Optimized Rate Allocation for State Estimation over Noisy Channels

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***Abstract*— Optimal rate allocation in a networked control system with limited communication resources is instrumental to achieve satisfactory overall performance. In this paper, a practical rate allocation technique for state estimation in linear dynamic systems over an erroneous channel is proposed. The method consists of two steps: (i) the overall distortion is expressed as a function of rates at all time instants by means of high-rate quantization theory, and (ii) a constrained optimization problem to minimize the overall distortion is solved by using Lagrange duality. Monte Carlo simulations illustrate the proposed scheme, which is shown to have good performance when compared to arbitrarily selected rate allocations.**

I. INTRODUCTION

Networked control systems based on limited sensor and actuator information have attracted increasing attention during the past decade. In these systems, it is important to encode the sensor measurements before sending them to the controller by using a few bits, because of the limited information that can be transmitted. However, the distortion introduced by the encoding should not reduce the performance of the controller. Hence, optimizing the rate allocation is essential to overcome the limited communication resources and to achieve a better performance.

The optimization of the encoder–controller mappings to improve the performance of control over finite-rate channels, with or without transmission errors, has been addressed in, e.g., [1], [2], [3]. How to assign bits among the elements of the state vector of the plant, while imposing a constraint on the number of bits over time, can be found in e.g., [4], [5]. In these works, it has been often assumed that bits (rates) are evenly distributed to sensor measurements. However, owing to the non-stationarity of the state observations, an even distribution of bits to sensor measurements is often not efficient for networked control. Hence, it is natural to expect considerable gains by employing a non-uniform rate allocation.

How to achieve the optimal rate allocation in control systems is a challenging task. The main obstacle to optimize the rates is the lack of tractable distortion functions, which we need to use as an objective function for the rate optimization problem. Furthermore, such a problem is often non-convex and non-linear, which implies that it is difficult to compute the optimal solution in practice. In this paper, given these difficulties, we focus on the special case of optimizing the rate allocation for *state estimation* as a first fundamental step in solving the rate allocation problem for state feedback control.

The main contribution of this paper is a novel method for optimal rate allocation for state estimation of a linear system over a noisy channel. By resorting to an approximation based on high-rate quantization theory, we are able to derive a computationally feasible scheme that minimizes the overall distortion over a finite time horizon. The resulting rate allocation is not necessarily evenly distributed. Practical considerations on integer rate constraints and the accuracy of the high-rate approximation are discussed and illustrated through numerical examples.

The problem we are addressing here is related to classical rate allocation problems in communications [6], [7]. To quantify the relation between rate and performance, we resort to high-rate quantization theory [6], [8], [9]. We also contribute to rate allocation based on high-rate theory by studying a general class of quantizers, while previous work has often focused on the special case of optimized quantizers. For example in [10], the problem is studied in the context of transform codes, where the objective function is convex, and the optimal solution can be derived in a closed-form. However, in our setting we will show that the overall distortion is a non-convex function of the rates, which makes more difficult the computation of the optimal solution.

The rest of the paper is organized as follows. In Section II, the overall system is described and the rate allocation problem is formulated. Thereafter, some useful results on high-rate quantization theory are given in Section III. In Section IV, we solve the rate constrained optimization problem by means of Lagrangian duality. Then, Section V is devoted to the practical issues such as the non-negativity and integer nature of the rates. Finally, numerical simulations are carried out in Section VI to demonstrate performance of the proposed bit-rate allocation scheme.

II. PROBLEM FORMULATION

The goal of this work is to arrive at a practical rate allocation scheme for state estimation of a dynamic system over an erroneous channel. We consider a scalar system, for which the plant is governed by the equation

$$x_{t+1} = ax_t + v_t, \quad a > 0, \tag{1}$$

where $x_t, v_t \in \mathbb{R}$. The initial state x_0 and the process noise v_t are mutually independent. They are i.i.d. zero-mean Gaussian with variances $\sigma_{x_0}^2$ and σ_v^2, respectively. The state measurement x_t

is encoded and transmitted to a decoder unit through an erroneous channel. The encoder is time-varying and memoryless, i.e., it takes only the current state x_t as the input,

$$i_t = f_t(x_t) \in \left\{0,\ldots,2^{R_t}-1\right\}. \tag{2}$$

The rate R_t is a non-negative integer. The index i_t will be mapped into a binary codeword before being fed to a binary channel. The mapping from an index to a codeword is commonly referred to as the *index assignment* (IA). Unlike in the error-free scenario where all IA's perform equally well, in the presence of channel errors, different IA's have a different impact on the system performance. Finding the optimal IA is a combinatorial problem which is known to be NP-hard [11]. In this paper, we therefore average out the dependence on a specific IA by randomization. At each transmission, a random assignment is generated and revealed to the encoder and decoder. Previous works that assumed a random IA to facilitate analysis include [12].

