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Transmission Strategies for Gaussian Relay Networks Obtained from Deterministic Models

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Abstract—A constant gap between the deterministic model of a class of a wireless relay network and its Gaussian model counterpart is derived. The method is constructive in the sense that a transmission solution in the deterministic model that obeys certain constraints can directly be translated into a transmission in the Gaussian model. We show that the rate in the Gaussian model is at most a constant gap below the rate in the deterministic model, and we derive an upper bound on this gap.

I. INTRODUCTION

In general, wireless communication is limited in time, power and bandwidth. The principle of conventional communication systems is to divide those resources among the transmissions that share the medium, e.g. in a TDMA or FDMA fashion, and thus creating orthogonal point-to-point links. Though, findings in information theory indicate, that there can be a large gain in transmission rate if interference is not avoided, but handled in a clever way. For instance, the rate of each user in a K -user interference channel can be shown to scale asymptotically with $1/2$, instead of with $1/K$ as one might expect [1].

However, capacity is known for few special cases and for small networks only. Even the capacity of the three-node relay channel is unknown, and so is the optimal transmission scheme. One way to proceed is to change the Gaussian channel model for a model that simplifies analysis, yet captures the main attributes of the problem. In [2] a deterministic model of a wireless network consisting of a single source, a single destination and a number of helping relays was introduced. The main result of this work is that the cut-set bound $\bar{C}_{\text{det}}$ is achievable in the deterministic model. Subsequent results show that the capacity of a Gaussian relay network is within a constant gap to the respective cut-set bound $\bar{C}_{\text{Gauss}}$ [3]. This suggests a way to approximate the capacity of a Gaussian relay network by the capacity of the corresponding deterministic relay network. Unfortunately, the gap between the two cut-set bounds can be unbounded [4] and therefore there is no immediate connection between the deterministic and the Gaussian model.

In this paper we use a constructive approach to establish such a connection for a class of partially coordinated deterministic models. Given a Gaussian relay network and the deterministic counterpart we propose a transmission scheme for the Gaussian network, that achieves a rate that is at most a constant below the rate of the deterministic network. In this way we show that in the cases where the Gaussian network

can be modeled as a partially coordinated deterministic network, the deterministic model correctly predicts the asymptotic throughput of the Gaussian network. Moreover, our scheme uses simple uncoded transmissions and is therefore suitable for implementation in a system that is constrained in detection complexity and transmission delay. We use a hierarchical type of modulation [5] to separate multiple information flows. Comparable approaches have been used in [6] on noiseless channels and in [7] on real-valued Gaussian channels.

Transmissions from different transmitters can be resolved if they are received on different power levels, resembling the different bit-levels in the deterministic model. The scheme used in this paper allows for multiple transmissions on the same power level and extends the work presented in [8]. The transmissions are resolved in a successive interference cancellation fashion that resembles the resolution of modulo-2 sums in the deterministic model.

The remainder of the paper is structured as follows. Sec. II and III introduce the channel models and the transmission scheme, respectively. In Sec. IV the main result, an upper bound on the gap G between the deterministic and the Gaussian channel model is derived, and Sec. V concludes the paper.

II. CHANNEL MODELS

A. Gaussian Channel Model

Consider a network of N nodes. The purpose of the network is to transmit information from a single source node 1 to a single destination node N (unicast). The remaining relay nodes $2, \dots, N-1$ are not interested in the information, but they can help in the transmission. In the Gaussian channel model, the received signal at node j is modeled as

$$y_j = \sum_{i=1, i \neq j}^N h_{ij}x_i + z_j, \quad j = 2, \dots, N. \quad (1)$$

The complex-valued signal consists of the sum of the signals $x_i \in \mathbb{C}$ transmitted at nodes i , weighted by the channel gains $h_{ij} \in \mathbb{C}$, and additive white Gaussian noise $z_j \sim \mathcal{CN}(0, 1)$. The transmit power of each node is constrained to $\mathbb{E}[|x_i|^2] = 1$, and we assume complete channel knowledge at all nodes.

