Master of Science. Thesis

Higher loop renormalization in large N-limit

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Abstract

In early 1970’s a breakthrough was achieved in Modern physics when t’ Hooft and Veltman proved the renormalizability of gauge theories. This idea was later used in perturbative Quantum field theories that we have described in the thesis. Amplitude of a Feynman diagram can be calculated by the Feynman rule of the theory. Every Feynman diagram give some divergences. The divergences of the Feynman amplitude are due to the slow decay of the propagators in momentum space (ultraviolet divergences). In order to get finite results for the correlation functions, we have to renormalize the theory. The procedure involves two steps. The first one is to regularize the theory by some regularization method. The renormalization consists of subtracting divergences by introducing some counter terms. We have used dimensional regularization and minimal subtraction scheme to renormalize the $\phi^4$ model. This theory of N-fields give rise to some special symmetry factors that accompanied the Feynman amplitudes. In the thesis, we do a detailed study of Feynman diagrams for N-component fields, and find some explicit formulae for general N-case. Our requirement is to perform the renormalization in the large N-limit. In the large N-limit, only higher order terms of symmetry factors are considered. Two-point diagrams give mass renormalization while four-point diagrams give vertex renormalization constants. In the large N-limit, we finally perform a three-loop calculation of the two and four-point functions.

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Quantum field theories are perceived with some suspicion by mathematicians. This is mainly due to the appearance of divergences which occur in computing probability amplitudes. These infinities have to be dealt with properly by an apparently obscure process called renormalization. Renormalization theory plays a major role in the perturbative approach to Quantum field theory. The study of renormalization is a crucial subject for the deep understanding of quantum field theory. Feynman graphs are interpreted as elements of a quantum field theory i.e., as an expansion of an interacting QFT in the neighborhood of a simple QFT. Hence, a better understanding of the Feynman graphs and their corresponding Feynman integrals lead to a better comprehension of true nature of QFTs. We describe the renormalization process on the level of One particle irreducible (1PI) Greens functions since these correspond to actual physical processes. This formalism of Feynman graphs is quite successful in the sense that the quantities obtained from it match extremely well with the quantities obtained in the experiment. The Feynman integrals are generically divergent. These have to be renormalized. The process of renormalization in perturbative QFT can be performed in many different ways. The most efficient one is dimensional regularization and minimal subtraction scheme. In this thesis, our main goal is to work on renormalization of higher order loop-diagrams. We perform renormalization of 1PI diagrams in two steps: 1) dimensional regularization, which isolates the singularities, 2) Minimal-subtraction scheme, which makes us to choose corresponding counterterms in order to cancel out the divergences. We have performed explicit calculations up to third-order diagrams. The complexity of perturbative calculations grows with the number of free internal momenta which are integrated over (loops), and the number of external particles of the process under consideration (legs). In renormalization process, one must have to choose some suitable counterterms to the original lagrangian. The divergences have to be compensated by the so-called counterterms. If the divergences can be compensated by adjusting only a finite number of parameters in the lagrangian (i.e., by leaving lagrangian invariant) [1], the theory is called renormalizable. A solved problem is given in ref. [2,3,4,5,6,7,8] to find a way to organise this correspondence between removing divergences and compensating counterterms in the Lagrangian for arbitrarily complicated graphs. In case of N-fields, each coupling constant becomes a symmetric tensor with four indices, each index represents one leg of a Feynman graph. Then, using simple tensor algebra one can find out symmetry factor for any Feynman loop diagram. One must not confuse the concept of general symmetry factors with these multi-field symmetry factors. These are special kind of symmetry factors obtained when one deal with the N-fields. Every Feynman diagram in the N-field has some symmetry factor. We have obtained general formulas to calculate n-loop O(N)-symmetry factors of two-point, four-point and vacuum diagrams. Therefore, when calculating the renormalization constant each Feynman integral has to be multiplied with Feynman integral of corresponding symmetry factor. In the large N-limit, only higher order terms (in N) of symmetry factors have to be considered and the lower order terms are to be neglected. This makes the calculations of renormalization constants much simpler. Mass and vertex renormalization constants are explicitly calculated up to three-loop diagrams in the large N-limit and then, conjecture a generalized form for n-loop diagrams.
Chapter 1

Introduction

Physics is the study of nature that involves study of matter and its motion through space-time. Since the twentieth century, physics has been divided into two branches: theoretical physics and experimental physics. Most experimental results in physics are numerical measurements, and theories in physics use mathematics to give numerical results to match these measurements. Physics relies upon mathematics to provide the logical framework in which physical laws may be precisely formulated and predictions quantified. Mathematics is the language in which Nature expresses its laws. Theoretical physicist Eugene Wigner has tried to give significance of Mathematics to simplify physical theories in his famous comment:

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning"[9].

Quantum field theory is one of the most vital areas of theoretical physics, providing a common language to many branches of physics. QFT is used in many contexts, most important is elementary particle physics and for the description of critical phenomenon and quantum phase transitions. One of the most important developments in QFT is Feynman graphs/diagrams. These were developed by the American theoretical physicist Richard.R.Feynman in 1940s and 50s and were named after him. A Feynman diagram is a representation of quantum field theory processes in terms of particle paths. There are many applications of Feynman diagrams in QFT and solid state theory and many other fields of physics. In this thesis, we are going to work on loop Feynman diagrams in order to understand the $\phi^4$ models. The $\phi^4$ model consists of N scalar fields with quartic interaction term which mainly is used as a prototype model to describe Feynman rules. It is very common in Modern physics to study ”Toy models” to illustrate an effect in order to make the phenomenon easier to visualize. The loop diagrams are divided in two classes: two point and four point Feynman diagrams. Some of the two-point diagrams are so-called tadpole diagrams, which are connected with rest of the Feynman diagram through two lines which meet at a vertex, like examples shown in Fig:1(a). Also, tadpole diagrams may appear anywhere as subdiagrams of Feynman diagrams. Any Feynman diagram can be decorated by adding tadpoles at all possible places, they occur in a nested way. Examples of such diagram are shown in Fig:1(b)

Occam’s razor is an important concept in science, which states that a simpler theory is always preferred over a complex one. In other words, if simpler theories have a high degree of validity, Occam’s razor would be a valid theory[10]. Albert Einstein makes this theory more reliable by stating that: ”Make things as simple as possible, but not simpler”

The thesis concentrate on two different things which are needed for calculating correlation func-
CHAPTER 1. INTRODUCTION

Fig:1(a)-A tadpole diagram. Fig:1(b)-Two point vertex scattering amplitude Feynman graph decorated with tadpoles.

tions for the $\phi^4$ models.
1. The O(N) symmetry factors which are special for these theories.
2. Renormalization, which takes place in two steps dimensional regularization and renormalization in minimal subtraction scheme.

O(N) symmetry factors: In the first chapter, we deal with Feynman diagrams of N-identical real fields where coupling constant becomes a tensor. To obtain O(N)-symmetry factors, we have to define a Feynman loop by the corresponding sum over indices of each line in a loop. Then, we get reduced matrix elements by applying tensor contractions. This gives general formulas for the symmetry factors of four-point, two-point and vacuum n-loop diagrams in order to discuss large N-limit case.

Renormalization: During 1960’s and 1970’s, QFT underwent a series of extraordinary theoretical developments when theoretical physicists gained the complete knowledge on the subject of renormalization[11]. The algorithm of renormalization takes place in two-steps: Regularization, where we regularize the divergences in Feynman integral and second step is to remove these divergences called renormalization[35]. Any person who first sees this method is somewhat unsettled because in Mathematics one cannot absorb or subtract infinities in this way. The early formulators of QFT were quite dissatisfied with this theory. Dirac’s criticism was the most persistent. As late as 1975, he was saying:

"Most physicists are very satisfied with the situation. They say: Quantum electrodynamics is a good theory, and we do not have to worry about it any more. I must say that I am very dissatisfied with the situation, because this so-called good theory does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small not neglecting it just because it is infinitely great and you do not want it!"[12],

Despite all this renormalization works fantastically well when it comes to predictions in experiments. These days, it has appeared as one of the hottest subject in Statistical physics and High energy physics[13].

Renormalization plays a central role in quantum theory of fields, as it provides a consistent scheme for extracting from divergent expressions finite values that can be matched to physical observed quantities. In the next chapter of renormalization, we work explicitly on dimensional regularization and minimal subtraction schemes of two-point and four-point diagrams up to three loops. The generalization to more loops is tedious. For higher orders, this can be done reliably only with the help of computer-algebraic calculations. First thing is to obtain correlation functions for two-point and four-point diagrams from vacuum diagrams. This will be done with the coordination of symmetry factors\(^1\) corresponding to each diagram. These symmetry factors can be evaluated with the help of formulae invented by Palmera and Carrington[14]. Using this formulae, one can calculate symmetry factors for loop Feynman diagrams up to nth

\(^1\)This is not the same type of symmetry factors mentioned earlier.
order. The calculation of Feynman integrals corresponding to the various diagrams is performed in dimensional regularization according to the rules of ’t Hooft and Veltman[15,16]. The divergences of the theory are removed by counterterms defined in a so-called minimal subtraction scheme (MS-scheme). The MS-scheme consists of absorbing only the divergent part into the counterterms. The renormalization constants obtained in this scheme are of maximal simplicity. The Renormalization group functions of the $\phi^4$ theory were first calculated analytically in 4 dimensions using dimensional regularization[17] and minimal subtraction scheme[15,16] in three and four-loop approximation in [18,19]. Field theoretic renormalization group techniques [20] in D=4-ε dimensions is described in ref [21,22,23,24].

For dimensions close to $D = 4$, Feynman integrals of two- and four-point functions in momentum space diverges. These divergences are divided into two categories: Ultraviolet- and infrared-divergences. Divergences arising from the short wavelength region of the integrals at large momenta are called Ultraviolet-divergences, while those arising from long wavelength at small momenta in zero-mass limit are called infrared-divergences. Infra-red divergences arise due to massless particles, like photon. Most significant divergences are ultraviolet one and we shall consider only these divergencies in our calculations. With the help of regularization processes these integrals can be made finite. There are several regularization procedures named by:

1. Momentum cut-off regularization,
2. Pauli-Villars regularization[25],
3. Analytic regularization[26],
4. Lattice regularization[27],
5. Dimensional regularization[15,16,28].

In subsection (3.2.2), we will work on dimensional regularization for the evaluation of integrals associated to one, two and three loop Feynman graphs in the large N-limit. The dimensional regularization procedure was invented by ’t Hooft and Veltman to regularize non-abelian gauge theories where all previous cut off methods failed. Some important properties of dimensional regularization is that it preserves all symmetries of the theory and allows an easy identification of divergences.

We know that integrals associated with a graph $G$ diverge at large momenta are called ultraviolet divergent. Here, we define a quantity $w(G)$ called superficial degree of divergence associated with a graph $G$. A superficial degree of divergence can be defined to control if the integral is convergent (degree negative) or divergent (degree non-negative). It also gives information about the possible valences of vertices and number of external legs allowed for the graph in that theory. For the scalar field, the superficial degree of divergence is defined by:

$$w(G) = DL - 2I = (D - 2)I + D - DP$$  \hspace{1cm} (1.1)

where $D$ is the space-time dimension of the theory, $I$ is the number of internal lines and $L$ is number of loops. A diagram with $I$ internal lines contains

$$L = I - p + 1,$$  \hspace{1cm} (1.2)

loop integrations. For $\phi^4$ theories, the number of internal lines in a Feynman diagram are expressed as:

$$I = 2p - \frac{n}{2},$$  \hspace{1cm} (1.3)

where $p$ stands for number of vertices and $n$ for number of external lines. Now, the superficial degree of divergence of the associated integral becomes

$$w(G) = D + n(1 - D/2) + p(D - 4).$$  \hspace{1cm} (1.4)

A diagram $G$ is said to be superficially divergent if $w(G) \geq 0$. In four dimensions, this becomes:

$$w(G) = 4 - n.$$  \hspace{1cm} (1.5)
Hence, only two-point and four-point 1PI diagrams are superficially divergent. If the integrals of two- and four-point functions are made finite, then from convergence theorem any n-point function will be finite. This property makes the theory renormalizable\cite{29}. Hence, a $\phi^4$ theory is renormalizable in four dimensions. A theory with dimensions less than four ($w(G) \leq 0$) is called super-renormalizable while for dimensions higher than four, the theory is said to be non-renormalizable\cite{30}. A super-renormalization theory is defined by $w(G) \leq 0$ and the integrals associated to the graphs with $w(G) \leq 0$ are called convergent integrals. The term ”non-renormalizable” does not mean that such theories cannot be made finite at all. These theories contain high number of divergences and hence, counterterms which make them unrealistic in the framework of perturbation theory. Renormalization is mostly done with the help of Minimal Subtraction Scheme, abbreviated as MS-Scheme. In QFT, MS-scheme is a particular renormalization scheme used to absorb the infinities that arise in perturbative calculations beyond leading order, introduced by ’t Hooft and Weinberg independently in 1973\cite{15,31} where it was applied to one-loop calculations in scalar electrodynamics, discussed the problem of overlapping divergences, the ward identities and anomalies. It quickly became the standard regularization and renormalization method for nonabelian gauge theories and the standard model. The method has since been applied widely to perturbative calculations in field theories.

1.1 $\phi^4$ Theory

In most of the quantum field theories, we study objects in terms of field $\phi$ of N identical components $\phi = (\phi_1(x) \cdots \phi_N(x))$ in space-time dimensions. The field components interact with each other via a fourth order term in the fields, $\lambda T_{\alpha\beta\gamma\delta} \phi^\alpha \phi^\beta \phi^\gamma \phi^\delta (\alpha, \beta, \gamma, \delta = 1, \cdots, N)$, where the parameter $\lambda > 0$ defines interaction strength and is called coupling constant of the theory. In renormalization section, we’ll set $\lambda = g\mu^\epsilon$ where $\epsilon = 4-D$. The quantity $T_{\alpha\beta\gamma\delta}$ is coupling tensor. Almost all of the quantum field theories starts with a Lagrangian density that determines the nature of the particle involved as well as their mutual interactions. The $\phi^4$ theory consists of adding a $\frac{\lambda}{4!} \phi^4$ interaction term in Klein Gordon Lagrangian, where $\lambda$ is a dimensionless coupling constant:

$$L = L_0 + \frac{\lambda}{4!} \phi^4.$$  
(1.6)

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4$$  
(1.7)

With N-particles we can have a $\phi^4$ model like

$$L = \sum_{i=1}^{N} \left[ \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{m^2}{2} \phi^i \phi^i + \frac{\lambda}{4!} (\phi^i \phi^i)^2 \right].$$  
(1.8)

Here, the $\phi$ represents a scalar field of zero spin. The parameter $m$ is called mass of the field. A $\phi^4$ theory is the simplest quantum field theory one can imagine. It is possible to learn a lot from a careful study of this theory e.g, one can extract general features which almost every quantum field theory has. Also, it provides different tools to get familiar with special formal techniques like renormalization.

Symmetry considerations play a key role in the understanding of nature. In quantum mechanics, symmetry techniques based on group theory is an important tool of theoretical physics. Spontaneous symmetry breaking defines a process by which a system described in a theoretically symmetric way ends up in a non-symmetric state. The $\phi^4$ theory exhibits the feature of spontaneous symmetry breaking and we make use of toy model to understand spontaneous symmetry breaking\cite{31}. In general, there is no reason why an invariance of a Hamiltonian of
1.1. $\phi^4$ THEORY

a quantum mechanical system should also be an invariance of the ground state of the system. Suppose that the Hamiltonian is invariant under a unitarity transformation $U$:

$$UHU^{-1} = H,$$

(1.9)

but the ground state isn’t,

$$U|0\rangle \neq |0\rangle.$$  \hspace{1cm} (1.10)

Hence, we can say that the state of lowest energy i.e., "the ground state" does not respect the symmetry of original Lagrange. That symmetry is said to be broken. For example, the nuclear forces are rotationally invariant, but this does not mean that the ground state of the nucleus is necessarily rotationally invariant.