Throughout the paper, the overall *erroneous channel* is composed by the combination of the random IA and a binary symmetric channel (BSC). The channel is completely specified by the symbol transition probabilities $\mathrm{P}_r(j_t|i_t)$. At the bit level, the channel is characterized by the crossover probability $\varepsilon = \mathrm{P}_r(0|1) = \mathrm{P}_r(1|0)$ of the BSC, while the overall symbol error probability $\mathrm{P}_r(j_t|i_t)$ is determined by both ε and the randomized IA, according to

$$\mathrm{P}_r(j_t|i_t) = \begin{cases} \alpha(R_t), & j_t \neq i_t, \\ 1-(2^{R_t}-1)\alpha(R_t), & j_t = i_t, \end{cases} \tag{3}$$

(cf., [12]), where $\alpha(R_t) = (1-(1-\varepsilon)^{R_t})/(2^{R_t}-1)$ is obtained by averaging over all possible index assignments. As revealed by (3), for this channel, all symbol errors are equally probable. Clearly, the error-free channel is a special case with $\varepsilon = 0$.

At the receiver side, the decoder takes j_t as the input, and produces d_t, an estimate of the state x_t,

$$d_t = D_t(j_t) \in \mathbb{R}, \tag{4}$$

where D_t is a deterministic function. The estimate d_t can take on one of 2^{R_t} values. Note that an encoder–decoder pair is functionally equivalent to a *quantizer*. Throughout this paper, *a bit-rate allocation* is the entire sequence $\mathbf{R}^{T-1} = \{R_0,\ldots,R_{T-1}\}$ of rates, and the total rate, R_{tot}, is the sum of all the instantaneous rates. Let J_t denote the instantaneous distortion at time t, where J_t is

$$\mathrm{J}_t = \mathbf{E}\left\{(x_t-d_t)^2\right\}. \tag{5}$$

Next, we specify the problem studied in this paper.

Problem 1. *Given the plant* (1)*, the channel* (3)*, and the encoder–controller pair* (2) *and* (4)*, find* $\mathbf{R}^{T-1}$ *that minimizes the distortion* (6)*, subject to a total rate constraint, namely,*

$$\min_{\mathbf{R}^{T-1}} \textstyle\sum_{t=0}^{T-1} \mathrm{J}_t, \quad \text{s.t.} \textstyle\sum_{t=0}^{T-1} R_t \leq R_{tot}, \tag{6}$$

where J_t *is given by* (5).

The performance measure in (6) represents an overall estimation error, and its implicit relation to the rate allocation $\mathbf{R}^{T-1}$ is specified by the channel and the coding scheme. Note that our criterion is motivated by the closed-loop control scenario [3], with a finite horizon T.

According to (1), the state x_t can be expressed in terms of the initial state x_0 and the process noises $\mathbf{v}_0^{t-1}$ as $x_t = a^t x_0 + \sum_{s=0}^{t-1} a^{t-1-s} v_s$. Since x_0 and $\mathbf{v}_0^{t-1}$ are i.i.d. zero-mean Gaussian distributed, consequently x_t is also zero-mean Gaussian with the variance $\sigma_{x_t}^2 = a^{2t}\sigma_{x_0}^2 + \sum_{s=0}^{t-1}\left(a^{t-1-s}\right)^2 \sigma_v^2$. We will use the distribution of the state x_t in the next section.