The capacity of this network is the maximum amount of information that can be sent from source to destination at an error rate that tends to zero. The maximization, however, is over all possible transmitted signals x_i , that is over all

Fig. 1. Example network in deterministic model with $n_{13} = 4$ and $n_{23} = 2$.

possible transmission strategies. Finding the optimum transmission strategy, that is finding the capacity for a general relay network, is extremely difficult, even if the network consists of only three nodes.

B. Linear Finite-Field Deterministic Model

A way to characterize capacity is to simplify the channel model. In [2] a deterministic model has been introduced, that captures the main effect in wireless networks (interference), but simplifies the effect of noise. The main idea is that a node i can send

$$n_{ij} = \left\lceil \log_2 |h_{ij}|^2 \right\rceil = \left\lceil \log_{\sqrt{2}} |h_{ij}| \right\rceil \quad (2)$$

bits to a node j , where $|h_{ij}|^2$ is the signal-to-noise ratio (SNR) of the link from node i to node j . Accordingly, it is useful to express the transmitted signals $\mathbf{x}_i \in \mathbb{F}_2^{n_{\text{mx}}}$ and the received signals $\mathbf{y}_j \in \mathbb{F}_2^{n_{\text{mx}}}$ as vectors of bits. Each vector has the same length $n_{\text{mx}} = \max_{ij} n_{ij}$.

The point-to-point connection from node i to node j is modeled such that the n_{ij} most significant bits in $\mathbf{x}_i$ are received at the n_{ij} least significant bits in $\mathbf{y}_j$. Consider only nodes 2 and 3 in Fig. 1 for an example. In this case $n_{23} = 2$ bits can be transmitted, and they are depicted as arrows.

More formally, the received signal is expressed as $\mathbf{y}_j = \mathbf{S}^{n_{\text{mx}} - n_{ij}} \mathbf{x}_i$, where the operation $\mathbf{S}^{n_{\text{mx}} - n_{ij}}$ shifts the bits in $\mathbf{x}_i$ down $n_{\text{mx}} - n_{ij}$ positions.

The interesting part is how to model interference. We extend the point-to-point channel in the natural way to the interference channel

$$\mathbf{y}_j = \sum_{i=1, i \neq j}^N \mathbf{S}^{n_{\text{mx}} - n_{ij}} \mathbf{x}_i, \quad j = 2, \dots, N. \quad (3)$$

Since all operations are in $\mathbb{F}_2$, the summation in (3) is modulo-2. That is the deterministic way of modeling interference is the modulo-2 summation of all bits that are received on the same level. Consider Fig. 1 again for an example. The two least significant bits of $\mathbf{y}_3$ consist of interfering bits sent by the nodes 1 and 2. The modulo-2 summation is denoted as $\oplus$.

As in the Gaussian model, the capacity in the deterministic model is the maximum number of bits, that the source 1 can transmit to the destination N . Again, the maximization is over all possible signals $\mathbf{x}_i$, but due to the simpler channel model, the optimal transmission is relatively easy to find. In [9] it has

been shown that this task can be solved by an algorithm that has polynomial complexity.

C. Generalized Deterministic Model

We can generalize the deterministic model by taking the logarithm in (2) to a general base $\vartheta \geq \sqrt{2}$,

$$n_{ij} = \lceil \log_{\vartheta} |h_{ij}| \rceil.$$

If we are interested in h_{ij} as a function of n_{ij} , we might formulate

$$h_{ij} = \vartheta^{n_{ij}} g_{ij}, \quad (4)$$

using the variable $g_{ij} \in \mathbb{C}$ to compensate for the absolute value operation, that is $\arg(g_{ij}) \in [0, 2\pi)$, and for the ceiling operation, that is $|g_{ij}| \in (\vartheta^{-1}, 1]$. We will need this back translation from the deterministic model to the Gaussian model in the next section.

III. TRANSMISSION SCHEME

We have argued that the simplified deterministic model can be used to determine the capacity of a wireless relay network, because the optimum transmission strategy is relatively simple to find. The main idea of this paper is to find a way to translate this optimum solution from the deterministic model into a transmission in the Gaussian model. Specifically, we show that this translation is possible, if the solution in the deterministic model fulfills certain constraints, and we show that the rate we achieve in the Gaussian model is at most a constant gap of G bits below the rate in the deterministic model.

