrino species and for matter density contrasts and baselines feasible in terrestrial experiments, matter-induced T violation effects are small and can safely be ignored. This is due to the fact that they are doubly suppressed by small factors $\sin\theta_{13}$ and $\Delta m_{21}^2/\Delta m_{31}^2 = \Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$. In particular, asymmetric matter effects cannot hinder the determination of the fundamental CP and T-violating phase δ_{CP} in the long baseline experiments as far as the error in the determination of δ_{CP} is larger than 1% at 99% C.L. This is the main result of our paper.

T-asymmetric matter effects for solar, supernova, and cosmological neutrinos, which propagate through larger density contrasts and over larger distances, can, in principle, be large in each individual oscillation channel. However, these effects are not observable because of the summation over channels inherent in the experimental detection of these neutrinos or in the observable quantities such as total number density of neutrinos of a given flavour in the case of neutrinos in the early universe.

The situation can be very different in four-neutrino schemes. In that case T-violating effects (both fundamental and matter-induced) are not, in general, suppressed by small factors like $\sin\theta_{13}$ or $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$. Therefore experimental detection of sizeable T violation effects would signify the existence of a fourth light neutrino, which, due to the LEP result on the invisible Z^0 boson width, must be a sterile neutrino. It should also be understood that our conclusion that matter-induced T violation cannot hamper the determination of

19

the fundamental CP and T-violating phases does not, in general, apply to the four-neutrino schemes; in this case one has to carefully take matter-induced T-violation into account in order to disentangle intrinsic T violation from the extrinsic one.

It is also interesting to note that the statement that the survival probability of electron neutrinos P_{ee} is invariant with respect to matter density profile reversal, discussed in Sec. 2, holds only in three-neutrino schemes. Indeed, it relies on the assumption that the matter-induced potentials of all neutrino species except ν_e are the same. This is no longer true if sterile neutrinos are present. Therefore an observation of non-invariance of P_{ee} under density profile reversal would be a signature of a sterile neutrino. It should be noted, however, that both this non-invariance of P_{ee} and large T-violating effects can only be observable in the (experimentally favored) 2+2 four-neutrino schemes, in which there are two pairs of nearly degenerate neutrino mass eigenstates separated by a large mass gap. In the 3+1 scheme with a lone neutrino mass eigenstate being predominantly a sterile neutrino these effects are expected to be small since this scheme can be considered as a small perturbation of the three-neutrino ones.

Acknowledgements

Appendix A: Analytic description of ΔP^T_{ab}

The evolution equation describing neutrino oscillations in the three-flavour neutrino system can be written as

$$i\frac{d}{dt}\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \left[U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \Delta \end{pmatrix} U^\dagger + \begin{pmatrix} V(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \qquad (A1)$$

where ν_a $(a = e, \mu, \tau)$ are the components of the neutrino vector of state in flavour basis, U is the leptonic mixing matrix, δ and Δ were defined in Eq. (21), and

$$V(t) = \sqrt{2} G_F N_e(t) \qquad (A2)$$

is the charged-current contribution to the matter-induced potential of electron neutrinos, G_F and N_e being the Fermi weak coupling constant and the electron number density of the medium, respectively. The neutral-current contributions to the potentials of ν_e, ν_μ, and ν_τ in matter are the same (up to tiny radiative corrections [24]) and therefore do not

affect the neutrino oscillations probabilities. The evolution matrix $S(t, t_0)$ satisfies the same Schrödinger equation (A1).