In Modern-day physics, the phenomenon of spontaneous symmetry breaking (SSB) appears in all branches of physics. It plays a crucial role in solid-state physics, statistical thermodynamics, condensed matter physics, theories of the fundamental interactions and cosmology. It has also started to appear in astrophysical settings. One can say that phenomenon of SSB is one of the cornerstone upon which modern physics rests.

Many physical systems are described by an $O(N)$-symmetric $\phi^4$ theory with $N$-identical fields. In Statistical mechanics, we define an $O(N)$-symmetric $\phi^4$ theory on a lattice and give a formulation of the $1/N$ expansion of the effective potential in a finite lattice system\[37\]. An $O(N)$-symmetric functional of $N$-component vector field $\phi(x) = (\phi_1(x), \cdots, \phi_N(x))$, takes the following form:

$$E[\phi] = \sum_a \partial_\mu \phi_a \partial^\mu \phi_a - \frac{m^2}{2} \phi_a \phi_a - \frac{\lambda}{4!} (\phi_a \phi_a)^2$$ \hspace{1cm} (1.11)

More explicitly, when we work on $N$-field Feynman diagrams, the probability amplitude is multiplied with corresponding symmetry factors. In this thesis, we are going to renormalize the loop-diagrams in the large $N$-limit. The large $N$-limit states that consider only higher order terms (in $N$) of the symmetry factors. All lower order terms will be ignored in this case. Then, using probability amplitudes and symmetry factors, one will be able to find out renormalization constants ($Z_\phi$, $Z_{m^2}$ and $Z_g$). The large $N$-limit gives only $Z_{m^2}$ and $Z_g$, while $Z_\phi$ contains sub leading terms and will be omitted.
Chapter 2

Reduction

2.1 Introduction

In this chapter, we will discuss $O(N)$-symmetric $\phi^4$ theory with $N$-identical real fields i.e., $\phi_\alpha$ with $\alpha = 1, 2, \cdots, N$. With such interactions, coupling constant for a Feynman diagram becomes a tensor $g_{\alpha \beta \gamma \delta}$ and momentum integral in a Feynman diagram is accompanied by corresponding sum over indices. The interaction energy of fourth-order in the field is:

$$E_{\text{int}}[\phi] = \int d^Dx \frac{\lambda}{4!} \phi^4(x) \tag{2.1}$$

In case of $N$-identical real fields, this becomes:

$$E_{\text{int}}[\phi] = \frac{1}{4!} \int d^Dx \sum_{\alpha, \beta, \gamma, \delta} \lambda_{\alpha \beta \gamma \delta} \phi_\alpha(x) \phi_\beta(x) \phi_\gamma(x) \phi_\delta(x) \tag{2.2}$$

for the case we will consider $\lambda_{\alpha \beta \gamma \delta}$ is the following combination of basis tensors:

$$\lambda_{\alpha \beta \gamma \delta} = \frac{\lambda}{3} (\delta_{\alpha \beta} \delta_{\gamma \delta} + \delta_{\alpha \gamma} \delta_{\beta \delta} + \delta_{\alpha \delta} \delta_{\beta \gamma}) = \lambda T_{\alpha \beta \gamma \delta} \tag{2.3}$$

One could also study the generalized form:

$$\lambda_{\alpha \beta \gamma \delta} = \sum_i \lambda_i T_{\alpha \beta \gamma \delta}. \tag{2.4}$$

Eq.(2.3) implies that the tensor $T_{\alpha \beta \gamma \delta}$ is defined as

$$T_{\alpha \beta \gamma \delta} = \frac{1}{3} (\delta_{\alpha \beta} \delta_{\gamma \delta} + \delta_{\alpha \gamma} \delta_{\beta \delta} + \delta_{\alpha \delta} \delta_{\beta \gamma}) \tag{2.5}$$

$\delta$-tensors are the basis tensors defined by:

$$\delta_{\alpha_1 \cdots \alpha_n} = \begin{cases} 1, & \alpha_1 = \cdots = \alpha_n, \\ 0, & \text{otherwise}. \end{cases} \tag{2.6}$$

They satisfy the identities

$$\sum_\gamma \delta_{\alpha_1 \cdots \alpha_n \gamma \gamma} = \delta_{\alpha_1 \cdots \alpha_n}. \tag{2.7}$$
\[ \sum_{\gamma} \delta_{\alpha_1 \cdots \alpha_n} \gamma \delta_{\beta_1 \cdots \beta_n} \gamma = \delta_{\alpha_1 \cdots \alpha_n} \delta_{\beta_1 \cdots \beta_n}, \]  
(2.8)

\[ \sum_{\gamma} \delta_{\gamma \gamma} = N. \]  
(2.9)

Our main focus is to find out O(N) symmetry factors for vacuum diagrams up to seven loops using above formulas. For this, we will make use of above relations and basic tensor properties to find out symmetry factors for higher loop four-point correlation function diagrams[32].

### 2.2 Symmetry Factors for O(N)-symmetry

The Tensor \( \lambda_{\alpha \beta \gamma \delta} \) is symmetric in all its indices and is some combination of basis tensor \( \delta_{\alpha \beta} \). The basis tensor has been chosen to be symmetric in all its indices. Let us considering the following Feynman graph:

Then we have the following O(N) symmetry factor

\[ \lambda_{\alpha \beta \sigma_1 \sigma_2}^{0(N)} \lambda_{\sigma_1 \sigma_2 \gamma \delta}^{0(N)} \big|_{\text{sym}} = \lambda^2 S \left( \begin{array}{c} \circ \vspace{0.5em} \circ \\ \circ \vspace{0.5em} \circ \end{array} \right) T_{\alpha \beta \gamma \delta}^{(1)} \]  
(2.10)

The external indices will always be labelled as \( \alpha, \beta, \gamma, \) and \( \delta \) in the remaining text. The symbol \( \big|_{\text{sym}} \) denotes symmetrization of these indices, which replaces each product \( \delta_{\alpha \beta} \delta_{\gamma \delta} \) by \( T_{\alpha \beta \gamma \delta} \)

For the evaluation of \( S \left( \begin{array}{c} \circ \vspace{0.5em} \circ \\ \circ \vspace{0.5em} \circ \end{array} \right) \), we will follow [33]:

\[ S \left( \begin{array}{c} \circ \vspace{0.5em} \circ \\ \circ \vspace{0.5em} \circ \end{array} \right) = \frac{N + 8}{9} \]  
(2.11)

Now, consider the two-loop diagram

\[ \lambda_{\alpha \beta \sigma_1 \sigma_2}^{0(N)} \lambda_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{0(N)} \lambda_{\sigma_3 \sigma_4 \gamma \delta}^{0(N)} \big|_{\text{sym}} = \lambda^3 S \left( \begin{array}{c} \circ \vspace{0.5em} \circ \vspace{0.5em} \circ \\ \circ \vspace{0.5em} \circ \vspace{0.5em} \circ \end{array} \right) T_{\alpha \beta \gamma \delta}^{(1)} \]  
(2.12)

For two-loop evaluation, we first get

\[ \lambda_{\alpha \beta \sigma_1 \sigma_2}^{0(N)} \lambda_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{0(N)} \lambda_{\sigma_3 \sigma_4 \gamma \delta}^{0(N)} \big|_{\text{sym}} = \lambda^3 \frac{N^2 + 6N + 12}{27} \left( \delta_{\alpha \beta} \delta_{\gamma \delta} + 4\delta_{\alpha \gamma} \delta_{\beta \delta} + 4\delta_{\alpha \delta} \delta_{\beta \gamma} \right) \]  
(2.13)

On symmetrizing, this gives

\[ S \left( \begin{array}{c} \circ \vspace{0.5em} \circ \vspace{0.5em} \circ \\ \circ \vspace{0.5em} \circ \vspace{0.5em} \circ \end{array} \right) = \lambda^3 \frac{N^2 + 6N + 20}{27} \]  
(2.14)

Now we would like to work out a general formula for any number of loops. These type of diagrams give the leading N-contributions.
2.3 Generalized Formalism

In the last section, we have evaluated one and two loop graphs in Eq.(2.11) and (2.13). In the same fashion, we can evaluate for diagrams with loops higher than two. For three loops, we get:

\[
\begin{align*}
\lambda^{0(N)}_{\alpha\beta\sigma_1\sigma_3} & \lambda^{0(N)}_{\sigma_1\sigma_2\sigma_3} \lambda^{0(N)}_{\sigma_1\sigma_2\sigma_3} \lambda^{0(N)}_{\sigma_1\sigma_2\sigma_3} \\
= \frac{\lambda^4}{3^4} \left[ (N^3 + 8N^2 + 24N + 32) \delta_{\alpha\beta\gamma} + 2^3 \delta_{\alpha\gamma\delta} + 2^3 \delta_{\alpha\delta\beta} \right]
\end{align*}
\]

and for four loops yields that:

\[
\begin{align*}
\lambda^{0(N)}_{\alpha\beta\sigma_1\sigma_3} & \lambda^{0(N)}_{\sigma_1\sigma_2\sigma_3} \lambda^{0(N)}_{\sigma_1\sigma_2\sigma_3} \lambda^{0(N)}_{\sigma_1\sigma_2\sigma_3} \\
= \frac{\lambda^5}{3^5} \left[ (N^4 + 10N^3 + 40N^2 + 80N + 80) \delta_{\alpha\beta\gamma\delta} + 2^4 \delta_{\alpha\gamma\delta\beta} + 2^4 \delta_{\alpha\delta\beta\gamma} \right]
\end{align*}
\]

A careful observation of Eq.(2.16) implies that it is possible to write it in more elegant way as:

\[
\begin{align*}
&= \lambda^{l+1+} \left[ \left( l + 1 \right) \frac{2^l N^0}{1!} + \left( l + 1 \right) l \frac{2^l N^1}{2!} \left( l - 1 \right) \frac{2^{l-2} N^2}{3!} \right. \\
&\quad + \left( l + 1 \right) l (l - 1) \frac{2^{l-3} N^3}{4!} + \left( l + 1 \right) l (l - 1) (l - 2) \frac{2^{l-4} N^4}{5!} \right] \delta_{\alpha\beta\gamma\delta} \\
&\quad + 2^l \delta_{\alpha\gamma\delta\beta} + 2^l \delta_{\alpha\delta\beta\gamma}
\end{align*}
\]

where \( l \) represents number of loops which are four here. In this section, our goal is to find out generalized formulas for four-point, two-point and vacuum diagram up to \( l \)-loops, \( l = 1, \ldots, n \). For this, we will simply generalize Eq.(2.17) for \( l \)-number of loops. This will yield an efficient formula for evaluation of any number of loops. Generalization for \( l \)-loops gives

\[
\lambda_{\alpha_0\beta_0\sigma_1\sigma_2} \lambda_{\sigma_2\sigma_3\sigma_4} \ldots \lambda_{\sigma_{n-1}\sigma_n \gamma \delta} = \lambda^{l+1} W_{\alpha\beta\gamma\delta}(l)
\]

where the dashed line in the above graph denotes that there are \( l \) number of loops and are denoted by letter \( l \). To find out an efficient formula, we need to simplify co-efficients of \( \delta_{\alpha\beta\delta\gamma} \).

First, let us leave the first part and concentrate on second (fractional) part of each term in Eq.(2.17), this looks like:

\[
\begin{align*}
&= \frac{2^l N^0}{1!} + \frac{2^{l-1} N^1}{2!} + \frac{2^{l-2} N^2}{3!} + \frac{2^{l-3} N^3}{4!} + \frac{2^{l-4} N^4}{5!}, \\
&= \frac{2^l N^0}{1!} + \frac{2^{l-1} N_{1-1}}{2!} + \frac{2^{l-2} N_{2-1}}{3!} + \frac{2^{l-3} N_{3-1}}{4!} + \frac{2^{l-4} N_{4-1}}{5!} + \frac{2^{l-5} N_{5-1}}{6!}, \\
&= \sum_{p=1}^{5} \frac{2^{l-p+1} N_{p-1}}{p!}.
\end{align*}
\]

where \( p \) stands for particle number. In the above case, we have 4 number of loops and then, \( p \) goes from 1 to 5. So, we can say that in case of \( l \) number of loops, \( p \) goes from 1 to \( (l+1) \). Hence for \( l \) number of loops this becomes

\[
\sum_{p=1}^{(l+1)} \frac{2^{l-p+1} N_{p-1}}{p!}.
\]
Now, consider rest part of Eq.(2.17) which includes product terms. For 4 loops, these are
\[(l + 1) + (l + 1)(l - 1)(l - 1) + (l + 1)l(l - 1) - (l + 1)(l = 1) p-1 \prod (l + 1 - n).
\]
This can easily be extended to \(l\) number of loops
\[
\sum_{p=1}^{(l+1) p-1} \prod (l + 1 - n).
\]
\[(2.21)
\]
A combination of Eq.(2.19) and (2.21), finally gives coefficient for \(\delta_{\alpha\beta\delta\gamma}\)
\[
f(l, N^l) = \sum_{p=1}^{(l+1) p-1} \prod (l + 1 - n) \frac{2^{l-p+1} N^{p-1}}{p!}.
\]
\[(2.22)
\]
Finally, we find out the formula to calculate \(O(N)\)-symmetry factors of any number of four-point loop diagrams
\[
W_{\alpha\beta\gamma\delta}(l) = \frac{f(l, N^l)}{3^{l+1}} \delta_{\alpha\beta\delta\gamma} + \frac{2^{l}}{3^{l+1}} \delta_{\alpha\gamma\beta\delta} + \frac{2^{l}}{3^{l+1}} \delta_{\alpha\beta\gamma\delta}.
\]
\[(2.23)
\]
On symmetrization, this gives the generalized form of \(O(N)\)-symmetry factor for four-point diagrams
\[
S_{l}^{O(N)} = \frac{f(l, N^l) + 2^{l+1}}{3^{l+1}}
\]
\[(2.24)
\]
where \(f(l, N^l)\) is defined in Eq.(2.22). We have verified up to 6 loops that the formula is correct. One could also prove it using mathematical induction.

The generalization for two-point diagrams is obtained by combining the two out-going lines named \(\beta\) and \(\delta\). After the combination of two outgoing lines, the number of loops are increased by one. So, now we deal with a diagram containing \(l + 1\) number of loops. If we substitute \(\beta = \gamma\) in Eq.(2.15) and make use of the relations \(\sum \delta_{\beta\beta} = N\) and \(\sum \delta_{\alpha\beta\delta\beta\gamma}\), we get
\[
\lambda_{\alpha\gamma\sigma_{1}}^{0(N)} \lambda_{\sigma_{1}\sigma_{2}\sigma_{3}\sigma_{4}}^{0(N)} \lambda_{\sigma_{3}\sigma_{4}\gamma\delta}^{0(N)} = \lambda^{4} \frac{(N + 2)(N^{2} + 6N + 20)}{3^{4}} \delta_{\alpha\delta}.
\]
\[(2.25)
\]
This equation describes that there are four number of loops. In the same fashion, Eq.(2.16) gives equation for five-loop two-point diagram
\[
\lambda_{\alpha\gamma\sigma_{1}\sigma_{2}\sigma_{3}}^{0(N)} \lambda_{\sigma_{1}\sigma_{2}\sigma_{3}\sigma_{4}}^{0(N)} \lambda_{\sigma_{4}\sigma_{5}\gamma\delta}^{0(N)} = \lambda^{5} \frac{(N + 2)(N^{3} + 8N^{2} + 24N + 48)}{3^{5}} \delta_{\alpha\delta}.
\]
\[(2.26)
\]
Now, the second term of Eq.(2.25) and Eq.(2.26) is equivalent to the coefficient of \(\delta_{\alpha\beta\delta\gamma}\) in two-loops and three-loops four-point function diagrams respectively plus a factor of \(2^{l-1}\), where \(l\) stands for current number of loops in two-point diagram. Using this fact, one can generalize this for \(l\) loop diagram of two-point function.

\[
W_{\alpha\beta\gamma\delta}(l) = (N + 2) \frac{f(l', N^l') + 2^{l' + 1}}{3^{l'}} \delta_{\alpha\delta}
\]
\[(2.27)
\]
2.4. **GENERAL FORMULAS UP TO SEVEN-LOOPS**

where \( l' = l - 2 \) and \( f(l, N^l) \) is defined in Eq.(2.22).