The gain of this approach is twofold. On the theoretical side, we show that the capacity of the Gaussian network can be approximated by the capacity of the deterministic network. The approximation gets accurate for high SNR, since the gap G becomes negligible. On the practical side, we provide a strategy to utilize the deterministic model for designing transmissions in the Gaussian model.

In this paper we analyze a relatively simple transmission strategy for the Gaussian model based on hierarchical modulation [5]. The strategy allows for instantaneous and low complexity encoding and decoding. Therefore it is suitable for implementation in networks, that are constrained in computational complexity and delay, such as wireless sensor networks or wireless control applications.

Due to the simplicity of the transmission strategy, the solution in the deterministic model has to be partially coordinated in order to allow for the translation into the Gaussian model. That is it has to fulfill a set of constraints. Partial coordination is a relaxation of full coordination that was analyzed in [8].

Definition 1: A solution in the deterministic model is referred to as *partially coordinated* if the following constraints are fulfilled:

- 1) Bits are transmitted in blocks. Each block consists of Q adjacent bits, and we call such a block a qit.
- 2) A receiver is allowed to receive interfering bits on the $Q(G + S)$ lowest bit-levels, where S is a design parameter. On the bit-levels above only single bits are allowed.

Fig. 2. Example of partially coordinated reception and transmission with $Q = 3$, $G = 1$ and $S = 3$. Qits and superpositions of qits are depicted as thin and thick arrows, respectively

- 3) The receiving node j resolves a number of ρ_j qits $q_1, q_2, \dots, q_{\rho_j}$ in a successive manner. Each qit is affected by a number of qits, and a qit can only be resolved if all affecting qits have been resolved before. Formally, the set

$$\mathcal{S}_j = \{q_i : q_i \text{ is affecting } q_j\}.$$

contains the qits q_i affecting q_j . In order to allow for the successive resolution, the sets have to fulfill

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \dots \subset \mathcal{S}_{\rho_j}. \quad (5)$$

We say that q_i is affecting q_j , if q_i is interfering with the highest occurrence of q_j . That is if any of the bits of q_i are received on the same or on a higher bit-level than the lowest bit of the highest occurrence of q_j (see example below).

- 4) Acting as a transmitter, the node i may transmit the ρ_i resolved qits at any position in the transmitted signal $\mathbf{x}_i$. The qits that were not resolved have to remain in the same order, but can be shifted up or down.
- 5) Transmissions form a layered network, that is each path from source to destination has an equal length of H hops.

Fig. 2 shows an example of a partially coordinated solution for a single node. Transmissions of qits approach from above, and each qit consists of $Q = 3$ bits. For this example we chose $G = 1$ and $S = 3$, hence reception of interfering bits is allowed on the lowest $Q(G + S) = 12$ bit-levels on the left side of the dashed line. The highest occurrence of each qit is marked by a circle. We find that $\mathcal{S}_1 = \emptyset$, because $\widehat{q_1}$ is not affected by any other qit. However, q_1 affects $\widehat{q_2}$, and therefore $\mathcal{S}_2 = \{q_1\}$. The same way, we find $\mathcal{S}_3 = \{q_1, q_2\}$, $\mathcal{S}_4 = \{q_1, q_2, q_3\}$ and $\mathcal{S}_5 = \{q_1, q_2, q_3, q_4\}$. Hence the sets include each other according to $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3 \subset \mathcal{S}_4 \subset \mathcal{S}_5$. Note that $\widehat{q_6}$ and $\widehat{q_7}$ affect each other, therefore $\mathcal{S}_6 \not\subset \mathcal{S}_7$ and $\mathcal{S}_7 \not\subset \mathcal{S}_6$. Thus, in this example the node can resolve $\rho = 5$ qits.

The node may forward the resolved qits on any position in its transmitted signal depicted in the lower part of Fig. 2. In this example, it forwards the qits q_2 and q_5 . The qits q_6 and q_7 could not be resolved, thus if the node choses to forward them, it can only shift them to a different position.

