

On Constructing a Market Consistent Economic Scenario Generator

Ebba K. Baldvinsdóttir & Lina Palmborg

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Abstract

Recently the insurance industry has started to realise the importance of properly managing options and guarantees embedded in insurance contracts. Interest rates have been low in the last few years, which means that minimum interest rate guarantees have moved from being far out-of-the money to expiring in-the-money. As a result, many insurance companies have experienced solvency problems. Furthermore, insurance firms operating within the European Union will, from the end of 2012, be subject to the Solvency II directive, which places new demands on insurance companies. For example, the valuation of assets and liabilities now needs to be market consistent. One way to accomplish a market consistent valuation is through the use of an economic scenario generator (ESG), which creates stochastic scenarios of future asset returns.

In this thesis, we construct an ESG that can be used for a market consistent valuation of guarantees on insurance contracts. Bonds, stocks and real estate are modelled, since a typical insurance company's portfolio consists of these three assets. The ESG is calibrated to option prices, wherever these are available. Otherwise the calibration is based on an analysis of historical volatility. An assessment of how well the models capture prices of instruments traded on the market is made, and finally the ESG is used to compute the value of a simple insurance contract with a minimum interest rate guarantee.

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Chapter 1

Introduction

1.1 Background

The financial products sold by insurance companies often contain guarantees and options of numerous varieties. Many life insurers, for example, offer products with minimum interest rate guarantees. Broadly speaking, two types of these guarantees exist [14, 16]. A *maturity guarantee* secures the policyholder a minimum rate of return over the holding period till the expiration of the contract. For *multi-period guarantees*, the contract period is divided into sub periods, for instance one year. The contract then specifies a binding guarantee for each sub period. This means that a good return in an earlier period is not lost in a period with a lower return on the investment portfolio. Such a guarantee is therefore very valuable for the policyholder and exposes the insurance company to a considerable amount of financial risk.

The pricing and management of guarantees embedded in insurance contracts is one of the most challenging problems faced by insurance companies today. Such options and guarantees are an obligation for the insurance company, affect the company's solvency and therefore need to be properly evaluated. In the past this has generally not been done for a number of reasons [18]. First, it is likely that some companies have not realised that their policies consist of several components which must be evaluated separately. Second, at the time of policy initiation, the options embedded in insurance contracts were so far out-of-the-money, that the companies disregarded their value as it was considered negligible compared with the costs associated with the valuation. Third, the valuation of these policies sometimes requires complex analytical methods, as well as a great deal of computer power and data.

In the light of current economic events and new legislations, insurance companies have realised the importance of properly managing their options and guarantees. Insurers have recently experienced significantly lower rates of return than before,

which means that minimum interest rate guarantees have moved from being far out-of-the-money to expiring in-the-money. As a result, companies have experienced solvency problems. Furthermore, insurance firms operating within the European Union will, from the end of 2012, be subject to the Solvency II directive. The directive is a set of regulatory requirements and is based on economic principles for the valuation of assets and liabilities. It is a risk-based system, as risk will be measured on consistent principles, and capital requirements will depend directly on this.

1.2 Valuation of Liabilities under Solvency II

Solvency II is a new legislation that affects the insurance industry. It is scheduled to come into effect on 31 December 2012. This new set of rules creates many new demands on insurance companies regarding for example capital requirement, risk management processes, and transparency.

The Solvency II framework consists of three pillars. Pillar I contains the quantitative requirements, i.e. how assets and liabilities should be valued and the capital that a company is required to hold. Pillar II covers the qualitative requirements, i.e. how risks should be governed and managed, as well as supervised. Finally, Pillar III sets out the requirements for disclosure and transparency, for example reporting to supervisory authorities.

Pillar I defines two levels of capital requirements: Minimum Capital Requirement (MCR) and Solvency Capital Requirement (SCR). The MCR is the absolute minimum capital that an insurance company has to hold. If the capital falls below this level the supervisory authorities will intervene. The SCR represents the required level of capital that an insurance company should hold, and it can be calculated either through a standard formula or through the use of an internal model which must be approved by the supervisory authorities.

In order to determine the capital requirements of an insurance company, one first has to calculate the technical provisions. Technical provisions is the amount that an insurance company must hold to ensure that it can meet its expected future obligations on insurance contracts. It consists of risks that can be hedged and risks that cannot be hedged. The value of the technical provisions for risks that cannot be hedged should be the sum of a best estimate of the expected liabilities and a risk margin.

1.2.1 QIS5

The insurance industry has been asked to participate in so-called Quantitative Impact Studies, where the 5th (QIS5) was undertaken in the autumn of 2010. The studies have given the European Commission an idea on how the proposed regulation will affect the industry and, in particular, the level of the required

capital the insurance companies need to hold. As a background document, the European Commission issued, in July 2010, the *QIS5 Technical Specifications* [12] and the *Annexes to the QIS5 Technical Specifications* [13]. These documents lay out the most recent details regarding the valuation of assets and liabilities under Solvency II.

It is important to stress that QIS5 is a study for testing purposes, to ensure that the Solvency II framework is as accurately formulated as possible when it comes into effect. This means that what is written in the QIS5 Technical Specifications might not be exactly what the framework will look like in a couple of years, depending on the results of the study. However, at this point in time, it is the best available description of how the required capital should be calculated, and it is generally believed that drastic changes to the methods proposed by the QIS5 Technical Specifications are unlikely.

1.2.2 Best Estimate

The QIS5 Technical Specifications states that the best estimate is the probability weighted average of future cash flows, discounted to its present value [12]. The Specifications suggests three different methods for the calculation of the best estimate; namely simulation techniques, deterministic techniques, and analytical techniques.

- Using an analytical technique means that the insurance company must be able to find a closed form solution for calculating the best estimate. One analytical technique can be to value guarantees by calculating the cost for fully hedging the guarantee. Another analytical technique is to make an assumption that future claims follow a given distribution.
- With a deterministic technique, the projection of the cash flows is based on a fixed set of assumptions. Examples of deterministic techniques are stress and scenario testing, and actuarial methods such as the Chain-Ladder method.
- A simulation approach means using a stochastic model to generate future scenarios. When using this method, there is no need to generate all possible future scenarios. However, one has to make sure that enough scenarios are generated so that they are representative of all possible future scenarios.

There may also be other methods that can be used to perform this calculation, but there are certain criteria that they need to fulfil, according to the QIS5 Technical Specifications. For example, they need to be actuarial or statistical methods that take into account the risks that affect the future cash flows.

The QIS5 Technical Specifications state that simulation methods can lead to a more robust valuation of the best estimate of insurance contracts with embedded

options and guarantees. The deterministic and analytical approaches are more appropriate for the best estimate of non-life liabilities, as well as for life insurance liabilities without options or guarantees [12]. Since insurance contracts with multi-period minimum interest rate guarantees are path dependent, i.e. the guaranteed rate is given at the end of each period, not just at the maturity of the insurance contract, the simulation approach is particularly suitable.

1.3 Aim and Scope

The valuation of insurance contracts with multi-period minimum interest rate guarantees requires complex mathematics. An economic scenario generator (ESG) can be used as a tool for this valuation. A “scenario” in this context is a stochastically generated economic simulation from a Monte-Carlo driven model. According to the Solvency II directive, ESG models are a key element of market consistent valuation for life insurance businesses.

The aim of this thesis is to describe and calibrate a market consistent ESG that can be used for the valuation of insurance contracts with multi-period minimum interest rate guarantees. The construction of the ESG requires a choice of assets or asset classes to be modelled, followed by a choice and calibration of a model for each asset class. When each asset has been calibrated and the correlation between them established, scenarios for each asset class can be generated.

There are many different assets that can be modelled in an ESG, for example, bonds, stocks, real estate, inflation, exchange rates and credit risk. Since a typical insurance company’s portfolio consists of bonds, stocks, and real estate, these are the three asset classes we choose to model in our ESG. The calibration will be based on data from the Swedish market.

The thesis is outlined as follows. Chapter 2 gives the theoretical background, focusing on the models for the asset classes bonds, stocks, and real estate. The analysis is presented in Chapter 3, where the results of the simulations are illustrated in various plots. In Section 3.4 a valuation of a simplified insurance contract based on the ESG is demonstrated. Finally, Chapter 4 offers a summary of the results and some concluding remarks.

Chapter 2

Theoretical Background

2.1 Economic Scenario Generators

In the insurance industry there are two types of ESGs with two different areas of application. A *real world* ESG is used to generate valid real world distributions for all the main risk factors, to support the calculation of the Solvency Capital Requirement (SCR). A *market consistent* ESG, on the other hand, is used in the calculation of the technical provisions for insurance contracts with financial options and guarantees [28].

Real world scenarios should be scenarios that reflect the expected future evolution of the economy by the insurance company, i.e. they should reflect the real world, hence the name. They should include risk premium and the calibration of volatilities and correlations is usually based on analysis of historical data.

Market consistent scenarios are of use during market consistent valuations in Solvency II, and the main function of these scenarios should be to reproduce market prices. Often the scenarios are risk neutral, i.e. they do not include risk premium, since this simplifies calculations without changing the valuation. Market consistent scenarios need to be arbitrage free.

It is important to stress that the market consistent scenarios are not intended to reflect real world expectations; for example ten-year long market consistent scenarios do not reflect how the insurer expects the world to look like in ten years. Instead they can be used to value a derivative with a maturity of ten years. Market consistent scenarios can help us calculate market prices today while real world scenarios can show us what the world might look like tomorrow.

2.2 Calibration of a Market Consistent ESG

The Solvency II directive places several requirements on a market consistent ESG. First, it must generate asset prices that are consistent with deep, liquid, and transparent financial markets; and second, it assumes no opportunity for arbitrage. To clarify this

- A liquid market means that assets can be easily bought and sold without causing significant movements in price.
- A deep market means that a large number of assets can be transacted without significantly affecting the price of the financial instruments.
- Transparency in the market means that current trade and price information is normally readily available to the public.

The requirements that need to be considered regarding the calibration of a market consistent asset model:

- It should be calibrated to instruments which in some way reflect the nature and term of the liabilities. It is especially important that they reflect liabilities that cause substantial guarantee cost.
- It should be calibrated to the current risk-free term structure used to discount cash flows.
- It should be calibrated to an appropriate volatility measure.

What constitutes an appropriate volatility measure is still debatable within the Solvency II framework. There are two possible approaches; calibration to market prices of different derivatives, and calibration based on analysis of historical volatilities of the assets themselves.

The purpose behind a market consistent valuation of an insurance company's liabilities is to reproduce the price the liabilities would be traded at, if they were in fact traded on the market. In order to do this it is important that the ESG can reproduce market prices of assets which have similar characteristics to the liability being valued. The liabilities stemming from insurance contracts with embedded options and guarantees have option-like features, which means that the ESG should be able to reproduce option prices on the market. An ESG calibrated to option prices would generally give a very good fit to the market prices of options.

However, there are also reasons why a calibration based on analysis of historical volatilities can be appropriate. To begin with, option prices are not available for all assets, for example for the asset class real estate. Also, a life insurance company generally has obligations stretching many years into the future, and

option prices with long maturities are often not available on the market. Another important advantage of calibration to historical volatilities is that they are more stable than the volatility implied by option prices. When the market is stressed, implied volatilities tend to be higher than real volatility leading to an overestimation of technical provisions. This type of pro-cyclical effect would be avoided by calibrating to historical volatilities.

In order to ensure market consistency, in a “normal” market situation, calibration to option prices seems more appropriate if these prices are available. This is also the conclusion in the QIS5 Annex G [13] where the definition of an appropriate volatility measure is discussed. This view was also supported by the majority of the insurance companies and associations who commented on CEIOPS Consultation Paper 39 [6, 7] where the best estimate calculation under Solvency II is discussed. However, many emphasised that the Solvency II directive should not limit insurance companies to calibrating only to volatilities implied by option prices or only to historical volatilities, but that it should be up to the individual company to choose which volatility measure to use, depending on the asset class and the market situation at the time of calibration.

To determine whether option prices are reliable and liquid and to establish what constitutes a stressed market is not always a straightforward task, and often requires expert judgement. Therefore, it would be very interesting to compare the results from a calibration to option prices with a calibration based on analysis of historical volatilities. However, this lies outside the scope of this thesis. For the reasons discussed above, we will focus on calibration to option prices, for the asset classes where these are available.

2.3 Overview of Models and Methods

The economic scenario generator described in this thesis consists of models for three asset classes; bonds, stocks, and real estate.

All of the asset classes are modelled as stochastic processes which satisfy the stochastic differential equation (SDE)

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t), \quad X(0) = x_0$$

where the function μ is called the *drift term*, the function σ is called the *diffusion coefficient*, and W is a *Wiener process*. A Wiener process is a process with independent increments, i.e. if $s < t \leq u < v$ then $W(v) - W(u)$ and $W(t) - W(s)$ are independent stochastic variables, and $W(t) - W(s)$ follow a Normal distribution $N(0, \sqrt{t - s})$.

As discussed in the previous section, the volatility for each model is determined from option prices on the market when these are available. If market data is not available, the estimation of volatility is based on historical data. A further

discussion on the calibration method is included in the section for the respective asset class.

- **Interest rates:** Calibration based on swaption prices, since these instruments resemble the behaviour of the interest rate guarantees embedded in insurance contracts. A swaption gives the holder the right, but not the obligation, to enter an interest-rate swap at a given future date, the maturity date of the swaption. In an interest rate swap, a number of floating rate payments are exchanged for the same number of fixed rate payments, at prespecified dates. Normally the rates are exchanged every six months, starting six months after the maturity date.
- **Equity:** Calibration based on equity option prices. An equity call (put) option gives the holder the right, but not the obligation, to buy (sell) the underlying stock or index at the maturity date of the option.
- **Real estate:** The calibration is here based on historical data since no options on real estate exist on the Swedish market.

The correlation between interest rates and equity is computed from market data, and the correlation between interest rates and real estate, as well as the correlation between equity and real estate is based on historical data.

2.4 Interest Rate Models

The interest rate model is a central part of the ESG, as most economic and financial series are related to interest rates in some way. A large number of interest rate models have been developed in the last few decades for estimating prices of interest rate derivatives. These models can be broadly divided into short rate models, forward rate models, and LIBOR and swap market models [1].

Short rate models, which describe the dynamics of the instantaneous spot rate, can further be classified as either equilibrium models or no-arbitrage models. Equilibrium models are also referred to as “endogenous term structure models” because the term structure of interest rates is an output of these models. The earliest and most well known equilibrium short rate model is the Vasicek model (1977), which was soon followed by other equilibrium short rate models, such as the Dothan model (1978) and the Cox-Ingersoll-Ross model (1985). The second class of short rate models is the no-arbitrage model, which is designed to exactly match the current term structure of interest rates. The Hull-White model (1990) and the Black-Karasinski model (1991) are generally regarded as two of the most important no-arbitrage short rate models. An example of a forward rate model is the Heath-Jarrow-Morton framework (1992), which chooses the instantaneous forward rate as a fundamental quantity to model. The problem with both short

rate models and forward rate models is that neither the instantaneous short rate nor the instantaneous forward rate is observable in the market. The LIBOR and swap market models are, on the other hand, a class of models that describe the evolution of rates that are directly observable in the market. However, they tend to be more complicated in their setup [26].