On symmetrization, this also gives the \( O(N) \)-symmetry factor of two-point diagrams

\[
S_2^{O(N)} = (N + 2) \frac{f(l', N^l') + 2l'^{+1}}{3l'^{-1}}
\]  

(2.28)

A **vacuum diagram** requires the combination of all out-going lines together. In order to obtain \( O(N) \)-symmetry factor for vacuum diagrams we have to combine the remaining two out-going lines named \( \alpha \) and \( \gamma \) in Eq.(2.27). Here, we shall substitute \( \alpha = \gamma \) and \( \sum_\alpha \delta_{\alpha\alpha} = N \) in Eq.(2.27) to obtain

![Diagram of a vacuum diagram](image)

\[
\lambda_{\alpha\beta\sigma_1\sigma_2} \cdots \lambda_{\sigma_n\sigma_{n+1}\beta\gamma} = \lambda^{l-1} W_{\alpha\beta\beta\alpha}(l)
\]

(2.29)

where \( l'^* = l - 3 \) and \( f(l, N^l) \) is given in Eq.(2.22).

The \( O(N) \)-symmetry factor for vacuum diagrams is

\[
S_0^{O(N)} = (N + 2) \frac{f(l'^*, N^{l'^*}) + 2l'^{+1}}{3l'^{-1}}
\]  

(2.30)

Vacuum diagrams can also be obtained by the product of two four-point or two-point diagrams having similar indices, e.g. consider a four-point diagram having \( l \)-loops get multiply with another diagram having \( l' \)-loops and both of them are having same external indices. Explicitly, this is written as

\[
W_{\alpha\beta\gamma\delta}(l)W_{\alpha\beta\gamma\delta}(l') = \frac{N^2 f(N, N^l)f(N, N^{l'}) + N(2l'^{+1}f(N, N^{l'}) + 2l'^{+1} f(N, N^l)) + N(N + 1)2l'^{+1} + 1}{3^{l'+l^2}}
\]  

(2.31)

where \( f(N, N^l) \) is defined in Eq.(2.22).

| \( W_{\alpha\beta\gamma\delta}(l_4) \) | \( \frac{(l(N^l))\delta_{\alpha\beta}\delta_{\gamma}\delta_{\delta}}{3^{l+1}} + \frac{2}{3^{l+1}} \delta_{\alpha\gamma} \delta_{\beta\delta} + \frac{2}{3^{l+1}} \delta_{\alpha\delta} \delta_{\beta\gamma} \) | \( S_4^{O(N)} \) | \( \frac{(l(N^l))^{2l'^{+1}}}{3^{l'+1}} \) |
| \( W_{\alpha\beta\beta\gamma}(l_2) \) | \((N + 2)\frac{f(l'^*, N^{l'}) + 2l'^{+1}}{3^{l' + 1}} \delta_{\alpha\gamma} \) | \( S_2^{O(N)} \) | \((N + 2)\frac{f(l'^*, N^{l'}) + 2l'^{+1}}{3^{l' + 1}} \) |
| \( W_{\alpha\beta\alpha}(l_0) \) | \( N(N + 2)\frac{f(l'^*, N^{l'}) + 2l'^{+1}}{3^{l' + 1}} \delta_{\alpha\gamma} \) | \( S_0^{O(N)} \) | \( N(N + 2)\frac{f(l'^*, N^{l'}) + 2l'^{+1}}{3^{l' + 1}} \) |

Table:1 - \( O(N) \)-Symmetry factors.

where \( l' = l - 2, l'^* = l - 3 \) and \( f(l, N^l) \) is defined in Eq.(2.22). In the above table, \( W(l_4) \), \( W(l_2) \) and \( W(l_0) \) are formulae to calculate four-point, two-point and vacuum diagrams while, \( S_4^{O(N)}, S_2^{O(N)} \) and \( S_0^{O(N)} \) are \( O(N) \)-symmetry factors for the respective diagrams.

2.4 **General Formulas up to Seven-Loops**

In this section, we will try to obtain general formulas for seven-loop vacuum diagrams using the formulas derived in above section. These calculations will be done in two steps. For simplicity,
we will first reduce the diagram as much as possible and then evaluate them.

**Diagram 2(a)** is a vacuum diagram with seven number of loops that is \( l = 7 \) and \( l^* = 4 \). Substituting these values in Eq.(2.29), we get

\[
W_{\alpha\beta\gamma\delta}(l) = N(N + 2)\frac{f(l, N^*) + 2l^* + 1}{3^{l - 1}} = N(N + 2)(N^4 + 10N^3 + 40N^2 + 80N + 112) \tag{2.32}
\]

**Diagram 2(b)** arise from the combination of two four-point function diagrams, one consists of three loops while other has only one-loop as shown in Fig:3(a). First, let us calculate four-point function diagrams for which substitute \( l = 3 \) in Eq.(2.23).

\[
W_{\alpha\beta\gamma\delta}(3) = \frac{f(l, N^*)}{3^4} \delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{2^3}{3^4} \delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{2^3}{3^4} \delta_{\alpha\delta}\delta_{\beta\gamma} \tag{2.33}
\]

where

\[
f(l, N^*) = \sum_{p=1}^{(4)} \prod_{n=0}^{p-1} (l + 1 - n) \frac{2^{l-p+1} N^{p-1}}{p!} = N^3 + 8N^2 + 24N + 32. \tag{2.34}
\]

Hence,

\[
W_{\alpha\beta\gamma\delta}(3) = N^3 = 8N^2 + 24N + 32 \tag{2.35}
\]

and for one-loop\(^1\)

\[
W_{\alpha\beta\gamma\delta}(1) = N + 4 \frac{2}{3^2} \delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{2}{3^2} \delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{2}{3^2} \delta_{\alpha\gamma}\delta_{\beta\delta} \tag{2.36}
\]

Now, whole seven-loop diagram in tensor notations can be can be obtained from the product of Eq.(2.35) and (2.36) where all indices will contract, leading to

\[
W_{\alpha\beta\gamma\delta}(3)W_{\alpha\gamma\beta\delta}(1) = \frac{N(N + 2)(3N^3 + 24N^2 + 80N + 136)}{3^6} \tag{2.37}
\]

**Diagram 2(c)** is much more complicated than previous two. Its explicit form is:

\[
W_{\alpha\gamma1 \delta1 \beta1 \gamma1 \delta1}(0)W_{\alpha1 \beta1 \gamma1 \delta1}(1)W_{\beta1 \alpha1 \gamma1}(0)W_{\alpha\beta\gamma\delta}(1) \tag{2.38}
\]

\[
U_{\alpha\delta\gamma\delta1 \beta1} = W_{\alpha\delta\gamma\delta1 \beta1}(0)W_{\alpha1 \beta1 \gamma1 \delta1}(1) \tag{2.39}
\]

\[
U_{\alpha\delta\alpha1 \beta1} = \frac{(3N + 10)}{3^4} \delta_{\alpha1 \beta1}\delta_{\alpha\delta} + \frac{(N + 6)}{3^4} \delta_{\alpha1 \alpha\delta}\delta_{\beta1 \delta} + \frac{(N + 6)}{3^4} \delta_{\alpha1 \delta\beta1 \alpha} \tag{2.40}
\]

\[
U_{\beta1 \alpha1 \alpha\delta} = W_{\beta1 \alpha1 \beta1}(0)W_{\alpha\beta\gamma\delta}(1) \tag{2.41}
\]

\[
U_{\beta1 \alpha1 \alpha\delta} = \frac{(3N + 10)}{3^4} \delta_{\alpha1 \delta\beta1 \alpha} + \frac{(N + 6)}{3^4} \delta_{\alpha1 \delta\alpha1 \beta} + \frac{(N + 6)}{3^4} \delta_{\alpha1 \delta\beta1 \alpha} \tag{2.42}
\]

\(^1\)is taken from [33].
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Fig: 3-Seven-loop Vacuum diagrams.
Product of Eqs.(2.40) and (2.42) give result for diagram (c) is\(^2\):

\[
U'_{\alpha\delta\alpha\delta} = U_{\alpha\delta\alpha\delta} U_{\beta_1\alpha_1\alpha_1} = \frac{N(N + 2)(11N^2 + 76N + 156)}{3^6}
\]  (2.43)

**Diagram 2(e)** contains two-point loop diagrams. Otherwise, this is quite similar to (b)( is shown in Fig:3(b)). Here, we will make use of Eq.(2.27). This reduces to two points, one represents two-point function diagram for four-loops and other for two-loop. First calculate three-loop two-point function diagram. For this, substitute \(l = 4\) and \(l' = 2\) in Eq.(2.27)

\[
W_{\alpha\beta\beta\gamma}(4) = (N + 2) \frac{f(l', N^l') + 2l'+1}{3^4} \delta_{\alpha\gamma}
\]  (2.44)

and for \(l = 2, l' = 0\) which gives

\[
W_{\alpha\beta\beta\gamma}(2) = (N + 2) \frac{f(l', N^l') + 2l'+1}{3^2} \delta_{\alpha\gamma}
\]  (2.45)

Putting Eqs.(2.44) and (2.45) together, all covariant indices get contracted, leading to a vacuum diagram

\[
W_{\alpha\beta\gamma}(3)W_{\alpha\delta\gamma}(1) = \frac{N(N + 2)^2(N^2 + 6N + 20)}{3^5}
\]  (2.46)

O(N)-Symmetry factors of all reduced matrix elements in Fig:2 are given in the following table.

\(^2U_{\alpha\beta\gamma\delta}\) is nothing but product of two tensors like \(W_{\alpha\beta\gamma\delta}\) (also \(U'\) is the product of two \(U\) tensors), all tensors \(U', U\) and \(W\) have similar properties.
2.5 LARGE N-LIMIT

Table:2 - Formulas for seven-loop vacuum diagrams

| a | \(N(N+2)(N^3+10N^4+40N^5+80N+112)/3^6\) |
| b | \(N(N+2)(3N^4+24N^5+80N+136)/3^6\) |
| c | \(N(N+2)(11N^2+76N+156)/3^6\) |
| d | \(N(N+2)(N^3+18N^4+80N+136)/3^6\) |
| e | \(N(N+2)^2(N^2+6N+20)/3^6\) |
| f | \(N(N+2)(N^3+14N^4+76N+152)/3^6\) |
| g | \(N(N+2)(11N^2+76N+132)/3^6\) |
| h | \(N(N+2)(2N^3+49N+152)/3^6\) |
| i | \(N(N+2)(7N^2+72N+164)/3^6\) |
| j | \(N(N+2)(7N^2+72N+164)/3^6\) |
| k | \(N(N+2)^3(N+8)^2/3^6\) |
| l | \((N+2)^2(5N+22)/3^6\) |
| m | \(N(N+2)(7N^2+72N+156)/3^6\) |
| n | \(N(N+2)(N^3+10N^2+72N+160)/N\) |
| o | \(N(N+2)(7N^2+72N+164)/3^6\) |
| p | \(N(N+2)^3/3^3\) |
| q | \(N(N+2)^3/3^3\) |

Table:2 - Formulas for seven-loop vacuum diagrams

In order to calculate the diagrams \(a\) - \(j\) and \(m\) - \(o\), we have used of Eq.(2.23). Ultimately, the results obtained for these diagrams look similar to \(O(N)\)-symmetry factors for four-point diagram in \cite{33} (tabulated on page-466), because all these diagrams are made from the contribution of four-point diagrams. While diagrams \(e\), \(k\), \(p\) and \(q\) in Fig:(2) contains two-point function loop diagrams and these are calculated using formula for two-point diagrams described in Eq.(2.27). [Detailed calculations for all these diagrams are given in Appendix A]. It is clear from the Table:2 that diagram(a) is a leading order diagram since each of its loop contains \(N\)-possible choices for the internal index. All other diagrams contains fixed internal indices in its loops which makes them next to leading order diagrams. Now, we are going to discuss this property in detail.

2.5 Large N-limit

In this section, we will discuss \(1/N\) expansion in \(O(N)\) version of \(\phi^4\) theory. The Lagrangian density for a theory of \(N\) real scalar fields, \(\phi^a\), \(a = 1 \cdots n\), with \(O(N)\)-symmetric quartic self interactions is

\[
L = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} \mu_0^2 \phi^a \phi^a - \frac{\lambda_0}{8} (\phi^a \phi^a)^2, \tag{2.47}
\]

Feynman diagrams are very important in the study of Large N-limit. The scattering process of two mesons of one type into two mesons of another type up to one-loop in the large N-limit is discussed in \cite{34,38}. We'll discuss the generalized formalism by expanding meson of type \(a\) and type \(b\) into a Feynman graph having \(m\) and \(n\) loops respectively as shown in Fig:4. Scattering in 4(a) is explicitly defined by

\[
U_{a_\beta_\gamma_\delta}(l_4) = \lambda_0^{m+1} W_{a_\beta_\sigma_1_\sigma_2}(m_4) \lambda_0^{n+1} W_{\sigma_1_\sigma_2_\gamma_\delta}(n_4) \tag{2.48}
\]

where \(l\) stands for total number of loops before and after scattering. In the above Eq.(2.15), \(\lambda_0\) is the coupling constant defined by

\[
g_0 = \lambda_0 N, \tag{2.49}
\]
We want large N-limit with fixed \( g_0 \). \( W_{\alpha\beta\gamma\delta}(l_4) \) is defined in Eq.(2.23) and the product of two \( W \) tensors with specific indices is

\[
U_{\alpha\beta\gamma\delta}(l_4) = \lambda_0^{m+1} W_{\alpha\beta\gamma\delta}(m_4) \lambda_0^{n+1} W_{\sigma_1\sigma_2\gamma\delta}(n_4) \\
= \lambda_0^{m+n+2} f(m, N^m)[f(n, N^n) + (N + 1)2^n] + 2^{m+n+1} f(n, N^n) \delta_{\alpha\beta}\delta_{\gamma\delta} \\
+ \frac{2^{m+n+1}}{3^{m+n+2}} \delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{2^{m+n+1}}{3^{m+n+2}} \delta_{\alpha\delta}\delta_{\beta\gamma} \\
(2.50)
\]

where

\[
f(n, N^n) = \sum_{p=1}^{n+1} \prod_{k=0}^{p-1} (n + 1 - k) \frac{2^{n-p+1}N^{p-1}}{p!} \sim \mathcal{O}(N^n) \\
(2.51)
\]

and

\[
f(m, N^m)f(n, N^n) \sim \sum_{p=1}^{(m+1)} \sum_{q=1}^{(n+1)} N^{p+q-2} \sim \mathcal{O}(N^{m+n}) \\
(2.52)
\]

substituting Eq.(2.46) and (2.49) in (2.45), we get

\[
U_{\alpha\beta\gamma\delta}(l_4) \sim \mathcal{O}\left(\frac{1}{N^{m+n+2}} N^{m+n+1}\right) \sim \mathcal{O}(1/N) \\
(2.53)
\]