As mentioned before, we translate the partially coordinated solution into a transmission in the Gaussian model. A qit is translated into a symbol from a quadrature phase-shift keying

(QPSK) constellation. Like in the deterministic model, a node can send multiple qits by sending a sum of QPSK symbols that have different amplitudes. This approach is known as hierarchical modulation and we specify it in the following definition.

Definition 2: Hierarchical modulation amounts to transmitting a signal

$$x_i = a \sum_{k=0}^{t_i-1} \vartheta^{-b_k} m_k + a \vartheta^{-b_{t_i}} M_i, \quad (6)$$

that consists of t_i QPSK symbols $m_k \in \{1 + j, 1 - j, -1 + j, -1 - j\}$ and one super-symbol M_i described in Lem. 1. The symbols are separated by different scaling factors ϑ^{b_k} with $2 < \vartheta \in \mathbb{R}$. The scaling exponents $b_k \in \mathbb{N}$ start at $b_0 = 0$ and have a spacing of $b_k - b_l \geq Q$ for $k > l$. The factor a is adjusted to meet the power constraint.

Consider a point-to-point connection first. The receiver receives the signal x_i in additive noise. The superimposed symbols are separated in amplitude by a factor of at least ϑ^Q . Therefore they can be resolved successively starting with the one that has largest amplitude. The resolution is successful if ϑ^Q is sufficiently large, that is if the interference imposed by the symbols at lower amplitudes is small enough.

The received signal y_j in a wireless relay network can be resolved accordingly, if the qits in the deterministic model are partially coordinated. Since nodes have complete channel knowledge, the interference caused by a symbol can be subtracted from the received signal once the symbol was resolved. Due to (5), partial coordination guaranties that the symbol that should be resolved is sufficiently stronger than the remaining interfering symbols on lower signal levels.

If the receiver cannot resolve all received symbols, it may forward the remainder of its received signal and hope for subsequent resolution by another node. The remainder forms the super-symbol M_i and the approach is equivalent to forwarding modulo-2 sums in the deterministic model. Hence, our scheme can be seen as adaptive decode-and-forward and amplify-and-forward.

In our system there are two parameters to choose. Firstly, the gap G determines a number of least significant QPSK symbols that the receivers do not resolve. The reason for this is to combat noise and to assure reception below a given probability of error. A higher G results in a higher SNR at the reception. However, those G symbols cannot be used for transmission, therefore G is the gap between the qits resolved by the nodes in the deterministic model and the symbols resolved in the Gaussian model, respectively.

Secondly, S is the number of qit-levels at which reception of interfering qits, that is reception of super-symbols is allowed. A higher S results in more degrees of freedom in finding the solution in the deterministic model, because the constraints due to Def. 1 are relaxed. Though, super-symbols carry some amount of noise, that means that a higher S results in lower SNR. Th. 1 shows the trade-off between S and G .

IV. PERFORMANCE ANALYSIS

In this section we formulate our main result, Th. 1. The proof requires a number of lemmata, that will be proved first.

Theorem 1: A partially coordinated transmission in the deterministic model can be translated into a transmission in the Gaussian model by using hierarchical modulation. The number of symbols that a node can resolve in the Gaussian model at a bit error rate below P_b is at most a constant gap

$$G \leq (S-1)(H-2) + \frac{1}{Q} \log_{\vartheta} \left(\frac{2Q^{-1}(P_b)\sqrt{N^{H-2}}}{\vartheta^{-(H-1)} - 4\sqrt{2}N^{H-1}\vartheta^{-Q}} \right)$$

below the number of qits that the node can resolve in the deterministic model. As mentioned before, H is the number of hops from source to destination, and S is the number of levels that may receive super-symbols. The inverse Q-function is denoted as $Q^{-1}(\cdot)$.

Before formulating the first lemma, we represent the received signal y_j in a layered structure. With (4) and (6) in (1),

$$y_j = a \sum_{l=-\infty}^{q_{\max}} \vartheta^{Ql} \sum_{t:l_t=l} \vartheta^{-c_t} g_t m_t + a \sum_{i,i \neq j} \vartheta^{Ql_i} \vartheta^{-c_i} g_i M_i + z_j. \quad (7)$$

We grouped the symbols with respect to their exponents $n_{ij} - b_k$. Specifically, for each symbol m_t find the layer $l_t = \lceil (n_{ij} - b_k)/Q \rceil$ at which the symbol is received, so that $n_{ij} - b_k = Ql_t - c_t$. The maximum number of symbols is $q_{\max} = \lfloor n_{\max}/Q \rfloor$.