In the standard parameterization, the leptonic mixing matrix U can be written in the form[8] $U = O_{23}V_{13}O_{12}$, where

$$O_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{CP}} & 0 & c_{13} \end{pmatrix}, \quad O_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}.$$
(A3)

Since the matrix of matter-induced neutrino potentials $\mathrm{diag}\,(V(t), 0, 0)$ commutes with O_{23}, it is convenient to perform the rotation according to $\nu' = O_{23}^T \nu$ (or $S' = O_{23}^T S O_{23}$), where $\nu = (\nu_e\ \nu_\mu\ \nu_\tau)^T$ and S are the neutrino vector of state and evolution matrix in flavour basis. The effective Hamiltonian $H'(t)$ governing the neutrino evolution in the rotated basis can be written as

$$H'(t) = \begin{pmatrix} c_{13}^2 s_{12}^2 \delta + s_{13}^2 \Delta + V(t) & c_{13}c_{12}s_{12}\delta & c_{13}s_{13}\left(\Delta - s_{12}^2\delta\right)e^{-i\delta_{CP}} \\ c_{13}c_{12}s_{12}\delta & c_{12}^2\delta & -s_{13}c_{12}s_{12}e^{-i\delta_{CP}}\delta \\ c_{13}s_{13}\left(\Delta - s_{12}^2\delta\right)e^{i\delta_{CP}} & -s_{13}c_{12}s_{12}e^{i\delta_{CP}}\delta & s_{13}^2 s_{12}^2 \delta + c_{13}^2 \Delta \end{pmatrix}.$$
(A4)

It can be decomposed as

$$H'(t) = H_0'(t) + H_I', \qquad H_I' = H_1' + H_2',$$
(A5)

where

$$H_0'(t) = \begin{pmatrix} \frac{1}{2}[-\cos 2\theta_{12}\,\delta + V(t)] & \frac{1}{2}\sin 2\theta_{12}\,\delta & 0 \\ \frac{1}{2}\sin 2\theta_{12}\,\delta & \frac{1}{2}[\cos 2\theta_{12}\,\delta - V(t)] & 0 \\ 0 & 0 & \tilde{\Delta}(t) \end{pmatrix} \equiv \begin{pmatrix} h(t) & & 0 \\ & & 0 \\ 0 & 0 & \tilde{\Delta}(t) \end{pmatrix}$$
(A6)

is of zeroth order in the small parameter s_{13}, H_1' is of the first order, and H_2' includes terms of the second and higher orders, namely

$$H_1' = \begin{pmatrix} 0 & 0 & s_{13}\left(\Delta - s_{12}^2\delta\right)e^{-i\delta_{CP}} \\ 0 & 0 & -s_{13}c_{12}s_{12}e^{-i\delta_{CP}}\delta \\ s_{13}\left(\Delta - s_{12}^2\delta\right)e^{i\delta_{CP}} & -s_{13}c_{12}s_{12}e^{i\delta_{CP}}\delta & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a^* & b^* & 0 \end{pmatrix}$$
(A7)

and $H_2' = \mathcal{O}(s_{13}^2)$. The function $\tilde{\Delta}(t)$ that enters into Eq. (A6) was defined in Eq. (21). We have subtracted from the Hamiltonian a term proportional to the unit matrix in order to make the upper-left 2×2 submatrix h of H_0' traceless. This amounts to multiplying all the components of the neutrino state by the same phase factor, which does not affect the neutrino oscillation probabilities.

We will look now for the solution to the Schrödinger equation for the evolution matrix S' in the rotated basis in the form

$$S'(t, t_0) = S_0'(t, t_0)\, S_1'(t, t_0),$$
(A8)

[8]Omitting possible Majorana phases which have no effect on neutrino oscillations.

where $S'_0(t, t_0)$ satisfies the equation

$$i\frac{d}{dt}S'_0(t, t_0) = H'_0(t)S'_0(t, t_0)\,,\qquad (A9)$$

with the initial condition $S'_0(t_0, t_0) = \mathbf{1}$. Its general solution can be written as

$$S'_0(t, t_0) = \begin{pmatrix} \alpha(t, t_0) & \beta(t, t_0) & 0 \\ -\beta^*(t, t_0) & \alpha^*(t, t_0) & 0 \\ 0 & 0 & f(t, t_0) \end{pmatrix}\,,\qquad |\alpha|^2 + |\beta|^2 = 1\,,\qquad (A10)$$