**Diagram 4(b)** can be written explicitly in tensor notations as

\[
U_{\alpha\beta\gamma\delta}(l_4) = \lambda_0^{m+1} W_{\alpha\beta\gamma\delta}(m_4) \lambda_0^{n+1} W_{\sigma_1\sigma_2\gamma\delta}(n_4) \\
= \lambda_0^{m+n+2} f(m, N^m)[f(n, N^n) + (N + 1)2^n] + 2^{m+n+1} f(n, N^n) \delta_{\alpha\beta}\delta_{\gamma\delta} \\
+ \frac{2^m f(n, N^n) + 2^2 n}{3^{m+n+2}} \delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{2^m f(n, N^n) + 2^n}{3^{m+n+2}} \delta_{\alpha\delta}\delta_{\beta\gamma} \\
(2.54)
\]

\[
\sim \mathcal{O}\left(\frac{1}{N^{m+n+2}} N^{m+n}\right) = \mathcal{O}(1/N^2)
\]

**Diagram 4(c)** is quite similar to 4(b) which leads to

\[
U_{\alpha\beta\gamma\delta}(l_4) = \lambda_0^{m+1} W_{\alpha\beta\gamma\delta}(m_4) \lambda_0^{n+1} W_{\sigma_2\sigma_2\gamma\delta}(n_4) \\
= \lambda_0^{m+n+2} \frac{2^n f(m, N^m) + 2^m f(n, N^n) + (N + 2)2^{m+n}}{3^{m+n+2}} \delta_{\alpha\beta}\delta_{\gamma\delta} \\
+ \frac{2^n f(m, N^m) + 2^m f(n, N^n)}{3^{m+n+2}} \delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{f(m, N^m)f(n, N^n) + 2^{m+n}}{3^{m+n+2}} \delta_{\alpha\delta}\delta_{\beta\gamma} \\
(2.55)
\]

\[
\sim \mathcal{O}\left(\frac{1}{N^{m+n+2}} N^{m+n}\right) = \mathcal{O}(1/N^2)
\]

We have seen that the first diagram 4(a) is proportional to \( \mathcal{O}(1/N) \), because there are \( N \) possible choices for the internal index. The second and third diagrams are \( \mathcal{O}(1/N^2) \), next order in \( 1/N^2 \), since there is no internal summation. Let us make a difference between structure of loops in algebra and topology. Algebraically, a loop is defined if there are \( N \) possible choices for the internal index. If two particles scattered and make a loop with fixed internal indices, it will not define a loop algebraically, but topologically it will consider as a loop. Diagrams 4(a), 4(b) and 4(c) have the same topological structure but, algebraically these are not similar. In
4(a), all loops has N-possible choices for the internal index that makes it a diagram in leading order. Diagrams 4(b) and 4(c) define same structure as both contains one loop with fixed internal indices that lowers the order of both diagrams by one factor. We shall define an auxiliary field, $\sigma$, to solve the problem of transformation of algebra into topology[34,38]. Elimination of auxiliary field into original lagrangian alters the lagrangian density by

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{1}{2} \left( \sigma - \frac{1}{2} \frac{\lambda_0}{N} \phi^\alpha \phi^\alpha \right)^2.$$  

The added term has no effects on the dynamics of the theory, but it alters Feynman rules. By elementary algebra,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^\alpha \partial^\mu \phi^\alpha - \frac{1}{2} \mu^2 \phi^\alpha \phi^\alpha + \frac{1}{2} \frac{N}{\lambda^2} \sigma^2 - \frac{1}{2} \sigma \phi^\alpha \phi^\alpha,$$  

The non-trivial interaction is $\phi \phi \sigma$ coupling in the above formalism. All factors of $1/N$ come from $\sigma$ propagator ($i\lambda_0/N$), and every closed $\phi$ loop gives a factor of $N$. 
Chapter 3

Renormalization

3.1 Introduction

In any non-trivial QFT, divergent integrals appear in perturbation expansion for Green’s functions. Renormalization is a process for removing all these divergences, order by order in perturbation theory, by adding extra terms, called counterterms, to the lagrangian that defines the theory. Renormalization scheme in QFT consists of two parts. First, there is a regularization procedure which isolates infinities that appear in individual feynman diagrams. In this section, we will make use of dimensional regularization procedure, which regularizes Feynman diagrams in space-time dimensions and isolates UV-divergencies at singularities. Second step consists of renormalization, where we remove all divergencies by adding counterterms in the original lagrangian. In $\phi^4$ theory, renormalization is defined with the help of renormalized lagrangian[33,36] as:

$$L_{\text{ren}} = L + L_{\text{c.t.}}$$  \hspace{1cm} (3.1)

$L_{\text{ren}}$ looks exactly the same as $L$ except for the parameters and fields. $L_{\text{ren}}$ leads to a finite theory while $L$ does not. First part in Eq.(3.1) is lagrangian for $\phi^4$ theory

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{g \phi^4}{4!},$$  \hspace{1cm} (3.2)

and counterterm lagrangian is:

$$L_{\text{c.t.}} = c_\phi \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + c_{m^2} \frac{1}{2} m^2 \phi^2 + c_g \frac{g \phi^4}{4!},$$  \hspace{1cm} (3.3)

counterterms are of the same type as original ones, so that we can write:

$$L_{\text{ren}} = (1 + c_\phi) \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + (1 + c_{m^2}) \frac{1}{2} m^2 \phi^2 + (1 + c_g) \frac{g \phi^4}{4!}.$$  \hspace{1cm} (3.4)

The quantities $\phi$, $m$, and $g$ in Eq.(3.3) are renormalized field, renormalized mass, and renormalized coupling constant and $\epsilon = 4 - D$. Renormalization constants are defined as:

$$Z_\phi \equiv 1 + c_\phi, \quad Z_{m^2} \equiv 1 + c_{m^2}, \quad Z_g \equiv 1 + c_g.$$  \hspace{1cm} (3.5)

The counterterms $c_\phi$, $c_{m^2}$, and $c_g$ produce additional vertices in the diagrammatic expansions and results in new Feynman rules indicated by:

$$\times = (-c_{m^2})m^2.$$  \hspace{1cm} (3.6)
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\[ (-c_\phi) k^2, \quad (3.7) \]

\[ (-c_g) g \mu^\epsilon. \quad (3.8) \]

By cleverly choosing all counterterms in Eq.(3.4), we can make theory renormalizable. The infinities are then absorbed by renormalization. Last step before we start regularization is to give a reasonable definition of one particle irreducible diagrams (abbreviated as 1PI). A one particle irreducible diagram is a connected Feynman diagram that cannot be disconnected by cutting a single internal line. These are the building blocks of the set of Feynman diagrams.

### 3.2 Regularization

In this section, we shall calculate two-point and four-point correlation function diagrams up to 3-loops. The sum of all possible two-point diagrams up to three loops are defined by the proper vertex function \( \Gamma^{(2)}(k) \). First of all, we shall calculate all 1PI diagrams explicitly.

\[
\Gamma^{(2)}(k) = k^2 + m^2 - \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{4} + \frac{1}{12} \right) + O(g^4) \quad (3.9)
\]

For four-point correlation function, we get too many diagrams that are quite difficult to handle. For simplicity, we will only consider the leading order diagrams (in \( N \)) and will omit all subleading diagrams. The sum of all possible four-point function diagrams in large \( N \)-limit are

\[
\Gamma^4(k) = - \left[ \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{8} \right] + O(g^5) \quad (3.10)
\]

#### 3.2.1 Symmetry factors

We use well-defined Feynman rules to represent a Feynman diagram as an integral over momentum. However, the overall numerical factor is calculated separately. This factor is called the symmetry factor of the diagram. In an interaction field theory that contains more than one factor of the same field, there will be diagrams for which the symmetry factor is not one[40]. Diagrammatically we can say that a diagram with one \( \phi^4 \) interaction (i.e., a tree level diagram) contains one symmetry factor while for a diagram with more than one \( \phi^4 \) interaction (e.g., loop diagrams) symmetry factor is not one. Palmera and Carrington has defined a formula to calculate symmetry factors for Feynman diagrams [14].

\[
S = \frac{1}{R} \left( \frac{1}{2} \right)^{D_1} \left( \frac{1}{2!} \right)^{D_2} \left( \frac{1}{3!} \right)^{D_3} \left( \frac{1}{4!} \right)^{D_4} . \quad (3.11)
\]

where:

\( R \) = Number of ways to permute the internal indices and produce an identical set of propagators,
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\[ D_1 = \text{Number of pairs of propagators of the form } D^2_{aa}, \]
\[ D_2 = \text{Number of pairs of propagators of form } D^2_{mn} \text{ plus number of factors of the form } D_{aa}, \]
\[ D_3 = \text{Number of triples of propagators of the form } D^3_{mn}, \]
\[ D_4 = \text{Number of sets of propagators of the form } D^4_{mn}. \]

The terms \( D_{aa} \) and \( D^2_{aa} \) represent one and two propagators respectively which start and end at the same vertex, \( a \). Similarly, \( D^2_{mn} \), \( D^3_{mn} \) and \( D^4_{mn} \) stand for two, three and four propagators respectively, each propagator starts at the same vertex named \( m \) and ends at another same vertex named \( n \). One can immediately find out symmetry factor for any number of loops using this formula. For example, consider a vacuum diagram of order \( g^3 \): \( S = \frac{1}{48} \) \( (R = 6, D_1 = 0, D_2 = 3, D_3 = 0, D_4 = 0) \). \( (3.12) \)

Using this technique, we can find out symmetry factors for any number of loop diagrams. Diagrams representing interactions corresponding to the functional \( Z[J] \) have no external lines, so that all solid lines end at a vertex[38].

\[ Z[J] = \int D\phi(x) e^{-E[\phi,J]}, \]

\( (3.13) \)

where \( E[\phi,J] \) is the energy functional defined by:

\[ E[\phi,J] \equiv E_0[\phi] + E_{int}[\phi] + \frac{1}{2} \int d^Dx \int d^Dy \phi(x,J(x,y))\phi(y). \]

\( (3.14) \)

Such diagrams with no external lines are called vacuum diagrams. These diagrams can further be classified into two types: so-called connected diagrams, where all graphical elements are connected via lines and vertices, and disconnected diagrams consisting of two or more connected parts. If we take the logarithm of functional \( Z[J] \) all disconnected diagrams cancel out. Therefore, we can say that logarithm of partition function \( Z[J] \) i.e. \( W[J] = \log Z[J] \) contains only connected diagrams. Diagrammatically, a derivative with respect to \( K \) corresponds to cutting one line of a vacuum diagram in all possible ways. Thus, all diagrams of the two-point function can be derived from such cuts multiplied by a factor of 2. For example, first consider first order vacuum diagram of \( W[K] \). Derivative of \( W[K] \) implies that we can cut one line in two possible ways, which implies the multiplication with factor 2:

\[ \frac{\partial W_1}{\partial K(x,y)} = 2 \times \frac{1}{8}, \]

\[ G^{(2)} = 2 \times \frac{\partial W_1}{\partial K(x,y)} = 2 \times 2 \times \frac{1}{8}, \]

\[ = \frac{1}{2}. \]

\( (3.15) \)

where \( S = \frac{1}{8} \) \( (R = 1, D_1 = 1, D_2 = 2, D_3 = 0, D_4 = 0) \) is the symmetry factor corresponds to vacuum diagram \( W_1[K] \). One can also find out all possible four-point correlation functions from vacuum diagrams. According to the above method, we will take second order derivative of \( W[K] \) with respect to \( K(x,y) \) to get four-point correlation functions, but it becomes quite complicated. It is also possible to find out four-point correlation function by taking derivative with respect to vertex function \( (-\lambda/4!) \). \( \partial W/\partial \lambda \) implies that removal of one vertex in all possible ways. To illustrate this, We’ll present an example. Let us take a diagram of order \( g^3 \) (given in Eq.(3.12)).

\[ W_2[0] = \frac{1}{48}, \]

\( (3.16) \)
Four-point correlation function is obtained by removing one vertex in three possible ways and multiplied by a factor of $4!$:

$$G^{(4)} = 4! \times \frac{1}{48} \times 3 \times \times \times$$ \hspace{1cm} (3.17)

where $S = \frac{1}{48}$ is the symmetry factor corresponding to the vacuum diagram.

In the similar fashion, we can find out two-point and four-point correlation functions up to higher order loops. In the next section, we are going to evaluate and dimensionally regularize all two-point and four-point diagrams up to three loops. So, we have to find out all correlation functions up to three loops. We will work only two-point and four-point correlation functions.

### 3.2.2 Calculation in Dimensional Regularization

When quantum-field amplitudes are constructed within perturbation theory, multi loop Feynman integrals arises. These are the integrals over so-called loop momenta. For a given graph, the corresponding Feynman amplitude is represented as a Feynman integral over loop momenta due to some Feynman rules. At higher orders this is a difficult task because some of the higher-loop diagrams appear to have overlapping divergences which arises because of a common propagator between two loops. The necessary integration over the loops are carried out in momentum space and divergent integrals are regularized dimensionally by shifting 4 to 4-$\epsilon$ dimensions of space-time. For a given Feynman integral the main task is then the derivation of an analytical expression in terms of known functions with well-defined properties. In the following text, we will give all the integrals contributing in the large N-limit.