The interest rate model chosen for the economic scenarios must fulfil the following criteria:

- For a market consistent calibration to be possible, the model must take the current term structure of interest rates as an input. This means that the endogenous term structure models, such as the Vasicek model, do not qualify.
- The calibration of the model to market prices of interest rate derivatives must be relatively simple and not too computationally expensive.

The Hull-White and the Black-Karasinski one-factor short rate models satisfy both of the above conditions, i.e. both models take the current yield curve as an input factor and both are relatively easy to calibrate. Moreover, these models are well known and popular among financial engineers for pricing interest rate derivatives, which makes them a natural first choice for modelling interest rates in the ESG.

The Hull-White one-factor model assumes that the instantaneous short-rate process r is normally distributed and has dynamics given by

$$dr(t) = [\theta(t) - a(t)r(t)]dt + \sigma(t)dW(t), \quad r(0) = r_0$$

where the function $a(t)$ is the rate at which $r(t)$ reverts towards its expected value, $\sigma(t)$ is the volatility of the instantaneous short rate, and $W(t)$ is a Wiener process.

The Black-Karasinski one-factor model assumes a lognormal distribution for the instantaneous short-rate process r , which has dynamics given by

$$d\ln(r(t)) = [\theta(t) - a(t)\ln(r(t))]dt + \sigma(t)dW(t), \quad r(0) = r_0$$

with $a(t)$ the rate at which the logarithm of $r(t)$ reverts towards its expected value, $\sigma(t)$ is the volatility of the logarithm of $r(t)$, and $W(t)$ is a Wiener process.

The two models thus involve different assumptions on the short rate distribution, as the Hull-White model assumes short rates to be normally distributed while the Black-Karasinski assumes lognormal short rates. When calibrated to the same data, these two models give roughly the same prices for at-the-money products. However, when pricing out-of-the-money instruments, the tails of the interest rates distributions become more relevant, so that one model tends

to amplify prices while the other underestimates them. The correct price lies somewhere between the Gaussian and the lognormal price [5].

The Hull-White model has the advantage that it is analytically tractable as explicit formulas exist for zero-coupon bonds and options on them. Because of this, the calibration process is fairly straightforward. The main disadvantage of this model is that the normally distributed short rate process can produce negative interest rates, although the probability of this happening is quite small under normal market conditions.

If the instantaneous short rate follows a lognormal distribution, such as in the Black-Karasinski model, interest rates cannot become negative. However, analytical formulas for the prices of zero-coupon bonds and bond options are not available, which makes the calibration of the Black-Karasinski model much more complex. Another drawback of this model is that the expected value of the money-market account is infinite, as a consequence of the lognormal distribution [5].

Finally, when choosing which interest rate model to use, one should consider the interest rate level at the time of calibration. When interest rates are low, the Hull-White model is problematic because the probability of negative interest rates is no longer negligible. The Black-Karasinski model can also cause problems because the volatility becomes much larger when interest rates are low than when they are high [19]. In such situations, a multi-factor short rate model might be necessary.

Calibration of the models involves finding the functions $a(t)$ and $\sigma(t)$ that give the best fit to the prices of interest rate derivatives on the market, as well as determining the function $\theta(t)$ so that the model exactly matches the current yield curve. The approach taken here during the calibration of the models, is to assume that $a(t)$ and $\sigma(t)$ are both independent of time. A better fit to market prices can be obtained by making either or both functions time-dependent. However, the disadvantage of this approach is that the volatility structure then becomes non-stationary, and the volatility implied by the model in the future can behave quite differently from the one existing in the market today [5, 19].

2.4.1 The Hull-White Extended Vasicek Model

The Hull-White model is an extension of the Vasicek model since some or all of the parameters are now allowed to vary with time. This ensures that the model can match the current term structure of interest rates exactly, which the Vasicek model cannot do.

Hull and White [20] assumed that the instantaneous short-rate process r under the risk-neutral measure has dynamics given by

$$dr(t) = [\theta(t) - a(t)r(t)] dt + \sigma(t)dW(t), \quad r(0) = r_0$$

where r_0 is a positive constant, and $\theta(t)$, $a(t)$ and $\sigma(t)$ are deterministic functions of time, and where $W(t)$ is a Wiener process. Setting $a(t) = a$ and $\sigma(t) = \sigma$ leads to the following model

$$dr(t) = [\theta(t) - ar(t)] dt + \sigma dW(t), \quad r(0) = r_0 \quad (2.4.1)$$

a and σ are now positive constants and $\theta(t)$ is chosen so that the model exactly matches the current term structure of interest rates.

It can be shown that the following holds for θ (see for example [1]).

$$\theta(t) = \frac{\partial f^M(0, t)}{\partial T} + af^M(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at})$$

where $\frac{\partial f^M(0, t)}{\partial T}$ is the partial derivative of f^M with respect to its second argument, and $f^M(0, t)$ is the market instantaneous forward rate at time 0 for maturity T , i.e.

$$f^M(0, T) = -\frac{\partial \ln P^M(0, T)}{\partial T}$$

where $P^M(0, T)$ is the zero-coupon price for maturity T .

By integrating equation (2.4.1) we get

$$\begin{aligned} r(t) &= r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma \int_s^t e^{-a(t-u)}dW(u) \\ &= r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)}dW(u) \end{aligned}$$

where,

$$\alpha(t) = f^M(0, t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2.$$

Defining the process x by

$$dx(t) = -ax(t)dt + \sigma dW(t), \quad x(0) = 0,$$

we see that

$$x(t) = x(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)}dW(u),$$

which yields $r(t) = x(t) + \alpha(t)$ for each t .

2.4.1.1 Bond and Option Pricing

The Hull-White model has an affine term structure (see for example [1]) which implies that

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)} \quad (2.4.2)$$

where $P(t, T)$ is the price at time t of a zero-coupon bond with maturity T , and

$$\begin{cases} B(t, T) = \frac{1}{a} [1 - e^{-a(T-t)}], \\ A(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \exp \left\{ B(t, T)f^M(0, t) - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t, T)^2 \right\}. \end{cases}$$

This explicit formula for zero-coupon prices is very useful since it leads to a formula for European options on bonds, which in turn leads to an explicit formula for swaption prices. Swaption prices are often used in the calibration of interest rate models.

Proposition 2.1. (Hull-White bond option) *In the Hull-White model the price at time t of a European call with strike price K , and time of maturity T on a bond maturing at S is given by the formula*

$$ZBC(t, T, S, K) = P(t, S)\phi(h) - KP(t, T)\phi(h - \sigma_p)$$

where

$$\sigma_p = \sigma \sqrt{\frac{1 - e^{-2a(T-t)}}{2a}} B(T, S),$$

$$h = \frac{1}{\sigma_p} \ln \frac{P(t, S)}{P(t, T)K} + \frac{\sigma_p}{2}$$

In the same way, the price at t of a European put with strike price K , and time of maturity T on a bond maturing at S is given by the formula

$$ZBP(t, T, S, K) = KP(t, T)\phi(-h + \sigma_p) - P(t, S)\phi(-h).$$

A European swaption can be seen as an option on a coupon bearing bond. This, together with Jamshidian's decomposition [23] which gives us the price of European options on coupon-bearing bonds, leads to the following formula for swaption prices

$$PS(t, T, t_1, \dots, t_n, N, K) = N \sum_{i=1}^n c_i ZBP(t, T, t_i, K_i) \quad (2.4.3)$$

where PS is the price of a payer swaption with strike rate K , maturity T and nominal value N , which gives the holder the right to enter at time T an interest rate swap with payment times $\{t_1, \dots, t_n\}$, $t_1 > T$ where he pays at the fixed rate K and receives LIBOR. τ_i is the year fraction from t_{i-1} to t_i , $i = 1, \dots, n$, $c_i = K\tau_i$ for $i = 1, \dots, n-1$ and $c_n = 1 + K\tau_n$.

$K_i = A(T, t_i) \exp(-B(T, t_i)r^*)$ where r^* is the value of the spot rate at time T for which

$$\sum_{i=1}^n c_i A(T, t_i) e^{-B(T, t_i)r^*} = 1$$

2.4.2 The Black-Karasinski Model

The short rate model proposed by Black and Karasinski assumes that short term interest rates are lognormal [2]. The model assumes that the logarithm of the instantaneous short rate evolves under the risk neutral measure according to

$$d \ln(r(t)) = [\theta(t) - a(t) \ln(r(t))] dt + \sigma(t) dW(t), \quad r(0) = r_0$$

where r_0 is a positive constant, $\theta(t)$, $a(t)$ and $\sigma(t)$ are deterministic functions of time, and where $W(t)$ is a Wiener process. Setting $a(t) = a$ and $\sigma(t) = \sigma$ leads to the following model

$$d \ln(r(t)) = [\theta(t) - a \ln(r(t))] dt + \sigma dW(t), \quad r(0) = r_0 \quad (2.4.4)$$

where $\theta(t)$ is chosen to exactly fit the current term structure observed in the market and the other two parameters are found by calibrating the model to swaption prices.

Using Ito's lemma and 2.4.4 we obtain

$$dr(t) = r(t) \left[\theta(t) + \frac{\sigma^2}{2} - a \ln r(t) \right] dt + \sigma r(t) dW(t)$$

which for each $s \leq t$ satisfies

$$r(t) = \exp \left\{ \ln r(s) e^{-a(t-s)} + \int_s^t e^{-a(t-u)} \theta(u) du + \sigma \int_s^t e^{-a(t-u)} dW(u) \right\}$$

Setting

$$\alpha(t) = \ln(r_0)e^{-at} + \int_0^t e^{-a(t-u)}\theta du \quad (2.4.5)$$

we get

$$\lim_{t \rightarrow \infty} E(r(t)) = \exp\left(\lim_{t \rightarrow \infty} \alpha(t) + \frac{\sigma^2}{4a}\right)$$

The original article on the lognormal short rate model by Fischer Black and Piotr Karasinski suggested using binomial trees to implement the model [2]. This method assumes that the probabilities of moving up and down in the tree are both equal to 0.5. Hull and White have suggested a different approach using trinomial trees. The construction procedure of trinomial trees used here for calculating bond and option prices is based on the one described by Brigo and Mercurio [5] which in turn is based on Hull and White's approach [21, 22]. The procedure is described in Appendix A.

2.5 Equity Models

The most widely used model for the equity process is the lognormal model proposed by Black and Scholes in 1973 [3]. The model assumes that the underlying stock $S(t)$ follows a geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad S(0) = S_0$$

with constant drift μ , volatility σ and where $W(t)$ is a Wiener process.

The price of a European call option, with strike K and a maturity T can be computed by the following standard Black-Scholes pricing formula

$$\mathcal{O}(t, T, K) = S_0\Phi(d_1) - KP(t, T)\Phi(d_2)$$

where $\Phi(\cdot)$ is the cumulative normal distribution, S_0 is the current spot price of the underlying stock or index, and where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

and

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

2.5.1 Volatility in the Black-Scholes Model

Although the Black-Scholes model assumes that implied volatilities of European options are constant and independent of the option's strike price and time to maturity, it is well known that the model does not realistically describe equity price dynamics and is not consistent with how options are priced in the market. Examination of market data reveals that implied volatility is generally lower for at-the-money options and becomes higher for both out-of-the-money and in-the-money options, resulting in a so-called volatility smile. Furthermore, volatility is also dependent of an option's time to maturity, as volatility tends to be an increasing function of maturity time. Combining volatility smiles and volatility term structure, we get a three-dimensional volatility surface where volatility is a function of both the strike and maturity of the option [19]. Additionally, in the past few years several models have emerged that assume that volatility can be modelled stochastically. This means that the volatility of equity changes from time to time in a random fashion.

2.5.2 Interest Rates in the Black-Scholes Model

The original Black-Scholes model assumes that interest rates are deterministic. In most cases this is a fair assumption to make as the variability in interest rates is usually much smaller than the variability observed in equity returns. However, the effect of stochastic interest rates is much more significant when pricing options with long maturities, and in such cases the stochastic feature of interest rates should not be disregarded [5]. As the objective here is to generate scenarios that can be used to price long term insurance liabilities, it is important that the equity model allows for stochastic interest rates. We have therefore chosen to model equity under Hull-White and Black-Karasinski interest rates, where the correlation between equity returns and interest rates can be derived from the models. This means that equity volatility must be assumed to be deterministic and constant over time. This is not the optimal way to model equity volatility, but building a model that assumes a more complex structure for equity volatility and/or where equity volatility is stochastic, while at the same time assuming stochastic interest rates is beyond the scope of this thesis.

For comparison, equity is also modelled under deterministic interest rates with the correlation between equity returns and interest rates computed from historical data.

Incorporating stochastic interest rates, where the short rate follows the Hull-White model, into the Black-Scholes equity model is relatively simple as explicit formulas for prices of European options can be derived. If the short rate follows a lognormal process, as in the Black-Karasinski model, it is not possible to derive explicit formulas for European option prices and they must instead be computed by an approximating tree. Constructing a trinomial tree for pricing derivatives makes the calibration of the model more complex and computationally heavy,

since in this case a two-dimensional trinomial tree must be constructed. The procedures described here are the ones used by Brigo and Mercurio [5] to model equity under stochastic interest rates.

2.5.2.1 The Equity Model under Hull-White Interest Rates

Assuming a Hull-White one-factor short rate model with a given mean reversion factor a and volatility σ_r , the price at time t of a European call option with maturity T , strike K , and written on the asset S with yield y is given by

$$\mathcal{O}(t, T, K) = S(t)e^{-y(T-t)}\Phi(d_1) - KP(t, T)\Phi(d_2)$$

where

$$d_1 = \frac{\ln \frac{S(t)}{KP(t, T)} - y(T-t) + \frac{1}{2}v^2(t, T)}{v(t, T)}$$

and

$$d_2 = \frac{\ln \frac{S(t)}{KP(t, T)} - y(T-t) - \frac{1}{2}v^2(t, T)}{v(t, T)}$$

$$v^2(t, T) = V(t, T) + \sigma_S(T-t) + 2\rho_{r,S} \frac{\sigma_r \sigma_S}{a} \left[T-t - \frac{1}{a} (1 - e^{-a(T-t)}) \right]$$

where

$$V(t, T) = \frac{\sigma_r^2}{a^2} \left[T-t + \frac{2}{a} e^{-a(T-t)} - \frac{1}{2a} e^{-2a(T-t)} - \frac{3}{2a} \right]$$

In this price formula the parameter $\rho_{r,S}$ is the instantaneous correlation between the two Wiener processes in the interest rate model on the one hand and the equity model on the other hand, i.e.

$$d\langle W_r, W_S \rangle_t := \rho dt$$

The instantaneous correlation is a parameter that describes the degree of dependence between changes in the two factors. The correlation is regarded as an endogenous parameter that is obtained by calibration to market prices.

