Analysis of Robot Navigation Schemes using Rantzer's Dual Lyapunov Theorem

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Abstract— When robots are driven by the negative gradient of a potential field that consists of the sum of an attractive and a repulsive term, convergence to the desired configuration cannot be guaranteed by traditional Lyapunov techniques. In this paper, sufficient conditions for convergence of such systems are provided instead with the use of Rantzer's Dual Lyapunov Theorem. In particular, a condition that involves the trace of the Hessian matrix of the potential function is derived and then applied to the cases of navigation of a single robot and of multi-robot formation stabilization. The main result of the paper states that a sufficient condition for convergence to a desired configuration in both cases is that the attractive potential admits a sufficiently large gain. A lower bound on the attractive potential is computed. Computer simulations that support the new results are provided.

I. INTRODUCTION

A common problem with sensor-based path planning algorithms in robotics is the existence of local minima when attractive and repulsive forces are combined. The pioneering work of Koditschek and Rimon [10] involved navigation of a single robot in an environment of spherical obstacles with guaranteed convergence. Their navigation function framework was later revisited by a number of researchers in the field of multi-robot systems to provide local-minima free algorithms for various related problems; examples include centralized [13],[19] and decentralized [3],[1],[2] multi-robot navigation algorithms. While the navigation function framework established in [10] was proven successful for some specific classes of potential functions, the problem of convergence for more general cases, when the potential function is the direct sum of an attractive and a repulsive term, is an problem. The use of the sum of an attractive and a repulsive term dates back to the work of Khatib [9]. The issue of local minima was highlighted as the main problem of that approach [11]. The sum of repulsive and attractive terms has been used extensively in multi-robot systems over the last few years; e.g., [12],[7],[16]. In this paper we attempt to characterize the local minima that arise in the negative gradient flows of the sum of a repulsive and an attractive term via Rantzer's Dual Lyapunov Theorem [17].

Rantzer [17] recently presented a new convergence criterion for nonlinear systems, that involved the divergence of the vector field with respect to a certain positive density function. A connection between Rantzer's density function and navigation functions appeared in [14] for a centralized and in [6] for a decentralized case. Both papers used density functions to examine known properties of vector fields: convergence that was proven in the past using other techniques. More precisely, [6] provided a connection between density functions and decentralized navigation functions, by applying Rantzer's Dual Lyapunov Theorem to the critical points of these particular navigation functions, and not the whole configuration space, as in the current paper. Furthermore, in [14] a density function was constructed for a system that is driven by a given navigation function, with pre-specified navigation properties. In the case of our paper, however, the potential field is not assumed to be a navigation function. Thus, no convergence guarantees are initially given. The fact that the closed-loop system actually converges to the goal is proven in the paper by the use of density functions. In essence, the current paper gives proof of convergence to a desired configuration for a class of systems that was not proved to converge in the literature.

In this paper we provide sufficient conditions for the convergence of a negative gradient system to a desired equilibrium point based on the new theory of density functions. After stating the problem in Section II, and revisiting Rantzer's Dual Lyapunov Theorem in Section III, a condition on the trace of the Hessian matrix of the potential function is derived in Section IV and the result is then applied to controller design in two navigation cases. In Section V we treat the case of a single robot navigating in an environment with static obstacles and then in Section VI, the case of a distributed formation stabilization problem for a multi-robot system. In both cases, the robots are driven by the negative gradient of a potential field given by the sum of an attractive and a repulsive term. The main result states that a sufficient condition for convergence to a desired configuration is that the attractive potential gain is larger than a finite lower bound. The lower bound depends on the underlying problem geometry and is computed based on a worst case approach. We illustrate the result through simulated examples in Section VII. We provide an example where a single robot is blocked in an undesirable local minimum when the gain of its attractive potential does not satisfy the bounds derived in the paper; hence, illustrating the tightness of the sufficient condition. The results are summarized in Section VIII.

II. PROBLEM STATEMENT

Consider a system with state $q \in W \subset \mathbb{R}^n$ and control input $u \in \mathbb{R}^n$. Denote by $\varphi : W \to [0, \infty)$ a potential

function whose negative gradient flow is responsible for driving the system to the desired configuration $q^* \in W$.

$$\dot{q} = u = -\nabla\varphi\left(q\right) \tag{1}$$

where $\nabla\varphi\left(q\right) = \left[\dfrac{\partial\varphi}{\partial q_1}, \ldots, \dfrac{\partial\varphi}{\partial q_n}\right]^T$. The potential function φ is defined as the sum of an attractive and a repulsive potential, $a, r : W \to [0, \infty)$, respectively: $\varphi\left(q\right) \triangleq a\left(q\right) + r\left(q\right)$. The attractive potential $a(q)$ has a global minimum at q^*. The repulsive potential $r(q)$ is responsible for collision avoidance, both with static and moving obstacles. The considered problem is to give conditions under which q converges to q^*.