#### Two-point correlation functions

Dimensional regularization is performed by the evaluation of Feynman integrals in $D$ dimensions, where the coupling constant $\lambda$ is no longer dimensionless. It is convenient to redefine it

$$\lambda = g\mu^{4-D} = g\mu^{\epsilon}$$

where $g$ is dimensionless and $\mu$ is an arbitrary constant with the dimensions of mass. Using the formulae from Appendix B, we proceed to evaluate Feynman diagrams for $\phi^4$ theory. In two-point functions, Feynman integrals associated to one and two-loop diagrams have already been calculated in [33,35]. For one-loop, we only have a simpler one so-called tadpole graph. Feynman integral associated to a tadpole graph is

$$\square = -\lambda \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2}.$$ \hspace{1cm} (3.18)

$$= m^2 g \left( \frac{2}{\epsilon} + \psi(2) + \log \left( \frac{4\pi\mu^2}{m^2} \right) + O(\epsilon) \right).$$

At two-loop level, we have two graphs: $\bigcirc$ and $\bigcirc\bigcirc$. First one is leading order diagram while, second one is subleading diagram. In the large N-limit, we must have to ignore second one. Now consider $\bigcirc\bigcirc$, the associated Feynman integral is written as

$$\bigcirc\bigcirc = \lambda^2 \int \frac{d^D p_1 \, d^D p_2}{(2\pi)^D \, (2\pi)^D} \frac{1}{p_1^2 + m^2} \frac{1}{(p_2^2 + m^2)^2}.$$ \hspace{1cm} (3.19)
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which shows that integral is factorizes into two parts, one is integral for tadpole graph (integral over $p_1$) and other is integral over $p_2$

$$\lambda \int \frac{d^D p_2}{(2\pi)^D} \frac{1}{(p_2^2 + m^2)^2} = \frac{g}{(4\pi)^2} \left[ \frac{2}{\epsilon} + \psi(1) + \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right],$$

(3.20)

which implies that

$$\begin{align*}
\begin{array}{c}
\cIRCLE\cIRCLE\cIRCLE = -m^2 g^2 \left[ \frac{4}{\epsilon^2} + 2 \frac{\psi(1) + \psi(2)}{\epsilon} - \frac{4}{\epsilon} \log \left( \frac{m^2}{4\pi \mu^2} \right) + \mathcal{O}(\epsilon^0) \right].
\end{array}
\end{align*}$$

(3.21)

At three-loop level, there are five digrams, \(\cIRCLE\cIRCLE\cIRCLE\cIRCLE\), \(\cIRCLE\cIRCLE\), \(\cIRCLE\), \(\cIRCLE\cIRCLE\), and \(\cIRCLE\cIRCLE\cIRCLE\). First two diagrams are leading order diagrams while last three are subleading diagrams. Our goal is to renormalize higher order diagrams in large $N$-limit. So, we will consider only leading order diagrams.

$$\begin{align*}
\begin{array}{c}
\cIRCLE\cIRCLE\cIRCLE\cIRCLE = -\lambda^3 \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \frac{1}{p_1^2 + m^2} \frac{1}{(p_2^2 + m^2)^2} \frac{1}{(p_3^2 + m^2)^2}.
\end{array}
\end{align*}$$

(3.22)

Integral over tadpole graph calculated in Eq.(3.18). For integrals over $p_2$ and $p_3$, we use results from Eq.(3.20). Product of all these integrations gives that

$$\begin{align*}
\begin{array}{c}
\cIRCLE\cIRCLE\cIRCLE\cIRCLE = m^2 g^3 \left[ \frac{8}{\epsilon^3} + 4 \frac{\psi(1) + \psi(2)}{\epsilon^2} + 12 \frac{\psi(1)}{\epsilon^2} \log \left( \frac{4\pi \mu^2}{m^2} \right) + \mathcal{O}(\epsilon^{-1}) \right].
\end{array}
\end{align*}$$

(3.23)

Feynman integral associated with second diagram also factorizes into three independent momentum integrals as given below:

$$\begin{align*}
\begin{array}{c}
\cIRCLE\cIRCLE\cIRCLE = -\lambda^3 \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \frac{1}{(p_1^2 + m^2)^3} \frac{1}{p_2^2 + m^2} \frac{1}{p_3^2 + m^2}.
\end{array}
\end{align*}$$

(3.24)

Here we have two tadpole graphs. What we need here is to evaluate integral over $p_1$. One can evaluate this using Appendix(B.1) with substitution of $\beta = 3$.

$$\begin{align*}
\begin{array}{c}
-\lambda \int \frac{d^D p_1}{(2\pi)^D} \frac{1}{(p_1^2 + m^2)^3} = -\frac{g \mu^\epsilon}{(4\pi)^{D/2}} \frac{\Gamma(3-D/2)}{\Gamma(3)} \frac{1}{(m^2)^{3-D/2}}.
\end{array}
\end{align*}$$

(3.25)

The Feynman integral is UV-divergent in six, eight, ten· · · dimensions, which is reflected by poles in the Gamma function at $D=6,8,10,\cdots$. Introduce the dimensionless coupling constant $g$

$$g \equiv \lambda \mu^{D-4} = \lambda \mu^{-\epsilon}$$

(3.26)

the above integral is now reads in terms of $g$ and $\epsilon$.

$$\begin{align*}
\begin{array}{c}
= -\frac{g}{(4\pi)^2} \left( \frac{4\pi \mu^2}{m^2} \right)^{\epsilon/2} \frac{1}{2m^2} \Gamma(1+\epsilon/2).
\end{array}
\end{align*}$$

(3.27)
The $\epsilon$-expansion of the Gamma function reads that
\[
\Gamma(1 + \epsilon/2) = \left[ 1 + \frac{\epsilon}{2} \psi(1) + O(\epsilon^2) \right]
\] (3.28)

The term underbraces contains IR-divergence in the limit $m^2 \to 0$. They are expanded in powers of $\epsilon$
like
\[
\left( \frac{4\pi \mu^2}{m^2} \right)^{\epsilon/2} = 1 + \frac{\epsilon}{2} \log \frac{4\pi \mu^2}{m^2} + O(\epsilon^2).
\] (3.29)

Substituting gamma and epsilon expansions in Eq. (3.25) gives
\[
-\lambda \int \frac{d^Dp_1}{(2\pi)^D} \frac{1}{(p_1^2 + m^2)^3} = -\frac{g}{(4\pi)^2} \frac{1}{2m^2} \left[ 1 + \epsilon \psi(1) + \frac{\epsilon}{2} \log \frac{4\pi \mu^2}{m^2} + O(\epsilon^2) \right]
\] (3.30)

Finally, product of Integration over $p_2$, $p_3$ (from (3.18)) and $p_1$ gives that
\[
\frac{g^3}{(4\pi)^6} \left[ \frac{2}{\epsilon^2} + 2 \frac{\psi(1)}{\epsilon} + 3 \frac{\log \frac{4\pi \mu^2}{m^2}}{m^2} + O(\epsilon) \right]
\] (3.31)

**Four-point correlation function**

Feynman integrals associated to one and two-loop four-point function diagrams are calculated in [33]. I’ll not go into the detail of these calculations but, will present some important results for the calculations. At one-loop level, we have only one graph
\[
\begin{aligned}
\begin{aligned}
&= -\lambda^2 \int \frac{d^Dp}{(2\pi)^D} \frac{1}{(p^2 + m^2)(p-k)^2 + m^2} \\
&= g \mu^\epsilon \frac{g}{(4\pi)^2} \left[ \frac{2}{\epsilon^2} + 2 \frac{\psi(1)}{\epsilon} + 4 \int_0^1 dx \log \frac{4\pi \mu^2}{k^2x(1-x) + m^2} + O(\epsilon) \right]
\end{aligned}
\end{aligned}
\] (3.32)

At two-loop level, we have two graphs
\[
\begin{aligned}
\begin{aligned}
&= -\lambda^3 \int \frac{d^Dp_1}{(2\pi)^D} \frac{1}{[(p_1 - k)^2 + m^2](p_1^2 + m^2)} \left[ (p_2 - k)^2 + m^2 \right] \frac{1}{(p_2^2 + m^2)} \\
&= -g \mu^\epsilon \frac{g^2}{(4\pi)^4} \left[ \frac{2}{\epsilon^2} + 4 \frac{\psi(1)}{\epsilon} + 4 \int_0^1 dx \log \frac{4\pi \mu^2}{k^2x(1-x) + m^2} + O(\epsilon) \right]
\end{aligned}
\end{aligned}
\] (3.33)

There are five three loop diagrams contributing to the four-point correlation function at large $N$-limit. These are:

\[
\begin{aligned}
\begin{aligned}
&= -\lambda^3 \int \frac{d^Dp}{(2\pi)^D} \frac{1}{[(p - k)^2 + m^2](p^2 + m^2)^2} \int \frac{d^Dp_1}{(2\pi)^D} \frac{1}{(p_1^2 + m^2)^2} \\
&= g \mu^\epsilon \frac{g^2}{(4\pi)^4} \left[ \frac{2}{\epsilon^2} + 4 \int_0^1 dx \frac{m^2(1-x)}{k^2x(1-x) + m^2} + O(\epsilon) \right]
\end{aligned}
\end{aligned}
\] (3.34)
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Feynman integral associated to first diagram of three-loops is:

\[
\int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \frac{1}{[(p_1-k)^2 + m^2][(p_1-k)^2 + m^2]}
\]

The pole term can easily be calculated using Eq. (3.33). Setting \( \lambda = g\mu' \), we find

\[
\begin{align*}
\lambda &= g\mu' \frac{g^3}{(4\pi)^6} \left[ \frac{2}{\epsilon} + \psi(1) + \int_0^1 dx \log \left( \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right) + O(\epsilon) \right]^3 \\
&= g\mu' \frac{g^3}{(4\pi)^6} \left[ \frac{8}{\epsilon^3} + \frac{12}{\epsilon^2} \psi(1) + \int_0^1 dx \log \left( \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right) + O(\epsilon^{-1}) \right]
\end{align*}
\]

The second diagram is product of two two-loop diagrams with a tadpole graph on one loop. Its explicit form is

\[
\int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \frac{1}{[(p_2-k)^2 + m^2][(p_2-k)^2 + m^2]}
\]

Pole term can easily be calculated using formula 3 in Appendix B and tadpole diagram (3.18). Replacing \( \lambda \) with \( g\mu' \):

\[
\begin{align*}
\lambda &= -m^2 \frac{g^2}{(4\pi)^4} \left[ \frac{2}{\epsilon} + \psi(1) + \log \left( \frac{4\pi \mu^2}{m^2} \right) + O(\epsilon) \right] \\
&\times \left[ \frac{1}{\epsilon^2} + \psi(1) + \int_0^1 dx \log \left( \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right) + O(\epsilon) \right] \\
&\times \left[ 1 + \frac{\epsilon}{2} \log 4\pi \mu^2 + O(\epsilon^2) \right] \left[ \int_0^1 dx \frac{1}{k^2 x(1-x) + m^2} - O(\epsilon) \right] \\
&= -g\mu' \frac{g^3}{(4\pi)^6} \left[ \frac{4}{\epsilon^2} \psi(1) + \psi(2) + \frac{2}{\epsilon^2} \log \frac{4\pi \mu^2}{m^2} + \frac{2}{\epsilon} \int_0^1 dx \log \left( \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right) + O(\epsilon^0) \right] \\
&\times \left[ \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right]
\end{align*}
\]

Integral associated to the third diagram is:

\[
\int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \frac{1}{[(p_1-k)^2 + m^2][(p_1-k)^2 + m^2]^3}
\]

Again, we make use of Appendix B.3 and tadpole graph to solve the above integral:

\[
\begin{align*}
\lambda &= m^4 \frac{g^2}{(4\pi)^4} \left[ \frac{2}{\epsilon} + \psi(2) + \log \left( \frac{4\pi \mu^2}{m^2} \right) + O(\epsilon) \right]^2 \\
&\times g\mu' \frac{g^3}{(4\pi)^6} \left[ \frac{4}{\epsilon^2} + \psi(2) + \frac{4}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + O(\epsilon^0) \right] \left[ \int_0^1 dx \left( \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right)^2 \right]
\end{align*}
\]
Integral associated to fourth diagram is:

\[
\lambda^4 \int \frac{d^D p_1}{(2\pi)^D} \frac{1}{(p_1 - k)^2 + m^2} \frac{1}{(p_1^2 + m^2)^2} \int \frac{d^D p_2}{(2\pi)^D} \frac{1}{p_2^2 + m^2} \int \frac{d^D p_3}{(2\pi)^D} \frac{1}{p_3^2 + m^2}
\]

This looks quite similar to previous one. Following the similar steps, one obtains:

\[
= m^4 \frac{g^2}{(4\pi)^4} \left[ \frac{2}{\epsilon} + \psi(2) + \log \left( \frac{4\pi \mu^2}{m^2} \right) + \mathcal{O}(\epsilon) \right]^2 \times g\mu' \frac{g}{(4\pi)^2} \Gamma(2 + \epsilon/2)
\]

\[
\times \left[ 1 + \frac{\epsilon}{2} \log 4\pi \mu^2 + \mathcal{O}(\epsilon^2) \right] \left[ \int_0^1 dx \frac{x(1-x)}{[k^2 x(1-x) + m^2]^2} - \mathcal{O}(\epsilon) \right]
\]

\[
= g\mu' \frac{g^3}{(4\pi)^6} \left[ \frac{4}{\epsilon^2} + \frac{4}{\epsilon} \psi(2) + \frac{4}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon^0) \right] \left[ \int_0^1 dx \frac{m^4 x(1-x)}{[k^2 x(1-x) + m^2]^2} - \mathcal{O}(\epsilon) \right]
\]

Finally, we shall calculate integral associated to the last diagram:

\[
\lambda^4 \int \frac{d^D p_1}{(2\pi)^D} \frac{1}{(p_1 - k)^2 + m^2} \frac{1}{(p_1^2 + m^2)^2} \int \frac{d^D p_2}{(2\pi)^D} \frac{1}{p_2^2 + m^2} \int \frac{d^D p_3}{(2\pi)^D} \frac{1}{p_3^2 + m^2}
\]

It is calculated using Eqs.(3.18), (3.20) and appendix B.3, Setting \( \lambda = g\mu' \), we get:

\[
= -m^2 \frac{g}{(4\pi)^2} \left[ \frac{2}{\epsilon} + \psi(2) + \log \left( \frac{4\pi \mu^2}{m^2} \right) + \mathcal{O}(\epsilon) \right]
\]

\[
\times \left[ 1 + \frac{\epsilon}{2} \log 4\pi \mu^2 + \mathcal{O}(\epsilon^2) \right] \left[ \int_0^1 dx \frac{1-x}{k^2 x(1-x) + m^2} - \mathcal{O}(\epsilon) \right]
\]

\[
= -g\mu' \frac{g^3}{(4\pi)^6} \left[ \frac{4}{\epsilon^2} + \frac{2}{\epsilon} \psi(1) + \psi(2) + \frac{2}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \frac{2}{\epsilon} \int_0^1 dx \log \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^0) \right]
\]

\[
\times \left[ \int_0^1 dx \frac{m^2 (1-x)}{k^2 x(1-x) + m^2} \right].
\]

The \( \mathcal{O}(\epsilon^0) [\mathcal{O}(\epsilon^{-1})] \) terms in the integrals are very complicated.

Renormalization is a two-step process: first is regularization and second is renormalization. There are several different regularization schemes and also several ways to renormalize. In renormalization, one way is to subtract off the divergent part that appears after renormalization. The choice of what to subtract off is called a "subtraction scheme", and the obvious minimal choice is known as "minimal subtraction". The subtraction schemes differ from one another in what finite parts to subtract known as "finite renormalization".

### 3.3 Minimal Subtraction Scheme

Over the years, field theories invented all kind of renormalization schemes. But since 1970’s, the most popular one is renormalization in Minimal subtraction scheme. It starts with dimensional
3.3. MINIMAL SUBTRACTION SCHEME

regularization technique in order to control all ultraviolet-divergences. Within the framework of dimensional regularization, one can implement renormalization by minimal subtraction, where all pole terms that appear at $\epsilon=0$ are subtracted at each order in loop expansion. The main idea of renormalization is to correct the original lagrangian of a quantum field theory by introducing a counter terms corresponding to each one particle irreducible diagram (1PI). These counter terms have the effect of cancelling the UV-divergences that appears in dimensional regularization. Hence, renormalized lagrangian is defined by

$$ L = \frac{1}{2} (\partial \phi)^2 + c \frac{1}{2} (\partial \phi)^2 + m^2 \phi^2 + c_{m^2} \frac{m^2}{2} \phi^2 + c_{\mu} \frac{\mu \epsilon}{4!} \phi^4 + c_{g} \frac{\mu \epsilon}{4!} \phi^4, $$

Our goal is to explicitly calculate counterterms corresponding to three-loop diagrams that leads to renormalization constants in the $\phi^4$ theory and in the MS-scheme. Counterterms corresponding to first and second order diagrams are calculated in [33] where the first order mass counterterm is chosen as pole term of tadpole diagram proportional to $m^2$ like

$$ \frac{N+2}{3} = \frac{N}{2} \, K^2, \quad S(0) = -m^2 g N (4\pi)^2 \, N^3 \epsilon. $$

where $N+2$ is the O(N)-symmetry factor for tadpole graph. Large N-limit ignores the factor $\frac{2}{3}$ giving $\frac{N}{3}$ as the symmetry factor of tadpole graph. Similarly, first order counterterm for coupling constant is chosen as pole term of one-loop four-point diagram proportional to $-\mu \epsilon g$

$$ \frac{N+8}{9} = \frac{N}{2} \, K^2, \quad S(0) = -\mu \epsilon g N (4\pi)^2 \, N^3 \epsilon. $$

again, $\frac{N+8}{9}$ is O(N)-symmetry factor of first order four-point diagram giving $\frac{N}{9}$ as leading order symmetry factor. These first-order counterterms helps us to find out next order counterterms. Diagrams contributing to the two-point and four-point functions are given in section (3.2). Note that four-point diagrams are taken in large N-limit.