2.5.2.2 The Equity Model under Black-Karasinski Interest Rates

As discussed earlier, the Black-Karasinski interest rate model is not analytically tractable. Therefore, numerical methods must be used to price equity derivatives under Black-Karasinski interest rates, which makes the calibration of the equity model a great deal more difficult than under Hull-White interest rates. To price equity options we need not just one trinomial tree for the interest rate, but

another trinomial tree for the stock price. This complicates the calibration of the equity model and also increases the computational burden.

To construct the two-dimensional trinomial tree, we first create two trinomial trees, one for the interest rate process r and one for the stock process S . Next, these two trees have to be linked together so that the correlation between the interest rate and the stock price is taken into account. The exact procedure for this can be found in Appendix A.

2.6 Real Estate Models

Real estate is a common asset type in an insurance company's portfolio and investing in real estate is generally regarded as a way of diversifying the portfolio. Here, real estate is modelled as an equity-type asset using the Black-Scholes model with constant volatility.

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad S(0) = S_0$$

The model assumes that the underlying asset $S(t)$ has constant drift μ , volatility σ and that $W(t)$ is a Wiener process.

The modelling of real estate differs from the modelling of equity and bonds as there is no central marketplace for real estate, and transactions occur mainly between private parties. This means that investment performance for this asset class cannot be as readily measured as for bonds and stocks. In most cases a real estate index, based on appraisals of individual properties, is therefore modelled. As options on a real estate index are generally not available, the calibration of the real estate model must be based on the historical volatility of the index. The calibration of the model is mainly based on the methods described in [11].

The market consistent calibration of an equity model means that the drift in the model is set to the risk-free interest rate. In real world modelling, however, the drift always consists of the sum of the risk-free interest rate and a risk premium. This means that the Black-Scholes model can be expressed as

$$dS(t) = (r_f + \mu_P)S(t)dt + \sigma S(t)dW(t)$$

where r_f is the risk-free interest rate and μ_P is the risk premium for the asset. As the aim is to construct a market consistent real estate model, the drift is set to the risk-free interest rate, even though the volatility is computed from historical data.

Djehiche and Hörfelt [11] also discuss the importance of keeping in mind the difference between the value of a real estate index and the real market value of the underlying real estate. The index is mainly based on appraisals rather than market values. This makes the index smoother than the true market value.

To recover the true market value from appraisal based indices, several authors, such as Geltner [17], Fisher, Geltner and Webb [15] and Cho, Kawaguchi and Shilling [9] have proposed unsmoothing methods.

2.6.1 Smoothing

In general, the term smoothing refers to the process of creating a function that captures the relevant changes in the data while disregarding the noise.

Smoothing can be explained by several separate factors. At the individual property level, the appraised value is the appraisers' estimates of the market value or the most likely transaction price of each individual property at some specified point in time. It is common for the appraiser to perform the valuation for certain properties year after year. The appraiser therefore knows the previous appraisal value and will combine that information with information about the most recent comparable transactions to reach the reported value, which is thus a weighted average of current information and historical appraisals. When doing this, the appraiser filters out transaction price noise which means that the systematic component to individual level real estate values will be smoothed over time in the appraisal. It is also generally recognized that the appraiser tends to be conservative in his beliefs of the change in the value of the property and will only partially adjust its value from one point in time to another [15, 17]. Furthermore, different properties are appraised at different points of time throughout the year, but the valuations are averaged to produce the index value for that year [17]. Finally, a real estate index includes valuations from different types of real estate which are aggregated to cause additional smoothing [4, 15].

It has been shown that smoothed real estate returns, i.e. returns derived directly from an appraisal-based real estate index, have a lower volatility and lower correlation with returns from stocks. Using the smoothed real estate returns might therefore result in a distortion in asset allocation as the diversifying effect of investing in real estate will be overestimated [4].

2.6.2 Unsmoothing

By defining a formal smoothing method, which represents the smoothing process described in the above section, it is then possible to invert the smoothing model to reveal the true, unobserved return series. The resulting unsmoothed index is thus a historical price series based on the published index, but with the smoothing removed.

The unsmoothing method chosen here is the Full-Information Value Index method proposed by Fisher, Geltner and Webb in 1994 [15]. This method assumes that the underlying real estate values behave stochastically as if real estate were

traded in an informationally efficient market. The smoothing model can be expressed as

$$r_t^* = w_0 r_t + w(B)r_{t-1}$$

where r_t^* is the inflation-adjusted smoothed index return during period t , r_t is the corresponding underlying true return during that period, w_0 is a weight between 0 and 1 and $w(B)$ is a polynomial function in the lag operator, B

$$w(B) = w_1 + w_2 B + w_3 B^2 + \dots$$

where B refers to one lag ($B r_{t-1} = r_{t-2}$), B^2 refers to two lags and so on.

It can easily be shown that the smoothing model can be represented by an autoregressive model of the form:

$$r_t^* = \phi(B)r_{t-1}^* + e_t \tag{2.6.1}$$

where $\phi(B)$ is the lag operator polynomial and e_t is given by

$$e_t = w_0 r_t$$

Equation 2.6.1 can be inverted to obtain an expression for the underlying return r_t as a function of the past and present values of the observable r_t^* :

$$r_t = \frac{(r_t^* - \phi(B)r_{t-1}^*)}{w_0}$$

The assumption of unpredictable true returns means that $w_0 r_t$ and hence e_t are white noise, so that the autoregressive parameters ϕ in the above equation can be estimated from the observable data on the r_t^* series.

To determine w_0 , a further assumption must be made about the behaviour of r_t . The assumption chosen here is that the “true volatility” of real estate returns of the type represented by appraisal based indices is approximately half the volatility of the of the stock market returns. This assumption, previously used by Geltner [17] and Fisher et al [15] is based on periodic surveys of professional investors’ expectations about real estate risk and returns. Denoting $\sigma^{Stock Market}$ as the yearly standard deviation of the relevant stock market index during the historical period considered, we obtain the following equation for w_0

$$w_0 = \frac{2 SD(r_t^* - \phi r_{t-1}^*)}{\sigma^{Stock Market}}$$

where $SD(r_t^* - \phi r_{t-1}^*)$ can be quantified based on the observable historical return series and the empirical estimation of ϕ .

When ϕ and w_0 have been determined, the true returns can be computed from the index returns. Finally, the real estate returns are corrected for the risk-free interest rate and then real estate volatility and the correlation between real estate, equity and interest rates can be determined from historical data.

2.7 Correlation and Simulation

2.7.1 Correlation

When building a financial model of several different assets or asset classes, it is not enough for the model to describe the rate of return and the volatility of each asset. It is also necessary to describe the way the assets move in relation to each other or, in other words, how they are correlated.

Correlation is a measure of the statistical relationship between two random variables or observed data values. In finance, correlation is most often used to describe how two assets move in relation to one another. The most common measure of correlation between two random variables, X and Y , is the so-called Pearson's correlation

$$\rho(X, Y) := \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

which is +1 in the case of perfect positive linear relationship, -1 in the case of perfect negative linear relationship and 0 if the variables are uncorrelated.

The simplest way of modelling the correlation between assets is to assume that it is constant over time. However, just as examination of market data reveals that volatility is neither constant over time nor constant over moneyness, it is also well known that correlation between assets or asset classes is not constant over time. For example, during a financial crisis volatility usually rises and all assets become highly correlated. To capture this kind of negative tail-dependence it is necessary to use a more complex correlation model, such as a copula, or use expert judgment as a way to estimate the correlation structure.

In the economic scenario generator discussed in this thesis, constant volatility is assumed for each asset class and the correlation between asset classes is also assumed to be constant. We then have a symmetric variance-covariance matrix Σ , where the variance of each asset class is on the diagonal while the covariance between asset classes i and j are in place (i, j) and (j, i) in the matrix. To obtain a d -dimensional correlated Wiener process $\tilde{W}(t) = (\tilde{W}_1(t), \tilde{W}_2(t), \dots, \tilde{W}_d(t))$, d independent one-dimensional Wiener processes are simulated. Multiplying by the independent Wiener process with a lower triangular matrix L generated by the Cholesky decomposition of Σ so that $\Sigma = L \cdot L^T$ yields the desired process [25].

2.7.2 Monte Carlo Simulation

When all parameters have been estimated for each of the models, and the correlation between the asset classes established, a sample space of economical scenarios can be generated. This is done with Monte Carlo simulation which is a statistical method used to study the behaviour of a system, as expressed in a mathematical model. It involves generating random samples of values for uncertain variables that are then plugged into the simulation model and used to calculate outcomes of interest. Monte Carlo methods have been applied to a large number of diverse problems within science, engineering and finance. Areas of application within finance are for example in the pricing of financial derivatives and risk analysis.

To simulate the returns of each asset, the stochastic differential equations must be discretized. Several discretization schemes exist, the most straightforward one being the Euler approximation [24]. Consider an Ito process $X(t)$ satisfying the stochastic differential equation

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t)$$

Discretizing the time interval $[t_0, T]$ with equidistant time steps $t_0 = \tau_0 < \tau_1 < \dots < \tau_n < \dots < \tau_N = T$. An Euler approximation is then a stochastic process Y which satisfies the iterative scheme

$$Y_{n+1} = Y_n + a(\tau_n, Y_n)(\tau_{n+1} - \tau_n) + b(\tau_n, Y_n)(W_{\tau_{n+1}} - W_{\tau_n})$$

where $Y_n = Y(\tau_n)$ and $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$ are random increments of the Wiener process $W = \{W_t, t \geq 0\}$. The increments are independent Gaussian random variables with mean

$$E(\Delta W_n) = 0$$

and variance

$$E((\Delta W_n)^2) = \tau_{n+1} - \tau_n$$

Other discretization schemes, such as the Milstein scheme, give a stronger convergence than the Euler scheme. However, when $b(\tau_n, Y_n)$ is a deterministic function of time, for example in the one-factor Hull-White and Black-Karasinski short rate models, the Euler and Milstein scheme coincide. It is then preferable to use the Euler scheme, as it simpler but gives the same degree of convergence as the Milstein scheme [24].

When each stochastic differential equation has been discretized, an appropriate time step and number of iterations (N) are chosen. For each time step, N realizations of a standard Gaussian distribution are generated and multiplied by $\sqrt{\tau_{n+1} - \tau_n}$ to simulate the distribution of $(W_{\tau_{n+1}} - W_{\tau_n})$. The simulated value of each asset class is then computed at every time point.

Chapter 3

Analysis

3.1 Interest Rate Modelling

3.1.1 Data

In order to do a market consistent calibration of the interest rate models, we use the prices of at-the-money swaptions traded in SEK on the over-the-counter (OTC) market (hereafter called Swedish swaptions) on 31-12-2009, since these instruments resemble the behaviour of the interest rate guarantees embedded in insurance contracts. We also calibrate the models to exactly fit the yield curve on the same date.

3.1.1.1 Yield Curve

We use the Swedish yield curve provided by CEIOPS for the latest qualitative impact study, QIS5 (see figure 3.1). The risk-free interest rate term structure is based on the swap curve which has been adjusted to remove credit risk. Where swaps are either not liquid or not reliable, the risk-free interest rate can instead be referenced to the government curve [10, 13]. The risk-free interest rate term structure also needs to be smooth, which means that both interpolation between data points and extrapolation beyond the last available liquid data point is required.

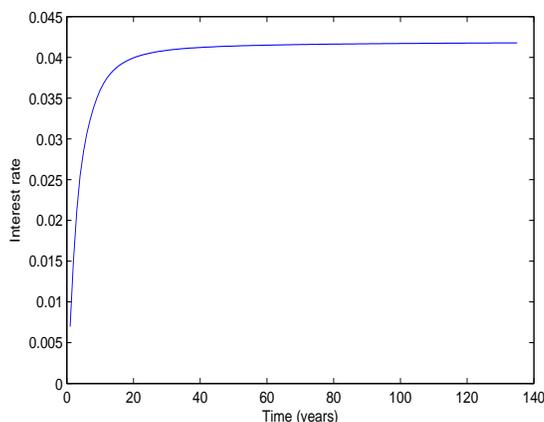


Figure 3.1: CEIOPS risk-free interest rate term structure based on the swap rate curve in SEK, 31-12-2009.

The interpolation and extrapolation technique used by CEIOPS is based on the Smith-Wilson approach, a macroeconomic extrapolation technique which assumes a long-term equilibrium interest rate. It is therefore important to determine an ultimate long-term forward rate, which should be stable over time and only change if there are fundamental changes in long term expectations. The CEIOPS estimate of this ultimate forward rate is based on estimates of the expected inflation and the expected short-term rate. CEIOPS estimates the expected annual inflation to 2% and the short-term return on risk free bonds to 2.2%. The estimated ultimate long-term forward rate is then the sum of these two estimates, i.e. 4.2% [8].

Other important factors in the extrapolation process is at what time the forward rate curve reaches the ultimate forward rate and the speed with which the curve converges towards the ultimate forward rate. CEIOPS sets the time this level is reached to 90 years, but the interval 70-120 years is considered appropriate. The convergence parameter is set to 0.1 since that ensures stable results and curves that look economically sound [8].

CEIOPS reasons for using the Smith Wilson approach are that it is easy to use, gives a forward rate curve which is relatively smooth, and gives an extrapolation of the spot rate curve which is also smooth. Furthermore, it fits input prices exactly.

3.1.1.2 Swaptions

Prices for at-the-money swaptions with maturity up to ten years are available on the OTC market traded in SEK. The prices quoted on the market are not actual prices, instead the swaptions are quoted as Black's implied volatilities (see figure 3.2).