III. High-Rate Approximation of the MSE

It should be observed that the state x_t does not depend on the communication over the erroneous link. Especially, x_t is not affected by the rate allocation, and the instantaneous distortion functions are separable. Hence, the major challenge lies in deriving a useful expression of the mean-squared error (MSE) for the instantaneous distortion (5). In general, it is hard to formulate closed-form expressions, even in the case of simple uniform quantizers. In order to proceed, we therefore resort to approximations based on *high-rate theory* [6]. For this reason, some useful results are reviewed briefly in this section. For further details, we refer the reader to [12] and [13]. Roughly speaking, the high-rate assumption requires the PDF of the source to be approximately constant within a quantization cell. Let P_{x_t} denote the PDF of the source, x_t, zero-mean with variance $\sigma_{x_t}^2$, following [13], at high-rate, the MSE $\mathbf{E}\left\{(x_t-d_t)^2\right\}$ can be approximated by the expression,

$$\begin{aligned}\mathbf{E}\left\{(x_t-d_t)^2\right\} \approx 2^{R_t}\alpha(R_t)\sigma_{x_t}^2 + \varphi_t\alpha(R_t)\int_y y^2\lambda_t(y)\mathrm{d}y \\ + \frac{G^{-2}}{3}\varphi_t^{-2}\int_x \lambda_t^{-2}(x)\mathrm{P}_{x_t}(x)\mathrm{d}x.\end{aligned} \tag{7}$$

The constant G is the volume of a unit sphere, so that for a scalar quantizer, $G=2$. The function $\lambda_t(x)$ is referred to as the *quantizer point density function*. This function is used to specify a quantizer in terms of the density of the reconstruction points. Resembling a probability density function, it holds that $\lambda_t(x) \geq 0$, for all x_t, and $\int \lambda_t(x)\mathrm{d}x = 1$. Finally, the parameter $1 \leq \varphi_t \leq 2^{R_t}$ specifies the number of codewords the encoder will chose. If the error probability ε is large, in order to protect against channel error, a good encoder may only use a part of the available codewords. In this paper, we consider only the encoder–decoders for which $\varphi_t = 2^{R_t}$.

Essentially, we are in need of a useful expression to describe the relation between the MSE and the rate R_t. We propose a further simplification of (7), in particular, $2^{R_t}\alpha(R_t) \approx 1-(1-\varepsilon)^{R_t}$, and we rewrite (7) and introduce $\hat{\mathrm{J}}_t(\beta_t,\kappa_t,R_t)$ as follows:

$$\begin{aligned}&\mathbf{E}\left\{(x_t-d_t)^2\right\} \approx \hat{\mathrm{J}}_t(\beta_t,\kappa_t,R_t) \triangleq \beta_t(1-(1-\varepsilon)^{R_t}) + \kappa_t 2^{-2R_t}, \\ &\beta_t \triangleq \sigma_{x_t}^2 + \int_y y^2\lambda_t(y)\mathrm{d}y, \quad \kappa_t \triangleq \bar{G}\int_x \lambda_t^{-2}(x)\mathrm{P}_{x_t}(x)\mathrm{d}x,\end{aligned} \tag{8}$$

where $\bar{G} \triangleq G^{-2}/3$. Such an expression of the distortion $\hat{\mathrm{J}}_t$ is rather general for a large variety of quantizers, described in term of the point density function, and derived under the high-rate assumption. For practical sources and quantizers, it holds that $0 < \beta_t < \infty$ and $0 < \kappa_t < \infty$, which is assumed throughout the paper. The distortion (8) has some useful properties that will allow us to solve the rate allocation problem. Next, we use a uniform quantizer to show the utility of (8).

Consider a uniform quantizer, for which the step size $\Delta_t = 2\nu_t/2^{R_t}$ is a function of the quantizer range $[-\nu_t, \nu_t]$ and the rate R_t. The point density function is then $\lambda_t(x) = 1/(2\nu_t)$. If the source signal and the uniform quantizer have the same support $[-\nu_t, \nu_t]$, a high-rate approximation of the MSE distortion according to (8) is

$$\hat{\text{J}}_t = \left(\sigma_{x_t}^2 + \nu_t^2/3\right)\left(1-(1-\varepsilon)^{R_t}\right) + 4\nu_t^2\bar{G}2^{-2R_t}. \tag{9}$$

It is important to remark that the channel error probability ε plays a significant role on the shape of the objective function $\hat{\text{J}}_t$. When $\varepsilon = 0$, $\hat{\text{J}}_t$ is monotonically decreasing with respect to R_t. In fact, $\hat{\text{J}}_t$ is a convex function of R_t, for all $0 < \kappa_t < \infty$. On the other hand, for erroneous channels, $\varepsilon \neq 0$, convexity only applies for certain $\{\beta_t, \kappa_t\}$ pairs. Regarding the general case of an arbitrary $\{\beta_t, \kappa_t\}$ pair, (8) is a quasiconvex function, as shown in the following lemma.