3.3.1 Two-point calculations

In this section, we deal with all two-point diagrams up to three loops (i.e, to order $g^3$). The corresponding counterterms are necessary to have the same order. We have to form all previous diagrams up to order $g^3$ plus those which arise from the second order ($g^2$) counterterms:

$$ \Gamma^{(2)} = (\epsilon - 1) \left[ \frac{1}{2} \bigcirc + \frac{1}{4} \bigcirc + \frac{1}{4} \bigcirc + \frac{1}{4} \bigcirc + \frac{1}{2} \bigcirc + \frac{1}{8} \bigcirc + \frac{1}{8} \bigcirc + \frac{1}{4} \bigcirc + \frac{1}{4} \bigcirc + \frac{1}{4} \bigcirc + \frac{1}{4} \bigcirc + \frac{1}{2} \bigcirc + \frac{1}{2} \bigcirc + \frac{1}{2} \bigcirc + \frac{1}{2} \bigcirc + \frac{1}{2} \bigcirc + O(g^4) \right], $$

where the second and third last counterterms are second order mass and vertex counterterms respectively. Renormalization constants up to $g^2$ is explicitly calculated in [31]. Here, I’ll work in large N-limit. On two-loop level, there is only one diagram $\bigcirc$ that contributes to the large N-limit. Also, there are two counterterm (mass and vertex) diagrams. The pole term of $\bigcirc$
is given in Eq. (3.21). In large N-limit, this get multiplied with corresponding O(N)-symmetry factor

\[
\frac{1}{4} \mathcal{K}(\square) = -\frac{m^2 g^2}{(4\pi)^4} \left[ \frac{1}{\epsilon^2} + \frac{\psi(1) + \psi(2)}{2\epsilon} - \frac{1}{\epsilon} \log \left( \frac{m^2}{4\pi\mu^2} \right) + O(\epsilon^0) \right] S(\square).
\]

(3.49)

The first counterterm arises from mass counterterm \( \square \) contains two vertices \( -\lambda = -g\mu^\epsilon \) for each vertex), which will be obtained by replacing one of the coupling constant \(-\mu^\epsilon g\) in \( \chi(\square)_{k^2=0} = \square \) by \(-m^2 c_{m^2}^1\) leading to

\[
\frac{1}{2} \mathcal{K}(\square) = \frac{m^2 g c_{m^2}^1}{(4\pi)^2} \left[ \frac{1}{\epsilon^2} + \frac{\psi(1) + \psi(2)}{2\epsilon} + \frac{1}{2} \log \left( \frac{4\pi\mu^2}{m^2} \right) + O(\epsilon^0) \right] S(\square).
\]

(3.50)

For \( \bigcirc \), we have to replace the coupling constant \(-\mu^\epsilon g\) in \( \square \) by \(-\mu^\epsilon g c_{g}^1\)

\[
\frac{1}{2} \mathcal{K}(\bigcirc) = \frac{m^2 g c_{g}^1}{(4\pi)^2} \left[ \frac{1}{\epsilon^2} + \frac{\psi(2)}{2\epsilon} + \frac{1}{2} \log \left( \frac{4\pi\mu^2}{m^2} \right) + O(\epsilon^0) \right] N^2 \frac{\epsilon^2}{3^2}.
\]

(3.51)

Now, we can obtain renormalization constants up to order \( g^2 \). Eq. (3.5) gives

\[
Z_{m^2}(g, \epsilon^{-1}) = 1 + c_{m^2} = c_{m^2}^1 + c_{m^2}^2,
\]

\[
= 1 + \frac{g}{(4\pi)^2} \frac{1}{\epsilon} + \frac{1}{m^2} \left[ \frac{1}{4} \mathcal{K}(\square) + \frac{1}{2} \mathcal{K}(\bigcirc) + \frac{1}{2} \mathcal{K}(\bigcirc) \right].
\]

(3.52)

where

\[
c_{m^2}^1 = \frac{g}{(4\pi)^2} \frac{N}{3\epsilon}, \quad c_{m^2}^2 = \frac{g^2}{(4\pi)^4} \frac{N^2}{9\epsilon^2}.
\]

(3.53)
The pole term of \( \frac{1}{8} K \) along with O(N)-symmetry factor is:

\[
\frac{1}{8} K \left( \frac{m^2 g^3}{(4\pi)^6} \right) = \frac{1}{8} \left( \frac{m^2 g^3}{(4\pi)^6} \right) \left[ \frac{1}{\epsilon^3} + \frac{2(1) + \psi(2)}{2\epsilon^2} + \frac{3}{2\epsilon^2} \log \left( \frac{4\pi\mu^2}{m^2} \right) + O(\epsilon^{-1}) \right] S \left( \frac{m^2 g^3}{(4\pi)^6} \right) N^3 \frac{3}{3^3}.
\]

The pole term of \( \frac{1}{8} K \) given in Eq. (3.31) along with O(N)-symmetry factor is:

\[
\frac{1}{8} K \left( \frac{m^2 g^3}{(4\pi)^6} \right) = \frac{1}{8} \left( \frac{m^2 g^3}{(4\pi)^6} \right) \left[ \frac{1}{\epsilon^3} + \frac{2(1) + \psi(2)}{2\epsilon^2} + \frac{3}{2\epsilon^2} \log \left( \frac{4\pi\mu^2}{m^2} \right) + O(\epsilon^{-1}) \right] S \left( \frac{m^2 g^3}{(4\pi)^6} \right) N^3 \frac{3}{3^3}.
\]

The expansion in (3.9) contains in addition seven counterterms of same order, arising from second-order counterterm diagrams, to be called counterterm diagrams. A counterterm diagram is calculated by replacing the coupling constant in the corresponding \( \phi^4 \) by the counterterm.

For \( g \), we replace one of the coupling constant \( -\mu' g \) in \( g \) by \( -\mu' g c_1^1 \) and find that:

\[
\frac{1}{4} = \frac{-m^2 g^2 c_1^1}{(4\pi)^3} \left[ \frac{1}{\epsilon^2} + \frac{\psi(1) + \psi(2)}{2\epsilon} - \frac{1}{\epsilon} \log \left( \frac{m^2}{4\pi\mu^2} \right) + O(\epsilon^0) \right] S \left( \frac{-m^2 g^2 c_1^1}{(4\pi)^3} \right) N^3 \frac{3}{3^3}.
\]

Similarly, the calculation for \( c_1^2 \) requires replacing the coupling constant \( -\mu' g \) in \( g \) by \( -\mu' g c_1^1 \):

\[
\frac{1}{4} = \frac{-m^2 g^2 c_1^2}{(4\pi)^3} \left[ \frac{1}{\epsilon^2} + \frac{\psi(1) + \psi(2)}{2\epsilon} - \frac{1}{\epsilon} \log \left( \frac{m^2}{4\pi\mu^2} \right) + O(\epsilon^0) \right] S \left( \frac{-m^2 g^2 c_1^2}{(4\pi)^3} \right) N^3 \frac{3}{3^3}.
\]

Now, turn over the first mass counterterm diagram \( \frac{m^2 g^3}{(4\pi)^6} \). This obtains by replacing one of
the coupling constant $-\mu^g$ in $\chi\chi\chi|_{k^2=0} = \mathcal{O}$ by $-m^2c_{m^2}$, leading to

$$
\frac{1}{4} \mathcal{O} = -\frac{m^2c_{m^2}g^2}{(4\pi)^4} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \psi(1) + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right] S_{(\mathcal{O})} \number{3.58}
$$

The next mass counterterm is $\mathcal{O}$ requires replacing one of the coupling constant $-\mu^g$ in $\chi\chi\chi|_{k^2=0} = \mathcal{O}$ by $-m^2c_{m^2}$:

$$
\frac{1}{4} \mathcal{O} = \frac{m^2g^2}{(4\pi)^4} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \psi(1) + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right] S_{(\mathcal{O})} \number{3.59}
$$

The second order vertex counterterm makes $\mathcal{O}$, which is obtained by replacing one of the coupling constant $-\mu^g$ in $\mathcal{O}$ by $-\mu^g c_g$ leading to

$$
\frac{1}{2} \mathcal{O} = \frac{m^2g^2}{(4\pi)^2} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \psi(1) + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right] S_{(\mathcal{O})} \number{3.60}
$$

Next is $\mathcal{O}$, which contains second order mass counterterm. This one is obtained by replacing one of the coupling constant $-\mu^g$ in $\chi\chi\chi|_{k^2=0} = \mathcal{O}$ by $-m^2c_{m^2}$ leading to

$$
\frac{1}{2} \mathcal{O} = \frac{m^2g^2}{(4\pi)^2} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \psi(1) + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right] S_{(\mathcal{O})} \number{3.61}
$$

In the end, we have tadpole diagram with both mass and vertex counterterms. For this, we have to replace one coupling constant $-\mu^g$ in $\chi\chi\chi|_{k^2=0} = \mathcal{O}$ by $-m^2c_{m^2}$ and other one by $-\mu^g c_g$, leading to:

$$
\frac{1}{2} \mathcal{O} = \frac{m^2g^2}{(4\pi)^2} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \psi(1) + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right] S_{(\mathcal{O})} \number{3.62}
$$
3.3. MINIMAL SUBTRACTION SCHEME

Now, renormalization constant up to $g^3$ is given by

$$Z_{m^2}(g, \epsilon^{-1}) = 1 + c_{m^2} = 1 + \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^2 + \frac{1}{m^2} \left[ \frac{1}{8} K\left(\frac{1}{4}\right) + \frac{1}{8} K\left(\frac{1}{8}\right) + \frac{1}{4} K\left(\frac{1}{4}\right) + \frac{1}{4} K\left(\frac{1}{8}\right) + \frac{1}{2} K\left(\frac{1}{2}\right) + \frac{1}{2} K\left(\frac{1}{8}\right) \right]$$

$$= 1 + \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^2 + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^3 + \cdots$$

(3.63)

We have seen that mass renormalization constant ($Z_{m^2}$) up to two-loop diagrams is of the order of $(N/3\epsilon)^2$. By increasing the number of loops, renormalization constant is increased by the same order the loops are increased. This leads us to conjecture a generalized form of $Z_{m^2}$ up to n-loops (i.e., renormalization constant for the perturbative expansion of n-loop Feynman graphs):

$$Z_{m^2}(g, \epsilon^{-1}) = 1 + c_{m^2} = 1 + \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^2 + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^3 + \cdots + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^n$$

(3.64)

If we have considered all leading and subleading diagrams instead of only leading order diagrams, we would also find out the subleading terms (in N) along with the leading order terms corresponding to each loop correlation function. One can still find out renormalization in large N-limit by ignoring the subleading terms. This also gives the similar result as we have found above.

3.3.2 Four-point Calculations

Four-point diagrams are considered at large N-limit. At three-loop level, we will perform all fourth order diagrams plus counterterms arising from lower order diagrams. Vertex function for four-point diagrams is

$$\Gamma^{(4)} = - \left[ \times + \frac{3}{2} \times + \bullet + \frac{3}{4} \times + \frac{3}{2} \times + 3 \bullet + 3 \times + \frac{3}{8} \times + \frac{3}{8} \times \right]$$

(3.65)

First, we will calculate renormalization constant for two-loop diagrams. In large N-limit, there are only two two-loop diagrams: $\times \times \times \times$ and $\times \times \times$. Pole term of both the diagrams and of counterterms are given below:
\[ \frac{3}{4} \begin{array}{c} \bigcirc \\ \bigcirc \end{array} = -3 g \frac{g^2}{(4\pi)^4} \left[ \frac{1}{e} + \frac{1}{e} \psi(1) + \frac{1}{e} \int_0^1 dx \log \left[ \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right] + O(\epsilon^0) \right] S_{\bigcirc \bigcirc} \]

\[ = -g \frac{g^2}{(4\pi)^4} \left[ \frac{1}{e} + \frac{1}{e} \psi(1) + \frac{1}{e} \int_0^1 dx \log \left[ \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right] + O(\epsilon^0) \right] \frac{N^2}{3^2} \] (3.66)

\[ \frac{3}{2} \begin{array}{c} \bigcirc \end{array} = 3 g \frac{g^3}{(4\pi)^6} \left[ \frac{1}{e} + \frac{1}{e} \psi(1) + \frac{1}{e} \int_0^1 dx \log \left[ \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right] + O(\epsilon) \right] S_{\bigcirc \bigcirc} \]

\[ = g \frac{g^3}{(4\pi)^6} \left[ \frac{1}{e} + \frac{1}{e} \psi(1) + \frac{1}{e} \int_0^1 dx \log \left[ \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right] + O(\epsilon^0) \right] \frac{N^2}{3^2} \] (3.67)

and the corresponding counterterms:

\[ 3 \begin{array}{c} \bigcirc \end{array} = 3 g \frac{g^3}{(4\pi)^2} \left[ \frac{2}{e} + \psi(1) + \int_0^1 dx \log \left[ \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right] + O(\epsilon) \right] S_{\bigcirc \bigcirc} \]

\[ = g \frac{g^3}{(4\pi)^2} \left[ \frac{2}{e} + \psi(1) + \int_0^1 dx \log \left[ \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right] + O(\epsilon^0) \right] \frac{N^2}{3^2} \] (3.68)

and

\[ 3 \begin{array}{c} \bigcirc \end{array} = 3 g \frac{g^3}{(4\pi)^2} \left[ \frac{1}{e} + \frac{1}{e} \psi(1) + \frac{1}{e} \int_0^1 dx \log \left[ \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right] + O(\epsilon^0) \right] \frac{N^2}{3^2} \] (3.69)

Renormalization constant upto order \( g^3 \) is:

\[ Z_g(g, \epsilon^{-1}) = 1 + c_g = 1 + c_g^1 + c_g^2 \]

\[ = 1 + g \frac{N}{(4\pi)^2} + \frac{1}{g \mu} \left[ \frac{3}{4} \mathcal{K}(\bigcirc \bigcirc \bigcirc) + \frac{3}{2} \mathcal{K}(\bigcirc \bigcirc) + 3 \mathcal{K}(\bigcirc \bigcirc) + 3 \mathcal{K}(\bigcircle \bigcircle) + O(g^3) \right] \]

\[ = 1 + g \frac{N}{(4\pi)^2} + \left[ \frac{g}{(4\pi)^2} \right]^2 \] (3.70)

where

\[ c_g^1 = g \frac{N}{(4\pi)^2} , \quad c_g^2 = \frac{g^2}{(4\pi)^2} \frac{N^2}{9\epsilon^2} \] (3.71)

At three-loop level, there are five diagrams. Feynman integral corresponding to each diagram has already been calculated and dimensionally regularized in section (3.2.2). Pole terms of all these diagrams along with the corresponding \( \mathcal{O}(N) \)-symmetry factors are

\[ \frac{3}{8} \mathcal{K}(\bigcirc \bigcirc \bigcirc) = g \frac{g^3}{(4\pi)^2} \left[ \frac{1}{e^3} + \frac{3}{2e^2} \psi(1) + \frac{3}{2e^2} \int_0^1 dx \log \left[ \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right] + O(\epsilon^{-1}) \right] \frac{N^3}{3^3} \] (3.72)

\[ \frac{3}{4} \mathcal{K}(\bigcircle \bigcircle) = -g \frac{g^3}{(4\pi)^2} \left[ \frac{1}{e^2} + \frac{1}{e} \psi(1) + \frac{1}{2e} \psi(2) + \frac{1}{2e} \log \frac{4\pi \mu^2}{m^2} + \frac{1}{2e} \int_0^1 dx \log \left[ \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} \right] + O(\epsilon^0) \right] \left[ \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right] \frac{N^3}{3^3} \] (3.73)
3.3. MINIMAL SUBTRACTION SCHEME

\[
\frac{3}{4} \mathcal{K} \left( \begin{array}{c}
\end{array} \right) = g \mu^f \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} \psi(2) + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon^0) \right] \left[ \int_0^1 dx \left( \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right)^2 \right] N^3 \frac{3}{3^3} \tag{3.74}
\]

\[
\frac{3}{4} \mathcal{K} \left( \begin{array}{c}
\end{array} \right) = g \mu^f \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} \psi(2) + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon^0) \right] \left[ \int_0^1 dx \left( \frac{m^4 x(1-x)}{k^2 x(1-x) + m^2} \right) \right] N^3 \frac{3}{3^3} \tag{3.75}
\]

\[
\frac{3}{4} \mathcal{K} \left( \begin{array}{c}
\end{array} \right) = -g \mu^f \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \frac{1}{2\epsilon} \int_0^1 dx \log \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^0) \right] \times \left[ \int_0^1 dx \left( \frac{m^2 (1-x)}{k^2 x(1-x) + m^2} \right) \right] N^3 \frac{3}{3^3}. \tag{3.76}
\]

and the corresponding counterterms having mass and vertex counterparts are evaluated below.