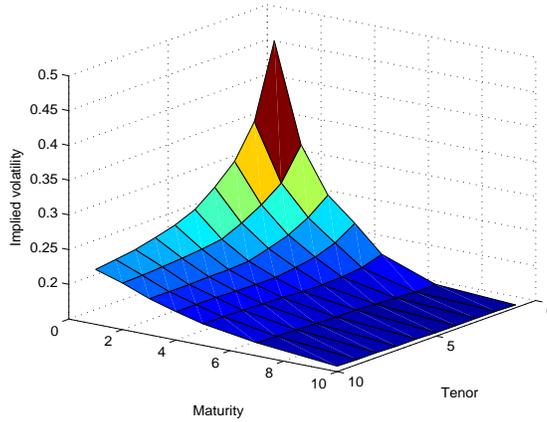


Figure 3.2: Swaption implied volatility surface from swaptions traded in SEK, 31-12-2009

In order to compute the prices, we use a Black-like formula. The price at time $t = 0$ of a payer swaption with maturity T_α , tenor $S = T_\beta - T_\alpha$, strike rate K , nominal value N , and implied volatility σ is given by

$$PS(0, T_\alpha, T_{\alpha+1}, \dots, T_\beta, N, K, \sigma) = NBl \left(K, S_{\alpha, \beta}(0), \sigma \sqrt{T_\alpha}, 1 \right) \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i). \quad (3.1.1)$$

Here, τ_i is the year fraction from T_{i-1} to T_i and $S_{\alpha, \beta}(0)$ is the forward swap rate at time $t = 0$ given by

$$S_{\alpha, \beta}(0) = \frac{P(0, T_\alpha) - P(0, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i)},$$

and Black's formula is as usual given by

$$Bl(K, F, \sigma \sqrt{T}, \omega) = F \omega \Phi(\omega d_1(K, F, \sigma \sqrt{T})) - K \omega \Phi(\omega d_2(K, F, \sigma \sqrt{T})),$$

$$d_1(K, F, \sigma \sqrt{T}) = \frac{\ln(F/K) + \sigma^2 T / 2}{\sigma \sqrt{T}},$$

$$d_2(K, F, \sigma\sqrt{T}) = \frac{\ln(F/K) - \sigma^2 T/2}{\sigma\sqrt{T}},$$

with $\omega = 1$ for a call option and $\omega = -1$ for a put option. Since the calibration is based on at-the-money swaptions, we can simply set $K = S_{\alpha,\beta}(0)$ to compute the market prices.

For the derivation of Black's swaption price formula, see for example Björk [1].

A life insurance company has obligations stretching much further into the future than ten years, which means that the swaption volatility surface needs to be extrapolated to create swaption prices for longer maturities. However, while there are several articles discussing how the interest rate curve should be extrapolated for longer maturities, as well as how to capture the volatility skew for swaptions with different strike prices, very little information can be found on extrapolating the swaption prices for longer maturities.

A possible solution to this problem is to analyse swaption volatility surfaces in economies that have swaptions with maturities longer than ten years, for example the swaption implied volatility surface from swaptions traded in EUR on the OTC market, where swaptions with maturities up to 30 years are available. By looking at the Euro-swaption implied volatility surface (figure 3.3) and comparing this to the implied volatility surface for Swedish swaptions (figure 3.2) we see that the shapes of the surfaces are very similar for maturities up to ten years.

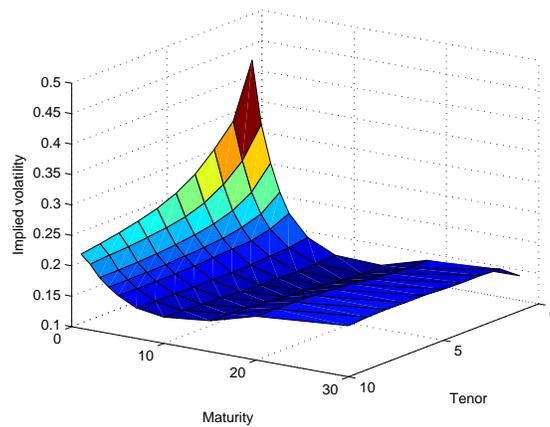


Figure 3.3: Swaption implied volatility surface for swaptions traded in EUR, 31-12-2009

Since the European and the Swedish market are generally highly correlated, the standard approach for financial practitioners in Sweden is to look at Euro-swaptions for longer maturities and adjust these to fit the implied volatility level

of the Swedish swaptions. This new implied volatility surface for the Swedish swaptions can be seen in figure 3.4. The “hump” that appears in the volatility surface for swaptions with maturities of about 20 years is a very interesting feature of the surface. It exists mainly because of the high demand from insurance companies who want to hedge their risks with interest rate derivatives with this maturity. This, in turn, drives the volatilities up in this area.

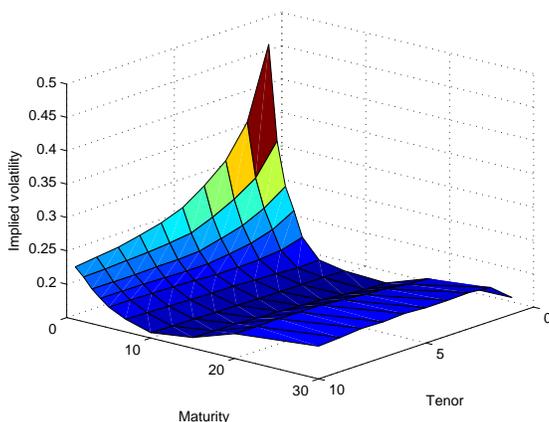


Figure 3.4: Swaption implied volatility surface for Swedish swaptions with implied volatilities for maturities over ten years taken from Euro-swaptions adjusted to fit the volatility level of the Swedish swaptions

However, the two interest rate models we have chosen cannot capture this more complex swaption implied volatility surface, with the hump at about 20 years maturity (see Appendix B). A calibration to this volatility surface leads to a very low, or even a negative a in both models, i.e. negative mean reversion rates. This goes against a fact that generally can be seen in the market; that interest rates are mean reverting. Instead it implies that interest rates in the long term will increase infinitely. The fact that the models assume mean reverting interest rates is a very attractive feature. This means that if we try to calibrate our models to this volatility surface, we will get a much less realistic simulation than if we simplify the volatility surface by assuming that the hump at about 20 years maturity does not exist.

Another option would be to use more complex interest rate models, for example two-factor, or even three-factor, models. However, these are vastly more complex to calibrate and also lead to difficulties for the equity modelling when stochastic interest rates are taken into account in the calibration. Since this thesis not only covers interest rate models, but also equity and real estate models, as well as how these three asset classes are correlated, we see no choice but to use one-factor models calibrated to a simplified version of the swaption volatility surface. Nevertheless, we need to stress that since it is the insurance

companies themselves that are mainly responsible for the hump in the volatility surface, it may be a very important feature to capture when valuing a liability portfolio where the bulk of the liabilities have very long maturities. For this type of portfolio our models will underestimate the value of the liabilities since the higher volatility area is not accounted for. For a liability portfolio where the bulk of the insurance contracts are paid out over the next 20 years, our more simplified volatility surface should be a reasonably good approximation.

For our simplified volatility surface, we use volatilities of Swedish swaptions with up to ten years maturity and extrapolate this surface to get swaptions with longer maturities. The extrapolation is based on the belief that very long-term interest rates should be very stable, i.e. the volatility should converge towards zero in the very long-term. We use the same assumption as CEIOPS do for their interest rate curve; the volatility converges towards zero after about 90 years. The speed with which the volatility surface goes towards zero is another very complex issue; here cubic splines have been used to interpolate between the last available volatility point and the convergence point after 90 years. Further studies are needed on how the swaption implied volatility surfaces might converge towards zero.

The volatility surface for swaptions with up to 50 years maturity which is used for the calibration of our interest rate models can be seen in figure 3.5.

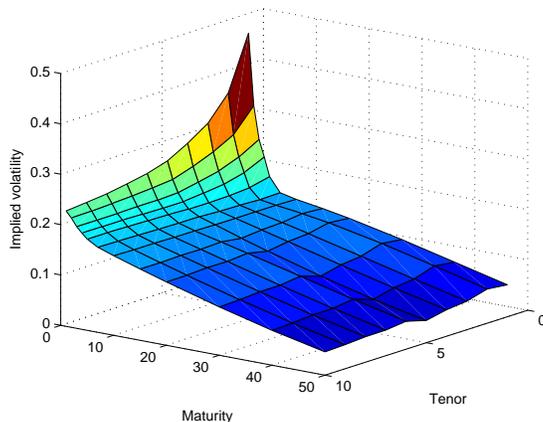


Figure 3.5: Swaption implied volatility surface for Swedish swaptions with market implied volatility for maturities up to ten years and extrapolated volatilities for maturities between ten and fifty years.

3.1.2 Model Calibration and Simulation

The interest rate models are calibrated by finding the deterministic function $\theta(t)$ so that the models exactly match the current term structure of interest rates

and the parameters a and σ which give the best fit of the models to market swaption prices. In other words, a and σ are the values for which the sum of squared differences between market prices and model prices is minimised. For the Hull-White model the model swaption price is given by equation 2.4.3, and for Black-Karasinski model we use the trinomial tree procedure discussed in Chapter 2.4.2. The values for a and σ obtained in the calibration can be seen in table 3.1.

	a	σ
Hull-White	0.0691	0.0113
Black-Karasinski	0.0594	0.273

Table 3.1: Interest rate parameter values calibrated to swaption prices

Next, future paths for the instantaneous short rate are simulated. The Euler scheme is used to discretize the stochastic differential equation. As we can see in figure 3.6 the volatility of the Black-Karasinski simulations is larger than for the Hull-White simulations. Interestingly enough, the mean of the two short rate model scenarios does not differ greatly (figure 3.7). The Black-Karasinski model leads to slightly higher short rates in the long run, but not as high as a quick glance on the scenarios might indicate. The reason for this is that, even though there are some extremely high short rate scenarios, the bulk of the Black-Karasinski short rates lies much closer to zero.

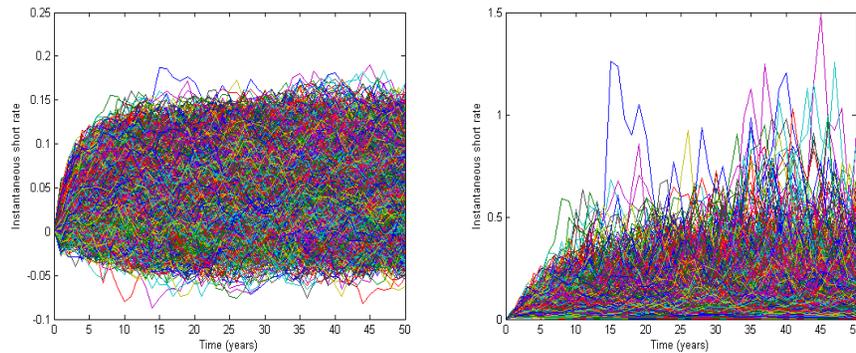


Figure 3.6: Simulated future paths for the instantaneous short rate for the Hull-White model (left) and the Black-Karasinski model (right)

We also clearly see the main drawback of the Hull-White model here; namely negative interest rates. Since the interest rate level was low on the date of calibration, 31-12-2009, the possibility of negative interest rates is quite large. In a more normal market situation this possibility would be very small. The fact that simulation of the Hull-White model results in many negative short

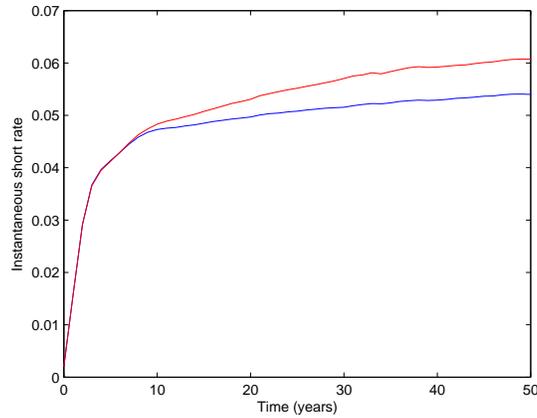


Figure 3.7: Comparison of the mean of the simulated future paths for the instantaneous short rate with Hull-White (blue) and Black-Karasinski (red)

rate scenarios and simulation of the Black-Karasinski model leads to scenarios where the short rate can be up to 150% means that neither of the models actually captures reality since negative and extremely high interest rates are not something generally observed on the Swedish market. This is the problem when choosing interest rate models - if they are simple enough to implement in a limited amount of time, they are generally not able to capture the market in a satisfying manner; and if they manage to capture the market adequately they are generally highly complex and difficult to implement.

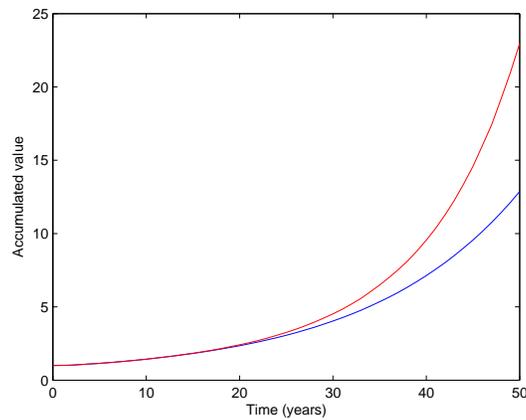


Figure 3.8: Investing yearly in a five year zero-coupon bond for 50 years with Hull-White interest rates (blue) and Black Karasinski interest rates (red)

Since the Hull-White model gives us an explicit formula for zero-coupon bond prices (2.4.2), the calculation of these prices is fairly straightforward. For the Black-Karasinski model the computation of zero-coupon prices is much more difficult, since no analytical formula exists. Instead, we have to resort to approximating the zero-coupon prices with trinomial trees. A comparison of the accumulated value of investing yearly in a five year zero-coupon bond can be seen in figure 3.8. We see that we get a higher return with the Black-Karasinski zero-coupon bond, which is not surprising since the mean of the short rate scenarios is higher for Black-Karasinski model than for Hull-White model.

3.1.3 Validation

The purpose of market consistent valuation is to put a value on insurance liabilities that is consistent with the value they would have if they were traded on the market. It should therefore be consistent with the prices we can observe on the market for financial instruments which are similar to the insurance liabilities. Therefore, we now check to see how close the theoretical model prices and the prices based on the simulations are to the real market prices for the swaptions we have used to calibrate the models.

In figure 3.9 we can see the prices for the 5 year and 10 year tenor swaptions. We see that both models fare reasonably well, they do not, however, capture the prices exactly. Since two volatility parameters (a and σ) are obtained from the calibration, two swaption prices can be captured exactly. But since the model is calibrated to 120 swaptions, it cannot capture all these prices exactly. Instead we get an average price curve over all the swaption prices.

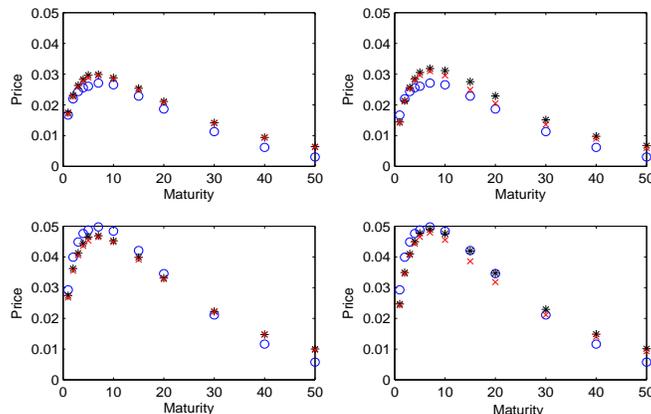


Figure 3.9: Comparison of market prices (blue), theoretical model prices (red), and simulated prices (black) for swaptions with 5 year tenor (above) and 10 year tenor (below). Hull-White prices on the left and Black-Karasinski prices on the right.