Lemma 1. *The distortion $\hat{\text{J}}_t = \beta_t(1-(1-\varepsilon)^{R_t}) + \kappa_t 2^{-2R_t}$, $\beta_t, \kappa_t > 0$, is quasiconvex and has a unique global minimum.*

Proof: Compute the derivative of $\hat{\text{J}}_t$ with respect to R_t,

$$\frac{\partial \hat{\text{J}}_t}{\partial R_t}(\beta_t, \kappa_t, R_t) = -\beta_t \ln(1-\varepsilon)(1-\varepsilon)^{R_t} - 2\ln(2)\kappa_t 2^{-2R_t}.$$

Since the first term, $-\beta_t \ln(1-\varepsilon)(1-\varepsilon)^{R_t}$, is strictly decreasing towards 0 as R_t grows, and the second term $-2\ln(2)\kappa_t 2^{-2R_t}$ is strictly increasing towards 0 as R_t grows, $\partial\hat{\text{J}}_t/\partial R_t$ has at most one critical point $R_t^\star$, which solves

$$\frac{\partial \hat{\text{J}}_t}{\partial R_t}(\beta_t, \kappa_t, R_t^\star) = -\beta_t \ln(1-\varepsilon)(1-\varepsilon)^{R_t^\star} - 2\ln(2)\kappa_t 2^{-2R_t^\star} = 0.$$

In the special case that $\varepsilon = 0$, the critical point is $R_t^\star = \infty$, because $\lim_{R_t\to\infty} \partial\hat{\text{J}}_t/\partial R_t = 0$. Compute the second order derivative of $\hat{\text{J}}_t$ with respect to R_t,

$$\frac{\partial^2 \hat{\text{J}}_t}{\partial R_t^2}(\beta_t, \kappa_t, R_t) = -\beta_t \ln^2(1-\varepsilon)(1-\varepsilon)^{R_t} + 4\ln^2(2)\kappa_t 2^{-2R_t}.$$

The critical point is a global minimum, since $\lim_{R_t\to 0} \partial^2\hat{\text{J}}_t/\partial R_t^2 > 0$, and it reveals that for all $R_t < R_t^\star$, $\hat{\text{J}}_t(\beta_t, \kappa_t, R_t)$ is strictly decreasing and for all $R_t > R_t^\star$, $\hat{\text{J}}_t(\beta_t, \kappa_t, R_t)$ is strictly increasing. ∎

Next, we use Lemma 1 to solve the rate allocation problems.

IV. Rate Allocation for State Estimate

Under the high-rate assumption, the distortion J_t in (5) can be approximated by the expression (8), i.e., $\hat{\text{J}}_t(\beta_t, \kappa_t, R_t)$. We reformulate Problem 1 and solve the rate allocation problem with respect to the instantaneous cost $\text{J}_t = \hat{\text{J}}_t(\beta_t, \kappa_t, R_t)$. In particular, the rate unconstrained and constrained optimization problems based on (8) are formulated as the following approximate versions of Problem 1.

Problem 2. *Find $\mathbf{R}^{T-1}$ which minimizes $\sum_{t=0}^{T-1}\hat{\text{J}}_t$, where $\hat{\text{J}}_t$ is as given in* (8)*.*

Problem 3. *Find $\mathbf{R}^{T-1}$ which solves the problem,*

$$\min_{\mathbf{R}^{T-1}} \textstyle\sum_{t=0}^{T-1}\hat{\text{J}}_t, \quad \text{s.t.} \sum_{t=0}^{T-1} R_t \le R_{tot},$$

where $\hat{\text{J}}_t$ is as given in (8)*.*

We solve the constrained optimization problem as shown in Theorem 1.