Convergence to q^* can be studied by using φ as a Lyapunov function. We have $\dot{\varphi}(q) = \left(\nabla\varphi\left(q\right)\right)^T \dot{q} = -\left\|\nabla\varphi\left(q\right)\right\|^2 \leq 0$. Unfortunately, even when there is a compact invariant set, LaSalle's principle only guarantees that the robots converge to the largest invariant subset of the set of critical points of φ, i.e., $\mathcal{C} \triangleq \{q \in W : \nabla\varphi\left(q\right) = 0\}$, and thus, system (1) may have stable equilibria in $\mathcal{C}$ that do not coincide with q^*.

In this paper we are able to guarantee convergence to q^* from almost everywhere in W by using Rantzer's Dual Lyapunov Theorem.

III. Rantzer's Dual Lyapunov Theorem

For a function $f : \mathbb{R}^n \to \mathbb{R}^n$, let $f \in C^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$ denote that f is continuously differentiable and introduce the notation $\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_n}{\partial x_n}$. We will use the following variation of Rantzer's Dual Lyapunov Theorem [17]:

Theorem 1: Given $\dot{x}(t) = f(x(t))$, where $f \in C^1\left(S, \mathbb{R}^n\right)$, $f(0) = 0$, and S is an a open, bounded subset of $\mathbb{R}^n$, and positively invariant and suppose $x^* = 0 \in S$ is a stable equilibrium point. Furthermore, suppose there exists a function $\rho \in C^1\left(S\backslash\{0\}, \mathbb{R}\right)$ such that $\rho\left(x\right)f\left(x\right) / \left\|x\right\|$ is integrable on $\{x \in S : \|x\| \geq 1\}$ and

$$\left[\nabla \cdot \left(f\rho\right)\right]\left(x\right) > 0 \text{ for almost all } x \in S \tag{2}$$

Then, for almost all initial states $x(0) \in S$, the trajectory $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$. $\square$
The main difference of Theorem 1 compared to the statement in [17] is that we restrict the domain S of f to be a bounded set, which is needed for the application of this paper. The proof of Theorem 1 is almost identical to the proof in [17], and can be found in the journal version of this paper [4].

A local version of Rantzer's Dual Lyapunov Theorem was used in [18]. Relaxed conditions for convergence to an equilibrium point in open subsets of $\mathbb{R}^n$ were given in [15].

IV. Convergence analysis via Rantzer's Dual Lyapunov Theorem

We use Theorem 1 to check the convergence of the negative gradient system (1). A general condition is provided that involves the trace of the Hessian of φ. It is applied to specific navigation problems in the next sections.

Consider Theorem 1 and let $\rho(q) = -\varphi(q)$ be a candidate density function. Assume that $q \in W \subset \mathbb{R}^n$, where W is

open, bounded and positively invariant. Suppose moreover that $q^* \in W$ is stable.

The divergence of $f\rho$ with $\dot{q} = -\nabla\varphi(q) = f(q)$ is now

$$\nabla \cdot (f\rho) = (\nabla\rho) \cdot f + \rho(\nabla \cdot f)$$
$$= \|\nabla\varphi\|^2 - \varphi(\nabla \cdot (-\nabla\varphi)) = \|\nabla\varphi\|^2 + \varphi(\nabla \cdot (\nabla\varphi))$$

Since $\|\nabla\varphi\|^2 \geq 0$ and φ is strictly positive at points that do not coincide with q^*, condition $\nabla \cdot (f\rho) > 0$ holds if $\nabla \cdot (\nabla\varphi(q)) > 0$. Note now that

$$\nabla \cdot (\nabla\varphi\left(q\right)) = \text{trace}\left(\nabla^2\varphi(q)\right) = \text{trace}\left(\nabla^2 a(q) + \nabla^2 r(q)\right)$$

so condition (2) is satisfied provided that $\text{trace}\left(\nabla^2 a(q) + \nabla^2 r(q)\right) > 0$ for almost all $q \in W$. The previous argument is summarized in the following theorem:

Theorem 2: Consider $\dot{q} = u$ with control $u = -\nabla\varphi\left(q\right)$. Let the desired configuration q^* be stable and such that $\nabla a\left(q^*\right) = 0$. Suppose $\dfrac{\varphi(q)\nabla\varphi(q)}{\|q\|}$ is integrable on $\{q \in W : \|q - q^*\| \geq 1\}$, where $W \subset \mathbb{R}^n$ is open, bounded and positively invariant. Then, for almost all initial states $q(0) \in W$ the trajectory $q(t)$ exists for $t \in [0, \infty)$ and tends to q^* as $t \to \infty$, provided that

$$\text{trace}\left(\nabla^2 a(q) + \nabla^2 r(q)\right) > 0 \tag{3}$$

for almost all $q \in W$. $\square$

Theorem 2 provides a quite general condition for convergence to the desired equilibrium points of a system that is driven by the negative gradient of a potential field, that is the sum of an attractive and a repulsive term. In the sequel, we show that Theorem 2 leads to conditions that are sufficient for closed-loop navigation.