where $f(t, t_0)$ is given in Eq. (23) and the parameters $\alpha(t, t_0)$ and $\beta(t, t_0)$ are to be found from the solution of the two-flavour neutrino problem that corresponds to the 2×2 submatrix h of H'_0. As follows from the evolution equation for S' and Eqs. (A8) and (A9), the matrix S'_1 satisfies the equation

$$i\frac{d}{dt}S'_1(t, t_0) = [S'_0(t, t_0)^{-1}H'_I S'_0(t, t_0)]S'_1(t, t_0)\,,\qquad S'_1(t_0, t_0) = \mathbf{1}\,.\qquad (A11)$$

Up to now everything is exact; we will determine now the evolution matrix S' to the first order in perturbation theory in the small parameter s_{13}. This leads to

$$S'(t, t_0) \simeq S'_0(t, t_0) - iS'_0(t, t_0)\int_{t_0}^t [S'_0(t', t_0)^{-1}H'_1 S'_0(t', t_0)]\,dt'\,.\qquad (A12)$$

A straightforward calculation then gives

$$S'(t, t_0) = \begin{pmatrix} \alpha(t, t_0) & \beta(t, t_0) & -iA \\ -\beta^*(t, t_0) & \alpha^*(t, t_0) & -iB \\ -iC & -iD & f(t, t_0) \end{pmatrix}\,,\qquad (A13)$$

where

$$A \equiv aA_a + bA_b\,,\qquad B \equiv aB_a + bB_b\,,\qquad (A14)$$
$$C \equiv a^*C_a + b^*C_b\,,\qquad D \equiv a^*D_a + b^*D_b\,,\qquad (A15)$$

the parameters a and b were defined in Eq. (A7) and

$$A_a \equiv \alpha(t, t_0)\int_{t_0}^t \alpha(t', t_0)^* f(t', t_0)\,dt' + \beta(t, t_0)\int_{t_0}^t \beta(t', t_0)^* f(t', t_0)\,dt'\,,\qquad (A16)$$

$$A_b \equiv \beta(t, t_0)\int_{t_0}^t \alpha(t', t_0)f(t', t_0)\,dt' - \alpha(t, t_0)\int_{t_0}^t \beta(t', t_0)f(t', t_0)\,dt'\,,\qquad (A17)$$

$$B_a \equiv \alpha(t, t_0)^*\int_{t_0}^t \beta(t', t_0)^* f(t', t_0)\,dt' - \beta(t, t_0)^*\int_{t_0}^t \alpha(t', t_0)^* f(t', t_0)\,dt'\,,\qquad (A18)$$

$$B_b \equiv \alpha(t, t_0)^*\int_{t_0}^t \alpha(t', t_0)f(t', t_0)\,dt' + \beta(t, t_0)^*\int_{t_0}^t \beta(t', t_0)f(t', t_0)\,dt'\,,\qquad (A19)$$

22

as well as

$$C_a \equiv f(t, t_0) \int_{t_0}^{t} \alpha(t', t_0) f(t', t_0)^* \, dt', \quad C_b \equiv -f(t, t_0) \int_{t_0}^{t} \beta(t', t_0)^* f(t', t_0)^* \, dt', \quad (A20)$$

$$D_a \equiv f(t, t_0) \int_{t_0}^{t} \beta(t', t_0) f(t', t_0)^* \, dt', \quad D_b \equiv f(t, t_0) \int_{t_0}^{t} \alpha(t', t_0)^* f(t', t_0)^* \, dt'. \quad (A21)$$