So, which model performs better, i.e. is more market-consistent? In figure 3.9 it is not possible to see clearly whether one model is better than the other since they both indicate prices that are slightly higher or slightly lower than the market prices. It is also important to remember that the figure only displays options on the 5-year swap and the 10-year swap; when in fact the models have been calibrated to options on the 1-year, 2-year, ..., 10-year swaps. In order to decide which model fares better over all the swaption prices we compute root mean squared errors (RMSE) between the market prices and model prices, as well as between the market prices and the simulated prices (see table 3.2). We see that the Hull-White model performs slightly better than the Black-Karasinski model, even though the difference between the models is very small. A possible reason why the RMSE for the simulated prices is larger for Black-Karasinski compared with the RMSE for the theoretical model prices, is that for the simulated Black-Karasinski interest rates we have to use the approximating trinomial trees to calculate the swaption prices, while there exists an analytical formula for the Hull-White model.

	RMSE theoretical prices	RMSE simulated prices
Hull-White	$1.97 \cdot 10^{-3}$	$2.00 \cdot 10^{-3}$
Black-Karasinski	$2.24 \cdot 10^{-3}$	$2.75 \cdot 10^{-3}$

Table 3.2: RMSE for the theoretical model prices and the simulated prices for Hull-White and Black-Karasinski.

Based on the RMSEs the Hull-White model performs slightly better than the

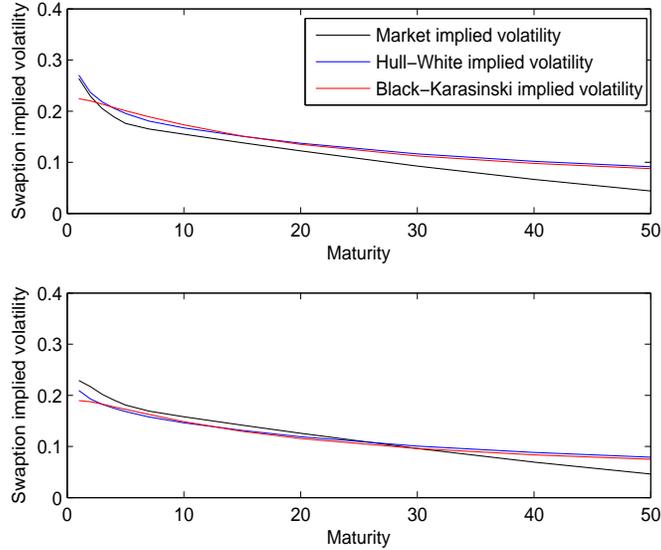


Figure 3.10: Hull-White model implied volatilities (blue) and Black-Karasinski implied volatilities (red) compared with the market implied volatilities (black). Implied volatilities from five year tenor swaptions above, and ten year tenor swaptions below.

Black-Karasinski model. Since the Hull-White model is far easier to calibrate and simulate, this would be our model of choice. However, it is important to remember that these results are based on market prices from one particular day; if the market situation changes even slightly it might mean that the Hull-White model is not the better of the two anymore. Therefore, no conclusion can be drawn from this validation on which model would generally perform better.

It is also interesting to examine the model implied volatilities compared with the market implied volatilities for the swaptions. To compute the model implied volatilities, the implied volatilities are computed from Black's swaption price formula (3.1.1). The model implied volatilities can be seen in figure 3.10. We see that the model implied volatilities for both models capture the shape of the swaption implied volatility quite well. We also see that the Hull-White implied volatility curves and the Black-Karasinski implied volatility curves are very similar, apart from at very short maturities where the Hull-White curve matches the market implied volatility curve slightly better.

3.2 Equity Modelling

3.2.1 Data

The OMX Stockholm 30 (OMXS30), which is a stock index for the Stockholm stock exchange, is modelled. This is a market value-weighted index that consists of the thirty most-traded stocks on the Swedish market. The index value reflects the total return of the weighted index constituents, as the value is adjusted by reinvesting all dividends in the index constituents in proportion to their respective weights. The prices of at-the-money options available for the index on 31-12-2009 are used to perform a market consistent calibration of the equity models.

3.2.2 Extrapolation of Option Volatility

In the calibration of the equity model, the same problem is encountered as in the calibration of the interest rate model, namely that the instruments available on the market have a shorter duration than the insurance liabilities. On the Swedish market, the at-the-money options on the OMXS30 index on the 31-12-2009 are liquid up to a maturity of only 18 months. Because of this, extrapolation of market data is necessary.

The extrapolation method used here is one that is discussed by CRO Forum, which is professional risk management group that develops and promotes best practices in risk management [10]. The method assumes that the forward volatility follows a mean reverting process towards the long-term forward volatility. This is in accordance with the Heston model which describes volatility as a mean reverting stochastic process, where the process reverts back to its long-term level with a certain speed. From the Heston model, the following simple and intuitive functional form for the forward variance is assumed

$$Var_{forward}(t) = IV_0^2 e^{-\alpha t} + IV_\infty^2 (1 - e^{-\alpha t})$$

This implies the following form for the implied volatility

$$IV_{model}(T) = \sqrt{\frac{1}{T} \int_{t=0}^T Var_{forward}(t) dt} = \sqrt{IV_\infty^2 + \frac{1}{\alpha T} (1 - e^{-\alpha T})(IV_0^2 - IV_\infty^2)}$$

To obtain values for IV_0 , IV_∞ and α we minimise the root mean square error between the available market data for all maturities and the model implied volatility:

$$RMSE = \sqrt{\frac{1}{N} \sum_T (IV_{model}(T) - IV_{market}(T))^2}$$

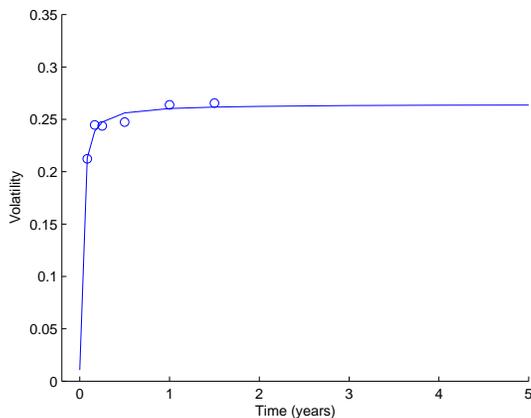


Figure 3.11: Option implied volatility of at-the-money call options on the OMXS30 index (circles) together with the extrapolation curve derived from the option volatilities on the market.

with the constraints that α should be non-negative and $1.05\sigma_{BE} \leq \sigma_{\infty} \leq 1.4\sigma_{BE}$ where σ_{BE} is the best-estimate assumption for volatility based on historical data for the relevant equity. This constraint is based on the observation that long-term implied volatility is consistently higher than long-term historical volatility over time. This can partly be explained by the capital cost of non-hedgeable market risk which is included in the estimation of implied volatility, and partly by uncertainty in the long-term best-estimate volatility.

The results of the extrapolation can be seen in figure 3.11.

3.2.3 Model Calibration and Simulation

The equity model is calibrated by minimising the sum of squares between the market price and model price of at-the-money European call options with up to 50 years maturity. Both model parameters (σ_S and $\rho_{r,S}$) are outputs of the calibration. To calibrate the model under Hull-White interest rates, exact formulas for the option prices exist, while the calibration of the model under Black-Karasinski interest rates requires the construction of a two-dimensional trinomial tree to obtain the option prices.

The values for the volatility and correlation between interest rates and equity are shown in table 3.3.

The results from the calibration show that the volatility is fairly independent of interest rate assumption, while the correlation varies more with the depending on which interest rate model is chosen. Nonetheless, both interest rate models result in a negative correlation between interest rates and equity. It should be

Interest rate model	σ_S	$\rho_{r,S}$
Hull-White	0.267	-0.224
Black-Karasinski	0.283	-0.414

Table 3.3: Equity parameter values from calibration to option prices under different interest rate models.

pointed out here that the calibration of the equity model under Black-Karasinski interest rates is extremely sensitive to the interest rate parameters a and σ . With the values of a and σ obtained from the calibration of the Black-Karasinski interest rate model to swaption prices on the 31-12-2009, the trinomial interest rate tree produces extremely high interest rates “far out in the branches” when using a time step of less than half a year. This is not problematic when calibrating the interest rate model itself, but when the interest rate tree is combined with the equity price tree, the high interest rates result in equity prices that, over a longer time period, are too high for the model to handle. Although the probability of reaching these very high interest rates is negligible, the existence of the corresponding extremely high equity prices causes the algorithm to fail. Because of this, the equity model under Black-Karasinski interest rates is only calibrated with a time step of half a year and the parameter values from the calibration are therefore not too reliable.

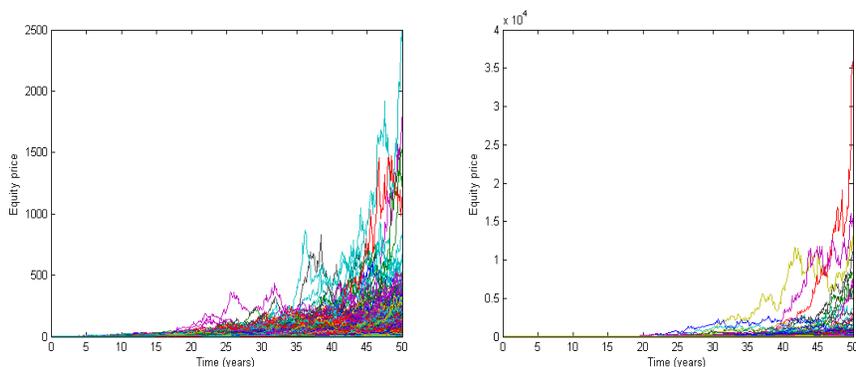


Figure 3.12: Simulated future paths of equity price under Hull-White interest rates (left) and Black-Karasinski interest rates (right).

When the models have been calibrated, future paths for the equity price are simulated. Figure 3.12 shows the simulated paths of the equity price under Hull-White interest rates (left) and under Black-Karasinski interest rates (right). Note the difference in scale between the simulations. As we saw in the previous section on the difference between the two interest rate models, the interest rates produced by the Black-Karasinski model are more volatile than the interest rates produced by the Hull-White model. The higher volatility interest rates

results in a few extremely high equity price scenarios when equity is modelled under Black-Karasinski interest rates.

The average of all simulations can be seen in figure 3.13. The difference between the two interest rates models is also clear when comparing the means of equity prices. The equity prices are much higher after 50 years of Black-Karasinski interest rates than after 50 years of Hull-White interest rates.

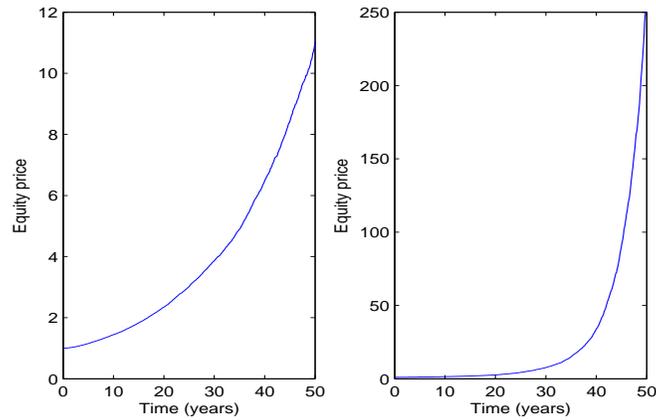


Figure 3.13: Comparison of the average of simulated future equity prices under Hull-White interest rates (left) and Black-Karasinski interest rates (right).

3.2.4 Validation

A simple validation of the equity models is performed in the same way as for the interest rate models, i.e. by using the models to calculate theoretical and simulated option prices and comparing the calculated prices to market prices. Figure 3.14 shows a comparison between market prices, theoretical model prices and simulated prices.

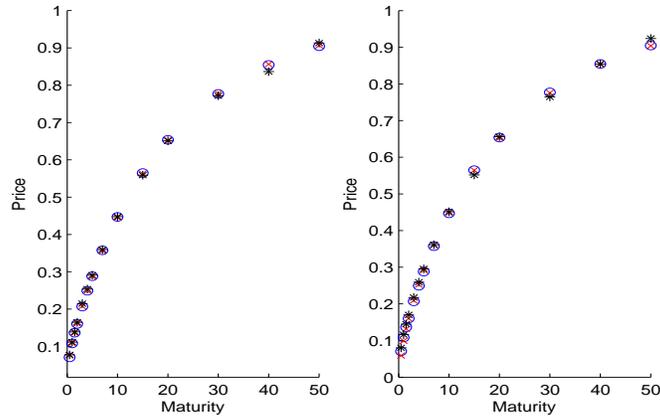


Figure 3.14: Comparison of market prices (blue circles), theoretical prices (red x) and simulated prices (black stars) for call options with maturity up to 50 years. The option prices under Hull-White interest rates are on the left, while the option prices under Black-Karasinski interest rates are on the right.

We see that both models manage to capture the market prices very well. The root mean square error (RMSE) between the market prices and model prices, as well as between the market prices and simulated prices is calculated and can be seen in table 3.4.

	RMSE theoretical prices	RMSE simulated prices
Hull-White	$1.9 \cdot 10^{-3}$	$6.7 \cdot 10^{-3}$
Black-Karasinski	$4.6 \cdot 10^{-3}$	$9.6 \cdot 10^{-3}$

Table 3.4: RMSE for the theoretical model prices and simulated prices.

The results from the validation show that the equity model calibrated to Hull-White interest rates gives option prices that are closer to market prices than the model calibrated to Black-Karasinski interest rates. As we noted in the previous section, the calibration of the Black-Karasinski equity model using trinomial trees was not stable for a time step of less than half a year. In light of this, the performance of the equity model under Black-Karasinski interest rates is surprisingly good.

Again, it is difficult to draw any final conclusion on the performance of the respective equity models. The current market situation makes the Black-Karasinski equity model unsuitable as a reliable calibration cannot be performed because of the interest rate parameters obtained from the calibration to swaption implied volatilities. Other market conditions would give other model parameters

for the interest rate model and this would in turn affect the calibration of the equity model. However, the two-dimensional trinomial tree, which is needed for the calibration under Black-Karasinski interest rates, is also quite complicated and time consuming compared with the relatively simple calibration methods which can be used for calibrating the equity model under Hull-White interest rates. This makes the Black-Karasinski equity model less attractive than the Hull-White model. Naturally, when evaluating performance of the two equity models one must also take into account the performance of the interest rate model, since one can only use the equity model under Hull-White interest rates if in fact the chosen interest rate model is the Hull-White model, and analogously for the Black-Karasinski model. This means that, for an insurance company investing in both interest rate derivatives and equity, the focus must lie on which interest rate model produces the overall best results during the validation of both interest rates and equity. However, since the stochastic feature of interest rates greatly affects long term equity option prices, a reasonable assumption is that the interest rate model that best captures the swaption prices also will be the best choice when calibrating the equity model.