V. Single robot with local sensing capabilities

The first navigation problem is a single robot navigating towards a target point in a planar environment with static obstacles. Treating this problem with the use of a potential field constituted by the sum of an attractive and a repulsive term has been shown to suffer from the existence of local minima [9]. Intuition, however, suggests that if the attractive potential is sufficiently large, critical points can be avoided. We provide a theoretical explanation of this based on Theorem 2.

Let the attractive potential that drives the robot to its desired destination be given by

$$a(q) = k_a \left\|q - q^*\right\|^2 \tag{4}$$

where $q \in W \subset \mathbb{R}^2$ denotes the robot's position, $q^* \in W$ its destination point and $k_a > 0$ a positive parameter to be appropriately tuned. The definition of W for this case will be given in the sequel. We have $\nabla a(q) = 2k_a\left(q - q^*\right)$, and thus $\nabla^2 a(q) = 2k_a I$, where I is identity matrix. The robot navigates in an environment with M static circular obstacles and has limited sensing capabilities. Specifically, the robot can sense obstacles located within a circle of radius d.

For each circular obstacle $o \in \{1, \ldots, M\}$, with center $q_o \in W$ and radius r_o, the repulsive potential is given by

$r = r\left(\beta_o(q)\right)$ where $\beta_o(q) = \|q - q_o\|^2 - r_o^2$. We then have $\nabla\beta_o(q) = 2\left(q - q_o\right)$, and thus $\nabla^2\beta_o(q) = 2I$. We moreover require that the obstacles are strictly disjoint with one another, i.e., $\left\|q_{o_i} - q_{o_j}\right\| > r_{o_i} + r_{o_j}$ for all $o_i, o_j \in \{1, \ldots, M\}$.

We can compute $\nabla\beta_o(q)\nabla\beta_o(q)^T = 4\left(q - q_o\right)\left(q - q_o\right)^T$ and $\mathrm{trace}\left(\nabla\beta_o(q)\nabla\beta_o(q)^T\right) = 4\|q - q_o\|^2 = 4\beta_o(q) + 4r_o^2$. A suitable repulsive potential is given by

$$r\left(\beta_o\right) = \begin{cases} \dfrac{k_r}{\beta_o}, 0 < \beta_o \leq c \\ \pi\left(\beta_o\right), c < \beta_o < d \\ 0, \beta_o \geq d \end{cases}$$

where d is the sensing radius and k_r, c are positive parameters. We require that r is strictly decreasing in $(0, d)$ which allows the robot to take into account obstacles located within its sensing zone at each time instant. Let π be a repulsive potential that is twice continuously differentiable at all points q with $\beta_o(q) > 0$, and strictly decreasing in $(0, d)$. For simplicity, let $\pi(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$. Such a function allows the control designer to treat the sensing radius d and the controller gain k_r as design parameters. The coefficients of π and the parameter c are chosen to satisfy the differentiability requirement $\varphi \in C^2$. We should emphasize here that the choices of functions π and r do not constrain the generality of the methodology. The analysis that follows holds for any π that renders r twice continuously differentiable and strictly decreasing at all points q with $\beta_o(q) > 0$. Moreover, the repulsive potential r is chosen to take into account the robot's limited sensing capabilities. The proposed framework holds for a general class of such repulsive potentials r.

We can now compute

$$\nabla r(\beta_o) = \begin{cases} -\dfrac{k_r}{\beta_o^2}\nabla\beta_o, 0 < \beta_o \leq c \\ \pi'\left(\beta_o\right)\nabla\beta_o, c < \beta_o < d \\ 0, \beta_o \geq d \end{cases}$$

and thus

$$\nabla^2 r(\beta_o) = \begin{cases} \dfrac{-k_r\beta_o^2\nabla^2\beta_o + 2k_r\beta_o\nabla\beta_o\nabla\beta_o^T}{\beta_o^4}, 0 < \beta_o \leq c \\ \pi'\left(\beta_o\right)\nabla^2\beta_o + \pi''\left(\beta_o\right)\nabla\beta_o\nabla\beta_o^T, c < \beta_o < d \\ 0, \beta_o \geq d \end{cases}$$

where π' and π'' are the first and second derivatives of π.

The total repulsive potential is defined as the sum of the repulsive potentials of all obstacles: $r_S \triangleq \sum_{o=1}^{M} r\left(\beta_o\right)$. The total potential φ driving the robot, is given by $\varphi \triangleq a + r_S$.

We first show that the subset $W \in \mathbb{R}^2$ in which the closed-loop system evolves can be rendered open, bounded and invariant by assuming an upper bound on the initial value of φ. This will also guarantee that a and r_S remain bounded. Thus, the robot remains at a positive distance from the obstacle boundary. In fact, we show in Lemma 3 that there is a strictly positive lower bound on the term β_o. This implies that the repulsive potential remains bounded and hence, the integrability condition of Theorem 2 holds.