For our analysis it is convenient to split the received signal y_j into a noiseless signal part $y_j^{\sim}$ and a noise part $y_j^{\mathcal{Z}}$.

Lemma 1: The signal part of the received signal y_j and the signal part of the super-symbol that the node j forwards after h transmissions have a layered structure as follows.

$$\begin{aligned} y_{j,h}^{\sim} &= a \sum_{l=-\infty}^{q_{\max}} \vartheta^{Ql} \sum_{t \in \mathcal{P}_{l,h}} m_t \prod_{k=1}^{h-1} \vartheta^{-c_{t,k}} g_{t,k}, \\ M_{j,h}^{\sim} &= \sum_{l=-\infty}^0 \vartheta^{Ql} \sum_{t \in \mathcal{P}_{l,h}^-} m_t \prod_{k=1}^{h-1} \vartheta^{-c_{t,k}} g_{t,k}, \end{aligned} \quad (8)$$

where $\mathcal{P}_{l,h}^-$ and $\mathcal{P}_{l,h}$ are sets of received symbols at layer l after h hops. We can bound their cardinality by $|\mathcal{P}_{l,h}^-| \leq |\mathcal{P}_{l,h}| \leq \sum_{i=1}^{h-1} N^i \leq 2N^{h-1}$.

Proof: By induction in h . The base case $h = 2$ follows from the fact that the source node sends no super-symbols, that means the nodes after the first hop can resolve all symbols and send no super-symbols on their part. Therefore $y_{j,2}^{\sim}$ equals the first term in (7). The receiver resolves all symbols on levels $l > \lambda$ with $G \leq \lambda \leq G + S$. The upper bound on λ is due to the fact that only single symbols are received on levels $l > G + S$, hence those symbols can always be resolved. The lower bound results from the definition of G , that is only symbols on levels $l > G$ are resolved. The remainder of the received signal $y_{j,2}$ is scaled by $\vartheta^{-Q\lambda}/a$ to form the super-symbol $M_{j,2}$, that node j forwards. The sum in l reaches to

λ , and a change of variables $l' = l - \lambda$ yields the result for $M_{j,2}^{\sim}$. Since the sum in t contains at most N terms, the base case is proved.

The inductive step follows from plugging (8) into (7) and ordering the symbols with respect to their level l ,

$$\begin{aligned} y_{j,h+1}^{\sim} &= a \sum_{l=-\infty}^{q_{\max}} \vartheta^{Ql} \left[\sum_{t:l_t=l} \vartheta^{-c_t} g_t m_t + \sum_{t:l_i+l'=l} m_t \prod_{k=1}^h \vartheta^{-c_{t,k}} g_{t,k} \right] \\ &= a \sum_{l=-\infty}^{q_{\max}} \vartheta^{Ql} \sum_{t \in \mathcal{P}_{l,h+1}} m_t \prod_{k=1}^h \vartheta^{-c_{t,k}} g_{t,k}, \end{aligned}$$

with $|\mathcal{P}_{l,h+1}| \leq N + N|\mathcal{P}_{l,h}^-| \leq \sum_{i=1}^h N^i \leq 2N^{h-1}$. Again, the receiver resolves all symbols on levels $l > \lambda$ with $G \leq \lambda \leq G + S$. The signal part of the forwarded super-symbol is

$$M_{j,h+1}^{\sim} = \vartheta^{-Q\lambda} \sum_{l=-\infty}^{\lambda} \vartheta^{Ql} \sum_{t \in \mathcal{P}_{l,h+1}^-} m_t \prod_{k=1}^h \vartheta^{-c_{t,k}} g_{t,k},$$

and a change of variables $l' = l - \lambda$ concludes the induction. Note that $|\mathcal{P}_{l,2}^-| \leq |\mathcal{P}_{l,2}|$, because some of the symbols m_t have possibly been resolved and subtracted. ■

The resolution of symbols is done successively starting with the symbol on the highest layer. The symbols on the lower layers act as interference. We find a lower bound on the distance d in the complex plane between the received signal and the boundary of the decision region.