Comparing the validation plots (figure 3.14) of the equity models with the validation plots (figure 3.9) of the interest rate models, we see that the equity models generally match the shape of the price curve better than the interest rate models. This reflects the higher complexity of the implied volatility structure of interest rates compared with the corresponding volatility structure of equity. The interest rate volatility structure is harder to capture in a relatively simple model and this is revealed in the validation of the models. It should however be noted that the calibration of the equity model is also very reliant on the extrapolation method used to generate option volatilities for longer maturities. The method used here, assumes that equity volatility reverts quickly to its long-term level as can be seen in figure 3.11. This means that when calibrating to options with maturities up to 50 years, a large part of them will have an implied volatility very close to the long-term level. In other words, there is little variation in the volatility of the instruments used for the calibration of the model and the option prices generated from a model that assumes constant volatility will therefore be close to the option prices used for the calibration.

In section 2.5.2 we discussed the necessity of incorporating stochastic interest rates into the equity model when pricing options with long maturities. Calibrating the standard Black-Scholes model from options on the market and then simulating equity price under deterministic interest rates does therefore not seem like an equity model worth considering. However, instead of calibrating the model under stochastic interest rates, an alternative approach might be to calibrate equity volatility from the standard Black-Scholes call-price formula. A simulation can then be run using the Hull-White or Black-Karasinski interest rates as before, but using the volatility obtained from calibration with the standard Black-Scholes formula and the historical correlation between equity returns and interest rates as approximations for the respective parameters in the equity model. The parameters obtained from this approach are shown in

table 3.5.

	σ_S	$\rho_{r,S}$
Alternative calibration	0.325	-0.337

Table 3.5: Equity parameter values. σ_S is obtained from calibration to option prices through the standard Black-Scholes call price formula, while $\rho_{r,S}$ is the historical correlation between equity returns and the one-month risk-free interest rate.

The results from simulating the equity prices using these parameters and then calculating the root mean square errors can be seen in table 3.6.

	RMSE simulated prices
Hull-White simulation	$3.7 \cdot 10^{-2}$
Black-Karasinski simulation	$4.1 \cdot 10^{-2}$

Table 3.6: RMSE for the option prices simulated using the alternative parameter values but under stochastic interest rates.

We see that the RMSE between the option prices from the simulations and the market prices is considerably larger than the RMSE when using the parameters from the calibration under stochastic interest rates. Nevertheless, this approach manages to capture the market prices relatively well and is simpler than the calibration of an equity model under stochastic interest rates. The calibration of the Black-Karasinski equity model is, as stated above, particularly complex and using this approach instead of building a two-dimensional trinomial tree might be considered a realistic alternative if one wishes to use the Black-Karasinski model for interest rates.

3.3 Real Estate

3.3.1 Model Calibration

The real estate index we have chosen to model is the IPD Sweden Annual Property Index (Svenskt Fastighetsindex, SFI), which has been published annually since 1997. The index measures the returns achieved on directly held real estate investments. It is compiled from valuation and management records for individual buildings in complete portfolios.

First, we unsmooth our real estate index, as discussed in Section 2.6.2. The lag polynomial $\phi(B)$ in (2.6.1) is chosen as ϕ_1 which makes the autoregressive model an AR(1) process. This is done partly because we wish to have a simple model as we have a limited amount of data, but it has also been shown that an

AR(1) model will capture optimal appraiser behaviour at the level of individual real estate valuations [27]. Applying this to our unsmoothing model we get

$$r_t^* = \phi r_{t-1}^* + e_t$$

where r_t^* is the inflation-adjusted unsmoothed real estate returns, $e_t = w_0 r_t$, and ϕ is determined with standard time series analysis methods. The true returns can now be expressed with a simple formula

$$r_t = \frac{(r_t^* - \phi r_{t-1}^*)}{w_0}$$

where w_0 is derived from the assumption that the true volatility of the IPD Sweden Annual Property Index is approximately half the historical volatility of the OMXS30 Index. Denoting σ^{OMXS30} as the yearly standard deviation of the OMXS30 Index during the historical period considered, we obtain the following equation for w_0

$$w_0 = \frac{2 SD(r_t^* - \phi r_{t-1}^*)}{\sigma^{OMXS30}}$$

where $SD(r_t^* - \phi r_{t-1}^*)$ can be quantified based on the observable historical return series of the real estate index and the empirical estimation of ϕ .

The returns of the real estate index can be seen together with the returns of the unsmoothed index in figure 3.15 (left hand side). The corresponding index values are on the right hand side of figure 3.15.

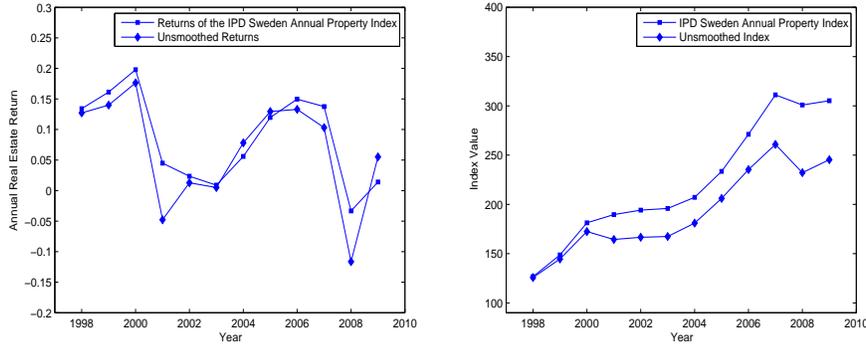


Figure 3.15: Annual returns (left) and index value (right) of the IPD Sweden Annual Property Index (squares) and the unsmoothed returns of the index (diamonds).

To calculate the volatility (σ_P) of real estate returns and the correlation between real estate returns and interest rate on the one hand and real estate returns and equity returns on the other hand ($\rho_{r,P}$ and $\rho_{S,P}$), the real estate returns are

first discounted with the appropriate historical risk-free interest rate. This is done to isolate the volatility of real estate as an asset class from the volatility due to the risk-free interest rate [11]. We set r_k as the discounted real estate log-returns over the time period $(k - 1) \Delta t \leq t < k \Delta t$ so that

$$r_k = \ln \frac{S_D(k \Delta t)}{S_D((k - 1) \Delta t)} \quad k = 1, 2, \dots, n$$

where S_D is the discounted real estate value

$$S_D(t) = S(t) \exp\left(-\int_0^t i(s) ds\right)$$

and $i(s)$ is the risk-free interest rate. Finally, by defining $r_k = \phi_S + \epsilon_k$, where ϕ_S is a constant and ϵ_k are i.i.d. normal random variables, the volatility of real estate can be calculated.

The results can be seen in table 3.7 together with the results from the original real estate index.

	σ_P	$\rho_{r,P}$	$\rho_{S,P}$
Unsmoothed index	0.0921	-0.456	0.719
SFI	0.0752	-0.181	0.537

Table 3.7: Real estate parameter values based on historical data using the returns of the unsmoothed IPD Sweden Annual Property Index and the returns of the index without unsmoothing.

These results show that real estate returns are negatively correlated with the risk free interest rate, but positively correlated with the equity. This is not surprising as real estate as an asset class is generally considered to behave in a similar manner as stocks. Comparing the parameters from the unsmoothed index with the parameters from the original real estate index, we find that the original real estate index returns have a lower volatility than the unsmoothed returns, although the difference is not large. Furthermore, we see that the returns from the original index are less positively correlated with the stock index returns than the unsmoothed real estate index returns and also less negatively correlated with interest rates. The reason for this is clear when the plot of the returns of the real estate index together with the unsmoothed returns, is examined. The general shape of the unsmoothed returns follows the shape of the index returns, but has sharper peaks. Thus, any correlation, whether positive or negative, will be larger when the index returns have been unsmoothed.

3.3.2 Simulation

When the model parameters for the real estate model have been computed, future paths of real estate prices can be simulated. The simulation is done both

under the assumption of Hull-White interest rates and Black-Karasinski interest rates. The results can be seen in figure 3.16.

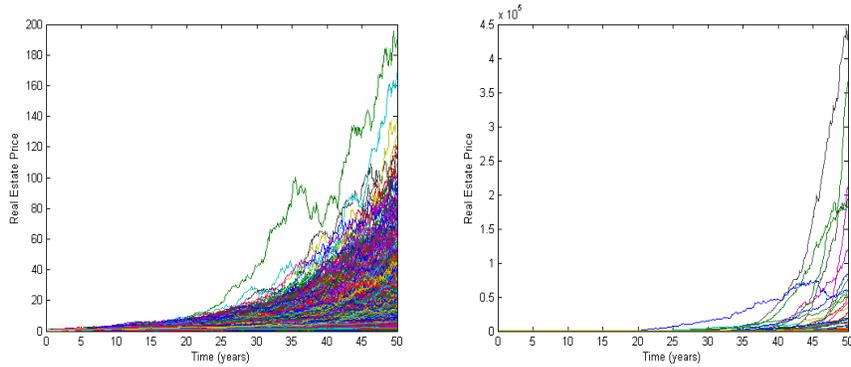


Figure 3.16: Simulated future paths of real estate prices under Hull-White interest rates (left) and under Black-Karasinski interest rates (right)

As in the simulation of equity, there is an obvious difference between the simulated values depending on whether they were simulated under Hull-White interest rates or Black-Karasinski interest rates. This same applies to the means of the simulated values, i.e. the mean real estate prices are much higher after 50 years of Black-Karasinski interest rates than after 50 years of Hull-White interest rates as seen in figure 3.17.

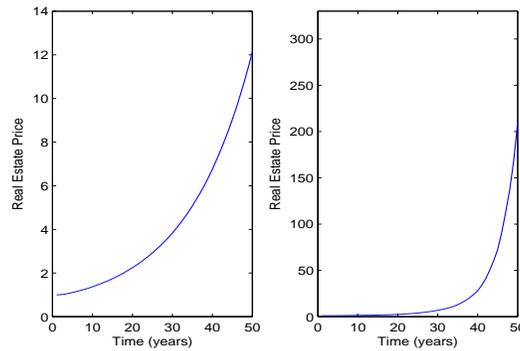


Figure 3.17: Comparison of the average of simulated future real estate prices under Hull-White interest rates (left) and Black-Karasinski (right).

The calibration of the real estate model is based on different principles than both the interest rate and the equity model as it is not calibrated to options on the market. Because of this, a formal validation of the results is not possible.

The construction of the real estate model exemplifies some of the difficulties

involved in building a market consistent ESG. For this asset class, there are no calibration instruments available on the Swedish market and the model parameters must therefore be obtained from the analysis of historical data. In this case, the data on historical returns is very short and may not be accurate as it is based on real estate appraisals and not market prices. Furthermore, the unsmoothing procedure that is considered necessary to obtain the true real estate returns is heavily dependent on the very broad assumption that the true volatility of real estate returns is half the historical volatility of the stock market. However, despite the assumptions made and the lack of data, this model should give some indication of the behaviour of real estate as an asset class.

3.4 An Example Valuation

The objective of this section is to compute the present value of the net cash flows (NCF) stemming from an insurance contract with an annual interest rate guarantee. We will see how different choices of asset portfolios and different levels of guaranteed interest rates change the value and compare the different outcomes depending on which interest rate model is used. The parameter values estimated previously in this chapter are used.

To simplify, we compute the present value the NCF stemming from one single insurance contract with an annual interest rate guarantee of g per cent. The policy holder pays a premium of 100 SEK at year 0. This is invested in equity, real estate, and zero-coupon bonds with a duration of one year. At the start of each new year the portfolio weights are adjusted to ensure that it consists of the original proportion of bonds, equity and real estate. The money is paid out at year 50.

This means that at year 0, the money is invested in a portfolio consisting of equity, real estate, and zero-coupon bonds. At the end of each year, the generated scenarios are used to compare the return on the portfolio with the interest rate guarantee. If the return on the portfolio is above the interest rate guarantee, the policy holder receives whichever is higher of 90 per cent of the return on the portfolio and the interest rate guarantee. What is left of the return on the portfolio goes to the insurance company. If the return of the portfolio is below the interest rate guarantee, the insurance company needs to make up the difference between the return and the guarantee, by injecting capital into the portfolio. The present value of the net cash flow is therefore the present value of the difference between the capital injections and the profit the insurance company makes when the return of the portfolio is above the guaranteed rate. A positive present value of net cash flows in this case means that the discounted capital injections are higher than the discounted gain of the company when returns are high.

Table 3.8 shows the present value of NCF for different portfolios and different levels of interest rate guarantees. We see that when the amount of equity and real estate in the portfolio increases, the present value of NCF increases. The

Guaranteed rate	Proportions			Hull-White	Black-Karasinski
	Bond	Equity	Real estate	Present value of net cash flows	Present value of net cash flows
0.03	0.90	0.05	0.05	7	4
0.03	0.70	0.20	0.10	74	82
0.03	0.50	0.30	0.20	174	192
0.04	0.90	0.05	0.05	34	34
0.04	0.70	0.20	0.10	122	133
0.04	0.50	0.30	0.20	251	275
0.05	0.90	0.05	0.05	81	85
0.05	0.70	0.20	0.10	194	208
0.05	0.50	0.30	0.20	361	392

Table 3.8: Present value of net cash flows for different choices of asset portfolios and different levels of interest rate guarantees.

reason for this is that the returns of a portfolio, with a higher proportion of equity and real estate, are more volatile than the returns of a portfolio with a higher proportion of bonds. Since the policy holder gets most of the excess return in high interest rate scenarios and the insurance company has to make up the whole difference in low interest rate scenarios, the insurance company loses more when the interest rate scenarios are very low compared to what it gains when the interest rate scenarios are very high. Therefore, the NCF is higher for a more volatile portfolio than for a less volatile portfolio even though the average rates of return on the portfolios are the same.

We also note that there is a difference between the NCF produced by the two interest rate models. For the portfolios with a lower proportion of bonds (70% and 50% respectively) Black-Karasinski interest rates results in a higher NCF than Hull-White interest rates. This can be explained by the fact that the equity model under Black-Karasinski interest rates has a higher volatility (28.3%) than the volatility under the Hull-White model (26.7%) and as the proportion of equity in the portfolio increases, the volatility of equity starts dominating.

For the portfolios with a high proportion of bonds (90%) we note that when the guaranteed rate is 3% Hull-White produces a higher present value of NCF than Black-Karasinski; when the guaranteed rate is 4% the present value of NCF under the two models are about the same; and when the interest rate guarantee is 5% Hull-White produces a lower present value of NCF than Black-Karasinski.