The calculation of the integrals in Eqs. (A16)-(A19) can be simplified considerably by noticing that their right-hand sides contain the expressions that are the products of the elements of the evolution matrix $S_0'(t_1, t)$. Indeed, one has

$$S_0'(t_1, t) = S_0'(t_1, t_0) S_0'(t, t_0)^\dagger = \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ -\beta_1^* & \alpha_1^* & 0 \\ 0 & 0 & f_1 \end{pmatrix} \begin{pmatrix} \alpha^* & -\beta & 0 \\ \beta^* & \alpha & 0 \\ 0 & 0 & f^* \end{pmatrix}, \quad (A22)$$

where we used the notation $\alpha \equiv \alpha(t, t_0)$, $\alpha_1 \equiv \alpha(t_1, t_0)$, $etc.$ On the other hand, we have

$$S_0'(t_1, t) = \begin{pmatrix} \alpha(t_1, t) & \beta(t_1, t) & 0 \\ -\beta(t_1, t)^* & \alpha(t_1, t)^* & 0 \\ 0 & 0 & f(t_1, t) \end{pmatrix}. \quad (A23)$$

Comparing Eqs. (A22) and (A23) one finds

$$(\alpha \alpha_1^* + \beta \beta_1^*) f_1 f^* = \alpha(t_1, t)^* f(t_1, t), \quad (\beta \alpha_1 - \alpha \beta_1) f_1 f^* = -\beta(t_1, t) f(t_1, t), \quad (A24)$$

$$(\alpha^* \beta_1^* - \beta^* \alpha_1^*) f_1 f^* = \beta(t_1, t)^* f(t_1, t), \quad (\alpha^* \alpha_1 + \beta^* \beta_1) f_1 f^* = \alpha(t_1, t) f(t_1, t). \quad (A25)$$

This allows a simplification of the integrals in Eqs. (A16)-(A19), making them similar in form to those in Eqs. (A20) and (A21):

$$A_a = f(t, t_0) \int_{t_0}^{t} \alpha(t', t)^* f(t', t) \, dt', \quad A_b = -f(t, t_0) \int_{t_0}^{t} \beta(t', t) f(t', t) \, dt', \quad (A26)$$

$$B_a = -f(t, t_0) \int_{t_0}^{t} \beta(t', t)^* f(t', t) \, dt', \quad B_b = f(t, t_0) \int_{t_0}^{t} \alpha(t', t) f(t', t) \, dt'. \quad (A27)$$

Rotating $S'(t, t_0)$ by O_{23} back to the original flavour basis, one finds from Eq. (A13)

$$S(t, t_0) = \begin{pmatrix} \alpha & c_{23}\beta - is_{23}A & -s_{23}\beta - ic_{23}A \\ -c_{23}\beta^* - is_{23}C & S_{22} & S_{23} \\ s_{23}\beta^* - ic_{23}C & S_{32} & S_{33} \end{pmatrix}, \quad (A28)$$

where

$$S_{22} \equiv c_{23}^2 \alpha^* + s_{23}^2 f - is_{23}c_{23}(B + D), \quad (A29)$$

$$S_{23} \equiv -s_{23}c_{23}(\alpha^* - f) - i(c_{23}^2 B - s_{23}^2 D), \quad (A30)$$

$$S_{32} \equiv -s_{23}c_{23}(\alpha^* - f) + i(s_{23}^2 B - c_{23}^2 D), \quad (A31)$$

$$S_{33} \equiv s_{23}^2 \alpha^* + c_{23}^2 f + is_{23}c_{23}(B + D). \quad (A32)$$

23

The parameters α, β, and f in Eqs. (A13) and (A28)-(A32) are of zeroth order in s_{13}, whereas A, B, C, and D are of the first order. From Eq. (A13) one can see that, if the mixing angle θ_{23} were zero, the T-odd probability difference $\Delta P_{\tau e}^T = |A|^2 - |C|^2$ would have scaled with θ_{13} as s_{13}^2 [9]. We have checked this by solving Eq. (A1) numerically. Since in vacuum there is no CP and T violation when any of the mixing angles is equal to zero, non-vanishing $\Delta P_{ab}^T = |A|^2 - |C|^2$ in the case $\theta_{23} = 0$ is a pure matter effect. Moreover, it is an effect of asymmetric matter: as we show below, in the case of matter with a symmetric density profile (and so also in vacuum) $|A|^2 = |C|^2$, and all ΔP_{ab}^T vanish. The atmospheric neutrino data indicate that θ_{23} is close to $45°$, and therefore the T-odd probability differences must scale linearly with s_{13}. From Eq. (A28) one finds

$$\Delta P_{e\mu}^T = |S_{21}|^2 - |S_{12}|^2 = -2s_{23}c_{23}\text{Im}\left[\beta^*(A - C^*)\right]. \qquad (A33)$$