Lemma 2: Consider the decision about the symbol m_r on layer l_r with $y_j^{\sim r}$ being the remainder of the signal $y_j^{\sim}$ after resolving all symbols on layers $l > l_r$. The distance d in the complex plane between $y_j^{\sim r}$ and the boundary of the decision region can be lower bounded as

$$d \geq \frac{\vartheta^{QG}}{2} \left(\vartheta^{-(H-1)} - 4\sqrt{2}N^{H-1}\vartheta^{-Q} \right).$$

Proof: We divide the remainder $y_j^{\sim r} = y_j^{\sim \text{res}} + y_j^{\sim \text{int}}$ into the part $y_j^{\sim \text{res}}$, that contains the symbol m_r , that should be resolved, and the part $y_j^{\sim \text{int}}$ that contains all interfering symbols. Due to Lem. 1,

$$|y_j^{\sim \text{res}}| = a \vartheta^{Ql_r} \left| \prod_{k=1}^{H-1} g_{r,k} \right| |m_r| \geq a \vartheta^{Ql_r} \vartheta^{-(H-1)} \sqrt{2}, \quad (9)$$

with $l_r = l - \sum_{k=1}^{H-1} c_{r,k}$ being the level of the symbol to resolve. The inequality is due to $|g_{r,k}| \geq \vartheta^{-1}$.

Since the transmission is partially coordinated, there is no symbol interfering with m_r , that is all symbols in $y_j^{\sim \text{int}}$ are at a level $l < l_r$. Hence with Lem. 1,

$$\begin{aligned} |y_j^{\sim \text{int}}| &\leq a \sum_{l=-\infty}^{l_r-1} \vartheta^{Ql} \sum_{t \in \mathcal{P}_{l,H} \setminus r} |m_t| \prod_{k=1}^{H-1} \vartheta^{-c_{t,k}} |g_{t,k}| \\ &\leq a\sqrt{2} |\mathcal{P}_{l_r,H}| \frac{\vartheta^{Ql_r}}{\vartheta^Q - 1} \leq 4a\sqrt{2} N^{H-1} \vartheta^{Q(l_r-1)}, \end{aligned} \quad (10)$$

where we have used $c_{t,k} \geq 0$, $|g_{t,k}| \leq 1$ and $\vartheta^Q \geq 2$.

Now, we turn to the decision distance d . Define $g_{\text{eff}} = \prod_{k=1}^{H-1} g_{r,k}$, so that $\arg(g_{\text{eff}})$ is the effective rotation of the

symbol m_r . The receiver compensates for this rotation before detecting the two bits represented by the real and imaginary part of m_r . Without loss of generality we consider the real part. The distance to the y-axis being the decision threshold is

$$d = \left| \Re \left((y_j^{\sim \text{res}} + y_j^{\sim \text{int}}) e^{-j \arg(g_{\text{eff}})} \right) \right| \geq \left| \frac{1}{\sqrt{2}} |y_j^{\sim \text{res}}| - |y_j^{\sim \text{int}}| \right| = \frac{1}{\sqrt{2}} |y_j^{\sim \text{res}}| - |y_j^{\sim \text{int}}|. \quad (11)$$

It remains to find a lower bound on a . To simplify analysis, we assume that the power of a super-symbol does not exceed the power of a symbol on the level above the super-symbol, that is $\mathbb{E}(|M_i|^2) \leq 2\vartheta^{2Q}$. This assumption is reasonable, since it is necessary for the reception of the symbol at a low probability of error. Since m_k and M_i in (6) are statistically independent, we find that

$$1 = \mathbb{E}(|x_i|^2) = a^2 \sum_{k=0}^{t_i-1} \vartheta^{-2b_k} m_k^2 + a^2 \vartheta^{-b_{t_i}} \mathbb{E}(|M_i|^2) \leq \sum_{k=0}^{t_i} \vartheta^{-2Qk} 2\vartheta^{2Q} \leq \frac{2a^2 \vartheta^{2Q}}{1 - \vartheta^{-2Q}}.$$