Recall that the closed-loop system is $\dot{q} = -\nabla\varphi$ and assume that the robot initial condition satisfies $\varphi(q(0)) < \varphi_0$ for some arbitrarily large yet finite φ_0. Then, since $\dot{\varphi} = -\|\nabla\varphi\|^2 \leq 0$, we have $\varphi\left(q(t)\right) < \varphi_0$ for all $t \geq 0$. Since a, r_S and r are all non-negative, we have $r\left(\beta_o\left(q(t)\right)\right) \leq r_S\left(q(t)\right) \leq \varphi\left(q(t)\right) < \varphi_0$ for all $t \geq 0$ and hence all repulsive potentials remain bounded. Condition $r\left(\beta_o\left(q(t)\right)\right) < \varphi_0$ implies that $\beta_o\left(q(t)\right) > \min\left\{c, k_r/\varphi_0\right\}$ for all $t \geq 0$ and all $o \in \{1, \ldots, M\}$, due to the definition of r. Moreover, we have $a\left(q(t)\right) < \varphi_0$ and thus $\|q(t) - q^*\| < \sqrt{\varphi_0/k_a}$, for all $t \geq 0$. The previous discussion is summarized as follows:

Lemma 3: The set

$$W_{\varphi_0} \triangleq \left\{ q \in \mathbb{R}^2 : \begin{array}{c} \left(\beta_o > \min\left\{c, k_r/\varphi_0\right\}\right) \\ \wedge\left(\|q - q^*\| < \sqrt{\varphi_0/k_a}\right) \end{array} \right\} \quad (5)$$

for all $o \in \{1, \ldots, M\}$ is an open, bounded, and invariant set for the trajectories of the closed-loop system.

Note that we make $W = W_{\varphi_0}$ explicitly dependent on φ_0, which can be viewed as an upper bound on the initial conditions $\varphi(q(0))$. Note also that the potential function remains bounded within W_{φ_0} and thus, the integrability condition of Theorem 2 holds. Moreover, collision avoidance is guaranteed, since β_o remains bounded from below by $\min\left\{c, k_r/\varphi_0\right\}$ for all obstacles $o \in \{1, \ldots, M\}$.

By the definition of r we have that $r(\beta_o(q)) = 0$ for $\beta_o(q) \geq d$. Define $B_\varepsilon(q^*) \triangleq \{q : \|q - q^*\| \leq \varepsilon\}$. We assume that for all $o \in \{1, \ldots, M\}$ we have $\beta_o(q) > d + \varepsilon, \forall q \in B_\varepsilon$. This implies that $r_S(q) = \sum_{o=1}^{M} r\left(\beta_o(q)\right) = 0$ for all $q \in B_\varepsilon$, i.e., the repulsive potential is identically zero in a neighborhood around the desired destination. In practice this means that the obstacles are not located too close to q^*. We have $\nabla\varphi\left(q^*\right) = \nabla a\left(q^*\right) = 0$ and thus $\nabla^2\varphi\left(q^*\right) = \nabla^2 a\left(q^*\right) = 2k_a I > 0$, which means that q^* is a non-degenerate local minimum of φ and, hence, a stable equilibrium point [10].

We now check the validity of (3) for $q \neq q^*$. For $0 < \beta_0(q) \leq c$, we have

$$\begin{aligned} \mathrm{trace}\left(\nabla^2 r\right) &= \frac{k_r}{\beta_o^4}\mathrm{trace}\left(-\beta_o^2\nabla^2\beta_o + 2\beta_o\nabla\beta_o\nabla\beta_o^T\right) \\ &= \frac{k_r}{\beta_o^4}\left(-4\beta_o^2 + 2\beta_o \cdot (4\beta_o + 4r_o^2)\right) \\ &= \frac{4k_r}{\beta_o^2} + \frac{8k_r r_o^2}{\beta_o^4} \end{aligned}$$

and for $c < \beta_o < d$, we have

$$\begin{aligned} \mathrm{trace}\left(\nabla^2 r\right) &= \mathrm{trace}\left(\pi'\left(\beta_o\right)\nabla^2\beta_o + \pi''\left(\beta_o\right)\nabla\beta_o\nabla\beta_o^T\right) \\ &= 4\pi'\left(\beta_o\right) + (4\beta_o + 4r_o^2)\pi''\left(\beta_o\right) \end{aligned}$$

so that

$$\text{trace}\left(\nabla^2 a + \nabla^2 r_S\right) = 4k_a + \sum_{\substack{o \in [1,\dots,M]: \\ 0 < \beta_o \le c}} \left(\frac{4k_r}{\beta_o^2} + \frac{8k_r r_o^2}{\beta_o^4}\right)$$
$$+ \sum_{\substack{o \in [1,\dots,M]: \\ \beta_o > c}} \left\{4\pi'\left(\beta_o\right) + (4\beta_o + 4r_0^2)\pi''\left(\beta_o\right)\right\}$$