Theorem 1. *Suppose $R_t \in \mathbb{R}$. The solution to Problem 3 is as follows.*
In case of an erroneous channel ($\varepsilon \neq 0$), it follows that

1) *If $R_{tot} \ge \sum_{t=0}^{T-1} R_t^\star$, where $\mathbf{R}^{\star T-1}$ is a solution to*

$$\begin{cases} 0 &= \frac{\partial\hat{\text{J}}_0}{\partial R_0}(\beta_0, \kappa_0, R_0^\star), \\ & \vdots \\ 0 &= \frac{\partial\hat{\text{J}}_{T-1}}{\partial R_{T-1}}(\beta_{T-1}, \kappa_{T-1}, R_{T-1}^\star), \end{cases} \tag{10}$$

then $\mathbf{R}^{\star T-1}$ also solves Problem 3.

2) *If $R_{tot} < \sum_{t=0}^{T-1} R_t^\star$, where $\mathbf{R}^{\star T-1}$ is a solution to* (10)*, then the solution $\{\mathbf{R}^{T-1}, \theta\}$ to the system of equations*

$$\begin{cases} \theta &= -\frac{\partial\hat{\text{J}}_0}{\partial R_0}(\beta_0, \kappa_0, R_0), \\ & \vdots \\ \theta &= -\frac{\partial\hat{\text{J}}_{T-1}}{\partial R_{T-1}}(\beta_{T-1}, \kappa_{T-1}, R_{T-1}), \\ R_{tot} &= \sum_{t=0}^{T-1} R_t. \end{cases} \tag{11}$$

solves Problem 3, where θ is the Lagrange multiplier.

In case of an error-free channel ($\varepsilon = 0$), the solution is

$$R_t = \frac{R_{tot}}{T} + \frac{1}{2}\log_2\left(\frac{\kappa_t}{\left(\prod_{t=0}^{T-1}\kappa_t\right)^{\frac{1}{T}}}\right), \; t = 0, \ldots, T-1. \tag{12}$$

To prove Theorem 1, we use Lemma 2–Lemma 5, as derived subsequently.

A. Erroneous Channels

We start with the general case that $\varepsilon \neq 0$. First, we note that the unconstrained problem for the erroneous scenario has a unique minimum that is not necessarily achieved at $R_t = \infty$, as stated in the following lemma.

Lemma 2. *In the presence of channel error $\varepsilon \neq 0$, Problem 2 has a global minimum, achieved at $\mathbf{R}^{\star T-1}$, which solves the system of equations* (10)*.*

Proof: Compute the critical point, at which the gradient is a zero vector, and (10) follows immediately. According to (10), the variables $\mathbf{R}^{\star T-1}$ are separable. Moreover, from Lemma 1 it follows that $\hat{\text{J}}_t(\beta_t, \kappa_t, R_t)$ is a quasiconvex function and has one unique minimum. Therefore, the overall distortion $\sum_{t=0}^{T-1}\hat{\text{J}}_t(\beta_t, \kappa_t, R_t)$ has a unique global minimum. ∎

Based on Lemma 2, we can state that if $R_{tot} \ge \sum_{t=0}^{T-1} R_t^\star$, where $\mathbf{R}^{\star T-1}$ is a solution to (10), $\mathbf{R}^{\star T-1}$ is simultaneously the solution to the constrained problem. On the other hand if $R_{tot} < \sum_{t=0}^{T-1} R_t^\star$, where $\mathbf{R}^{\star T-1}$ is a solution to (10), we need to solve (11), as shown in the following lemma.

Lemma 3. *The solution to* (11) *solves Problem 3.*

Proof: The proof is based on Lagrange dual theory. We note that strong duality holds, because the constraint is a positive linearly independent combination of R_t, the Mangasarian-Fromowitz constraint qualification applies [14]. Next, we minimizes the Lagrangian, $\eta = \sum_{t=0}^{T-1} \hat{\mathrm{J}}_t + \theta\left(\sum_{t=0}^{T-1} R_t - R_{tot}\right)$, where $\hat{\mathrm{J}}_t$ is as given in (8). The straightforward calculation of the derivatives of η with respect to R_t and θ yields (11). ■