In the first inequality we have used the fact that $b_k = Qk$ maximizes the number of transmitted symbols. Hence,

$$a^2 \geq \frac{\vartheta^{-2Q}}{2} (1 - \vartheta^{-2Q}) \geq \frac{\vartheta^{-2Q}}{4}. \quad (12)$$

We conclude the proof by plugging (9), (10) and (12) into (11) and observing that $l_r \geq G + 1$. ■

Next, we turn to the noise part $y_j^{\sim}$ of the received signal.

Lemma 3: The variance of the noise contained in the received signal y_j after h transmission hops can be upper bounded as

$$\mathbb{E}(|y_j^{\sim}|^2) = \sigma_{y,h}^2 \leq \sum_{i=0}^{h-2} \left(N \vartheta^{2Q(S-1)} \right)^i \quad (13)$$

$$\leq 2 \left(N \vartheta^{2Q(S-1)} \right)^{h-2}. \quad (14)$$

Proof: By induction in h . For the base case consider once more, that after two hops we receive no super-symbols, hence $\sigma_{y,2}^2 = \sigma_z^2 = 1$. The inductive step follows from (7),

$$y_{j,h+1}^{\sim} = \sum_{i,i \neq j}^N \vartheta^{Ql_i} \vartheta^{-c_i} g_i \vartheta^{-Q\lambda_i} y_{i,h}^{\sim} + z_j.$$

Since $y_{i,h}^{\sim}$ are statistically independent,

$$\begin{aligned} \sigma_{y,h+1}^2 &\leq \sum_{i,i \neq j}^N \vartheta^{2Ql_i} \vartheta^{-2Q\lambda_i} \sum_{i=0}^{h-2} \left(N \vartheta^{2Q(S-1)} \right)^i + 1 \\ &\leq \sum_{i=0}^{h-1} \left(N \vartheta^{2Q(S-1)} \right)^i. \end{aligned}$$

In the first inequality we have used (13) and the fact that $c_i \geq 0$ and $|g_i| \leq 1$. In the second inequality we have used

that $l_i \leq G + S$ and $\lambda_i \geq G + 1$. Note that we defined $\lambda_i \geq G$ before, but $\lambda_i = G$ would imply that node i resolves all possible symbols and forwards no super-symbol. The step from (13) to (14) follows since $N \vartheta^{2Q(S-1)} \geq 2$. ■

Proof of Th. 1: Since the noise part is $y_j^{\sim} \sim \mathcal{CN}(0, \sigma_y^2)$, we know that real and imaginary part are zero-mean Gaussian with variance $\sigma_y^2/2$. Hence, the bit error rate is

$$P_b = \int_d^\infty \frac{1}{\sqrt{\pi \sigma_y^2}} \exp\left(-\frac{x^2}{\sigma^2}\right) dx = \mathcal{Q}\left(\sqrt{2} \frac{d}{\sigma_y}\right).$$

Plugging in Lem. 2 and Lem. 3 and solving for G yields the desired expression. ■

V. CONCLUSION

In this paper we showed that a partially aligned transmission solution found in the deterministic model of a wireless relay network can be translated into a transmission in the respective Gaussian model. The proposed transmission scheme for the Gaussian model uses hierarchical modulation to separate different layers of transmission. The scheme allows for instantaneous and relatively simple detection and is therefore suitable for implementation in delay and computational constrained networks like wireless sensor networks.

To guaranty a bit error rate below a given threshold P_e , the receiver has to refrain from resolving a certain amount of G symbols. Hence, G is the gap between the rate in the deterministic model and the rate in the Gaussian model. We derived an upper bound on G that is dependent on the number H of hops between source and destination, the super-symbol margin S due to partial coordination as well as the system parameters ϑ and Q . Since G is independent on the channel gains, the gap becomes negligible in the high SNR regime. Therefore, the deterministic model can both provide rate approximations and serve as a tool to design transmission schemes for the Gaussian model.

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