First, we have first order mass counterterm in two-loop diagram:

\[
\frac{3}{2} \mathcal{K} \left( \begin{array}{c}
\end{array} \right) = g \mu^f \frac{g^2}{(4\pi)^4} \left[ \frac{2}{\epsilon} + \psi(1) + \int_0^1 dx \log \frac{4\pi \mu^2}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon) \right] \times \left[ \int_0^1 dx \left( \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right) \right] N^2 \frac{3^2}{3^2}
\]

\[
= g \mu^f \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} + \frac{1}{2\epsilon} \log \frac{4\pi \mu^2}{m^2} \right] \times \left[ \int_0^1 dx \left( \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right)^2 \right] N^3 \frac{3}{3^3}. \tag{3.77}
\]

Next counterterm also contains first order mass counterterm:

\[
3 \mathcal{K} \left( \begin{array}{c}
\end{array} \right) = -g \mu^f \frac{g^2}{(4\pi)^4} \left[ \frac{2}{\epsilon} + \psi(2) + \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon^0) \right] \left[ \int_0^1 dx \left( \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right)^2 \right] N^2 \frac{3^2}{3^2}
\]

\[
= -g \mu^f \frac{g^3}{(4\pi)^6} \left[ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon^0) \right] \left[ \int_0^1 dx \left( \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right) \right] N^3 \frac{3}{3^3}, \tag{3.78}
\]

This term is the combination of two terms

\[
3 \mathcal{K} \left( \begin{array}{c}
\end{array} \right) = \frac{3}{2} \mathcal{K} \left( \begin{array}{c}
\end{array} \right) + \frac{3}{2} \mathcal{K} \left( \begin{array}{c}
\end{array} \right) \tag{3.79}
\]

In order to cancel out all divergences, we must have to consider a counterterm with two independent first order mass counterterms in one propagator of one-loop diagram

\[
3 \mathcal{K} \left( \begin{array}{c}
\end{array} \right) = g \mu^f \frac{g^2}{(4\pi)^4} \left[ \int_0^1 dx \left( \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right)^2 \right] N \frac{3}{3^3} \tag{3.80}
\]

\[
= g \mu^f \frac{g^2}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} \int_0^1 dx \left( \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right)^2 + \mathcal{O}(\epsilon^{-1}) \right] N^3 \frac{3}{3^3}
\]
\[ 3\mathcal{K}(\bigotimes) = -g\mu^2 \frac{g^2}{(4\pi)^4} \epsilon_{\mu}^2 \left[ \frac{2}{\epsilon} + \psi(2) + \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right] \left[ \int_0^1 dx \frac{m^4 x(1-x)}{k^2 x(1-x) + m^2} \right] N^2 \]
\[ = -g\mu^2 \frac{g^3}{(4\pi)^6} \left[ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \psi(2) + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon^0) \right] \left[ \int_0^1 dx \frac{m^4 x(1-x)}{k^2 x(1-x) + m^2} \right] N^3 \]

(3.81)

Similarly, this one is the combination of two counterterms where both of them have similar amplitudes

\[ 3\mathcal{K}(\bigotimes) = \frac{3}{2} \mathcal{K}(\bigotimes) + \frac{3}{2} \mathcal{K}(\bigotimes) \]

(3.82)

Again, we have a diagram with two independent counterterms, each propagator carries one mass counterterm as shown below:

\[ 3\mathcal{K}(\bigotimes) = g\mu^2 \frac{g^2}{(4\pi)^4} \epsilon_{\mu}^2 \left[ \frac{1}{\epsilon} + \frac{1}{2} \psi(1) + \frac{1}{2} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right] \left[ \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right] N^2 \]
\[ = g\mu^2 \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} + \frac{1}{2\epsilon} \psi(2) + \frac{1}{2\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon^0) \right] \left[ \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right] N^3 \]

(3.83)

\[ \frac{3}{2} \mathcal{K}(\bigotimes) = g\mu^2 \epsilon_{\mu} m^2 \frac{g^2}{(4\pi)^4} \left[ \frac{1}{\epsilon} + \frac{1}{2} \psi(1) + \frac{1}{2} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right] \left[ \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right] N^2 \]
\[ = g\mu^2 \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} + \frac{1}{2\epsilon} \psi(2) + \frac{1}{2\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon^0) \right] \left[ \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right] N^3 \]

(3.84)

Now, we have first order vertex counterterm in two-loop diagram

\[ 3\mathcal{K}(\bigotimes) = g\mu^2 \epsilon_{\mu}^2 \frac{g^2}{(4\pi)^4} \left[ \frac{2}{\epsilon} + \psi(2) + \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon) \right] \left[ \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right] N^2 \]
\[ = g\mu^2 \frac{g^3}{(4\pi)^6} \left[ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \psi(2) + \frac{1}{\epsilon} \log \frac{4\pi \mu^2}{m^2} + \mathcal{O}(\epsilon^0) \right] \left[ \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} \right] N^3 \]

(3.85)

\[ 3\mathcal{K}(\bigotimes) = \frac{3}{2} \mathcal{K}(\bigotimes) + \frac{3}{2} \mathcal{K}(\bigotimes) \]

(3.86)

The following counterterm carries both mass and vertex counterterms of first order:

\[ 3\mathcal{K}(\bigotimes) = -g\mu^2 \epsilon_{\mu} \frac{g}{(4\pi)^2} \left[ \frac{1}{\epsilon} \left( \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon) \right) \right] N \]
\[ = -g\mu^2 \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} \left( \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^{-1}) \right) \right] N^3 \]

(3.87)

Next, we find out a second order mass counterterm in one-loop diagram which makes it a counterterm diagram of order \( g^5 \):

\[ 3\mathcal{K}(\bigotimes) = -g\mu^2 \epsilon_{\mu} m^2 \frac{g}{(4\pi)^2} \left[ \frac{1}{\epsilon} \left( \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon) \right) \right] N \]
\[ = -g\mu^2 \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} \left( \int_0^1 dx \frac{m^2(1-x)}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^{-1}) \right) \right] N^3 \]

(3.88)
Again, we have a first order vertex counterterm in two-loop diagram

\[
\frac{9}{4} \mathcal{K}(\bullet \bullet \bullet) = -g \mu^2 \frac{g^2}{(4\pi)^2} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \psi(1) + \frac{1}{\epsilon^2} \int_0^1 dx \log \frac{4\pi\mu^2}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^0) \right] \frac{N^2}{3^2} 
\]

\[= -g \mu^2 \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon^2} \psi(1) \right] \int_0^1 dx \log \frac{4\pi\mu^2}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^{-1}) \right] \frac{3N^3}{3^3}.
\]

(3.89)

Similarly, we have second order vertex counterterm in one-loop diagram

\[
\frac{6}{2} \mathcal{K}(\bullet \bullet) = g \mu^2 \frac{g^2}{(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{1}{\epsilon^2} \psi(1) + \frac{1}{2} \int_0^1 dx \log \frac{4\pi\mu^2}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^0) \right] \frac{2N}{3} 
\]

\[= g \mu^2 \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^3} + \frac{1}{2\epsilon^2} \psi(1) \right] \int_0^1 dx \log \frac{4\pi\mu^2}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^{-1}) \right] \frac{2N^3}{3^3}.
\]

(3.91)

and finally, we have one loop diagram with two vertex counterterms of order \(g^2\) at both vertices:

\[
\frac{3}{2} \mathcal{K}(\bullet \bullet) = g \mu^2 \frac{g^2}{(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{1}{\epsilon^2} \psi(1) + \frac{1}{2} \int_0^1 dx \log \frac{4\pi\mu^2}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^0) \right] \frac{N}{3} 
\]

\[= g \mu^2 \frac{g^3}{(4\pi)^6} \left[ \frac{1}{\epsilon^3} + \frac{1}{2\epsilon^2} \psi(1) \right] \int_0^1 dx \log \frac{4\pi\mu^2}{k^2 x(1-x) + m^2} + \mathcal{O}(\epsilon^{-1}) \right] \frac{N^3}{3^3}.
\]

(3.92)

Renormalization constant up to order \(g^4\) is:

\[
Z_0(g, \epsilon^{-1}) = 1 + c_9 = 1 + \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^2 + \frac{1}{g \mu^2} \left[ \frac{3}{8} \mathcal{K}(\bullet \bullet \bullet \bullet) + \mathcal{K}(\bullet \bullet \bullet \bullet \bullet) \right] + \frac{3}{4} \mathcal{K}(\bullet \bullet \bullet) + 3 \mathcal{K}(\bullet \bullet \bullet) + 3 \mathcal{K}(\bullet \bullet \bullet) + 3 \mathcal{K}(\bullet \bullet \bullet) + 3 \mathcal{K}(\bullet \bullet \bullet) + \mathcal{K}(\bullet \bullet \bullet) 
\]

(3.94)

Like mass renormalization constant \((Z_{m^2})\), vertex renormalization constant \((Z_0)\) up to two-loop diagrams is of the order of \((N/3\epsilon)^2\). By increasing the number of loops, renormalization constant is increased by the same order the loops are increased. In the large N-limit, the conjectured form of \(Z_{m^2}\) and \(Z_0\) are quite similar. But, it won’t be similar if one considers all leading and subleading four-point diagrams (in N). For vertex renormalization, we have

\[
Z_0(g, \epsilon^{-1}) = 1 + c_9 = 1 + \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^2 + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^3 + \cdots + \left[ \frac{g}{(4\pi)^2} \frac{N}{3\epsilon} \right]^n
\]

(3.95)
We have seen that the counterterms are used to eliminate the divergences that occur at very large momentum scales. Every loop integral depends on parameters $m/\mu$ and $\mu^\epsilon$. As long as the counterterms have order by order no finite part, we do not expect the residues of their poles to depend on $m$. The technical difficulty in finding out the solutions lies in the dependence of the loop-integrals on both $m/\mu$ and $\mu^\epsilon$. 
Chapter 4

Conclusion

A common prominent feature of QFT is the emergence of the divergence in calculations beyond the tree level. To handle these divergences, various regularization and renormalization methods have been proposed which are discussed in this thesis in the N-field $\phi^4$ theory. A QFT is defined by a lagrangian functional of the fields $L[\phi, \lambda, m]$ which depends on parameters: the coupling constants and mass parameters. In the N-field $\phi^4$ theory, coupling constant $\lambda$ is no more a constant term, it turns out as a tensor. In this thesis, every vertex of a loop Feynman diagram is represented by a tensor like $\lambda_{\alpha\beta\gamma\delta}$, where $\alpha, \beta, \gamma$ and $\delta$ are indices corresponding to each loop line. Then, using tensor algebra, we have constructed O(N) symmetry factors for special four-point, two-point and vacuum diagrams up to N-loops. Then, we turned to solve vacuum diagrams containing two-point and four-point loops as subdiagrams. Such vacuum diagrams up to order $g^7$ (containing eight loops) are given in [31] on page:258. We have chosen diagram of order $g^6$ to solve. Results of diagrams arising from four-point diagrams looks quite similar to four-point four-loop diagrams in [31] on page:466. In N-field $\phi^4$-theory, each propagator is represented by summation indices. In section (2.5), we have seen that a Feynman diagram containing N-summation indices (where all tensor indices are contracted) in all of its loops is a leading order diagram. If, there is some loop with fixed internal indices, it will make the diagram subleading (or next to leading order) diagram. Our goal is to renormalize the higher loop diagrams in large N-limit. In renormalization, we subtract infinite constants and absorb infinite constants into parameters. We wanted to perform the two-point renormalization to all leading and subleading diagrams. The subleading diagrams especially, contains overlapping divergences. We tried to solve \[\text{\phantom{0000}\phantom{0000}\phantom{0000}}\] by lowering the degree of divergences via partial integration in which surface term was discarded. Because of some technical difficulties arising, we were not able to move ahead along with this diagram. So, we neglected all subleading diagrams and performed two-point renormalization in the large N-limit. We can say that for simplicity, we did renormalization for only the leading order diagrams. In large N-limit, renormalization is done by choosing only leading order diagrams in perturbative expansion of 1PI diagrams. We have presented analytical calculations in dimensional regularization of all two-point and four-point leading order diagrams (in N) up to order $g^3$ and $g^4$ respectively in the large N-limit. One can observe that for increasing number of loops L, the Feynman integral corresponding to that loop diagram posses singularities of the type $1/\epsilon^i$; $(i=1, \cdots, N)$. These divergences turn out to contain all information on critical exponent in $4-\epsilon$ dimensions. These divergences have been removed using Minimal subtraction scheme, where we have to choose counterterms $c_{m^2}$ and $c_g$ arising from lower order diagrams. These counterterms depends on $g$, $\epsilon$ and $m^2/\epsilon$. All terms containing logarithms and $\psi(n)$ will be cancelled as explicitly shown in section (3.3.2). When, working
in large N-limit, finite observables can be obtained by multiplying each correlation function by compensating \( O(N) \)-symmetry factors. Mass and vertex renormalization constants obtained are exactly similar to each other as given in Eqs. (3.64) and (3.95). It would also be interesting to consider next to leading N. The general formulas for \( O(N) \)-symmetry factors would tell us which diagrams to contribute in next to leading N.
Appendix A

Seven-loop O(N)-symmetry factors

Diagram 2(d) consists of two two-loop four-point functions which can be calculated using Eq.(2.23)

\[ W_{\alpha\beta\gamma\delta}(2) = \frac{f(l, N)}{3^3} \delta_{\alpha\beta} \delta_{\gamma\delta} + \frac{2^2}{3^3} \delta_{\alpha\gamma} \delta_{\beta\delta} + \frac{2^2}{3^3} \delta_{\alpha\delta} \delta_{\beta\gamma} \] (A.1)

where

\[ f(l, N) = \sum_{p=1}^{3} \prod_{n=0}^{p-1} (l + 1 - n) \frac{n!}{p!} = N^2 + 6N + 12 \] (A.2)