In case of an erroneous channel, we do not have a closed-form solution to (11). The system of non-linear equations (11) can be solved by numerical methods [15]. Below, we briefly present a numerical methods based on Newton's method. Define the system of non-linear equations,

$$Z \triangleq \begin{cases} Z_0 &= \frac{\partial \hat{\mathrm{J}}_0}{\partial R_0}(\beta_0, \kappa_0, R_0) + \theta, \\ & \vdots \\ Z_{T-1} &= \frac{\partial \hat{\mathrm{J}}_{T-1}}{\partial R_{T-1}}(\beta_{T-1}, \kappa_{T-1}, R_{T-1}) + \theta, \\ Z_T &= \sum_{t=0}^{T-1} R_t - R_{tot}. \end{cases}$$

Define the vector constructed by all unknown variables $\Phi = [R_0 \ \ldots \ R_{T-1} \ \theta]'$, where $(\cdot)'$ denotes the matrix transpose. We are looking for Φ that gives $Z(\Phi) = 0$. Netwon's method derives the solution iteratively, and the results of the k^{th} and $(k-1)^{th}$ iterations are related by the following equation,

$$\Phi(k) = \Phi(k-1) - J_F^{-1} Z(\Phi(k-1)),$$

where J_F denotes the Jacobian matrix.

B. Error-Free Channels

For error-free channels, we can show that the system of equations (11) has a closed-form solution, because when $\varepsilon = 0$, $\beta_t \ln(1-\varepsilon)(1-\varepsilon)^{R_t} = 0$, for all t. Hence, Problem 2 has the global minimum at $R_t = \infty$, as shown below in Lemma 4.

Lemma 4. *When $\varepsilon = 0$, Problem 2 is convex and the minimum is achieved at $R_t = \infty$.*

Proof: When $\varepsilon = 0$, the instantaneous distortion becomes $\hat{\mathrm{J}}_t = \kappa_t 2^{-2R_t}$. Taking the first order derivative of the overall distortion $\sum_{t=0}^{T-1} \hat{\mathrm{J}}_t$ with respect to R_t, we obtain $\partial/\partial R_s \sum_{t=0}^{T-1} \hat{\mathrm{J}}_t = -2\ln(2)\kappa_s 2^{-2R_s}$. This function is monotonically increasing, and especially, $\lim_{R_t \to \infty} -2\ln(2)\kappa_t 2^{-2R_t} = 0$. Compute the second order derivatives, and the Hessian of the overall distortion $\sum_{t=0}^{T-1} \hat{\mathrm{J}}_t$ is always positive definite, because all the elements on the diagonal are positive. Therefore, the optimization problem is convex. The minimum is achieved at $R_t^\star = \infty$. ■

Moving on to the constrained optimization problem, the solution to (11) is summarized in Lemma 5.

Lemma 5. *Let $\varepsilon = 0$, the solution $\{\mathbf{R}^{T-1}, \theta\}$ to the system of equations* (11) *is given by* (12)*.*

Proof: Based on (11), write R_t as a function of θ,

$$R_t = -\frac{1}{2}\log_2 \frac{\theta}{2\ln(2)\kappa_t} = \frac{1}{2}\log_2(2\ln(2)\kappa_t) - \frac{1}{2}\log_2 \theta. \tag{13}$$

First, solve θ by means of the total bits constraint, and then substitute R_t into (13), (12) follows immediately. ■

Now we are in the position to prove Theorem 1:

Proof of Theorem 1: The proof follows from Lemma 2–Lemma 5. ■

Finally, consider the special case that the instantaneous distortion can be written in the form

$$\hat{\mathrm{J}}_t = \sigma_{x_t}^2 \tilde{\mathrm{J}}_t(\tilde{\beta}, \tilde{\kappa}, R_t) \triangleq \sigma_{x_t}^2 (\tilde{\beta}(1-(1-\varepsilon)^{R_t}) + \tilde{\kappa} 2^{-2R_t}), \tag{14}$$

where $\tilde{\beta}$ and $\tilde{\kappa}$ are time-invariant. The instantaneous distortion is a linear function of the variance of the source signal. This property is very useful to solve the control problem. By applying Lemma 2 and Theorem 1 we can show that the unconstrainted rate allocation problem has a global minimum at $R_t = R^\star$, which is the solution to the following equation,

$$0 = \tilde{\beta} \ln(1-\varepsilon)(1-\varepsilon)^{R^\star} + 2\ln(2)\tilde{\kappa} 2^{-2R^\star}. \tag{15}$$

If $R_{tot} \geq T R^\star$, with $R^\star$ given by (15), then $\mathbf{R}^{\star T-1}$ also solves Problem 3, otherwise, we should solve the system of equations (11).