Since the polynomial function π has bounded coefficients and β_o attains values in the bounded set $(c, d]$, we deduce that there are finite upper and lower bounds of $4\pi'\left(\beta_o\right) + (4\beta_o + 4r_0^2)\pi''\left(\beta_o\right)$. Moreover, the term $\frac{4k_r}{\beta_o^2} + \frac{8k_r r_o^2}{\beta_o^4}$ is strictly positive for all $\beta_o \in (0, c]$. Therefore, denoting $m_1 \triangleq \max_{\beta_o \in (c,d]} \left\{-\pi'\left(\beta_o\right) - (\beta_o + r_o^2)\pi''\left(\beta_o\right)\right\}$ and choosing k_a to satisfy

$$k_a > M m_1 \tag{6}$$

we finally get $\text{trace}\left(\nabla^2 a + \nabla^2 r_S\right) > 0$. Hence condition (3) is satisfied. By virtue of Lemma 3, we finally get that all conditions of Theorem 2 are satisfied. The previous derivations are summarized in the following theorem:

Theorem 4: Consider a robot $\dot{q} = u$ with control law $u(q) = -\nabla\varphi(q)$ where $\varphi(q) = a(q) + r_S(q)$ fulfills (6) and $\beta_o\left(q\right) > d + \varepsilon$ for some $\varepsilon > 0$ and all $q \in B_\varepsilon(q^*)$ and $o \in \{1, \dots, M\}$. Then, for any $0 < \varphi_0 < \infty$ and almost all $q(0) \in W_{\varphi_0}$, with W_{φ_0} as in (5), the trajectory $q(t)$ of the closed-loop system exists for all $t \in [0, \infty)$ and $q(t)$ tends to q^* as $t \to \infty$.

The theorem states that the robot is driven to its destination point provided that the attractive potential gain is large enough. A lower bound on the gain is provided by (6). Note that k_a does not depend on φ_0. Thus, it does not constrain the invariant subset W_{φ_0} which depends on φ_0.

VI. DECENTRALIZED FORMATION STABILIZATION OF MULTIPLE ROBOTS

In this section, we consider the problem of formation stabilization. A similar result to the one in the previous section is derived for a multi-robot system driven by an attractive term, responsible for formation convergence, and a repulsive term, responsible for collision avoidance. In particular, it is shown that decentralized navigation to a desired formation is possible, provided that the attractive potential gains of the robots are sufficiently large.

Consider a system of N circular robots of radius r_a operating in a workspace W and let q_i denote the position of robot i. Let $q = [q_1^T, \dots, q_N^T]^T \in W \subset \mathbb{R}^{2N}$ be the stack vector of all robot positions. The motion of each robot i is given by $\dot{q}_i = u_i, i \in V = \{1, \dots, N\}$, where u_i is the velocity of i. The definition of W is provided in the sequel.

Each robot's objective is to go to a desired relative position with respect to a subset of the rest of the team, while avoiding colliding other robots. Specifically, each robot is assigned a specific subset $N_i \subset V$, called the robot i's *communication set*, with which it can communicate. The desired formation is encoded in terms of an undirected graph, the *formation graph* $G = \{V, E\}$, whose set of vertices $V = \{1, \dots, N\}$

is indexed by the team members, and whose set of edges $E = \{(i, j) \in V \times V | j \in N_i\}$ contains pairs of vertices that represent inter-robot formation specifications. A vector $c_{ij} \in \mathbb{R}^2$ is associated to each edge $(i, j) \in E$, in order to specify the desired relative robot positions in the goal formation. Graph G is undirected, so $(i, j) \in E$ if and only if $(j, i) \in E$ and moreover $c_{ij} = -c_{ji}$ for all $(i, j) \in E$.

We now recall some tools from algebraic graph theory [8] used in the sequel. For the graph G the $N \times N$ *adjacency matrix* $A = A(G) = (a_{ij})$ is given by $a_{ij} = 1$, if $(i, j) \in E$ and $a_{ij} = 0$, otherwise. If $(i, j) \in E$, then i, j are called *adjacent*. A *path* of length r from a vertex i to a vertex j is a sequence of $r + 1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of G, then G is called *connected*. The *degree* d_i of vertex i is the number of its adjacent vertices, i.e., $d_i = |N_i|$. Let Δ be the $N \times N$ diagonal matrix of d_i's. The *Laplacian* of G is the symmetric positive semidefinite matrix $L = L(G) = \Delta - A$. For a connected graph, L has a simple eigenvalue in zero with corresponding eigenvector being the vector of ones, $\overrightarrow{\mathbf{1}}$.

Collision avoidance means that the robot regions never overlap. The collision avoidance procedure is distributed in the sense that each robot only needs local knowledge of the robots that are close. We assume that each robot can sense robots that are within a disc of radius d around it. This disc is called the *sensing zone i*. The robots that belong to sensing zone i are denoted $M_i = \{j \in V, j \ne i : \|q_i - q_j\| \le d\}$. Each robot i requires knowledge of q_j for robots j belonging to $N_i \cup M_i$ at each time instant.

Formation stabilization is pursued with the use of a total attractive potential

$$a(q) = \sum_{i=1}^{N} \gamma_i(q), \quad \gamma_i(q) \triangleq \frac{k_f}{2} \sum_{j \in N_i} \|q_i - q_j - c_{ij}\|^2 \tag{7}$$

The formation objective is realized if and only if $a(q) = 0$. Each γ_i represents a cost function that is minimized whenever the formation is realized with respect to i.