From the definition of the parameters a and b in Eq. (A7) it follows that the ratio $|b/a| \simeq \Delta m_{21}^2/\Delta m_{31}^2$ is small; it can also be shown that $|A_b/A_a| \sim |C_b/C_a| \sim \Delta m_{21}^2/\Delta m_{31}^2$. Therefore, the contributions of A_b and C_b to Eq. (A33) are suppressed by the factor $(\Delta m_{21}^2/\Delta m_{31}^2)^2$, and one finally arrives at Eq. (24).

We will now show that our expression (A33) satisfies the requirement that in the absence of the fundamental CP and T violation (*i.e.*, in the case $\delta_{CP} = 0$), the T-odd probability differences must vanish for any symmetric matter density profile. To do so we will show that in this case the expression $\beta^*(A - C^*)$ is real.

Consider the evolution of the neutrino system over the symmetric time interval $[-t, t]$. First, from the evolution equations (9) and (10) we notice that in the case of symmetric matter the matrix $S_0'(0, -t)^T$ coincides with $S_0'(t, 0)$ (the phase δ_{CP} does not affect the evolution equations for S_0'). Together with Eq. (8) this leads to the following symmetry properties of α, β, and f in symmetric matter:

$$\alpha(-t, 0) = \alpha(t, 0)^*, \qquad \beta(-t, 0) = \beta(t, 0)^*, \qquad f(-t, 0) = f(t, 0)^*. \qquad (A34)$$

Consider now the evolution matrix $S'(t, -t) = S'(t, 0)S'(0, -t)$. It can be parameterized in the form similar to that of Eq. (A13), and its entries can be found in terms of those of $S'(t, 0)$ and $S'(0, -t)$. Using Eq. (A34) it is then straightforward to show that in symmetric matter the entries of $S(t, -t)$ satisfy

$$A_a = C_a, \qquad A_b = C_b. \qquad (A35)$$

In the case of $\delta_{CP} = 0$ the parameters a and b are real, and from Eqs. (A15) and (A35) one finds $A = C$. This means that $A - C^*$ is pure imaginary; since β is also pure imaginary in this case, the right-hand side of Eq. (A33) vanishes, which completes the proof.

As follows from the definition of the parameters a and b in Eq. (A7), they can be written as $a = e^{-i\delta_{CP}}a'$, $b = e^{-i\delta_{CP}}b'$ with real a' and b'. Therefore in the general case of $\delta_{CP} \neq 0$ one finds that in symmetric matter $|A| = |C|$.

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[9]Due to Eq. (20) so would $\Delta P_{e\mu}^T$ and $\Delta P_{\mu\tau}^T$ do. However, the former cannot be found directly from Eq. (A13) as this would require the calculation to be done in the next order in perturbation theory.

Appendix B: Two special cases

We shall now calculate $\Delta P_{e\mu}^T$ in two special cases – matter consisting of two layers of constant densities and the adiabatic approximation in the case of an arbitrary matter density profile.

Consider first matter consisting of two layers of constant electron number densities N_1 and N_2 and widths L_1 and L_2, respectively. The corresponding matter-induced potentials (A2) are V_1 and V_2. The values θ_1 and θ_2 of the mixing angle in the (1,2)-subsector in matter of densities N_1 and N_2 are given by

$$\cos 2\theta_1 = \frac{\cos 2\theta_{12}\delta - V_1}{2\omega_1}, \qquad \cos 2\theta_2 = \frac{\cos 2\theta_{12}\delta - V_2}{2\omega_2}, \tag{B1}$$