V. Practical Considerations

In this section we deal with the assumption of Theorem 1 that R_t is allowed to be real. In practice, of course, R_t is integer-valued and positive.

If Problem 2 and Problem 3 give negative rates, we set them to zero, which is equivalent to excluding the corresponding instantaneous distortions from the overall distortion. Then, we resolve Problem 2 and Problem 3 with respect to the new overall distortion.

The proposed algorithms in Section IV result in real-valued rates. As a simple approach, we round the solutions to the nearest integer. A more sophisticated rounding algorithm can be formulated as a binary optimization problem, where the rounded rate $\tilde{R}_t$ is related to the real-valued rate R_t as,

$$\tilde{R}_t = b_t \lceil R_t \rceil + (1-b_t)\lfloor R_t \rfloor, \ b_t \in \{0,1\}, \tag{16}$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denotes the rounding upwards and downwards to the nearest integer, respectively. We optimize the rounding by finding the binary sequence $\mathbf{b}_0^{T-1}$ which minimizes $\sum_{t=0}^{T-1} \hat{\mathrm{J}}_t(\tilde{R}_t)$, subject to the total rate constraint. A solution to the binary rounding problem can always be obtained by applying exhaustive search or combinatorial algorithms [15].

VI. Numerical Experiment

In this section, numerical experiments are conducted to verify the performance of the proposed bit-rate allocation algorithm. The system parameters are chosen in the interest of demonstrating non-uniform rate allocations, in particular, $a = 0.5$, $T = 10$, $R_{tot} = 30$, $\varepsilon = 0.005$. The initial state and the process noise are i.i.d. Gaussian with zero-mean and the variances $\sigma_{x_0}^2 = 10$ and $\sigma_v^2 = 0.1$. A time-varying uniform encoder–decoder is employed. The quantizer range is specified by $v_t = 4\sigma_{x_t}$, and the high-rate approximation (9) is used by assuming the distortion outside the range of the quantizer

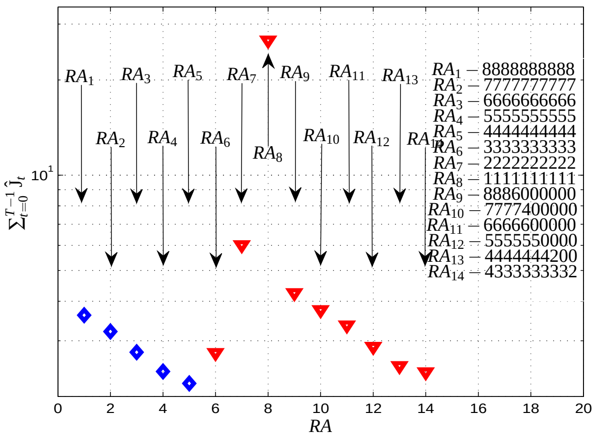

Fig. 1. Performance of various rate allocations. The x-axis is associated to the allocation, whereas the y-axis is the the overall distortion. Notice that allocations marked with a diamond do not satisfy the total rate constraint.

is negligible. In addition, the binary rounding algorithm described in Section V is applied.

In Fig. 1, we compare the optimized allocation, denoted by RA_{14}, which was obtained by the method proposed in this paper, with 13 other allocations, denoted by RA_1–RA_{13}. In particular, the allocation RA_5 was achieved with our method by solving the unconstrained rate allocation problem. Performance in Fig. 1 is measured by the distortion (5). The distortion is obtained by averaging over 50 IA's and each IA 150 000 samples.

Regarding the optimized allocation RA_{14}, R_t is rather evenly distributed over t. Compared with the uniform allocation RA_6, which only differs 1 bit at $t=0$ and $t=9$, we see that our method gives an evident gain. The uniform allocations RA_1–RA_8 have a time-invariant rate, R_t, varying from 8-bits to 1-bit. Among these allocations, RA_8, for which $R=1$, has the worst performance, while RA_5, for which $R_t=4$, has better performance. In fact, we can show that $\tilde{\beta}_t=\tilde{\beta}$, $\tilde{\kappa}_t=\tilde{\kappa}$, and the unconstrained global minimum is achieved at $R_t^\star=4$. This is consistent with the simulation result that RA_5 is even superior to allocations with higher total rates. In the presence of the channel errors, more bits can sometimes do more harm than good. However, RA_5 does not satisfy the total rate constraint, therefore, (11) is solved which yields RA_{14}. It should be mentioned that due to all simplifications and approximations, a solution given by (11) is an approximation for Problem 1, but our experiments showed that the resulting performance degradation is often insignificant.