For collision avoidance, the following repulsive potential is used: $r_F(q) = \sum_{i=1}^{N} \sum_{j \in M_i} V_{ij}\left(\beta_{ij}(q)\right)$, with $V_{ij}(\beta_{ij}(q)) = r(\beta_{ij}(q))$ and $\beta_{ij}(q) \triangleq \|q_i - q_j\|^2 - 4r_a^2$. Each V_{ij} represents the repulsion between robots i and j. Note that this repulsion has the same form as the repulsion for the single robot case. The control law of robot i is defined as:

$$u_i = -\sum_{j \in M_i} \frac{\partial V_{ij}}{\partial q_i} - \frac{\partial \gamma_i}{\partial q_i} \tag{8}$$

The closed-loop system is written in stack vector form as $\dot{q} = -\nabla\varphi(q)$, where the total potential φ is given by $\varphi = a + r_F$. See [5] for this derivation.

Similarly to the previous section, we will show that W can be defined as an open, bounded, invariant subset in the space of the robots *relative* positions. Assume that $\varphi\left(q(t)\right) \le \varphi(q(0)) < \varphi_0 < \infty$ for all $t \ge 0$. This implies that $r\left(\beta_o\left(q(t)\right)\right) \le r_F\left(q(t)\right) \le \varphi\left(t\right) < \varphi_0$ for

all $t \geq 0$, and thus, $\beta_{ij}\left(q(t)\right) > \min\{c, k_r/\varphi_0\}$ for all $t \geq 0$ and all $(i,j) \in E$. Since for all i, $\gamma_i\left(q(t)\right) \leq a\left(q(t)\right) \leq \varphi(q(t)) < \varphi_0$, then for all $(i,j) \in E$ we also have $\frac{k_f}{2}\|q_i - q_j - c_{ij}\|^2 < \varphi_0$ which implies that $\|q_i - q_j\| < \sqrt{2\varphi_0/k_f} + c_{\max}$, where $c_{\max} \triangleq \max\limits_{(i,j)\in E}\{\|c_{ij}\|\}$. Assuming a connected graph we finally get $\|q_i - q_j\| < (N-1)\left(\sqrt{2\varphi_0/k_f} + c_{\max}\right)$, for any pair of robots, since the maximum length of a path between any two robots is $N-1$. The following counterpart of Lemma 3 holds:

Lemma 5: The set

$$W_{\varphi_0} \triangleq \left\{ q \in \mathbb{R}^2 : \begin{array}{c} (\beta_{ij} > \min\{c, k_r/\varphi_0\}) \\ \wedge (\|q_i - q_j\| < C_{\varphi_0}) \end{array} \right\} \quad (9)$$

where $C_{\varphi_0} \triangleq (N-1)\sqrt{2\varphi_0/k_f} + c_{\max}$ and $i, j \in V$, is an open, bounded, and invariant set for the closed-loop system.

Note that we make $W = W_{\varphi_0}$ explicitly dependent on φ_0, as in the previous section. Recall now that $r(\beta_{ij}(q)) = 0$ for $\beta_{ij}(q) \geq d$. Hence, the repulsive potential is identically zero in a neighborhood $B_\varepsilon^F \triangleq \{q : \gamma_i(q) \leq \varepsilon, \forall i \in V\}$ around any desired formation configuration, provided that $\beta_{ij}(q) > d + \varepsilon, \forall q \in B_\varepsilon^F$. We define by $\Phi \triangleq \{q \in W \,|\, q_i - q_j = c_{ij}, \,\forall\, (i,j) \in E\}$ the set of feasible formation configurations. In the framework of Theorem 2, $q \in \Phi$ if and only if $q = q^*$. The following holds:

Lemma 6: Assume that $\beta_{ij}(q) > d + \varepsilon, \forall q \in B_\varepsilon^F$ holds. Then each $q \in \Phi$ is a stable equilibrium point of the closed-loop system.

Proof: Since $\beta_{ij}(q) > d + \varepsilon, \forall q \in B_\varepsilon^F$ holds, for each $q \in \Phi$ we have $\varphi(q) = a(q)$. Hence, for each $q \in \Phi$, $\nabla\varphi\,(q \in \Phi) = \nabla a\,(q \in \Phi) = 0$ and $\nabla^2\varphi(q) = \nabla^2 a(q) = \nabla^2\sum\limits_i \gamma_i = \sum\limits_i k_f |N_i| I$ where $|.|$ denotes cardinality. It follows that each $q \in \Phi$ is a non-degenerate local minimum of φ [10] and hence, a stable equilibrium point. $\diamond$