That Hull-White should produce a higher present value of NCF, as it does when the guaranteed rate is 3%, seems logical. After all, Hull-White produces a number of scenarios where interest rates are negative, while the Black-Karasinski model always produces positive interest rates. Then why does the Hull-White

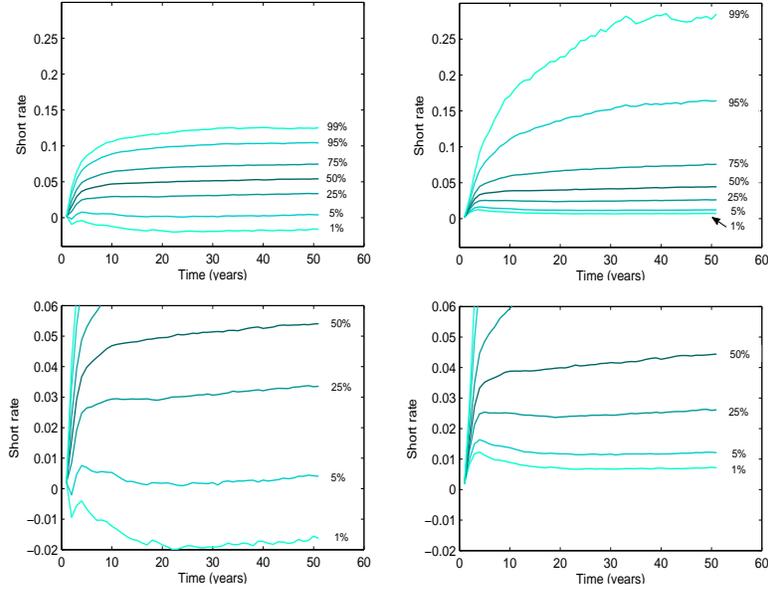


Figure 3.18: The percentiles for the Hull-White interest rate simulations (left) and for the Black-Karasinski interest rate simulations (right). Below is a zoomed in plot of the percentiles for Hull-White (left) and Black-Karasinski (right).

model imply a lower NCF for a higher guaranteed rate? The explanation for this can be seen in the percentile plots for the simulated interest rates for the two models (figure 3.18). Note that for the Hull-White model, the 50th percentile lies far higher than the 50th percentile for the Black-Karasinski model. In fact, the 50th percentile for the Black-Karasinski interest rates is on the same level as the 25th percentile for the Hull-White interest rates which explains why Hull-White results a lower NCF for higher guaranteed rates. The 5th percentile for the Hull-White scenarios is lower than the 5th percentile for the Black-Karasinski scenarios, which means that the very low rates for Hull-White are lower than the very low Black-Karasinski rates. However, when we look at the 25th percentile for the two models, we see the opposite relation; the Hull-White 25th percentile is clearly higher than the corresponding percentile for Black-Karasinski. This means that, even though the very low rates are lower in the Hull-White model, there is a larger number of scenarios in the Black-Karasinski model which lie below the guaranteed rate of 5%, and this causes the NCF to be higher in the Black-Karasinski model. For the 3% guaranteed interest rate case we can still see that there is a larger number of scenarios below the guaranteed rate for Black-Karasinski than for Hull-White, but Black-Karasinski still produces a lower NCF since the very low interest rates in the Hull-White case now makes up a higher proportion of the scenarios that fall below the guaranteed rate.

Chapter 4

Summary and Conclusions

We have described the construction of a market consistent economic scenario generator which can be used to value insurance liabilities, specifically those with multi-period minimum interest rate guarantees. The asset classes we modelled were interest rates, equity, and real estate. The ESG was calibrated to option prices, wherever these were available. Otherwise the calibration was based on an analysis of historical volatility.

Two one-factor short rate models were implemented. The validation of the models showed that the Hull-White interest rate model was more market consistent than the Black-Karasinski model, since the swaption prices implied by the model were closer to the market prices. However, the difference between the swaption prices implied by the two models was very small, hence we cannot draw any conclusions regarding which model is generally more market consistent. It is also important to remember that the calibration was based on market data from one specific date; data from a different date might not produce the same results. The Hull-White model has one main drawback; namely, the possibility of negative interest rates. Nevertheless, it is far easier to calibrate and simulate from than the Black-Karasinski model since it gives analytical formulas for both bond prices and swaption prices. When using the Black-Karasinski model, one has to resort to using approximating trinomial trees, a procedure which is very computationally expensive. For this reason, when the implied model prices are so similar, the Hull-White model is preferable.

The equity model was calibrated under the assumption of constant equity volatility and stochastic interest rates, using both the Hull-White and the Black-Karasinski one-factor models. Validation of the models showed that both captured option prices on the market quite well, though as before the model under Hull-White interest rates performed better than the model under Black-Karasinski interest rates. Again, using Black-Karasinski interest rates makes the calibration of the model much more complicated and time consuming, which makes the equity model under Hull-White interest rates the equity model of

choice, at least in the current market situation. An alternative approach to the calibration of the equity model was also implemented. Here, equity volatility was calibrated from the standard Black-Scholes call price formula, and the correlation between interest rates and equity was based on historical data. Using the parameters from this calibration did not produce model prices that capture market prices as well as the models that assume stochastic interest rates, but this model performed well enough to make it an alternative worth considering if one would want a simpler calibration procedure for the equity model.

The calibration of the real estate model followed different principles than the calibration of the interest rate models and the equity models. The reason for this is that no options on real estate prices are available on the market. Therefore the calibration of the real estate model was based on the historical returns of a real estate index, which had to be unsmoothed to better reflect the volatility of real estate, and the correlation between real estate and the other two asset classes. Because of the lack of calibration instruments on the market, no validation was performed on the real estate model.

A simple valuation example finally showed how the proportions of the three asset classes in an insurance company's portfolio affect the valuation of an insurance contract with an annual interest rate guarantee. This example demonstrated that for the insurance company, the least risky portfolio was the one that invested the highest proportion of the capital in bonds.

A market consistent calibration of an ESG presents many challenges. One of the most prominent of these is to decide whether to calibrate to option prices or to base the calibration on historical volatilities. The aim of this thesis is to value the liabilities stemming from insurance contracts with guarantees which have option-like features. Therefore it is important that the ESG can reproduce market prices of instruments with similar characteristics as the liabilities being valued, i.e. options. However, since option prices are not always reliable or available, and since a calibration to these prices can lead to pro-cyclicality in stressed markets, there are situations where calibrating to historical volatility is more appropriate. For this reason it would be interesting to compare the results from a calibration to option prices with a calibration based on analysis of historical volatilities.

A difficulty when calibrating to option prices is the lack of long-term calibration instruments. In a perfect world, instruments with maturities up to the duration of the insurance liabilities would be available on the market. This is of course not the case, especially not on the relatively small Swedish market. Because of this, methods need to be developed for the extrapolation of market implied volatility. Relatively little research has been done in this area so far, but this will certainly be a research focus in the next years as the insurance companies begin implementing market consistent valuation methods to comply with Solvency II.

The ESG could be improved in several directions depending on the asset portfolio of the insurance company. One way of improving the ESG would be to include models that describe inflation, credit risk and exchange rates. There

are also other, more complex, interest rate models that capture the market better, for example two-factor, or even three-factor models. However, the focus here will be different for different insurance companies and different insurance contracts. If the company mainly invests in interest rate derivatives, the focus should be on improving the interest rate models. A portfolio of insurance contracts with a fairly low average duration might not need an ESG calibrated to long-maturity options, while the extrapolation of option implied volatilities for longer maturities is more important when valuing insurance contracts with longer durations.

In general, a market consistent ESG should be able to capture the complexity of the market and how different asset classes are correlated. However, it is also important that it is simple, transparent and robust; in fact, one important feature of Solvency II is that the models and methods used within the company should be well understood by senior management. Therefore, one must always ensure that the benefit of implementing more advanced models outweighs the increased complexity they bring.

Appendix A

The Construction of a Trinomial Tree

A.1 Approximating the Interest Rate with a Trinomial Tree

The following section describes the procedure of constructing a trinomial tree of interest rates used to value bond and interest rate derivative prices. The procedure can be divided into two stages, where in the first stage a trinomial tree is constructed for the process x and in the second stage the tree for x is converted into a tree for r by displacing the nodes on the tree for x to perfectly match the current structure of interest rates.

In the first stage we build a tree for x which is initially zero and follows the process

$$dx = -axdt + \sigma dW(t)$$

This process is symmetrical around $x = 0$. Notice that for the Black-Karasinski model we can write

$$r(t) = e^{\alpha(t)+x(t)}$$

where α is given by (2.4.5).

We begin by fixing a finite set of times $0 = t_0 < t_1 < \dots < t_n = T$ and set $\Delta t_i = t_{i+1} - t_i$. The tree nodes are denoted by (i, j) where the time index i ranges from 0 to N and the space index j ranges from some $\underline{j}_i < 0$ to some $\bar{j}_i > 0$. Thus, the process value at node (i, j) is $x_{i,j}$.

The mean and variance of the process are

$$E \{x(t_{i+1})|x(t_i) = x_{i,j}\} = x_{i,j}e^{-a\Delta t_i} = M_{i,j}$$

$$Var \{x(t_{i+1})|x(t_i) = x_{i,j}\} = \frac{\sigma^2}{2a} [1 - e^{-2a\Delta t_i}] = V_i^2$$

At each time t_i , we have a finite number of equispaced states with a constant vertical step Δx_i . We set $x_{i,j} = j\Delta x_i$, where

$$\Delta x_i = \sigma \sqrt{\frac{3}{2a} [1 - e^{-2a\Delta t_{i-1}}]}$$

Assume that at time t_i we are on the j -th node with associated value $x_{i,j}$. The process can now move to $x_{i+1,k-1}$, $x_{i+1,k}$ and $x_{i+1,k+1}$ in the next time step t_{i+1} with probabilities p_u , p_m and p_d respectively. The central node is therefore the k -th node at time t_{i+1} where the level k is chosen as close as possible to $M_{i,j}$, i.e.

$$k = \text{round} \left(\frac{M_{i,j}}{\Delta x_{i+1}} \right)$$

The probabilities of moving from node $x_{i,j}$ at time t_i to one of the three nodes $x_{i+1,k-1}$, $x_{i+1,k}$ and $x_{i+1,k+1}$ in the next time step must now be calculated. Since $x_{i+1,k+1} = x_{i+1,k} + \Delta x_{i+1}$ and $x_{i+1,k-1} = x_{i+1,k} - \Delta x_{i+1}$ we can derive the constants p_u , p_m and p_d which sum up to one and satisfy

$$M_{i,j} = p_u(x_{i+1,k} + \Delta x_{i+1}) + p_mx_{i+1,k} + p_d(x_{i+1,k} - \Delta x_{i+1})$$

$$V_{i,j}^2 + M_{i,j}^2 = p_u(x_{i+1,k} + \Delta x_{i+1})^2 + p_mx_{i+1,k}^2 + p_d(x_{i+1,k} - \Delta x_{i+1})^2$$

Setting $\eta_{j,k} = M_{i,j} - x_{i+1,k}$ and rearranging the equations we obtain

$$p_u = \frac{V_{i,j}^2}{2\Delta x_{i+1}^2} + \frac{\eta_{i,j,k}^2}{2\Delta x_{i+1}^2} + \frac{\eta_{i,j,k}}{2\Delta x_{i+1}}$$

$$p_m = 1 - \frac{V_{i,j}^2}{\Delta x_{i+1}^2} - \frac{\eta_{i,j,k}^2}{\Delta x_{i+1}^2}$$

$$p_d = \frac{V_{i,j}^2}{2\Delta x_{i+1}^2} + \frac{\eta_{i,j,k}^2}{2\Delta x_{i+1}^2} - \frac{\eta_{i,j,k}}{2\Delta x_{i+1}}$$

To ensure that the probabilities remain positive we exploit the available degrees of freedom and make the variance $V_{i,j}^2$ dependent only on time and not on the

state j . In other words instead of writing $V_{i,j}^2$ we now write V_i^2 . This simplifies the probability equations

$$\begin{aligned} p_u &= \frac{1}{6} + \frac{\eta_{i,j,k}^2}{6V_i^2} + \frac{\eta_{i,j,k}}{2V_i\sqrt{3}} \\ p_m &= \frac{2}{3} - \frac{\eta_{i,j,k}^2}{3V_i} \\ p_d &= \frac{1}{6} + \frac{\eta_{i,j,k}^2}{6V_i^2} - \frac{\eta_{i,j,k}}{2V_i\sqrt{3}} \end{aligned}$$

The second stage in the construction of the interest rate tree involves displacing the nodes on the x -tree so that the initial term structure of interest rates is exactly matched. The displacement at time t_i is denoted by α_i and is common to all nodes at a certain time. Furthermore, we denote by $Q_{i,j}$ the present value of an instrument paying 1 if node (i, j) is reached and zero otherwise. First the current value of α is calculated from the correct discount factor for maturity t_1 , i.e.

$$\alpha_0 = \ln \left[-\ln \left(\frac{P^M(0, t_1)}{t_1} \right) \right]$$

When each α has been calculated, we can compute the values of $Q_{i+1,j}$ for all values of j .

$$Q_{i+1,j} = Q_{i,h} q(h, j) \exp(-\exp(\alpha - i + h\Delta x - i\Delta t_i))$$

where $q(h, j)$ is the probability of moving from node (i, h) to node $(i + 1, j)$. After calculating the value of $Q_{i,j}$ for each j , the value of α_i is calculated by solving numerically the following equation

$$\psi(\alpha_i) = P(0, t_{i+1}) - \sum_{j=\underline{j}_i}^{\overline{j}_i} Q_{i,j} \exp(-\exp(\alpha_i + j\Delta x_i) \Delta t_i) = 0$$

This equation can be solved by for example using the Newton-Raphson procedure, as the first derivative of ψ is known

$$\psi'(\alpha_i) = \sum_{j=\underline{j}_i}^{\overline{j}_i} Q_{i,j} \exp(-\exp(\alpha_i + j\Delta x_i) \Delta t_i) \exp(\alpha_i + j\Delta x_i) \Delta t_i$$

Finally we construct an interest rate tree where each node (i, j) has associated value $r_{i,j} = \exp(x_{i,j} + \alpha_i)$.

A.1.1 Bond and Option Pricing

To obtain bond prices from the trinomial interest rate tree, we work backwards from the date of maturity of the bond, where its value is known. In each node of the tree the price of the bond is calculated using the appropriate discounting factor and the probabilities that have already been computed. Denoting the price of a bond with maturity T at time t in node j where $t < T$ as $P_{i,j}(T)$ we get

$$P_{i,j}(T) = e^{-r_{i,j}(t_{i+1}-t)} [p_u P_{i+1,k+1}(T) + p_m P_{i+1,k}(T) + p_d P_{i+1,k-1}(T)]$$

This gives us the price at time t at node (i, j) for a bond with maturity at time T . By calculating the price at every node at each time point, we get a tree over prices where the price in each node corresponds to the short rate in the same node in the interest rate tree. Finally, the trinomial trees can also be used to calculate the price of a European swaption.