The allocations RA_9–RA_{13} represent the strategies when more bits are assigned to the initial states. These allocations are not suitable in the current example, because, as discussed previously, the additional bits exceeding the critical point, $R^\star=4$, do more harm than good. Finally, we have also applied the rate allocation algorithm to two other ε values. At $\varepsilon=0.001$, the optimized rate allocation is $\mathbf{R}^9=[5433333222]$, while at $\varepsilon=0.01$, the optimized rate allocation is $\mathbf{R}^9=[3333333333]$. Here we see that, as ε increases, the optimized allocation becomes more uniform and uses lower rates. It is also worth mentioning that the random index assignment used in this paper is neither efficient in protecting against channel errors, nor practical in implementation. In the next step, more efficient and practical coding–control scheme should be studied.

VII. CONCLUSION

In this paper, we studied the bit allocation problem for state estimation of a dynamic system over erroneous channels. First, we approximated the overall distortion function by means of the high-rate approximation theory. Second, we showed that the unconstrained optimization problem has a global minimum, which solves the rate allocation problem if such a global minimum does not violate the rate constraint. On the other hand, if the global minimum violates the rate constraint, we solved the rate constrained optimization problem by means of Lagrangian duality for non-linear non-convex problems. Finally, numerical simulations showed good performance of the proposed scheme. Based on the result in this paper, we will in the next step solve the analogous problem of bit allocation for controlling a dynamic system.

REFERENCES

[1] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1549–1561, Sept. 2004.

[2] G. N. Nair, F. Fagnani, S. Zampieri, and R. Evans, "Feedback control under data rate constraints: an overview," in *Proc. of the IEEE*, Jan. 2007, pp. 108–137.

[3] L. Bao, M. Skoglund, and K. H. Johansson, "On the separation principle in optimal control over noisy channels," in *IFAC World Congress 2008*, 2008.

[4] Q. Ling and M. D. Lemmon, "Optimal dynamic bit assignment in noise-free quantized linear control systems," in *Proc. of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005*, Dec. 2005, pp. 8191–8196.

[5] L. Xiao, M. Johansson, H. Hindi, S. Boyd, and A. Goldsmith, "Joint optimization of wireless communication and networked control systems," *Chapter in Switching and Learning, Springer Lecture Notes in Computer Science 3355*, pp. 248–272, Sept. 2005.

[6] A. Gersho and R. M. Gray, *Vector quantization and signal compression*. Kluwer, 1992.

[7] B. Farber and K. Zeger, "Quantization of multiple sources using nonnegative integer bit allocation," *IEEE Transactions on Information Theory*, vol. 52, pp. 4945–4964, Jan. 2006.

[8] S. W. Mclaughlin and D. L. Neuhoff, "Asymptotic quantization for noisy channels," in *Proc. of 1993 IEEE International Symposium on Information Theory*, Jan. 1993, p. 442.

[9] C. V. Rao, J. B. Rawlings, and D. Q. Mayne, "Constrained state estimation for nonlinear discrete-time systems: stability and moving horizon approximations," *IEEE Transactions on Computers*, vol. 48, no. 2, pp. 246–258, 2003.

[10] J. Lim, "Optimal bit allocation for noisy channels," *Electronics Letters*, vol. 41, no. 7, Mar. 2005.

[11] N. Farvardin, "A study of vector quantization for noisy channels," *IEEE Transactions on Information Theory*, vol. 36, no. 4, pp. 799–809, 1990.

[12] K. Zeger and V. Manzella, "Asymptotic bounds on optimal noisy channel quantization via random coding," *IEEE Transactions on Information Theory*, vol. 40, no. 6, Nov. 1994.

[13] C. R. Murthy and B. D. Rao, "High-rate analysis of source coding for symmetric error channels," in *Data Compression Conf. (DCC)*, Mar. 2006.

[14] *Handbook of global optimization*. Kluwer Academic Publishers, 1995.

[15] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.