Hence,

\[ W_{\alpha\beta\gamma\delta}(2) = \frac{N^2 + 6N + 12}{3^3} \delta_{\alpha\beta} \delta_{\gamma\delta} + \frac{4}{3^3} \delta_{\alpha\gamma} \delta_{\beta\delta} + \frac{4}{3^3} \delta_{\alpha\delta} \delta_{\beta\gamma} \] (A.3)

and the other two-point loop looks like

\[ W_{\alpha\gamma\beta\delta}(2) = \frac{N^2 + 6N + 12}{3^3} \delta_{\alpha\gamma} \delta_{\beta\delta} + \frac{4}{3^3} \delta_{\alpha\beta} \delta_{\gamma\delta} + \frac{4}{3^3} \delta_{\alpha\delta} \delta_{\gamma\beta} \] (A.4)

and their product gives result for Diagram 2(d)

\[ W_{\alpha\beta\gamma\delta}(2)W_{\alpha\gamma\beta\delta}(2) = \frac{N(N + 2)(N^3 + 18N^2 + 80N + 136)}{3^6} \] (A.5)

Diagram 2(f) takes the following form in tensor notations

\[ W_{\alpha\beta\gamma\delta}(2)W_{\gamma\alpha\delta\beta}(0)W_{\beta\gamma\delta\alpha}(1) = W_{\alpha\beta\gamma\delta}(2)U_{\beta\gamma\delta\alpha}(1) \] (A.6)

where

\[ W_{\alpha\beta\gamma\delta}(2) = \frac{(N^2 + 6N + 12)}{3^3} \delta_{\alpha\beta} \delta_{\gamma\delta} + \frac{4}{3^3} \delta_{\alpha\gamma} \delta_{\beta\delta} + \frac{4}{3^3} \delta_{\alpha\delta} \delta_{\beta\gamma} \] (A.7)

and

\[ U_{\beta\gamma\delta\alpha}(1) = \frac{3N + 10}{3^3} \delta_{\beta\gamma} \delta_{\delta\alpha} + \frac{N + 6}{3^3} \delta_{\beta\alpha} \delta_{\delta\gamma} + \frac{N + 6}{3^3} \delta_{\beta\delta} \delta_{\alpha\gamma} \] (A.8)

which gives the result for 2(f)

\[ U_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta}(2)U_{\beta\gamma\delta\alpha} = \frac{N(N + 2)(N^3 + 14N^2 + 76N + 152)}{3^6} \] (A.9)
Diagram 3(g) contains the following form

\[
W_{\alpha\beta\gamma\delta_1}(1)W_{\gamma_1\gamma\delta_2}(1)W_{\delta\delta_1}(1) = U_{\alpha\beta\gamma}(1)W_{\delta\delta_1}(1)
\]

(A.10)

\[
U_{\alpha\beta\gamma}(2) = W_{\alpha\beta\gamma\delta_1}(1)W_{\gamma_1\gamma\delta_2}(1) = \frac{3(N + 4)(N + 2)}{34}\delta_{\alpha\beta}\delta_{\gamma}\delta_{\delta_1} + 2\frac{(N + 6)}{34}\delta_{\alpha\beta}\delta_{\gamma}\delta_{\delta_2}
\]

(A.11)

this leads to

\[
U_{\alpha\beta\gamma}(2)W_{\delta\delta_1}(1) = \frac{N(N + 2)(11N^2 + 76N + 132)}{34}
\]

(A.12)

Diagram 2(h) We are interested in finding the symmetry factors for loop diagrams, while diagram 2(h) contains no loop so, we will skip this diagram.

Diagram 2(i) Now, turn to the next diagram 2(i) which contains only one loop that makes it quite complicated. One loop can be calculated using the formula described in Eq.(2.23). For all other interactions, we have to perform calculations to get the final result. The explicit form of this diagram in tensor notations is:

\[
\lambda^{(3)}_{\alpha\beta\sigma\alpha'}\lambda^{(4)}_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}\lambda^{(5)}_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}\lambda^{(1)}_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}\lambda^{(6)}_{\sigma_6\sigma_7\sigma_8\sigma_9\sigma_3}
\]

(A.13)

where the numbers in the upper indices stand for vertex positions in vacuum diagram. We can see that the tensor pair (1,6) has one loop that means it will contract. So, it becomes

\[
\lambda^{(1)}_{\sigma_7\sigma_8\sigma_9\sigma_3}\lambda^{(6)}_{\sigma_0\sigma_1\sigma_2\sigma_3} = \frac{\lambda^2}{9}(N + 4)\delta_{\sigma_7}\delta_{\sigma_8} + 2\delta_{\sigma_7}\delta_{\sigma_9} + \delta_{\sigma_7}\delta_{\sigma_8} + \delta_{\sigma_8}\delta_{\sigma_9}
\]

(A.14)

Now, the next contraction with the next vertex looks like

\[
W^{(2,1,6)}_{\sigma_7\sigma_8\sigma_9\sigma_3} = \lambda^{(3)}_{\lambda^{(2)}_{\sigma_0\sigma_1\sigma_2\sigma_3}} = \frac{\lambda^3}{9}(N + 4)(\delta_{\sigma_7}\delta_{\sigma_8}\delta_{\sigma_9} + \delta_{\sigma_7}\delta_{\sigma_8}\delta_{\sigma_9} + \delta_{\sigma_7}\delta_{\sigma_9}\delta_{\sigma_9})
\]

(A.15)

Interaction with the next vertex named (5) gives too many terms as given below

\[
\lambda^{(5)}_{\sigma_2\sigma_4\sigma_6\sigma_7}\lambda^{(1,6)}_{\sigma_7\sigma_8\sigma_9\sigma_3} = \lambda^{(2)}_{\sigma_8\sigma_9\sigma_3} = \\
= (N + 4)\left[\delta_{\sigma_7}\delta_{\sigma_8}\delta_{\sigma_9} + \delta_{\sigma_7}\delta_{\sigma_8}\delta_{\sigma_9} + \delta_{\sigma_7}\delta_{\sigma_8}\delta_{\sigma_9} + \delta_{\sigma_7}\delta_{\sigma_8}\delta_{\sigma_9} + \delta_{\sigma_7}\delta_{\sigma_8}\delta_{\sigma_9}ight]
\]

(A.16)

1 Note that we use notation $W$ for the product of two $\lambda$ tensors and $U$ for the product of two $W$ tensors.
Now, when it interacts with the next vertex (4), lots of terms get contracted and we get a much simplified expression:

\[
\lambda^{(4)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \lambda^{(5)}_{\sigma_5 \sigma_6 \sigma_7} W^{(1,6)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} W^{(2)}_{\sigma_5 \sigma_6 \sigma_7} \lambda^{(3)}_{\sigma_8 \sigma_9 \sigma_{10}} = \\
\lambda^5 \left[ (3N^2 + 26N + 64)\delta_{\alpha_1} \delta_{\sigma_1} + (N^2 + 18N + 56)\delta_{\alpha_1} \delta_{\beta_2} + (N^2 + 18N + 56)\delta_{\alpha_2} \delta_{\beta_1} \right]
\]

(A.17)

Final interaction with vertex (3) makes it a vacuum diagram and

\[
= \lambda^6 N(N+2)(5N^2 + 18N + 62N + 184)
\]

(A.18)

Hence, the O(N) symmetry factor for 3(i) is

\[
= \frac{N(N+2)(5N^2 + 18N + 62N + 184)}{3^6}
\]

(A.19)

**Diagram 2(j)** is almost similar to 2(c) with a bit change of indices. It takes the following form:

\[
W_{\alpha \beta \gamma \delta}(0) W_{\gamma_1 \delta_1}(1) W_{\delta_1 \alpha_1 \beta_1}(0) W_{\alpha_1 \gamma_1 \beta_1}(1) = U_{\alpha \beta \gamma_1 \delta_1}(1) U_{\delta_1 \beta \gamma_1 \alpha}(1)
\]

(A.20)

where

\[
U_{\alpha \beta \gamma_1 \delta_1}(1) = \frac{3N + 10}{3^3} \delta_{\alpha_1} \delta_{\gamma_1} + \frac{N + 6}{3^3} \delta_{\alpha_1} \delta_{\gamma_1} + \frac{N + 6}{3^3} \delta_{\alpha_1} \delta_{\gamma_1}
\]

(A.21)

and

\[
U_{\delta_1 \beta \gamma_1 \alpha}(1) = \frac{3N + 10}{3^3} \delta_{\delta_1} \delta_{\beta \gamma_1} + \frac{N + 6}{3^3} \delta_{\delta_1} \delta_{\beta \gamma_1} + \frac{N + 6}{3^3} \delta_{\delta_1} \delta_{\beta \gamma_1}
\]

(A.22)

Eq.(A.13) becomes

\[
U_{\alpha \beta \gamma_1 \delta_1}(1) U_{\delta_1 \beta \gamma_1 \alpha}(1) = \frac{N(N+2)(7N^2 + 72N + 164)}{3^6}
\]

(A.23)

**Diagram 2(k)** is similar to (e) except number of loops are changed. For this, we use Eq.(2.27) with \(l = 3\) and \(l' = 1\)

\[
W_{\alpha \beta \gamma \delta}(3) W_{\alpha \delta \beta}(3) = \frac{N(N+2)^2(N+8)^2}{3^6}
\]

(A.24)

**Diagram 2(l)**. In this diagram, we are able to extract two-point one-loop diagram .

\[
\frac{6}{3} = \frac{N + 2}{3}
\]

(A.25)

Rest part of the diagram becomes:

\[
W_{\alpha \beta \gamma \delta}(1) W_{\delta \alpha \beta \gamma}(1) = \frac{(N+2)(5N+22)}{3^4}
\]

(A.26)

product of Eq.(A.18),(A.19) gives

\[
= \frac{(N+2)^2(5N+22)}{3^5}
\]

(A.27)
Here N comes because two terms get multiply with N internal indices. Diagram 2(m) is similar to that of 2(l) with small change of indices that makes some change in symmetry factor.

\[
W_{\alpha\beta\gamma\delta}(1)W_{\gamma\delta_1\delta_1\gamma_1}(1)W_{\gamma_1\beta\delta_1\alpha}(1) = U_{\alpha\beta\delta_1\gamma_1}(2)W_{\gamma_1\beta\delta_1\alpha}
\]  
(A.28)

\[
U_{\alpha\beta\delta_1\gamma_1}(2) = \frac{3N^2 + 18N + 32}{3^4}\delta_{\alpha\beta}\delta_{\delta_1\gamma_1} + \frac{2(N + 6)}{3^4}\delta_{\alpha\delta_1}\delta_{\beta\gamma_1} + \frac{2(N + 6)}{3^4}\delta_{\alpha\gamma_1}\delta_{\beta\delta_1}
\]  
(A.29)

substituting this in A.21 and performing simpler calculations, gives

\[
= \frac{N(N + 2)(11N^2 + 76N + 156)}{3^6}
\]  
(A.30)

**Diagram 2(n)** Here, we have three one-loops which are independent from each other. Mathematical expression in tensor notations is:

\[
W_{\alpha\beta\gamma\delta}(1)W_{\delta_\sigma_1\sigma_2\beta}(1)W_{\sigma_2\alpha\sigma_1\gamma}(1) = U_{\alpha\sigma_2\sigma_1\gamma}(1)W_{\delta_\gamma_1\gamma_1\alpha}(1)
\]  
(A.31)

\[
U_{\alpha\sigma_2\sigma_1\gamma} = \frac{N^2 + 8N + 20}{3^4}\delta_{\alpha\sigma_2}\delta_{\sigma_1\gamma} + \frac{4(N + 4)}{3^4}\delta_{\alpha\sigma_1}\delta_{\sigma_2\gamma} + \frac{8(N + 3)}{3^4}\delta_{\alpha\gamma}\delta_{\sigma_1\sigma_2}
\]  
(A.32)

substitute the expression for U in (A.31) and take the product with W tensor gives for 3(n)

\[
= \frac{N(N + 2)(N^3 + 10N^2 + 72N + 160)}{3^6}
\]  
(A.33)

**Diagram 2(o)** we will use Eq.(2.23) to evaluate this diagram. First, let us write it in explicit form:

\[
W_{\alpha\beta\delta_1\gamma_1}(1)W_{\gamma_1\delta_1\beta}(1)W_{\delta_\gamma_1\gamma_1\alpha}(1) = U_{\alpha\gamma_1\beta\delta}(2)W_{\delta_\gamma_1\gamma_1\alpha}(1)
\]  
(A.34)

\[
U_{\alpha\gamma_1\beta\delta}(2) = \frac{8(N + 3)}{3^4}\delta_{\alpha\gamma_1}\delta_{\beta\delta} + \frac{4(N + 4)}{3^4}\delta_{\alpha\gamma_1}\delta_{\gamma_1\delta} + \frac{N^2 + 8N + 20}{3^4}\delta_{\alpha\delta}\delta_{\gamma_1\gamma}
\]  
(A.35)

Finally, we get

\[
= \frac{N(N + 2)(7N^2 + 72N + 164)}{3^6}
\]  
(A.36)

**Diagram 2(p)** is simply the cube of two-point two-loops multiplied by N.

\[
= \frac{N(N + 2)^3}{3^3}
\]  
(A.37)

**Diagram 2(q)** is almost similar to 2(p). Here, we can extract two two-point two-loops. Then, we will have a closed three loop diagram. Make a cut on outer loop, it becomes two point two-loop diagram. Now, we have three two-loop diagrams multiplied by $N^2$ (one N for closed loop and one for closing cut in 4-loop diagram):

\[
= \frac{N^2(N + 2)^3}{3^3}
\]  
(A.38)
Appendix B

Important Formulas

\[
\int \frac{d^D p}{(2\pi)^D (p^2 + m^2)^\alpha} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\alpha - D/2)}{\Gamma(\alpha)} \frac{1}{(m^2)^{\alpha - D/2}}. \quad (B.1)
\]

\[
\int \frac{d^D p}{(2\pi)^D (p^2 + 2pq + m^2)^\alpha} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\alpha - D/2)}{\Gamma(\alpha)} \frac{1}{(m^2 - q^2)^{\alpha - D/2}}. \quad (B.2)
\]

\[
\int \frac{d^D p}{(2\pi)^D (p^2 + m^2)^\alpha \left[(p-k)^2 + m^2\right]^\beta} = \frac{\Gamma(\alpha + \beta - D/2)}{(4\pi)^{D/2} \Gamma(\alpha) \Gamma(\beta)} \times \int_0^1 dx \frac{x^{\beta-1} (1-x)^{\alpha-1}}{[k^2 x (1-x + m^2)]^{\alpha + \beta - D/2}} \quad (B.3)
\]

\[
\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left[ \frac{1}{\epsilon} + \psi(n+1) + \frac{\epsilon}{2} \left( \frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1) \right) + O(\epsilon^2) \right]. \quad (B.4)
\]

\[
\Gamma(n + 1 + \epsilon) = n! \left[ 1 + \epsilon \psi(n+1) + \frac{\epsilon^2}{2} \left( \psi'(n+1) + \psi(n+1)^2 \right) + O(\epsilon^3) \right]. \quad (B.5)
\]

\[
B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^1 dy \, y^{\alpha-1} (1-y)^{\beta-1}. \quad (B.6)
\]

\[
\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1} (1-x)^{\beta-1}}{(Ax + B(1-x))^{\alpha+\beta}} \quad (B.7)
\]

\[
\frac{1}{A_1^{\alpha_1} \cdots A_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 + \cdots + \alpha_n)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} \int_0^1 dx_1 \cdots \int_0^1 dx_n \frac{\delta(1 - x_1 + \cdots + x_n) x_1^{\alpha_1-1} x_n^{\alpha_n-1}}{(x_1 A_1 + \cdots + x_n A_n)^{\alpha_1+\cdots+\alpha_n}} \quad (B.8)
\]
Bibliography


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