We thus can apply Theorem 2 in this case as well. In our setting, (3) is again equivalent to trace $\left(\nabla^2 a + \nabla^2 r_F\right) > 0$. We evaluate the left hand-side similarly as in Section V. For all $q \in W$, the attractive potential a satisfies trace $\left(\nabla^2 a\right) = 2\sum\limits_i k_f |N_i|$. For the repulsive potential r_F, we can compute trace $\left(\nabla^2 V_{ij}(\beta_{ij})\right) = $ trace $\left(\nabla^2 r(\beta_{ij})\right) = \frac{4k_r}{\beta_{ij}^2} + \frac{32 k_r r_a^2}{\beta_{ij}^4} > 0$, for $0 < \beta_{ij} \leq c$ and trace $\left(\nabla^2 V_{ij}(\beta_{ij})\right) = 4\pi'(\beta_{ij}) + (4\beta_{ij} + 16 r_a^2)\pi''(\beta_{ij})$, for $c < \beta_{ij} < d$. Thus,

$$\begin{aligned} \text{trace}\left(\nabla^2 a + \nabla^2 r_F\right) &= 2\sum\limits_i k_f |N_i| \\ &+ \sum\limits_i \sum\limits_{j \neq i: 0 < \beta_{ij} \leq c} \left\{\frac{4k_r}{\beta_{ij}^2} + \frac{32 k_r r_a^2}{\beta_{ij}^4}\right\} \\ &+ \sum\limits_i \sum\limits_{j \neq i: \beta_{ij} > c} \left\{4\pi'(\beta_{ij}) + (4\beta_{ij} + 16 r_a^2)\pi''(\beta_{ij})\right\} \end{aligned}$$
$$(10)$$

Using the same analysis as in the single robot navigation case, the last expression is always positive for sufficiently large attractive potential gains k_f. In fact, using the notation $m_2 \triangleq \max\limits_{\beta_{ij}\in(c,d]} \left\{-\pi'(\beta_{ij}) - (\beta_{ij} + 4 r_a^2)\pi''(\beta_{ij})\right\}$ the term

trace $\left(\nabla^2 a + \nabla^2 r_F\right)$ is guaranteed to be positive provided that the gains k_f satisfy

$$k_f > \frac{2 m_2 N (N-1)}{\sum\limits_{j \in V} |N_j|} \quad (11)$$

The result of this section is stated in the following Theorem:

Theorem 7: Consider the robots $\dot{q}_i = u_i, i \in V = \{1, \dots, N\}$, with control law $u(q) = -\nabla\varphi(q)$, where $\varphi(q) = a(q) + r_F(q)$ fulfills (11) and $\beta_{ij}(q) > d + \varepsilon$ for some $\varepsilon > 0$ and all $q \in B_\varepsilon^F$. Also assume that the desired formation is feasible, i.e., $\Phi \neq \emptyset$ and that the formation graph is connected. Then for any $0 < \varphi_0 < \infty$ and almost all $q(0) \in W_{\varphi_0}$, with W_{φ_0} as in (9), the trajectory $q(t)$ of the closed-loop system exists for all $t \in [0, \infty)$, and $q(t)$ tends to Φ as $t \to \infty$.

Proof: Theorem 2 gives $q(t) \to \bar{q}$, with $\nabla a(\bar{q}) = 0$ because (i) trace $\left(\nabla^2 a(q) + \nabla^2 r_F(q)\right) > 0$ for all $q \notin \Phi$, due to (10),(11) and (ii) $\bar{q}$ is stable, by Lemma 6. It remains to show that $\bar{q} \in \Phi$.

Equation $\nabla a(\bar{q}) = 0$ implies $(L \otimes I_2)\bar{q} + z = 0$ [5], where L is the Laplacian of the formation graph, $z \triangleq \left[c_{11}^T, \dots, c_{NN}^T\right]^T$ and $c_{ii} \triangleq -\sum\limits_{j \in \mathcal{N}_i} c_{ij}$. For all $i \in V$, let c_i denote the configuration of robot i in a desired formation configuration with respect to the global coordinate frame. Then, $c_{ij} = c_i - c_j \;\forall(i,j) \in E$ for all possible desired formations. Define $\tilde{q}_i \triangleq \bar{q}_i - c_i$ so that $\bar{q}_i - \bar{q}_j - c_{ij} = \bar{q}_i - \bar{q}_j - (c_i - c_j) = \tilde{q}_i - \tilde{q}_j$ for all $(i,j) \in E$. Then, $(L \otimes I_2)\bar{q} + z = (L \otimes I_2)\tilde{q} = 0$, so that $L\tilde{x} = L\tilde{y} = 0$ where $\tilde{q}$ is the stack vector of $\tilde{q}_i, i = 1, \dots, N$ and $\tilde{x}, \tilde{y}$ the stack vectors of $\tilde{q}$ in the x, y directions. G being connected implies that L has a simple zero eigenvalue with corresponding eigenvector the vector of ones, $\overrightarrow{\mathbf{1}}$ [8]. This guarantees that both $\tilde{x}, \tilde{y}$ are eigenvectors of L satisfying $\tilde{x} = x_c \overrightarrow{\mathbf{1}}$ and $\tilde{y} = y_c \overrightarrow{\mathbf{1}}$ where $x_c, y_c \in \mathbb{R}$. Therefore all $\tilde{q}_i$ are equal to a common vector value $q_c = [x_c, y_c]$. Hence $\tilde{q}_i = q_c \;\forall i \Rightarrow \bar{q}_i - \bar{q}_j = \tilde{q}_i + c_i - (\tilde{q}_j + c_j) = q_c + c_i - q_c - c_j = c_i - c_j = c_{ij} \forall i, j, j \in N_i$ which implies that $\bar{q} \in \Phi$. $\diamond$

VII. SIMULATIONS

To illustrate the derived results, we provide the results of a series of computer simulations.