To obtain the zero coupon bond prices from the simulated interest rates, where the simulated value does not exactly match a value found at the relevant time point in the tree, we find the two nodes at time t_i in the interest rate tree which have the interest rates closest to the simulated interest rate. The bond price is then computed by interpolation of the bond prices in the two relevant nodes in the bond price tree.

A.2 Approximating the Stock Price with a Two-Dimensional Trinomial Tree

First, an approximating tree for r is constructed, using the procedure outlined in the previous section. Next, an approximating tree for \bar{S} is constructed. We start with the assumption that interest rates are constant and equal to \bar{r} .

At each time t_i , we have a finite number of equispaced states with a constant vertical step $\Delta\bar{S}_i$. We set $\bar{S}_{i,l} = l\Delta\bar{S}_i$, where

$$\Delta\bar{S}_{i+1} = \eta\sqrt{3\Delta t_i}$$

Assume that at time t_i we are on the l -th node with associated value $\bar{S}_{i,l}$. The process can now move to $\bar{S}_{i+1,h-1}$, $\bar{S}_{i+1,h}$ and $\bar{S}_{i+1,h+1}$ in the next time step t_{i+1} with probabilities q_u , q_m and q_d respectively. The central node is therefore the h -th node at time t_{i+1} where the level h is chosen as

$$h = \text{round} \left(\frac{\bar{S}_{i,l} + (\bar{r} - y - \frac{1}{2}\eta^2)\Delta t_i}{\Delta\bar{S}_{i+1}} \right)$$

The probabilities of moving from node $\bar{S}_{i,l}$ at time t_i to one of the three nodes $\bar{S}_{i+1,h-1}$, $\bar{S}_{i+1,h}$ and $\bar{S}_{i+1,h+1}$ in the next time step must now be calculated. Since $\bar{S}_{i+1,h+1} = \bar{S}_{i+1,h} + \Delta\bar{S}_{i+1}$ and $\bar{S}_{i+1,h-1} = \bar{S}_{i+1,h} - \Delta\bar{S}_{i+1}$ we can derive the constants q_u , q_m and q_d which sum up to one in the same way as for the approximating tree for r , and we get

$$\begin{cases} q_u(i,l) = \frac{1}{6} + \frac{\xi_{l,h}^2}{6\eta^2\Delta t_i} + \frac{\xi_{l,h}}{2\sqrt{3}\eta\sqrt{\Delta t_i}}, \\ q_m(i,l) = \frac{2}{3} - \frac{\xi_{l,h}^2}{3\eta^2\Delta t_i}, \\ q_d(i,l) = \frac{1}{6} + \frac{\xi_{l,h}^2}{6\eta^2\Delta t_i} - \frac{\xi_{l,h}}{2\sqrt{3}\eta\sqrt{\Delta t_i}}, \end{cases} \quad (\text{A.2.1})$$

where

$$\xi_{l,h} = \frac{\bar{S}_{i,l} + (\bar{r} - y - \frac{1}{2}\eta^2)\Delta t_i}{\Delta\bar{S}_{i+1}}$$

To get the approximating tree for S we use the relation $S(t) = S_0 \exp(\bar{S}(t))$ at every node.

However, in this approximating tree for S we have assumed that interest rates are constant equal to \bar{r} , and thus we have not taken the correlation between interest rates and the stock price into account. To adjust for correlation we need to create a two-dimensional tree. Since interest rates are no longer constant the marginal probabilities in (A.2.1) will vary when r varies in the two-dimensional tree.

We start by constructing a preliminary two-dimensional tree where the correlation is assumed to be zero. We denote the tree node at time t_i by (i, j, l) , and here $x = x_{i,j} = j\Delta x_i$ and $\bar{S} = \bar{S}_{i,l} = l\Delta\bar{S}_i$. We now have nine different nodes we can move to from each node, and each direction is associated with a probability:

- π_{uu} is the probability of moving from node (i, j, l) to node $(i+1, k+1, h+1)$;
- π_{um} is the probability of moving from node (i, j, l) to node $(i+1, k+1, h)$;
- π_{ud} is the probability of moving from node (i, j, l) to node $(i+1, k+1, h-1)$;
- π_{mu} is the probability of moving from node (i, j, l) to node $(i+1, k, h+1)$;
- π_{mm} is the probability of moving from node (i, j, l) to node $(i+1, k, h)$;
- π_{md} is the probability of moving from node (i, j, l) to node $(i+1, k, h-1)$;
- π_{du} is the probability of moving from node (i, j, l) to node $(i+1, k-1, h+1)$;

- π_{dm} is the probability of moving from node (i, j, l) to node $(i+1, k-1, h)$;
- π_{dd} is the probability of moving from node (i, j, l) to node $(i+1, k-1, h-1)$;

where k and h are chosen as before.

Since we have assumed zero correlation, the probabilities are the product of the corresponding marginal probabilities, with \bar{r} replaced by $\exp(x_{i,j} + \alpha_i)$ in (A.2.1).

Now we need to ensure that the probabilities also take the correlation into account. Define Π_0 as the matrix of these probabilities, i.e.,

$$\Pi_0 := \begin{pmatrix} \pi_{ud} & \pi_{um} & \pi_{uu} \\ \pi_{md} & \pi_{mm} & \pi_{mu} \\ \pi_{dd} & \pi_{dm} & \pi_{du} \end{pmatrix}.$$

If we shift each probability in Π_0 such that the following holds:

$$\begin{aligned} \Delta x_{i+1} \Delta \bar{S}_{i+1} (-\pi_{ud} - \varepsilon_{ud} + \pi_{uu} + \varepsilon_{uu} + \pi_{dd} + \varepsilon_{dd} - \pi_{du} - \varepsilon_{du}) \\ = Cov \{x(t_{i+1}), \bar{S}(t_{i+1}) | \mathcal{F}_{t_i}\} \\ = \sigma \eta \rho_{r,S} \Delta t_i, \end{aligned} \quad (\text{A.2.2})$$

at first order in Δt_i . ε_{ud} here denotes the shift for π_{ud} . There are no middle “ m ” terms here since such terms correspond to displacing at least one of the marginal probabilities by 0, giving no contribution to the covariance. The shift of the probabilities must also be done in such a way that the sum of the shifts in each row and each column is zero to maintain the marginal distributions.

From the construction of the interest rate tree, we know that

$$\Delta x_{i+1} = \sqrt{3V_i} = \sqrt{3 \frac{\sigma^2}{2a} (1 - e^{-2a\Delta t_i})} \approx \sigma \sqrt{3\Delta t_i}$$

and from the construction of the approximating tree for the stock price we have that

$$\Delta \bar{S}_{i+1} = \eta \sqrt{3\Delta t_i}.$$

With these two expressions equation (A.2.2) becomes, in the limit,

$$-\varepsilon_{ud} + \varepsilon_{uu} + \varepsilon_{du} - \varepsilon_{du} = \frac{\rho_{r,S}}{3}.$$

We can solve this under the constraint that the shift of the probabilities must also be done in such a way that the sum of the shifts in each row and each column is zero, and one possible solution proposed by Brigo and Mercurio [5], is

$$\begin{aligned}\Pi_{\rho_{r,S}} &= \Pi_0 + \rho_{r,S} (\Pi_1^l - \Pi_0^l), \text{ if } \rho_{r,S} > 0 \\ \Pi_{\rho_{r,S}} &= \Pi_0 - \rho_{r,S} (\Pi_{-1}^l - \Pi_0^l), \text{ if } \rho_{r,S} < 0\end{aligned}$$

where $\Pi_{\rho_{r,S}}$ is the probability matrix when the correlation $\rho_{r,S}$ between the interest rate and the stock price has been taken into account, and Π_1^l , Π_0^l and Π_{-1}^l are the probability matrices in the limit ($\Delta t_i = 0$) respectively for correlation 1, 0, and -1. This gives us

$$\Pi_{\rho_{r,S}} = \Pi_0 + \frac{\rho_{r,S}}{36} \begin{pmatrix} -1 & -4 & 5 \\ -4 & 8 & -4 \\ 5 & -4 & -1 \end{pmatrix} \text{ if } \rho_{r,S} > 0$$

and

$$\Pi_{\rho_{r,S}} = \Pi_0 - \frac{\rho_{r,S}}{36} \begin{pmatrix} 5 & -4 & -1 \\ -4 & 8 & -4 \\ -1 & -4 & 5 \end{pmatrix} \text{ if } \rho_{r,S} < 0.$$

Appendix B

Interest Rate Model Implied Volatility Surfaces

Calibrating the one-factor models to the swaption implied volatility surface with a humped shape at maturities of about 20 years (figure 3.4 on page 26), leads to mean reversion rates very close to zero, or even negative mean reversion rates. For the Hull-White model we get the parameter values seen in table B.1.

a	σ_R
0.00651	0.00794

Table B.1: Hull-White parameter values calibrated to a swaption implied volatility surface with a hump at maturities of about 20 years.

This in turn leads to scenarios where the mean of the short rate scenarios increases quickly over time instead of converging to a stable long-term value (see figure B.1). As can be seen in the figure, a mean reversion rate very close to zero produces unrealistic scenarios.

The model implied volatility surface compared to the market implied volatility surface for swaptions with tenor five years can be seen in figure B.2. We clearly see that even if we would accept the very low mean reversion rate, the Hull-White model cannot capture this type of swaption volatility curve.

Let's look at what type of volatility curves the Hull-White model and the Black-Karasinski model can imply. We calibrate the implied volatility surface for a swaption with a five year tenor. For Hull-White, we first vary a between 0.03 and 0.33 with time step 0.03 and keep σ fixed. Next, we vary σ between 0.002 and 0.022 with time step 0.002 and keep a fixed. For Black-Karasinski, we first vary a between 0.03 and 0.53 with time step 0.05 and keep σ fixed. Next, we vary σ between 0.05 and 0.55 with time step 0.05 and keep a fixed. In figure

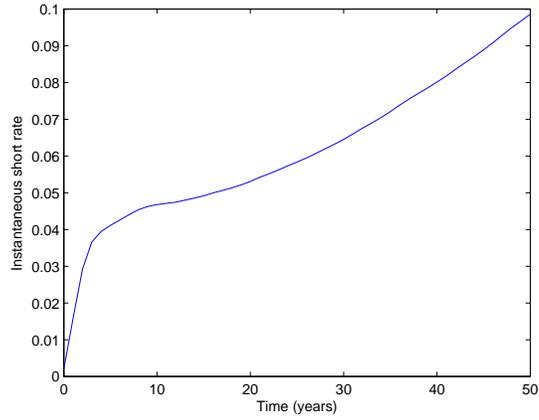


Figure B.1: Mean of the Hull-White short rate scenarios, with $a = 0.00651$ and $\sigma = 0.00794$.

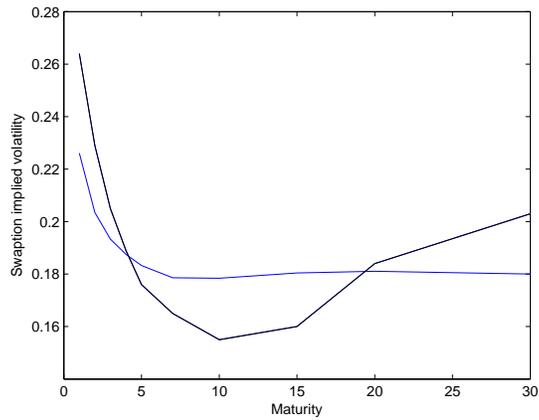


Figure B.2: Model implied volatility surface for Hull-White (blue) compared with market implied volatility surface (black) for swaptions with an underlying five year swap when $a = 0.00651$ and $\sigma = 0.00794$.

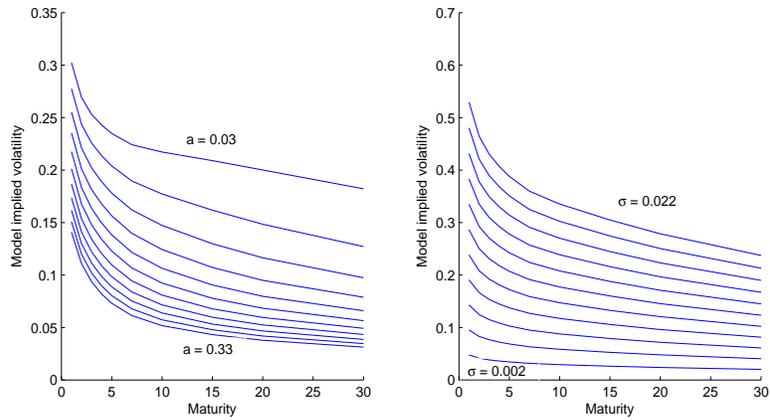


Figure B.3: Fixed-tenor volatility curves implied by the Hull-White model. In the left plot, a varies ceteris paribus from 0.03 to 0.33 with time step 0.03. In the right plot, σ varies ceteris paribus from 0.002 to 0.022 with time step 0.02.

B.3 we see the Hull-White fixed tenor volatility curves and in figure B.4 we see the same curves for Black-Karasinski. Clearly, none of the implied volatility curves looks anything like the market implied volatility curve in figure B.2, which means the models seem to be unable to capture this type of volatility curve for any reasonable values of a and σ .

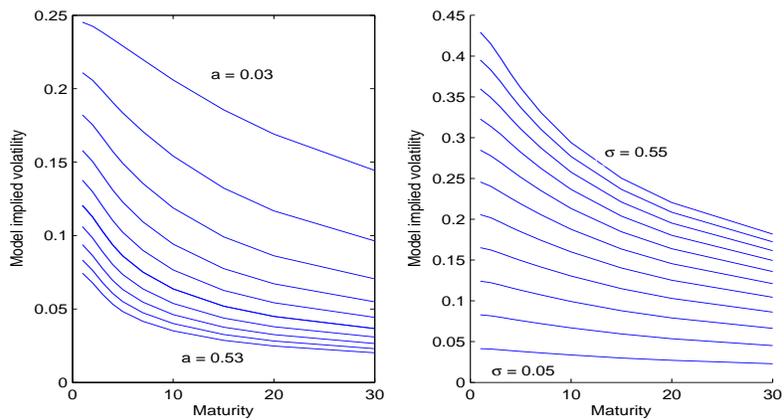


Figure B.4: Fixed-tenor volatility curves implied by the Black-Karasinski model. In the left plot, a varies ceteris paribus from 0.03 to 0.53 with time step 0.05. In the right plot, σ varies ceteris paribus from 0.05 to 0.55 with time step 0.05.

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