where ω_1 and ω_2 were defined in Eq. (26). The time interval of the evolution of the neutrino system can be divided into two parts: (I) $0 \leq t' < L_1$ and (II) $L_1 \leq t' \leq L_1 + L_2 \equiv L$. The parameters α, β, and f for the first interval are given by the well-known evolution in matter of constant density:

$$\alpha(t', 0) = \cos(\omega_1 t') + i \cos 2\theta_1 \sin(\omega_1 t'), \qquad \beta(t', 0) = -i \sin 2\theta_1 \sin(\omega_1 t'), \tag{B2}$$

$$f(t', 0) = \exp(-i\Delta_1 t'), \qquad\qquad \Delta_{1,2} \equiv \Delta - \frac{1}{2}(\delta + V_{1,2}), \tag{B3}$$

whereas for the second interval they can be expressed through the elements of the two-flavour neutrino evolution matrix in the two-layer medium [30]:

$$\alpha(t', 0) = c_1 c_2' - s_1 s_2' \cos(2\theta_1 - 2\theta_2) + i(s_1 c_2' \cos 2\theta_1 + s_2' c_1 \cos 2\theta_2), \tag{B4}$$

$$\beta(t', 0) = -i(s_1 c_2' \sin 2\theta_1 + s_2' c_1 \sin 2\theta_2) + s_1 s_2' \sin(2\theta_1 - 2\theta_2), \tag{B5}$$

$$f(t', 0) = \exp\{-i[\Delta_1 L_1 + \Delta_2(t' - L_1)]\}. \tag{B6}$$

Here

$$s_2' = \sin(\omega_2 \tau), \qquad c_2' = \cos(\omega_2 \tau), \qquad \tau = t' - L_1, \tag{B7}$$

and $s_{1,2}$ and $c_{1,2}$ were defined in Eq. (27). Direct calculation using Eqs. (24), (A26), and (25) [or Eqs. (24), (A16), and (25)] gives the result presented in Eqs. (28)-(30).

Next, we consider the adiabatic regime in the case of an arbitrary matter density profile. In this regime, the change of the matter density along the neutrino path is slow compared to the oscillation frequency. The effective Hamiltonian of the neutrino system is approximately diagonal in the basis of instantaneous matter eigenstates, and the evolution of the eigenstates amounts to a mere multiplication by phase factors. The evolution in the flavour basis is obtained by rotating the evolution matrix in the instantaneous eigenstates basis $S(t, t_0)_{\text{eigen}}$ by the matrices $U(t_0)$ and $U(t)$ of leptonic mixing in matter, which correspond to the initial and final times of neutrino evolution: $S(t, t_0) = U(t)S(t, t_0)_{\text{eigen}}U(t_0)^\dagger$. Applying this procedure to the two-flavour neutrino evolution described by the Hamiltonian H_0', one arrives at Eqs. (36) and (37), while the parameter f is always given by Eq. (23).

We will be assuming that $V(t) \lesssim \delta \ll \Delta$, and also that the oscillations governed by the large $\Delta = \Delta m_{31}^2/2E$ are in the averaging regime (i.e., that $\Delta \cdot L \gg 1$). The latter

assumption allows us to obtain simple approximate expressions for the relevant integrals. We first note that

$$\int_{t_0}^{t} g(t')e^{-i\Delta t'}\,dt' \simeq \frac{i}{\Delta}[g(t)e^{-i\Delta t} - g(t_0)e^{-i\Delta t_0}] + \mathcal{O}(1/\Delta^2)\,, \tag{B8}$$

where $g(x)$ is an arbitrary regular function which changes slowly on the time intervals $\sim 1/\Delta$, and the integration by parts has been used. Direct calculation then yields

$$A_a - C_a^* \simeq i\frac{2}{\Delta}\{\cos[\Delta(t - t_0)] - \cos(\theta - \theta_0)\cos\Phi\}\,. \tag{B9}$$

Eqs. (24), (36), (37), and (B9) lead to Eq. (39).

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27