The first simulation involves a single robot navigating in an environment of three static cyclic obstacles. The robot is visualized as a triangle and it starts at the bottom of the left plot of Figure 1. The objective of the robot is to move to its target configuration, which is denoted by a small circle in the top of the plot. In this example, we have $M = 3, m_1 = 0.017$, and thus, $Mm_1 = 0.051$. As can be witnessed in the plot, the robot is blocked at a configuration from which it cannot escape, due to that the attractive potential gain k_a is given by $0.02 < Mm_1$. In contrast, in the second simulation of the right plot of Figure 1 the attractive potential gain is ten times larger than the previous example. The robot is now successfully driven to its desired target, avoiding at the same time collisions with
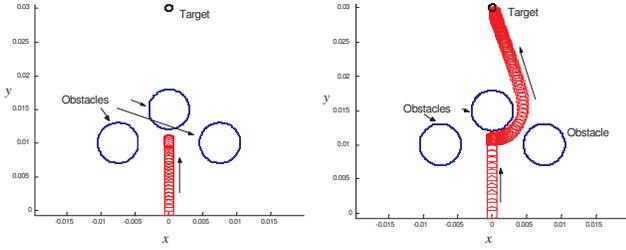
Fig. 1. A robot navigates in an environment with three static obstacles. In the left plot, the robot gets "trapped" in a local minimum and cannot navigate to its target configuration. The gain k_a does not satisfy (6) and hence Theorem 4 does not hold. In contrast, in the right plot k_a has been increased adequately in order to satisfy (6) and hence Theorem 4 holds. The robot navigates successfully to its target avoiding obstacles. The tightness of the control law is witnessed by these two examples.

the obstacles. The parameter $k_a = 0.2 > M m_1$ is now large enough to satisfy the requirement imposed by Theorem 4, encoded by the inequality (6).

In the last simulation, four single integrator robots converge to a rectangular formation. Robots navigate under the control (8). The attractive potential gains k_f are chosen so that they satisfy (11) and hence the requirement imposed by Theorem 7 is satisfied. The communication sets are defined as $N_1 = \{2,4\}, N_2 = \{1,3\}, N_3 = \{2,4\}, N_4 = \{1,3\}$ and the inter-robot desired relative positions as $c_{12} = -c_{21} = c_{43} = -c_{34} = -[0.02,0], c_{14} = -c_{41} = c_{23} = -c_{32} = [0.02,0]$. In Figure 2, Ai denotes the initial and Ti the final positions of robot i, $i = 1, \ldots, 4$, respectively. Moreover the trajectory of each robot is the line that connects its initial and final configurations. The purpose of including two plots is to show the influence of the volume of the sensing radius in the robot trajectories. In particular, in the left plot we have set the sensing radius five times smaller than the right one. The robots navigate towards the rectangular formation without colliding with each other in both cases. The difference in the trajectory form is due to the difference in the value of the sensing radius in the two cases.
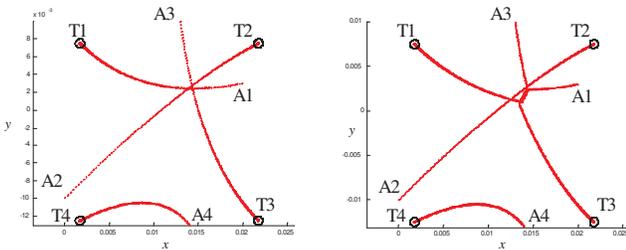

Fig. 2. Four single integrator robots converge to a rectangular formation. Ai denotes the initial and Ti the final positions of robot i, $i = 1, \ldots, 4$. The trajectory of each robot is the line that connects its initial and final position. In the left plot, the sensing radius is five times smaller than the right. The robots navigate towards the rectangular formation without colliding with each other in both cases. The gains k_f satisfy the bounds (11) and hence the requirement imposed by Theorem 7 is also satisfied. The difference in the trajectory form is due to the difference in the value of the sensing radius in the two cases.

VIII. CONCLUSIONS

A sufficient condition for navigation of a negative gradient system was derived based on Rantzer's Dual Lyapunov Theorem. In particular, a condition that involves the trace of the Hessian matrix of the potential function was derived and the result was applied to controller design in two different navigation cases. The main result of the paper states that a sufficient condition for guaranteed convergence to a desired configuration in both cases is that the attractive potential admits a sufficiently large gain, and a lower bound for this is computed. Computer simulations that support the derived results were also provided.

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